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THREE PAPERS ON THE WIRE
ABOVE EARTH PROBLEM

translated with corrections from the Russian by

Edward F. Kuester

Department of Electrical Engineering
University of Colorado

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TABLE OF CONTENTS

FOREWORD

Ever since the appearance of Carson's classic paper [1] in 1926,
much attention has been devoted to the theory of waves propagated along
a thin wire parallel to the earth's surface. While the original interest
arose from the problem of overhead power lines, telecommunications problems
such as signal propagation along wires, behavior of antennas of finite
radius above the earth's surface, and even propagation of transients along
power lines, called attention to the need for a theory more accuracte
than those available at the time. Although quite a few papers have appeared
on the subject within the last 20 years (see the bibliography in [2]), the
first two papers of this collection seem to be the first serious attempts
in this direc ion, and for a long time have remained apparently unknown
to researchers in the West. It was therefore considered appropriate to
produce these translations as a technical report in order to make them more
readily available to others working in this field, Also included is a
paper by L. S. Perel'man generalizing the theory to an arbitrary N-wire
system. In common with the other papers, this one deals with the question
of approximate evaluation of the Sommerfeld integrals which arise in the
course of the investigation.

[1] J. R. Carson, "Wave propagation in overhead wires with ground return,"
     Bell Syst. Techn. J. v. 5, pp. 539-554 (1926).

[2] E. F. Kuester and D. C. Chang, "Modal representation of a horizontal
     wire above a finitely conducting earth," Sci. Rept. No. 21 (RADC-
     TR-76-287), Dept. of Elec. Eng., Univ. of Colorado, Boulder, CO (1976).

                              -Edward F. Kuester

A.A. Pistol'kors
corresponding member of AN SSSR

On the theory of a wire parallel to the plane interface between two media*

In this paper an equation is derived permitting determination

of the propagation constants of cylindrical electromagnetic waves

along a thin wire parallel to the plane interface between two media,

and it is shown that, if both media are lossless dielectrics,

cylindrical waves can take place only in the case of a wire located

in the medium of higher dielectric permittivity.

I.   Introduction

The problem of a wire parallel to the plane interface between two

media has usually been solved in the context of a wire over the earth

[1-7]. The solution is carried out by an approximate method, substituting

for the wire an equivalent transmission line, whose parameters are given

in terms of a total impedance per unit length and a total conductance of

the wire with the earth's influence taken into account. To calculate

this influence the current is assumed to be uniform along the length of

the wire and either synphasal or propagating with the speed of light,

and the charge uniform along its length and static. Obviously, neither

of these assumptions corresponds to reality, but they do permit one to

determine approximately the propagation constants of electromagnetic

waves along the wire.

One of the more rigorous methods of solving the problem consists of finding an equation which accounts for the effect of the plane interface right from the start, and thus permits one to determine the exact value of the propagation constant. This method, considered in the present paper, leads to a rather complex equation, which we use for some inferences of a general nature involving cases when both media are lossless dielectrics.

2. The electromagnetic field of a wire parallel to the plane
    interface between two media

We shall take a cylindrical coordinate system and orient the axis of the wire along the z-axis (Fig. 1). We will take the surface of the wire, i.e., the surface of a circular cylinder of radius a, to be perfectly conducting. We set up a means of excitation of oscillations in the wire as follows: let us locate in the plane z = 0 a perfectly conducting screen of infinite extent in which we cut a narrow annular slot of radius ρ concentric with the wire. Between the edges of the slot we will maintain an everywhere synphasal voltage U.

Such a means of exciting oscillations is similar to the excitation of a wire by means of a magnetic ring current considered by Noether [8]. Using and rearranging somewhat the quoted formulas, we obtain an expression for the z component of the electric field induced by the annular slot:

$$E_{z1} = \frac{U\rho}{\pi} \int_{-\infty}^{\infty} \beta I_0(\beta r) K_1(\beta\rho) e^{-i\alpha z} d\alpha \qquad r < \rho \qquad (1)$$

$$E_{z1} = -\frac{U\rho}{\pi} \int_{-\infty}^{\infty} I_1(\beta\rho) K_0(\beta r) e^{-i\alpha z} d\alpha \qquad r < \rho \qquad (2)$$

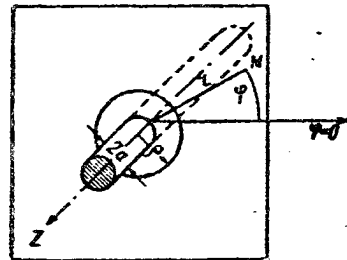
Fig. 1

Here $\beta = \sqrt{\alpha^2 - k^2}$, where $k = 2\pi/\lambda$ is the wavenumber in the medium surrounding the screen. $I_0(x)$, $I_1(x)$, $K_0(x)$ and $K_1(x)$ are Bessel functions of imaginary argument.

The secondary field induced by currents on the surface of the wire, is represented in the form

$$E_{z2} = \frac{U\rho}{\pi} \int\limits_{-\infty}^{\infty} P(\alpha) K_0(\beta r) e^{-i\alpha z} d\alpha \qquad r \geq a. \qquad (3)$$

We now assume that the wire, whose diameter is small compared to a wavelength, is situated parellel to a plane interface between two media at a distance h from it (Fig. 2). In this case there will be, in the half-space containing the wire, an additional field which we will denote as "reflected" from the interface and thus, as will be shown below, this field can be represented as the superposition of an infinite number of plane waves reflected from the plane interface. Therefore in order to calculate it, it is convenient to use cartesian coordinates, oriented as shown in Fig. 2.

In the exterior region $(r > \rho)$ the longitudinal electric field E_z has 4 components:

Fig. 2

1) a component of the source oscillations

$$E_1 = \frac{U\rho}{\pi} \int\limits_{-\infty}^{\infty} \beta \; I_1(\beta\rho) K_0(\beta r) e^{-i\alpha z} d\alpha; \qquad (4)$$

2) a reflected source field

$$E_2 = -\frac{U\rho}{\pi} \int\limits_{-\infty}^{\infty} \beta I_1(\beta\rho) R(\alpha,x,y) e^{-i\alpha z} \; d\alpha; \qquad (5)$$

3) a component induced by the current on the wire

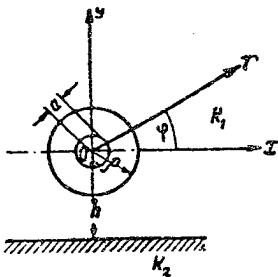$$E_3 = \frac{U\rho}{\pi} \int\limits_{-\infty}^{\infty} P(\alpha) K_0(\beta r) e^{-i\alpha z} \; d\alpha; \qquad (6)$$

and 4) a reflected wire field

$$E_4 = \frac{U\rho}{\pi} \int_{-\infty}^{\infty} P(\alpha)R(\alpha,x,y)e^{-i\alpha z}d\alpha. \tag{7}$$

Here $R(\alpha,x,y)e^{-i\alpha z}$ is the reflected field corresponding to an elementary cylindrical wave $K_0(\beta r)e^{-i\alpha z}$. The function $\rho(\alpha)$ must be found from the condition that the resulting field strength E_z vanish at the surface of the wire. However, for $a\leq r<\rho$

$$E_1 = \frac{U\rho}{\pi} \int_{-\infty}^{\infty} \beta I_0(\beta r)K_1(\beta\rho)e^{-i\alpha z}d\alpha. \tag{8}$$

For a small wire diameter one can write the equality

$$\beta I_0(\beta a)K_1(\beta\rho) - \beta R I_1(\beta\rho) + P(\alpha)K_0(\beta a) + RP(\alpha) = 0. \tag{9}$$

Here $R = R(\alpha,a,0)$ is the strength of the reflected field at the wire surface for $y=0$, $x=a$.

Solving equation (9), we obtain

$$P(\alpha) = \beta\frac{RI_1(\beta\rho)-I_0(\beta a)K_1(\beta\rho)}{R+K_0(\beta a)} \tag{10}$$

Adding all the components of E_z and substituting the value found for $P(\alpha)$, we obtain:

$$E_z = -\frac{U\rho}{\pi} \int_{-\infty}^{\infty} \beta[K_0(\beta a)I_1(\beta\rho) + I_0(\beta a)K_1(\beta\rho)]\frac{K_0(\beta r)+R(\alpha,x,y)}{K_0(\beta a)+R} e^{-i\alpha z}d\alpha. \tag{11}$$

Letting the radius ρ of the slot tend to the radius a of the wire and using the relation $K_0(\beta a)I_1(\beta a)+I_0(\beta a)K_1(\beta a) = 1/\beta a$, we obtain:
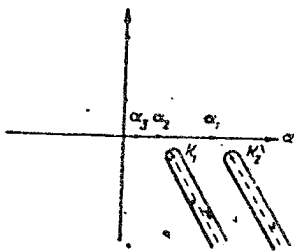

$$E_z = -\frac{U}{\pi} \int_{\infty}^{\infty} \frac{K_0(\beta r)+R(\alpha,x,y)}{K_0(\beta a)+R} e^{-i\alpha z}d\alpha. \tag{12}$$

Fig. 3

Usually one can evaluate integrals of a similar kind by integrating in the complex plane of the variable α (Fig. 3). Here it is necessary to account for two branch cuts determinable from the branch points $\alpha=k_1$ and $\alpha=k_2$, where k_1 and k_2 are the wavenumbers of the first and second media. The poles of the integrand (the points α_1, α_2, α_3...) are of practical interest inasmuch as only in their presence will there exist cylindrical waves providing a transfer of energy down the wire. The values of α corresponding to the poles are determined by the equation:

$$K_0(\beta a) + R = K_0 (a\sqrt{\alpha^2-k_1^2}) + R(\alpha,a,0) = 0; \qquad (13)$$

whereas, as far as the branch cut integrals are concerned, these give an additional field whose amplitude usually decays quickly along the wire. We will now derive the formula for the elementary reflected field $R(\alpha,x,y)$.

3. Derivation of the expression for the reflected field

As before, we will assume that the diameter of the wire is small (compared to a wavelength) and that the wire is located sufficiently far from the interface ($a \ll h$). Under these conditions we can neglect the proximity effect and the higher order waves, and represent the primary longitudinal electric field E_z of the wire in the form of a cylindrical function of zeroth order, for example, $K_0(r\sqrt{\alpha^2-k_1^2})e^{-i\alpha z}$. For a primary field distribution given in this manner we will seek the corresponding secondary or "reflected" field.

We use an expression representing the function $K_0(r\sqrt{\alpha^2-k_1^2})e^{-i\alpha z}$ as the superposition of an infinite number of plane waves[1]

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[1] Cf., for example, the book of G.A. Grinberg [9]. A related expression was used by F. Pollaczek [2] to calculate the coefficients of mutual induction of a wire in the presence of the earth; in the latter case α was taken as given, assuming $\alpha=k_1$. An analogous expression for the case $\alpha=0$ was also given by M.I. Kontorovich [6].

$$K_0(r\sqrt{\alpha^2-k_1^2})e^{-i\alpha z} = -\frac{i}{2}\int_{-\infty}^{\infty}\frac{e^{-i|y|\sqrt{k_1^2-\alpha^2-m^2}-imx-i\alpha z}}{\sqrt{k_1^2-\alpha^2-m^2}}\,dm. \tag{14}$$

Each elementary plane wave is characterized by the angles u, v, and γ, which generate its propagation direction with respect to the x,y, and z axes. Here

$$k_1 \cos u = m,$$

$$k_1 \cos v = +\sqrt{k_1^2-\alpha^2-m^2},$$

$$k_1 \cos \gamma = \alpha.$$

Real values of the angles u, v and γ will occur when

$$k_1^2 > \alpha^2 + m^2. \tag{15}$$

If y is replaced by its modulus, expression (14) is valid for all points in the xy-plane.

The corresponding expression for E_x can be found by realizing that

$$E_x = E_r \cos \phi = -\frac{i\alpha}{\sqrt{\alpha^2-k_1^2}}\,K_1(r\sqrt{\alpha^2-k_1^2})\,\frac{x}{r}\,e^{-i\alpha z}. \tag{16}$$

This expression can also as represented as a sum of plane waves

$$E_x = \frac{i\alpha}{2(k_1^2-\alpha^2)}\int_{-\infty}^{\infty}\frac{e^{-i|y|\sqrt{k_1^2-\alpha^2-m^2}-imx-i\alpha z}}{\sqrt{k_1^2-\alpha^2-m^2}}\,m\,dm. \tag{17}$$

In a quite similar manner we find

$$E_y = \frac{i\alpha}{2(k_1^2-\alpha^2)}\int_{-\infty}^{\infty}e^{-i|y|\sqrt{k_1^2-\alpha^2-m^2}-imx-i\alpha z}\,dm. \tag{18}$$

It is not difficult to convince oneself of the validity of the equation

$$\text{div } \overline{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

for each elementary wave.

As far as the magnetic field is concerned, $H_z = 0$, and of the two remaining components we need H_x, for which we find

$$H_x = -\frac{i}{2\omega\mu} \frac{k_1^2}{k_1^2 - \alpha^2} \int_{-\infty}^{\infty} e^{-i|y|\sqrt{k_1^2 - \alpha^2 - m^2} - imx - i\alpha z} \, dm \qquad (19)$$

Having expanded the field induced by the wire into a sum of plane waves, one can solve the problem of reflection from the plane interface for each elementary incident wave. Then summing the elementary reflected plane waves, we obtain the secondary field, induced by the charges and currents of the second medium, due to the current of the wire. Now to shorten the calculations one can apply the ready-made solution to the plane wave reflection problem to a separate elementary wave and insert the Fesenel coefficients. The solution obtained in this manner has this intuitive meaning only for real values of the angles u, v and γ, but as is well-known, it is also valid for imaginary values of the angles.

In order to use the Fresnel coefficients, it is necessary to expand the incident wave into perpendicularly and non-perpendicularly polarized components and for each of them find the reflected field.

The mutual arrangement of the wire and the plane interface is shown in Fig. 2. The plane of incidence of the wave is perpendicular to the xz-plane, its equation is

$$x \cos \gamma - z \cos u = 0. \qquad (20)$$
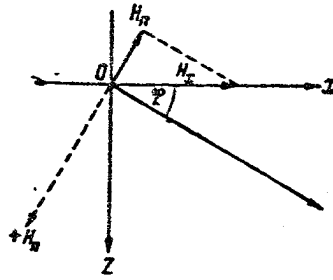
Fig. 4

In Fig. 4, a top view of the plane of incidence of the wave at the plane interface (y=-h) is depicted, along with the angle ϕ with respect to the x axis*. For perpendicular polarization the electric vector lies in the plane of incidence, and the magnetic vector H_n is normal to it and directed as shown in Fig. 4 (+H_n). From this figure and expression (20) we find

$$\frac{z}{x} = \tan \phi = \frac{\cos \gamma}{\cos u} \; ,$$

whence

$$\sin \phi = \frac{\cos \gamma}{\sqrt{\cos^2 u + \cos^2 \gamma}} = \frac{\alpha}{\sqrt{m^2 + \alpha^2}}$$

and for each elementary wave

$$H_n^{el} = - H_x^{el} \sin \phi = - H_x^{el} \frac{\alpha}{\sqrt{m^2 + \alpha^2}} .$$

Using expression (19) we obtain

$$H_n = \frac{i}{2\omega\mu} \frac{k_1^2}{k_1^2 - \alpha^2} \int_{-\infty}^{\infty} \frac{\alpha}{\sqrt{m^2 + \alpha^2}} \; e^{-i|y|\sqrt{k_1^2 - \alpha^2 - m^2} - imx - i\alpha z} \; d\alpha \tag{21}$$

For the reflected field H_n^{refl} we obtain

$$H_n^{rfl} = \frac{i}{2\omega\mu} \frac{k_1^2}{k_1^2 - \alpha^2} \int_{-\infty}^{\infty} \frac{\alpha R_1}{\sqrt{m^2 + \alpha^2}} \; e^{-i(2h+y)\sqrt{k_1^2 - \alpha^2 - m^2} - imx - i\alpha z} \; d\alpha \tag{22}$$

*ϕ is apparently not the same angle as shown in Fig. 2 (Transl.).

where y>-h and the Fresnel coefficient is

$$R_1 = \frac{k_2 \cos v_1 - k_1 \cos v_2}{k_2 \cos v_1 + k_1 \cos v_2} = \frac{k_2^2 \sqrt{k_1^2 - \alpha^2 - m^2} - k_1^2 \sqrt{k_2^2 - \alpha^2 - m^2}}{k_2^2 \sqrt{k_1^2 - \alpha^2 - m^2} + k_1^2 \sqrt{k_2^2 - \alpha^2 - m^2}} \quad . \tag{23}$$

Since $E^{refl} = rot_z \overline{H}^{refl}/i\omega\varepsilon = -(1/i\omega\varepsilon)\partial H_x^{refl}/\partial y$ and $H_x^{refl} = -H_n^{refl} \sin \phi = -H_n^{refl}(\alpha/\sqrt{m^2+\alpha^2})$ (for the elementary waves), then the perpendicularly polarized component E_{z1} of the reflected field E_z is equal to

$$E_{z1}^{rfl} = \frac{i\alpha^2}{2(k_1^2-\alpha^2)} \int_{-\infty}^{\infty} \frac{\sqrt{k_1^2-\alpha^2-m^2}}{\alpha^2+m^2} R_1 e^{-i(2h+y)\sqrt{k_1^2-\alpha^2-m^2}} d\alpha. \tag{24}$$

To verify the expression obtained we consider an elementary wave normally incident on the plane interface: for it, $m=k_1 \cos u = 0$ and $\alpha = k_1 \cos \gamma = 0$

Using formulas (19) and (24) we obtain:

$$H_x^{el} = -\frac{i}{2\omega\mu} e^{-jk_1|y|}$$

and

$$H_x^{refl,el} = -\frac{i}{2\omega\mu} R_1 e^{-ik_1(y+2h)} = \frac{-i}{2\omega\mu} \frac{k_2-k_1}{k_2+k_1} e^{-ik_1(y+2h)} \quad ,$$

i.e., the correct result was obtained, in particular for $k_2 \to \infty$, the reflected magnetic field near the interface interferes constructively with the incident field as must be the case.

To calculate the component E_a corresponding to a perpendicularly incident wave, we turn to Fig. 5. The positive direction of the electric field vector lies here in the region of positive z^2, Thus for an elementary

[2] Cf., for wxample, Stratton [10], p. 493, fig. 90.

wave

$$E_a = E_{21} + E_{22} = -E_x \sin\phi + E_z \cos\phi = -E_x \frac{\alpha}{\sqrt{m^2+\alpha^2}} + E_z \frac{m}{\sqrt{m^2+\alpha^2}}$$

and

$$E_\alpha^{refl} = \frac{-ik_1^2}{2(k_1^2-\alpha^2)} \int_{-\infty}^{\infty} \frac{m}{\sqrt{m^2+\alpha^2}} \frac{R_2 e^{-i(2h+y)\sqrt{k_1^2-m^2+\alpha^2}-imx-i\alpha z}}{\sqrt{k^2-\alpha^2-m^2}} \, dm, \quad (y>-h)$$

where

$$R_2 = \frac{k_1 \cos\nu_1 - k_2 \cos\nu_2}{k_1 \cos\nu_1 - k_2 \cos\nu_2} = \frac{\sqrt{k_1^2-\alpha^2-m^2} - \sqrt{k_2^2-\alpha^2-m^2}}{\sqrt{k_1^2-\alpha^2-m^2} + \sqrt{k_2^2-\alpha^2-m^2}}. \tag{25}$$

But for an elementary wave $E_{az}^{refl} = E_a^{refl} \cos\phi = E_a^{refl} \frac{m}{\sqrt{m^2+\alpha^2}}$.

Thus the total component E_z^{refl} of the reflected field will be

$$E_z^{refl} = R(\alpha,x,y)e^{-i\alpha z} = \frac{i\alpha^2}{2(k_1^2-\alpha^2)} \int_{-\infty}^{\infty} \frac{dm}{\alpha^2+m^2} \left(\alpha^2 R_1 \sqrt{k_1^2-\alpha^2-m^2} - \right.$$

$$\left. - \frac{k_1^2 m^2 R_2}{\sqrt{k^2-\alpha^2-m^2}} \right) e^{-i(2h+y)\sqrt{k_1^2-\alpha^2-m^2}-imx-i\alpha z} \tag{26}$$

where R_1 and R_2 are defined by expressions (23) and (25). This solution is valid for arbitrary heights of suspension of the wire (including those large compared to a wavelength), subject only to the conditions $a \ll h$ and $a \ll \lambda$.

At the surface of the wire the strength of the reflected field will be considered everywhere identical and equal to its value at the point $x=a$, $y=0$; thus

$$R = \frac{i}{2(k_1^2-\alpha^2)} \int_{-\infty}^{\infty} \frac{dm}{\alpha^2+m^2} \left(\alpha^2 R_1 \sqrt{k_1^2-\alpha^2-m^2} - \frac{k_1^2 m^2 R_2}{\sqrt{k_1^2-\alpha^2-m^2}} \right) e^{-i2h\sqrt{k_1^2-\alpha^2-m^2}-ima}. \tag{27}$$
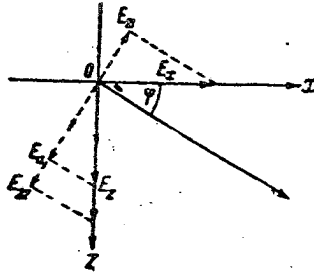
Fig. 5

Substituting this expression into eqn. (13), we obtain:

$$\frac{i}{2(k_1^2-\alpha^2)} \int\limits_{-\infty}^{\infty} \frac{dm}{\alpha^2+m^2} \, (\alpha^2 R_1 \sqrt{k_1^2-\alpha^2-m^2} - \frac{k_1^2 m^2 R_2}{\sqrt{k_1^2-\alpha^2-m^2}}) e^{-i2h\sqrt{k_1^2-\alpha^2-m^2}-im}$$

$$= K_0(\alpha\sqrt{\alpha^2-k_1^2}). \tag{28}$$

Upon solving this equation for α, we find the propagation constants of cylindrical waves which have circular symmetry.

4. Two special cases

In view of the complexity of eqn. (28), the treatment of methods of its solution is a problem in its own right. However there are special cases in which the solution is simplified. We will consider two such cases.

a) The case of weak influence of the boundary

This will occur for sufficiently large distances between the wire and the plane interface, or for small differences in the values k_1 and k_2. In this case α must be near k_1 and the quantity $a\sqrt{\alpha^2-k_1^2}$ must be small.

Taking $K_0(a\sqrt{\alpha^2-k_1^2}) = -\ln \frac{\gamma a}{2}\sqrt{\alpha^2-k_1^2}$, where $\gamma = 1.781...$, we cast eqn. (28) in the form

$$-ia^2 J/2 = v^2 \ln \frac{\gamma v}{2} \tag{29}$$

Here   J   represents the integral on the left-hand side of eqn. (28) and $v = a\sqrt{\alpha^2-k_1^2}$.*   The equation obtained is related to the Sommerfeld equation for a cylindrical wire of finite conductivity.   In the absence of reflection it goes over into the equation for a wire in free space.[3]

Equation (19) can be put into the form

$$\xi \ln \xi = \eta,$$

where ξ is proportional to v, and η involves the integral on the left-hand side of eqn. (28).   When k_1h is large, η will be small, and for sufficiently different values of k_1 and k_2 will depend only weakly on α. To solve eqn. (30) in this case, the method of successive approximations can be applied, analogous to Sommerfeld's method of continued fractions. If the nth approximation is known, then the $(n+1)^{st}$ can be determined from the equation:

$$\xi_{n+1} \ln \xi_n = \eta_n$$

An approximate value ξ_0 must be given; the integral in η_n has to be evaluated graphically.   In many cases η_n can be reckoned a constant, by substituting therein the value of the integral   J   for $\alpha = k_1$.

b)   The case when one of the media is a good conductor

Let for instance, $|k_2| \gg |k_1|$ and $k_2^2 \approx -i\mu\sigma\omega$.   Separating out unity from R_1 and minus one[4] from R_2, the integral on the left hand side of equ. (28) is separated in two.   The first of these is equal to $K_0(2h\sqrt{\alpha^2-k_1^2})$, since, substituting $R_1=1$ and $R_2=-1$ we obtain:

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*Not the angle v considered earlier (Transl.).

[3]Cf., for example, eqns. (21) and (23) on p. 528 of Stratton [10].

[4]This method was proposed by G.A. Lavrov.

$$\frac{i}{2(\alpha^2-k_1^2)} \int\limits_{-\infty}^{\infty} \frac{dm}{\alpha^2+m^2} \frac{\alpha^2(k_1^2-\alpha^2-m^2)+k_1^2 m^2}{\sqrt{k_1^2-\alpha^2-m^2}} e^{-i2h\sqrt{k_1^2-\alpha^2-m^2}-im\alpha} =$$

$$= -\frac{i}{2} \int\limits_{-\infty}^{\infty} \frac{e^{-i2h\sqrt{k_1^2-\alpha^2-m^2}-im\alpha}}{\sqrt{k_1^2-\alpha^2-m^2}} dm = K_0(\sqrt{(4h^2+a^2)(\alpha^2-k_1^2)}) \approx$$

$$\approx K_0(2h\sqrt{\alpha^2-k_1^2}).$$

The second integral for the given values of k_2 goes over into

$$J_2 = -\frac{\sqrt{-i}}{\sqrt{\mu\sigma\omega}} \frac{2k_1^2}{\alpha^2-k_1^2} \int\limits_{0}^{\infty} \cos ma\, e^{-i2h\sqrt{k_1^2-\alpha^2-m^2}} dm.$$

Putting $\cos ma = 1$ (for a small radius of the wire) and neglecting $k_1^2-\alpha^2$ compared to m^2 (for small heights h), we obtain:

$$J_2 = -\frac{\sqrt{-i}}{\sqrt{\mu\sigma\omega}} \frac{1}{h} \frac{k_1^2}{\alpha^2-k_1^2},$$

whence

$$\frac{1-i}{\sqrt{2\mu\sigma\omega}} \frac{k_1^2}{h} = [K_0(2h\sqrt{\alpha^2-k_1^2}) - K_0(a\sqrt{\alpha^2-k_1^2})](\alpha^2-k_1^2). \tag{31}$$

For good conductivity of the second medium α is close to k_1 and $a\sqrt{\alpha^2-k_1^2}$ is a small quantity. For small suspension height the argument $2h\sqrt{\alpha^2-k_1^2}$ can likewise be taken to be a small quantity. Therefore eqn. (31), as in the preceding case, can be cast in the form $\xi \ln\xi = \eta$. Here η does not depend on α and to solve the equation one can apply Sommerfeld's method of continued fractions.

5. On a wire near the interface between two lossless dielectrics

When both media are lossless dielectrics, eqn. (28) can be used for some inferences of a general nature.

Since neither medium introduces damping, k_1 and k_2 are real and the α being sought must also be real. But for real α the left side of expression (28) will be complex, while the right side will be so only under the condition $\alpha < k_1$. If $k_2 < k_1$, this condition is natural, since it means that in the presence of the interface, the propagation velocity increases under the influence of the less dense medium. Hence it follows that in the case of a wire located in the denser medium (having larger ε), cylindrical waves can occur. If however $k_1 < k_2$, i.e., the wire is located in the medium with smaller ε, the existence of cylindrical waves is impossible.

In fact, under the influence of the neighboring dielectric $\alpha \neq k_1$ and α must be either larger than k_1 (propagation velocity smaller than in the medium surrounding the wire) or smaller than k_1 ($\alpha < k_1 < k_2$, or a phase velocity larger than that of either medium). The first, most natural, assumption is eliminated because in this case the right side of eqn (28) will be real. At the same time (as shown in the appendix) the imaginary part of the left side of the equation will, under the given conditions, be different from zero. The second case corresponds to a waveguide kind of propagation and must be excluded in view of the absence of suitable physical hypotheses for this in the problem being considered.*

Thus, in the given case ($k_1 < k_2$) the variation (decrease of the electromagnetic field along the wire bears a specific character and is determined by the branch cut integrals. Only in the presence of loss in the dielectric are cylindrical waves possible, which clearly occurs for a wire above a real earth.

An analogous route is taken for the propagation of electromagnetic waves along a dielectric cylinder.

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*Apparently, this means the absence of confining walls (Transl.).

In this case the propagation constants of the electromagnetic waves are determined from the equation of Hondros and Debye [11], which has the form:

$$\frac{\xi}{\varepsilon} \frac{H_0^{(2)}(\xi)}{H_1^{(2)}(\xi)} = \frac{\eta}{\varepsilon_1} \frac{J_0(\eta)}{J_1(\eta)} , \tag{32}$$

where

$$\xi = a\sqrt{k_2^2 - a^2} ; \quad \eta = a\sqrt{k_1^2 - \alpha^2}, \quad a \text{ is the radius of the cylinder}$$

k_1 is the wavenumber of the cylinder material,

k_2 is the wavenumber of the exterior dielectric.

For $k_1 > k_2$, when of the cylinder is larger than the ε of the surrounding medium, this equation, as is well known [12], can have only real roots. In this case the right side of equation (32) is real, and the left as well, since $\alpha > k_2$, ξ is imaginary and the Hankel functions go over into the purely real Macdonald functions, multiplied by $(-i)^{p+1}\pi/2$, where p is the order of the function.

When the permittivity of the cylinder is smaller than that of the medium, i.e., $k_1 < k_2$, the following inequalities for the propagation constant α are possible:

$$k_2 > \alpha > k_1; \quad k_2 > k_1 > \alpha; \quad \alpha > k_2 > k_1 .$$

In the first two cases ξ is real and η is imaginary or real. Here the right side of eqn. (32) is always real, and the left complex. Consequently, if $\alpha < k_2$, the equation has no roots. If $\alpha > k_2$ (the 3rd case) both sides of the equation are real, but the left is always smaller than the right.

Thus, for $k_1 < k_2$, the equation of Hondros and Debye has no solutions and cylindrical waves along a cylindrical cavity in a dielectric are impossible (even for $\varepsilon_2 \to \infty$).

It can also be shown that in a space with smaller ε between two planar boundaries with two dielectrics (Fig. 6), the propagation of planar waves as well as of radial cylindrical waves is impossible.
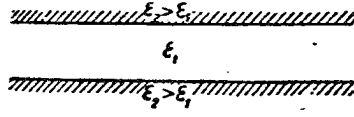
Fig. 6

Thus, from the point of view of the possibility of cylindrical waves arising, the cases when the wire is located near an interface with a dielectric of larger or smaller permittivity (than that of the medium surrounding the wire) are fundamentally different. If energy transfer using cylindrical waves is possible in the first case, then in the second case such transfer turns out to be impossible. Consequently, approximate methods of solution, starting from the hypothesis that cylindrical waves exist near a wire parallel to the plane interface between two media, cannot be used in all cases.

Appendix

For $k_1 < \alpha < k_2$, $\sqrt{k_1^2 - \alpha^2 - m^2} = i\sqrt{\alpha^2 + m^2 - k_1^2}$ and the left side of eqn.(28) will have the form:

$$\frac{1}{\alpha^2 - k_1^2} \int_0^\infty \frac{dm}{\alpha^2 + m^2} \; \alpha^2 R_1 \sqrt{\alpha^2 + m^2 - k_1^2} + \frac{k_1^2 m^2 R_2}{\sqrt{\alpha^2 + m^2 - k_1^2}} \; e^{-2h\sqrt{\alpha^2 + m^2 - k_1^2}} \cos ma.$$

The integrand will be complex in the interval $0 \leq m \leq \sqrt{k_2^2 - \alpha^2}$ since in this interval R_1 and R_2 will be complex.

We split off the imaginary parts of R_1 and R_2:

$$\text{Im } R_1 = \text{Im } \frac{ik_2^2\sqrt{\alpha^2+m^2-k_1^2} + k_1^2\sqrt{k_2^2-\alpha^2-m^2}}{ik_2^2\sqrt{\alpha^2+m^2-k_1^2} - k_1^2\sqrt{k_2^2-\alpha^2-m^2}} = \frac{-2k_1^2k_2^2\sqrt{\alpha^2+m^2-k_1^2}\,\sqrt{k_2^2-\alpha^2-m^2}}{k_2^4(\alpha^2+m^2-k_1^2)+k_1^4(k_2^2-\alpha^2-m^2)} \; ;$$

$$\text{Im } R_2 = \text{Im } \frac{i\sqrt{\alpha^2+m^2-k_1^2} +\sqrt{k_2^2-\alpha^2-m^2}}{i\sqrt{\alpha^2+m^2-k_1^2} -\sqrt{k_2^2-\alpha^2-m^2}} = - \frac{-2\sqrt{\alpha^2+m^2-k_1^2}\sqrt{k_1^2-\alpha^2-m^2}}{k_2^2-k_1^2} \; .$$

In both cases the signs of the numerator and denominator do not change over the entire interval $0 \leq m \leq \sqrt{k_2^2-\alpha^2}$ and the imaginary part of the integral will be different from zero and negative, at least for $ma < \pi/2$ or $a\sqrt{k_2^2-\alpha^2} < \pi/2$, or approximately, $k_2 a < \pi/2$, i.e., for smaller wire diameters than a half wavelength in the second medium, which agrees with the assumptions made in the paper and values of the ratio k_2/k_1 realizable in practice.

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G.A. Grinberg and B.E. Bonshtedt

FOUNDATIONS OF AN EXACT THEORY OF TRANSMISSION LINE FIELDS*

In this paper, starting from a rigorous formulation of the problem, the propagation of electromagnetic waves along a single wire located above a plane homogeneous earth is determined, and a method of approach is indicated for the solution of the analogous problem for a multiconductor line which accounts for the "proximity effect" on the conductors.

Virtually exact integral representations for the electromagnetic fields in all space are obtained; with the aid of these, equations for the determination of the attenuation and propagation velocity of the wave are obtained. For a given range of values for the parameters of the problem, relatively simple formulas for determining these quantities are found, as well as formulas for the effective parameters of the line. The transcendental function which enters into these formulas has been tabulated for the most interesting range of values of its argument. Comparing these formulas with the existing approximate ones in the literature, the limits of applicability of these formulas are found. In the second part of the paper, using the approximate Leontovich boundary conditions, approximate formulas for the field are found, and in comparing these with the exact solution, the limits of applicability of the former are demonstrated. Relatively simple computational formulas for the field near the surface of the earth are given.

Introduction

The problem of electromagnetic wave propagation along a system of conductors situated at some height above the earth's surface is of fundamental interest from a theoretical as well as a practical standpoint, and has been the object of study by a long series of authors.

In the limit of an idealized model wherein the conductor as well as the earth are assumed to be perfectly conducting, the solution to the problem is obtained relatively easily. In this case the problem, as is well known, reduces to a planar electrostatic problem, the wave propagation velocity along the line is equal to the wave velocity in air, and its attentuation vanishes.

The calculation for finite conductivity of the wire or the earth introduces a great deal of complication into the solution of the problem. Even in the case of the simple problem of wave propagation along a uniform circular wire of finite conductivity located in a uniform unbounded medium, solved first by Sommerfeld [1], the determination of the complex propagation constant requires the solution of a complicated transcendental equation.

For a system of two parallel wires of finite conductivity, the problem was first studied by Mie [2]. In this case, special difficulties arise because of the fact that in the bipolar coordinate system, in which the conductor surfaces coincide with coordinate surfaces, the variables in the wave equation are not separable. Thus Mie applied an approximate method of solution, in which the field outside the conductors is considered as quasistatic. Nevertheless, even when this hypothesis is made, the boundary conditions are rather difficult.

By far the most interesting problem, that of electromagnetic wave propagation along a wire situated above the surface of the earth (possessing finite conductivity), has been studied by a long series of authors. Of these we mention the work of Carson [3] and Pollaczek [4]. The problem of the

fields of a wire located near the interface between two media has also been treated by Pistol'kors [13] and Kostenko [14].

Of these, in reference [3] a series of substantial assumptions is made in order to solve the problem, namely: the transverse electric field component in the earth is neglected in comparison with the longitudinal one, and likewise the displacement current is neglected relative to the conduction current; the fields in the air are taken to be quasistatic. Assuming the radius of the wire to be small, and the distribution of current on it to be axially symmetric, the author obtains an equation for determining the propagation constant (the wave propagation velocity along the line and the attenuation). As we show below, this equation is valid for a rather wide range of parameter values for the problem, but not, however, for very high frequencies. As for the formula for the field in the air, it can obviously be useful only ⌐ ⌐ to distances small in terms of a wavelength.

Pollaczek [4] in his papers does not place the same restrictions on the problem of the theoretical determination of the propagation constant since he solves the corresponding electrodynamic problem, but he assumes that some value for this constant is given, having been found in some other manner. As far as the field in the air is concerned, in spite of the general expressions given in the form of complex integrals deduced by considerations of the wave nature of the field, in these expressions, on one hand, the propagation constant figures as some given quantity, but on the other hand the simplification of these formulas toward the end of reducing them to a practical, useful form leads to the assumption that the wave number in the air is equal to zero.

As can be seen from the above discussion, all existing solutions to the problem of the electromagnetic field due to a wire situated over a finitely conducting earth, lead to expressions for the field which are useless at

large distances from the wire. Besides that, all the formulas for deter-
mining the attenuation of the wave along the wire and the change in the
wave propagation velocity relative to its value in the air, were obtained
under definite assumptions either during the formulation of the problem,
or in arriving at its solution. These assumptions permit neither an
evaluation of the accuracy of values obtained in this manner, nor a pre-
cise statement of the range of applicability of the deduced formulas.
In the meantime it has turned out that a knowledge of the wave field
of the line in the air is necessary to enable one to calculate the inductive
effect of a transmission line in coupled lines, separated by distances
on the order of a wavelength or more, and for determining the radio
interference which they generate. In this connection, the fundamental
goal of the present work is to find a practical, sufficiently accurate
solution to the problem of the wave fields of the transmission line in
the air. At the same time, the formulas for finding both the complex
wave propagation velocity along the line, as well as its effective
parameters (self-inductance, resistance and capacitance per unit length)
which result from the previously obtained engineering solution of the
problem are made precise and properly formulated.

In the present work all calculations are carried out for a single-
wire line and for purely sinusoidal time dependence. The earth is here
assumed to be planar and homogeneous in its properties, while the wire is
rectilinear, situated at some height h above the surface of the earth,
and possesses a circular cross-section. However, the method used can
be immediately extended both to the case of a multiconductor system,
which presents no difficulty, and to the calculation of the mutual
effects of the conductors on the current distribution over their cross-
sections.

In sections 1-5 the solution of the indicated problem in a rigorous formulation is carried out. In sections 6-9 the same problem is studied in an approximate formulation - namely, the effect of the earth on the field in the air is accounted for using the approximate Leontovich boundary conditions and a comparison with the first, virtually rigorous, solution is made.

On the basis of the results obtained in sections 1-3 and 7, formulas for finding the complex wave propagation velocity along the wire (section 4), the effective line parameters (section 5), as well as the wave field of the line near the surface of the earth (section 9) are deduced. A table of values of the auxiliary function $E_0(\varkappa)$ which enters into the obtained formulas is appended to the paper.

1. Formulation of the problem

The problem of the electromagnetic wave propagation along a wire extending over the earth's surface, or a system of such wires in parallel, leads to the problem of integrating Maxwell's equations for the strengths of the electric and magnetic fields $\overline{E}$ and $\overline{H}$ inside each of these media, wherein it is also required that the components of $\overline{E}$ and $\overline{H}$ satisfy certain boundary conditions at the interfaces and a condition at infinity which in our problem leads to the requirement that all field components tend to zero as the distance from the wire axis approaches infinity.

For sinusoidal processes, if the time dependence is taken in the form

$$e^{i\omega t},$$

where ω is the angular frequency, and the complex dielectric constant

$$\varepsilon' = \varepsilon - \frac{4\pi i \sigma}{\omega} \tag{1}$$

is introduced (ε is the dielectric constant and σ the conductivity), Maxwell's equations take the form

$$\left.\begin{array}{l} \text{rot } \overline{H} = \dfrac{i\varepsilon'\omega}{c} \overline{E}, \\[4ex] \text{rot } \overline{E} = -\dfrac{i\mu\omega}{c} \overline{H}, \end{array}\right\} \tag{2}$$

whence the two other equations - $\text{div } \overline{E} = 0$ and $\text{div } \overline{H} = 0$ -- are satisfied automatically.[1]

On each interface between some i^{th} and k^{th} medium, the tangential components of $\overline{E}$ and $\overline{H}$ must satisfy the continuity conditions

$$E_t^{(i)} = E_t^{(k)}, \quad H_t^{(i)} = H_t^{(k)}. \tag{3}$$

When condition (3) is fulfilled, the two boundary conditions

$$\mu_i H_n^{(i)} = \mu_k H_n^{(k)}, \tag{4}$$

$$\varepsilon_i' E_n^{(i)} = \varepsilon_k' E_n^{(k)}, \tag{5}$$

where the subscript n indicates the component normal to the interface, are satisfied automatically.

We introduce the right-handed xyz coordinate system (Fig. 1) so that the plane z=0 coincides with the surface of the earth, the x-axis is parallel to the wire, and the z-axis passes through the axis of the wire and is directed upwards. The height of the wire above the earth is denoted by h, and the radius of the wire by a. The distance from the wire axis to an arbitrary paoint with coordinates y, z is denoted by r, where

$$r = \sqrt{y^2 + (z-h)^2}, \tag{6}$$

and the angle between the y-axis and the direction of the vector r is called θ, so that

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[1] We use the Gaussian system of units in all calculations.
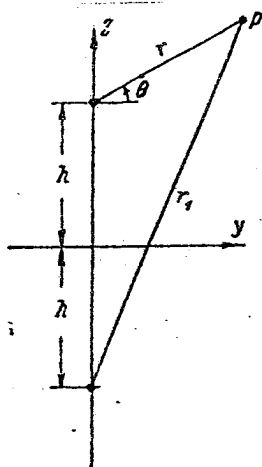
Fig. 1

$$\cos\ \theta = \frac{y}{r}\ ; \quad \sin\theta = \frac{z-h}{r}\ . \tag{7}$$

We introduce one further geometrical quantity, which we shall need in what follows, namely

$$r_1 = \sqrt{y^2+(z+h)^2} \tag{8}$$

the distance from the image of the center of the wire in the plane z=0 to an arbitrary point with coordinates y,z.

We will assume that all quantities (field components, current, charge) depend on x according to the law

$$e^{-iqx}, \tag{9}$$

as is usually done in such cases. This hypothesis corresponds to an assumption that only waves propagating towards the positive side of the x-axis exist, and that opposite ones are absent. The (complex) quantity q, which we will call the propagation constant, is general in all media which form the system.

From equations (2) with (9) taken into account, we obtain the equation

$$\Delta \Psi + m_i^2 \Psi = 0, \tag{10}$$

where by Ψ is to be understood any of the cartesian components of $\overline{E}$ or $\overline{H}$, and the notations

$$\left.\begin{array}{l}
\Delta = \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \ , \\[3mm]
m_i = \sqrt{k_i^2 - q^2}, \ \text{Im } mi > 0, \\[3mm]
k_i = - \ \varepsilon'_i \mu_i, \ \text{Im } k_i > 0.
\end{array}\right\} \tag{10a}$$

have been introduced.

The symbol i here denotes the medium. In what follows we shall understand medium 1 to be the wire, medium 2 the air and medium 3 the earth; we shall correspondingly place superscripts or subscripts on quantities relating to thes media.

Assuming that the distance between the wire and the earth is large compared to the radius of the wire, we shall solve the problem by the method of successive approximations. Neglecting the effect of the field due to a current source in the earth on the current distribution over the wire, we will assume that the latter is axially symmetric, after which we show how the effect of the secondary fields on the current distribution over the wire and on the field as a whole can be accounted for.

2. Fields of a wire in an unbounded homogeneous medium with a symmetric current distribution over its cross-section.

The solution of the problem of fields on a wire in an unbounded homogeneous medium qas obtained by Sommerfeld [1] and has the following form

$$\left.\begin{array}{l}
E_r = - \dfrac{iq}{m^2} \dfrac{\partial E_x}{\partial r} \ , \\[4mm]
H_\theta = - \dfrac{i\omega\varepsilon'}{cm^2} \dfrac{\partial E_x}{\partial r} \ ,
\end{array}\right\} \tag{11}$$

in which the component E_x satisfies the equations

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_x}{\partial r} \right) + m_k^2 E_x = 0, \quad (k=1,2). \tag{12}$$

Its solution inside the wire must be bounded at $r=0$, and therefore

$$E_x^{(1)} = A J_0(m_1 r), \tag{13}$$

where A is a constant, $J_0(\xi)$ is the Bessel function of the first kind and zeroth order. The solution of equation (12) outside the wire must tend to zero for $r \to \infty$. For our choice of sign of the imaginary part of m_2, this condition leads to

$$E_x^{(2)} = C H_0^{(1)}(m_2 r), \tag{13a}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and zeroth order, and C is the constant. In what follows the sign (1) in $H_0^{(1)}$ will be everywhere dropped, since in our solution the Hankel function of the second kind will never occur.

In what follows we will denote the fields of the wire with a symmetric current distribution as $\bar{E}^0$ and $\bar{H}^0$. From (13), (13a) and (11) we have

Inside wire   Outside wire

$$E_x^{(1)0} = A J_0(m_1 r) \qquad\qquad E_x^{(2)0} = C H_0(m_2 r)$$

$$E_r^{(1)0} = -\frac{iq}{m_1} A J_0'(m_1 r) \qquad\qquad E_r^{(2)0} = -\frac{iq}{m_2} C H_0'(m_2 r) \qquad \left.\right\} \tag{14}$$

$$H_\theta^{(1)0} = -\frac{i\omega\varepsilon_1'}{cm_1} A J_0'(m_1 r) \qquad\qquad H_\theta^{(2)0} = -\frac{i\omega\varepsilon_2'}{cm_2} C H_0'(m_2 r)$$

Here $J_0'(\xi) = \frac{d}{d\xi} J_0(\xi)$ and, analogously, $H_0'(\xi) = \frac{d}{d\xi} H_0(\xi)$.

The constant A can be easily expressed in terms of the total current.

On the surface of the wire, continuity of the components E_x and H_θ must be satisfied, which in the case of the wire in an unbounded homogeneous medium leads to two equations which determine the constant C and propagation constant q. In our case, to satisfy the boundary conditions at the wire surface it is necessary to take into account the secondary fields of the earth which lead to modified equations for the propagation constant. Therefore any process for determining the constants will depend ultimately on our determination of the secondary fields of the earth.

The corresponding fields outside the wire in cartesian coordinates on the basis of (14), (6), (7) will be, evidently,

$$
\left.
\begin{aligned}
E_x^{(2)0} &= CH_0(m_2 r); & H_x^{(2)0} &= 0, \\
E_y^{(2)0} &= -\frac{iq}{r^2} C \frac{\partial}{\partial y} H_0(m_2 r); & H_y^{(2)0} &= \frac{i\omega\varepsilon_2}{cm_2^2} C \frac{\partial}{\partial z} H_0(m_2 r), \\
E_z^{(2)0} &= -\frac{iq}{m_2^2} C \frac{\partial}{\partial z} H_0(m_2 r); & H_z^{(2)0} &= -\frac{i\omega\varepsilon_2'}{cm_2^2} C \frac{\partial}{\partial y} H_0(m_2 r).
\end{aligned}
\right\} \quad (15)
$$

The fields determined from formulas (15) will be considered as the primary fields. We now pass to the determination of the secondary fields due to the presence of the earth.

3. Fields of the earth and satisfaction of the boundary conditions
   at its surface.

We begin by determining the vertical component of electric field E_z, which satisfies an equation of the form (10), and for which the boundary conditions can be written independently of the remaining fields. The first of these is condition (5):

$$
\varepsilon_2' E_z^{(2)} = \varepsilon_3' E_z^{(3)}, \quad (z=0). \tag{16}
$$

The second condition is obtained at once from the equation div $\overline{E} = 0$, if condition (3) is also taken into account, namely,

$$\frac{\partial E_z^{(2)}}{\partial z} = \frac{\partial E_z^{(3)}}{\partial z} \ , \quad (z=0). \tag{17}$$

We represent $\overline{E}^{(2)}$ in the form

$$\overline{E}^{(2)} = \overline{E}^{(2)0} - \tilde{\overline{E}}^{(2)}, \tag{18}$$

where $\overline{E}^{(2)0}$ is the field of the wire of axially symmetric current distribution which was determined above, while $\tilde{\overline{E}}^{(2)}$ denotes the secondary field of the earth.

Since $E_z^{(2)0}$ is a solution of equation (10), then $\tilde{\overline{E}}_z^{(2)}$ must also be a solution of this equation. The general solution to the latter which vanishes at infinity (for $z>0$) and satisfies a symmetry condition with respect to y will be

$$\tilde{\overline{E}}_z^{(2)} = \frac{2qC}{\pi m_2^2} \int_0^\infty M_2(\nu) \, e^{-\eta_2(z+h)} \cos\nu y \ d\nu, \tag{19}$$

where

$$\eta_2 = \sqrt{\nu^2 - m_2^2}, \quad \text{Re } \eta_2 > 0, \tag{19a}$$

in which the multiplier in front of the integral sign and $e^{-\eta_2 h}$ were separated out to simplify the subsequent calculations. Completely analogously we can write

$$E_z^{(3)} = \frac{2qC}{\pi m_2^2} \int_0^\infty M_3(\nu) e^{-\eta_2 h} e^{\eta_3 z} \cos \nu y \ d\nu \ , \tag{20}$$

where

$$\eta_3 = \sqrt{\nu^2 - m_3^2} \ , \quad \text{Re } \eta_3 > 0 \ . \tag{20a}$$

Using the known formula[2]

$$\frac{\pi i}{2} H_0^{(1)}\left(m\sqrt{y^2+z^2}\right) = \int_0^\infty e^{-|z|\sqrt{\nu^2-m^2}} \cos \nu y \ \frac{d\nu}{\sqrt{\nu^2-m^2}} \tag{21}$$

---

[2]Cf., for example, [11], p. 99.

we can likewise express $E_z^{(2)0}$, as defined by equation (15), for $z<h$ in the form

$$E_z^{(2)0} = -\frac{2qC}{\pi m_2^2} \int_0^\infty e^{\eta_2(z-h)} \cos \nu y \, d\nu. \qquad (22)$$

Substituting (19), (20) and (22) into (16), and their derivatives into (17), we obtain a system of equations for $M_2(\nu)$ and $M_3(\nu)$. Solving for these quantities and substituting them into (19) and (20) we obtain

$$\tilde{E}_z^{(2)} = \frac{2qC}{\pi m_2^2} \int_0^\infty \frac{\varepsilon_3'\eta_2 - \varepsilon_2'\eta_3}{\varepsilon_3'\eta_2 + \varepsilon_2'\eta_3} e^{-\eta_2(z+h)} \cos \nu y \, d\nu, \quad (z \geq 0) \qquad (23)$$

$$E_z^{(3)} = -\frac{4qC}{\pi m_2^2} \int_0^\infty \frac{\varepsilon_2'\eta_2}{\varepsilon_3'\eta_2 + \varepsilon_2'\eta_3} e^{-\eta_2 h + \eta_3 z} \cos \nu y \, d\nu, \quad (z \leq 0). \qquad (24)$$

Knowing E_z we can now write two boundary conditions for E_y in which E_z enters only as a known function. In particular, considering the equation

$$\frac{\partial E_x^{(2)}}{\partial y} - \frac{\partial E_y^{(2)}}{\partial z} = -\frac{i\omega\mu_2}{c} H_x^{(2)},$$

in which we allow $z \to 0$, an analogous equation for the third medium (earth), and bearing condition (3) in mind, we obtain the relation

$$\frac{1}{\mu_2} \frac{\partial E_y^{(2)}}{\partial z} - \frac{1}{\mu_3} \frac{\partial E_y^{(3)}}{\partial z} = \frac{\partial}{\partial y}\left(\frac{1}{\mu_2} E_z^{(2)} - \frac{1}{\mu_3} E_z^{(3)}\right), \quad (z=0).$$

Hence, considering (16) and (23), we obtain

$$\left(\mu_3 \frac{\partial E_y^{(2)}}{\partial z} - \mu_2 \frac{\partial E_y^{(3)}}{\partial z}\right)_{x=0} = \frac{4qC}{\pi m_2^2}(\varepsilon_3'\mu_3 - \varepsilon_2'\mu_2)\int_0^\infty \frac{\eta_2 e^{-\eta_2 h}}{\varepsilon_3'\eta_2 + \varepsilon_2'\eta_3} \nu\sin \nu y \, d\nu$$

$$\qquad (25)$$

and from the first of conditions (3)

$$E_y^{(2)} = E_y^{(3)}, \ (z=0). \tag{26}$$

According to (18), $E_y^{(2)} = E_y^{(2)0} - \tilde{E}_y^{(2)}$. Having taken $E_y^{(2)0}$ from (15) and using (21), we obtain for $z<h$.

$$E_y^{(2)0} = \frac{2qC}{\pi m_2^2} \int_0^\infty e^{\eta_2(z-h)} \frac{\nu}{\eta_2} \sin \nu y \ d\nu. \tag{27}$$

If now, in analogy with the preceeding, we write $\tilde{E}_y^{(2)}$ and $E_y^{(3)}$ in the form

$$\tilde{E}_y^{(2)} = \frac{2qC}{\pi m_2^2} \int_0^\infty N_2(\nu) e^{-\eta_2(z+h)} \nu \sin \nu y \ d\nu \tag{28}$$

and

$$E_y^{(3)} = \frac{2qC}{\pi m_2^2} \int_0^\infty N_3(\nu) e^{-\eta_2 h + \eta_3 z} \nu \sin \nu y \ d\nu, \tag{29}$$

then by substituting (27)-(29) into (25)-(26) we obtain for $N_2(\nu)$ and $N_3(\nu)$ a system of two equations, from which $N_2(\nu)$ and $N_3(\nu)$ can be found and substituted into (28) and (29).

For

$$\mu_2 = \mu_3 = 1, \tag{30}$$

which corresponds to situations of practical interest to us, we thus obtain

$$\tilde{E}_y^{(2)} = \frac{2qC}{\pi m_2^2} \int_0^\infty \left(\frac{1}{\eta_2} - \frac{2\varepsilon_2'}{\varepsilon_3'\eta_2 + \varepsilon_2'\eta_3} \right) e^{-\eta_2(z+h)} \nu \sin \nu y \ d\nu \tag{31}$$

and

$$E_y^{(3)} = \frac{4qC}{\pi m_2^2} \int_0^\infty \frac{\varepsilon_2'}{\varepsilon_3'\eta_2 + \varepsilon_2'\eta_3} e^{-\eta_2 h + \eta_3 z} \nu \sin \nu y \ d\nu. \tag{32}$$

Now we can determine E_x from the equation div $\overline{E} = 0$, which takes the form

$$iqE_x = \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{33}$$

and then, knowing E_x, E_y and E_z we can, using the second of equations (2), find H_x, H_y and H_z as well, by simply differentiating. Thus, the formulas found for $\bar{E}$ and $\bar{H}$ define the fields completely for all space.

We further write out in explicit form the component $\tilde{E}_x^{(2)}$, which we shall need in what follows. From (33), (31) and (23) we obtain

$$\tilde{E}_x^{(2)} = CH_0(m_2 r_1) + \frac{4\varepsilon_2' C}{\pi i m_2^2} \int_0^\infty \frac{\eta_2 \eta_3 \nu^2}{\varepsilon_3' \eta_2 + \varepsilon_2' \eta_3} e^{-\eta_2(z+h)} \cos \nu y \; d\nu. \tag{34}$$

4. Boundary conditions at the surface of the wire. The equation for

the propagation constant.

At the surface of the wire we enforce the following boundary conditions

$$E_x^{(2)} = E_x^{(1)}; \quad H_x^{(2)} = H_x^{(1)}; \quad E_\theta^{(2)} = E_\theta^{(1)}; \quad H_\theta^{(2)} = H_\theta^{(1)} \quad \text{at} \quad r = \alpha. \tag{35}$$

In this paragraph we shall satisfy these conditions in a first approximation, neglecting the variation of the secondary fields over the wire cross-section, in which we take the values of $\tilde{\bar{E}}^{(2)}$ and $\tilde{\bar{H}}^{(2)}$ as equal to their values along its axis. In the Appendix we will show how these boundary conditions can be satisfied more accurately, and estimate the order of the correction terms. As shown there, the discarded terms are of order $(a/2h)^2$ compared to the main ones, that is, these may be unconditionally neglected in our solution, since these quantities are, in realistic situations, always small.

As is evident from symmetry considerations, $\tilde{E}_\theta^{(2)} = \tilde{H}_x^{(2)} = 0$ for $y = 0$, and since in addition $E_\theta^0 \equiv H_x^0 \equiv 0$, then the second and third conditions in (35) are, in our approximation, satisfied identically. The two other conditions in (35) in our approximation take the form

$$\left(E_x^{(2)0}\right)_{r=a} - \left(\tilde{E}_x^{(2)}\right)_{y=0,\ z=h} = \left(E_x^{(1)}\right)_{r=a}, \tag{36}$$

$$\left(H_\theta^{(2)0}\right)_{r=a} = \left(H_\theta^{(1)0}\right)_{r=a}.$$

It should be noted that equations (36) are retained also for the satisfaction of the boundary conditions in the second approximation as shown in the Appendix. This point is essential, since from these equations an equation for the propagation constant is obtained.

Substituting the expressions for the field components from (14) and (34) into equation (36), we obtain

$$\begin{cases} CH_0(m_2 a) - CH_0(2m_2 h) - \dfrac{2}{\pi i}\ \dfrac{C}{m_2^2}\ F' = AJ_0(m_1 a), \\[4mm] \dfrac{C}{m_2} H_0'(m_2 a) = \dfrac{\varepsilon_1'}{m_1} AJ_0'(m_1 a), \end{cases} \tag{37}$$

where

$$F' = 2\varepsilon_2' \int_0^\infty \frac{\eta_2 \eta_3 - \nu^2}{\varepsilon_3' \eta_2 + \varepsilon_2' \eta_3}\ e^{-2\eta_2 h}\ d\nu. \tag{38}$$

We now express the constants A and C in terms of the total current flowing along the wire

$$I = \int_0^{2\pi} \int_0^a \sigma_1 E_x^{(1)} r\,dr\,d\theta = 2\pi a\ \frac{\sigma_1}{m_1}\ AJ_1(m_1 a),$$

where by I we should understand the amplitude of the total current flowing through the wire cross-section at $x=0$. Hence

$$A = \frac{Im_1}{2\pi a \sigma_1 J_1(m_2 a)}. \tag{39}$$

after which from the second of equations (37) we obtain

$$C = \frac{Im_2\varepsilon_1'}{2\pi a\sigma_1 H_1(m_2 a)} \quad . \tag{40}$$

Dividing the first of equations (37) by the second, we obtain after simple rearrangements the following equation for the propagation constant

$$m_2 \frac{H_0(m_2 a) - H_0(2m_2 h)}{H_0'(m_2 a)} = \frac{F'}{m_2[\frac{\pi i}{2}H_0'(m_2 a)]} + \frac{m_1 J_0(m_1 a)}{\varepsilon_1' J_0'(m_1 a)} \quad , \tag{41}$$

where F' is given by equation (38).

Equation (41) is valid for arbitrary frequencies and medium parameters. As it stands, it involves the complex integral F', into which the unknown quantity q (which is to be determined) enters. Because of this, finding q from equation (41) is, generally speaking, a complex and difficult process. As shown below, it is important to give relatively simple formulas for determining q over a definite and sufficiently wide range of the parameters of the problem.

From physical considerations it is clear that, for sufficiently large conductivities of the wire and of the earth, the propagation velocity should not differ significantly from ω/c, In fact, from the known approximate solution [3], it is clear that $|q|$ differs from ω/c under ordinary conditions by not more than 15-25% of this value. To simplify formula (41) obtained above we shall use, in the following estimates, this fact from the very beginning, and shall convince ourselves that the results obtained justify this assumption.

Considering that $|m_2| < \omega/c$, we have, for frequencies $\omega < 10^6$ and for wire heights $h \leq 10m$ above the earth, with an error of less than 0.5%

$$\frac{\pi i}{2} H_0(m_2 2h) = -\ln \frac{\gamma m_2 2h}{2i},$$

where $\gamma = 1.781\ldots$ is Euler's constant.[3] With even greater accuracy

$$\frac{\pi i}{2} H_0(m_2 a) = -\ln \frac{\gamma m_2 a}{2i}$$

and

$$\frac{\pi i}{2} H_0'(m_2 a) = -\frac{1}{m_2 a}.$$

In addition, since $|k_1| >> \omega/c$, one can for all practical purposes reckon accurately that $m_1 = k_1$.

Substituting these approximate values into equation (41) we obtain the formula

$$\varepsilon_2' \left(\frac{m_2}{k_2}\right)^2 \ln \frac{2h}{a} = F - \frac{1}{ak_1} \frac{J_0(ak_1)}{J_0(ak_1)}, \tag{42}$$

where

$$F = 2\varepsilon_2' \int_0^\infty \frac{\eta_2 \eta_3 - \nu^2}{k_3^2 \eta_2 + k_2^2 \eta_3} e^{-2h\eta_2} d\nu. \tag{43}$$

The function F can be expressed as the sum of three integrals

$$F = 2\varepsilon_2'\{K + R + G\}, \tag{44}$$

in which

$$K = -\int_0^\infty \frac{\eta_2}{k_3^2 - k_2^2} e^{-2h\eta_2} d\nu, \qquad R = \int_0^\infty \frac{\eta_3}{k_3^2 - k_2^2} e^{-2h\eta_2} d\nu,$$

$$G = q_i^2 \int_0^\infty \frac{1}{k_3^2 \eta_2 + k_2^2 \eta_3} e^{-\eta_2 2h} d\nu.$$

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[3] Cf., for example, [12], pp. 224 ff.

All the following calculations will be carried out under the same assumptions and to the same accuracy used in obtaining equation (42). The first integral is calculated at once with the help of formula (21)

$$K = - \frac{\pi i}{2} \frac{m_2^2}{k_3^2 - k_2^2} H_0''(m_2 2h) \simeq - \frac{1}{(2hk_3)^2} . \tag{45}$$

To calculate the integral R we use the fact that $|m_2| << |k_3|$, and assume $m_3 = k_3$ as well.[4] Carrying out the change of variables $2h\nu = \xi$ and introducing the notation

$$\varkappa^2 = - (2hk_3)^2, \quad \text{Re } \varkappa > 0, \tag{46}$$

$$\delta^2 = (2hm_3)^2,$$

we can write

$$R = - \frac{1}{\varkappa^2} \int_0^\infty \sqrt{\xi^2 + \varkappa^2} \, e^{-\sqrt{\xi^2 - \delta^2}} \, d\xi = - \frac{1}{\varkappa^2} \int_0^\infty e^{-\xi} \sqrt{\xi^2 + \varkappa^2} \, d\xi$$

$$- \frac{1}{\varkappa^2} \int_0^\infty \left(e^{-\sqrt{\xi^2 - \delta^2}} - e^{-\xi} \right) \sqrt{\xi^2 + \varkappa^2} \, d\xi \tag{47}$$

An estimate of the second integral in (47) shows that

$$\left| \frac{1}{\varkappa^2} \int_0^\infty \left(e^{-\sqrt{\xi^2 - \delta^2}} - e^{-\xi} \right) \sqrt{\xi^2 + \varkappa^2} \, d\xi \right| < \left| \frac{\delta}{\varkappa} \right| \left| \delta \ell n 2n + \frac{1}{2n} \right|, \tag{48}$$

where n is some number chosen such that $|n\delta| << |\varkappa|$. In the frequency interval below $\omega = 10^6$ for conductivity of the earth $\sigma_3 \geq 10^7$, neglecting the integral (48) gives an error of less than 0.25%.

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[4] With an accuracy under 0.1% in all ranges of parameters studied below.

Now carrying out the change of variable $\xi = \varkappa\rho$, we can represent R in the form

$$R = -\int_0^\infty e^{-\varkappa\rho}\sqrt{\rho^2+1}\, d\rho \qquad (49)$$

We use the formula[5]

$$\frac{\pi}{2}Y_0(x) = \int_0^1 \frac{\sin\varkappa t\, dt}{\sqrt{1-t^2}} - \int_0^\infty \frac{e^{-\varkappa\rho}d\rho}{\sqrt{1+\rho^2}}$$

Differentiating it twice with respect to $\varkappa$ (which we are permitted to do by reason of the absolute convergence of all integrals) and combining it with the original formula we obtain, by the use of known relations between Bessel functions

$$\frac{\pi}{2\varkappa}Y_1(\varkappa) = \int_0^1 \sqrt{1-t^2}\,\sin\varkappa t\, dt - \int_0^\infty e^{-\varkappa\rho}\sqrt{1+\rho^2}d\rho,$$

and hence in place of (49)

$$R = \frac{\pi}{2\varkappa}Y_1(\varkappa) - \int_0^1 \sqrt{1-t^2}\,\sin\varkappa t\, dt. \qquad (50)$$

Finally, expanding the $\sin\varkappa t$ in the integrand into a series, we obtain finally

$$R = \frac{1}{\varkappa}\left\{\frac{\pi}{2}Y_1(\varkappa) - \sum_{n=1}^\infty (-1)^{n-1}\frac{\varkappa^{2n}}{1^2 3^2 \ldots (2n-1)^2(2n+1)}\right\}. \qquad (51)$$

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[5]Cf., for example, [11], p. 702.

The final integral G which enters into formula (44) will, for all frequencies under $\omega=10^6$, be a small correction to the first two.

To evaluate this integral we use the fact that $|k_3^2\eta_2| \gg |k_2^2\eta_3|$ over the entire integration interval and expand the integrand in powers of the ratio of these quantities. Thus we obtain

$$G = q^2 \int_0^\infty \frac{e^{-2h\eta_3}}{\eta_2 k_3^2} \left(1 - \frac{k_2^2 \eta_3}{k_3^2 \eta_2} + \ldots \right) d\nu = \frac{q^2}{k_3^2} \frac{\pi i}{2} H_0(2hm_2) = \ldots,$$

where the neglected terms are much smaller than the one written out, and since G itself is a correction, we can take

$$G = -\frac{q^2}{k_3^2} \, \ell n \, \frac{2\gamma m_2 h}{2i} . \tag{52}$$

We now sub_titute (45), (51) and (52) into (44)

$$F = 2\varepsilon'_2 \left\{ \frac{1}{\varkappa^2} + \frac{1}{\varkappa} \left[\frac{\pi}{2} Y_1(\varkappa) - \sum_{n=1}^\infty (-1)^{n-1} \frac{\varkappa^{2n}}{1^2 3^2 \ldots (2n-1)^2 (2n+1)} \right] - \left(\frac{q}{k_3} \right)^2 \ell n \, \frac{\gamma m_2 h}{i} \right\}, \tag{53}$$

where $\varkappa$ is given by (46).

The substitution of (53) into (42) gives a practical, convenient form of the equation for finding the propagation constant. To solve this equation one is led to the method of successive approximations.[6]

In the first approximation we obtain from (42), neglecting in expression (53) for the function F the (small) last term in the curly brackets,

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[6]This should not be surprising since even in the simplest case of a single wire in an infinite space one is also led to the method of successive approximations (cf., for example, [1]). The present case is incomparably more complex.

$$\varepsilon_2' \left(\frac{m_2}{k_2}\right)^2 \ln \frac{2h}{a} = F_0 - \frac{1}{ak_1} \frac{J_0(ak_1)}{J_0'(ak_1)} , \qquad (54)$$

in which

$$F_0 = 2\varepsilon_2' \left\{ \frac{1}{\varkappa^2} + \frac{1}{\varkappa} \left[\frac{\pi}{2} Y_1(\varkappa) - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\varkappa^{2n}}{1^2 3^2 \ldots (2n-1)^2 (2n+1)} \right] \right\} . \qquad (55)$$

For low frequencies, when $|\varkappa| < 0.3$ and

$$\varkappa \equiv \varkappa_3 \sqrt{i} = \sqrt{i} \frac{2h}{c} \sqrt{4\pi\sigma_3\omega}, \qquad (56)$$

it is more convenient to use the formulas

$$ReF_0 = \varepsilon_2' \left[\ln\frac{\gamma\varkappa_3}{2} - \frac{1}{2} - \frac{\varkappa_3\sqrt{2}}{3} + \frac{\pi}{32} \varkappa_3^2 - \frac{\varkappa_3^3\sqrt{2}}{45} \right] ,$$

$$ImF_0 = \varepsilon_2' \left[\frac{\pi}{4} - \frac{\varkappa_3\sqrt{2}}{3} - \frac{\varkappa_3^3}{8} \ln\left(\frac{\gamma\varkappa_3}{2} - \frac{5}{4}\right) + \frac{\varkappa_3^3\sqrt{2}}{45} \right] \qquad (57)$$

The values of m_2 and q found from equation (54) should be substituted into (53) and (42), however it should be pointed out that similar corrections need to be inserted for conductivities σ_3 on the order of 10^8 and higher, only beginning with frequencies ω on the order of 10^5. For poorer soil conductivity the correction will become substantial for much lower frequencies.

It should be noted that equation (54) coincides with the equation for the propagation constant given by Carson [3], if we take $\varepsilon_2' = 1$ and neglect ε_3, while in Carson's solution the leakage conductivity to the earth is set equal to zero. Thus, carrying out the above analysis points out the limits of applicability of the solution in [3].

We note also that the quantity $J_0(ak_1)/[ak_1 J_0'(ak_1)]$ which enters into equations (42) and (54), is equal to $Z/2i\omega$ where Z is the total impedance of the wire to alternating current.

Table 1

	$\mathrm{Im}F_0(\varkappa)$	$-\mathrm{Re}F_0(\varkappa)$	$\varkappa_3$	$\mathrm{Im}F_0(\varkappa)$	$-\mathrm{Re}F_0(\varkappa)$
0.1	0.743	2.965	1.4	0.451	0.799
0.2	0.706	2.516	1.5	0.438	0.760
0.3	0.674	1.953	1.6	0.426	0.726
0.4	0.644	1.706	1.7	0.414	0.693
0.5	0.618	1.524	1.8	0.402	0.662
0.6	0.594	1.380	1.9	0.392	0.633
0.7	0.571	1.261	2.0	0.383	0.608
0.8	0.550	1.165	2.1	0.373	0.585
0.9	0.531	1.081	2.2	0.363	0.563
1.0	0.512	1.010	2.3	0.354	0.543
1.1	0.496	0.948	2.4	0.346	0.527
1.2	0.480	0.893	2.5	0.338	0.510
1.3	0.465	0.844			

In order to carry out the calculation of the propagation constant the function $F_0(\varkappa)$ has been tabulated for a range of values of the variables of most interest to us. In doing this we neglected the displacement current in the earth. The results of these calculations are presented in Table 1.

Table 2

ω	$\dfrac{qc}{\omega}$	$q \cdot 10^3 (\mathrm{cm}^{-1})$
$3 \cdot 10^2$	$1.246 - i\,0.0907$	$1.246 - i\,0.0907$
10^3	$1.211 - i\,0.0594$	$4.036 - i\,0.198$
10^4	$1.143 - i\,0.0453$	$38.09 - i\,1.51$
10^5	$1.081 - i\,0.0363$	$360.5 - i12.1$
10^6	$1.037 - i\,0.0236$	$3455 - i78.5$

In Table 2 we carry out as an example the evaluation of the quantities q and qc/ω in the concrete case where $\sigma_3 = 9 \times 10^7$, $\sigma_1 = 5.14 \times 10^{17}$, h=10m, a=1cm for the frequency interval of ω from 3×10^2 to 10^6 sec^{-1}. The results of the calculations, as has already been pointed out above, confirm the preliminary estimates, which we used to arrive at the equation for the

propagation constant.

5. Effective parameters of the line.

The usual approach to the solution of the problem of the propagation of alternating current along an electric transmission line is contained in the integral of the so-called telegraphist's equations in which are substituted the line parameters, determined from the approximate solution of Maxwell's equations without allowing for attenuation along the line or distinguishing the propagation velocity from that of the air. It is therefore of interest to obtain the values for the effective line parameters from the relatively simple solution of the problem found above.

We define the potential between the wire and the earth by the formula

$$\phi = - \int_0^{h-a} E_z^{(2)} \Big|_{y=0} dz, \tag{58}$$

which agrees with the sense of this quantity in the usual transmission line equations, which have the form

$$\left. \begin{aligned} \frac{\partial \phi}{\partial x} + R_0 I &= \frac{L_0}{c^2} \frac{\partial I}{\partial t} = 0, \\[2mm] C_0 \frac{\partial \phi}{\partial t} + G_0 \phi + \frac{\partial I}{\partial x} &= 0, \end{aligned} \right\} \tag{59}$$

where R_0, L_0, C_0 and G_0 are the effective line parameters - resistance, self-inductance, capacitance and leakage conductance per unit length of line respectively.

We transcribe (59) into the form

$$\left. \begin{aligned} iq\phi &= \left(R_0 + \frac{i\omega}{c^2} L_0 \right) I, \\[2mm] (i\omega C_0 + G_0)\phi &= iqI. \end{aligned} \right\} \tag{60}$$

From (60) follows additionally the known relation

$$- q^2 = \left(R_0 + \frac{i\omega}{c^2} L_0 \right) (i\omega C_0 + G_0). \tag{61}$$

Thus, to determine the effective line parameters, it is necessary only to calculate the integral (58), starting from our solution in which $E_z^{(2)}$ is proportional to the total current I in the wire, and to substitute the value obtained into equation (60). Calculating the integral (58) using formulas (15) and (23) shows that

$$\phi = \frac{2qC}{\pi m_2^2} \ln \frac{2h}{a} \tag{62}$$

with an error of less than 0.5% for frequencies $\omega \leq 10^6$ and conductivities $\sigma_3 \geq 10^7$; the accuracy improves for lower frequencies.

Substituting formula (40) for the constant C into (62) and taking into account that $|m_2 a| \ll 1$, we obtain

$$\phi = \frac{2qI}{\omega} \ln \frac{2h}{a} . \tag{63}$$

The second of equations (60) takes the form

$$i\omega C_0 + G_0 = \frac{i\omega}{2\ln \frac{2h}{a}}$$

i.e.,

$$G_0 = 0, \tag{64}$$

as must hold under our hypothesis $(\sigma_2 = 0)$, and

$$C_0 = \frac{1}{2 \ln \frac{2h}{a}} \tag{65}$$

For the capacitance per unit length of line we have obtained the very same value which is obtained in the elementary theory; this follows also because the distance between the wire and the earth is very small in terms of a wavelength.

Substituting (63) into the first of equations (60) gives

$$R_0 + \frac{i\omega}{c^2} L_0 = - q^2 \frac{2\ell n \frac{2h}{a}}{i\omega}$$

and if here is substituted the value of q^2 from equation (42), then after some straightforward manipulations we obtain

$$i\omega L_0 + c^2 R_0 = 2i\omega \ell n \frac{2h}{a} - 2i\omega F + \frac{2i\omega}{ak_1} \frac{J_0(ak_1)}{J_0'(ak_1)} . \qquad (66)$$

Equation (66) can also be written in the form

$$Z_0 = i\omega L_1 + Z_3 + Z_1, \qquad (67)$$

where Z_0 is the total effective impedance,

$$L_1 = 2\ell n \frac{2h}{a} \qquad (68)$$

is the self-inductance under the assumption of an ideally conducting earth,

$$Z_1 = \frac{2i\omega}{ak_1} \frac{J_0(ak_1)}{J_0'(ak_1)} \qquad (69)$$

is the total impedance of the wire to alternating current, and finally

$$Z_3 = - 2i\omega F \qquad (70)$$

is the total impedance attributable to the finite conductivity of the earth.

Formula (68) differs from the formula given by Carson [3] in the

possibility that F differs from F_0 and in that the displacement current of the earth is taken into account, which can be shown to be substantial for sufficiently high frequencies.

6. <u>Approximate boundary conditions at the earth's surface.</u>

The formulas obtained in section 3 for the field components in the air are hardly suitable for practical calculation of the field in view of their complexity. One can try to obtain directly approximate formulas for the fields by applying the approximate boundary conditions at the earth's surface proposed by Leontovich [8].

Considering a body possessing a complex dielectric constant of large modulus within which the wavelength is much smaller than the wavelength in air, Leontovich introduced approximate boundary conditions which exterior tangential field components must satisfy on the surface of such bodies. We write these conditions for the field in the air at the earth's surface in the form

$$E_x^{(2)} + \alpha H_y^{(2)} = 0 \tag{71}$$

$$E_y^{(2)} - \alpha H_x^{(2)} = 0, \tag{72}$$

where

$$\alpha = \sqrt{\frac{\mu_3}{\varepsilon_3'}} \, , \ \text{Re } \alpha > 0 \tag{73}$$

and $z = 0$.

In reference [8] Leontovich carries out an analysis of the limits of applicability of conditions (71), (72), using results of the work of Rytov [5], and shows that for bodies possessing large losses they are accurate to a quantity of order α^2 under the following conditions: the wavelength in the body and the penetration depth into the body are small compared to the wavelength in the surrounding medium, the distance to the source of the field, and the radius of curvature of the surface.

One can by considerations of a similar type connected with the exponential decay of the field towards the interior of a good conductor, deduce approximate boundary conditions for the electric field component normal to the interface [6]. We write this condition in the form

$$\frac{\partial E_z^{(2)}}{\partial z} + \beta E_z^{(2)} = 0 \quad \text{at} \quad z=0,$$
(74)

where

$$\beta = -\frac{i\omega}{c}\alpha.$$
(75)

A condition analogous to (74) can be obtained as well for the normal magnetic field component

$$\frac{\partial H_z^{(2)}}{\partial z} + \gamma H_z^{(2)} = 0 \quad \text{at} \quad z=0,$$
(76)

where

$$\gamma = -\frac{i\omega}{c}\frac{1}{\alpha}.$$
(77)

We now consider the possibility of applying conditions (71), (72) to the problem which interests us.[7] In our case the interface is planar; thus the condition regarding the curvature drops out. The condition of smallness of the wavelength in the earth and the penetration depth there compared to the wavelength in air for soil conductivity $\sigma_3 \geq 10^7$ is satisfied rather well for every case with frequency $\omega \leq 10^6$.

As for the condition of smallness of the wavelength in the earth and the penetration depth compared to the distance from the source of the field,

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[7]Conditions (74) and (76) can be obtained from Maxwell's equations and conditions (71) and (72) without such additional considerations. It is therefore sufficient to study only the applicability of conditions (71) and (72).

this is not satisfied on the portion of the surface which lies directly

under the wire for realistic earth conductivities and heights of the wire

above the surface, when the frequency $\omega < 10^6$. However a similar statement

holds true in a series of radiowave propagation problems and nevertheless,

the solutions of these problems obtained through application of the appro-

ximate boundary conditions turn out to be valid at moderate distances

from the source. Thus, for example, Leontovich [7] solved the problem of

the field of a vertical dipole situated above the plane surface of the

earth, by applying the approximate boundary conditions, and obtained a

solution which agreed with the Weyl-van der Pol formula, which in turn

is obtained from Sommerfeld's rigorous solution.

We note here that the application of approximate boundary conditions

in radiowave propagation problems has turned out to be highly fruitful

because of the practicability of approach to the solution of wave propag-

ation problems which account for medium inhomogeneity or roughness of the

interface. Thus in the already cited work of Grinberg [6] the solution

was obtained for the problem of interface refraction of radiowaves; the

work of Fock [9] solved the problem of wave propagation around the earth

taking atmospheric refraction into account; Feinberg in the same collection

[10] studied the problem taking into account the influence of surface

roughness and inhomogeneity of its electrical characteristics on radio-

wave propagation.

All the above permits us to hope that the solution to our problem

using approximate boundary conditions will be valid in all cases at suffic-

ient distances from the wire. In the present work we confine ourselves

to a solution of the problem only for the case of a smooth and homogeneous

earth, in which we deduce the validity of the approximate solution compared

to the accurate one, obtained in sections 3-4 and we find the limits of

applicability of the approximate solution.

7. <u>Solution of the problem of the secondary field of the earth,</u>

   <u>applying the approximate boundary conditions.</u>

In the present paragraph we will solve the problem of the secondary

fields of the earth $\tilde{\bar{E}}^{(2)}$ and $\tilde{\bar{H}}^{(2)}$ by applying the approximate boundary con-

ditions (71), (72), (74) and (76) and assuming that the current distrib-

ution over the wire cross-section possesses axial symmetry.[8] This problem

reduces to integrating Maxwell's equations in the air for $\tilde{\bar{E}}^{(2)}$ and $\tilde{\bar{H}}^{(2)}$

and satisfying conditions (71), (72), (74) and (76) at the earth's surface

for the total fields $\bar{E}^{(2)} = \bar{E}^{(2)0} - \tilde{\bar{E}}^{(2)}$ and $\bar{H}^{(2)} = \bar{H}^{(2)0} - \tilde{\bar{H}}^{(2)}$ in which

$\bar{E}^{(2)0}$ and $\bar{H}^{(2)0}$ are given by formula (15).

We rewrite the boundary conditions in the form

$$\tilde{E}_x^{(2)} + \alpha \tilde{H}_y^{(2)} = E_x^{(2)0} + \alpha H_y^{(2)0}, \tag{78}$$

$$\tilde{E}_y^{(2)} - \alpha \tilde{H}_x^{(2)} = E_y^{(2)0} - \alpha H_x^{(2)0}, \tag{79}$$

$$\text{at} \quad z=0$$

$$\frac{\partial \tilde{E}_z^{(2)}}{\partial z} + \beta \tilde{E}_z^{(2)} = \frac{\partial E_z^{(2)0}}{\partial z} + \beta E_z^{(2)0}, \tag{80}$$

$$\frac{\partial \tilde{H}_z^{(2)}}{\partial z} + \gamma \tilde{H}_z^{(2)} = \frac{\partial H_z^{(2)0}}{\partial z} + \gamma H_z^{(2)0}. \tag{81}$$

The constants α, β and γ are determined from formulas (73), (75)

and (77).

The components $\tilde{E}_z^{(2)}$ and $\tilde{H}_z^{(2)}$ can be determined relatively simple since

for these we have independent boundary conditions and an equation like (10).

The boundary conditions for the four other components are connected with

each of the two components in pairs. However one can, using conditions

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[8] As shown in the Appendix this assumption is justified in our case for
practical purposes.

(78) and (79) easily determine two combinations of these four components, namely

$$S^{(2)} = E_x^{(2)} + \alpha H_y^{(2)}, \tag{82}$$

$$Q^{(2)} = E_y^{(2)} - \alpha H_x^{(2)}. \tag{83}$$

In fact, since each of the components $E_x^{(2)}$, $E_y^{(2)}$, $H_x^{(2)}$ and $H_y^{(2)}$ satisfies equation (10), each of $S^{(2)}$ and $Q^{(2)}$ also satisfies this equation (as a linear combination of them). On the other hand, we have for $z=0$ the boundary conditions

$$S^{(2)} = 0, \tag{84}$$

$$Q^{(2)} = 0. \tag{85}$$

Further,

$$S^{(2)} = S^{(2)0} - \tilde{S}^{(2)} \qquad Q^{(2)} = Q^{(2)0} - \tilde{Q}^{(2)},$$

where

$$S^{(2)0} = CH_0(m_2 r) + C\frac{i\omega\alpha}{cm_2^2}\frac{\partial}{\partial z} H_0(m_2 r), \tag{86}$$

$$Q^{(2)0} = -\frac{iqC}{m_2^2}\frac{\partial}{\partial y} H_0(m_2 r) \tag{87}$$

on the basis of formula (15). It is immediately obvious that the expressions

$$\tilde{S}^{(2)} = \tilde{E}_x^{(2)} + \alpha\tilde{H}_y^{(2)} = CH_0(m_2 r_1) - \frac{i\omega\alpha}{cm_2^2} C\frac{\partial}{\partial z} H_0(m_2 r_1), \tag{88}$$

$$\tilde{Q}^{(2)} = \tilde{E}_y^{(2)} - \alpha\tilde{H}_y^{(2)} = -\frac{iqC}{m_2^2}\frac{\partial}{\partial y} H_0(m_2 r_1) \tag{89}$$

satisfy equation (10), boundary conditions (84) and (85) (by taking (86) and (87) into account), and the condition at infinity. By virtue of the uniqueness of the solution of this problem, formulas (88) and (89) give the

desired representation for the two indicated combinations of secondary field components.

To determine $\tilde{E}_z^{(2)}$ we have equation (10) and boundary condition (80). As in section 3, we express the general solution to equation (10) which vanishes at infinity for $z>0$ and satisfies symmetry conditions in y in the form

$$\tilde{E}_z^{(2)} = \int_0^\infty M(\nu)\cos\nu y\, e^{-\eta_2 z}\, d\nu, \tag{90}$$

where $M(\nu)$ is an unknown function to be determined from the boundary conditions (80).

Using formulas (15) and (21), we represent $E_z^{(2)0}$ for $z<h$ in the form

$$E_z^{(2)0} = -\frac{2qC}{\pi m_2^2} \int_0^\infty e^{\eta_2(z-h)} \cos\nu y\, d\nu.$$

From condition (80) we now easily obtain

$$M(\nu) = \frac{2qC}{\pi m_2^2} e^{-\eta_2 h}\left(1 + \frac{2\beta}{\eta_2-\beta}\right). \tag{91}$$

Substituting (91) into (90) and using formula (21) once again, we obtain $\tilde{E}_z^{(2)}$ in the form

$$\tilde{E}_z^{(2)} = -\frac{iqC}{m_2^2} \frac{\partial}{\partial z} H_0(m_2 r_1) + \frac{4q\beta C}{\pi m_2^2} J, \tag{92}$$

where

$$J = \int_0^\infty \frac{e^{-\eta_2(z+h)}}{\eta_2 - \beta} \cos\nu y\, d\nu. \tag{93}$$

The second term in formula (92) appears as a consequence of β differing from zero, i.e., of the finite conductivity of the earth.

Formulas (92) and (93) already give the solution for the field $\overset{(2)}{E_z}$ but the integral J in the form (93) is hardly suitable for calculations. Therefore we shall transform it into a more convenient form. Multiplying equation (93) by $e^{\beta(z+h)}$ and differentiating with respect to z, we obtain

$$\frac{\partial}{\partial z}(e^{\beta(z+h)}J) = -\int_0^\infty e^{(\beta-\eta_2)(z+h)}\cos\nu y\,d\nu = \frac{\pi i}{2}e^{\beta(z+h)}\frac{\partial}{\partial z}H_0(m_2 r_i),$$

where once again we have used formula (21). Integrating the latter equation, we find

$$e^{\beta z}J = \frac{\pi i}{2}\int_0^\infty e^{\beta z}\frac{\partial}{\partial z}\cdot H_0(m_2 r_1)dz + P(y), \qquad (94)$$

where $P(y)$ is independent of z.

To determine $P(y)$ we proceed in the following manner. We represent J in the form

$$J = \frac{1}{2}\int_{-\infty}^\infty e^{i\nu y - \eta_2(z+h)}\frac{d\nu}{\eta_2-\beta}$$

Adding to J a contour integral (as shown in Fig. 2) we obtain[9]

$$J = -\frac{1}{2}\int_{CDE} e^{i\nu y - \eta_2(z+h)}\frac{d\nu}{\eta_2-\beta} + P_1, \quad (y>0),$$

in which P_1 is πi times the residue at $\nu = \nu_0$, where ν_0 is the zero of the denominator of the integrand, and is equal to

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[9] The cut in our contour leads from the branch point of the function $\eta_2(\nu)$ (at $\nu=m_2$) which lies in the upper half plane since Im $m_2>0$, along the curve Re $\eta_2=0$, to $\nu=i\infty$. In this case Re $\eta_2>0$ everywhere within the contour.

$$\nu_0 = \sqrt{m_2^2 + \beta^2}, \quad \text{Im } \nu_0 > 0. \tag{95}$$

The point ν_0 in fact, lies in our sheet of the ν-plane, since Re $\eta_2 > 0$ within our contour, while Re $\beta > 0$ as determined from (75). On the path CD, $\eta_2 = i|\eta_2|$, while on ED $\eta_2 = -i|\eta_2|$. Now it is easy to represent J in the form

$$J = \int_{ED} e^{i\nu|y|} \frac{i|\eta_2|\cos[|\eta_2|(z+h)] - i\beta\sin[|\eta_2|(z+h)]}{|\eta_2|^2 + \beta^2} d\nu + \frac{\pi i \beta}{\nu_0} e^{i\nu_0|y| - \beta(z+h)}.$$

$$\tag{96}$$
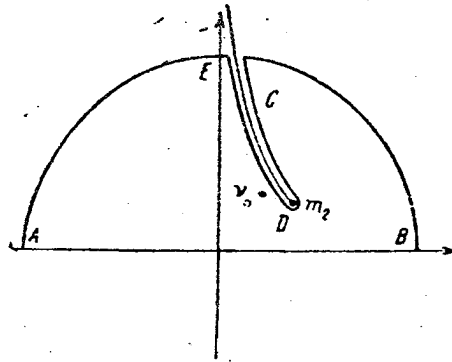
Fig. 2

In this last formula we have written everywhere $|y|$, since for $y < 0$ by taking an analogous contour of integration in the lower half-plane, we arrive at the very same formula (96). From (96) it follows at once that[10]

$$\lim_{z \to -\infty} (e^{\beta z} J) = \frac{i\pi\beta}{\nu_0} e^{i\nu_0|y| - \beta h},$$

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[10] The condition $J \to 0$ as $z \to +\infty$ cannot determine $P(y)$, since Re $\beta > 0$ and $|e^{\beta z}| \to \infty$ as $z \to \infty$. Therefore it is necessary to find $\lim_{z \to -\infty} (e^{\beta z} J)$.

and then, on the basis of (94), after some elementary manipulation,

$$J = \frac{\pi i}{2} H_0(m_2 r_1) + \frac{\pi i \beta}{\nu_0} e^{i\nu_0|y|-\beta(z+h)} -$$

$$- \frac{\pi i \beta}{2} \int_{-\infty}^{z+h} e^{\beta(u-z-h)} H_0(m_2\sqrt{y^2+u^2})du, \qquad (97)$$

where ν_0 is given by formula (95).

Substituting (97) into (92), we finally find

$$\hat{E}_z^{(2)} = - \frac{iqC}{m_2^2} \left[\frac{\partial}{\partial z} H_0(m_2 r_1) - 2\beta H_0(m_2 r_1) + \right.$$

$$\left. + 2\beta \left\{ \int_{-\infty}^{z+h} \beta e^{\beta(u-z-h)} H_0(m_2\sqrt{y^2+u^2})du - \frac{2\beta}{\nu_0} e^{i\nu_0|y|-\beta(z+h)} \right\} \right], \qquad (98)$$

A simple approximate expression for the integral which enters into this formula will be given below for a wide range of values for the co-ordinates y and z of interest.

We pass now to the determination of the vertical component $\hat{H}_z^{(2)}$ of the magnetic field, which satisfies equation (10) and condition (84) at the earth's surface. Taking account of the obvious condition of anti-symmetry in y, we express the general solution of equation (10) for $\hat{H}_z^{(2)}$ in the form

$$\hat{H}_z^{(2)} = \int_0^\infty e^{-\eta_2 z} B(\nu) \sin\nu y d\nu. \qquad (99)$$

Taking $H_z^{(2)0}$ from (15) and applying (21), we obtain for $z<h$

$$H_z^{(2)0} = \frac{2\omega C}{\pi c m_2^2} \int_0^\infty \frac{e^{\eta_2(z-h)}}{\eta_2} \nu \sin\nu y d\nu. \qquad (100)$$

---

[10] The condition $J \to 0$ as $z \to +\infty$ cannot determine $p(y)$, since $\mathrm{Re}\beta > 0$ and $|e^{\beta z}| \to \infty$ as $z \to \infty$. Therefore it is necessary to find $\lim_{z \to 0} (e^{\beta z} J)$.

Substituting (99) and (100) into the boundary condition (81), we find

$$\tilde{H}_z^{(2)} = \frac{2\omega C}{\pi c m_2^2} \int_0^\infty \frac{e^{-\eta_2(z+h)}}{\eta_2} \left(1 + \frac{2\eta_2}{\gamma-\eta_2}\right) v \sin v y \, dv. \tag{101}$$

Using formula (21), one can rewrite (101) in the form

$$\tilde{H}^{(2)} = -\frac{i\omega C}{c m_2^2} \frac{\partial}{\partial y} \left\{ H_0(m_2 r_1) + \frac{4i}{\pi} K \right\}, \tag{102}$$

where

$$K = \int_0^\infty \frac{e^{-\eta_2(z+h)}}{\eta_2-\gamma} \cos v y \, dv. \tag{103}$$

To transform the integral K we apply the same method as we used for the integral J entering into $\tilde{E}_z^{(2)}$. As a matter of fact, we find

$$e^{\gamma z} K = \frac{\pi i}{2} \int_0^\infty e^{\gamma z} \frac{\partial}{\partial z} H_0(m_2 r_1) dz + R(y),$$

and since $\mathrm{Re}\gamma < 0$ and $\lim_{z\to\infty}(e^{\gamma z}K)=0$, we obtain at once

$$K = -\frac{\pi i}{2} \int_{z+h}^\infty e^{\gamma(u-z-h)} \frac{\partial}{\partial u} H_0(m_2 \sqrt{y^2+u^2}) du.$$

Substituting this expression for K into formula (102), we find

$$\tilde{H}_z^{(2)} = -\frac{i\omega C}{c m_2^2} \frac{\partial}{\partial y} \left\{ H_0(m_2 r_1) + 2 \int_{z+h}^\infty e^{\gamma(u-z-h)} \frac{\partial}{\partial u} H_0(m_2\sqrt{y^2+u^2}) du \right\}. \tag{104}$$

In the following we shall show how, using the quantities $S^{(2)}$, $\tilde{Q}^{(2)}$, $\tilde{E}_z^{(2)}$ and $\tilde{H}_z^{(2)}$, one can determine the remaining field components through simple differentiations. But our immediate concern will be the problem of the limits of applicability of the obtained solution.

8. <u>Transformation of the rigorous solution. Limits of applicability</u>
   <u>of the approximate solution.</u>

In this paragraph, transforming the integrals which represent the rigorous

solution to the problem (here applying a method analogous to that used by

V.A. Fock to obtain the Weyl-van der Pol formula from A. Sommerfeld's

rigorous solution to the problem of the field of a vertical dipole) and

comparing the formulas obtained with the solution of the previous paragraph,

we determine the limits of applicability of the latter and are able to

estimate its accuracy.

Figuring that at moderate distances from the wire, near the earth's

surface, E_z is the dominant electric field component, while the two others

are always small, we shall study only the formulas for this field component.

However, one can carry out in an analogous fashion calculations for the other

components as w l.

We rewrite formula (23) in the form[11]

$$E_z^{(2)} = \frac{iqC}{m_2^2} \, J, \tag{105}$$

where

$$J = \frac{1}{\pi i} \int_{-0}^{\infty} \frac{k_3^2 \eta_2 - k_2^2 \eta_3}{k_3^2 \eta_2 + k_2^2 \eta_3} \, e^{-\eta_2(z+h) + i\nu|y|} d\nu . \tag{106}$$

The integrand has two branch points and a pole in the upper half plane.

We draw cuts from the branch point $\nu = m_2$ along the curve Re η_2 to $\nu = i\infty$ and

from the branch point $\nu = m_3$ along the curve Re $\eta_3 = 0$ also to $\nu = i\infty$. In this

case η_2 and η_3 have a positive real part in the upper half plane. The

pole is located at the point $\nu = \nu_0$ where ν_0 is determined from the equation

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[11]In the following calculations we take $\varepsilon_2' = 1$ and $k_2 = \omega/c$.

$$\sqrt{\frac{\nu_0^2 - m_3^2}{\nu_0^2 - m_2^2}} = -\left(\frac{k_3}{k_2}\right)^2 , \tag{107}$$

which gives

$$\nu_0^2 = \frac{k_2^2 k_3^2}{k_3^2 + k_2^2} - q^2, \quad \text{Im}\,\nu_0 > 0. \tag{108}$$

The point ν_0 in fact does lie in our sheet of the plane, since we can write

$$\sqrt{\nu_0^2 - m_3^2} = \frac{i\nu_0 k_3^2}{k_2 \sqrt{\epsilon_3' m_2^2 - q^2}} \;;\; \sqrt{\nu_0^2 - m_2^2} = - \frac{i\nu_0 k_2}{\sqrt{\epsilon_3' m_2^2 - q^2}} \quad ,$$

in which, for $\text{Im}\sqrt{\epsilon_3' m_2^2 - q^2} > 0$ both expressions written out have real parts greater than zero, and equation (107) is satisfied.

Taking int. account that the integral over an arc of infinite radius in the upper half-plane vanishes, we can represent the integral J in the form

$$J = 2P + Q_1 + Q_2, \tag{109}$$

where P is the residue at the point $\nu = \nu_0$, while Q_1 and Q_2 are integrals over the cuts $m_2 \to i\infty$ and $m_3 \to i\infty$ respectively.

We easily obtain that

$$P = -2 \frac{\chi^2 k_3^4}{(k_3^4 - k_2^4)\nu_0} e^{i\chi(z+h) + i\nu_0 |y|} . \tag{110}$$

where the notation

$$\chi = i \sqrt{\nu_0^2 - m_2^2} = - \frac{k_2^2}{\sqrt{k_3^2 + k_2^2}} \quad . \tag{111}$$

has been introduced.

Transforming the integrand of Q_1, we find

$$Q_1 = \frac{1}{\pi i} \int\limits_{i\infty}^{m_2^+} \left(1 + \frac{2}{\epsilon_3'^2 - 1} - \frac{2k_3^4\chi^2}{(k_3^4 - k_2^4)(\nu^2 - \nu_0^2)} - \frac{2k_2^2 k_3^2 \eta_2 \eta_3}{(k_3^4 - k_2^4)(\nu^2 - \nu_0^2)}\right) e^{-\eta_2(z+h)+i\nu|y|} \, d\nu.$$

(112)

From formula (21) it follows that

$$\frac{1}{\pi i} \int\limits_{i\infty}^{m_2^+} e^{-\eta_2(z+h)+i\nu|y|} \, d\nu = -\frac{\partial}{\partial z} H_0(m_2 r_1).$$

Taking into account that the point ν_0 is very close to m_2, we expand $\sqrt{\nu^2 - m_3^2}$ in powers of $\nu^2 - \nu_0^2$. We obtain

$$\frac{\sqrt{\nu^2 - m_3^2}}{\nu^2 - \nu_0^2} = \frac{i\chi k_3^2}{k_2^2} \frac{1}{\nu^2 - \nu_0^2} + \frac{ik_2^2}{2\chi k_3^2} + \ldots$$

We substitute this expansion into the last term of formula (112), which gives

$$\int\limits_{i\chi}^{m_2^+} \frac{\eta_2 \eta_3}{\nu^2 - \nu_0^2} e^{-\eta_2(z+h)+i\nu|y|} \, d\nu = -\frac{i\chi k_3^2}{k_2^2} \frac{\partial U}{\partial z} - \frac{\pi k_2^2}{2\chi k_3^2} \frac{\partial^2}{\partial z^2} H_0(m_2 r_1) + \ldots$$

where the notation

$$U = \int\limits_{i\infty}^{m_2^+} e^{-\eta_2(z+h)+i\nu|y|} \frac{d\nu}{\nu^2 - \nu_0^2} .$$

has been introduced.

Formula (112) takes now the form

$$Q_1 = - \left(1 + \frac{2}{\epsilon_3'^2 - 1}\right) \frac{\partial}{\partial z} H_0(m_2 r_1) + \frac{2}{\pi} \frac{\chi k_3^4}{k_3^4 - k_2^4} V - \frac{i k_2^4}{(k_3^4 - k_2^4)\chi} \frac{\partial^2}{\partial z^2} H_0(m_2 r_1) + \ldots,$$

$$(113)$$

where

$$V = \frac{\partial U}{\partial z} + i\chi U = - \int_{i\infty}^{m_2^+} \frac{\eta_2 - i\chi}{\nu^2 - \nu_0^2} e^{-\eta_2(z+h) + i\nu|y|} \, d\nu = - \int_{i\infty}^{m_2^+} e^{-\eta_2(z+h) + i\nu|y|} \frac{d\nu}{\eta_2 + i\chi}$$

Through a direct differentiation we obtain a differential equation for the quantity V

$$\frac{\partial V}{\partial z} - i\chi V = - \pi i \frac{\partial}{\partial z} H_0(m_2 r_1),$$

while from the integral representation it is clear that

$$V \to 0 \text{ as } z \to -\infty.$$

Assuming $V = - \pi i H_0(m_2 r_1) + e^{i\chi(z+h)} V_1$, we find

$$\frac{\partial V_1}{\partial z} = \pi \chi e^{-i\chi(z+h)} H_0(m_2 r_1),$$

and thus

$$V = - \pi i H_0(m_2 r_1) + \pi \chi e^{i\chi(z+h)} \int_{-\infty}^{z+h} e^{-i\chi\xi} H_0(m_2 \sqrt{y^2 + \xi^2}) \, d\xi.$$

In the integral Q_2 the integrand will be as it was in Q_1 in formula (112), however here only the integral in the last term does not vanish, and it can be shown that Q_2 will be on the order of $e^{ik_3 r_1}/\epsilon_3' r_1$.

In the following, since $|\epsilon_3'| \gg 1$, we will neglect unity compared with

ε_3'. Then we can neglect the quantity Q_2 and formula (109) in this approximation becomes

$$J = - \frac{\partial}{\partial z} H_0(m_2 r_1) - 2i\chi H_0(m_2 r_1) + 2\chi^2 \left[\int_{-\infty}^{z+h} e^{-i\chi(\xi-z-h)} H_0(m_2\sqrt{y^2+\xi^2})d\xi -\right.$$

$$\left. - \frac{2}{v_0} e^{i\chi(z+h)+iv_0|y|} \right] - \frac{i}{\varepsilon_3'^2 \chi} \frac{\partial^2}{\partial z^2} H_0(m_2 r_1) + \dots \,, \qquad (114)$$

and formula (105)

$$\tilde{E}_z^{(2)} = - \frac{iqC}{m_2^2} \left\{ \frac{\partial}{\partial z} H_0(m_2 r_1) + 2i\chi H_0(m_2 r_1) - \right.$$

$$- 2\chi^2 \left[\int_{-\infty}^{z+h} e^{-i\chi(\xi-z-h)} H_0(m_2\sqrt{y^2+\xi^2})d\xi - \frac{2}{v_0} e^{i\chi(z+h)+iv_0|y|} \right] +$$

$$\left. + \frac{i}{\varepsilon_3'^2 \chi} \frac{\partial^2}{\partial z^2} H_0(m_2 r_1) + \dots \right\} \qquad (115)$$

In this same approximation $-i\chi$ coincides with the constant β of the previous paragraph, and likewise the definitions of v_0 by formulas (108) and (95) coincide. Formula (115) can now be written in the form

$$\tilde{E}_z^{(2)} = - \frac{iqC}{m_2^2} \left\{ \frac{\partial}{\partial z} H_0(m_2 r_1) - 2\beta H_0(m_2 r_1) + \right.$$

$$+ 2\beta^2 \left[\int_{-\infty}^{z+h} e^{\beta(\xi-z-h)} H_0(m_2\sqrt{y^2+\xi^2})d\xi - \frac{2}{v_0} e^{-\beta(z+h)+iv_0|y|} \right] +$$

$$\left. + \frac{1}{\varepsilon_3'^2 \beta} \frac{\partial^2}{\partial z^2} H_0(m_2 r_1) + \dots \right\} . \qquad (116)$$

This last formula coincides with formula (98) of the preceeding paragraph if we can neglect the last term, which is possible for

$$\left| \varepsilon_3' k_3 (z+h) \right| \gg 1. \tag{117}$$

This inequality is satisfied rather well for frequencies $\omega \lesssim 10^6 \sec^{-1}$ and earth conductivities $\sigma_3 \gtrsim 10^7$ for $(z+h) \gtrsim 2 \times 10^3$ cm. An estimate shows that, at the earth's surface, for the most unfavorable conditions in the parameters for the range we are studying -- $\sigma_3 = 9 \times 10^6$, $\omega = 10^6$ and a height of the wire above the earth $h = 10$m -- neglecting the last term in formula (116) gives an error not exceeding 2%. The error decreases very fast as the conductivity increases from here.

In a quite similar manner one could also check the formulas for the component $\tilde{E}_y^{(2)}$ (all the rest of the components are calculated, as pointed out in section 3, from these two, but we will not do this, in order not to prolong this extensive presentation).

9. Computational formulas.

We will now find expressions for the components $\tilde{E}_x^{(2)}$, $\tilde{E}_y^{(2)}$, $\tilde{H}_x^{(2)}$, $\tilde{H}_y^{(2)}$ in terms of the quantities $S^{(2)}$, $\tilde{Q}^{(2)}$ defined in section 7 and the components $\tilde{E}_z^{(2)}$ and $\tilde{H}_z^{(2)}$.

From Maxwell's equations it is easy to obtain the following formulas

$$\tilde{H}_y^{(2)} = \frac{1}{1-\alpha^2} \left\{ \frac{ic}{\omega} \frac{\partial S^{(2)}}{\partial z} - \frac{qc}{\omega} \tilde{E}_z^{(2)} - \alpha S^{(2)} - \frac{i\alpha c}{\omega} \frac{\partial \tilde{H}_z^{(2)}}{\partial y} \right\},$$

$$\tilde{H}_y^{(2)} = \frac{1}{1-\alpha^2} \left\{ \frac{ic}{\omega} \frac{\partial \tilde{E}_z^{(2)}}{\partial y} - \frac{ic}{\omega} \frac{\partial \tilde{Q}^{(2)}}{\partial z} + \alpha \tilde{Q}^{(2)} - \frac{q\alpha c}{\omega} \tilde{H}_z^{(2)} \right\},$$

$$\tilde{E}_y^{(2)} = \frac{1}{1-\alpha^2} \left\{ \tilde{Q}^{(2)} - \frac{i\alpha c}{\omega} \frac{\partial \tilde{Q}^{(2)}}{\partial z} + \frac{i\alpha c}{\omega} \frac{\tilde{E}_z^{(2)}}{\partial y} - \frac{q\alpha^2 c}{\omega} \tilde{H}_z^{(2)} \right\}, \tag{118}$$

$$\tilde{E}_x^{(2)} = \frac{1}{1-\alpha^2} \left\{ S^{(2)} - \frac{\alpha q c}{\omega} \tilde{E}_z^{(2)} - \frac{i\alpha c}{\omega} \frac{\partial S^{(2)}}{\partial z} + \frac{i\alpha^2 c}{\omega} \frac{\partial \tilde{H}_z^{(2)}}{\partial y} \right\},$$

In formulas (118) the quantity α^2 in the coefficients should be neglected in comparison with unity, since the approximate boundary conditions themselves are only valid to an accuracy on the order of α^2

(we note that $|\alpha^2| < 0.01$ for all ranges of parameters of the problem of interest to us).

Applying formulas (88), (89), (98) and (104) and discarding quantities of the order of α^2, we obtain in place of (118)

$$\hat{H}^{(2)} = \frac{ikC}{m_2^2} \frac{\partial}{\partial z} H_0(m_2 r_1) + \frac{2\alpha C}{k^2} \frac{\partial^2}{\partial z^2} H_0(m_2 r_1) - \frac{2\alpha q^2 C}{m_2^2} H_0(m_2 r_1) +$$

$$+ \frac{2\alpha q^2 C}{m_2^2} \left\{ \beta \int_{-\infty}^{z+h} e^{\beta(u-z-h)} H_0(m_2\sqrt{y^2+u^2}) du - \frac{2\beta}{\nu_0} e^{-\beta(z+h)+i\nu_0|y|} \right\}, \quad (119)$$

$$\hat{H}_x^{(2)} = - \frac{2i\alpha q C}{m_2^2} \frac{\partial}{\partial y} H_0(m_2 r_1) \qquad\qquad (120)$$

$$\hat{E}_y^{(2)} = - \frac{iqC}{m_2^2} \frac{\partial}{\partial y} H_0(m_2 r_1) \qquad\qquad (121)$$

$$\hat{E}_x^{(2)} = CH_0(m_2 r_1) - \frac{2i\alpha C}{k} \frac{\partial}{\partial z} H_0(m_2 r_1) \qquad\qquad (122)$$

In formulas (119)-(122) $k=\omega/c$. Formula (104) can be rewritten in the form

$$\hat{H}_z^{(2)} = - \frac{ikC}{m_2^2} \frac{\partial}{\partial y} \left\{ H_0(m_2 r_1) - \frac{2i\alpha}{k} \frac{\partial}{\partial z} H_0(m_2 r_1) \right\} , \qquad (123)$$

and from formula (98) one can obtain for points near the earth's surface, sufficiently removed from the wire axis,

$$\tilde{E}_z^{(2)} = - \frac{iqC}{m_2} \left\{ \frac{z+h}{y} H_0'(m_2|y|) + 2[1-\beta(z+h)]\frac{\beta}{m_2} H_0(m_2|y|) - \right.$$

$$\left. - 2\left(\frac{\beta}{m_2}\right)^2 e^{im_2|y|} - 2\left(\frac{\beta}{m_2}\right)^3 m_2|y| \cdot H_0'(m_2|y|) \right\} . \tag{124}$$

For $y \geq 10(z+h)$ formula (124) gives an error not exceeding 1%. In formulas (119)-(124) it is also necessary to substitute the value of C from formula (40), which can be written approximately in the form

$$C = m_2^2 \frac{\pi I}{\omega} . \tag{125}$$

For points near the earth's surface and sufficiently far from the wire (119) can be rewritten, analogously to (124), in the form

$$\tilde{H}_y^{(2)} = \frac{ikC}{m_2^2} \left[\frac{\partial}{\partial z} H_0(m_2 r_1) + \frac{2\beta m_2^2}{k^4} \frac{\partial^2}{\partial z^2} H_0(m_2 r_1) - \right.$$

$$- 2\beta \left(\frac{q}{k}\right)^2 \left\{ [1-\beta(z+h)]H_0(m_2 r_1) + \frac{\beta}{m_2} e^{im_2|y|} + \right.$$

$$\left. \left. + \left(\frac{\beta}{m_2}\right)^2 m_2|y|H_0'(m_2|y|) \right\} \right] . \tag{126}$$

Formulas (120)-(126) for the secondary fields of the earth determine them completely near its surface at a sufficient distance from the wire, and from here the calculation of the field will present no further difficulty.

We note additionally that the rigorous integral representations of the fields obtained in section 3 can be transformed in the same way as was done in section 8 to compare the approximate solution with the rigorous one, but without neglecting the quantity $\frac{1}{\varepsilon_3'}$ compared to one; in this way

comparatively simple formulas for the fields in the case of higher frequencies than were studied here can be obtained.

Accounting for the influence of the secondary fields on the current
distribution over the wire.

In section 4 we satisfied approximate boundary conditions at the
surface of the wire without taking into account the change in current
distribution over its cross-section under the action of the secondary
field. Here we will show how to account for this change to a desired
degree of accuracy and estimate this quantity, and subsequently also the
error involved in satisfying the approximate boundary conditions at the
surface of the wire.

We will take into account this disturbance of symmetry of the current
distribution over the wire by introducing into the solution of Maxwell's
equations terms proportional to $\sin\theta$ and $\cos\theta$, neglecting further
terms in the Fourier series expansion of the solution. Simultaneously we
will expand the secondary field about the point $(y=0, z=h)$ in a power
series in y and $(z-h)$ and neglect all powers higher than the first.
To reduce the calculations we assume at once that E_x, E_z and H_y are
symmetric with respect to the z-axis, while H_x, H_z and E_y are anti-
symmetric. We easily obtain the components E_θ and H_θ from E_x and H_x,
using Maxwell's equations.

Thus, we find the field inside the wire in the form

$$E_x^{(1)} = AJ_0(m_1 r) + A_1 \sin\theta J_1(m_1 r),$$

$$H_x^{(1)} = B_1 \cos\theta \, J_1(m_1 r),$$

$$E_\theta^{(1)} = \frac{i\mu_1 \omega}{cm_1} B_1 \cos\theta \, J_1'(m_1 r) - \frac{iq}{rm_1^2} A_1 \cos\theta \, J_1(m_1 r), \qquad (a)$$

$$H_\theta^{(1)} = \frac{iq}{m_1^2 r} B_1 \sin\theta \, J_1(m_1 r) - \frac{i\omega\varepsilon_1'}{cm_1} [AJ_0'(m_1 r) + A_1 \sin\theta \, J_1'(m_1 r)].$$

The field outside the wire in the neighborhood of its surface will have the components

$$E_x^{(2)} = CH_0(m_2 r) + C_1 \sin\theta \, H_1(m_2 r) - \left(\tilde{E}_x^{(2)}\right) - r \sin\theta \left(\frac{\partial \tilde{E}_x^{(2)}}{\partial y}\right) \quad,$$

$$H_x^{(2)} = D_1 \cos\theta \, H_1(m_2 r) - r \cos\theta \left(\frac{\partial \tilde{H}_x^{(2)}}{\partial y}\right) \quad.$$

$$E_\theta^{(2)} = \frac{i\omega}{cm_2} \left[D_1 \cos\theta \; H_1'(m_2 r) - \frac{1}{m_2} \cos\theta \left(\frac{\partial \tilde{H}_x^{(2)}}{\partial y}\right) \right] - \qquad \text{(b)}$$

$$- \frac{iq}{rm_2^2} \left[C_1 \cos\theta \, H_1(m_2 r) - r \cos\theta \left(\frac{\partial \tilde{E}_x^{(2)}}{\partial z}\right) \right] ,$$

$$H_\theta^{(2)} = \frac{iq}{rm_2^2} \left[D_1 \sin\theta \, H_1(m_2 r) - r \sin\theta \left(\frac{\partial \tilde{H}_x^{(2)}}{\partial y}\right) \right] -$$

$$- \frac{i\omega}{cm_2} \left[CH_0'(m_2 r) + C_1 \sin\theta \; H_1'(m_2 r) - \frac{1}{m_2} \sin\theta \left(\frac{\partial \tilde{E}_x^{(2)}}{\partial z}\right) \right] \quad,$$

in which $\left(\tilde{E}_x^{(2)}\right)$, $\left(\frac{\partial \tilde{E}_x^{(2)}}{\partial z}\right)$ and $\left(\frac{\partial \tilde{H}_x^{(2)}}{\partial y}\right)$ denote the corresponding

quantities at the center of the wire, and ε_2' and μ_2 are taken equal

to unity.

For $r=a$, the conditions

$$E_x^{(2)} = E_x^{(1)}, \quad H_x^{(2)} = H_x^{(1)}, \quad E_\theta^{(2)} = E_\theta^{(1)}, \quad H_\theta^{(2)} = H_\theta^{(1)}.$$

should be satisfied.

Substituting therein (a) and (b), we obtain a system of six equations which decompose into the two separate subsystems

$$AJ_0(m_1 a) = CH_0(m_2 a) - \left(\tilde{E}_x^{(2)}\right) ,$$

$$\left. \frac{\varepsilon_1'}{m_1} AJ_0'(m_1 a) = \frac{1}{m_2} CH_0'(m_2 a) \right\} \quad \text{(c)}$$

and

$$A_1 J_1(am_1) - C_1 H_1(m_2 a) = - a \left(\frac{\partial \tilde{E}_x^{(2)}}{\partial z}\right) ,$$

$$B_1 J_1(am_1) - D_1 H_1(m_2 a) = - a \left(\frac{\partial \tilde{H}_x^{(2)}}{\partial y}\right) ,$$

$$B_1 \frac{i\omega\mu_1}{cm_1} J_1'(m_1 a) - A_1 \frac{iq}{am_1^2} J_1(am_1) - D_1 \frac{i\omega}{cm_2} H_1'(m_2 a) +$$

$$+ C_1 \frac{iq}{am_2^2} H_1(m_2 a) = - \left(\tilde{E}_z^{(2)}\right) ,$$

$$B_1 \frac{iq}{am_1^2} J_1(am_1) - A_1 \frac{i\omega\varepsilon_1'}{cm_1} J_1'(am_1) - D_1 \frac{iq}{am_2^2} H_1(am_2) +$$

$$+ C_1 \frac{i\omega}{cm_2} H_1'(m_2 a) = \left(\tilde{H}_y^{(2)}\right) .$$

$$\left. \right\} \quad \text{(d)}$$

Here we have used the relations

$$- \frac{\partial \tilde{H}_x^{(2)}}{\partial y} = \frac{i\omega}{c} \tilde{E}_z^{(2)} + iq\tilde{H}_y^{(2)}; \qquad - \frac{\partial \tilde{E}_x^{(2)}}{\partial z} = iq\tilde{E}_z^{(2)} + \frac{i\omega}{c} \tilde{H}_y^{(2)} ,$$

which follow immediately from Maxwell's equations.

The system (c) coincides with system (36), and consequently the equation for the propagation constant obtained in section 4 and the expressions for the constants A and C in terms of the total current remain, to this approximation, unchanged.

In the formulas for the fields, additional terms must be included, the coefficients of which are determined from the system (d) after the constants from system (c) which enter into it have been determined. Having solved this system and substituted the values of the secondary field components, we obtain expressions for all four coefficients. However, we will not carry out these calculations here, but only indicate estimates for the coefficients. For the coefficients of the external fields we have

$$C_1 \simeq -\frac{2i}{\pi} C \left(\frac{a}{2h}\right)^2 \cdot 2m_2 h, \tag{e}$$

$$D_1 \simeq -\frac{2}{\pi} C \left(\frac{a}{2h}\right)^2 \cdot 2h \; k_2 \frac{q}{m_2 a k_1}, \tag{f}$$

It is immediately obvious that the error involved in neglecting these quantities is, in our case, extremely small.

It should be remarked that the method presented here for satisfying the boundary conditions at the surface of the wire represents only the first step in a rigorous solution to the problem, whose execution can be important in the case of a wire situated at a relatively small height above the earth, or likewise in the case of a multiconductor line wherein the distances between separate conductors are not large compared to their radii.

For such problems as stated, the current flowing through each conductor, as well as the characteristic field of each, should be expanded in a Fourier series in sines and cosines of multiples of the angles, and thereupon the secondary field of the earth due to each of these harmonics determined. In satisfying the boundary conditions at the surface of each conductor, the characteristic field of each separate conductor and the secondary field of the earth should be represented also in the form of such a series expansion. As a result of this process, a linear (and, generally speaking, infinite) system of equations is obtained, truncating which at some point, we can

obtain a solution of the problem to the desired degree of accuracy.

Such a process will be very complex, if it is necessary to account

for very many terms in the series.

However, for practical problems when the radii of the wires is small

compared to the distances between them and to the earth, it can be every-

where assumed that the secondary field of the earth is determined only

by the axially-symmetric part of the cross-sectional current distribution

in the conductors;[12] analogously, instead of finding at an arbitrary

conductor the proper field of all the other conductors, one may likewise

usually consider it due only to the axially symmetric part of the current

distribution over the cross-section. Finally, in practical problems of

wave propagation along multiconductor lines, it is clear that it will

suffice always to consider the disturbance to symmetry of the current

distribution ove. the wire cross-section as due only to terms proportional

to sin θ and cos θ and neglecting higher angular dependences. For these

simplifications the problem becomes sufficiently simple that it might be

relatively easily brought to a solution.

---

[12]Since computation of the successive terms of the Fourier series is
analogous to the computation of the dipole wave and gives terms which
at large distances compared to the radii of the wires will be very small
compared to the first term.

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DETAILS OF THE THEORY OF WAVE PROPAGATION ALONG

MULTICONDUCTOR TRANSMISSION LINES IN CONNECTION WITH

SOME ENGINEERING PROBLEMS[†]

by L.S. Perel'man

Introduction

The problem of electromagnetic wave propagation along multiconductor

lines is usually solved via the telegraphist's equations [1-4]. For such

a solution the electromagnetic field in the air is taken to be quasi-

stationary, while the influence of the earth (with the exception of [3])

is accounted for by means of Carson's integrals [5], which figure in the

self and mutual impedances of the wires of the line. In accounting for

the influence of the earth Carson neglected the transverse electric field

components in he earth and the longitudinal displacement currents in

the earth. Wise [6] refined Carson's integrals by taking into account

the longitudinal displacement currents in the earth.

Solving the system of telegraphist's equations determines propagation

constants and relationships among the currents in the wires for the wave

channels of a multiconductor line (each wave channel is a definite system

of currents propagating along the line with a single propagation constant).

The assumptions made in such approximate methods of solution can, at high

frequencies, in certain situations lead to appreciable errors in determin-

ing the parameters of the wave channels and the electromagnetic fields,

particularly at large distances from the wires. Thus for a study of the

electromagnetic field of coronal radio interference from power lines, of

the parameters of high frequency communication channels over power lines

and of the influence of power lines on communication lines, a more accurate

solution must be considered.

The existing theoretical methodology for computing coronal radio interference due to power line wires, worked out by Adams [7-9], is based on a number of assumptions: the electromagnetic field of the radio interference is assumed to be that for ideally conducting wires and earth, and the parameters of the wave channels are considered on the basis of equal wave impedances of all wires for a given wave channel, mutually orthogonal potential gradients for different wave channels, and a determination of losses in the conducting media using their wave impedances and the magnetic fields on the boundary between the conducting media and the air.

G.A. Grinberg and B.E. Bonshtedt [10] worked out a rigorous theory of electromagnetic wave propagation along a single-wire line over the earth. M.V. Kostenko [11] used this theory to consider wave propagation over a three wire line with horizontally spaced wires; in the course of the solution he introduced several simplifications and obtained a system of equations for determining the parameters of the wave channels, analogous to Carson's system of equations [1]. With a rigorous solution of the problem of wave propagation along a multiconductor line, the error in the approximate solutions of this problem could clearly be established.

In the present paper, the method proposed in [10] is expanded and used to investigate wave propagation along multiconductor lines under the following assumptions. It is assumed that the multiconductor line consists of n long parallel uniform wires of circular cross sections and smooth surfaces. The earth is taken to be uniform, and there is no ground cable. In order to solve the problem we neglect the "proximity effect" and the "end effect" in the wires, and consider the wires to be parallel to the place of the earth. The problem is solved for the so-called "fundamental" sinusoidal waves [12] which propagate along the wires without radiation.

The MKSA system of units is used in this paper.

1. Solution of the problem in general form

A diagram of the multiconductor line and the notation for its geometrical parameters is given in Fig. 1.
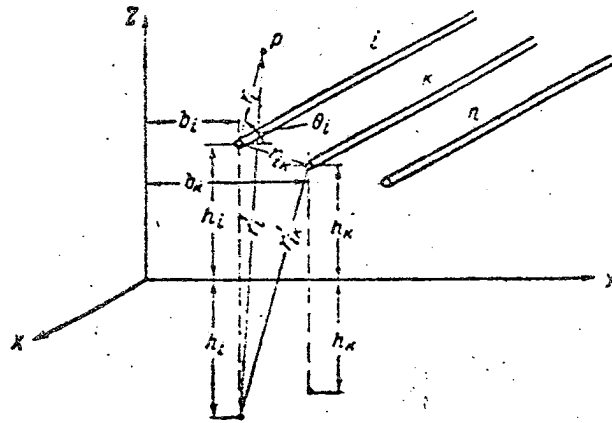
Fig. 1. Diagram of multiconductor line

To determine the electromagnetic field in the air, earth and wires, it is necessary to solve Maxwell's equations and satisfy the boundary conditions at the media interfaces. We take the time dependence of the electromagnetic field, as well as the variation along the x coordinate (parallel to the wires) to be of the form $e^{j(\omega t - \gamma x)}$, where γ is the propagation constant $(\gamma = \alpha - j\beta)$. Here Maxwell's equations in each medium can be written in the form:

$$\left. \begin{aligned} \operatorname{rot} \dot{H} &= j\omega\varepsilon'\dot{E}, \\ \operatorname{rot} \dot{E} &= -j\omega\mu\dot{H}, \end{aligned} \right\} \tag{1}$$

where

$$\varepsilon' = \varepsilon - j\frac{\sigma}{\omega} ;$$

ε, μ and σ are respectively the absolute dielectric permittivity, absolute magnetic permeability and conductivity of the medium.

Since we are neglecting the proximity effect, the superposition method can be used to solve Maxwell's equations, taking for a general solution for the fields in the air and the earth a sum of the partial solutions for each wire above the earth obtained in [10], and for the fields inside the wires the Sommerfeld solution for an isolated wire.

We designate the electric and magnetic field strengths in the i$\underline{^{th}}$

medium induced by the current in the kth wire by $E_k^{(i)}$ and $H_k^{(i)}$; for the media the index 0 denotes air, g the earth, and for the wires the indices run from 1, 2, ..., k, ...n. In its own polar coordinate system (r, θ, x) the field induced by each wire (without the presence of the earth) in agreement with Sommerfeld's solution is determined by the formulas [10]:

inside the kth wire

outside the kth wire

$$
\begin{aligned}
\dot{E}_x^{(k)} &= A_k J_0(m_k r_k); & \dot{E}_{xk}^{(0)} &= C_k H_0(m_0 r_k); \\
\dot{E}_r^{(k)} &= -\frac{j\gamma}{m_k} A_k J_0'(m_k r_k); & \dot{E}_{rk}^{(0)} &= -\frac{j\gamma}{m_0} C_k H_0'(m_0 r_k); \\
\dot{H}_\theta^{(k)} &= -\frac{j\omega\varepsilon'_k}{m_k} A_k J_0'(m_k r_k); & \dot{H}_{\theta k}^{(0)} &= -\frac{j\omega\varepsilon_0}{m_0} C_k H_0'(m_0 r_k); \\
\dot{E}_\theta^{(k)} &= \dot{H}_x^{(k)} = \dot{H}_r^{(k)} = 0; & \dot{E}_{\theta k}^{(0)} &= \dot{H}_{xk}^{(0)} = \dot{H}_{rk}^{(0)} = 0.
\end{aligned} \tag{2}
$$

Here we have made the designations:

$$
\begin{aligned}
m_i &= \sqrt{k_i^2 - \gamma^2}; & \text{Im } m_i > 0; \\
k_i &= \sqrt{\mu_i \varepsilon'_i}; & \text{Im } k_i > 0;
\end{aligned}
$$

A_k and C_k are integration constants;

J_0 and J_0' are the Bessel function of the 1st kind and zeroth order and its derivative;

H_0 and H_0' are the Hankel function of the 1st kind and zeroth order and its derivative.

Below we will neglect displacement currents in the wires and the quantity γ^2 compared to k_k^2:

$$
\varepsilon'_k = -j\frac{\sigma_k}{\omega}; \quad m_k = k_k = \sqrt{-j\omega\mu_k\sigma_k} \quad (k=1, 2, \ldots, n).
$$

Adding the longitudinal electric field components in the air at an arbitrary point p (Fig. 1), in accordance with the superposition

principle and the results obtained in [10] we have

$$\dot{E}_x^{(0)} = \sum_{k=1}^{n} C_k \left[H_0(m_0 r_k) - H_0(m_0 r_k') + \frac{2jk_0^2}{\pi m_0^2} F_{kp} \right], \qquad (3)$$

where

$$F_{kp} = 2 \int_0^\infty \frac{n_0 n_g - \nu^2}{k_g^2 \eta_0 + k_0^2 \eta_g} e^{-\eta_0(z_p + h_k)} \cos\nu(y_p - b_k) d\nu;$$

z_p and y_p are the coordinates of the point p;

$$\eta_0 = \sqrt{\nu^2 - m_0^2}; \quad \text{Re } \eta_0 > 0;$$

$$\eta_g = \sqrt{\nu^2 - m_g^2}; \quad \text{Re } \eta_g > 0.$$

In the derivation of expression (3) we have set $\mu_g = \mu_0$.

The propagation constants and current relationships in the wires are determined by enforcing the boundary conditions at the surfaces of all the wires. On the surface of the kth wire, when the proximity effect is neglected, the following boundary conditions must be satisfied [10]:

$$\dot{E}_x^{(k)} = \dot{E}_{xk}^{(0)}; \quad \dot{H}_\theta^{(k)} = \dot{H}_{\theta k}^{(0)} \qquad (4)$$

In the first of conditions (4) the magnitude of the field on the surface of the kth wire due to currents on the other wires and in the earth is replaced by the magnitude of this field on the axis of the kth wire.

Expressing A_k and C_k in terms of the wire currents, and considering that usually, up to very high frequencies,

$$H_0' (m_0 a_k) = \frac{2j}{\pi} \cdot \frac{1}{m_0 a_k} \quad ,$$

where a_k is the radius of the kth wire, we obtain from the first of conditions (4) and expression (3) the following equation for each kth wire:

$$\dot{I}_k \left[H_0(m_0 a_k) - H_0(m_0 \cdot 2h_k) + \frac{2jk_0^2}{\pi m_0^2} (F_k + M_k) \right] +$$

$$+ \sum_{i=1}^{n} \dot{I}_i \left[H_0(m_0 r_{ik}) - H_0(m_0 r_{ik}') + \frac{2jk_0^2}{\pi m_0^2} F_{ik} \right] = 0 \ (i \neq k), \ (5)$$

where $\dot{I}_i$ is the complex amplitude of the current in the ith wire:

$$M_k = - \frac{\mu_k J_0 (k_k a_k)}{\mu_0 \ k_k a_k J_0' (k_k a_k)} \quad ; \tag{6}$$

$$F_{ik} = 2 \int_0^\infty \frac{\eta_0 \eta_g - \nu^2}{k_g^2 \eta_0 + k_0^2 \eta_g} \ e^{-\eta_0 (h_i + h_k)} \ \cos \nu b_{ik} \ d\nu; \tag{7}$$

$$b_{ik} = b_i - b_k;$$

F_k is a particular case of the integral F_{ik} for $i=k$ and $b_{ik}=0$.

We introduce the notations:

$$\left. \begin{aligned} B_{kk} &= H_0(m_0 a_k) - H_0(m_0 \cdot 2h_k) + \frac{2jk_0^2}{\pi m_0^2} (F_k + M_k); \\ \\ B_{ik} &= H_0(m_0 r_{ik}) - H_0(m_0 r_k') + \frac{2jk_0^2}{\pi m_0^2} F_{ik} \\ & \hspace{4cm} (B_{ik} = B_{ki}) \end{aligned} \right\} \tag{8}$$

We then obtain from equation (5) the system of equations:

$$\dot{I}_1 B_{11} + \dot{I}_2 B_{12} + \ldots + \dot{I}_n B_{1n} = 0;$$

$$\dot{I}_1 B_{12} + \dot{I}_2 B_{22} + \ldots + \dot{I}_n B_{2n} = 0; \qquad \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \qquad (9)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\dot{I}_1 B_{1n} + \dot{I}_2 B_{2n} + \ldots + \dot{I}_n B_{nn} = 0.$$

The system (9) of equations has a nonzero solution in the currents

if its determinant Δ vanishes:

$$\Delta = \begin{vmatrix} B_{11}B_{12} \cdots B_{1n} \\ B_{12}B_{22} \cdots B_{2n} \\ \ldots\ldots\ldots\ldots \\ B_{1n}B_{2n} \cdots B_{nn} \end{vmatrix} = 0 \qquad (10)$$

Expression (10) leads to the following equation for m_0:

$$m_0^{2n} f_n(m_0) + m_0^{2(n-1)} f_{n-1}(m_0) + \ldots + m_0^2 f_1(m_0) + f_0(m_0) = 0, \qquad (11)$$

where $f_0(m_0)$, $f_1(m_0), \ldots,$ $f_n(m_0)$ are functions which, as follows from

expression (8), depend on differences of zeroth order Hankel functions

and on the integrals F_{ik}.

For the fundamental waves in the frequency range usually encountered

in practice, the differences of the zeroth order Hankel functions in

expressions (8) and the integrals $F_i k$ vary much more slowly than m_0;

consequently, $f_0(m_0), f_1(m_0), \ldots,$ $f_n(m_0)$ likewise vary more slowly than

m_0. Therefore equation (11) can be solved the the method of successive

approximations, in which the first approximation for the functions

$f(m_0)$ should be evaluated at $m_0 = 0$. In a certain range of the frequency

and of other parameters, the first approximation already gives a suffic-

iently accurate solution which, as will be shown below, corresponds to

Carson's solution [1].

It should be noted that the solution of the problem with the earth absent is carried out analogously, and the coefficients B_{kk} and B_{ik} will be determined by formulas (8), in which the second term on the right hand side and the integrals F_{ik} must be set equal to zero. In this case in order to solve equation (11) by the method of successive approximations, one can give beforehand an approximate order of magnitude for m_0 in the first approximation by solving a system of two identical wires similar in parameters to the ones under consideration, at the given frequency and with equal and opposite currents flowing in the wires, since the solution of such a system is obtained easily.

In solving equation (11) under the conditions indicated above, n distinct roots are obtained in general, corresponding to n wave channels:

$$m_{0(1)}^2, \ m_{0(2)}^2 \ \cdots \ , \ m_{0(s)}^2, \ \cdots \ , \ m_{0(n)}^2.$$

The roots of equation (11) determine 2n propagation constants:

$$\pm\gamma_{(1)}, \ \pm\gamma_{(2)}, \ \cdots \ , \ \pm\gamma_{(s)}, \ \cdots \ , \ \pm\gamma_{(n)},$$

since

$$\gamma(s) = \pm\sqrt{k_0^2 - m_{0(s)}^2}.$$

The solutions with the "+" signs correspond to direct waves, while those with "-" signs are backward waves. In the following we will consider only direct waves for simplicity.

The existence of n distinct roots $m_{0(s)}^2$ attests to the fact that for arbitrary $m_{0(s)}$ the rank of the matrix of the system (9) of equations is equal to n-1. The solution of the system (9) of equations for each wave channel then determines the relationships of the currents in the wires which are caluclated by the formula

$$\lambda_{ki(s)} = \frac{\dot{I}_{k(s)}}{\dot{I}_{i(s)}} = \frac{\Delta_{pk(s)}}{\Delta_{pi(s)}} , \tag{12}$$

where $\Delta_{pk(s)}$ and $\Delta_{pi(s)}$ are the algebraic complements of the elements of an arbitrary p$\underline{^{th}}$ row and respectively the k$\underline{^{th}}$ and i$\underline{^{th}}$ columns of the determinant Δ with $m_0 = m_{0(s)}$.

It should be noted that in problems with the earth absent, for multi-conductor systems possessing certain wire symmetries, equation (11) can have multiple roots and consequently the number of wave channels will be fewer than the number of wires.

And so, consideration of the system (9) of equations shows that an n-wire line has in general n wave channels, each of which is characterized by its own propagation constant and relationship between the curents in the wires. The complete solution for the electromagnetic field is the sum of the partial solutions for each of the wave channels. So, for example, for the vertical component of the electric field strength of an n-wire line in air at an arbitrary point p (Fig. 1), using the solutions obtained in [10] for the single wire, the following expression can be constructed on the basis of the foregoing analysis:

$$\dot{E}_z^{(0)} = - \frac{j}{4\varepsilon_0\omega} \sum_{s=1}^{n} \gamma_{(s)}\dot{I}_{i(s)} e^{-j\gamma(s)x_p} \sum_{k=1}^{n} \lambda_{ki(s)} \left[\frac{\partial}{\partial z} H_0(m_{0(s)}r_k) - \right.$$

$$- \frac{\partial}{\partial z} H_0(m_{0(s)}r'_k) + 2\delta H_0(m_{0(s)}r'_k) -$$

$$-2\delta^2 \int_{-\infty}^{z_p+h_k} e^{\delta(u-z_p-h_k)} H_0(m_{0(s)} \sqrt{(y_p-b_k)^2+u^2})du +$$

$$\left. + \frac{4\delta^2}{\nu_{0(s)}} e^{j\nu_{0(s)}|y_p-b_k|} {}^{-\delta(z_p+h_k)} \right] , \tag{13}$$

where $\dot{I}_{i(s)}$ is the complex amplitude of the current in the $i\underline{\text{th}}$ wire for the $s\underline{\text{th}}$ wave channel at the initial point $x=0$; this wire can be arbitrarily chosen for each wave channel,

$$\delta = - j\omega\varepsilon_0 \sqrt{\frac{\mu_0}{\varepsilon'_g}} \; ; \; \text{Re } \delta > 0; \quad \nu_{0(s)} = \sqrt{m^2_{0(s)} + \delta^2}; \quad \text{Im } \nu_{0(s)} \; 0.$$

We will show that an arbitrary given system of currents in the wires of an n-wire line is uniquely expanded in the n-channel system of currents. We form the following equations to determine the currents $\dot{I}_{i(s)}$:

$$\left.\begin{aligned}
\dot{I}_1 &= \lambda_{1i(1)}\dot{I}_{i(1)} + \lambda_{1i(2)}\dot{I}_{i(2)} + \cdots + \lambda_{1i(n)}\dot{I}_{i(n)}, \\
\dot{I}_2 &= \lambda_{2i(1)}\dot{I}_{i(1)} + \lambda_{2i(2)}\dot{I}_{i(2)} + \cdots + \lambda_{2i(n)}\dot{I}_{i(n)}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{I}_n &= \lambda_{ni(1)}\dot{I}_{i(1)} + \lambda_{ni(2)}\dot{I}_{i(2)} + \cdots + \lambda_{ni(n)}\dot{I}_{i(n)},
\end{aligned}\right\} \quad (14)$$

where $\dot{I}_1$, $\dot{I}_2$, ..., $\dot{I}_n$ are the given system of currents in the wires of the line.

The system (14) of equations consists of 2n equations in general (considering that each equation of the system consists of two equations: one for the real parts and the other for the imaginary parts) and has 2n unknowns (real and imaginary parts of $\dot{I}_{i(s)}$). Since in the given n-wire line there exist n different wave channels, each of equations (14) is independent; consequently, this system has a unique solution in the currents $\dot{I}_{i(s)}$.

Thus, to solve the problem of electromagnetic wave propagation along a multiconductor line we must know the total current in the wires of this line at the point of its excitation. The propagation constants

and relationships between the currents in the wires for each wave channel are found using equation (11) and formula (12). The currents in one of the wires for each wave channel are determined from the system (14) of equations, while the electromagnetic fields are determined by formulas of the general character of (13)*.

The important difficulty in the determination of the parameters of the wave channels of a multiconductor line lies in the evaluation of the integrals F_{ik}.

In [10] the integral F_k is evaluated in a general form in the region $h \leq 10m$, $\omega \leq 10^6$, $\sigma_g \geq 10^{-2}$ ohm^{-1} m^{-1} to within an accuracy of 0.5%.

S.P. Belousov and B.G. Yampol'skii [13] proposed an approximate method of evaluating the integral F_k in the region $\omega/\sigma_g \leq 10^{12}$ ohm m/sec to within an accuracy of a few percent, however for the determination of the parameters of phase-phase wave channels such an accuracy is not sufficient.

In the region $r'_{ik} \leq 30m$, $\omega \leq 10^6$, $\sigma_g \geq 10^{-2}$ ohm^{-1} m^{-1} the integral F_{ik} can be evaluated to a sufficient degree of accuracy, analogously to the evaluation of the integral F_k in [10]. We represent F_{ik} in the form of a sum of three integrals:

$$F_{ik} = 2(K_{ik} + R_{ik} + G_{ik}), \qquad (15)$$

$$K_{ik} = -\int_0^\infty \frac{\eta_0}{k_g^2 - k_0^2} e^{-\eta_0(h_i + h_k)} \cos\nu b_{ik} \, d\nu =$$

$$= -\frac{j\pi}{2(k_g^2 - k_0^2)} \cdot \frac{\partial^2}{\partial(h_i + h_k)^2} H_0(m_0 r'_{ik}); \qquad (16)$$

---

*To calculate the fields at large distances from the wire axes using formulas from [10], errors in the latter should be corrected, which arose because of the incorrect sign in front of the second term of the right side of the fourth equation of (118).

(Translator's note: For this translation, equation (118) has been left as is in reference [10])

$$R_{ik} = \int_0^\infty \frac{\eta_g}{k_g^2 - k_0^2}\, e^{-\eta_0(h_i + h_k)} \cos\nu b_{ik}\; d\nu; \qquad (17)$$

$$G_{ik} = \int_0^\infty \frac{\gamma^2}{k_0^2\eta_0 + k_0^2\eta_g}\, e^{-\eta_0(h_i + h_k)} \cos\nu b_{ik}\; d\nu. \qquad (18)$$

To solve the problem in the first approximation the integrals R_{ik} and K_{ik} are evaluated in general form, if we put $m_0 = 0$, i.e. $\gamma^2 = k_0^2$. In this case

$$K_{ik} = \frac{b_{ik}^2 - (h_i + h_k)^2}{(k_g^2 - k_0^2)(r'_{ik})^4} = \frac{1}{2}\left(\frac{1}{\varkappa_1^2} + \frac{i}{\varkappa_2^2}\right), \qquad (19)$$

where

$$\varkappa_1 = \xi(h_i + h_k + jb_{ik}); \quad \varkappa_2 = \xi(h_i + h_k - jb_{ik});$$

$$\xi = \sqrt{-k_g^2 + k_0^2}; \quad \mathrm{Re}\,\xi > 0.$$

$$R_{ik} = \frac{1}{2(k_g^2 - k_0^2)} \int_0^\infty \eta_g\, e^{-\nu(h_i + h_k + jb_{ik})} + e^{-\nu(h_i + h_k - jb_{ik})}\; d\nu =$$

$$= \frac{1}{2} - \left[\int_0^\infty e^{-\varkappa_1 u}\sqrt{u^2 + 1}\; du - \int_0^\infty e^{-\varkappa_2 u}\sqrt{u^2 + 1}\; du\right] =$$

$$= \frac{n}{4\varkappa_1} Y_1(\varkappa_1) + \frac{n}{4\varkappa_2} Y_1(\varkappa_2) +$$

$$+ \frac{1}{2} \sum_{n=1} (-1)^n \frac{\varkappa_1^{2n-1} + \varkappa_2^{2n-1}}{1^2 \cdot 3^3 \cdots (2n-1)^2(2n+1)}, \qquad (20)$$

where Y_1 is the Weber function of first order.

To evaluate the integrals R_{ik} one can make use of tables of the function Y_1 for complex arguments [14].

It should be noted that for phase-phase wave channels, formulas (19) and (20) allow evaluation of the integrals K_{ik} and R_{ik} with a degree of accuracy sufficient for a wider range of values r'_{ik}, ω and σ_g than indicated above, since for these channels the quantity γ^2 differs much less from the quantity k_0^2 than for a phase-earth channel.

In the region $|\varkappa_1| = |\varkappa_2| > 5$ the evaluation of the integrals R_{ik} simplifies. Considering the analysis of the integral

$$\int_0^\infty e^{-\varkappa u}\sqrt{u^2+1}\, du$$

carried out in [15], for $|\varkappa| > 5$ we obtain to within an accuracy of 0.3%:

$$\int_0^\infty e^{-\varkappa u}\sqrt{u^2+1}\, du = \frac{1}{\varkappa} + \frac{1}{\varkappa^3} - \frac{3}{\varkappa^5} \quad . \tag{21}$$

From expressions (19), (20) and (21) we have

$$2(K_{ik}+R_{ik}) = -\frac{1}{\varkappa_1} - \frac{1}{\varkappa_2} + \frac{1}{\varkappa_1^2} + \frac{1}{\varkappa_2^2} - \frac{1}{\varkappa_1^3} - \frac{1}{\varkappa_2^3} + \frac{3}{\varkappa_1^5} + \frac{3}{\varkappa_2^5} \quad . \tag{22}$$

The integrals G_{ik} are evaluated, under the condition $|k_g^2\eta_0| \gg |k_0^2\eta_g|$ over the whole integration interval, by the formula

$$G_{ik} \approx \frac{\gamma^2}{k_g^2} \cdot \frac{j\pi}{2} H_0(m_0 r'_{ik}) \quad . \tag{23}$$

To evaluate the integrals G_{ik} by this formula it is necessary to know the quantity m_0; therefore to solve the problem in the first approximation G_{ik} should be set equal to zero, and in the second approximation the value of m_0 found in the first approximation should be used.

In the range of values of r'_{ik}, ω and σ_g indicated above, the accuracy of the evaluation of the integral G_{ik} by formula (23) has practically no effect on the overall accuracy of the solution, since the integral G_{ik} in this region constitutes a small part of the integral F_{ik}.

For a wider range of values of ω and σ_g the integrals R_{ik} and G_{ik}, in the second approximation to the solution of the problem, should be evaluated more accurately by other methods (for example, by way of a numerical integration).

For $\sigma_g \gg \omega\varepsilon_g$ the displacement currents in the earth can be neglected. Then

$$k_g^2 - k_0^2 \approx - j\omega\mu_0\sigma_g \; . \tag{24}$$

In this case the sum of the integrals $K_{ik} + R_{ik}$ evaluated by formulas (19), (20) and (22) will coincide with Carson's integral:

$$K_{ik} + R_{ik} = jJ(p,q),$$

where

$$p = (h_i+h_k)\sqrt{\omega\mu_0\sigma_g}; \quad q = b_{ik}\sqrt{\omega\mu_0\sigma_g}.$$

Thus to solve the problem in the first approximation one can use results of Carson's computations [5], represented graphically, however it should be kept in mind that for phase-phase channels the accuracy of the solution may be very poor because the graphical determination of Carson's integrals is inaccurate.

Using the known expressions for the Hankel functions in terms of Bessel functions of the first and second kinds, one can obtain the following approximate expression for the Hankel function of small argument, if one limits oneself to terms of order no higher than fourth in the argument:

$$H_0(x) = \frac{2j}{\pi} \left(\ell n \, \frac{\gamma'x}{2j} - \frac{x^2}{4} \ell n \, \frac{\gamma'x}{2je} + \frac{x^4}{64} \ell n \, \frac{\gamma'x}{2je^{\frac{3}{2}}} \right) \tag{25}$$

where $\gamma'=1.781$ is Euler's constant;

e is the base of the natural logarithms.

To solve equation (11) in the first approximation, one can use the first term in the series expansion (25) to determine the Hankel functions in expressions (8). Here, reducing all terms of the system of equations (9) by the common factor $-2jk_0^2/\pi m_0^2$, we obtain instead of the coefficients B_{kk} and B_{ik} the following coefficients:

$$\left. \begin{aligned} B'_{kk} &= \frac{m_0^2}{k_0^2} \ell n \, \frac{2h_k}{a_k} - F_k - M_k; \\ \\ B'_{ik} &= \frac{m_0^2}{k_0^2} \ell n \, \frac{r'_{ik}}{r_{ik}} - F_{ik}. \end{aligned} \right\} \tag{26}$$

It can be shown that the transition to the coefficients B'_{kk} and B'_{ik} introduces negligible error for a wider range of values for r'_{ik}, ω and σ_g than indicated above, especially for the phase-phase channels.

The system of equations of the type (9) with coefficients B'_{kk} and B'_{ik} using the integrals F_{ik} evaluated in the first approximation and putting $G_{ik} = 0$, is analogous to that obtained by Carson [1] and Kostenko [4] in the case of sinusoidal waves and zero conductivity of the air. The solution of the problem in the second approximation makes possible an estimate of the error in Carson's solution, which arises because of

a) the assumption of quasistationarity of the fields in the air, which corresponds to condition (26) and evaluation of the integrals F_{ik} at $m_0 = 0$;

b) neglecting the transverse currents in the earth, which corresponds to setting the integrals G_{ik} equal to zero;

c) neglecting displacement currents in the earth, which corresponds to condition (24).

2. Examples of solutions of the problem for some multiconductor lines

a) Split-phase multiconductor systems

We shall extend the obrained general solution to split-phase multi-conductor transmission lines in which each phase bundle is identical and electrically interconnected. To solve this problem we make the following assumptions.

a) we neglect the proximity effect in the split-phase wires;

b) we take the electromagnetic field at the surface of each phase bundle due to currents of the other phases and the earth to be equal to the electromagnetic field of these currents at the center of the split phase;

c) we assume that the currents of the split-phase bundles are equal to each other.

In this case the boundary conditions at the surface of an arbitrary $\ell\underline{^{th}}$ bundle of the $k\underline{^{th}}$ phase lead to the following equation:

$$\dot{I}_k \left[\frac{1}{q_k} H_0(m_0 a_k) + \frac{1}{q_k} \sum_{u=1}^{a_k} H_0(m_0 d_{u\ell}) - H_0(m_0 \cdot 2h_k) + \right.$$

$$\left. + \frac{2jk_0^2}{\pi m_0^2} F_k + \frac{M_k}{q_k} \right] + \sum_{i=1}^{n} \dot{I}_i \left[H_0(m_0 r_{ik}) - H_0(m_0 r'_{ik}) + \right.$$

$$\left. + \frac{2jk_0^2}{\pi m_0^2} F_{ik} \right] = 0 \quad (u \neq \ell; \quad i \neq k), \qquad (27)$$

where q_k is the number of bundles of the $k\underline{^{th}}$ phase;

a_k is the radius of the bundles of the $k\underline{th}$ phase;

$d_{u\ell}$ is the distance between bundles u and ℓ;

r_{ik} is the distance between centers of split phases;

h_k is the distance from the center of a split phase to the earth.

Equation (27) leads to a system of equations analogous to the system (9). To solve this system in the first approximation the coefficients B'_{ik} are determined from expression (26), while instead of the coefficients B'_{kk} the following coefficients are obtained:

$$B''_{kk} = \frac{m_0^2}{k_0^2} \ln \frac{2h_k}{r_{eq,k}} - F_k - \frac{M_k}{q_k} , \qquad (28)$$

$$r_{eq,k} = (a_k d_{1\ell} d_{2\ell} \cdots d_{qk\ell})^{\frac{1}{q_k}} ,$$

where $r_{eq,k}$ is the equivalent radius of the $k\underline{th}$ phase.

b) Two-wire system

From expression (10) we have

$$B_{11}B_{22} - B_{12}^2 = 0. \qquad (29)$$

1. Case of identical wires in one horizontal plane.

In this case $B_{11} = B_{22}$ and from expression (29) there follows:

$$(B_{11} - B_{12})(B_{11} + B_{12}) = 0. \qquad (30)$$

Equation (30) determines two wave channels:

1st wave channel

$$B_{11(1)} = B_{12(1)}; \quad \frac{\dot{i}_{2(1)}}{\dot{i}_{1(1)}} = -1; \qquad (31)$$

in the first approximation

$$\frac{m_{0(1)}^2}{k_0^2} = \frac{F_1 + M_1 - F_{12}}{\ln \dfrac{2h_1 r_{12}}{a_1 r_{12}'}} \quad ; \tag{32}$$

2nd wave channel

$$B_{11(2)} = - B_{12(2)} ; \quad \frac{\dot{i}_{2(2)}}{\dot{i}_{1(2)}} = 1 ; \tag{33}$$

in the first approximation

$$\frac{m_{0(2)}^2}{k_0^2} = \frac{F_1 + M_1 + F_{12}}{\ln \dfrac{2h_1 r_{12}'}{a_1 r_{12}}} \tag{34}$$

To solve the problem in the absence of the earth we obtain in the

first approximation:

$$\frac{m_{0(1)}^2}{k_0^2} = \frac{M_1}{\ln \dfrac{r_{12}}{a_1}} \quad ; \tag{35}$$

$$\frac{m_{0(2)}^2}{k_0^2} = - \frac{M_1}{2\ln \dfrac{\gamma' m_{0(2)} \sqrt{r_{12} a_1}}{2j}} \quad . \tag{36}$$

The solution (35) can also be obtained using the telegraphist's

equations. In this connection the exact solution of a given system of

wires permits one to estimate the region within which the telegraphist's

equations give the solution to a given accuracy. To find this region

we determine $B_{12(1)}$ using two terms of the series expansion for the Hankel

function of (25). Now instead of the solution (35) we obtain:

$$\frac{m_{0(1)}^2}{k_0^2} - \frac{M_1}{\ln \dfrac{r_{12}}{a_1} - \dfrac{m_{0(1)}^2 r_{12}^2}{4} \ln \dfrac{\gamma' m_{0(1)} r_{12}}{2je}} \quad . \tag{37}$$

The relative error of solution (35) compared to solution (37) is equal to:

$$\delta_{rel} = - \frac{m_{0(1)}^2 r_{12}^2 \ln \frac{\gamma' m_{0(1)} r_{12}}{2je}}{4 \ln \frac{r_{12}}{a_1}} . \tag{38}$$

Suppose we specify an error in solution (35) equal to $|\delta_{rel}| = 1\%$. Then solving (38) for $m_{0(1)} r_{12}$ in the region $r_{12}/a_1 \geq 10^2$, we obtain

$$|m_{0(1)}^2 r_{12}^2| \leq 0.08. \tag{39}$$

From this condition and solution (37), considering (6) and the fact that at high frequencies usually $J_0(k_1 a_1) \simeq jJ_0'(k_1 a_1)$, we obtain a range of parameters in which the telegraphist's equations give a solution for the quantity $|m_{0(1)}^2|$ within an accuracy of 1%:

$$\frac{r_{12}^2 \varepsilon_0 \omega}{a_1 \ln \frac{r_{12}}{a_1}} \sqrt{\frac{\omega \mu_1}{\sigma_1}} \leq 0.08. \tag{40}$$

A region within which the electromagnetic field can be considered quasistationary to a given accuracy is determined analogously.

Let us consider some numerical examples.

IN THE ABSENCE OF THE EARTH. Copper wire: $\sigma_1 = 57 \times 10^6$ mho/m; $\mu_1 = \mu_0$; $a_1 = 10^{-2}$ m; $r_{12} = 10$ m; $f = 10^6$ Hz.

Results of calculations using formulas (31), (33), (35) and (36) are presented in Table 1.

From condition (40) we determine the range of frequencies in which one can, for a given system of wires, determine the attenuation coeffic-

ient $\beta_{(1)}$ using the telegraphist's equations to an accuracy of the order
of 1%: $f \leq 1.9 \times 10^8$ Hz.

System	Number of wave channel	$\dfrac{\dot{i}_{2(s)}}{\dot{i}_{1(s)}}$	$\dfrac{\dot{i}_{3(s)}}{\dot{i}_{1(s)}}$	$\dfrac{\alpha}{k_0} - 1$	$\dfrac{\beta}{k_0}$	β, dB/km
Two identical wires	1	-1	$-$	$2.4 \cdot 10^{-4}$	$2.4 \cdot 10^{-4}$	0.044
	2	$+1$	$-$	$9 \cdot 10^{-5}$	$9 \cdot 10^{-5}$	0.0165
Two different wires	1	$-2.68+$ $+j0.0014$	$-$	$4.5 \cdot 10^{-3}$	$4.5 \cdot 10^{-3}$	0.82
	2	$0.0235+$ $+j0.0014$	$-$	$1.34 \cdot 10^{-4}$	$1.34 \cdot 10^{-4}$	0.0245
Three identical wires	1	0	-1	$2.2 \cdot 10^{-4}$	$2.2 \cdot 10^{-4}$	0.040
	2	-1.92	$+1$	$2.5 \cdot 10^{-4}$	$2.5 \cdot 10^{-4}$	0.046
	3	$+1.04$	$+1$	$6.9 \cdot 10^{-5}$	$6.9 \cdot 10^{-5}$	0.0126

Table 1. Results of calculations in the absence of the earth.

Thus, in the present case the telegraphist's equations are valid even
when the wavelength is several times smaller than the distances between
wires.

IN THE PRESENCE OF THE EARTH. The wires and their separation are
the same; $\omega=10^6$; $\sigma_g = 10^{-2}$ mho/m; $h_1=h_2=10$m.

The results of calculations using formulas (15), (19), (20), (23),
(31)-(34) are presented in Table 2; as are results of computations for
a single wire-earth system for similar parameters for comparison.

System	Number of wave channel	$\dfrac{\dot{I}_2(s)}{\dot{I}_1(s)}$	$\dfrac{\dot{I}_3(s)}{\dot{I}_1(s)}$	$\dfrac{\alpha}{k_0}$	$\dfrac{\beta}{k_0}$	β, dB/km	Influence of G_{ik} on β, %
Wire-earth	1	–	–	1.0368	0.0239	0.69	+1.9
Two identical wires-earth	1	-1	–	1.0064	0.0024	0.069	+0.6
	2	+1	–	1.061	0.0404	1.17	+1.9
Two different wires-earth	1	-0.78- -j0.03	–	1.013	0.0088	0.255	–
Three identical wires-earth	1	0	-1	1.016	0.0062	0.180	+0.7
	2	-1.83+ +j0.04	1	1.0026	0.0009	0.026	+1
	3	1.05+ +j0.03	1	1.081	0.0528	1.53	+1.8

Table 2. Results of calculations in the presence of the earth.

2. Case of different wires

In this case the propagation constants are determined from equation (29).

In Tables 1 and 2 results of calculations for a two-wire system are given in the absence and in the presence of the earth, when the second wire is steel; $\sigma_2 = 9 \times 10^6$ mho/m and the mean permeability of the steel is $\mu_2 = 100\mu_0$; the first wire is copper; the size of the wires and the other parameters are just as in the above examples.

It should be noted that in the examples in the absence of the earth the attenuation of the antiphase wave channel is considerably larger than that of the cophase channel, and the currents in the wires for each wave channel differ considerably from one another in absolute value. Thus, in the case of different wires, one may not construct telegraphist's equations on the basis of identical magnitudes of currents in the wires for the antiphase wave channel, as is done by some authors (cf., for example, [16]).

c) Three-wire system symmetrical with respect to the center wire

In this case the end wires have identical parameters.

$$a_1 = a_3; \quad h_1 = h_3; \quad r_{12} = r_{23}; \quad r'_{12} = r'_{23}; \quad F_1 = F_3; \quad F_{12} = F_{23}; \quad M_1 = M_3.$$

Consequently

$$B_{11} = B_{33}; \quad B_{12} = B_{23}.$$

Then from expression (10) we obtain:

$$\Delta = \begin{vmatrix} B_{11} & B_{12} & B_{13} \\ B_{12} & B_{22} & B_{12} \\ B_{13} & B_{12} & B_{11} \end{vmatrix} = (B_{11} - B_{13})[B_{22}(B_{11} + B_{13}) - 2B_{12}^2]. \tag{41}$$

From expression (12) it follows that:

$$\frac{\dot{i}_{2(s)}}{\dot{i}_{1(s)}} = \frac{B_{12}(B_{13} - B_{11})}{B_{11}B_{22} - B_{12}^2}; \quad \frac{\dot{i}_{3(s)}}{\dot{i}_{1(s)}} = \frac{B_{12}^2 - B_{22}B_{13}}{B_{11}B_{22} - B_{12}^2}. \tag{42}$$

Formulas (41) and (42) determine three wave channels:

1st wave channel

$$B_{11(1)} = B_{13(1)}; \quad \frac{\dot{i}_{2(1)}}{\dot{i}_{1(1)}} = 0; \quad \frac{\dot{i}_{3(1)}}{\dot{i}_{1(1)}} = -1; \tag{43}$$

in the first approximation in the presence of the earth

$$\frac{m_{0(1)}^2}{k_0^2} = \frac{F_1 + M_1 - F_{13}}{\ln \dfrac{2h_1 r_{13}}{a_1 r'_{13}}} \quad ; \qquad (44)$$

2nd and 3rd wave channels. For these channels the quantity m_0^2 is determined from the equation

$$B_{22}(B_{11} + B_{13}) - 2B_{12}^2 = 0. \qquad (45)$$

Using (45), we obtain from (42):

$$\left. \begin{array}{l} \dfrac{\dot{I}_{3(2)}}{\dot{I}_{1(2)}} = 1; \quad \dfrac{\dot{I}_{2(2)}}{\dot{I}_{1(2)}} = -\dfrac{2B_{12(2)}}{B_{22(2)}} = -\dfrac{B_{11(2)} + B_{13(2)}}{B_{12(2)}} \quad ; \\[4mm] \dfrac{\dot{I}_{3(3)}}{\dot{I}_{1(3)}} = 1; \quad \dfrac{\dot{I}_{2(3)}}{\dot{I}_{1(3)}} = -\dfrac{2B_{12(3)}}{B_{22(3)}} = -\dfrac{B_{11(3)} + B_{13(3)}}{B_{12(3)}} \quad . \end{array} \right\} \qquad (46)$$

In Tables 1 and 2 computational results are given for a three-wire system with identical copper wires, lying in one horizontal plane, in the absence and in the presence of the earth. The sizes of the wires, distances between neighboring wires and the other parameters are the same as in the examples with two identical wires.

From Table 2 it follows that the influence of transverse currents in the earth (the integrals G_{ik}) on the attenuation coefficient is insignificant in all the given examples.

Conclusion

1. A rigorous solution in general form has been obtained for the problem of electromagnetic wave propagation along multiconductor transmission lines in the presence or in the absence of the earth, neglecting the proximity effect in the wires and end effects.

2. Under specific assumptions results can be obtained from the rigorous solution accounting for the earth which correspond to a solution obtained from the telegraphist's equations. The identical assumptions characterize the accuracy of the solution found by the telegraphist's equations.

3. In the numerical examples it is shown that for $\omega \leq 10^6$ and $\sigma_g \geq 10^{-2}$ mho/m the influence of transverse currents in the earth on the parameters of all the wave channels is insignificant.

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