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ANALYTICAL BASES FOR ELECTROMAGNETIC SENSING
OF COAL PROPERTIES

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ABSTRACT

A number of theoretical problems have been studied that are relevant to the electromagnetic probing of the environment in coal mines. Particular emphasis was given to methods of determining the roof structures in underground workings. Essentially the idea is that the structure is illuminated by a source or primary field and the secondary response is detected. This secondary field is then related to the roof structure. The following approaches were employed: i) low frequency induction wherein the induced eddy-currents give rise to a secondary measurable field, ii) direct current excited by probes that interact with the inhomogeneous roof structure, iii) microwave coupling to the roof by resonant loops, iv) microwave reflection and scattering by horn antennas. The collected results presented here are intended to provide an analytical framework for further work in this area.

EXECUTIVE SUMMARY

(including project summaries, conclusions and recommendations)

In Chapter I, an analysis of low frequency electromagnetic coupling is presented in an effort to devise a simple conductivity measurement scheme for probing of coal mine environments. Two possible methods are investigated using a planar two-layer model of a coal seam over a region of homogeneous rock. The first of these is the induction method that uses mutual impedance characteristics of transmitting and receiving loop antennas to deduce the conductivity. Two different configurations are treated: (1) Horizontal coplanar, in which the loop axes are vertical, and (2) Perpendicular, in which the receiving loop is perpendicular to the transmitting loop. A second measurement method, based on potential theory, involves the use of two or four electrode probe arrays. Included are previous results for the Wenner, Eltran, and Right Angle arrays. Also, results for planar and dipping interfaces are presented using the Theta array. This is a novel four electrode configuration, feasible for use in a mining environment. Implementation and actual testing of such devices are recommended.

In Chapter II, the theory of current flow in homogeneous and layered conducting media is developed for the case where an axial conductor or cable is present. This cable, which is characterized by a specified axial impedance, is assumed to be infinite in length. Various configurations are chosen such as a current point source in an infinite, semi-infinite, and layered region where the cable is taken parallel to the interface(s). The resulting formulas for the potentials reduce to known cases in the absence of the cable. Using these formulations presented, we present some concrete calculated examples that are relevant to resistivity probing of perturbed homogeneous and layered structures. Only the two-electrode array is treated but various cable orientations are considered. In general, it is found that a long axial conductor such as a bare cable will distort the potential distribution of the current in a major way. This leads to profound departures from the apparent resistivity curves calculated for idealized homogeneous and layered structures. These analytical results should be considered by those involved in probing layered roof structures in mine environments when the presence of rails, cables and other metallic

that the performance of a horizontal loop operating near its first resonance can be altered significantly due to the presence of a roof structure consisting of a thin layer of uncut coal seam in front of a bulk or draw slate. Questions concerning the inversion from a set of physically measurable qualities of the loop to information leading to the unique determination of the roof thickness, are discussed. The implementation and possible simplification of these interpretation tools would seem to be worthy of further study.

In Chapter IV, the use of an annular, coaxially-driven slot antenna for probing of the roof thickness in a coal mine is discussed. For the case when the thickness of the roof and the slot width are both small compared with free-space wavelength, the inversion from the measured data on radiated power to roof thickness proves to be a very simple process. It appears that such a sensor not only can be rugged in construction, simple to install, but also has the additional advantage of being less sensitive to scatterings from surrounding environment. The practical application of such a scheme should be attempted at the earliest opportunity.

In Chapter V, an analysis is given for scattering of plane waves by a conducting layer, the electrical properties of which vary in a periodic manner. The fields may be expressed in terms of Floquet functions or, alternatively, in terms of a related set of functions described below. The latter approach seems to have computational advantages. Both the transverse electric (TE) and the transverse magnetic (TM) cases are considered for arbitrary incident angle. Numerical examples for the TE case, relevant to remote probing of coal seams, are presented. The results are primarily intended to illustrate the limitations of lateral non-uniformities in probing structures that are otherwise uniform in the sense that the media properties vary only in a direction transverse to the interface.

In Chapter VI, the use of broad-band electromagnetic measurements in the remote probing of geological structures is analyzed. A possible application is the measurement of "header" thickness in a coal mine. Here a thin layer of coal must be left for safety reasons, as the overlying slate is unstable when exposed to air. Currently, mining operations must be stopped periodically while a hole is manually drilled into the roof. The ability to monitor "header" thickness remotely would be a definite advantage for use with automatic mining machinery. Also useful in mine applications would be the capability to look ahead of the shaft to detect and identify hazards such as large water or gas pockets. The techniques discussed make use of a large number of reflection measurements made at various microwave frequencies. The data are processed via computer to simulate the effect of an incident pulse. When this pulse encounters a discontinuity, part of it is reflected. By examining the time of flight for the reflected pulse, it is possible to determine the distance to the interface. The amplitude and phase of the pulse give information on the electrical properties of the materials at the interface and just beyond. Further work on this interesting analogue technique should probably wait for the completion of field tests.

In Chapter VII, an analytical technique is given for determining the thickness and the electrical constitutive parameters of a planar layered medium, such as a coal seam in a mine environment. Time-domain experimental data are analyzed with Prony's method to determine the natural frequencies of the layered medium. Explicit relations are given (for dielectric layers) for determining the thicknesses and dielectric constants from the experimentally determined natural fre-

swept frequency excitation. Extensions of the technique to a non-planar medium and practical implications of the method are discussed. Actually, the basic concept is also applicable in acoustical probing so further research in this area should be carefully coordinated to benefit from these findings. From a theoretical standpoint, the generality and limitations of the natural frequency concept should be re-examined for open (i.e. unbounded) structures.

In Chapter VIII, the effect of lateral inhomogeneities on electromagnetic remote probing of layered structures is considered. The excitation is taken to be a plane wave incident at any angle. We determine the variation of the electric and magnetic fields, the surface admittance, and the wave tilt as function of the following parameters: angle of incidence, distance from the air-coal interface, electrical contrast, and the layer profile (e.g. sinusoidal, step, and slant profiles). Both low frequency and high frequency cases are illustrated and compared with corresponding results with no lateral inhomogeneities. We find that the surface admittance and the tangential electric and tangential magnetic fields are "good" indicators of the local structure, whereas the magnetic field normal to the interface and the wave tilt at the interface are not. However, the wave tilt is a good indication of anomalies within the seam.

As the observation point is moved away from the air-coal interface, the "information content" regarding the lateral inhomogeneities rapidly decreases. Thus, measurements performed on the surface, or measurements dependent upon the fields at the surface, e.g. reflection coefficients, are more sensitive to lateral inhomogeneities than measurements performed away from the surface. Field tests in controlled environments should definitely be carried to confirm the significance of these theoretical predictions of lateral non-uniformities.

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THEORY OF LOW FREQUENCY CONDUCTIVITY PROBING OF ROOF STRUCTURES IN COAL MINES

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An analysis of low frequency electromagnetic coupling is presented in an effort to devise a simple conductivity measurement scheme. Two possible methods are investigated using a planar two-layer model of a coal seam over a region of homogeneous rock. The first of these is the induction method that uses mutual impedance characteristics of transmitting and receiving loop antennas to deduce the conductivity. Two different configurations are treated: (1) Horizontal coplanar, in which the loop axes are vertical, and (2) Perpendicular, in which the receiving loop is perpendicular to the transmitting loop.

A second measurement method, based on potential theory, involves the use of two or four electrode probe arrays. Included are previous results for the Wenner, Eltran, and Right Angle arrays. Also, results for planar and dipping interfaces are presented using the Theta array. This is a novel four electrode configuration, feasible for use in a mining environment.

SECTION I

INTRODUCTION

Determining electrical ground constants of the earth is an important problem in geophysical probing [Keller, 1971], and underground communication system design [Aldredge, 1973]. Several different methods are used to determine the relative permittivity ϵ_r , and resistivity ρ , or conductivity σ of a two-layered earth.

As noted in the Institute of Electrical and Electronics Engineers (IEEE) Standard 356 [1974], techniques for determining the electrical constants of the earth are roughly divided into two categories: (1) Methods where drill-hole access into the earth does not exist, or is not required, (2) Techniques requiring drill-hole access into the earth.

Methods using drill holes are primarily for determining electrical ground constants in a local area. Where a single drill hole is used, values of the conductivity and permittivity can be found in the immediate vicinity of the hole. Propagation between two holes can be measured to give bulk values of the electrical constants of the intervening medium.

The one drill-hole method, described by *de Bettencourt and Frazier* [1963], uses the depth attenuation ratio of the field between transmitting and receiving antennas to measure ground conductivity. *Tsao and de Bettencourt* [1967] also discussed experimental results of obtaining the phase angle of the mutual impedance between two vertical electric dipoles in drill holes.

To determine electrical constants over broad areas, techniques using probes at or near the surface are often used. One technique in this category is the measurement of wave tilt. This is usually a high frequency

technique, although, in principle, it may be applicable to low frequencies. *Maley* [1971] describes the method which requires launching an electromagnetic wave by an antenna near the earth's surface and measuring the tilts of the wavefronts at the surface of the earth. The vertical field strength E_v and a radial or longitudinal component E_h are measured. Then, the wave tilt, W is the ratio of E_h/E_v .

Lytle and Lager [1976] considered the applicability of the wave tilt technique in probing a two-layer, coal mine roof structure where the coal layer has lateral inhomogeneities. *Chang and Wait* [1976] also investigated the use of wave tilt measurement of a homogeneous coal layer over a laterally inhomogeneous layer of slate. Normally, the slate is a good conductor compared with the coal layer so that it can be described locally in terms of a surface impedance or can be synthesized from periodic variations due to the inhomogeneity.

In local areas, one probing technique that has been used extensively to obtain electrical ground constants is the magneto-telluric method. *Cagniard* [1953] presented the basic theory behind this method which involves comparison of the horizontal components of the magnetic and electric fields associated with the flow of telluric (natural earth) currents. The horizontal electric field E_h is observed using a horizontal center FED wire with ends grounded through electrodes. Wire arms are then arranged in different dipole geometries. As noted by *Cantwell and Madden* [1960], the magneto-telluric method requires very low frequencies, usually around 1 Hz. Although the method is simple, when used over a stratified earth, a thin layer becomes indistinguishable unless it is highly conducting [*Vozoff*, 1972]. A comprehensive evaluation and critical assessment of the

magneto-telluric theory has been given by *Wait* [1962a).

One active method of deducing earth constants is the AC, low frequency measurement of the mutual impedance between a transmitting and a receiving loop at or above the surface of the earth. Actually, *Sommerfeld* [1909, 1926] initiated research of this method with his classic papers on an alternating current dipole over a homogeneous, semi-infinite earth. *Baños* [1966] extended *Sommerfeld's* work to give a systematic treatment of the electromagnetic radiation from elementary Hertzian dipoles. More complicated cases of a Hertzian dipole in the earth have been investigated by *Wait* [1969] and by *Fuller and Wait* [1976].

There has been extensive study of numerical techniques for the evaluation of the field components of a dipole over a stratified earth. For example, using quasi-static theory, *Anderson* [1974] developed a Fortran IV program based on *Wait's* [1966] formulation for computing the cartesian components of the electric and magnetic fields about a grounded horizontal electric dipole (HED), or a finite wire source. Digital linear filtering programs [*Anderson*, 1973; *Daniels*, 1974; *Johnston*, 1975] also were developed to compute the field components of a dipole over a stratified earth.

Spies, Wait and Fuller [1972, 1973] presented quantitative and graphical results for the mutual impedance between small horizontal coplanar loops on a two-layered earth. They displayed the results graphically using Argand plots of the real and imaginary parts of the mutual impedance for different conductivity ratios of two homogeneous layers. Such presentations form the basis of direct interpretation of measured impedance data in terms of the electrical and geometrical properties of the layered region.

Another active method of deducing earth constants involves essentially a DC measurement of the mutual resistance between a transmitting and a receiving antenna on the surface of the earth. Four electrode DC measurement schemes have been discussed extensively in the literature [Mooney and Wetzel, 1952; Wait and Conda, 1958; Grant and West, 1965; Sunde, 1966; Kunetz, 1966]. In this context, DC techniques refer to AC if the frequency is sufficiently low (e.g. frequencies much lower than 1 kHz).

Typically, in a four electrode array, current is introduced into the earth through electrodes C and C'. The potential difference resulting at another point is measured between electrodes P and P'. In the Western literature, the configurations most frequently used are the Wenner and Eltran arrays. A modification of the Wenner configuration is the Lee array, described by Frischknecht and Keller [1966]. Wait and Conda [1958] also described a modification of the Eltran array, which they called the Right Angle Array. This array has very low mutual inductance.

Knowing the electrode current length and spacing between the primary and secondary electrode circuits, it is possible to determine the apparent conductivity or resistivity using a particular array. The geometric mean conductivity is determined by matching field data to master curves obtained by varying the interelectrode spacing.

Al'pin *et al* [1966] presented several master curves for different four electrode arrays over two, three, and four-layered earths. Curves for dipping as well as horizontally stratified earth models were included.

In Section II, we consider mutual coupling of loops as a basis for conductivity measurements of a coal mine roof structure. Graphical and analytic justification is included for the use of low frequency quasi-static approxi-

mations in determining the mutual impedance response between two small loops. Further mutual impedance results are shown in Sections III through V for various layered roof models. In Section IV, we consider the case where the coal layer conductivity increases linearly, due to water saturation. As noted by *Hoekstra et al* [1974], and *Ames et al* [1963], when water is introduced into the pore spaces of a rock or element, the resistivity decreases (i.e. conductivity increases). Results for two other profiles, describing piecewise linear conductivity variations are included in Section V.

In Sections VI and VII, attention is focused on two and four electrode arrays over two-layer planar and dipping earth structures. Section VI summarizes previously available information on resistivity probing using the Wenner and Eltran arrays. Also, apparent conductivity curves for these configurations and the Right Angle Array are given in Section VI.

The Theta Array, a novel four electrode (quadripole) configuration is introduced in Section VII. A discussion of this array begins with two electrodes at infinity. Master curves for different interelectrode spacings of this array on two-layer planar and dipping earth structures can be used to determine apparent conductivity of a coal seam over homogeneous rock or slate.

SECTION II

FREQUENCY DEPENDENCE OF ELECTROMAGNETIC COUPLING BETWEEN SMALL SPACED LOOPS NEAR A TWO-LAYER ROOF

Perhaps the earliest consideration of using antenna theory to solve environmental problems is found in the classic work of *Sommerfeld* [1909, 1926]. He used a homogeneous conducting half space of dielectric constant ϵ , conductivity σ , and magnetic permeability μ . In his formulation, an elementary magnetic or electric dipole was oriented either horizontally or vertically and located in the free space region above the conducting half space. (A more accessible summary of Sommerfeld's work appears in the "Lectures on Theoretical Physics" series [1964]).

Others have extended Sommerfeld's formal integral solutions to encompass an n-layered earth model. One very recent application of the solution to this problem occurs in mining where a two-layered earth model is used to describe a coal seam over an infinite layer of homogeneous rock or slate. Although the formal solution to the problem is in terms of infinite integrals which are difficult to evaluate without a computer, *Wait* [1958] expressed the solutions in a form amenable to numerical evaluation.

In *Wait's* formulation [1958, 1972] for a two-layer, homogeneous model, a small current loop, representable by an oscillating vertical magnetic dipole (VMD) is located at $z = h$ with respect to the earth's surface at $z = 0$. The upper layer ($0 > z > -d$) has a conductivity σ_1 , or σ (1972 formulation), while the lower layer ($z < -d$) has a conductivity σ_2 or $K\sigma$. In all regions, the magnetic permeability $\mu = \mu_0 = 4\pi \times 10^{-7}$ H/m.

Wait noted that if all significant distances in the upper half space are assumed much smaller than a free space wavelength, quasi-static theory is valid. In this case, the fields for a magnetic dipole at (0,0,h) form a solution to Laplace's equation and can be derived from a scalar magnetic potential Φ .* Thus, for $z > 0$,

$$\bar{H} = -\nabla\Phi \quad (2.1)$$

where,

$$\Phi = -c \frac{\partial}{\partial z} \left[\frac{1}{r} + \int_0^{\infty} R(\lambda) e^{-\lambda(z+h)} J(\lambda\rho) d\lambda \right] \quad (2.2)$$

$$c = i\omega\mu_0 IdA/4\pi \quad (2.3)$$

and

$$r = \{\rho_0^2 + (z-h)^2\}^{1/2} \quad \rho = (x^2 + y^2)^{1/2}.$$

Now,

$$R(\lambda) = \frac{(u_1 + \lambda)(u_1 - u_2)e^{-2u_1 d} - (u_1 - \lambda)(u_1 + u_2)}{(u_1 + \lambda)(u_1 + u_2) - (u_1 - \lambda)(u_1 - u_2)e^{-2u_1 d}} \quad (2.4)$$

where,

$$u_1 = (\lambda^2 + \gamma_1^2)^{1/2}, \quad u_2 = (\lambda^2 + \gamma_2^2)^{1/2}$$

and,

$$\gamma_1^2 = i\omega\mu_0\sigma_1 - \omega^2\mu_0\epsilon_1 \quad \gamma_2^2 = i\omega\mu_0\sigma_2 - \omega^2\mu_0\epsilon_2$$

Using *Wait's* [1958] formulation in *Frischknecht and Keller's* [1966, 1967] notation, the primary magnetic fields of a VMD in the absence of the earth are given by:

* NOTE: There are several fundamental texts on electromagnetic fields which describe the use of scalar and vector potentials in Maxwell's equations. A complete discussion is given by *Rand, Whinnery, Van Duzer* [1965] and *Johnk* [1975].

$$H_x^p = \frac{3mx(z-h)}{4\pi r^5} \quad (2.5a)$$

$$H_y^p = \frac{3my(z-h)}{4\pi r^5} \quad (2.5b)$$

$$H_z^p = \frac{3m(z-h)^2}{4\pi r^5} - \frac{m}{4\pi r^3} \quad (2.5c)$$

With the magnetic dipole approximation, $m = NAI$, where N is the number of turns, A is the loop area and I is the current. In the cylindrical coordinate system (ρ, ϕ, z) , the corresponding electric field has a tangential component which is given for completeness.

$$E_{\phi, \rho} = - \frac{i\omega\mu_0 \rho m}{4\pi r^3} \quad (2.5d)$$

In the region below the roof, the three components of the magnetic field are written in terms of the integrals T_0 and T_1 which are functions of the dimensionless quantities A , B , and K . At low frequencies, the displacement currents in the ground are usually negligible in comparison with conduction currents in the ground are usually negligible in comparison with conduction currents (i.e., $\sigma_1 \gg \epsilon_1 \omega$ and $\sigma_2 \gg \epsilon_2 \omega$). Thus, A , B , and K are written as follows:

$$A = (z + h)/\delta = 2h/\delta \quad (2.6a)$$

$$B = \rho/\delta = \rho_0/\delta \quad (2.6b)$$

where

$$\rho_0 = 1m$$

$$K = \frac{\sigma_2 + i\omega\epsilon_2}{\sigma_1 + i\omega\epsilon_1} \approx \frac{\sigma_2}{\sigma_1} \quad (2.6c)$$

$$\delta = \left[\frac{2}{(\sigma_1 + i\omega\epsilon_1)\mu_o\omega} \right]^{1/2} \approx \left(\frac{2}{\sigma_1\mu_o\omega} \right)^{1/2} \quad (2.6d)$$

The secondary magnetic fields scattered by the earth are given below:

$$H_x^S = - \frac{m}{4\pi\delta^3} T_1 \left(\frac{x}{\rho} \right) \quad (2.7a)$$

$$H_y^S = - \frac{m}{4\pi\delta^3} T_1 \left(\frac{y}{\rho} \right) \quad (2.7b)$$

$$H_z^S = - \frac{m}{4\pi\delta^3} T_o \quad (2.7c)$$

where

$$T_o = \int_0^\infty R(D,g) g^2 e^{-gA} J_o(gB) dg \quad (2.8a)$$

$$T_1 = \int_0^\infty R(D,g) g^2 e^{-gA} J_1(gB) dg \quad (2.8b)$$

$$R(D,g) = 1 - 2g \frac{(U+V) + (U-V)e^{-ud}}{(U+g)(U+V) - (U-g)(U-V)e^{-ud}} \quad (2.8c)$$

$$U = (g^2 + 2i)^{1/2}, \quad V = (g^2 + 2ik)^{1/2}, \quad D = 2d/\delta$$

J_o is a Bessel function of the first kind and zeroth order.

J_1 is a Bessel function of the first kind and first order.

As we shall indicate below, ratios of mutual impedances are often used in dealing with two-loop electromagnetic measurement. Here the mutual impedance Z is defined as the ratio of the open-circuit voltage induced in the receiving loop to the current I in the transmitting loop. Thus,

$$Z = V/I = -i\omega\mu_o NAH/I \quad (2.9)$$

where H denotes the axial field at the receiving loop.

For two-loop configurations, the transmitting antenna is represented by a circular loop of area A_1 with N_1 turns. The receiving loop has area A_2 with N_2 turns. If it is assumed that the separation ρ between these loops is significantly less than a wavelength, but large compared with the loop diameters, the free space mutual impedance for two coplanar loops is:

$$Z_o = \frac{i\omega\mu_o N_1 N_2 A_1 A_2}{4\pi\rho^3} \quad (2.10)$$

This also assumes that the loops are small enough so that the current distribution around the loops is essentially uniform.

The normalized mutual impedance, Z_m , between the loops can be defined by the ratio:

$$Z_m = \frac{Z}{Z_o} = \frac{\text{Mutual Impedance of Loops}}{\text{Mutual Impedance of Coplanar Loops in free space}} \quad (2.11)$$

Wait [1954a, 1954b, 1955, 1956, 1958], *Frischknecht and Keller* [1966, 1967], and *Sinha* [1976] are among those who have treated the mutual impedance as a function of frequency for several different "two-loop" configurations. The most commonly treated cases for a homogeneous, multilayered earth are:

1. Horizontal Coplanar Loops (axes vertical)
2. Perpendicular Loops (the first loop axis is vertical while the second loop axis is horizontal)
3. Vertical Coplanar Loops (coplanar loops with horizontal axes)
4. Vertical Coaxial Loops (coaxial loops with horizontal axes).

Using *Wait's* [1958] formulation and notation, *Frischknecht* [1967] considered the case of a horizontal coplanar loop and wire element over an earth with two homogeneous layers. He published computed and tabulated values for all five configurations. For the horizontal coplanar and perpendicular loop configurations, plots of the magnitudes and phases of these tabulated values over a wide frequency range are shown in Figures 1a, 1b, 2a, and 2b.

Using the ratios $A/B = 0.25$, $D/B = 0.5$, and the parameters $K = 0.1$, 0.3 , 1 , 3 , 10 , the following values for roof thickness, distance of loops to the roof, and separation between loops, are used when referring to the ω scale: $d = 0.25\text{m}$, $h = 0.125\text{m}$, $\rho = \rho_o = 1\text{m}$. Actually, the curves for this example are certainly not valid for $\omega > 10^8$ rads, since displacement currents are not negligible. However, there may be other situations where the displacement currents are negligible over the entire range of B plotted. The quasi-static theory given by *Wait* [1958] is still valid at low frequencies when displacement currents are significant. However, as noted by *Sinha* [1976], most of the numerical work appearing in the literature is done under the assumption that displacement currents are negligible.

The magnitude of Z_m is plotted in the (a) portions of Figures 1 and 2, while the (b) portions are for the phases. For the horizontal coplanar loop configuration,

$$Z_m = 1 + B^3 T_o = 1 + S \quad (2.12)$$

Similarly, for the perpendicular loop configuration,

$$Z_m = B^3 T_1 = T \quad (2.13)$$

In most instances, the tabulated values of mutual impedance appearing in *Frischknecht's* [1967] report were in the range $0.2 < B < 10$. Since B is proportional to the square root of frequency, expansions have been developed for large and small values of B . Treating the case for large A or B first, *Wait* [1958] used a Taylor's series expansion for the integrands in T_0 and T_1 given in (2.8a) and (2.8b). Retaining only the first three terms, these asymptotic expansions are given by:

$$T_0 \approx \frac{2A^2 - B^2}{(A^2 + B^2)^{5/2}} - \left(\frac{2}{i}\right)^{1/2} Q \frac{6A^3 - 9AB^2}{(A^2 + B^2)^{7/2}} - iQ^2 \frac{24A^4 - 72A^2B^2 + 9B^4}{(A^2 + B^2)^{9/2}} \quad (2.12a)$$

$$T_1 \approx \frac{3AB}{(A^2 + B^2)^{5/2}} - \left(\frac{2}{i}\right)^{1/2} Q \frac{12A^2B - 3B^3}{(A^2 + B^2)^{7/2}} - iQ^2 \frac{60A^3B - 45AB^3}{(A^2 + B^2)^{9/2}} \quad (2.12b)$$

$$Q = \frac{(1+K^{1/2}) + (1-K^{1/2}) \exp \{-(2i)^{1/2} D\}^*}{(1+K^{1/2}) - (1-K^{1/2}) \exp \{-(2i)^{1/2} D\}} \quad (2.12c)$$

At low frequencies (i.e., less than 1 MHz), another approximate expansion may be used for computing the mutual impedance of the horizontal coplanar and perpendicular loop configurations. Details of these expansions for conductivity profiles of two homogeneous layers, as well as two layers with continuous and piecewise linear conductivity variations are presented in Section III through V. For now, it is sufficient to write:

*NOTE: Q appears in several different forms throughout the literature. At high frequencies, $|Q|$ oscillates around 1. For sample plots showing the nature of Q at low and high frequencies, see *Wait* [1970], or *Jackson, Wait, and Walters* [1962].

FIG. 1a. Magnitude of the mutual impedance for the horizontal coplanar loop configuration. Drawn for $A/B = 0.25$, $D/B = 0.5$, $\rho = \rho_o = 1\text{m}$, $d = 0.25\text{m}$, $h = 0.125\text{m}$.

FIG. 2a. Magnitude of the mutual impedance for the perpendicular loop configuration. Drawn for $A/B = 0.25$, $D/B = 0.5$, $\rho = \rho_0 = 1\text{m}$, $d = 0.25\text{m}$, $h = 0.125\text{m}$.

FIG. 2b. Phase of the mutual impedance for the perpendicular configuration. Drawn for $A/B = 0.25$, $D/B = 0.5$, $\rho = \rho_0 = 1\text{m}$, $d = 0.25\text{m}$, $h = 0.125\text{m}$.

FIG. 3. Magnitude of S at very low frequencies. Actual versus approximate values for horizontal coplanar loops. Drawn for $A/B = 0.25$, $D/B = 0.5$, $\rho = \rho_0 = 1\text{m}$, $d = 0.25\text{m}$, $h = 0.125\text{m}$.

FIG. 4. Magnitude of T at very low frequencies. Actual versus approximate values for perpendicular loops. Drawn for $A/B = 0.25$, $D/B = 0.5$, $\rho = \rho_0 = 1\text{m}$, $d = 0.25\text{m}$, $h = 0.125\text{m}$.

$$Z_m = \frac{Z}{Z_0} \approx 1 - \frac{iB^2 \rho_0^2}{2} \left[\frac{1}{[1 + (A/B)^2]^{1/2}} - \frac{1}{[1 + ((A+B)/D)^2]^{1/2}} + K \frac{1}{[1 + ((A+B)/D)^2]^{1/2}} \right] \quad (2.13)$$

For horizontal coplanar loops, and:

$$Z_m \approx \frac{iB^2 \rho_0^2}{2} \left[\frac{((A+D)/B)}{[1 + ((A+D)/B)^2]^{1/2}} - \frac{(A/B)}{[1 + (A/B)^2]^{1/2}} + K \left(1 - \frac{((A+D)/B)}{[1 + ((A+D)/B)^2]^{1/2}} \right) \right] \quad (2.14)$$

for perpendicular loops. Although succeeding terms are proportional to B^4 , B^6 ... etc., they are ignored here.

In Figures 3 and 4, it is shown how the tabulated values of *Frischknecht* [1967] using the actual integral expressions for T_0 and T_1 approach the approximations for S and T at low frequencies. The same parameters are used in determining these approximations as appear in Figures 1 and 2.

In some sense, the preceding figures lend justification to the low frequency approximations appearing in the following sections. Succeeding sections will show derivations working with S and T directly. These are defined to be the low frequency response functions and will be subject to various normalizations.

SECTION III

LOW FREQUENCY ELECTROMAGNETIC COUPLING OF LOOPS NEAR A ROOF WITH TWO HOMOGENEOUS LAYERS

One simple non-resonant method for determining the thickness of the upper coal layer after tunneling through a seam is to measure the mutual coupling between two small loops kept at a constant spacing. It is proposed that these loops be rigidly mounted on a fiberglass boom. As this pair of loops is brought into proximity of the roof, the mutual impedance response is related to the thickness of the coal layers and to the electrical properties of the media.

A two-layer model of the coal mine roof structure is idealized as a homogeneous layer or slab of coal of thickness d and conductivity σ_1 with the homogeneous region above the coal (assumed to be rock or slate) having a conductivity σ_2 . This model is illustrated in Figures 5a and 5b.

Using a cylindrical coordinate system (ρ, ϕ, z) , the coal layer corresponds to the region $h < z < h + d$, while the rock beyond the coal layer corresponds to $z > h + d$. In the air region below the roof $z < 0$, the influence of the floor and tunnel walls is neglected. This model definition gives a more physical interpretation to a coal mine "roof" structure than the one in the preceding section.

Although several authors, including *Wait, Frischknecht and Keller* [1966, 1967], and *Sinha* [1976] have considered many different loop configurations, only two will be discussed here in detail. These were chosen for their flexibility in a mining environment. As indicated in Figures 5a and 5b, the source loop in both cases is a vertical magnetic dipole (VMD) located at $z = 0$. In the first case, the receiving loop is also a VMD and is located at $\rho = \rho_0$, $z = 0$. This model is often referred to as the horizon-

FIG. 5a. Two horizontal coplanar loops below a two-layer, homogeneous roof.

FIG. 5b. Two perpendicular loops below a two-layer, homogeneous roof.

tal coplanar configuration in the literature.

The second case uses the receiving loop in a "null" or perpendicular arrangement. In this case, the receiving loop is a horizontal magnetic dipole (HMD) located at $\rho = \rho_0$ and $z = 0$.

Although the loops are treated as magnetic dipoles, they may actually be finite sized wire loops if the frequency is sufficiently low. For simplicity, the separation ρ_0 between the loops is assumed much greater than the mean radius of both loops. In practice, however, this ratio need only be greater than five. Additionally, the loops may have many turns and be wound on ferrite cores which serve to increase the effective loop areas.

For the horizontal coplanar loops shown in Figure 5a, it is possible to define the dimensionless quantity S as:

$$S = \frac{Z - Z_0}{Z_0} \quad (3.1)$$

where Z is the mutual impedance of the loops in the presence of the roof while Z_0 is the mutual impedance of the loops if they were located in free space. In the case of the perpendicular configuration in Figure 5b, a dimensionless quantity T is also defined as:

$$T = \frac{Z}{Z_0} \quad (3.2)$$

where Z is the mutual impedance between the loops for this configuration and Z_0 is the mutual impedance of the corresponding coplanar configuration located in free space. For the ideal perpendicular configuration, Z vanishes when the loops are located in free space. In Section II, it was shown that S is equal to $B^3 T_0$, and T is equal to $B^3 T_1$ at high frequencies. Also, in Section II it was indicated that when the ground is assumed homogeneous, or comprised of conducting layers, the formal solutions at high

frequencies are quite complicated and require numerical integration since closed form solutions cannot be obtained.

At very low frequencies, closed form solutions may often be obtained in a straight forward manner. This low frequency approximation is valid when all significant dimensions of the problem are electrically small. Thus, d , ρ_o and h are all much less than skin depths δ_1 and δ_2 , where $\delta_1 = (2/\sigma_1\mu_o\omega)^{1/2}$ and $\delta_2 = (2/\sigma_2\mu_o\omega)^{1/2}$; ω is the angular frequency and $\mu_o = 4\pi \times 10^{-7}$ H/m. For conductivity values of $\sigma_1 = 10^{-1}$ S/m and $\sigma_2 = 10^{-2}$ S/m, the frequency is typically restricted to less than 1 or 2 MHz.

Wait [1953, 1962] presented the theory for the quasi-static, low frequency case of a thin plane sheet of conducting material. This theory, leading up to a derivation of S and T is briefly summarized below.

For the perpendicular configuration, the vector magnetic field outside the sheet is the gradient of a scalar function which is a solution of Laplace's equation. Thus,

$$i\omega\mu\bar{H} = \nabla(\nabla\cdot\bar{F}) \quad (3.3)$$

where $\bar{F}$ is a vector having only a z component. The magnetic field components are then written as:

$$i\omega\mu H_{m\rho} = \frac{\partial^2 F_m}{\partial\rho\partial z} \quad (3.3a)$$

$$i\omega\mu H_{m\phi} = 0, \quad (3.3b)$$

$$i\omega\mu H_{mz} = \frac{\partial^2 F_m}{\partial z^2} \quad (3.3c)$$

where m takes the values 1 or 2 to designate the upper or lower region.

The vector electric field $\bar{E}$ is also written in terms of the auxiliary vector potential $\bar{F}$ as follows:

$$\bar{E} = -\nabla \times \bar{F}, \quad (3.4)$$

so that:

$$E_{m\rho} = E_{mz} = 0, \quad E_{m\phi} = \frac{\partial F_m}{\partial \rho}.$$

Within the sheet, the scalar function F_m is a solution of the following Helmholtz equation:

$$(\nabla^2 - \gamma^2)F_m = 0 \quad (3.5)$$

in cylindrical coordinates, where $\gamma^2 = i\omega\mu_0\sigma$ since displacement currents are considerably smaller than the conduction currents at low frequencies.

After applying boundary conditions, the formal solution for $z > h$ is given by:

$$F = F_0 - ciq \int_0^\infty (\lambda + iq)^{-1} \cdot e^{-\lambda(z-2h)} J_0(\lambda\rho) d\lambda \quad (3.6)$$

while, for $z < h$:

$$F = F_0 - ciq \int_0^\infty (\lambda + iq)^{-1} \cdot e^{\lambda z} J_0(\lambda\rho) d\lambda. \quad (3.7)$$

Now, F_0 is the potential function that gives rise to the primary fields of the magnetic dipole and is written in terms of the Lipschitz integral (see *Watson*, 1944):

$$F_0 = c \int_0^\infty e^{-\lambda z} J_0(\lambda\rho) d\lambda = \frac{c}{(\rho^2 + z^2)^{1/2}}. \quad (3.8)$$

Also:

$$q = \frac{\sigma(z)\mu_o\omega(dz)}{2}, \quad \text{and} \quad c = \frac{IdA\omega\mu_o}{4\pi}$$

is the moment of the magnetic dipole.

For the perpendicular configuration, the H_ρ component in the plane $z = 0$ is of interest. Thus, a series expansion of T may be obtained as follows:

$$T = \left. \frac{4\pi\rho^3 H_\rho}{IdA} \right|_{z=0} = i\rho C^3 + \frac{\rho^2 C^2}{1+S} + \frac{i\rho^3 C}{1+S} + \dots \quad (3.9)$$

where,

$$C = \frac{\rho}{\{\rho^2 + (2z)^2\}^{1/2}} \quad \text{and} \quad S = (1-C^2)^{1/2}$$

$$\rho = \rho q = \sigma(z)\mu_o\omega\rho(dz)/2.$$

At sufficiently low frequencies, it is necessary to retain only the first term approximation to T so that

$$T = \frac{i\omega\mu_o}{2} \int_0^\infty \frac{\rho^4 \sigma(z) dz}{\{\rho^2 + 4z^2\}^{3/2}} \quad (3.10)$$

For $z < h$, $\sigma(z) = \sigma_o$. (Conductivity of air is assumed to be effectively 0.) Consequently, it is only necessary to integrate T over the interval (h, ∞) . Hence, letting $\rho = \rho_o$,

$$\begin{aligned} T &= \frac{i\omega\mu_o\rho_o^4}{2} \left\{ \sigma_1 \int_h^{h+d} \frac{dz}{(\rho_o^2 + 4z^2)^{3/2}} + \sigma_2 \int_{h+d}^\infty \frac{dz}{(\rho_o^2 + 4z^2)^{3/2}} \right\} \\ &= \frac{i\omega\mu_o\rho_o^2}{2} \left\{ \frac{\sigma_1 z}{(\rho_o^2 + 4z^2)^{1/2}} \right|_h^{h+d} + \frac{\sigma_2 z}{(\rho_o^2 + 4z^2)^{1/2}} \right|_{h+d}^\infty \end{aligned}$$

or

$$T = \frac{i\omega\mu_o\rho_o^2}{2} \left\{ \sigma_1 \left[\frac{(h+d)}{\{\rho_o^2 + 4(h+d)^2\}^{1/2}} - \frac{h}{(\rho_o^2 + 4h^2)^{1/2}} \right] + \sigma_2 \left[\frac{1}{2} - \frac{(h+d)}{\{\rho_o^2 + 4(h+d)^2\}^{1/2}} \right] \right\} \quad (3.11)$$

Deriving this expression for T uncovered an error in *Wait's* [1962b] formulation. In his equation (8), h_1 and h_2 should be replaced by $2h_1$ and $2h_2$. To compare his formulation with equation (3.10), $h_1 = h$ and $h_2 = h + d$.

Beginning with equation (3.3), a similar procedure can be used for determining S . In this case, F has a component in two directions. Thus, H_z and H_z^P (primary field) are of interest. Now,

$$S = \frac{i\omega\mu_o (H_z - H_z^P)}{c} \rho^3 \Big|_{z=0} \quad (3.12)$$

where

$$c = \frac{IdA i\omega\mu_o}{4\pi},$$

and

$$H_z = c \frac{\partial^2}{\partial z^2} \frac{1}{(\rho^2 + z^2)^{1/2}} + ciq \frac{\partial}{\partial z_o} \int_0^\infty \frac{\lambda}{\lambda + iq} e^{-\lambda(z-2h)} J_o(\lambda\rho) d\lambda \quad (3.13)$$

At low frequencies, q is much less than λ so that,

$$i\omega\mu_o H_z \approx i\omega\mu_o H_z^P + ciq \frac{\partial}{\partial z} \frac{1}{\{\rho^2 + (2h+z)^2\}^{1/2}}$$

Thus,

$$S = \frac{iq (2h+z)}{\{\rho^2 + (2h+z)^2\}^{3/2}} \Big|_{z=0} = \frac{iq (2h)}{\{\rho^2 + (2h)^2\}^{3/2}} \quad (3.14)$$

In (3.14), replacing h by z , and substituting $q = \sigma(z)\mu_o\omega dz/2$, S can be written as an integral from 0 to ∞ as follows:

$$S = \frac{i\omega\mu_o\rho_o^3}{2} \int_0^{\infty} \frac{2\sigma(z)z}{(\rho_o^2 + 4z^2)^{3/2}} \quad (3.15)$$

where $\rho = \rho_o$. Again, for $z < h$, $\sigma(z) = \sigma_o = 0$. Thus, S is written as:

$$\begin{aligned} S &= i\omega\mu_o\rho_o^3 \left[\sigma_1 \int_{+h}^{+(h+d)} \frac{zdz}{(\rho_o^2 + 4z^2)^{3/2}} + \sigma_2 \int_{+(h+d)}^{+\infty} \frac{zdz}{(\rho_o^2 + 4z^2)^{3/2}} \right] \\ &= -\frac{i\omega\mu_o\rho_o^3}{4} \left\{ \frac{\sigma_1}{(\rho_o^2 + 4z^2)^{1/2}} \Big|_{+h}^{+(h+d)} + \frac{\sigma_2}{(\rho_o^2 + 4z^2)^{1/2}} \Big|_{+(h+d)}^{+\infty} \right\} \end{aligned}$$

or

$$\begin{aligned} S &= \frac{i\omega\mu_o\rho_o^3}{4} \left\{ \sigma_1 \left[\frac{1}{(\rho_o^2 + 4h^2)^{1/2}} - \frac{1}{\{\rho_o^2 + 4(h+d)^2\}^{1/2}} \right] \right. \\ &\quad \left. + \sigma_2 \frac{1}{\{\rho_o^2 + 4(h+d)^2\}^{1/2}} \right\} \quad (3.16) \end{aligned}$$

To illustrate the nature of these results, $S/\sigma_2\omega$ and $T/\sigma_2\omega$ are plotted in Figures 6 to 9 for a range of the parameters included below. The separation ρ_o is fixed at 1 meter, and the distance h from the loops to the air/coal interface is allowed to vary from 0 to 1.2 meters. On each figure, the roof thickness d is assigned a sequence of values from 0 to 50 m, which is considered effectively infinite. The ratio,

$$\frac{\sigma_2}{\sigma_1} = \frac{\text{conductivity of overlying rock}}{\text{conductivity of coal}} \quad (3.17)$$

takes on the values 0.1, 0.3, 3 and 10 as indicated in Figures 6, 7, 8, and 9, respectively. The (a) portion of these figures is for $S/\sigma_2\omega$ and the (B) portion is for $T/\sigma_2\omega$. Although S and T are proportional to i or $\exp(i\pi/2)$, only magnitudes are plotted. Within the low frequency approximation, the real part of the response functions S and T is neglected.

Diagnostic information should be obtained from the plotted curves. In concept, the normalized mutual impedance Z_m can be determined as a function of h for a particular measurement scheme. Then, the deduced response functions S and T would be matched to these response curves. Even if the conductivity ratio σ_2/σ_1 is not known, an estimate can be made for the "roof thickness" d .

FIG. 6a. The horizontal coplanar loop response function versus distance to the roof for $\sigma_2/\sigma_1 = 0.1$.

FIG. 6b. The perpendicular loop response function versus distance to the roof for $\sigma_2/\sigma_1 = 0.1$.

FIG. 7a. The horizontal coplanar loop response function versus distance to the roof for $\sigma_2/\sigma_1 = 0.3$.

FIG. 7b. The perpendicular loop response function versus distance to the roof for $\sigma_2/\sigma_1 = 0.3$.

FIG. 8a. The horizontal coplanar loop response function versus distance to the roof for $\sigma_2/\sigma_1 = 3.0$.

FIG. 8b. The perpendicular loop response function versus distance to the roof for $\sigma_2/\sigma_1 = 3.0$.

FIG. 9a. The horizontal coplanar loop response function versus distance to the roof for $\sigma_2/\sigma_1 = 10.0$.

FIG. 9b. The perpendicular loop response function versus distance to the roof for $\sigma_2/\sigma_1 = 10.0$.

APPENDIX

An alternative derivation for equations (3.11) and (3.16) is presented below. Equation (3.10) is repeated for convenience.

$$T = \frac{i\omega\mu_0}{2} \int_0^{\infty} \frac{\sigma(z)\rho_0^4}{(\rho_0^2+4z^2)^{3/2}} dz . \quad (1)$$

Utilizing integration by parts, equation (1) is rewritten as:

$$T = \frac{i\omega\mu_0\rho_0^2}{2} \int_0^{\infty} \sigma(z) \frac{d}{dz} \left(\frac{z}{(\rho_0^2+4z^2)^{1/2}} \right) dz \quad (2)$$

where,

$$u = \sigma(z), \quad dv = \frac{d}{dz} \left(\frac{z}{(\rho_0^2+4z^2)^{1/2}} \right) dz$$

and

$$v = \frac{z}{(\rho_0^2+4z^2)^{1/2}} .$$

Noting that as $z \rightarrow \infty$, $\sigma(\infty) \rightarrow \sigma_{\infty}$, T is rewritten as:

$$T = \frac{i\omega\mu_0\rho_0^2}{2} \left[\frac{\sigma_{\infty}}{2} - \int_0^{\infty} \frac{z}{(\rho_0^2+4z^2)^{1/2}} \sigma'(z) dz \right] . \quad (3)$$

For the two-layered case described in this chapter, the conductivity profile may be written as:

$$\begin{aligned} \sigma(z) = & \sigma_0 \{u(z) - u(z-h)\} + \sigma_1 \{u(z-h) \\ & - u(z-d-h)\} + \sigma_2 u(z-d-h) \end{aligned} \quad (4)$$

where

$$u(z) = \begin{cases} 1 & z > 0 \\ 0 & z < 0 \end{cases}$$

Since $\sigma_o = 0$, the derivative of the conductivity is written as:

$$\begin{aligned} \frac{d\sigma(z)}{dz} = \sigma'(z) = \sigma_1 \{ \delta(z-h) - \delta(z-d-h) \} \\ + \sigma_2 \{ \delta(z-d-h) \} \end{aligned} \quad (5)$$

where $\delta(z)$ is the Dirac delta function. In this case, $\sigma_2 = \sigma_\infty$. Substituting (5) into (3) yields,

$$\begin{aligned} T = \frac{i\omega\mu_o\rho_o^3}{2} \left[\frac{\sigma_2}{2} - \frac{\sigma_1 z}{(\rho_o^2 + 4z^2)^{1/2}} + \frac{\sigma_1(h+d)}{\{\rho_o^2 + 4(h+d)^2\}^{1/2}} \right. \\ \left. - \frac{\sigma_2(h+d)}{\{\rho_o^2 + 4(h+d)^2\}^{1/2}} \right] \end{aligned} \quad (6)$$

which, after rearranging, is identical to (3.10).

A parallel development for horizontal coplanar loops begins with (3.15), which is repeated below:

$$S = \frac{i\omega\mu_o\rho_o^3}{2} \int_0^\infty \sigma(z) \frac{2zdz}{(\rho_o^2 + 4z^2)^{3/2}} \quad (7)$$

After integrating by parts, (7) becomes:

$$\begin{aligned} S = - \frac{i\omega\mu_o\rho_o^3}{4} \left[\sigma(z) \frac{1}{(\rho_o^2 + 4z^2)^{1/2}} \right]_0^\infty - \int_0^\infty \frac{1}{(\rho_o^2 + 4z^2)^{1/2}} \\ \left[\frac{d\sigma(z)}{dz} \right] dz \end{aligned} \quad (8)$$

As $z \rightarrow \infty$, $\sigma(\infty) \rightarrow \sigma_\infty$, and

$$\lim_{z \rightarrow \infty} \frac{\sigma(z)}{(\rho_o^2 + 4z^2)} = 0 \quad (9)$$

Using (9), (8) is rewritten as:

$$S = \frac{i\omega\mu_o\rho_o^3}{4} \left[\int_0^\infty \frac{1}{(\rho_o^2 + 4z^2)} \sigma'(z) dz + \frac{\sigma(0)}{\rho} \right] \quad (10)$$

For the two-layer roof model, (5) is substituted into (10) to obtain,

$$S = \frac{i\omega\mu_o\rho_o^3}{4} \left[\frac{\sigma_1}{(\rho_o^2 + 4h^2)^{1/2}} - \frac{\sigma_1}{\{\rho_o^2 + 4(h+d)^2\}^{1/2}} + \frac{\sigma_2}{\{\rho_o^2 + 4(h+d)^2\}^{1/2}} \right] \quad (11)$$

After rearranging, it is easily seen that (11) is identical to (3.16).

Assuming that all layers are thin, and maintaining the same low frequency restrictions set forth in this section, (4) may be rewritten for N-layers as follows:

$$\sigma(z) = \sigma_1 u(z-h) - \sigma_1 u(z-d_1-h) + \sum_{k=2}^n \sigma_k (u_{k-1} - u_k) , \quad (12)$$

where

$$u_{k-1} = u(z - \sum_{k=2}^n d_{k-1} - h), \quad u_k = u(z - \sum_{k=2}^n d_k - h) ,$$

and $d_k (k=1,2,\dots,n)$ denotes the thicknesses of the different layers.

Since $\sigma_n = \sigma_\infty$, (3) is rewritten in the horizontal coplanar case as:

$$T = \frac{i\omega\mu_o\rho_o^2}{2} \left[\frac{\sigma_n}{2} - \frac{\sigma_1 h}{(\rho_o^2 + 4h^2)^{1/2}} + \sum_{k=2}^n \frac{(\sigma_{k-1} - \sigma_k) h_{k-1}}{r} \right] \quad (13)$$

where

$$h_{k-1} = (h + \sum_{k=2}^n d_{k-1}) \quad \text{and} \quad r = (\rho_o^2 + 4h_{k-1}^2)^{1/2} .$$

Similarly, using (12) in (10) produces the following result for perpendicular loops:

$$S = \frac{i\omega\mu_0\rho_0^3}{4} \left[\frac{\sigma_1}{(\rho_0^2 - 4h^2)^{1/2}} - \sum_{k=2}^n \frac{(\sigma_{k-1} - \sigma_k)}{r} \right]$$

where h_{k-1} and r are defined above.

SECTION IV

LOW FREQUENCY LOOP COUPLING NEAR AN INHOMOGENEOUS ROOF

In the previous section the simple geophysical probing technique of measuring the mutual coupling between two loops at low frequencies was discussed. The mutual impedance curves presented were based on a two-layer earth model where both conducting regions were homogeneous.

Here, another model having merit in the interpretation of data where the roof structure cannot be idealized in terms of homogeneous layers is discussed. In this case, the roof conductivity varies linearly over a finite vertical distance. Beyond this range, the conductivity is constant. Although this is a very idealized assumption about the roof structure, the results provide some insight into geophysical probing where there may be a continuous, as opposed to an abrupt change of medium properties.

As mentioned by *Frischknecht and Keller* [1966], *Wait* [1971], *Hoekstra et al* [1974], and many horticulturists, moisture content of a coal or soil layer may vary as the distance from the air/material interface. This situation can be treated adequately by the conductivity profile described above.

In Figure 10a, the roof structure is now idealized as an inhomogeneous, but laterally uniform coal slab of thickness ℓ . The conductivity of this slab varies linearly from the starting, or initial value σ_s at the coal/air interface to the final value σ_t . Beyond this level, the rock or slate is assumed to be homogeneous. This conductivity variation is shown in Figure 10b.

Using the cylindrical coordinate system (ρ, ϕ, z) indicated for the two configurations in Figure 10a, the conductivity $\sigma(z)$ as a function of z is given by:

FIG. 10a. Coplanar and perpendicular loop configurations over an inhomogeneous roof.

FIG. 10b. Conductivity profile as a function of height (drawn for $\sigma_t > \sigma_s$).

$$\sigma(z) = \left[(\sigma_t - \sigma_s) \frac{z - h}{\ell} + \sigma_s \right] u(z-h) \\ + (\sigma_t - \sigma_s) \left[1 - \frac{z - h}{\ell} \right] u(z-h-\ell) \quad (4.1)$$

where

$$u(z) = 1 \quad \text{if } z > 0 \\ = 0 \quad \text{if } z < 0.$$

The use of the unit step function in describing the conductivity profile and determining the response functions is described in the Appendix to Section III. As in the case of two homogeneous layers, the influence of the tunnel floor or haulageway is neglected in the air region $z < 0$.

Considering the horizontal coplanar and perpendicular loop configurations shown in Figure 10a, normalized response functions S and T may be computed. As in Section III, $S = (Z - Z_0)/Z_0$ and $T = Z/Z_0$ where Z_0 is the mutual impedance of the coplanar configuration if the two loops were located in free space.

Using the "low frequency" approximation ($\omega < 1$ MHz, with the loop size, spacing, distance of loops to the roof, and coal layer thickness all small in comparison with the effective coal layer skin depth) for any conductivity profile $\sigma(z)$, which vanishes for $z < h$, S and T are written:

$$S = i\omega\mu_0\rho_0^3 \int_h^\infty \sigma(z) \frac{z}{(\rho_0^2 + 4z^2)^{3/2}} dz \quad (4.2)$$

and

$$T = \frac{i\omega\mu_0\rho_0^4}{2} \int_h^\infty \sigma(z) \frac{1}{(\rho_0^2 + 4z^2)^{3/2}} dz. \quad (4.3)$$

Alternate forms of (4.2) and (4.3) are easily obtained when utilizing integration by parts. This derivation is shown in the Appendix to Section III and leads to the following results:

$$S = \frac{i\omega\mu_o\rho_o^3}{4} \int_h^\infty \sigma'(z) \frac{1}{(\rho_o^2 + 4z^2)^{1/2}} dz \quad (4.4)$$

and

$$T = \frac{i\omega\mu_o\rho_o^2}{2} \left[\frac{\sigma_\infty}{2} - \int_h^\infty \frac{z}{(\rho_o^2 + 4z^2)^{1/2}} \sigma'(z) dz \right] \quad (4.5)$$

where $\sigma'(z) = d\sigma(z)/dz$ and $\sigma_\infty = \sigma(\infty)$.

Taking the derivative of (4.1) yields,

$$\sigma'(z) = \sigma_s \delta(z-h) + \frac{\sigma_t - \sigma_s}{\ell} \{u(z-h) - u(z-h-\ell)\} \quad (4.6)$$

for the conductivity profile indicated in Figure 10b. The dirac delta function $\delta(z-h)$ is an impulse function of strength σ_s , located at $z = h$. Substituting (4.6) into (4.4) yields:

$$S = \frac{i\omega\mu_o\rho_o^3}{4} \left[\frac{\sigma_s}{(\rho_o^2 + 4h^2)^{1/2}} + \frac{\sigma_t - \sigma_s}{\ell} \int_h^{h+\ell} \frac{1}{(\rho_o^2 + 4z^2)^{1/2}} dz \right] \quad (4.7)$$

This integration is easily performed to give:

$$S = \frac{i\omega\mu_o\rho_o^3}{4} \left[\frac{\sigma_s}{(\rho_o^2 + 4h^2)^{1/2}} + \frac{\sigma_t - \sigma_s}{2\ell} \times \ln \left[\frac{2(h+\ell) + \{\rho_o^2 + 4(h+\ell)^2\}^{1/2}}{2h + \{\rho_o^2 + 4h^2\}^{1/2}} \right] \right] \quad (4.8)$$

Similarly, substituting (4.6) into (4.5) yields:

$$T = \frac{i\omega\mu_0\rho_0^2}{2} \left[\frac{\sigma_t}{2} - \frac{\sigma_s h}{(\rho_0^2 + 4h^2)^{1/2}} - \frac{(\sigma_t - \sigma_s)}{\ell} \right. \\ \left. \times \int_h^{h+\ell} \frac{z}{(\rho_0^2 + 4z^2)^{1/2}} dz \right]. \quad (4.9)$$

Performing the indicated integration produces the following result:

$$T = \frac{i\omega\mu_0\rho_0^2}{2} \left[\frac{\sigma_t}{2} - \frac{\sigma_s h}{(\rho_0^2 + 4h^2)^{1/2}} - \frac{(\sigma_t - \sigma_s)}{4\ell} \right. \\ \left. \times [\{\rho_0^2 + 4(h+\ell)^2\}^{1/2} - \{\rho_0^2 + 4h^2\}^{1/2}] \right] \quad (4.10)$$

To illustrate the results, $S/\sigma_t\omega$ and $T/\sigma_t\omega$ are plotted in Figures 11 to 14. As in Section III, the loop separation ρ_0 is fixed at 1 meter and the distance h of the loops from the roof varies from 0 to 1.2 meters. On each curve the transition distance ℓ is assigned a sequence of values from 0 to 50 m, which is assumed infinite. The ratio:

$$\frac{\sigma_t}{\sigma_s} = \frac{\text{conductivity of upper homogeneous region}}{\text{Initial conductivity of linear transition region}}$$

acquires the values 0.1, 0.3, 3 and 10 as indicated in Figures 11, 12, 13 and 14, respectively. Within the low frequency approximation S and T are again proportional to i or $\exp(i\pi/2)$ as the real parts of the response functions are neglected.

FIG. 11a. The horizontal coplanar loop response function versus distance to the roof for $\sigma_t/\sigma_s = 0.1$.

FIG. 11b. The perpendicular loop response function versus distance to the roof for $\sigma_t/\sigma_s = 0.1$.

FIG. 12a. The horizontal coplanar loop response function versus distance to the roof for $\sigma_t/\sigma_s = 0.3$.

FIG. 12b. The perpendicular loop response function versus distance to the roof for $\sigma_t/\sigma_s = 0.3$.

FIG. 13a. The horizontal coplanar loop response function versus distance to the roof for $\sigma_t/\sigma_s = 3.0$.

FIG. 13b. The perpendicular loop response function versus distance to the roof for $\sigma_t/\sigma_s = 3.0$.

FIG. 14a. The horizontal coplanar loop response function versus distance to the roof for $\sigma_t/\sigma_s = 10$.

FIG. 14b. The perpendicular loop response function versus distance to the roof for $\sigma_t/\sigma_s = 10$.

APPENDIX

In the limits as $\ell \rightarrow 0$ and $\ell \rightarrow \infty$, the response functions S and T behave as they should for homogeneous half-spaces. To prove this, it is necessary to begin with equation (4.8), which is repeated below for convenience.

$$S = \frac{i\omega\mu_o\rho_o^3}{4} \left[\frac{\sigma_s}{(\rho_o^2+4h^2)^{1/2}} + \frac{\sigma_t - \sigma_s}{2\ell} \times \ln \left[\frac{2(h+\ell) + \{\rho_o^2 + 4(h+\ell)^2\}^{1/2}}{2h + \{\rho_o^2 + 4h^2\}^{1/2}} \right] \right] \quad (1)$$

Case A: $\ell \rightarrow \infty$.

$$\lim_{\ell \rightarrow \infty} (1) S = \frac{i\omega\mu_o\rho_o^3}{4} \left[\frac{\sigma_s}{(\rho_o^2+4h^2)^{1/2}} + \frac{\sigma_t - \sigma_s}{2\ell} \ln(4\ell) \right] .$$

Now,

$$\lim_{\ell \rightarrow \infty} S = \frac{i\omega\mu_o\rho_o^3}{4} \frac{\sigma_s}{(\rho_o^2+4h^2)^{1/2}} \quad (2)$$

Case B: $\ell \rightarrow 0$.

This case requires evaluation of the following:

$$\lim_{\ell \rightarrow 0} \frac{\sigma_t - \sigma_s}{2\ell} \ln \left[\frac{2(h+\ell) + \{\rho_o^2 + 4(h+\ell)^2\}^{1/2}}{2h + (\rho_o^2+4h^2)^{1/2}} \right]$$

or

$$\lim_{\ell \rightarrow 0} \frac{\sigma_t - \sigma_s}{2\ell} [\ln[2(h+\ell) + \{\rho_o^2 + 4(h+\ell)^2\}^{1/2}] - \ln\{2h + (\rho_o^2+4h^2)^{1/2}\}], \quad (3)$$

which is of the indeterminate form $0/0$. As a result, it is necessary to use L'Hospital's rule to evaluate (3). Now,

$$\lim_{\ell \rightarrow 0} \frac{d}{d\ell} \left[\frac{\ln[2(h+\ell) + \{\rho_o^2 + 4(h+\ell)^2\}^{1/2}]}{2\ell} \right]$$

or

$$\lim_{\ell \rightarrow 0} \frac{2}{\{\rho_o^2 + 4(h+\ell)^2\}^{1/2}} = \frac{2}{(\rho_o^2 + 4h^2)^{1/2}} \quad (4)$$

Substituting (4) into (1) yields:

$$\lim_{\ell \rightarrow 0} S = \frac{i\omega\mu_o\rho_o^3}{4} \frac{\sigma_t}{(\rho_o^2 + 4h^2)^{1/2}} \quad (5)$$

Similarly, using (4.9), the limiting cases for T may be derived.

$$\begin{aligned} T = & \frac{i\omega\mu_o\rho_o^2}{2} \left[\frac{\sigma_t}{2} - \frac{\sigma_s h}{(\rho_o^2 + 4h^2)^{1/2}} \right. \\ & \left. - \frac{\sigma_t - \sigma_s}{4\ell} [\{\rho_o^2 + 4(h+\ell)^2\}^{1/2} - (\rho_o^2 + 4h^2)^{1/2}] \right] \end{aligned} \quad (6)$$

Case A: $\ell \rightarrow \infty$.

$$\lim_{\ell \rightarrow \infty} (6) = \frac{i\omega\mu_o\rho_o^2}{2} \left[\frac{\sigma_t}{2} - \frac{\sigma_s h}{(\rho_o^2 + 4h^2)^{1/2}} + \frac{(\sigma_s - \sigma_t)}{2} \right].$$

Thus,

$$\lim_{\ell \rightarrow \infty} T = \frac{i\omega\mu_o\rho_o^2}{2} \sigma_s \left[\frac{1}{2} - \frac{h}{(\rho_o^2 + 4h^2)^{1/2}} \right] \quad (7)$$

Case B: $\ell \rightarrow 0$.

This case also requires evaluation of the indeterminate form given below:

$$\lim_{\ell \rightarrow 0} \frac{\{\rho_o^2 + 4(h+\ell)^2\}^{1/2} - (\rho_o^2 + 4h^2)^{1/2}}{4\ell} \quad (8)$$

Again, using L'Hospital's rule on (8),

$$\lim_{\ell \rightarrow 0} \frac{d}{d\ell} \left[\frac{\{\rho_o^2 + 4(h+\ell)^2\}^{1/2} - (\rho_o^2 + 4h^2)^{1/2}}{4\ell} \right] \quad (8a)$$

$$\lim_{\ell \rightarrow 0} (8a) = \frac{h}{(\rho_o^2 + 4h^2)^{1/2}} \quad (9)$$

Substituting (9) into (7) yields:

$$\begin{aligned} \lim_{\ell \rightarrow 0} T = \frac{i\omega\mu_o\rho_o^2}{2} & \left[\frac{\sigma_t}{2} - \frac{\sigma_s h}{(\rho_o^2 + 4h^2)^{1/2}} \right. \\ & \left. + \frac{(\sigma_s - \sigma_t)h}{(\rho_o^2 + 4h^2)^{1/2}} \right] \quad (10) \end{aligned}$$

which may be simplified to

$$\lim_{\ell \rightarrow 0} T = \frac{i\omega\mu_o\rho_o^2}{2} \sigma_t \left[\frac{1}{2} - \frac{h}{(\rho_o^2 + 4h^2)^{1/2}} \right] \quad (11)$$

SECTION V

LOW FREQUENCY LOOP COUPLING NEAR A ROOF WITH PIECEWISE LINEAR VARIATIONS IN CONDUCTIVITY

Previous sections outlined measurements of mutual coupling between two small loop antennas using two different models of a coal mine roof structure. Other models, combining the theory of those previously discussed are described here.

First, consider the problem where the conductivity is constant throughout a portion of the first layer. Following a certain point, the conductivity varies linearly throughout the remainder of the first layer. Beyond the first layer, the conductivity is constant. The roof structure consists of a homogeneous slab of coal of thickness $h_1 - h$, followed by an inhomogeneous slab of thickness ℓ . From the coal/air interface until depth h_1 , the conductivity of the slab is of uniform value σ_1 . Beyond h_1 , the conductivity varies linearly with z to the final value σ_2 at $z = h_1 + \ell$. The rock or slate is assumed to be homogeneous beyond this point. This conductivity profile is shown in Figure 15b.

The conductivity $\sigma(z)$ is written with respect to the cylindrical coordinate system (ρ, ϕ, z) as follows:

$$\begin{aligned} \sigma(z) = & \sigma_1 u(z-h) + (\sigma_2 - \sigma_1) u(z-h_1) \\ & + (\sigma_2 - \sigma_1) \left[1 - \frac{z - h_1}{\ell} \right] u(z-h_1-\ell) \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} u(z) = & 1 \quad \text{if } z > 0 \\ & 0 \quad \text{if } z < 0 \end{aligned}$$

FIG. 15a. Horizontal coplanar and perpendicular loop configurations below a roof with piecewise linear variations.

FIG. 15b. Conductivity profile as a function of height.

As usual, the influence of the tunnel floor or haulageway in the portion of the air region $z < 0$ is neglected.

Using the "low frequency" approximation where $\sigma(z)$ vanishes for $z < h$, it is necessary to substitute (5.1) into (4.4) and (4.5), repeated below for convenience.

$$S = \frac{i\omega\mu_o\rho_o^3}{4} \int_0^{\infty} \sigma'(z) \frac{1}{(\rho_o^2 + 4z^2)^{1/2}} dz \quad (5.2)$$

$$T = \frac{i\omega\mu_o\rho_o^2}{2} \left[\int_h^{\infty} \frac{\sigma_{\infty}}{2} - \int_h^{\infty} \frac{z}{(\rho_o^2 + 4z^2)^{1/2}} \sigma'(z) dz \right] \quad (5.3)$$

where $\sigma'(z) = d\sigma(z)/dz$, $\sigma_{\infty} = \sigma(\infty)$, and S and T are defined in Section III.

Taking the derivative of (5.1) yields:

$$\sigma'(z) = \sigma_1 \delta(z-h) + \frac{\sigma_2 - \sigma_1}{\ell} [u(z-h_1) - u(z-h_1-\ell)]. \quad (5.4)$$

The limiting case

$$\lim_{h \rightarrow h_1} \sigma'(z) = \sigma_1 \delta(z-h_1) + \frac{\sigma_2 - \sigma_1}{\ell} [u(z-h_1) - u(z-h_1-\ell)] \quad (5.5)$$

is identical with (4.6) if $\sigma_s = \sigma_1$ and $\sigma_t = \sigma_2$. Using (5.4) in (5.3) and (5.2) produces the following results:

$$S = \frac{i\omega\mu_o\rho_o^3}{4} \left[-\frac{\sigma_1}{(\rho_o^2 + 4h^2)^{1/2}} + \frac{\sigma_2\sigma_1}{2\ell} \times \ln \left[\frac{2(h_1+\ell) + \{\rho_o^2 + 4(h_1+\ell)^2\}^{1/2}}{2h + (\rho_o^2 + 4h^2)^{1/2}} \right] \right] \quad (5.6a)$$

and

$$T = \frac{i\omega\mu_o\rho_o^2}{4} \left[\frac{\sigma_2}{2} - \frac{\sigma_1 h}{(\rho_o^2 + 4h^2)^{1/2}} - \frac{(\sigma_2 - \sigma_1)}{4\ell} \right. \\ \left. \times [\{\rho_o^2 + 4(h_1 + \ell)^2\}^{1/2} - (\rho_o^2 + 4h_1^2)^{1/2}] \right] \quad (5.6b)$$

The methods given in the Appendix to Section IV are used to compute the extreme cases where the transition region from σ_2 to σ_1 is either zero or infinite.

$$\lim_{\ell \rightarrow 0} S = \frac{i\omega\mu_o\rho_o^3}{4} \left[\frac{\sigma_1}{(\rho_o^2 + 4h^2)^{1/2}} + \frac{(\sigma_2 - \sigma_1)}{(\rho_o^2 + 4h_1^2)^{1/2}} \right] \quad (5.7)$$

$$\lim_{\ell \rightarrow \infty} S = \frac{i\omega\mu_o\rho_o^3}{4} \frac{\sigma_1}{(\rho_o^2 + 4h^2)^{1/2}} \quad (5.8)$$

$$\lim_{\ell \rightarrow 0} T = \frac{i\omega\mu_o\rho_o^2}{2} \left[\frac{\sigma_2}{2} - \frac{\sigma_1 h}{(\rho_o^2 + 4h_1^2)^{1/2}} + \frac{(\sigma_1 - \sigma_2)h_1}{(\rho_o^2 + 4h_1^2)} \right] \quad (5.9)$$

$$\lim_{\ell \rightarrow \infty} T = \frac{i\omega\mu_o\rho_o^2\sigma_1}{2} \left[\frac{1}{2} - \frac{h}{(\rho_o^2 + 4h^2)^{1/2}} \right] \quad (5.10)$$

when $h \rightarrow h_1$, (5.7) and (5.9) reduce to the case given in the Section IV Appendix. These limiting cases show that when $\ell \rightarrow 0$ and $h \rightarrow h_1$, the conductivity is σ_1 , or, simply the responses for homogeneous half-spaces.

The magnitudes of $S/\sigma_2\omega$ and $T/\sigma_2\omega$ are plotted in Figures 16 and 17 for h , the distance of the loops from the roof, varying from 0 to 1.2m, and the loop separation ρ_o equal to 1m. Again, in plotting the magnitudes of $S/\sigma_2\omega$ and $T/\sigma_2\omega$, the real parts of these functions are neglected at

low frequencies. The ratio

$$\frac{\sigma_2}{\sigma_1} = \frac{\text{conductivity of upper homogeneous region}}{\text{conductivity of initially homogeneous region}}$$

is equal to 10.

Defining d to be the distance from the beginning of nonzero conductivity to the middle of the transition region, it is possible to write:

$$d = h_1 - h + \frac{\ell}{2} . \quad (5.11)$$

where d is assigned the values 0, 0.1, 0.3, and 0.7m. The ratio ℓ/d determines the "sharpness" of the transition region. This ratio takes on the values 0 (instantaneous transition), 1.0, 2.0, and 5.0.

The curves in Figures 16a and 16b show that there is very little resolution for transition regions less than 1 meter. Comparing the curves of Figure 16 with those of Figure 17, it is seen that resolution improves as the transition region becomes larger.

A second case of piecewise linear variation in conductivity may be treated using the methods of this section. In this case, the roof conductivity varies linearly over a finite vertical distance, say from h to $h + \ell$. At $h + \ell$, the conductivity jumps instantaneously and becomes homogeneous.

Mathematically, this conductivity profile is written:

$$\begin{aligned} \sigma(z) = & \left[(\sigma_2 - \sigma_1) \frac{z - h}{\ell} + \sigma_1 \right] u(z-h) + \left[(\sigma_2 - \sigma_1) \right. \\ & \times \left[1 - \frac{z - h}{\ell} \right] - \sigma_2 + \sigma_3 \left. \right] u(z-h-\ell) . \end{aligned} \quad (5.12)$$

Taking the derivative of (5.12) yields:

$$\begin{aligned} \sigma'(z) = & \sigma_1 \delta(z-h) + \frac{\sigma_2 - \sigma_1}{\ell} [u(z-h) - u(z-h-\ell)] \\ & + (\sigma_3 - \sigma_2) \delta(z-h-\ell) . \end{aligned} \quad (5.13)$$

After substituting (5.13) into (5.2) and (5.3), the following results are obtained.

$$\begin{aligned} S = & \frac{i\omega\mu_o\rho_o^3}{4} \left[\frac{\sigma_1}{(\rho_o^2 + 4h^2)^{1/2}} + \frac{\sigma_2 - \sigma_1}{2\ell} \right. \\ & \times \ln \left[\frac{2(h+\ell) + \{\rho_o^2 + 4(h+\ell)^2\}^{1/2}}{2h + (\rho_o^2 + 4h^2)^{1/2}} \right] + \left. \frac{\sigma_3 - \sigma_2}{\{\rho_o^2 + 4(h+\ell)^2\}^{1/2}} \right] \end{aligned} \quad (5.14)$$

$$\begin{aligned} T = & \frac{i\omega\mu_o\rho_o^2}{4} \left[\frac{\sigma_3}{2} - \frac{\sigma_1 h}{(\rho_o^2 + 4h^2)^{1/2}} + \frac{\sigma_1 - \sigma_2}{4\ell} \right. \\ & \left. [\{\rho_o^2 + 4(h+\ell)^2\}^{1/2} - (\rho_o^2 + 4h^2)^{1/2}] \right. \\ & \left. + \frac{(\sigma_3 - \sigma_2)(h+\ell)}{\{\rho_o^2 + 4(h+\ell)^2\}^{1/2}} \right] \end{aligned} \quad (5.15)$$

In the limiting cases $\sigma_3 \rightarrow \sigma_2$, $\ell \rightarrow 0$, and $\ell \rightarrow \infty$, (5.14) and (5.15) reduce to the responses for homogeneous half spaces.

The magnitudes of $S/\sigma_2\omega$ and $T/\sigma_2\omega$ are shown in Figures 18 and 19. Again, h ranges from 0 to 1.2m, and $\rho_o = 1$ m. The curves are plotted for $\sigma_3/\sigma_2 = 2$, and $\sigma_2/\sigma_1 = 0.1$ and 10.

Additional conductivity profiles are treated in the literature. For example, *Wait* [1962] presents results for conductivity having exponential variations using the perpendicular loop configuration.

FIG. 16a. The horizontal coplanar loop response function vs roof distance for $\sigma_2/\sigma_1 = 10$ and $\ell/d = 0$, with curves for $\ell/d = 1$ and 2 superimposed.

FIG. 16b. The perpendicular loop response function vs roof distance for $\sigma_2/\sigma_1 = 10$ and $\ell/d = 0$, with curves for $\ell/d = 1$ and 2 superimposed.

FIG. 17a. The horizontal coplanar loop response function vs roof distance for $\sigma_2/\sigma_1 = 10$ and $l/d = 5$.

FIG. 17b. The perpendicular loop response function vs roof distance for $\sigma_2/\sigma_1 = 10$ and $l/d = 5$.

Others, such as *Lytle and Lager* [1976] have studied low frequency probing techniques for different profiles. In particular, these two authors study sinusoidal, step, and dipping profiles.

FIG. 18a. The horizontal coplanar loop response function vs roof distance for $\sigma_3/\sigma_2 = 2$ and $\sigma_2/\sigma_1 = 0.1$.

FIG. 18b. The perpendicular loop response function vs roof distance for $\sigma_3/\sigma_2 = 2$ and $\sigma_2/\sigma_1 = 0.1$.

FIG. 19a. The horizontal coplanar loop response function vs roof distance for $\sigma_3/\sigma_2 = 2$ and $\sigma_2/\sigma_1 = 10$.

FIG. 19b. The perpendicular loop response function vs roof distance for $\sigma_3/\sigma_2 = 2$ and $\sigma_2/\sigma_1 = 10$.

SECTION VI

DC GROUND CONDUCTIVITY METHODS

As indicated earlier, DC electrical conductivity methods may be used to determine the geologic structure of a coal seam to one or several meters. *Mooney and Wetzel* [1956] noted that this is one geophysical probing technique that can be used to obtain ground conductivity without mandatory drilling. However, drill holes can be used to calibrate measuring devices and to corroborate data.

Typically, four electrode (quadripole) arrays are used in DC ground conductivity measurement. Using a four electrode array, current is passed between two electrodes at the earth's surface and voltage is measured between two potential electrodes. Usually, the conductivity over a broad area is required. Thus, the dimensions of the electrodes are small so that the current distribution is essentially the same as when electrodes are considered as point sources.

Where current flows between two nearby electrodes, most of it remains closer to the surface. Using greater electrode spacing, deeper materials will affect the current distribution, with resulting changes in the measured voltage between the potential electrodes.

Ground conductivity is obtained by varying electrode spacing and using Ohm's law to obtain the mutual impedance. At DC, however, the mutual impedance is real and resistive.

Summarizing the derivations given by *Sunde* [1966] for the two-layered, planar earth, a simple case of conductivity measurement using a quadripole can be treated if one current and potential electrode are assumed at infinity. Since one current electrode is assumed far away, the current is radial about

the other surface point electrode. The radial current density at the surface of the coal at a distance ρ is then given by $J = I/2\pi\rho^2$. Since $\bar{J} = \sigma\bar{E}$, the electric intensity at ρ is:

$$E(\rho) = \frac{I}{2\pi\sigma\rho^2} \quad (6.1)$$

The potential at the distance ρ from the electrode becomes:

$$V = \int_0^{\infty} E(\rho)ds = \frac{I}{2\pi\sigma\rho} \quad (6.2)$$

Taking the ratio of potential to current, the mutual resistance of the point under consideration is:

$$R(\rho) = \frac{V}{I} = \frac{1}{2\pi\sigma\rho} \quad (6.3)$$

For a two-layered earth structure, such as a coal seam over homogeneous rock or slate, the mutual resistance function $R(\rho)$ between points may be expressed as a function F of two ratios, $\alpha = b/d$, where d = depth of the coal layer, b is the separation between electrodes, and $K = \sigma_2/\sigma_1$. As a result,

$$R(\rho) = \frac{1}{2\pi\sigma_1\rho} F(\alpha, K) . \quad (6.4)$$

when $b \ll d$, the mutual resistance approaches $1/2\pi\sigma_1 b$, and when b is large compared to d , it approaches $1/2\pi\sigma_2 b$. The earth conductivity derived from the mutual resistance varies between σ_1 and σ_2 when measurements are made with increasing electrode spacing. How the mutual resistance varies with d depends on the b/d ratio.

Comparing the observed variation with theoretical curves for various assumed values of d , an approximate value of d may be obtained. In

As in previous discussions of mutual coupling between two loop antennas over a two-layered earth, the field is assumed to consist of primary and secondary components. The resulting potentials in the upper and lower layers are written:

$$V_m = V_m^P + V_m^S \quad (6.5)$$

where,

p denotes primary field
 s denotes secondary field
 $m = 1, 2$ depending on layer.

Since it is assumed that the z axis extends into the earth through the electrode, there is circular symmetry about the z axis. The primary field becomes:

$$V^P = \frac{I}{2\pi\sigma_1} \frac{1}{(z^2 + \rho^2)^{1/2}} \quad (6.6)$$

Using a Fourier transformation to put V^P into a form where z and ρ are separate functions:

$$\frac{1}{(z^2 + \rho^2)^{1/2}} = \int_0^\infty e^{-\lambda z} f(\lambda \rho) d\lambda \quad (6.7a)$$

where,

$$f(\lambda \rho) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{\lambda z}}{(z^2 + \rho^2)^{1/2}} dz = J_0(\lambda \rho) \quad (6.7b)$$

Thus,

$$V^P = \frac{I}{2\pi\sigma_1} \int_0^\infty e^{-\lambda z} J_0(\lambda \rho) d\lambda \quad (6.8)$$

The secondary potentials for the two-layered earth are written:

$$V_m^s = \int_0^{\infty} \{f_m(\lambda)e^{-\lambda z} + g_m(\lambda)e^{\lambda z}\} J_0(\lambda \rho) d\lambda \quad (6.9)$$

where $m = 1, 2$.

To satisfy the boundary condition that requires the normal current flow at the insulator is zero, except at the source ($z=0$),

$$E_z^s = \frac{-dV}{dz} = 0. \quad (6.10)$$

Thus, $f_1 = g_1$. To assure that the potential vanishes as $z \rightarrow \infty$ requires that $g_2 = 0$. The potentials and the current densities in the z direction must be equal at $z = d$ so that:

$$V_1 = V_2 \quad (6.11a)$$

$$J_1 = -\sigma_1 \frac{dV}{dz} = J_2 = -\sigma_2 \frac{dV}{dz} \quad (6.11b)$$

Applying the boundary conditions to (6.9),

$$\frac{I}{2\pi\sigma_1} e^{-\lambda d} + f_1(\lambda)(e^{-\lambda d} + e^{\lambda d}) = \frac{I}{2\pi\sigma_1} e^{-\lambda d} + f_2(\lambda)e^{-\lambda d} \quad (6.12a)$$

and

$$\lambda\sigma \left[\frac{-I}{2\pi\sigma_1} e^{-\lambda d} + f_1(\lambda)(e^{-\lambda d} + e^{\lambda d}) \right] = \lambda\sigma_2 \left[\frac{-I}{2\pi\sigma_1} e^{-\lambda d} + f_2(\lambda)e^{-\lambda d} \right] \quad (6.12b)$$

Solving (6.12a) and (6.12b) yields:

$$f_1(\lambda) = \frac{I}{2\pi\sigma_1} \frac{-\text{Re}^{-2\lambda d}}{1 + \text{Re}^{-2\lambda d}} \quad (6.13)$$

where the reflection coefficient

$$R = \frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1}.$$

Returning to (6.5) and (6.8), the mutual resistance V/I on the surface of the ground becomes:

$$R(\rho) = \int_0^{\infty} \left[\frac{1}{2\pi\sigma_1} + 2f_1(\lambda) \right] J_0(\lambda\rho) d\lambda \quad (6.14)$$

$$= \frac{1}{2\pi\sigma_1} \int_0^{\infty} \frac{1 - \operatorname{Re}^{-2\lambda d}}{1 + \operatorname{Re}^{-2\lambda d}} J_0(\lambda\rho) d\lambda \quad (6.15)$$

$$= \frac{1}{2\pi\sigma_1} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(-R)^n}{\{1 + (2nd/b)^2\}^{1/2}} \right] \quad (6.16)$$

Now, lettering $R' = -R$ so that

$$R' = \frac{1 - K}{1 + K},$$

where $K = \sigma_2/\sigma_1$, (6.16) is written:

$$\frac{\sigma_a}{\sigma_1} = \sum_{n=0}^{\infty} \frac{\epsilon_n R'^n}{[1 + (2nd/b)^2]^{1/2}} \quad (6.17)$$

where

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \neq 0 \end{cases}$$

where σ_a is the apparent conductivity when $\rho = b$.

Apparent conductivity curves showing σ_a/σ_1 for different K and d/b ratios are shown in Figure 23. While $.002 < K < 1000$, d/b ranges from 10^{-2} to 10^2 . These curves are directly comparable to the resistivity curves given by *Sunde* [1966].

Now, if it is assumed that both current and potential electrodes are spaced a finite distance apart, various four electrode configurations can be used to determine the apparent conductivity of a coal mine roof structure.

The mutual resistance for a general four electrode configuration consisting of two current electrodes C and C', and two potential electrodes P and P' on the surface of the coal are given by:

$$2\pi\sigma \frac{V}{I} = \frac{V(CP)}{CP} + \frac{V(C'P')}{C'P'} - \frac{V(CP')}{CP'} - \frac{V(C'P)}{C'P} \quad (6.18)$$

In Western countries, the most commonly used four electrode configuration is the Wenner array. This configuration uses equi-spaced current and potential electrodes. The current electrodes are at the ends of the array, while the potential electrodes are centrally located.

Now, using (6.18), the apparent conductivity is given by:

$$\sigma_a = \frac{I}{2\pi bV}$$

and

$$\frac{\sigma_a}{\sigma_1} = 2 \sum_{n=0}^{\infty} \epsilon_n R^n \left\{ \frac{1}{[1 + (2nd/b)^2]^{1/2}} - \frac{1}{[4 + (2nd/b)^2]^{1/2}} \right\} \quad (6.19)$$

where

$$R = \frac{1 - K}{1 + K} \quad \text{and} \quad \epsilon_n = \begin{matrix} 1, & n = 0 \\ 2, & n \neq 0 \end{matrix}$$

Equation (6.19) is used to compute the curves appearing in Figure 24. Again, d/b ranges from 10^{-2} to 10^2 . Now, however, K takes on values between .05 and 1000.

In another electrode arrangement, the probes are in line and equi-spaced, but the current electrodes C and C' are at one end of the array and the potential electrodes P - P' are at the other. This quadri-pole is called the Eltran array.

FIG. 20. The four electrode Wenner array ($b_1 = b_2 = b_3$).

FIG. 21. The four electrode Eltran array (z axis into page.)

The mathematical expressions for apparent conductivity are given as follows:

$$\sigma_a = \frac{I}{6\pi bV}$$

and,

$$\frac{\sigma_a}{\sigma_1} = 3 \sum_{n=0}^{\infty} \epsilon_n R^n \left\{ \frac{1}{[1 + (2nd/b)^2]^{1/2}} + \frac{1}{[9 + (2nd/b)^2]^{1/2}} - \frac{2}{[4 + (2nd/b)^2]^{1/2}} \right\} \quad (6.20)$$

Apparent conductivity curves for d/b between 10^{-2} and 10^2 , and $.01 < K < 1000$ are shown in Figure 25.

A modification of the Eltran array, known as the Right Angle array, has been described by *Wait and Conda* [1958]. The current electrodes $C - C'$ are on a line which is normal to the line joining the potential electrodes. The interelectrode spacing is equal to the linear distances CP , CP' and $C'P'$.

The formulas for σ_a and σ_a/σ_1 are now given by:

$$\sigma_a = \frac{0.053052I}{bV}^*, \quad (6.21a)$$

and,

$$\frac{\sigma_a}{\sigma_1} = 3.014 \sum_{n=0}^{\infty} \epsilon_n R^n \left\{ \frac{1}{[1 + (2nd/b)^2]^{1/2}} + \frac{1}{[(2.914)^2 + (2nd/b)^2]^{1/2}} - \frac{2}{[(1.849)^2 + (2nd/b)^2]^{1/2}} \right\} \quad (6.21b)$$

*NOTE: Numerical discrepancies occur between these equations and those given by *Wait and Conda* [1958]. Specifically, in (6.21a), *Wait and Conda* obtained 0.05605 in the numerator. In (6.21b), they obtain 2.84 instead of 3.014. These discrepancies are probably due to slide rule inaccuracies.

FIG. 22. The four electrode Right Angle array ($\theta = 45^\circ$).

Apparent conductivity curves are shown in Figure 26 for $10^{-2} < d/b < 10^2$ and K between .01 and 1000.

Other four electrode configurations have been proposed in the literature. For example, Carpenter proposed [Kunetz, 1966] a quadripole in line array which alternates current and potential electrodes. In this array, the current and potential electrodes need not be equi-spaced. One useful array in which the four electrodes are not in line has the potential electrodes displaced laterally, but parallel from the current electrodes. This configuration is used in the rectangle ground conductivity measurement method.

Arrays having more than four electrodes are often used in DC ground conductivity measurements. For example, in drill holes, the "Laterolog 7" [Roy, 1975], a seven electrode, in line configuration having three current probes is typical of the specialized arrays used in electrical soundings.

FIG. 23. Apparent conductivity curves for a two electrode array vs d/b .

FIG. 24. Apparent conductivity curves for the four electrode Wenner array vs d/b .

FIG. 25. Apparent conductivity curves for the four electrode Eltran array vs d/b .

FIG. 26. Apparent conductivity curves for the four electrode Right Angle array vs d/b .

SECTION VII

THE THETA ARRAY IN DC GROUND CONDUCTIVITY MEASUREMENTS

The preceding section showed several different quadripole arrangements and their use in DC ground conductivity measurements. Here, a new four electrode configuration called the Theta array is considered.

FIG. 27. The Theta array

This array consists of two current electrodes which are fixed and symmetric about a central point O . Potential electrodes are on a rotating boom which is also centered at O . The array gets its name from the angle the potential electrode P' makes with C' and P makes with C .

Three parameters are used to describe the Theta array, a , b , and d . In this array, a is the distance from the current electrode to the central point O , while b denotes the distance from the potential electrode to O .

A simple case of using this array in DC ground conductivity measurements of a planar roof structure with two homogeneous layers is described if P' and C are assumed remote. As a result, $C'P$, $C'P'$, and CP' all approach infinity in (6.18). Now,

$$\frac{V}{I} = \frac{1}{2\pi\sigma_1} \frac{V(CP)}{CP} \quad (7.1)$$

$$\frac{\sigma_a}{\sigma_1} = \sum_{n=-\infty}^{\infty} \frac{\epsilon_n R^n}{r_1 \cdot 2nd^2 \gamma^{1/2}} \quad (7.2)$$

where,

$$\epsilon_n = 1, n = 0$$

$$2, n \neq 0$$

and

$$R = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} = \frac{1 - K}{1 + K}.$$

The apparent conductivity relationships given by (6.17) and (7.2) are of the same form. However, the length CP in (7.1) is angularly dependent. Using the law of cosines,

$$CP = (a^2 + b^2 - 2ab \cos\theta)^{1/2} \quad (7.3)$$

Substituting (7.3) in (7.1) and (7.2), a normalized mutual resistance response function $V(\theta)$ may be defined as:

$$V(\theta) = \frac{V\pi\sigma_1 a}{I} = \frac{a}{2} \sum_{n=0}^{\infty} \frac{\epsilon_n R^n}{\{a^2 + b^2 - 2ab\hat{C} + (2nd)^2\}^{1/2}} \quad (7.4)$$

where $\hat{C} = \cos\theta$ and $0^\circ \leq \theta \leq 180^\circ$. After normalizing by b, (7.4) becomes:

$$V(\theta) = \frac{V\pi\sigma_1 a}{Ib} = \frac{1}{2} \left(\frac{a}{b}\right) \sum_{n=0}^{\infty} \frac{\epsilon_n R^n}{[(a/b)^2 + 1 - (2a\hat{C}/b) + (2nd/b)^2]^{1/2}} \quad (7.5)$$

To illustrate the results, Figures 32 and 33 show $V(\theta)$ curves for different d/b ratios from 0 - 10 for a given conductivity ratio $K = \sigma_2/\sigma_1 = 0.1$ or 10, and $a/b = 2.0$.

If the coal layer is not planar, but dipping or slanted, image theory can be used to derive an approximation for $V(\theta)$. However, as noted by *Kunetz* [1966], image theory is only viable when the dipping angle δ is small, and/or the conductivity ratios of the two layers are extremely large or small. In this derivation, it is assumed that $\delta = 2.5^\circ$ and the images

can be terminated at $\pi/2$. Thus, images are located at $C_1, C_2, \dots, C_n$ with strengths $RI, R^2I, \dots, R^nI$.

FIG. 28. The Theta array on a dipping coal/rock interface ($OA = d \cot \delta = D$, $C_O = a$, $PO = b$).

The coordinates of P are $(b\hat{C}, b\hat{S}, 0)$, where $\hat{C} = \cos\theta$, and $\hat{S} = \sin\theta$.

Mathematically, $V(\theta)$ is determined as follows:

$$\begin{aligned} \frac{V}{I} = & \frac{1}{2\pi\sigma_1} \left\{ \frac{1}{C_O P} \right\} + \frac{R}{\pi\sigma_1} \left\{ \frac{1}{C_1 P} \right\} + \frac{R^2}{\pi\sigma_1} \left\{ \frac{1}{C_2 P} \right\} \dots \\ & + R^j \left\{ \frac{1}{C_j P} \right\} \end{aligned} \quad (7.6)$$

The coordinates of the image terms are given by:

$$\begin{aligned}
& C_1; (S_1 0, 0, S_1 C_1) \\
& C_2; (S_2 0, 0, S_2 C_2) \\
& \dots\dots\dots \\
& C_j; (S_j 0, 0, S_j C_j)
\end{aligned}
\tag{7.7}$$

Now, the radius $AC_1 = D + a$ so that $AS_1 = (D+a)\cos(2\delta)$. Writing the image terms,

$$\begin{aligned}
S_{10} &= AS_1 - OA = (D+a)\cos 2\delta - D \\
S_1 C_1 &= (D+a)\sin 2\delta \\
S_2 0 &= (D+a)\cos 4\delta - D \\
S_2 C_2 &= (D+a)\sin 4\delta \\
&\dots\dots\dots \\
&\dots\dots\dots \\
S_j 0 &= (D+a)\cos 2j\delta. \\
S_j C_j &= (D+a)\sin 2j\delta
\end{aligned}
\tag{7.8}$$

Letting $\hat{S} = \sin\theta$, $\hat{C} = \cos\theta$ and using the law of cosines,

$$V(\theta) = \frac{\pi \sigma_1 V}{I} = \frac{1}{2} \sum_{j=0}^J R^j \epsilon_j \left\{ \frac{1}{C_{jP}} \right\}
\tag{7.9}$$

where,

$$C_{jP} = \{(S_j 0 - b\hat{C})^2 + (b\hat{S})^2 + (S_j C_j)^2\}^{1/2}
\tag{7.10}$$

As a result,

$$V(\theta) = \frac{1}{2} \sum_{j=0}^J \frac{R^j \epsilon_j}{\{(D+a)\cos 2j\delta - D - b\hat{C}\}^2 + (b\hat{S})^2 + \{(D+a)\sin 2j\delta\}^2}^{1/2}
\tag{7.11}$$

where $D = d \cot \delta$.

When δ becomes small (i.e., 2.5°), sufficient accuracy can be obtained using J terms such that $j \leq \pi/(4\delta)$. Image terms beyond $j = \pi/(4\delta)$ are sufficiently small that they may be neglected.

With the planar case superimposed, Figures 34 and 35 show $V(\theta)$ curves for $\delta = 2.5^\circ$, and $0.25 \leq d/b \leq 10$, where $a/b = 2.0$, and $K = \sigma_2/\sigma_1 = 0.1$ or 10 . Physically, it is impossible to describe a coal seam that dips but has zero depth. Thus, d/b ratios smaller than 0.25 are meaningless in the dipping case. This means the current electrode must be to the right of the dipping angle apex in Figure 28. Mathematically, this constraint is written:

$$(1/2)\tan^{-1}(d/a) \geq \delta \quad \text{or} \quad d/a \geq \tan 2\delta \quad (7.12)$$

In Figures 36 and 37, normalized $V(\theta)$ curves for $\delta = 2.5^\circ$, and the previous values of d/b , a/b , and K show how the curves depart from each other for various theta angles. The normalization used is to divide $V(\theta)$ by $V(\theta=0^\circ)$. This normalization shows that the Theta array can clearly distinguish between different coal layer depths for a given potential electrode spacing.

Returning to the four electrode Theta array and the case of a planar, two-layered earth (illustrated in Figure 27), equation (6.18) is written:

$$\frac{V}{I} = \frac{1}{2\pi\sigma_1} \left\{ \frac{V(CP)}{CP} + \frac{V(C'P')}{C'P'} - \frac{V(CP')}{CP'} - \frac{V(C'P)}{C'P} \right\} \quad (7.13)$$

Since the cosine is an even function,

$$\begin{aligned} C'P' &= CP = (a^2 + b^2 - 2ab\hat{C})^{1/2} \\ C'P &= CP' = (a^2 + b^2 + 2ab\hat{C})^{1/2} \end{aligned} \quad (7.14)$$

Substituting (7.14) into (7.13), the normalized response function $V(\theta)$ is written as:

$$V(\theta) = \frac{V\pi\sigma_1 a}{Ib} = \frac{a}{b} \left[\frac{V(C'P')}{C'P'} - \frac{V(C'P)}{C'P} \right] \quad (7.15)$$

Now, the complete expression for $V(\theta)$ is:

$$V(\theta) = \frac{a}{b} \sum_{n=0}^{\infty} \epsilon_n R^n \left\{ \frac{1}{[(a/b)^2 + 1 - (2a\hat{C}/b) + (2nd/b)^2]^{1/2}} - \frac{1}{[(a/b)^2 + 1 + (2a\hat{C}/b) + (2nd/b)^2]^{1/2}} \right\} \quad (7.16)$$

$$0 \leq \theta \leq 90^\circ$$

In Figures 38 and 39, curves for different conductivity ratios $\sigma_2/\sigma_1 = K$ from 0 to 100 for a fixed d/b ratio are shown. The a/b ratio is 2 in Figure 38. In Figure 39, the a/b ratio is 8. Normalized $|V(\theta)|/|V(\theta=0^\circ)|$ curves are included in Figures 40 and 41 for the parameters given above. As the current electrodes are moved farther away from 0, the shapes of these curves become increasingly similar. This indicates a loss of resolution if the current electrodes are extended over long distances.

Figures 42 and 43 show $V(\theta)$ curves for $K = 0.1$ and 10, where $a/b = 2$ and d/b ranges between 0 and 10. These may be compared with Figures 32 and 33 for the two electrode case.

In (7.16), it is interesting to note that if $a = 3b$, $\theta = 0^\circ$, the result reduces to a Wenner array of interelectrode spacing $2b$. Substituting for a and θ .

$$V(\theta)_{\substack{a=3b \\ \theta=0^\circ}} = 3 \sum_{n=0}^{\infty} \epsilon_n R^n \left\{ \frac{1}{[4 + (2nd/b)^2]^{1/2}} - \frac{1}{[16 + (2nd/b)^2]^{1/2}} \right\}$$

This result is easily compared with (6.19). Now,

$$\begin{aligned} \frac{V}{I} &= \frac{1}{\pi\sigma_1 a} V(\theta) = \frac{1}{\pi\sigma_1 3b} V(\theta) \\ &= \frac{1}{\pi\sigma_1} \sum_{n=0}^{\infty} \epsilon_n R^n \left\{ \frac{1}{\{(2b)^2 + (2nd)^2\}^{1/2}} - \frac{1}{\{(4b)^2 + (2nd)^2\}} \right\} \end{aligned} \quad (7.18)$$

If $K = \sigma_2/\sigma_1 = 1$ and $n = 0$, (7.18) becomes:

$$\frac{V}{I} = \frac{1}{\pi\sigma_1} \left[\frac{1}{2b} - \frac{1}{4b} \right] = \frac{1}{2\pi\sigma_1(2b)} \quad (7.19)$$

Again, it is possible to use the array on a dipping, two-layered earth. The image theory assumptions used in the two electrode case are used here also.

FIG. 29. The four electrode Theta array over a dipping interface ($OA = d \cot \delta = D$, $C_0O = OC'_0 = a$, $PO = OP' = b$).

Images are now located at:

$$\begin{aligned} &C_1, C_2 \dots C_n, \text{ with strengths } RI, R^2I \dots R^nI. \\ &C_1^1, C_2^1 \dots C_n^1, \text{ with strengths } -RI, -R^2I \dots -R^nI. \end{aligned}$$

FIG. 30. A cross-sectional view of the coal mine roof structure showing the four electrode Theta array ($\hat{S} = \sin\theta$, $\hat{C} = \cos\theta$).

Rewriting (7.6) for the four electrode array,

$$\begin{aligned}
 V(\theta) = & \frac{1}{2\pi\sigma_1} \left\{ \frac{1}{C_0P} + \frac{1}{C'_0P'} - \frac{1}{C_0P'} - \frac{1}{C'_0P} \right\} \\
 & + \frac{R}{\pi\sigma_1} \left\{ \frac{1}{C_1P} + \frac{1}{C'_1P'} - \frac{1}{C_1P'} - \frac{1}{C'_1P} \right\} \\
 & + \dots \\
 & + \frac{R^j}{\pi\sigma_1} \left\{ \frac{1}{C_jP} + \frac{1}{C'_jP'} - \frac{1}{C_jP'} - \frac{1}{C'_jP} \right\}
 \end{aligned} \tag{7.20}$$

Now, the image coordinates are:

$$\begin{aligned}
 & C_1; (S, 0, 0, S_1 C_1) \\
 & C'_1; (-S, 0, 0, S_1 C_1) \\
 & \dots \\
 & \dots \\
 & C_j; (S_j, 0, 0, S_j C_j) \\
 & C'_j; (-S_j, 0, 0, S_j C_j)
 \end{aligned} \tag{7.21}$$

The radial distances are $AC_1 = D + a$, and $AC'_1 = D - a$, so that

$$AS_1 = (D+a)\cos 2\delta \quad (7.22a)$$

$$AS'_1 = (D-a)\cos 2\delta \quad (7.22b)$$

Thus, the j^{th} image term is given by:

$$\begin{aligned} S_j O &= AS_j - OA = (D+a)\cos 2j\delta - D \\ S_j^1 O &= OA - AS_j = D - (D-a)\cos 2j\delta \\ S_j C_j &= (D+a)\sin 2j\delta \\ S_j C_j^1 &= (D-a)\sin 2j\delta \end{aligned} \quad (7.23)$$

Using the law of cosines,

$$C_j P = \{(S_j O - b\hat{C})^2 + (b\hat{S})^2 + (S_j C_j)^2\}^{1/2} \quad (7.24a)$$

$$= \{(D+a)\sin 2j\delta - D - b\hat{C}\}^2 + (b\hat{S})^2 + \{(D+a)\sin 2j\delta\}^2^{1/2}$$

$$C_j' P' = \{(S_j^1 O - b\hat{C})^2 + (b\hat{S})^2 + (S_j^1 C_j^1)^2\}^{1/2} \quad (7.24b)$$

$$= \{D - (D-a)\cos 2j\delta - b\hat{C}\}^2 + (b\hat{S})^2 + \{(D-a)\sin 2j\delta\}^2^{1/2}$$

$$C_j P' = \{(S_j O + b\hat{C})^2 + (b\hat{S})^2 + (S_j C_j)^2\}^{1/2} \quad (7.24c)$$

$$= \{(D+a)\cos 2j\delta - D + b\hat{C}\}^2 + (b\hat{S})^2 + \{(D+a)\sin 2j\delta\}^2^{1/2}$$

$$C_j' P = \{(S_j^1 O + b\hat{C})^2 + (b\hat{S})^2 + (S_j^1 C_j^1)^2\}^{1/2} \quad (7.24d)$$

$$= \{D - (D-a)\cos 2j\delta + b\hat{C}\}^2 + (b\hat{S})^2 + \{(D-a)\sin 2j\delta\}^2^{1/2}$$

Now,

$$V(\theta) = \frac{V}{I} = \frac{1}{2} \sum_{j=0}^J R_j^j \epsilon_j \left\{ \frac{1}{C_j P} + \frac{1}{C_j' P'} - \frac{1}{C_j P'} - \frac{1}{C_j P} \right\} \quad (7.25)$$

This is the contribution from the source ($j=0$) and J images. Note, if $\delta \rightarrow 0$, $D = d \cot \delta \rightarrow \infty$. However, $(D \pm a) \sin 2j\delta \rightarrow d/\delta \sin 2j\delta \rightarrow 2j\delta$ and, $(D+a) \cos 2j\delta - D = D \cos 2j\delta - D + a \cos 2j\delta$. (7.26)

Also,

$$\lim_{\delta \rightarrow 0} (7.26) = \left(\frac{\cos 2j\delta - 1}{\tan \delta} + a \cos 2j\delta \right) = a \quad (7.27)$$

Thus, $V(\theta)$ reduces to:

$$\frac{\pi \sigma_1 V}{I} = \sum_{j=0}^J R^j \epsilon_j \left\{ \frac{1}{\{(a-b\hat{C})^2 + b\hat{S} + (2j\delta)^2\}^{1/2}} - \frac{1}{\{(a+b\hat{C})^2 + (b\hat{S})^2 + (2j\delta)^2\}^{1/2}} \right\} \quad (7.28)$$

As $J \rightarrow \infty$, this becomes identical to the four electrode Theta array on a planar, two-layered coal mine roof structure.

Using (7.25), curves are computed for $K = \sigma_2/\sigma_1 = 0.1, 10, a/b = 2$, and d/b between 0 and 10. In Figures 44 and 45, curves show $V(\theta)$ for $\delta = 2.5^\circ$.

If one of the potential electrodes (P') is allowed to return to infinity, a three electrode Theta array can be described. It should be noted that the potential electrode P' can be either at 0 or ∞ since both points have zero potential.

In this case, the mutual resistance of the Theta array on a planar, two-layered roof structure becomes:

$$\frac{V}{I} = \frac{1}{2\pi\sigma_1\rho} Q(\rho) = \frac{1}{2\pi\sigma_1} \left\{ \frac{Q(CP)}{CP} - \frac{Q(C'P)}{C'P} \right\} \quad (7.29)$$

where,

$$CP = (a^2 + b^2 - 2ab\hat{C})^{1/2}$$

$$C'P = (a^2 + b^2 + 2ab\hat{C})^{1/2}$$

$$\text{and } \hat{C} = \cos\theta.$$

FIG. 31. The three electrode Theta array.

Writing the normalized response $V(\theta)$

$$V(\theta) = \frac{V\pi\sigma_1 a}{I} = \frac{a}{2} \left[\frac{Q(CP)}{CP} - \frac{Q(C'P)}{C'P} \right],$$

which becomes:

$$V(\theta) = \frac{1}{2} \frac{a}{b} \sum_{n=0}^{\infty} \epsilon_n R^n \left\{ \frac{1}{[(a/b)^2 + 1 - 2a\hat{C}/b + (2nd/b)^2]^{1/2}} - \frac{1}{[(a/b)^2 + 1 + (2a\hat{C}/b) + (2nd/b)^2]^{1/2}} \right\} \quad (7.30)$$

$$0^\circ \leq \theta \leq 90^\circ$$

This demonstration shows that

$$V(\theta) \Big|_{3 \text{ electrode}} = \frac{1}{2} V(\theta) \Big|_{4 \text{ electrode}} \quad (7.31)$$

Sensitivity of the three electrode Theta array is reduced from that obtained using four electrodes. However, using the three electrode configuration,

only one potential probe needs to be moved to make measurements.

Furthermore, if $\delta > 0^\circ$, as in the case of a dipping two-layered roof structure, $V(0^\circ) \neq V(180^\circ)$ since the Theta array is no longer symmetrical. Also, mutual resistance measurements must be taken in the range $90^\circ < \theta^\circ \leq 180^\circ$. This is another reason why the four electrode configuration is a superior measuring device. However, the dip-angle can be inferred from the 3-electrode measurement.

In Sections VI and VII, everything discussed relates to direct current ground conductivity measurements. As noted by *Wait and Conda* [1958] and *Kunetz* [1966], alternating current ground conductivity measurements offer several advantages such as ease of power production, elimination of most electrode polarization effects, facility to amplify potentials, and the capacity to distinguish between signal and noise. However, there is one overwhelming disadvantage to using AC; skin effect. This problem occurs due to the concentration of AC near the contact between materials of different conductivities. As the frequency is increased, or the conductivity difference becomes greater, the concentration is heavier. The resulting skin effect phenomenon causes a rapid, exponential decrease of current density as a function of depth, and a decreased depth of penetration. At low frequencies, however, this effect need not be of any concern in completely penetrating a coal layer.

FIG. 32. The normalized mutual resistance response $V(\theta)$ vs θ for a two electrode Theta array ($a/b = 2$, $K = 0.1$).

FIG. 33. The normalized mutual resistance response $V(\theta)$ vs θ for a two electrode Theta array ($a/b = 2$, $K = 10$).

FIG. 34. The normalized mutual resistance response $V(\theta)$ vs θ for a two electrode Theta array over a dipping interface ($a/b = 2$, $K = 0.1$, $\delta = 2.5^\circ$). Differences for the planar case are superimposed with dotted lines.

FIG. 35. The normalized mutual resistance response $V(\theta)$ vs θ for a two electrode Theta array over a dipping interface ($a/b = 2$, $K = 10$, $\delta = 2.5^\circ$). Differences for the planar case are superimposed with dotted lines.

FIG. 36. The magnitude of $V(\theta)/V(\theta=0^\circ)$ vs θ for a two electrode Theta array over a dipping interface ($a/b = 2$, $K = 0.1$, $\delta = 2.5^\circ$).

FIG. 37. The magnitude of $V(\theta)/V(\theta=0^\circ)$ vs θ for a two electrode Theta array over a dipping interface ($a/b = 2$, $K = 10$, $\delta = 2.5^\circ$).

FIG. 38. $V(\theta)$ vs θ for the four electrode Theta array over a planar interface ($d/b = 0.5$, $a/b = 2$).

FIG. 40. The magnitude of $V(\theta)/V(\theta=0^\circ)$ vs θ for the four electrode

FIG. 41. The magnitude of $V(\theta)/V(\theta=0^\circ)$ vs θ for the four electrode Theta

FIG. 42. $V(\theta)$ vs θ for the four electrode Theta array over a planar interface ($a/b = 2$, $K = 0.1$).

FIG. 43. $V(\theta)$ vs θ for the four electrode Theta array over a planar interface ($a/b = 2$, $K = 10$).

FIG. 44. $V(\theta)$ vs θ for the four electrode Theta array over a dipping interface ($a/b = 2$, $K = 0.1$, $\delta = 2.5^\circ$). Differences for the planar case are superimposed with dotted lines.

FIG. 45. $V(\theta)$ vs θ for the four electrode Theta array over a dipping interface ($a/b = 2$, $K = 10$, $\delta = 2.5^\circ$). Differences for the planar case are superimposed with dotted lines.

SECTION VIII

CONCLUSIONS

Several different techniques for obtaining environmental data in a coal mine have been described. Sections II through V demonstrate how low frequency mutual coupling between two loop antennas can be used to determine the thickness of a coal seam and/or its conductivity. In Section VI, apparent conductivity curves for the Wenner, Eltran, and Right Angle arrays are presented. Additional ground conductivity curves for the Theta array appear in the last section.

The concept in using the horizontal coplanar or perpendicular loop configurations is that the normalized mutual impedance response function would be measured as a function of the distance from the coal mine roof. Considerable care must be taken, however, in this proposed measurement since $|S|$ and $|T|$ defined in Section III are $\ll 1$ at low frequencies. Thus, the loop system needs to be calibrated and tested in a simulated free space environment before beginning underground tests.

Several conclusions are derived from the low frequency mutual impedance curves for different conductivity variations in a coal seam. In Section III, the homogeneous seam's depth is easily determined when it is less than or equal to 1m. Also, mutual impedance measurements are made at distances less than 0.6m from the roof. Even if the coal/slate conductivity ratio is known, matching field data to the curves is easiest if mutual impedance measurements are made at small distances from the roof.

In Section IV, it is assumed that the conductivity varies linearly due to an anomaly such as changing water content. The resolution of either two-loop configuration is better for small transition regions.

Generally, it is easier to determine the coal seam depth if it is less than 2m, and measurements of normalized T are made at distances from the roof less than 0.6m. However, field data for S can be easily matched to the computed curves for measurements taken up to 1.2m from the roof. Either the scale for plotting normalized T should be expanded, or the horizontal coplanar configuration offers more accuracy than the perpendicular arrangement when probing a coal seam with linear conductivity variations.

Section V shows that it is difficult to determine the depth of a coal seam when it contains piecewise linear conductivity variations. Both normalized response functions S and T are incapable of determining coal seam depth unless the conductivity transition is at least twice as great as the distance from the air/coal interface to the coal/rock interface. Loops cannot be used effectively for measuring these conductivity profiles unless the coal layer is very thin.

Apparent conductivity curves, obtained from DC quadripole arrays show that the depth of a homogeneous coal seam is easily determined if the current and voltage probes are spaced large distances apart. Adequate resolution between curves for different conductivity ratios can be obtained using two electrodes if the probes are spaced $0.1d$. However, for the four electrode Wenner, Eltran and Right Angle arrays, the condition $b \geq d$ should be used to obtain accurate information about the conductivity ratio. Knowing the conductivity ratio, the coal seam depth is determined most accurately when probes are spaced even farther apart.

The Theta array also can be used successfully in probing a planar, homogeneous coal seam. One requirement for lower conductivity ratios is that the probes need to be spaced far apart in order to determine accurately

the coal seam depth. For higher conductivity ratios, the probe spacing should be small in comparison with the coal seam depth.

An approximate depth of a dipping coal seam can be determined using the Theta array. However, the dipping angle δ must be small and the coal/slate conductivity very high or low. As seen in Section VII, image theory is not exact unless the conductivity ratio is zero or infinite.

The curves presented in Section VII for $\delta = 2.5^\circ$ show very little departure from those for a planar coal layer. As might be expected, when $\delta > 2.5^\circ$, this departure becomes greater. Yet, the four electrode Theta array cannot be used to detect dipping angle accurately since image theory breaks down for large values of δ . However, the asymmetry of the three electrode results should show significant differences between planar and dipping interfaces, even for small δ .

Previous chapters include a broad survey of low frequency AC and DC ground conductivity measurement techniques in a coal mine operation. The response of the horizontal coplanar and perpendicular loop configurations over a coal layer with (piecewise) linear conductivity variations at higher frequencies should be investigated further. Also, results for the Wenner, Eltran, and Right Angle arrays in probing a coal seam with (piecewise) linear conductivity variations would be useful.

Several authors [*Frischknecht*, 1967; *Wait and Spies*, 1972, 1973] have drawn Argand diagrams showing the real versus imaginary portions of the normalized mutual impedance responses for two or more homogeneous layers. This presentation technique could be useful in showing S and T for different K ratios and transition region lengths.

The Theta array also warrants further investigation. One possible conductivity profile that can be treated is the case of a vertical contact [Kunetz, 1966]. When it is possible to ignore the substratum, which assumes that the contact extends vertically downward to infinity, it is possible to use the same computation methods as for horizontal conductivity profiles. Accordingly, it is possible to compute apparent conductivity curves for a multilayered, horizontally stratified earth in the manner of *Mooney and Wetzel* [1956], and *Al'pin et. al.* [1966].

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PART 1

SOME EARTH RESISTIVITY PROBLEMS
INVOLVING BURIED CABLES
PART 1 - ANALYSIS

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Abstract—The theory of current flow in homogeneous and layered conducting media is developed for the case where an axial conductor or cable is present. This cable, which is characterized by a specified axial impedance, is assumed to be infinite in length. Various configurations are chosen such as a current point source in an infinite, semi-infinite, and layered region where the cable is taken parallel to the interface(s). The resulting formulas for the potentials reduce to known cases in the absence of the cable. Part II will contain various examples based on numerical integration of the rapidly convergent integral formulas developed here.

1: INTRODUCTION

There are many instances when electrical probing of the environment is influenced unduly by the presence of man-made conductors such as wires, cables and pipes. These axial conductors channel the current flow and distort the potential field distribution. A quantitative understanding of these effects is largely lacking. It is our purpose

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here to analyze some specific geometries that provide relevant information. Here we will particularly emphasize the electrical potential or so called resistivity methods. In these techniques the current flow is governed primarily by potential theory even though alternating currents are used. There is now a vast literature on the subject (for example, see *Al'pin et al*, 1966; *Grant and West*, 1965; *Keller and Frischknecht*, 1966, *Kunetz*, 1966, *Mooney and Wetzel*, 1956; *Sunde*, 1968).

Our main objective is to provide the theoretical framework for a class of problems that permit an analytical approach to be used. In Part II, we will present numerical examples based on the theory developed here.

2: CHARACTERIZATION OF THE CABLE

We deal first with a description of the axial conductor that for convenience is referred to as the "cable". As indicated in Fig. 1, it is modelled as a concentric cylindrical structure of infinite length. With respect to a cylindrical coordinate system (ρ, ϕ, z) , the solid center conductor is defined by $-\infty < z < \infty$ and $\rho < c_0$. This is surrounded by a coating of conductivity σ_c that is defined by $-\infty < z < \infty$ and $c_0 < \rho < c$. For the moment, we do not need to specify the characteristics of the region $\rho > c$ external to the cable. However, we invoke the assumption that the fields in the vicinity of the cable are locally uniform in the sense that azimuthal symmetry prevails (i.e. $\partial/\partial\phi = 0$). Also, since we are concerned primarily with axial currents on the cable, only the TM or transverse magnetic fields are important. These have non-vanishing electric field components E_ρ and E_z that, in general, are functions

that is also, in general, a function of ρ and z .

Without further restricting the validity of our conclusions, we can assume that the cable is being excited by an external current source such that E_z is an odd function about $z = 0$. This suggests the Fourier sine integral representation

$$E_z(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \hat{E}_z(\lambda) \sin \lambda z \, d\lambda \quad (1)$$

and its inverse

$$\hat{E}_z(\lambda) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} E_z(z) \sin \lambda z \, dz \quad (2)$$

The possible ρ dependence here is understood. Similar representations can be used for $E_\rho(z)$ and $H_\phi(z)$ with corresponding inverse transforms $\hat{E}_\rho(\lambda)$ and $\hat{H}_\phi(\lambda)$.

Now we have stated that our interest is primarily in potential fields but, in any practical measurement scheme, the time-varying nature of the fields is utilized. Thus we assume that the fields vary as $\exp(i\omega t)$ where ω the angular frequency can be allowed to vanish.

Dealing first with the fields in the central conductor (i.e. $\rho < c_0$), we can derive these from an electric vector with only a z component $\Pi_z(z)$. Thus

$$E_z(z) = \left(-\gamma_w^2 + \frac{\partial^2}{\partial z^2}\right) \Pi_z(z) \quad (3)$$

$$E_\rho(z) = \frac{\partial^2 \Pi_z(z)}{\partial \rho \partial z} \quad (4)$$

and

$$H_\phi(z) = -\sigma_w \frac{\partial \Pi_z(z)}{\partial \rho} \quad (5)$$

where $\gamma_w^2 = i\sigma_w \mu_w \omega$ and μ_w is the magnetic permeability. Displacement currents here have been neglected. If they need be considered, we merely replace σ_w by $\sigma_w + i\epsilon_w \omega$ where ϵ_w is the permittivity of the central conductor. Similarly,

$$\Pi_z(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \hat{\Pi}_z(\lambda) \sin \lambda z \, d\lambda \quad (6)$$

$$\text{Now, since } (\nabla^2 - \gamma_w^2) \Pi_z(z) = 0 \quad (7)$$

for the region $\rho < \rho_0$, we can write

$$\hat{\Pi}_z(\lambda) = C(\lambda) I_0(u\rho) \quad (8)$$

where $u = (\lambda^2 + \gamma_w^2)^{\frac{1}{2}}$, $C(\lambda)$ is an arbitrary function and I_0 is the modified Bessel function that remains finite at $\rho = 0$.

Without difficulty, we can now deduce the following expression for the axial impedance $Z_w(\lambda)$ of the center conductor:

$$Z_w(\lambda) = \frac{\hat{E}_z(\lambda)}{2\pi\rho\hat{H}_\phi(\lambda)} \bigg|_{\rho=c_0} = \frac{u I_0(uc_0)}{2\pi\sigma_w c_0 I_1(uc_0)} \quad (9)$$

In cases where $|\gamma_w^2| \gg \lambda^2$, we can write

$$Z_w(\lambda) \approx Z_w = \frac{\gamma_w I_0(\gamma_w c_0)}{2\pi\sigma_w c_0 I_1(\gamma_w c_0)} \quad (10)$$

In the case where $|\gamma_w c_0| \ll 1$ this reduces to

$$Z_w \approx 1/(\pi\sigma_w c_0^2) \quad (11)$$

which is the expected D.C. result.

In dealing with the concentric region $c_0 < \rho < c$, we may proceed

$$\hat{\Pi}_z(\lambda) = A(\lambda)I_0(v\rho) + B(\lambda)K_0(v\rho) \quad (12)$$

where $v = (\lambda^2 + \gamma_c^2)^{1/2}$, $\gamma_c^2 = i\sigma_c \mu_c \omega$ and where A and B are unknown functions.

Here μ_c is the magnetic permeability of the coating. To simplify the discussion here, we utilize the fact that the coating is a relatively poor conductor so that, for all conceivable situations, $|v\rho| \ll 1$.

Then the simpler form is relevant

$$\hat{\Pi}_z(\lambda) \approx A(\lambda)[1 + \chi(\lambda)\ln \rho] \quad (13)$$

where $\chi(\lambda)$ is yet to be determined. Then, for this same region,

$$c_0 < \rho < c,$$

$$(2/\pi)^{1/2} \hat{E}_z(\lambda) \approx -A(\lambda)\lambda^2[1 + \chi(\lambda)\ln \rho] \quad (14)$$

and

$$(2/\pi)^{1/2} \hat{H}_\phi(\lambda) \approx -\sigma_c \chi(\lambda) A(\lambda) / \rho \quad (15)$$

Now requiring that (9) hold for $\rho = c_0$, we easily deduce that

$$\left. \frac{\hat{E}_z(\lambda)}{2\pi\rho\hat{H}_\phi(\phi)} \right|_{\rho=c} = Z_c(\lambda) \approx Z_w(\lambda) + \frac{\lambda^2}{2\pi\sigma_c} \ln \frac{c}{c_0} \quad (16)$$

This quantity $Z_c(\lambda)$ is the effective series impedance (per unit length) of the cable for fields that have a z dependence of the form $\exp(\pm i\lambda z)$ even though we derived it for a z dependence of the form $\sin\lambda z$. As indicated before, σ_c can be replaced by $\sigma_c + i\epsilon_c \omega$ to account for the displacement currents in the coating whose permittivity is ϵ_c . It is also worth pointing out that this same approach [Wait and Hill, 1975 and Hill and Wait, 1977] can be used for characterizing

an outer metallic sheath.

A significant feature of (16) is that it is a function of λ the axial wave number. In this sense, we say that the impedance is spatially dispersive, although, in many cases of practical interest the λ dependence can be ignored. In that case, of course, Z_c is given by $Z_c = E_z/I$, at $\rho = c$, where I is the total current carried by the cable and E_z is the actual tangential electric field in the z direction.

3: RESPONSE OF THE CABLE TO A POINT CURRENT SOURCE IN AN INFINITE MEDIUM

We now consider a basic prototype field problem that involves the response of a cable for a prescribed excitation. The specific situation is shown in Fig. 2 where we have located a current source of I_0 amperes at $(\rho_0, \phi_0, 0)$ in a homogeneous region of infinite extent of conductivity σ . The problem is to deduce an expression for the fields at (ρ, ϕ, z) due to the source in the presence of the cable that is coincident with the z axis. We deal here with potential fields in the region external to the cable since all dimensions such as ρ and ρ_0 are small compared with the skin depth in the external medium [i.e. ρ and $\rho_0 \ll \delta = (2/\sigma\mu_0\omega)^{1/2}$]. Also, in this analysis, we do not consider the inductive coupling of the source current circuit (e.g. leads to the generator) with the cable [Wait and Conda, 1958].

For the conditions stated, the primary potential Ω^P of the source current, in the absence of the cable, is given by

$$\Omega^P = I_0/4\pi\sigma R \quad (17)$$

Using the known integral formula [Wheelon, 1968]

$$\frac{2}{\pi} \int_0^{\infty} K_0(\beta x) \cos \alpha x \, dx = \frac{1}{(\alpha^2 + \beta^2)^{1/2}} \quad (19)$$

for $\beta > 0$, we can write

$$\Omega^P = \frac{I_0}{2\pi^2\sigma} \int_0^{\infty} K_0[\lambda[\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)]^{1/2}] \cos \lambda z \, d\lambda \quad (20)$$

Now the total potential $\Omega = \Omega^P + \Omega^S$, where Ω^S the secondary potential due to the cable current, is postulated to have the form

$$\Omega^S = \frac{I_0}{2\pi^2\sigma} \int_0^{\infty} K_0(\lambda\rho) P(\lambda) \cos \lambda z \, d\lambda \quad (21)$$

Clearly, $\nabla^2 \Omega^S = 0$ for $\rho > c$ and the ϕ independence of Ω^S is appropriate when $\rho_0 \gg c$. Here $P(\lambda)$ is yet to be determined.

The corresponding Hertz potential Π_z^S has the form

$$\Pi_z^S = \int_0^{\infty} F(\lambda) K_0(\lambda\rho) \sin \lambda z \, d\lambda \quad (22)$$

If $\lambda F(\lambda) = -I_0 P(\lambda) / (2\pi^2\sigma)$, this is equivalent to

$$\Pi_z^S = - \frac{I_0}{2\pi^2\sigma} \int_0^{\infty} P(\lambda) K_0(\lambda\rho) \frac{\sin \lambda z}{\lambda} \, d\lambda \quad (23)$$

which can be verified by noting that $\Omega^S = -\partial \Pi_z^S / \partial z$. The magnetic field of the cable currents is now obtained from

$$H_{\phi}^S(z) = -\sigma \frac{\partial \Pi_z^S}{\partial \rho} = - \frac{I_0}{2\pi^2} \int_0^{\infty} P(\lambda) \sin \lambda z K_1(\lambda\rho) \, d\lambda \quad (24)$$

where we have used the identity $K_1'(x) = -K_0(x)$. The actual cable

$$I(z) = 2\pi\rho H_{\phi}^S(z) \Big|_{\rho=c} = - \frac{I_o c}{\pi} \int_0^{\infty} P(\lambda) \sin\lambda z K_1(\lambda c) d\lambda \quad (25)$$

This has the form

$$I(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \hat{I}(\lambda) \sin\lambda z d\lambda \quad (26)$$

where

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{I}(\lambda) = -I_o (c/\pi) P(\lambda) K_1(\lambda c) \quad (27)$$

The total electric field in the axial direction is obtained from

$$E_z(z) = - \frac{\partial}{\partial z} (\Omega^P + \Omega^S) \quad (28)$$

Using (20) and (21), we see this has the representation

$$E_z(z) = \frac{I_o}{2\pi^2\sigma} \int_0^{\infty} \left\{ K_o(\lambda [\rho^2 + \rho_o^2 - 2\rho\rho_o \cos(\phi - \phi_o)]^{\frac{1}{2}}) \right. \\ \left. + P(\lambda) K_o(\lambda\rho) \right\} \lambda \sin\lambda z d\lambda \quad (29)$$

And, in particular,

$$E_z(z) \Big|_{\rho=c} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \hat{E}_z(\lambda) \Big|_{\rho=c} \sin\lambda z d\lambda \quad (30)$$

where

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{E}_z(\lambda) \Big|_{\rho=c} = \frac{I_o \lambda}{2\pi^2\sigma} [K_o(\lambda [c^2 + \rho_o^2 - 2\rho_o c \cos(\phi - \phi_o)]^{\frac{1}{2}}) \\ + P(\lambda) K_o(\lambda c)] \approx \frac{I_o \lambda}{2\pi^2\sigma} [K_o(\lambda\rho_o) + P(\lambda) K_o(\lambda c)] \quad (31)$$

The impedance condition at the cable is

where $Z_c(\lambda)$ is defined by (16). On using (27) and (31), we can solve (32) for the unknown $P(\lambda)$. Thus, without difficulty, we find that

$$P(\lambda) = - \frac{K_o(\lambda \rho_o) \lambda}{[K_o(\lambda c) \lambda + Z_c(\lambda) 2\pi \sigma c K_1(\lambda c)]} \quad (33)$$

Then our desired solution is

$$\Omega = \frac{I_o}{4\pi\sigma} \left[\frac{1}{R} + \frac{2}{\pi} \int_0^{\infty} P(\lambda) K_o(\lambda \rho) \cos \lambda z \, d\lambda \right] \quad (34)$$

Furthermore, for the important part of the integration range in (34), we can say that $\lambda c \ll 1$. In this case

$$P(\lambda) \approx \frac{K_o(\lambda \rho_o) \lambda^2}{[K_o(\lambda c) \lambda^2 + Z_c(\lambda) 2\pi\sigma]} \quad (35)$$

Strictly speaking, the obtained solution for the resultant potential at (x, y, z) is valid only at zero frequency in which case Z_c is the resistance per unit length of the cable. However, for low frequency A.C., the general form given by (34) is still valid but the Z_c should be regarded as the effective series impedance per unit length.

4: THE CABLE LOCATED IN THE HALF-SPACE FOR POINT SOURCE EXCITATION

The next order of complexity is to locate the cable within a homogeneous conducting half space of conductivity σ . The situation is illustrated in Fig. 3 in both perspective and plan view. Here rectangular coordinates are chosen with the injected current I_0 located at the origin and the cable located at $x = -h$ and $y = d$. The cable is parallel to the z axis.

The primary potential Ω^P for the space $x < 0$ is

$$\Omega^P = I_0 / (2\pi\sigma R) \quad (36)$$

where $R = (x^2 + y^2 + z^2)^{1/2}$. Using the integral formula (19) this is written in equivalent form

$$\Omega^P = \frac{I_0}{\pi^2\sigma} \int_0^\infty K_0[\lambda(x^2 + y^2)^{1/2}] \cos\lambda z \, d\lambda \quad (37)$$

Now the secondary potential Ω^S due to the cable at $x = -h$, $y = d$ must include the contribution to its image at $x = +h$, $y = d$. This suggests writing

$$\Omega^S = - \int_0^\infty F(\lambda) \{ K_0[\lambda[(x+h)^2 + (y-d)^2]^{1/2}] + K_0[\lambda[(x-h)^2 + (y-d)^2]^{1/2}] \} \times \lambda \cos\lambda z \, d\lambda \quad (38)$$

which clearly satisfies the condition $\partial\Omega^S/\partial x = 0$ at $x = 0$. The corresponding Hertz potential Π_z^S , due to the cable currents, is then

$$\begin{aligned} \Pi_z^S = \int_0^\infty F(\lambda) \{ K_0[\lambda[(x+h)^2 + (y-d)^2]^{\frac{1}{2}}] \\ + K_0[\lambda[(x-h)^2 + (y-d)^2]^{\frac{1}{2}}] \} \sin \lambda z \, d\lambda \end{aligned} \quad (39)$$

As before $F(\lambda)$ is proportional to the transform $\hat{I}(z)$ of the cable current. This is checked by noting, in terms of local cylindrical coordinates (ρ', ϕ', z) about the cable, that

$$H_{\phi'}^S = -\sigma \partial \Pi_z^S / \partial \rho' \quad (40)$$

where, in fact, $\rho' = [(x+h)^2 + (y-d)^2]^{\frac{1}{2}}$. Then it is not difficult to show that

$$\begin{aligned} I(z) &= -\sigma \int_0^{2\pi} \rho' \frac{\partial \Pi_z^S}{\partial \rho'} d\phi' \bigg|_{\rho'=c} \\ &\approx -2\pi \rho' \sigma \frac{\partial}{\partial \rho'} \int_0^\infty F(\lambda) K_0(\lambda \rho') \sin \lambda z \, d\lambda \bigg|_{\rho'=c} \end{aligned} \quad (41)$$

provided h is somewhat greater than c (i.e. cable not closer than several radii from the interface). Thus we find that

$$(2/\pi)^{\frac{1}{2}} \hat{I}(\lambda) = 2\pi c \sigma F(\lambda) K_1(\lambda c) \lambda \quad (42)$$

The axial field $E_z(z)$ is now obtained from

$$E_z(z) = -\partial / \partial z (\Omega^P + \Omega^S) \quad (43)$$

Then we find that

$$\hat{E}_z(\lambda) = \hat{E}_z^P + \hat{E}_z^S \quad (44)$$

where

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{E}_z^P = \frac{I_0}{\pi^2 \sigma} K_0[\lambda(x^2+y^2)^{\frac{1}{2}}] \lambda \quad (45)$$

and

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{E}_z^S = -F(\lambda) \{K_0[\lambda(x+h)^2 + (y-d)^2]^{\frac{1}{2}} + K_0[\lambda[(x-h)^2 + (y-d)^2]^{\frac{1}{2}}\} \lambda^2 \quad (46)$$

The boundary equation

$$[\hat{E}_z^S + \hat{E}_z^P]_{\rho=c} = z_c(\lambda) \hat{I}(\lambda) \quad (47)$$

in combination with (42), (45), and (46) can now be used to yield

$$F(\lambda) = \frac{I_0}{\pi^2 \sigma} \frac{K_0[\lambda(h^2+d^2)^{\frac{1}{2}}] \lambda}{\{[K_0(\lambda c) + K_0(2\lambda h)] \lambda^2 + 2\pi \lambda c K_1(\lambda c) \sigma z_c(\lambda)\}} \quad (48)$$

Using this value for $F(\lambda)$ in (48), we then have an explicit expression for the secondary potential Ω^S due to the currents in the cable. Adding this to the primary potential Ω^P we then have the desired solution for the total potential Ω :

$$\Omega(x, y, z) = \frac{I_0}{2\pi\sigma} \left[\frac{1}{R} - \frac{2}{\pi} \int_0^\infty \frac{K_0[\lambda(h^2+d^2)^{\frac{1}{2}}] [K_0[\lambda[(x+h)^2 + (y-d)^2]^{\frac{1}{2}}] + K_0[\lambda[(x-h)^2 + (y-d)^2]^{\frac{1}{2}}]}{\{[K_0(\lambda c) + K_0(2\lambda h)] \lambda^2 + 2\pi \lambda c \sigma K_1(\lambda c) z_c(\lambda)\}} \right. \\ \left. \times \lambda^2 \cos \lambda z \, d\lambda \right] \quad (49)$$

Again this expression can be simplified by noting that $\lambda c K_1(\lambda c) \approx 1$ for most cases of interest.

5: THE CABLE LOCATED IN A LAYERED HALF-SPACE FOR POINT SOURCE EXCITATION

We now consider a slightly more complicated situation. As indicated in Fig. 4, we locate the cable in the upper layer of a two layer half-space. We treat this problem in a slightly more systematic fashion since we may later adapt the same solution to a half-space of any number of layers.

The starting point is again based on the integral representation

$$\frac{1}{R} = \frac{2}{\pi} \int_0^{\infty} K_0[\lambda(x^2+y^2)^{\frac{1}{2}}] \cos \lambda z \, d\lambda \quad (50)$$

where $R = (x^2+y^2+z^2)^{\frac{1}{2}}$. Then, using the spectral definition

$$K_0[\lambda(x^2+y^2)^{\frac{1}{2}}] = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{u} e^{-ux} e^{-i\beta y} \, d\beta \quad (51)$$

where $u = (\beta^2 + \lambda^2)^{\frac{1}{2}}$, we see that

$$\frac{1}{R} = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{+\infty} \frac{1}{u} e^{-u|x|} e^{-i\beta y} \cos \lambda z \, d\lambda d\beta \quad (52)$$

The "primary" potential of the current source I_0 in the two-layer half space, ignoring the presence of the cable is now written in the indicated double spectral form. Thus, for the region $0 > x > -s$,

$$\Omega_1^p = \frac{I_0}{2\pi^2 \sigma_1} \int_0^{\infty} \int_{-\infty}^{+\infty} \frac{1}{u} [e^{ux} + f(\lambda, \beta) e^{ux} + g(\lambda, \beta) e^{-ux}] e^{-i\beta y} \cos \lambda z \, d\lambda d\beta \quad (53)$$

while, for the region $x < -s$,

Here the functions f , g , and h are to be determined from the boundary conditions.

Now we know that for a homogeneous half-space (i.e. $s \rightarrow \infty$) $\Omega_1^P \rightarrow I_0/(2\pi\sigma_1 R)$ which is singular as $R \rightarrow 0$. The right hand side of (53) has the same behavior. A condition on the problem is that the non-singular part of the potential satisfies

$$\left\{ \frac{\partial}{\partial x} \left[\Omega_1^P - \frac{I_0}{2\pi\sigma_1 R} \right] \right\}_{x=0} = 0 \quad (55)$$

As a consequence,

$$f(\lambda, \beta) = g(\lambda, \beta). \quad (56)$$

The other two required conditions are that

$$\left[\Omega_1^P - \Omega_2^P \right]_{x=-s} = 0 \quad \text{and} \quad \left[\sigma_1 \frac{\partial \Omega_1^P}{\partial x} - \sigma_2 \frac{\partial \Omega_2^P}{\partial x} \right]_{x=-s} = 0 \quad (57)$$

These lead to

$$e^{-us} + f e^{-us} + g e^{us} = h e^{-us} \quad (58)$$

$$(\sigma_1/\sigma_2)[e^{-us} + f e^{-us} - g e^{us}] = h e^{-us} \quad (59)$$

On eliminating h we find that

$$f = g = R e^{-2us} / [1 - R e^{-2us}] \quad (60)$$

where

$$R = (\sigma_1 - \sigma_2) / (\sigma_1 + \sigma_2) \quad (61)$$

It is now a simple matter to deduce the following expressions for the "primary" potential of the current source at the surface of the two-

$$\Omega_1^p \Big|_{x=0} = \frac{I_o}{2\pi^2\sigma_1} \int_0^\infty \int_{-\infty}^{+\infty} \frac{1}{u} \left[1 + \frac{2Re^{-2us}}{1 - Re^{-2us}} \right] \cos\lambda z e^{-i\beta y} d\lambda d\beta \quad (62)$$

$$= \frac{I_o}{2\pi^2\sigma_1} \int_0^\infty \int_{-\infty}^{+\infty} \frac{1}{u} \left[1 + 2 \sum_{n=1}^\infty R^n e^{-2nus} \right] \cos\lambda z e^{-i\beta y} d\lambda d\beta \quad (63)$$

$$= \frac{I_o}{\pi^2\sigma_1} \int_0^\infty \left[K_o(\lambda y) + 2 \sum_{n=1}^\infty R^n K_o[\lambda[y^2 + (2ns)^2]^{\frac{1}{2}}] \right] \cos\lambda z d\lambda \quad (64)$$

$$= \frac{I_o}{2\pi\sigma_1} \left[\frac{1}{(y^2+z^2)^{\frac{1}{2}}} + 2 \sum_{n=1}^\infty R^n \frac{1}{[y^2 + z^2 + (2ns)^2]^{\frac{1}{2}}} \right] \quad (65)$$

The latter version has the well known form and it is the basis for apparent resistivity calculations for a uniform two layer conducting earth [e.g. *Keller*, 1968; *Wait*, 1970]. This serves as an interim check on our analysis.

The general form of the primary potential for the region $0 > x > -s$ that is needed for our purpose is

$$\Omega_1^p = \frac{I_o}{\pi^2\sigma_1} \int_0^\infty \left\{ K_o[\lambda(x^2+y^2)^{\frac{1}{2}}] + \sum_{n=1}^\infty R^n [K_o[\lambda[(2ns-x)^2 + y^2]^{\frac{1}{2}}] + K_o[\lambda[(2ns+x)^2 + y^2]^{\frac{1}{2}}]] \right\} \cos\lambda z d\lambda \quad (66)$$

Using a very similar argument to the above, we may write down general expressions for the secondary fields due to the current in the cable. Thus, for $0 > x > -s$ we have

$$\Omega_1^s = -\frac{1}{2} \int_0^\infty \int_{-\infty}^{+\infty} F(\lambda) \frac{1}{u} [e^{-u|x+h|} + f_c(\lambda, \beta) e^{ux} + g_c(\lambda, \beta) e^{-ux}]$$

while for $x < -s$,

$$\Omega_2^s = -\frac{1}{2} \int_0^{\infty} \int_{-\infty}^{+\infty} F(\lambda) \frac{1}{u} h_c(\lambda, \beta) e^{ux} e^{-i\beta(y-d)} \lambda \cos \lambda z \, d\lambda d\beta \quad (68)$$

where $u = (\beta^2 + \lambda^2)^{\frac{1}{2}}$ and where f_c , g_c and h_c are to be determined by the boundary conditions. In writing (67), we have used the integral representation (52). Also, $F(\lambda)$ has the same meaning as before and is connected to the Fourier sine transform $\hat{I}(\lambda)$ of the cable current by

$$(2/\pi)^{\frac{1}{2}} \hat{I}(\lambda) = 2\pi c \lambda K_1(\lambda c) \sigma_1 F(\lambda) \quad (69)$$

In a straight forward fashion we can now apply the following boundary conditions

$$\left. \frac{\partial \Omega_1^s}{\partial x} \right|_{x=0} = 0 \quad (70)$$

$$\left[\Omega_1^s - \Omega_2^s \right]_{x=-s} = 0 \quad (71)$$

and

$$\left[\sigma_1 \frac{\partial \Omega_1^s}{\partial x} - \sigma_2 \frac{\partial \Omega_2^s}{\partial x} \right]_{x=-s} = 0 \quad (72)$$

to yield

$$f_c = \frac{e^{-uh} + R e^{uh} e^{-2us}}{1 - R e^{-2us}} \quad (73)$$

and

$$g_c = \frac{R[e^{-u(2s-h)} + e^{-u(2s+h)}]}{1 - R e^{-2us}} \quad (74)$$

Using the obtained expressions for the coefficients we can deduce the following expressions for the secondary potential due to the cable

$$\begin{aligned}
\Omega_1^S = & - \int_0^\infty F(\lambda) K_0[\lambda[(x+h)^2 + (y-d)^2]^{\frac{1}{2}}] \lambda \cos \lambda z \, d\lambda \\
& - \frac{1}{2} \int_0^\infty F(\lambda) \int_{-\infty}^{+\infty} \frac{1}{u} \left\{ \frac{[e^{-uh} + R e^{u(h-2s)}] e^{ux} + R[e^{-u(2s-h)} + e^{-u(2s+h)}] e^{-ux}}{1 - R e^{-2us}} \right\} \\
& \times e^{-i\beta(y-d)} \, d\beta \lambda \cos \lambda z \, d\lambda
\end{aligned} \tag{75}$$

$$\begin{aligned}
& = - \int_0^\infty F(\lambda) K_0(\lambda \rho') \lambda \cos \lambda z \, d\lambda \\
& - \frac{1}{2} \int_0^\infty F(\lambda) \int_{-\infty}^{+\infty} \frac{1}{u} \sum_{n=0}^\infty R^n e^{-2uns} [e^{-u(h-x)} + R e^{-u(2s-h-x)} \\
& \quad + R e^{-u(2s-h+x)} + R e^{-u(2s+h+x)}] \\
& \times e^{-i\beta(y-d)} \, d\beta \lambda \cos \lambda z \, d\lambda
\end{aligned} \tag{76}$$

$$\begin{aligned}
& = - \int_0^\infty F(\lambda) \left\{ K_0(\lambda \rho') + \sum_{n=0}^\infty R^n [K_0[\lambda[(2ns+h-x)^2 + (y-d)^2]^{\frac{1}{2}}] \right. \\
& \quad + R K_0[\lambda[(2(n+1)s-h-x)^2 + (y-d)^2]^{\frac{1}{2}}] + R K_0[\lambda[(2(n+1)s-h+x)^2 + (y-d)^2]^{\frac{1}{2}}] \\
& \quad \left. + R K_0[\lambda[(2(n+1)s+h+x)^2 + (y-d)^2]^{\frac{1}{2}}] \right\} \lambda \cos \lambda z \, d\lambda
\end{aligned} \tag{77}$$

where

$$\rho' = [(x+h)^2 + (y-d)^2]^{\frac{1}{2}}.$$

The boundary condition at the cable again has the form

$$[\hat{E}_{1z}^S(\lambda) + \hat{E}_{1z}^P(\lambda)] = Z_-(\lambda) \hat{I}(\lambda). \tag{78}$$

Since

$$E_{1z}^P = - \frac{\partial \Omega_1^P}{\partial z} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \hat{E}_{1z}^P \sin \lambda z \, d\lambda \quad (79)$$

and

$$E_{1z}^S = - \frac{\partial \Omega_1^S}{\partial z} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} \hat{E}_{1z}^S \sin \lambda z \, d\lambda \quad (80)$$

it follows from (66) and (77) that

$$\begin{aligned} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{E}_{1z}^P(\lambda) \Big|_{\rho'=c} &\cong \frac{I_0 \lambda}{\pi^2 \sigma_1} \{K_0[\lambda(x^2+y^2)^{\frac{1}{2}}] \\ &+ \sum_{n=1}^{\infty} R^n [K_0[\lambda[(2ns-x)^2 + y^2]^{\frac{1}{2}}] \\ &+ K_0[\lambda[(2ns+x)^2 + y^2]^{\frac{1}{2}}]\} \\ &\quad \begin{matrix} x=-h \\ y=d \end{matrix} \end{aligned}$$

and

$$\begin{aligned} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \hat{E}_{1z}^S(\lambda) &\cong -F(\lambda) \lambda^2 \left\{ K_0(\lambda c) + \sum_{n=0}^{\infty} R^n \{K_0[\lambda(2ns+2h)] \right. \\ &\quad \left. + 2 R K_0[2\lambda(n+1)s] + R K_0[2\lambda((n+1)s-h)] \} \right\} \end{aligned} \quad (82)$$

Inserting (81) and (82) into (78) we may solve for $F(\lambda)$ to yield

$$\begin{aligned} F(\lambda) &= \frac{I_0 \lambda}{\pi^2 \sigma_1} \{K_0[\lambda(h^2+d^2)^{\frac{1}{2}}] + \sum_{n=1}^{\infty} R^n [K_0[\lambda[(2ns+h)^2 + d^2]^{\frac{1}{2}}] \\ &+ K_0[\lambda[(2ns-h)^2 + d^2]^{\frac{1}{2}}]\} \left\{ \lambda^2 [K_0(\lambda c) + \sum_{n=0}^{\infty} R^n [K_0[2\lambda(ns+h)] \right. \\ &\quad \left. + R K_0[2\lambda((n+1)s-h)] + 2 R K_0[2\lambda(n+1)s]] + 2\pi \sigma_1 \lambda c K_1(\lambda c) Z_c(\lambda) \right\}^{-1} \end{aligned} \quad (83)$$

The resultant surface potential is thus given by

$$\begin{aligned}
 \Omega_1 \Big|_{x=0} = & \frac{I_o}{2\pi\sigma_1} \left[\frac{1}{(y^2+z^2)^{\frac{1}{2}}} + 2 \sum_{n=1}^{\infty} R^n \frac{1}{[y^2 + z^2 + (2ns)^2]^{\frac{1}{2}}} \right] \\
 & - \int_0^{\infty} F(\lambda) \left\{ K_0[\lambda[h^2 + (y-d)^2]^{\frac{1}{2}}] + \sum_{n=0}^{\infty} R^n [K_0[\lambda[(2ns+h)^2 + (y-d)^2]^{\frac{1}{2}}] \right. \\
 & \left. + 2 R K_0[\lambda[(2(n+1)s-h)^2 + (y-d)^2]^{\frac{1}{2}}] + R K_0[\lambda[(2(n+1)s+h)^2 + (y-d)^2]^{\frac{1}{2}}] \right\} \\
 & \times \lambda \cos \lambda z \, d\lambda
 \end{aligned} \tag{84}$$

where $F(\lambda)$ is given explicitly by (83). As it should this rather complicated expression for the resultant potential over the two layer earth reduces to the corresponding half-space problem when the upper layer thickness $s \rightarrow \infty$ and/or when the conductivity ratio $\sigma_2/\sigma_1 \rightarrow 1$.

Actually, the expressions derived above for the fields are also valid if the two-layer half-space is replaced by an N-layered half-space as indicated in Fig. 5. Here the n'th layer is bounded by $x = -s_{n-1}$ and $x = -s_n$ has a thickness d_n with a conductivity σ_n where $n = 1, 2, \dots, N$. The potential Ω_{n-1} within the n'th layer as the form

$$\Omega_{n-1} = f_{n-1} e^{ux} + g_{n-1} e^{-ux} \tag{85}$$

where, on the right hand side, we have omitted the operator

$$\frac{I_o}{2\pi^2\sigma_1} \int_0^{\infty} \int_{-\infty}^{+\infty} \frac{1}{u} [\dots] \cos \lambda z \, d\lambda$$

The λ dependence of f_{n-1} and g_{n-1} are also understood.

Now there is no reason why we cannot write (85) in the form

$$\Omega_{n-1} = f_{n-1} [e^{ux} + R_{n-1} e^{-2us_{n-1}} e^{-ux}] \quad (86)$$

where $R_{n-1} = (g_{n-1}/f_{n-1}) e^{2us_{n-1}}$ is yet to be determined. But in the N 'th region, the potential must vanish as $x \rightarrow -\infty$ thus $R_N = 0$ and thus

$$\Omega_N = f_N e^{ux} \quad (87)$$

Then, in the $(N-1)$ 'th layer,

$$\Omega_{N-1} = f_{N-1} [e^{ux} + R_{N-1} e^{-2us_{N-1}} e^{-ux}] \quad (88)$$

The boundary conditions

$$\left. \begin{aligned} \Omega_N &= \Omega_{N-1} \\ \sigma_N \frac{\partial \Omega_N}{\partial x} &= \sigma_{N-1} \frac{\partial \Omega_{N-1}}{\partial x} \end{aligned} \right\} \text{ at } x = -s_{N-1} \quad (89)$$

may now be applied to (87) and (88) to yield

$$R_{N-1} = \frac{\sigma_{N-1} - \sigma_N}{\sigma_{N-1} + \sigma_N} \quad (90)$$

The next step is note that

$$\Omega_{N-2} = f_{N-2} [e^{ux} + R_{N-2} e^{-2us_{N-2}} e^{-ux}] \quad (91)$$

The boundary conditions

$$\left. \begin{aligned} \Omega_{N-1} &= \Omega_{N-2} \\ \sigma_{N-1} \frac{\partial \Omega_{N-1}}{\partial x} &= \sigma_{N-2} \frac{\partial \Omega_{N-2}}{\partial x} \end{aligned} \right\} \text{ at } x = -s_{N-2} \quad (92)$$

may now be applied to (88) and (91) to yield

$$R_{N-2} = \frac{\sigma_{N-2} - \sigma_{N-1}(1 - R_{N-1}e^{-2ud_{N-1}})(1 + R_{N-1}e^{-2ud_{N-1}})^{-1}}{\sigma_{N-2} + \sigma_{N-1}(1 - R_{N-1}e^{-2ud_{N-1}})(1 + R_{N-1}e^{-2ud_{N-1}})^{-1}} \quad (93)$$

where $d_{N-1} = s_{N-1} - s_{N-2}$ is the thickness of the (N-1)'th layer. In fact, by induction, we find that

$$R_{n-1} = \frac{\sigma_{n-1} - \sigma_n(1 - R_n e^{-2ud_n})(1 + R_n e^{-2ud_n})^{-1}}{\sigma_{n-1} + \sigma_n(1 - R_n e^{-2ud_n})(1 + R_n e^{-2ud_n})^{-1}} \quad (94)$$

In particular,

$$R_1 = \frac{\sigma_1 - \sigma_2(1 - R_2 e^{-2ud_2})(1 + R_2 e^{-2ud_2})^{-1}}{\sigma_1 + \sigma_2(1 - R_2 e^{-2ud_2})(1 + R_2 e^{-2ud_2})^{-1}} \quad (95)$$

Thus, we see that (84), the expression for the surface potential, still holds for an N-layered half-space (with the cable within the upper layer) if R is replaced by R_1 as defined by (95).

6: EXTENSION OF THE THEORY TO MULTIPLE CABLES

In the above analyses, we have treated only the case of a single buried cable with a specified series impedance per unit length. The formal extension to more than one cable is considered here; specifically we consider two parallel cables located at $x = -h_1$, $y = d_1$ and at $x = -h_2$, $y = d_2$ in a homogeneous half-space of conductivity σ . The situation is indicated in Fig. 6.

The derivation is outlined only briefly since it is a straight-forward generalization of the case for a single cable located in the homogeneous half-space. Following the earlier prescription we write down expressions in

suitable form, for the primary potential Ω^P and the respective secondary potentials $\Omega^{S,1}$ and $\Omega^{S,2}$ due to the cable currents $I_1(z)$ and $I_2(z)$ respectively:

$$\Omega^P = \frac{I_0}{\pi^2 \sigma} \int_0^\infty K_0[\lambda(x^2+y^2)^{\frac{1}{2}}] \cos \lambda z \, d\lambda \quad (96)$$

$$\Omega^{S,1} = - \int_0^\infty F_1(\lambda) \{ K_0[\lambda[(x+h_1)^2 + (y-d_1)^2]^{\frac{1}{2}}] + K_0[\lambda[(x-h_1)^2 + (y-d_1)^2]^{\frac{1}{2}}] \cdot \lambda \cos \lambda z \, d\lambda \quad (97)$$

and

$$\Omega^{S,2} = - \int_0^\infty F_2(\lambda) \{ K_0[\lambda[(x+h_2)^2 + (y-d_2)^2]^{\frac{1}{2}}] + K_0[\lambda[(x-h_2)^2 + (y-d_2)^2]^{\frac{1}{2}}] \cdot \lambda \cos \lambda z \, d\lambda \quad (98)$$

As before, the functions $F_1(\lambda)$ and $F_2(\lambda)$ are related to the sine transforms $\hat{I}_1(\lambda)$ and $\hat{I}_2(\lambda)$ of the currents by

$$(2/\pi)^{\frac{1}{2}} \hat{I}_1(\lambda) = 2\pi\sigma \lambda c_1 K_1(\lambda c_1) F_1(\lambda) \quad (99)$$

and

$$(2/\pi)^{\frac{1}{2}} \hat{I}_2(\lambda) = 2\pi\sigma \lambda c_2 K_1(\lambda c_2) F_2(\lambda) \quad (100)$$

where c_1 and c_2 are the respective radii of the two cables.

The impedance boundary conditions are that

$$(\hat{E}_z^P + \hat{E}_z^{S,1} + \hat{E}_z^{S,2}) \Big|_{\rho'_1=c_1} = Z_{c,1}(\lambda) \hat{I}_1(\lambda) \quad (101)$$

and

$$(\hat{E}_z^P + \hat{E}_z^{S,1} + \hat{E}_z^{S,2}) \Big|_{\rho'_2=c_2} = Z_{c,2}(\lambda) \hat{I}_2(\lambda) \quad (102)$$

where $Z_{c,1}$ and $Z_{c,2}$ are the respective series impedance of the cables. Here $\rho'_1 = [(x+h_1)^2 + (y-d_1)^2]^{\frac{1}{2}}$ and $\rho'_2 = [(x+h_2)^2 + (y-d_2)^2]^{\frac{1}{2}}$

c_1 and c_2 are small compared with other transverse dimensions. Thus, for example, at $\hat{\rho}_1' = c_1$, we can replace $[(x-h_1)^2 + (y-d_1)^2]$ by $(2h_1)^2$ and $[(x\pm h_2)^2 + (y-d_2)^2]$ by $[(h_2\mp h_1)^2 + (d_2-d_1)^2]$. Then (101) and (102) yield the two algebraic equations

$$F_1(\lambda) \{ [K_0(\lambda c_1) + K_0(2\lambda h_1)]\lambda^2 + 2\pi\sigma\lambda c_1 K_1(\lambda c_1) \} \quad (103)$$

$$+ F_2(\lambda) [K_0(\lambda r) + K_0(\lambda r')] \lambda^2 = [I_0/(\pi^2\sigma)] K_0[\lambda(h_1^2 + d_1^2)^{1/2}] \lambda$$

and

$$F_2(\lambda) \{ [K_0(\lambda c_2) + K_0(2\lambda h_2)]\lambda^2 + 2\pi\sigma\lambda c_2 K_1(\lambda c_2) \} \quad (104)$$

$$+ F_1(\lambda) [K_0(\lambda r) + K_0(\lambda r')] \lambda^2 = [I_0/(\pi^2\sigma)] K_0[\lambda(h_2^2 + d_2^2)^{1/2}] \lambda$$

where $r = [(h_2-h_1)^2 + (d_2-d_1)^2]^{1/2}$ and $r' = [(h_2+h_1)^2 + (d_2-d_1)^2]^{1/2}$.

These immediately tell us that

$$\frac{F_2(\lambda)}{F_1(\lambda)} = \frac{K_0(\lambda c_1) + K_0(2\lambda h_1) - K_0(\lambda r) - K_0(\lambda r')}{K_0(\lambda c_2) + K_0(2\lambda h_2) - K_0(\lambda r) - K_0(\lambda r')} = \chi(\lambda) \quad (105)$$

Then, either (103) or (104) may be used to determine $F_1(\lambda)$ and $F_2(\lambda)$.

In the case where λc_1 and $\lambda c_2 \ll 1$, $\lambda c_1 K_1(\lambda c_1) \approx \lambda c_2 K_1(\lambda c_2) \approx 1$

and

$$\chi(\lambda) = \hat{I}_2(\lambda)/\hat{I}_1(\lambda) \quad (106)$$

being the ratio of the sine transforms of the respective cable currents.

The manner in which one considers any finite number M of parallel cables buried in the half-space should be clear. We would then end up with M linear simultaneous equations for the functions $F_m(\lambda)$ for $m = 1, 2, \dots, M$. These could be solved in a straight-forward fashion to yield explicit

The further extension to the case of M parallel cables located in the upper layer of the N layered half-space is also straight-forward but the algebra becomes rather tedious. This exercise is left to the diligent reader and/or those who have a specific need for such results.

7: A RELATED TUNNEL CONFIGURATION INCLUDING THE EFFECT OF A BURIED CABLE

A rather interesting problem arises when one wishes to probe the wall of an air-filled tunnel cut through a conducting medium. Not only will the curvature of the tunnel wall be a factor, but there may be rails, pipes and other axial conductors present that will distort the current flow. It is our purpose here to consider a relevant problem of this type since it is quite analogous to the case of a cable located in a conducting half-space. In fact, the half-space solution obtained earlier is really a limiting case of the solution for the tunnel model when the wall radius approaches infinity and other dimensions remain finite.

The situation we consider is shown in Fig. 7. A cylindrical coordinate system (ρ, ϕ, z) is chosen such that the tunnel wall is defined by $\rho = a$ and the current source I_0 is at $(\rho_0, 0, 0)$ where $\rho_0 \geq a$. The "cable" parallel to the z axis is located at $\rho = \rho_c$ and $\phi = \phi_c$. The radius c of the cable is assumed to be small compared with other transverse dimensions of the problem. This again permits us to deal only with the filamental current $I(z)$ on the cable and argue that azimuthal variations of the current density within the cable can be ignored.

The primary potential Ω^P of the current source I_0 is again defined to be the solution, in the region $\rho > a$, that satisfies the boundary condition at the tunnel wall but ignores the presence of the cable. Actually, Ω^P can be written as the sum of two parts

$$\Omega^P = \Omega^{P'} + \Omega^{P''}$$

$$\Omega^{P'} = I_0 / (4\pi\sigma R) \quad (108)$$

where

$$R = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\phi + z^2]^{\frac{1}{2}} \quad (109)$$

is the potential for an infinite medium of conductivity σ . Then $\Omega^{P''}$ is the effect of the tunnel.

We now write

$$\Omega^{P'} = \frac{I_0}{2\pi^2\sigma} \int_0^\infty K_0[\lambda(\rho' + \rho_0' - 2\rho\rho_0 \cos\phi)^{\frac{1}{2}}] \cos\lambda z \, d\lambda \quad (\text{for } \rho > a) \quad (110)$$

$$= \frac{I_0}{2\pi^2\sigma} \int_0^\infty \sum_{m=0}^\infty \epsilon_m K_m(\lambda\rho_0) I_m(\lambda\rho) \cos m\phi \cos\lambda z \, d\lambda \quad (\text{for } \rho_0 > \rho > a) \quad (111)$$

where we have made use of the addition theorem for the modified Bessel function K_0 . The summation extends over all integers including zero.

Also, note that $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m = 1, 2, 3, \dots$. Now, since $\nabla^2 \Omega^{P''} = 0$ in the region $\rho > a$ and $\Omega^{P''}$ must vanish as $\rho \rightarrow \infty$,

solutions should be superpositions of $K_m(\lambda\rho) \cos m\phi \cos\lambda z$ that are also even about $\phi = 0$ and $z = 0$. Then we also invoke the condition of normal current flow at the tunnel wall or what is equivalent

$$\frac{\partial}{\partial \rho} [\Omega^{P'} + \Omega^{P''}] \Big|_{\rho=a} = 0 \quad (112)$$

This all leads to the desired result for the primary potential

$$\Omega^P = \Omega^{P'} + \Omega^{P''} = \frac{I_0}{2\pi^2\sigma} \int_0^\infty \sum_{m=0}^\infty \epsilon_m K_m(\lambda\rho_0) \left[I_m(\lambda\rho) - \frac{I'_m(\lambda a)}{K'_m(\lambda a)} K_m(\lambda\rho) \right] \cos m\phi \cos\lambda z \, d\lambda \quad (\text{for } \rho > \rho > a) \quad (113)$$

$$= \frac{I_o}{4\pi\sigma} \left[\frac{1}{R} - \frac{2}{\pi} \int_0^\infty \sum_{m=0}^\infty \epsilon_m K_m(\lambda \rho_o) \frac{I'_m(\lambda a)}{K'_m(\lambda a)} K_m(\lambda \rho) \cos m\phi \cos \lambda z \, d\lambda \right]$$

(for $\rho > a$) (114)

Next we deal with the secondary potential Ω^S that is the contribution due to the cable current. It has the form

$$\Omega^S = - \int_0^\infty F(\lambda) \left\{ K_o[\lambda(\rho^2 + \rho_c^2 - 2\rho\rho_c \cos(\phi - \phi_c))]^{\frac{1}{2}} \right. \\ \left. - \sum_{m=0}^\infty \epsilon_m K_m(\lambda \rho_c) \frac{I'_m(\lambda a)}{K'_m(\lambda a)} K_m(\lambda \rho) \cos m(\phi - \phi_c) \right\} \lambda \cos \lambda z \, d\lambda$$

(115)

which is valid for all $\rho > a$. Here $F(\lambda)$ is related to the Fourier sine transform $\hat{I}(\lambda)$ of the cable current by

$$(2/\pi)^{\frac{1}{2}} \hat{I}(\lambda) = 2\pi\sigma\lambda c K_1(\lambda c) F(\lambda) \quad (116)$$

To obtain $F(\lambda)$ we use the well worn impedance boundary condition

$$\hat{E}_z^S + \hat{E}_z^P = \hat{I}(z) Z_c(\lambda) \quad (117)$$

that may be applied at $\phi = \phi_c$ and $\rho = \rho_c + c$. Here we readily arrive at the explicit solution

$$F(\lambda) = \frac{I_o \lambda}{2\pi^2 \sigma} \left[K_o[\lambda(\rho_o^2 + \rho_c^2 - 2\rho_o \rho_c \cos \phi_c)]^{\frac{1}{2}} - \sum_{m=0}^\infty \epsilon_m K_m(\lambda \rho_o) \frac{I'_m(\lambda a)}{K'_m(\lambda a)} K_m(\lambda \rho_c) \cos m\phi_c \right] \\ \times \left\{ \left[K_o(\lambda c) - \sum_{m=0}^\infty \epsilon_m [K_m(\lambda \rho_c)]^2 \frac{I'_m(\lambda a)}{K'_m(\lambda a)} \right] \lambda^2 + 2\pi\sigma Z_c(\lambda) \lambda c K_1(\lambda c) \right\}^{-1}$$

(118)

This is the final result. The form for $\rho_o \rightarrow a$ is analagous to (48)

in the half-space solution discussed earlier.

8: CONCLUDING REMARKS

It has been our purpose here to derive sequentially a number of idealized current flow problems which are relevant to resistivity of homogeneous and layered media that contain an axial conductor. The resulting expressions for the potential of the specified current source involve an axial integral that has a highly convergent form in most cases. In Part II (to be coauthored with K. Umashankar), we will present numerical results for the various situations analyzed here.

APPENDIX A

EXCITATION OF THE CABLE BY AN A.C. SOURCE

In the present analysis, we have used potential theory which is strictly valid only for zero frequency or D.C. (Direct Current) excitation. In the context of the present work, it is useful to formulate the basic problem of the single cable excited by a time varying source in order to illustrate the nature of the D.C. limit.

As indicated in Fig. 8, we consider first an electric dipole located at $(\rho_0, \phi_0, 0)$ with respect to the cable located at $\rho = 0$ of the cylindrical coordinate system (ρ, ϕ, z) . In terms of the coaxial cartesian system (x, y, z) , the dipole is located at $(x_0, y_0, 0)$. We locate the dipole in the transverse plane $z=0$ and, without further loss of generality, we orient it in the x direction. The current moment of the elemental dipole being considered is $I ds$. The surrounding medium is homogeneous and of infinite extent with conductivity σ , permittivity ϵ , and permeability μ .

In view of the symmetry of the problem, it is evident that the excited cable current $I(z)$ is an odd function of z . This is compatible with choosing the following spectral representation for the corresponding Hertz vector that has only a z component $\Pi_z^S(\rho, z)$:

$$\Pi_z^S(\rho, z) = \int_0^{\infty} F(\lambda) K_0(u\rho) \sin \lambda z \, d\lambda \quad (A-1)$$

where $u = (\lambda^2 + \gamma^2)^{1/2}$, $\gamma^2 = i\mu\omega(\sigma + i\epsilon\omega)$ and $F(\lambda)$ is identified below.

Proceeding as before,

$$I(z) = 2\pi c H_{\phi}^S \Big|_{\rho=0} = -2\pi c \sigma \frac{\partial \Pi_z^S}{\partial z} \Big|_{\rho=0} = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{\infty} \hat{I}(\lambda) \sin \lambda z \, d\lambda \quad (A-2)$$

where

$$(2/\pi)^{1/2} \hat{I}(\lambda) = 2\pi\sigma uc K_1(uc) F(\lambda) \quad (A-3)$$

Also,

$$E_z^S = \left(-\gamma^2 + \frac{\partial^2}{\partial z^2} \right) \Pi_z^S = \left(\frac{2}{\pi} \right)^{1/2} \int_0^\infty \hat{E}_z^S(\lambda) \sin \lambda z \, d\lambda \quad (A-4)$$

where

$$(2/\pi)^{1/2} \hat{E}_z^S(\lambda) = -u^2 F(\lambda) K_0(u\rho) \quad (A-5)$$

We now consider the primary excitation of the dipole. The corresponding Hertz vector has only an x component Π_x^P given by

$$\Pi_x^P = \left\{ I ds / [4\pi(\sigma + i\epsilon\omega)] \right\} \exp(-\gamma R) R^{-1} \quad (A-6)$$

where $R = [\rho^2 + \rho_o^2 - 2\rho\rho_o \cos(\phi - \phi_o) + z^2]^{1/2}$. Now we make use of the Sommerfeld integral representation

$$\frac{e^{-\gamma R}}{R} = \frac{2}{\pi} \int_0^\infty K_0[u[\rho^2 + \rho_o^2 - 2\rho\rho_o \cos(\phi - \phi_o)]^{1/2}] \cos \lambda z \, d\lambda \quad (A-7)$$

which clearly reduces to (20) when $\gamma \rightarrow 0$.

Using (A-6) and (A-7), we determine that

$$E_z^P = \frac{\partial^2 \Pi_x^P}{\partial x \partial z} = \left(\frac{2}{\pi} \right)^{1/2} \int_0^\infty \hat{E}_z^P(\lambda) \sin \lambda z \, d\lambda \quad (A-8)$$

where

$$\left(\frac{2}{\pi} \right)^{1/2} \hat{E}_z^P(\lambda) = \frac{I ds (x - x_o)}{2\pi^2 (\sigma + i\epsilon\omega) r} \lambda u K_1(ur) \quad (A-9)$$

and where $r = [\rho^2 + \rho_o^2 - 2\rho\rho_o \cos(\phi - \phi_o)]^{1/2} = [(x - x_o)^2 + (y - y_o)^2]^{1/2}$.

The cable impedance boundary condition

$$\left(\frac{2}{\pi} \right)^{1/2} \left[\hat{E}_z^P(\lambda) + \hat{E}_z^S(\lambda) \right]_{\rho=c} = Z_c(\lambda) \hat{I}(\lambda) \left(\frac{2}{\pi} \right)^{1/2} \quad (A-10)$$

is now applied to field

$$F(\lambda) = - \frac{Ids}{2\pi^2\sigma} \left(\frac{x_o}{\rho_o} \right) \frac{\lambda u K_1(u\rho_o)}{[u^2 K_o(uc) + 2\pi(\sigma+i\epsilon\omega)uc K_1(uc) Z_c(\lambda)]} \quad (A-11)$$

where $\rho_o = (x_o^2 + y_o^2)^{1/2}$. Thus the Hertz potential due to the cable current, induced by the external dipole, is given by

$$\Pi_z^s = - \frac{Ids \cos\phi_o}{2\pi^2(\sigma+i\epsilon\omega)} \int_0^\infty \frac{\lambda u K_1(u\rho_o) K_o(u\rho) \sin\lambda z d\lambda}{u^2 K_o(uc) + 2\pi(\sigma+i\epsilon\omega)uc K_1(uc) Z_c(\lambda)} \quad (A-12)$$

where $\cos\phi_o = x_o/\rho_o$. This, of course, can also be written

$$\Pi_z^s = \frac{Ids}{2\pi^2\sigma} \frac{\partial}{\partial x_o} \int_0^\infty \frac{K_o(u\rho_o) K_o(u\rho) \lambda \sin\lambda z d\lambda}{u^2 K_o(uc) + 2\pi(\sigma+i\epsilon\omega)uc K_1(uc) Z_c(\lambda)} \quad (A-13)$$

To achieve our purpose, we now consider a finite linear current element extending from (x_1, y_o) to (x_2, y_o) in the plane $z = 0$ as indicated in Fig. 9. The current distribution at the variable point x_o is $I(x_o)$ so that the elemental dipole has a current moment $I(x_o)dx_o$ at this point.

It is now a simple matter to show that the current $I_f(z)$ induced on the cable by this transverse current element is

$$I_f(z) = \left(\frac{2}{\pi} \right)^{1/2} \int_0^\infty \hat{I}_f(\lambda) \sin\lambda z d\lambda \quad (A-14)$$

where

$$\left(\frac{2}{\pi} \right)^{1/2} \hat{I}_f(\lambda) = 2\pi(\sigma+i\epsilon\omega)uc K_1(uc) F_f(\lambda) \quad (A-15)$$

where

$$F_f(\lambda) = \frac{1}{2\pi^2(\sigma+i\epsilon\omega)} \int_{x_1}^{x_2} I(x_o) dx_o \frac{\partial}{\partial x_o} \left[\frac{K_o(u\rho_o)\lambda}{[u^2 K_o(uc) + 2\pi(\sigma+i\epsilon\omega)uc K_1(uc) Z_c(\lambda)]} \right] \quad (A-16)$$

$$F_f(\lambda) = \frac{I(x_o)}{2\pi^2(\sigma+i\epsilon\omega)} \frac{K_o(u\rho_o)\lambda}{[u^2K_o(uc) + 2\pi(\sigma+i\epsilon\omega)ucK_1(uc)Z_c(\lambda)]} \Big|_{x_1}^{x_2} - \frac{1}{2\pi^2(\sigma+i\epsilon\omega)} \int_{x_1}^{x_2} \frac{\partial I(x_o)}{\partial x_o} \frac{K_o(u\rho_o)\lambda dx_o}{[u^2K_o(uc) + 2\pi(\sigma+i\epsilon\omega)ucK_1(uc)Z_c(\lambda)]} \quad (A-17)$$

If we considered $I(x_o)$ to be a constant I , then, of course, the integral term vanishes. Also, if ω was sufficiently small $\sigma + i\epsilon\omega \rightarrow \sigma$, $u \rightarrow \lambda$, we would have

$$F_f(\lambda) = F_{f,2}(\lambda) - F_{f,1}(\lambda) \quad (A-18)$$

where

$$F_{f,j} = \frac{I}{2\pi^2\sigma} \frac{K_o(\lambda\rho_j)\lambda}{[\lambda^2K_o(\lambda c) + 2\pi\sigma\lambda cK_1(\lambda c)Z_c(\lambda)]} \quad (A-19)$$

and

$$\begin{aligned} \rho_j &= \rho_1 = (x_1^2 + y_o^2)^{\frac{1}{2}} \quad \text{for } j = 1 \\ &= \rho_2 = (x_2^2 + y_o^2)^{\frac{1}{2}} \quad \text{for } j = 2 \end{aligned}$$

Clearly the result is consistent with the D.C. excitation of the cable by a current source I_o at (x_2, y_o) and a sink $-I_o$ at (x_1, y_o) .

APPENDIX B

RESISTIVITY ANALYSIS FOR TUNNEL WITH LAYERED WALLS

An auxiliary problem that provides some further insight is the current flow from a point electrode in the conducting medium adjacent to an air filled tunnel. Here the tunnel has a circular cross section of radius a_1 and the immediate homogeneous region of conductivity σ_1 extends out to a radius a_2 beyond which the conductivity is σ_2 . The situation is illustrated in Fig. 10 where, with respect to a cylindrical coordinate system (ρ, ϕ, z) , the air-wall boundary of the tunnel is $\rho = a_1$. A current source I_0 is located at ρ_0 where $a_1 < \rho_0 < a_2$. The immediate task is to find the resultant potential.

Here we begin with the fact that the potential Ω_1 in the region $a_1 < \rho < a_2$ should behave as $I_0/4\pi\sigma_1 R$ as R the distance to the source tends to zero. Because of the cylindrical geometry of the problem, this suggests using the representation

$$\frac{1}{R} = \frac{2}{\pi} \int_0^{\infty} K_0[\lambda(\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\phi)^{1/2}] \cos\lambda z \, d\lambda \quad (\text{B-1})$$

where, of course, $R = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\phi + z^2]^{1/2}$. Now by employing the addition theorem for the modified Bessel function K_0 , it follows that

$$\frac{1}{R} = \Gamma \begin{cases} K_m(\lambda\rho_0)I_m(\lambda\rho); & \rho < \rho_0 \\ I_m(\lambda\rho_0)K_m(\lambda\rho); & \rho > \rho_0 \end{cases} \quad (\text{B-2})$$

where Γ is the integral-summation operator

$$\Gamma = \frac{I_0}{2\pi^2\sigma_1} \int_0^\infty \sum_{m=-\infty}^{\infty} (\dots) e^{-im\phi} \cos\lambda z \, d\lambda \quad (B-3)$$

Here we sum over all integers from $-\infty$ to $+\infty$ including zero.

Appropriate solutions for the concentric conducting regions are then

$$\Omega_1 = \begin{cases} \Gamma[K_m(\lambda\rho_0)I_m(\lambda\rho) + A_m(\lambda)I_m(\lambda\rho) + B_m(\lambda)K_m(\lambda\rho)] & (\text{for } a_1 < \rho < \rho_0) \\ \Gamma[I_m(\lambda\rho_0)K_m(\lambda\rho) + A_m(\lambda)I_m(\lambda\rho) + B_m(\lambda)K_m(\lambda\rho)] & (\text{for } \rho_0 < \rho < a_2) \end{cases} \quad (B-4)$$

and

$$\Omega_2 = \Gamma C_m(\lambda)K_m(\lambda\rho) \quad (\text{for } \rho > a_2) \quad (B-5)$$

The functions $A_m(\lambda)$, $B_m(\lambda)$, and $C_m(\lambda)$ are to be determined by the boundary conditions that read

$$\partial\Omega_1/\partial\rho = 0 \quad \text{at } \rho = a_1 \quad (B-6)$$

and

$$\left. \begin{aligned} \sigma_1 \partial\Omega_1/\partial\rho &= \sigma_2 \partial\Omega_2/\partial\rho \\ \Omega_1 &= \Omega_2 \end{aligned} \right\} \quad \text{at } \rho = a_2 \quad (B-7)$$

On applying (B-6) and (B-7) to (B-4) and (B-5), we readily deduce that

$$A_m(\lambda) = \frac{[K(\lambda\rho_0)I'(\lambda a_1) - I(\lambda\rho_0)K'(\lambda a_1)][K'(\lambda a_2) + (\lambda/\sigma_1 Z)K(\lambda a_2)]}{[I'(\lambda a_2) + (\lambda/\sigma_1 Z)I(\lambda a_2)]K'(\lambda a_1) - [K'(\lambda a_2) + (\lambda/\sigma_1 Z)K(\lambda a_2)]I'(\lambda a_1)} \quad (B-8)$$

and

$$B_m(\lambda) = \frac{\{I(\lambda\rho_o)[K'(\lambda a_2) + (\lambda/\sigma_1 Z)K(\lambda a_2)] - K(\lambda\rho_o)[I'(\lambda a_2) + (\lambda/\sigma_1 Z)I(\lambda a_2)]\}I'(\lambda a_1)}{[I'(\lambda a_2) + (\lambda/\sigma_1 Z)I(\lambda a_2)]K'(\lambda a_1) - [K'(\lambda a_2) + (\lambda/\sigma_1 Z)K(\lambda a_2)]I'(\lambda a_1)} \quad (B-9)$$

where, for brevity, we have dropped the subscript m on I and K . Also, in the above

$$Z = Z_m = - \frac{\lambda^2}{\sigma_1} \left. \frac{\Omega_1}{\partial \Omega_1 / \partial \rho} \right|_{\rho=a_2} = - \frac{\lambda^2}{\sigma_2} \left. \frac{\Omega_2}{\partial \Omega_2 / \partial \rho} \right|_{\rho=a_2} \quad (B-10)$$

$$= -(\lambda/\sigma_2)K_m(\lambda a_2)/K'_m(\lambda a_2) \quad (B-11)$$

For a homogeneous external region (i.e. $\sigma_1 = \sigma_2$ and/or $a_2 \rightarrow \infty$) we see that $A_m(\lambda) = 0$ and

$$B_m(\lambda) = -K_m(\lambda\rho_o)I'_m(\lambda a_1)/K'_m(\lambda a_1) \quad (B-12)$$

An important special case of the general formulation is to let the current electrode approach the tunnel wall and at the same time let the potential electrode (i.e. the observer) also approach the tunnel wall. Then the relevant form for the potential (at a, ϕ, z) for the source at $(a, 0, 0)$ is

$$\Omega_1 = \Gamma \{I_m(\lambda a)K_m(\lambda a) + A_m(\lambda)I'_m(\lambda a) + B_m(\lambda)K'_m(\lambda a)\} \quad (B-13)$$

Making use of the Wronskian

$$I'_m(\lambda a)K_m(\lambda a) - I_m(\lambda a)K'_m(\lambda a) = 1/(\lambda a) \quad (B-14)$$

(B-13) reduces to

$$\Omega_1 = -\Gamma \frac{K(\lambda a_1)}{\lambda a_1 K'(\lambda a_1)} \left\{ \frac{1 - \frac{I(\lambda a_1)}{K(\lambda a_1)}}{1 - \frac{I'(\lambda a_1)}{K(\lambda a_1)}} \left[\frac{K'(\lambda a_2) + (\lambda/\sigma_1 Z)K(\lambda a_2)}{I'(\lambda a_2) + (\lambda/\sigma_1 Z)I(\lambda a_2)} \right] \right\} \quad (B-15)$$

where we have used the same abbreviated notation. The curly bracket term, of course, approaches 1 for a homogeneous external medium.

To discuss the limiting case of a planar boundary, we make use of the asymptotic approximations [Abramowitz and Stegun, 1964] for the modified Bessel functions that are valid for large order and large argument. Thus on replacing m by βa we can write

$$\lim_{\lambda a_1 \rightarrow \infty} \frac{K'_m(\lambda a_1)}{K_m(\lambda a_1)} = \frac{K'_{\beta a_1}(\lambda a_1)}{K_{\beta a_1}(\lambda a_1)} \approx - \frac{[1 + (\lambda/\beta)^2]^{\frac{1}{2}}}{\lambda/\beta} = - \frac{u}{\lambda} \quad (B-16)$$

where $u = (\beta^2 + \lambda^2)^{\frac{1}{2}}$. Now the use of such approximate limiting forms in (B-15) is justified because, for the significant range of λ in the integration the parameters λa_1 and λa_2 are indefinitely large. Thus in this limiting asymptotic sense

$$\Omega_1 \approx \Gamma \frac{1}{u a_1} \left[\frac{1 + R e^{-2us}}{1 - R e^{-2us}} \right] \quad (B-17)$$

where $s = a_2 - a_1$ and $R = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$.

We are now also required to replace the summations over m by an integral over β . Thus

$$\sum_{m=-\infty}^{+\infty} (\quad) \exp(-i\beta a_1 \phi) \approx a_1 \int_{-\infty}^{+\infty} (\quad) \exp(-i\beta a_1 \phi) d\beta \quad (B-18)$$

which becomes more accurate as $a_1 \rightarrow \infty$. Then identifying $a_1 \phi$ with the transverse coordinate y we see that

$$\Omega_1 = \frac{I_o}{2\pi^2\sigma_1} \int_0^\infty \int_{-\infty}^{+\infty} \frac{e^{-i\beta y}}{u} \left[\frac{1 + R e^{-u2s}}{1 - R e^{-u2s}} \right] d\beta \cos\lambda z \, d\lambda \quad (B-19)$$

$$= \frac{I_o}{2\pi^2\sigma_1} \int_0^\infty \int_{-\infty}^{+\infty} \frac{e^{-i\beta y}}{u} \left[1 + 2 \sum_{n=1}^{\infty} R^n e^{-2nus} \right] \cos\lambda z \, d\lambda \quad (B-20)$$

$$= \frac{I_o}{\pi^2\sigma_1} \int_0^\infty \left\{ K_0(y) + 2 \sum_{n=1}^{\infty} R^n K_0[\lambda(y^2 + (2ns)^2)^{\frac{1}{2}}] \right\} \cos\lambda z \, d\lambda \quad (B-21)$$

$$= \frac{I_o}{2\pi\sigma_1} \left[\frac{1}{(y^2 + z^2)^{\frac{1}{2}}} + 2 \sum_{n=1}^{\infty} R^n \frac{1}{[y^2 + z^2 + (2ns)^2]^{\frac{1}{2}}} \right] \quad (B-22)$$

This clearly has the correct limiting form.

The generalization of the cylindrical tunnel model to any number of concentric layers is straight forward. The general form for the potential Ω_1 and the expressions for the coefficients A_m and B_m (as given by (B-8) and (B-9)) are identical but now the more general form for the impedance function as defined by (B-10) rather than (B-11) should be used.

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FIGURE CAPTIONS

- Fig. 1 Enlarged view of the "cable" that is a coaxial structure consisting of a solid center conductor with a concentric sheath. The cylindrical coordinate system (ρ, ϕ, z) is also indicated.
- Fig. 2 Current source I_o with cylindrical coordinates (ρ_o, ϕ_o, z_o) located outside the "cable" that is coaxial with the z axis.
- Fig. 3 Perspective and end-on view of "cable" located within a homogeneous conducting half-space for a current point source at the origin of a cartesian coordinate system (x, y, z) . Note that the cable is directed parallel to the z axis.
- Fig. 4 Perspective and end-on view of "cable" located within a two-layer half-space.
- Fig. 5 Current point source over a N-layered half-space.
- Fig. 6 End-on view of two cables located in a the conducting half-space.
- Fig. 7 Cable located in the conducting medium around an air-filled tunnel. The current source I_o is in the adjacent conducting medium.
- Fig. 8 The geometry for an AC (Alternating Current) dipole source that is contained in a plane perpendicular to the cable.
- Fig. 9 Finite length current element adjacent to the cable.
- Fig. 10 Current source I_o in cylindrical concentric region.

Fig. 1

Fig. 2

Cable at
 $x = -h$ and
 $y = d$

Fig. 4

Fig. 5

Fig. 7

Fig. 8

Fig. 9

Fig. 10

SOME EARTH RESISTIVITY PROBLEMS
INVOLVING BURIED CABLES
PART 2 - NUMERICAL EXAMPLES

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Abstract-Using the formulations presented in Part I, we present some concrete calculated examples that are relevant to resistivity probing of perturbed homogeneous and layered structures. Only the two-electrode array is treated but various cable orientations are considered. In general, it is found that a long axial conductor such as a bare cable will distort the potential distribution of the current in a major way. This leads to profound departures from the apparent resistivity curves calculated for idealized homogeneous and layered structures.

1. INTRODUCTION

In Part I we presented analyses for a number of closely related current-flow problems involving axial conductors excited by a current point source. In most cases, the results were left in the form of integrals and/or infinite summations. But the physical significance of the results and relationships with previous work were pointed out.

Here we describe some numerical applications and discuss their significance to geophysical probing. The selection of the parameters is governed to some extent by the possible relevance to mining tech-

of roof structures in coal mines. In many cases these can be represented as uniformly layered media with horizontal bedding planes. However, there are instances where the structure may possess inhomogeneities such as metal conductors, e.g. pipes, cables, and rails. It is our purpose here to consider a few concrete situations based on the earlier general analysis. We restrict attention to the so called two-electrode resistivity array wherein the second current and second potential electrode are removed effectively to infinity. As is well known to practitioners in geophysical prospecting, the results for two electrode configurations can be superimposed to give corresponding results for three and four electrode arrays [see references in Part I, particularly *Keller and Frischknecht*, 1966].

2: CHARACTERIZATION OF THE CABLE

First of all, we describe the axial conductor that here is regarded as a perturbation (i.e. annoyance) to the ideal structure. As indicated by (I-9), i.e. equation (9) of Part I, the axial impedance Z_w of the bare cable of radius c_o is

$$Z_w = \frac{\gamma_w I_o(\gamma_w c_o)}{2\pi\sigma_w c_o I_1(\gamma_w c_o)} \quad (1)$$

where $\gamma_w \approx (i\sigma_w \mu_w \omega)^{1/2}$, σ_w is the conductivity of the cable conductor, $\mu_w (= \mu_o)$ = permeability of the cable conductor, ω is the frequency in radians/sec., I_o and I_1 are the modified Bessel functions. For the limiting case where $|\gamma_w c_o| \ll 1$, we obtain the expected D.C. resistance $R_w \rightarrow (\pi c_o^2 \sigma_w)^{-1}$.

The cable characteristics for a typical copper conductor $\sigma_w = 5.65 \times 10^{+7}$ are shown in Figures 1 and 2 as a function of frequency for different radii of the cable. Figure 1 gives $R_w = \text{Real}(Z_w)$ the resistance of the cable per meter length and Figure 2 gives $X_w = \text{Imag}(Z_w)$ the self reactance of the cable per meter length.

As the frequency is increased, both R_w and X_w increase linearly on the logarithmic scale. Also, at very low frequencies, X_w approaches zero, while R_w approaches the D.C. result. The attainment of the D.C. limit is a function of the radius of the cable; for $c_o = 0.001$ meter, $Z_w \approx R_w$ up to a frequency range of 10^4 Hz. But this applicable D.C. range dramatically diminishes as the radius of the cable increases. In fact, for $c_o = 0.01$ meter, $Z_w \approx R_w$ up to a frequency range of only 10^2 Hz.

For concentric cables, one can use these results along with an additional term; then the axial impedance, as indicated by (I-16), per unit length of an infinitely long cable with a concentric coating, is given by

$$Z_c(\lambda) = Z_w(\lambda) + \frac{\lambda^2}{2\pi\sigma_c} \ln \frac{c}{c_o} \quad (2)$$

where λ is the axial wavenumber, σ_c = conductivity of the concentric coating on the cable conductor, c_o = radius of conductor = internal radius of the coating, c = external radius of the cable. In fact, $Z_c(\lambda)$ is then spatially dispersive in the sense that the series impedance is a function of the spatial wavelength $2\pi/\lambda$ of the external field.

Another factor here is that σ_c , if regarded as a complex con-

the surface of the cable. However, we do not delve into this problem other than to indicate that the formulation allows for this possibility. Even with this very simple example, it is evident that the "cable" and many similar axial structures can introduce profound effects. In what follows, we just deal with a bare cable of radius c_0 that is in intimate contact with the medium.

3: RESPONSE OF THE CABLE TO A POINT CURRENT SOURCE IN AN INFINITE MEDIUM

The geometry of the first case considered is shown in Fig. 3. The infinitely long cable conductor is oriented along the z-axis and is excited by a current point source I_0 located at $(\rho_0, \phi_0, 0)$ in an isotropic homogeneous medium $\sigma, \mu_0, \epsilon_0$. The cable has an impedance $Z_w(\lambda)$ per meter length. The total potential at any point is given by the expression (I-34) which we write here in the following form

$$\Omega(\rho, \phi, z) = \frac{I_0}{4\pi\sigma} \left[\frac{1}{R} + \frac{2}{\pi} \int_0^{\infty} P(\lambda) K_0(\lambda\rho) \cos\lambda z \, d\lambda \right] \quad (3)$$

where

$$R = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0) + z^2]^{\frac{1}{2}} \quad (4)$$

$$P(\lambda) = - \frac{K_0(\lambda\rho_0)\lambda}{K_0(\lambda c) \lambda + Z_w(\lambda) 2\pi\sigma c K_1(\lambda c)} \quad (5)$$

K_0 and K_1 are the modified Bessel functions of the second kind of zero and first order, respectively.

The total potential is evaluated by numerical integration. In Figure 4 we show the potential distribution in the vicinity of the cable as a function of x for different values of $|z|$. The cable

is an even function with respect to z and very gradually decays as z approaches very large values. Similarly, for large values of x , the potential drops off very slowly.

As we see, there is a significant contribution from the secondary potential in the vicinity of the cable. Hence the effect of the cable is important, particularly in resistivity calculations. Actually, one is normally interested in the ratio of the apparent resistivity to the resistivity of the medium. This is defined in the following fashion

$$\frac{\rho_a}{\rho} = \frac{4\pi a \Omega(\rho) \sigma}{I_0} \quad (6)$$

where $\rho = 1/\sigma$ is the resistivity of the ambient medium.

For a two electrode array, as shown in Figure 5, the total potential $\Omega(\rho)$ at any point ρ is calculated assuming the cable to be a copper conductor of radius $c_0 = 0.005$ meter and whose impedance characteristics are as given in Figures 1 and 2. In Figure 6, we give the magnitude of ρ_a/ρ as a function of the b/a ratio for array angles of $\psi = 0^\circ, 45^\circ$ and 90° . These were actually calculated for a frequency of 10 kHz. The source is assumed to be along the x-axis at a distance b from the cable. The value of a , the distance between source and the observation point is assumed to be 1 meter so that for large values of b/a , the potential is normalized to unity and the phase approaches zero. But even for the b/a values indicated, the phase angle of ρ_a/ρ is less than a few degrees so it is not plotted. Actually, calculations show that the variation of frequency has a minor effect on the results (at least for frequencies ≤ 100 kHz).

4: RESPONSE OF THE CABLE IN THE HALF-SPACE FOR POINT SOURCE EXCITATION

In Figure 7 we indicate the geometry for the case when the cable is located within the half-space medium $x < 0$. It is oriented along the axis parallel to the z -axis (also parallel to the air-earth interface $x = 0$) and located at a distance $x = -h$ and $y = d$. The exciting point current source is placed at the origin having a magnitude I_0 .

The total potential $\Omega(x, y, z)$ for the region $x \leq 0$, as given by (I-49), is

$$\Omega(x, y, z) = \frac{I_0}{2\pi\sigma} \left[\frac{1}{R} - \frac{2}{\pi} \int_0^{\infty} K_0[\lambda(h^2+d^2)^{1/2}] [K_0(\lambda\rho_1) + K_0(\lambda\rho_2)] \frac{\lambda \cos \lambda z \, d\lambda}{[K_0(\lambda c) + K_0(2\lambda h)]\lambda + 2\pi c_0 \sigma K_1(\lambda c_0) Z_w(\lambda)} \right] \quad (7)$$

where

$$\begin{aligned} R &= (x^2 + y^2 + z^2)^{1/2} \\ \rho_1 &= [(x+h)^2 + (y-d)^2]^{1/2} \\ \rho_2 &= [(x-h)^2 + (y-d)^2]^{1/2} \end{aligned}$$

The above complex integral has been evaluated numerically and the magnitude resistivity ratio ρ_a/ρ is plotted in Figures 8a, 8b, and 8c for the two electrode array on the surface $x = 0$. As we indicated above, the variation of the potential in the low-frequency ranges can be ignored as a function of frequency. In Figures 8a, 8b, and 8c we show ρ_a/ρ as a function of d/a for different depths of the location of cable from the interface and array angles of 0° , 45° , and 90° . The

the normalized value of unity. But in the vicinity of the cable, particularly for smaller depth locations, its effect is very significant.

5: RESPONSE OF THE CABLE LOCATED IN A LAYERED HALF SPACE FOR POINT CURRENT SOURCE EXCITATION

This particular situation is a further extension of the one discussed previously. The region $x < 0$, is now a two layer half space and Figure 9 gives the geometry of the problem. The upper layer is of thickness s and conductivity σ_1 and the layer $x < -s$ has the conductivity σ_2 . The cable is located in the upper layer $0 < x < -s$, at $x = -h$ and $y = d$. The exciting point current source is again located at the origin.

The total potential at the surface $x = 0$ was given by (I-84) and is here written in summarized form

$$\Omega_1(0, y, z) = \frac{I_0}{2\pi\sigma_1} \left[\left[\frac{1}{(y^2 + z^2)^{1/2}} + 2 \sum_{n=1}^{\infty} R^n \frac{1}{[y^2 + z^2 + (2ns)^2]^{1/2}} \right] - \frac{2}{\pi} \int_0^{\infty} F(\lambda) \left[K_0(\lambda\rho_1) + \sum_{n=0}^{\infty} R^n \{ K_0(\lambda\rho_2) + 2R K_0(\lambda\rho_3) + R K_0(\lambda\rho_4) \} \right] \lambda \cos \lambda z \, d\lambda \right] \quad (8)$$

where

$$\begin{aligned} \rho_1 &= [h^2 + (y-d)^2]^{1/2} \\ \rho_2 &= [(2ns+h)^2 + (y-d)^2]^{1/2} \\ \rho_3 &= [[2(n+1)s - h]^2 + (y-d)^2]^{1/2} \\ \rho_4 &= [[2(n+1)s + h]^2 + (y-d)^2]^{1/2} \end{aligned}$$

$$R = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}$$

$$F(\lambda) = \frac{K_o [\lambda(h^2+d^2)^{\frac{1}{2}}] + \sum_{n=1}^{\infty} R^n [K_o [\lambda[(2ns+h)^2 + d^2]^{\frac{1}{2}}] + K_o [\lambda[(2ns-h)^2 + d^2]^{\frac{1}{2}}]]}{\lambda [K_o (\lambda c_o) + \sum_{n=0}^{\infty} R^n [K_o [2\lambda(ns+h)] + R K_o [2\lambda((n+1)s-h)] + 2R K_o [2\lambda(n+1)s]]]} + 2\pi\sigma_1 c_o K_1 (\lambda c_o) Z_w(\lambda)$$

as defined in equation (I-83).

Some examples for the apparent resistivity are given in Figures 10a to 10d to indicate how the total potential actually varies with spacing a for a fixed depth of layer, for various cable locations. Here we assume that $\sigma_2/\sigma_1 = 0.1$ corresponding to a relatively resistive basement layer. Two array angles are considered $\psi = 0^\circ$ and 90° for $h/s = 0.25, 0.75$. Each of the plots has ρ_a/ρ_1 plotted as a function of a/s for different cable locations d/a . Again, the results are virtually independent of frequency, although the calculations were performed for a frequency of 10 kHz with $a = 1$ meter. Obviously, the results can be scaled to other dimensions provided our basic assumptions about the validity of potential theory are not violated.

The results illustrated in Figure 10a, b, c, and d show very dramatically that the cable completely changes the character of the resistivity response curves. In fact, it is significant to note that for $d/a = 0$ (i.e. cable underneath the array), the apparent resistivity is shielded from the effect of the basement layer.

Additional calculations are shown in Figs. 11a to 11f for the same

CONCLUDING REMARKS

Here we have given some specific numerical examples for the apparent resistivity of a two-electrode resistivity array in the vicinity of a long and slender metal conductor. Not surprisingly, the results show that the metal conductor or cable channels a major portion of the current flow even if the cable is located at a distance as great as $3a$ (where a is the spacing between the current and potential electrode). A quantitative understanding of this effect seems to have been lacking hitherto.

FIGURE CAPTIONS

- Fig. 1 Resistance R_w in ohms per meter of cable as a function of frequency in Hz. Radius of cable c_o indicated on curves for copper conductor; $\sigma_w = 5.65 \times 10^7$, $\mu_w = \mu_o = 4\pi \times 10^{-7}$.
- Fig. 2 Reactance X_w in ohms per meter of cable as a function of frequency in Hz for same conditions as Fig. 1.
- Fig. 3 Excitation of cable in an infinite homogeneous medium by a current source I_o .
- Fig. 4 Potential in plane $y = 0$ as a function of distance x from cable for various axial distances $|z|$ for source location $x_o = 1$, $y_o = 1$, $z_o = 0$.
- Fig. 5 Geometry is applicable for cable and source in infinite medium or when both are on the surface of a homogeneous half space.
- Fig. 6 Apparent resistivity for situation indicated in Fig. 5. Calculated for $\rho = 1/\sigma = 10^2$, $a = 1$, and variable b . Curves

Fig. 7 Cable buried in homogeneous conducting half-space for current source I_o on the surface and observer P also on surface.

Fig. 8a, Apparent resistivity for situation indicated in Fig. 7. Calculated for $\rho = 1/\sigma = 10^2$, $a = 1$ while d and h are varied. Angle ψ takes the value 0° , 45° or 90° in (a), (b) and (c), respectively.

Fig. 9 Three orthogonal views of cable buried in upper layer of a two-layer conducting half-space. Both current source I_o and observer P are at the surface $x = 0$.

Fig. 10 Apparent resistivity for situation indicated in Fig. 9.

Values of ψ , d/a , and h/s are as indicated. Calculations carried out for $c_o = 0.005m$, $\sigma_w = 5.65 \times 10^7$, $\sigma_2/\sigma_1 = 0.1$, and $\sigma_1 = 1/\rho_1 = 10^{-2}$ and $a = 1 m$.

Fig. 11 Apparent resistivity for situation indicated in Fig. 9. Values of ψ , d/s and h/s are as indicated. Calculations carried out for $c_o = 0.005 m$, $\sigma_w = 5.65 \times 10^7$, $\sigma_2/\sigma_1 = 0.1$, and $\sigma_1 = 1/\rho_1 = 10^{-2}$ and $s = 1 m$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8a

Fig. 8b

Fig. 8c

Fig. 9

CHAPTER III

AN ANALYSIS OF A RESONANT LOOP AS AN ELECTROMAGNETIC SENSOR OF COAL SEAM THICKNESS

by

David C. Chang and James R. Wait

AN ANALYSIS OF A RESONANT LOOP AS AN ELECTROMAGNETIC SENSOR OF COAL SEAM THICKNESS

The use of a resonant loop as a sensor for the detection of roof thickness in a typical coal mine environment is considered. It is shown that the performance of a horizontal loop operating near its first resonance can be altered significantly due to the presence of a roof structure consisting of a thin layer of uncut coal seam in front of a bulk of draw slate. Questions concerning the inversion from a set of physically measurable qualities of the loop to information leading to the unique determination of the roof thickness, are discussed.

DESCRIPTION OF THE PROBLEM

In this report we consider the use of a resonant loop as a probe for the determination of roof thickness in a coal mine operation. The roof structure we are dealing with typically consists of a thin slab of uncut coal seam of the order of 10 cm in thickness, situated in front of a thicker layer of slate which usually allows very little penetration of electromagnetic waves because of its high conductivity and permittivity. This means we can model the structure by a two-layer half-space, with the "top layer consisting of a slightly lossy dielectric material (coal) and a "bottom" layer of infinite extent consisting of a highly conducting substrate (slate). To determine thickness of the top layer, i.e. the so-called roof thickness, we propose that a horizontal loop be placed near the coal surface and the self impedance of this loop be measured over a range of frequencies. The effectiveness of such a probing scheme is then determined, to a large extent, by our analytical ability to interpret such data in terms of the layer thickness. For a loop which is operating near its first resonance (where the cosinusoidal current is predominant), the measured change in the resonance characteristics provides additional information from which the layer parameters can be extracted. It would seem that the construction of such a remote sensor should be relatively simple and would present no particular alignment problem.

The results presented here show the effectiveness of such a probing scheme. The numerical data are obtained from a thin-wire formulation of a horizontal loop, located in air above a multi-layered, non-magnetic half-space, driven by a voltage generator of amplitude V_0 across an infinitesimal break at $\phi = 0$ on the loop [Chang, 1973]. The geometry of the

problem is depicted in Fig. 1. For the thin-wire assumption to be valid, we have to require that the radius of the wire, a , is very small as compared with both the radius of the loop, b , and the height h above the layered half-space so that current around the wire is nearly uniform. Provided the wire is perfectly-conducting, it has been shown before that the expression for current $I_\phi(\phi)$ in amperes per unit volt, is:

$$I_\phi(\phi)V_o = I_o + 2 \sum_{m=1}^{\infty} I_m \cos m\phi; \quad I_m^{-1} = i120\pi^2(a_m^p + a_m^s). \quad (1)$$

Here, the term a_m^p is designated as the primary contribution of an isolated loop in the absence of the layered half-space, and the term a_m^s is the scattering contribution due to the half-space. Expression for a_m^p is given as follows [King, 1969]:

$$a_m^p = (k_o b)^2 [\mu_{m+1}^p + \mu_{m-1}^p]/2 - m^2 \mu_m^p \quad (2)$$

$$\mu_m^p = (\pi k_o b)^{-1} [K_o(ma/b) I_o(ma/b) + c_m - Q_m]; \text{ for } m \neq 0, \quad (3)$$

$$\mu_o^p = (\pi k_o b)^{-1} [\ln 8b/a - Q_o], \quad (4)$$

$$Q_m = (\pi/2) \int_0^{2k_o b} [\Omega_{2m}(x) - i J_{2m}(x)] dx \quad (5)$$

$$c_m = 0.577216 - 2 \sum_{m'=0}^{m-1} (2m'+1)^{-1} + \ln 4m \quad (6)$$

and, $k_o = 2\pi f/c$ is the wave number in air, $c = 3 \times 10^8$ m/sec is the assumed velocity of light in air, f is the operating frequency (a time convention of $\exp(-i2\pi ft)$ is assumed but suppressed); $J_{2m}(x)$ is the Bessel function of order $2m$ and Ω_{2m} is the Lommel-Weber function of the same order [Abramowitz and Stegun, 1964]. Numerical values of a^p for $m = 0.1.2$ are tabulated in

Table 1, for an electrically small loop ($k_0 b = 0.3$), and loops which are near resonance ($k_0 b = 1, 1.06, 1.12$). For an operating frequency of 300 MHz, this means that radius of the loop is 4.8 cm for $k_0 b = 0.3$, and 16 cm for $k_0 b = 1$. It is seen that, in the case of a small loop ($k_0 b = 0.3$), absence of the layered half-space, the magnitude of $|a_0^p|^{-1}$ is the largest among all the a_m^p 's. Hence, the angularly uniform current, i.e. $I_\phi \approx I_0$, is the dominant term for an electrically small loop. But for the case of a resonant loop where $k_0 b = 1.06$, $|a_1^p|^{-1}$ is the largest one and hence, current on such a loop is basically cosinusoidal, i.e. $I_\phi \approx 2I_1 \cos\phi$. For a loop with a fixed radius $b = 16$ cm, the time-average radiated power, defined as $P_{\text{rad}} = \text{Re}\{I(\phi=0)V_0\}$, is shown in Fig. 2 over a wide range of operating frequency. A sharp resonance is seen to have occurred at a frequency equal to 318 MHz, or $k_0 b = 1.06$. The radiated power at this frequency is typically several hundred times greater than the radiated power at a lower frequency, say 95.4 MHz or $\beta_0 b = 0.3$. Such a drastic increase most certainly would have to be modified with the presence of a dissipative layered half-space. Thus, from the viewpoint of remote probing, a resonant loop may very well be a better sensor for the determination of the layer or roof thickness than an electrically small loop which does not possess such a resonant behavior.

To account for the influence of the layered half-space, a scattering contribution, a_m^s needs to be included into the expression for current in (1). Following a similar derivation by *Wait* [1966] for a multi-layered earth, it can be shown that a_m^s is given by the following integral representation [*Chang*, 1973]:

$$a_m^s = (k b)^2 [\mu_{m,1}^s + \mu_{m,2}^s] / 2 - m^2 \mu_{m,0}^s ; \quad (7)$$

$$\mu_{m,j}^s = \int_0^{\infty} \Gamma_j(\alpha) \exp[-2k_o \gamma_o] J_m^2(k_o b \alpha) \gamma_o^{-1} \alpha d\alpha, \quad \text{for } j = 1, 2; \quad (8)$$

$$\Gamma_{\perp}(\alpha) = R_{\perp}(\alpha); \quad \Gamma_2(\alpha) = \alpha^{-2} [R_{\perp}(\alpha) - \gamma_o^2 R_{\parallel}(\alpha)], \quad (9)$$

$$R_{\perp}(\alpha) = (\gamma_o - Y)/(\gamma_o + Y); \quad R_{\parallel}(\alpha) = (\gamma_o - Z)/(\gamma_o + Z), \quad (10)$$

$$Y = \frac{\gamma_1(\gamma_2 + \gamma_1 \tanh \gamma_1 k_o d)}{\gamma_1 + \gamma_2 \tanh \gamma_1 k_o d}; \quad (11)$$

$$Z = \frac{\lambda_1(\lambda_2 + \lambda_1 \tanh \gamma_1 k_o d)}{\lambda_1 + \lambda_2 \tanh \gamma_1 k_o d}, \quad (12)$$

$$\lambda_{\ell} = \gamma_{\ell}/n_{\ell}^2; \quad \gamma_{\ell} = (\alpha^2 - n_{\ell}^2)^{\frac{1}{2}} = -i(n_{\ell}^2 - \alpha^2)^{\frac{1}{2}} \quad \text{for } \ell = 0, 1, 2, \quad (13)$$

Here $n_{\ell} = (\epsilon_{\ell r} + i1.8 \times 10^{10} f^{-1} \sigma_{\ell}) =$ refractive index of coal ($\ell=1$) and slate ($\ell=2$); $(\epsilon_{\ell r}, \sigma_{\ell}) =$ relative permittivity and conductivity of coal ($\ell=1$), and slate ($\ell=2$); d is the thickness of the first layer in the roof, and h is the height of the loop from the coal-air interface. Integrals such as those given in (8) generally have to be computed numerically. A numerical scheme for the evaluation of these semi-infinite integrals, yielding five digits accuracy has been reported previously [Chang, 1971].

Without undue difficulty, the foregoing theory can be generalized to any number of layers by using the appropriate expressions for Y and Z as given, for example, by Wait [1966]. In what follows, we will restrict attention to the two layer case.

NUMERICAL INVESTIGATION

In this section, we shall investigate numerically the change in radiation characteristics of a horizontal, resonant loop placed near the air-coal interface. Unless otherwise specified, the following antenna and coal seam parameters are used*.

$$b(\text{radius of the thin-wire loop}) = 16 \text{ cm},$$

$$a(\text{wire radius}) = 0.25 \text{ cm},$$

$$\epsilon_{1r}(\text{relative permittivity of coal}) = 3,$$

$$\sigma_{1r}(\text{conductivity of coal}) = 3 \times 10^{-3} \text{ mhos/m},$$

$$\epsilon_{2r}(\text{relative permittivity of slate}) = 100,$$

$$\sigma_{2r}(\text{conductivity of slate}) = 3 \text{ mhos/m}.$$

A twenty-term Fourier series expansion instead of an infinite series as given in (1) is used in the determination of current along the loop. At the feed point $\phi = 0$, we define the admittance of the loop antenna as the current at the feed point divided by the applied voltage, i.e.,

$$Y = I_{\phi}(\phi=0)/V_o = G + iB = I_o + 2 \sum_{m=1}^{20} I_m \text{ mhos}. \quad (14)$$

This quantity usually is one which can be measured with ease. We note that for a voltage generator of unit amplitude, the value of G (designated as the

*Using the concept of frequency scaling, results presented in this report are equally applicable to other frequency ranges, provided that the physical dimensions are proportionally adjusted so that normalized distance as measured in terms of the free-space wavelength of the waves, and the loss-tangents of both coal and slate (i.e. $\tan^{-1}(\sigma_j/\omega\epsilon_{jr})$, for $j=1,2$) remain unchanged. For instance, if the operating frequency is scaled up by a factor of 2, numerical results will remain unchanged when we reduce every physical dimension by half, and increase the assumed conductivity of coal

input conductance of a loop antenna) corresponds exactly to the power radiated from the loop, P_{rad} . On the other hand, the value B , (designated as the input susceptance of a loop antenna) is proportional to the amount of reactive energy stored in the vicinity of the antenna. Now since we have assumed the loop is driven by a voltage source across a gap of zero thickness, the input susceptance would have to be theoretically infinite in order to reflect local knife edge capacitive effect at the gap. This means although we have a finite value of B in our computation because we have truncated the infinite series in (1) to only twenty terms, the susceptance value thus obtained is not very meaningful unless an empirical correction is made for representing the true capacitance of a realistic excitation scheme which is always finite. For this reason, we shall confine ourself to the input conductance rather than the input susceptance in the following discussion.

To begin with our investigation, we first demonstrate the kind of change in each current component due to the scattering contribution. In Table 2, the first few a_m^S 's over a range of roof thickness, along with the value of a_m^P 's for a resonant loop are tabulated. Operating frequency is chosen as 318 MHz in this case, and the height of the loop is 10 cm. It is seen that the amplitude of a_1^S is consistently larger than both a_0^S and a_2^S , while the amplitude of a_1^P is substantially smaller than both a_0^P and a_2^P . Thus, the influence of the layered coal seam for the $m=1$ component which represents a cosinusoidal current, is strongest when the loop is operating near its resonance.

We now proceed to study the change in input conductance as a result of different roof thicknesses. This is shown in Fig. 3 for an antenna height h fixed at 10 cm above the coal surface, and an operating frequency

corresponds exactly to a loop above a slate surface. As the roof thickness increases, G oscillates around the value $G_c = 6.55$ millimhos, with a decaying amplitude. In the limit of $d \rightarrow \infty$, the value of G therefore converges to G_c , which is the input conductance of a loop above a coal surface in the absence of a slate surface. The period for which G changes from a minimum to a maximum is 14 cm which is approximately one quarter of the effective wavelength of waves propagating in coal. Thus, for a typical uncut coal seam of 5 ~ 35 centimeters in thickness, the input conductance can vary from a minimum value of 3.7 millimhos to a maximum of 9.8 millimhos. Such a change is easily detectable in a given experimental situation.

One important question is whether such a contrast in the input conductance can be expected over a wide range of frequency. In order to examine this, we have also included in Fig. 3 the values of G computed for roof thickness varying from 5 cm to 30 cm at two other frequencies, one at 318 MHz and the other at 336 MHz. It is apparent that similar oscillations exist for both cases.

For a loop which is operating near its first resonance, as in the present case, it is also desirable to know whether the resonance characteristics of a loop can be alternated substantially due to the presence of the roof. In Fig. 4, the change in input conductance over a range of frequency is shown for a roof thickness d ranging from 5 to 15 centimeters, and in Fig. 5, d varies from 20 to 30 centimeters. The height h is again chosen as 10 cm above the coal surface. For comparison, we have also included, in dashed lines, the case of an isolated loop in the absence of the entire roof structure. It is seen that the resonance phenomenon indeed can be either enhanced or suppressed depending upon the specific value of roof

maximum, i.e. $G = G_{\max}$, and Δf as the bandwidth between the maximum frequency f_{res} , and the frequency for which G is at the halfway point (i.e., $G|_{f+\Delta f} = \frac{1}{2} G_{\max}$), we can then use the Q-factor of a loop-antenna defined as

$$Q = \frac{f_{\text{res}}}{2\Delta f}, \quad (15)$$

to describe the sharpness of a given resonance curve. In Fig. 6, we have shown how these quantities, f_{res} , $G_{\max}$ and Q vary as a function of roof thickness (f_0 , G_0 and Q_0 are the corresponding values of an isolated loop in the absence of the roof structure). One observes that the Q-factor can be as small as 1.6 when the roof thickness is about 12.5 centimeters.

Judging from the results in Fig. 4, the resonance phenomenon is almost completely destroyed when Q is so small. On the other hand, the Q-factor can be as large as 12.2 when the roof thickness is about 25 centimeters, which according to Fig. 5, indeed yields a much sharper resonance than the case of an isolated loop. Now since the dependence of the Q-factor on roof thickness is completely different from that of the resonant frequency over the same range, it is possible, at least in principle, to determine uniquely the roof thickness by measuring these two qualities. However, such a conclusion is valid only because of the assumption that we have prior knowledge concerning the electric properties of both coal and slate.

Next we shall examine the change in radiation characteristics of a loop antenna over a range of antenna height h . In Fig. 7, the input conductance

G is shown as a function of height ranging from 15 centimeters to 150 centimeters for a horizontal loop above a homogeneous half-space with an operating frequency of 300 MHz*. As h increases, the input conductance again oscillates with a damping amplitude similar to the situation described in Fig. 3 for varying roof thickness. However, the half-period in this case is about 25 centimeters instead of 14 centimeters as in Fig. 3 reflecting the fact that the wavelength in air is longer than the wavelength in coal by a factor of about 1.74. For a typical operating height of 5 centimeters to 25 centimeters, one therefore does not expect to observe any oscillation as in the previous case.

The frequency response of the loop near resonance is shown in Fig. 8 for an uncut coal seam of 10 centimeters in thickness, and with the loop height ranging from 5 centimeters to 25 centimeters. Here again, a similar observation can be made that both the resonant frequency and the Q-factor of the loop change very substantially from the case when the loop is very close to the coal surface, say $h = 5$ centimeters to the case when the loop is moderately away from the surface, say $h = 25$ cm. For a fixed operating frequency, the input conductance or the power radiated from the loop can either decrease or increase depending upon the specific frequency chosen when the loop is moving away from the coal surface. It is of particular interest to note that at around 310 MHz, the input conductance becomes very insensitive to the change in the antenna height from $h = 10$ centimeters

* This figure is taken from a previous report by *Chang* [1971] for a homogeneous half-space representing a wet soil. Conductivity and relative permittivity are chosen as 3×10^{-2} mhos/m and 10, which are somewhat different from the electric constants for coal in the present investigation.

to 30 centimeters. This same situation no longer exists for adjacent frequencies, as shown in Fig. 9.

SOME THEORETICAL CONSIDERATIONS

From the numerical results we described above, it is quite apparent that performance of a loop near resonance can be strongly influenced by the nearby environment factors. However, whether it can be used advantageously as a sensor for the determination of roof thickness, depends very much upon our ability to obtain a simple and explicit relationship with which the thickness parameters is thereby unambiguously determined by one of the physical qualities we actually measure. This quality can be either the resonance frequency, or the Q-factor, or the functional dependence on antenna height, or even the complete frequency response as one sweeps the operating frequency over a narrow bandwidth centered around the resonant frequency.

We recall that the dependence of loop performance on the coal seam thickness is formally expressed by the integrals $\mu_{m,j}^s$ in (8). These integrals then combine together according to (7) to form the scattering term a_m^s for each Fourier component current. Now, since the input admittance formula given in (14) is the sum of all Fourier component currents at the feed point, the dependence on the roof thickness is generally a very complicated one. However, for a loop which is near resonance, we have seen in Table 2 that $|a_m^p|^2$ is typically much greater than $|a_m^s|^2$, except the case $m = 1$. Thus, within the first-order approximation, we can replace the term $(a_m^p + a_m^s)$ by a_m^p for $m \neq 1$ in (1) to obtain

$$I_{\phi}^{(1)}(\phi) = \frac{I_{\phi}^{(0)}(\phi)}{1 + (i60\pi^2)^{-1} \left(\frac{1}{\cos\phi} - \frac{1}{\sin\phi} \right)} \quad (16)$$

where $I_{\phi}^{(0)}$ denotes the current on an isolated loop in the absence of the roof structure. Change in the input admittance from the value of an isolated loop, say Y_o , is therefore given as

$$\Delta Y = (Y - Y_o) = (i60\pi^2)^{-1} \left(\frac{1}{a_1^p + a_1^s} - \frac{1}{a_1^p} \right),$$

Now since ΔY must depend upon the operating frequency, the antenna height as well as the coal seam parameters, this means we could, in principle, determine the value of a_1^s over a range of operating frequency and/or height indirectly by measuring the change in input admittance of a horizontal loop near the coal surface from its free-space value as follows:

$$a_1^s = a_1^p \left[1 + \frac{1}{i60\pi^2 \Delta Y a_1^p} \right]^{-1} \quad (17)$$

With the expression for a_1^p explicitly given in (2), the problem of remote probing using a resonant loop is then reduced to one of determining the roof thickness d from a set of "measurables", a_1^s .

Based upon the values tabulated in Table 1, one can make a further observation that a_m^p is basically real except for $m = 1$. Thus, the input conductance of a loop near resonant is approximately given as

$$G \approx (60\pi^2)^{-1} \times \text{Imaginary part of } (a_1^p + a_1^s)^{-1} \quad (18)$$

As one now proceeds to vary the operating frequency, the value of G typically goes through a maximum (resonance) as evident from those figures presented in the previous section. The resonant frequency, as we now know, varies from case to case, depending upon the specific roof thickness and antenna height chosen. However, it appears for most cases, the functional behavior of

G can be well represented by the following form*

$$G(\omega) = \frac{A_r}{(\omega - \omega_r)^2 + \omega_i^2} \quad (19)$$

where $\omega = 2\pi f$ is the angular frequency; A_r , ω_r , ω_i are real parameters which can be determined simply from the resonant frequency f_{res} , the G-factor and the maximum value of G at resonance, G_{max} as follows:

$$\omega_r = 2\pi f_{res}, \quad \omega_i = \omega_r / 2Q \quad \text{and} \quad A_r = \omega_i G_{max} \quad (20)$$

Equation (19) is a direct consequence of a simple pole in the expression for G at the location $\omega = \omega_r + i\omega_i$ in a complex angular frequency plane for which the term $a_1^p + a_1^s = 0$. Now since the value a_1^s is determined by both the roof thickness and antenna height, location of this pole also has to vary corresponding to the change in either the roof thickness or the antenna height[†]. If we denote $\hat{\omega}_0^{(0)} = \omega_r^{(0)} + i\omega_i^{(0)}$ as the

* For the case where the resonance curve exhibits a significant degree of asymmetry, a more general expression of $G = [A_r \omega_i + A_i (\omega - \omega_r)] / [(\omega - \omega_r)^2 + \omega_i^2]$ can be used. However, expressions similar to (20) can still be expected, if one matches the measured curve with the analytical expression at the two half-power points on both sides of the resonance.

[†] The fact that G should exhibit simple poles of this kind in a complex frequency domain is well established in the context of the singular expansion method (SEM) for an arbitrary finite radiating body in free-space [Baum, 1971], and for an isolated loop in particular [Umashankov, et al, 1973]. Often these poles are referred to as the (complex) natural resonant frequencies of a radiating structure.

location of such a complex pole for an isolated loop, and $\hat{\omega}_0 = \omega_r + i\omega_i$, with the presence of the roof structure, we have

$$a_1^p(\omega = \hat{\omega}_0^{(0)}) = 0, \quad \text{and} \quad a_1^p(\omega = \hat{\omega}_0) + a_1^s(\omega = \hat{\omega}_0) = 0 \quad (21)$$

Then, since the shift in resonance frequency and consequently the distance between the two poles is usually small, one can express a_1^p evaluated at $\omega = \hat{\omega}_0^{(0)}$ in terms of a two term taylor expansion of the a_1^p evaluated at $\omega = \hat{\omega}_0$, i.e. $a_1^p(\omega = \hat{\omega}_0^{(0)}) = a_1^p(\omega = \hat{\omega}_0) + (\omega_0^{(0)} - \omega_0) \left[\frac{\partial}{\partial \omega} a_1^p \right]_{\omega_0}$. Subtraction of the first equation from the second one in (21) therefore yields the result

$$a_1^s(\hat{\omega}_0) = \Delta\omega \left[\frac{\partial}{\partial \omega} a_1^p \right]_{\omega_0}; \quad \Delta\omega = \hat{\omega}_0^{(0)} - \hat{\omega}_0. \quad (22)$$

Thus, the information concerning a_1^s at the resonant frequency can be again uncovered, except this time we only need to measure the resonant frequency and the Q-factor of an input conductance rather than the actual value of both input conductance and input susceptance of a near resonant loop.

From the above discussion, it is clear that the remote-probing problem is now one of determining the roof thickness from a set of "measurable" a_1^s . The inversion generally involves the solution of those integral representations described in (7) - (12) with the unknown parameter d imbedded in the expression of $\Gamma_j(\alpha)$ in (8) under the integration sign. The difficulty in obtaining an explicit inversion scheme is further compounded by the fact that the electric property of coal seams could, in principle, vary from site to site. Therefore, we need to determine the roof thickness as well as the electric properties of both coal and slate if the latter can not be determined prior to the roof test by some other measurement techniques. Generally speaking, a numerical search routine based upon the concept of

be uncovered with a numerical or even a graphical search procedure. However, the merit of such a sensor would be greatly reduced if extensive computation and complicated electronic circuit designs are required. From this point of view, one may want to place the loop sufficiently remote from the coal surface in order to eliminate the numerical computation of time consuming integrals as discussed in the previous section. Alternatively, one can employ other sensing techniques to establish the electric properties of the roof structure while using the loop exclusively for the determination of roof thickness. As we have concluded earlier, measurements of the Q-factor and the resonant frequency f_{res} alone can be used to determine unambiguously the roof thickness in that case. Thus, the inversion would involve nothing more than a simple search from a tabulated value of Q and f_{res} over a range of known roof parameters.

parameter optimization may be necessary in order to achieve a certain degree of accuracy. However, in some special cases when the location of the loop is sufficiently removed from the coal surface, the integral representation which relates the measurable with the unknowns in (8) can be approximated in closed form. Only then one might expect a simple and explicit relationship from which information about the roof thickness can be extracted.

CONCLUDING REMARKS

In this report, we have presented some numerical data on the performance of a resonant loop in the presence of a roof structure in a coal mine operation which consists of a thin layer of uncut coal seam in front of a bulk of draw slate. We have shown that the impedance characteristics of a loop operating near its first resonance is strongly influenced by the roof parameters. This leads us to conclude that a resonant loop can indeed be used advantageously as a sensor for the detection of these parameters. The loop, being a mono-static probing scheme, is an attractive one in that only the self-impedance is needed. It largely eliminates the instrumentation and alignment problems commonly encountered in a more complicated bi-static scheme.

However, the horizontal resonant loop antenna is only one of the many radiating structures possessing these qualities. Other linear antennas such as a horizontal dipole antenna, a vertical monopole antenna, and a vertical loop antenna, may also be used in a similar fashion.

A problem, critical to the use of a loop as a sensor, is concerned with the effective inversion of those physically-measurable qualities of a loop to information about the thickness of the coal seam in the above mentioned roof structure. There is little doubt that this information can

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TABLE 1

$$a = 2.48 \times 10^{-3}$$

$$\Omega = 2\ln 2\pi b/a = 12$$

$$f = 300 \text{ MHz}$$

$k_o b$	a_o^p	a_1^p	a_2^p
0.30	$0.411 + i0.001$	$-4.101 + i0.029$	$-14.88 + i0.000$
1.00	$1.488 + i0.136$	$-0.154 + i0.224$	$-3.500 + i0.039$
1.06	$1.586 + i0.168$	$-0.019 + i0.240$	$-3.176 + i0.048$
1.12	$1.683 + i0.203$	$0.108 + i0.254$	$-2.88 + i0.058$

TABLE

$$b = 0.16\text{m}, \quad a = 2.48 \times 10^{-3}, \quad \Omega = 2\ln 2\pi b/a = 12$$

$$f = 318 \text{ MHz}; \quad k_o b = 1.06$$

$$\sigma_1 = 3 \times 10^{-3}, \quad \epsilon_1 = 3, \quad n_1 = 1.73 + i0.05$$

$$\sigma_2 = 3, \quad \epsilon_2 = 100, \quad n_2 = 12.4 + i7.28$$

$$h = 10 \text{ cm}$$

	$a_o^p = 1.586+i0.168$	$a_1^p = -0.019+i0.240$	$a_2^p = -3.176+i0.048$
roof thickness d (cm)	a_o^s	a_1^s	a_2^s
0.05	$-0.029 - i0.084$	$0.210 + i0.019$	$0.087 + i0.000$
0.10	$0.022 - i0.023$	$0.105 + i0.146$	$0.081 + i0.034$
0.15	$0.001 + i0.049$	$-0.065 + i0.077$	$0.048 + i0.046$
0.20	$-0.059 + i0.053$	$-0.047 - i0.068$	$0.028 + i0.030$
0.25	$-0.078 + i0.018$	$0.049 - i0.105$	$0.030 + i0.012$

Fig. 1 Geometry of a horizontal loop above a roof structure consisting of a top layer of uncut coal seam situated in front of a slate bed.

Fig. 2 Frequency response of a thin wire loop in free-space.

$$k_0 b = 1.0, \quad \Omega = 2 \ln 2 \pi b / a = 12$$

$$\epsilon_1 = 3, \quad \sigma_1 = 10^{-3} \text{ mho/m}, \quad (n_1 = 1.73 + i0.05)$$

$$\epsilon_2 = 10^2, \quad \sigma_2 = 1 \text{ mho/m}, \quad (n_2 = 12.4 + i7.28)$$

Fig. 3 Input conductance of a horizontal loop antenna over a roof structure consisting of a slab of uncut coal seam in front of a slate bed. Height of the loop is fixed at $h = 10$ cm above the coal surface.

Fig. 4 Input conductance of the loop vs operating frequency. Height of the loop is fixed at 10 cm. Three different values of roof thickness, 5 cm, 10 cm, 15 cm, are used. The one with the dash-line is the response of a loop in free-space.

Fig. 5 Input conductance of the loop vs operating frequency. Height of the loop is fixed at 10 cm. Three different values of roof thickness, $d = 20$ cm, 25 cm and 20 cm are used. The one with the dash-line is the response of a loop in free-space.

Fig. 6 Change in the resonance frequency (f_{res}), the Q-factor, and the peak value of G at resonance (G_{max}) over a range of roof thickness. The corresponding values for a loop in free-space are: $f_{\text{res}} = 106.4$ MHz, $G_{\text{max}} = 7.1$ mhos, and $Q = 5.4$.

Fig. 7 The change in input conductance ($G - G_0$) of a horizontal loop over a homogeneous earth having a reflective permittivity of 10 and conductivity $\sigma = 10 \text{ mhos/m}$; G_0 is the input conductance of a loop in free-space.

$b = 16 \text{ cm}, a = 0.25 \text{ cm}$

$d = 10 \text{ cm}$

① $h = 5 \text{ cm}$

② $h = 10$

③ $h = 15$

④ $h = 20$

⑤ $h = 25$

⑥ $h = \infty$

Fig. 8 Frequency response of a loop over a roof structure for different antenna height. The γ thickness is chosen at 10 cm.

$d = 10\text{ cm}$, $b = 16\text{ cm}$, $a = 0.25\text{ cm}$

① $f = 290\text{ MHz}$

② $f = 310\text{ MHz}$

③ $f = 330\text{ MHz}$

Fig. 9 Input conductance as a function of antenna height with a fixed roof thickness ($d = 10\text{ cm}$) and for three different operating frequencies.

CHAPTER IV

A NEW EM SENSOR FOR THE DETECTION OF ROOF THICKNESS IN A COAL MINE

by

David C. Chang

ABSTRACT

The use of an annular, coaxially-driven slot antenna for probing of the roof thickness in a coal mine is discussed. For the case when the thickness of the roof and the slot width are both small compared with free-space wavelength, the inversion from the measured data on radiated power to roof thickness proves to be a very simple process. It appears that such a sensor not only can be rugged in construction, simple to install, but also has the additional advantage of being less sensitive to scatterings from surrounding environment.

A new EM sensor for the detection of roof thickness in a coal mine

In the development of automated mining machines, it is sometimes important to have sensors which are capable of detecting the thickness of a roof structure in a coal mine. Typically, we are talking about the sensing of a thin slab of coal, say ten centimeters in thickness, backed by a highly-reflecting overburden material such as draw slate. An ideal sensor in this case is one that not only is rugged in construction, and simple to install, but also one that yields measurements easy to interpret. Two types of non-contact sensors have been reported previously. The first is a two-loop method based upon the mutual coupling of two small co-planar or perpendicular loops placed in front of the coal surface. It is basically a low frequency technique (below 1 MHz) which measures the contrast in conductivity between coal and slate [Ralston and Wait, 1977]. The second one, on the other hand, is a high frequency technique (say, 300 MHz) which utilizes the resonant property of a horizontal loop antenna and measures basically the contrast in permittivity between coal and slate [Chang and Wait, 1977]. Both methods, however, employ loop antennas whose electric properties are assumed to be influenced only by the roof structure with no significant disturbance due to scattering from near-by mining and measuring equipment. Thus in order to meet this requirement, these sensors ideally should be mounted onto a non-metallic boom structure which can extend out alone to the coal surface. Such an arrangement of course would complicate the mechanical design of these sensors in a typical mining environment.

One antenna structure which does not appear to have such a disadvantage

Figure 1

Electromagnetic interference from surrounding objects is minimized in this case because of the isolation inherently provided by the ground plane. If it operates at a frequency above a few MHz, the small physical size of such a sensor may even allow its installation onto the surface of the rotating drum of a mining machine. The question is then whether this type of antenna can be used effectively as a sensor for the detection of the roof thickness.

To answer this question, we have included in Appendix A an analysis of a narrow, coaxial-driven, annular slot antenna with an infinite ground plane in the presence of a lossless coal slab, together with a perfectly-reflecting overburden. The radiated power measured in the coaxial line is found in (A.11) and (A.12) of the Appendix to be

$$G = \frac{2k_0 a}{\zeta_0} \left\{ \frac{\pi}{2} - \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} [1 - i\pi a R(\alpha) J_1(\alpha a) H_1^{(2)}(\alpha a)] d\alpha / \zeta_0 \right\}, \quad (1)$$

where

$$R(\alpha) = [\varepsilon_0 \xi_1 \tan(\xi_0 h) \tan(\xi_1 t) - \varepsilon_1 \xi_0] / [\varepsilon_0 \xi_1 \tan(\xi_1 t) + \xi_1 \xi_0 \tan(\xi_0 h)] \quad (2)$$

$$\xi_0 = (k_0^2 - \alpha^2)^{1/2}, \quad \xi_1 = (k_1^2 - \alpha^2)^{1/2} \quad \text{with } \operatorname{Im}(\xi_0), \operatorname{Im}(\xi_1) \leq 0,$$

$$k_0 = \omega(\mu_0 \varepsilon_0)^{1/2}; \quad k_1 = \omega(\mu_0 \varepsilon_1)^{1/2},$$

$\omega = 2\pi f$ is the angular operating frequency,

ε_0 and ε_1 is, respectively, the permittivity of air and coal; $\varepsilon_1 = \varepsilon_r \varepsilon_0$,

μ_0 is the permeability of a non-magnetic material,

$\zeta_0 = (\mu_0 / \varepsilon_0)^{1/2} = 120\pi$ ohm, is the intrinsic impedance of air,

h is the antenna height as measured from the coal surface,

t is the thickness of the coal slab (roof),

(a, b) is the inner, outer radius of the coaxial-line.

the information regarding roof thickness in principle can be retrieved by comparing graphically the measured data with theoretical curves for different thickness, in very much the same way as the two loop methods. As shown in the Appendix A, however, substantial simplification of the result can be achieved if one employs an operating frequency low enough so that both the antenna height and roof thickness are small with respect to the effective wavelength. (Typically, this means the operating frequency should be in the neighborhood of 100MHz for a roof thickness of 10 centimeters or less.) In that case, the expression for the radiated power reduces to (A.18) and (A.19) which are repeated here.

$$G \approx \pi^2 (\zeta_0 S)^{-1} (\alpha_1 a) J_1^2(\alpha_1 a), \quad (2)$$

where

$$\alpha_1 = k_0 [(h+t)/(h+\epsilon_r^{-1}t)]^{1/2}, \quad S = [(h+t)(h+\epsilon_r^{-1}t)]^{1/2} / a \quad (3)$$

Here, α_1 actually corresponds to the propagation constant of the dominant TM_{00} - mode in the composite waveguide consisting of the air and coal slab located between the ground plane and the reflecting overburden material. In Fig. 2, the normalized radiated power, $g = (\zeta_0 G)\pi^{-2}$, is plotted as a function of $\alpha_1 a$ with S as a parameter. Since only α_1 varies explicitly as a function of frequency, we can first measure g by sweeping the operating frequency and superpose the plot of g with frequency onto the precalculated theoretical curves in Fig. 2. The best match of the measured result would then yield the values for $(\alpha_1 f^{-1})$ and S . Now since the produce of these two numbers depends only upon the ratio $(h+t)/a$, we have from (3),

Figure 2

knowledge of the height and radius of the slot antenna. For instance, if the measured data matches the theoretical curve for $s = 0.5$ as postulated in Fig. 2, we have from (4) the thickness of 7.3 cm for a slot antenna located at a distance of 4.7 cm from the coal surface. Furthermore, one can show from (3) that $\epsilon_r = 3$ if such a measured data is obtained for a slot sensor of radius $a = 18.4$ cm.

From the above discussion, it is seen that the annular slot sensor not only can minimize the electromagnetic interference from near-by equipment, but also can provide measured results that are easy to interpret, which by and large, is not true for most other high frequency antenna structures. While this analysis necessarily needs to be modified when applied to a realistic environment because of the non-perfect reflection at the overburden, the finite size of the ground plane, and the dissipation inherent in the coal medium, it does help to point out a new type of sensor which in our opinion, warrants a closer and more comprehensive examination.

Appendix A

A.1 Formulation of the problem:

Consider the structure of an annular slot antenna as consisting of a semi-infinitely long, perfectly conducting coaxial line of inner and outer radii of a, b with its open-end flush-mounted onto a perfectly conducting ground screen of infinite extent as shown in Figure 1. The antenna is then placed at a distance h away from the coal slab which has a permittivity ϵ_1 , thickness t and is backed up by a highly-reflecting slate of complex permittivity $\hat{\epsilon}_2$. A cylindrical coordinate system (ρ, θ, z) is chosen so that the z -axis coincides with the axis of the annular slot and the plane $z=0$ with the ground plane. A current wave of the form $\exp(i[\omega t - k_0 z])$ on the inner conductor of the coaxial line is assumed to be incident from below. Here, ω is the angular frequency and k_0 , the free-space wave number. We seek in particular the dependance of the reflected current wave, and hence the apparent input admittance of the slot antenna, on the coal seam thickness.

Solution of the problem can be obtained by first solving for the tangential electric field distribution at the aperture. As shown in Appendix B, the transverse magnetic field component H_ϕ in air is related to the electric field E_ρ by the following expression,

$$H_\phi(\rho, z) = i\omega\epsilon_0 \int_a^b E_\rho(\rho'; 0) G_0(\rho, \rho'; z, 0) \rho' d\rho' \quad 0 < z < h \quad \text{and} \quad \rho > 0' \quad (A1)$$

where the kernel $G_0(\rho, \rho'; z, 0)$ of the integral is given by

$$R(\alpha) = \frac{\epsilon_0 \xi_1 \tan \xi_0 h - Y_2(\alpha) \xi_0 \epsilon_1}{\epsilon_0 \xi_1 + \xi_0 \epsilon_1 Y_2(\alpha) \tan \xi_0 h} ; Y_2(\alpha) = \frac{i \xi_1 \epsilon_2 - \epsilon_1 \xi_2 \tan \xi_1 t}{\epsilon_1 \xi_2 + i \xi_1 \epsilon_2 \tan \xi_1 t} \quad (A3)$$

and $\xi_j = (k_j^2 - \alpha^2)^{1/2}$, $\text{Im}(\xi_i) \leq 0$ and $k_j = \omega(\mu \epsilon_j)^{1/2}$ for $j = 0, 1, 2$; J_0 is the Bessel function of order zero. We note that because of the symmetry inherent in the present problem, only the field components E_ρ , E_z , H_ϕ exist and they all have no angular dependence. A similar expression for the transverse magnetic field component in the coaxial region is known to be of the following form [Chang, 1970].

$$H_\phi(\rho, z) = -i\omega\epsilon_0 \int_a^b E_\rho(\rho', 0) G_c(\rho, \rho'; z, 0) \rho' d\rho' + (\pi\rho)^{-1} \cos k_0 z$$

$$a \leq \rho \leq b, z \leq 0 \quad (A4)$$

with the kernel $G_c(\rho, \rho'; z, 0)$ given as

$$G_c(\rho, \rho'; z, 0) = -i \int_{-\infty}^{\infty} d\alpha \exp(i\alpha z) M(\alpha, \rho_>, b) M(\alpha, \rho_<, a) \Delta^{-1}(\alpha; b, a) \quad (A5)$$

where

$$M(\alpha, X, Y) = J_1(\xi_0 X) H_0^{(2)}(\xi_0 Y) - J_0(\xi_0 Y) H_1^{(2)}(\xi_0 X) \quad (A6)$$

$$\Delta(\alpha; b, a) = H_0^{(2)}(\xi_0 b) J_0(\xi_0 a) - H_0^{(2)}(\xi_0 a) J_0(\xi_0 b) \quad (A7)$$

and $H_0^{(2)}$ is the Hankel function of the second kind and order zero [Abramovitz and Stegun, 1965]. The term $(\pi\rho)^{-1} \cos k_0 z$ in (A4) is explicitly related to

$$\int_a^b E_\rho(\rho';0) [G_o(\rho,\rho';0,0) + G_c(\rho,\rho';0,0)] \rho' d\rho' = -i 120(k_o \rho)^{-1}, \quad a \leq \rho \leq b \quad (A8)$$

Once the aperture field $E_\rho(\rho',0)$ is known, the expression for the reflected current is then obtained directly from (A5) and the relationship

$I(z) = 2\pi a H_\phi(a;z)$ in the coaxial line. As pointed out in Chang [1970], the solution of this seemingly very complicated integral equation indeed can be given analytically in closed form once we allow the assumption of a narrow gap, i.e., $(b-a) \ll a$. In that case, the kernel G_o is shown in Appendix C to be approximately

$$G_o(\rho,\rho';0,0) = -(\pi a)^{-1} [\ln k_o |\rho - \rho'|/2 + \gamma + iC_o], \quad (A9)$$

where $\gamma = 0.577216$ is Euler's constant, and

$$C_o = \pi/2 - \int_0^\infty [1 - i\pi\alpha R(\alpha) J_1^2(\alpha a)] d\alpha / \xi_o \quad (A10)$$

Except for the expression of C_o , equation (A10) and hence the approximate form of the integral equation (A8) is identically the same as the one discussed in [Chang, 1970] for an annular slot antenna radiating into an unbounded free-space. Without much ado, we can write down the explicit expression of the input admittance as

$$Y_a = G + iB;$$

$$G = \frac{2k_o a}{\xi_o} \operatorname{Re}(C_o), \quad B = \frac{2k_o a}{\xi_o} [-\gamma + 1 + \ln \pi/2 + \ln k_o (b-a)/2 + \operatorname{Im}(C_o)] \quad (A11)$$

that the input conductance is related directly to the radiated power of the antenna.

A.2 Evaluation of the Input Conductance G :

Utilizing the relationships that $2J_1(\alpha a) = H_1^{(2)}(\alpha a) + H_1^{(2)}(-\alpha a)$ and $J_1(\alpha a) = -J_1(-\alpha a)$, we can rewrite the expression for C_0 as

$$C_0 = \pi/2 - \frac{1}{2} \int_{-\infty}^{\infty} [1 - i\pi\alpha R(\alpha)J_1(\alpha a)H_1^{(2)}(\alpha a)] d\alpha/\xi_0 \quad (A12)$$

where the electric parameters of the coal seam and the slate are contained implicitly in the expression for $R(\alpha)$ given by (A4). Hence, the information regarding the thickness of the coal seam can be retrieved, at least in principle, from the measurement of the radiated power, and the comparison with theoretical results calculated from (A11) and (A12) for various thickness t . However, in order to further ease the computation, the slate is now assumed to be perfectly reflective so that $R(\alpha)$ reduces to

$$R(\alpha) = \frac{\epsilon_0 \xi_1 \tan(\xi_0 h) \tan(\xi_1 t) - \epsilon_1 \xi_0}{\epsilon_0 \xi_1 \tan(\xi_1 t) + \epsilon_1 \xi_0 \tan(\xi_0 h)} \quad (A13)$$

and the second term in the integrand in (A12) contains only simple poles, while the first term has only a pair of branch cuts at $\alpha = \pm k_0$, in the complex α -plane. A subsequent deformation in the lower half-plane would then allow us to express C_0 in the form of

$$C = \lim \{ \pi^2 a \sum_{n=1}^M (\alpha_n / F_n) J_1(\alpha_n a) H_1^{(2)}(\alpha_n a) - \lim_{M \rightarrow \infty} (\alpha - \alpha_n) R(\alpha) \}$$

where α_m is the m^{th} root of the secular equation

$$\epsilon_0 \xi_{1m} \tan(\xi_{1m} t) + \epsilon_1 \xi_{0m} \tan(\xi_{0m} h) = 0 \quad (\text{A15})$$

and $\xi_{jm} = (k_j^2 - \alpha_m^2)^{1/2}$ for $j = 0, 1$. We note that, provided the thickness t and height h are sufficiently small (more specifically, $k_0 h < \pi/2$ and $k_1 t < \pi/2$), only one of these roots is located on the real axis between k_0 and k_1 in the complex α -plane, while the rest are on the imaginary axis. With some manipulation it is then not difficult to show that all the terms except $m = 1$ in the summation in (A14) are purely imaginary, so that the input conductance (or the radiated power) as given by (A11) now takes the form

$$G = 2\pi^2 k_0^2 a F_1 / \zeta_0 ; \quad (\text{A16})$$

where

$$\begin{aligned} F_1 &= \lim_{\alpha \rightarrow \alpha_1} (\alpha - \alpha_1) R(\alpha) \alpha_1 / \xi_{01} \\ &= \epsilon_1 \xi_{11} \xi_{01} \sec^2 \xi_{01} h [\xi_{01} \epsilon_1 (\tan \xi_{11} t + \xi_{11} t \sec^2 \xi_{11} t) \\ &\quad + \xi_{11} \epsilon_1 (\tan \xi_{01} h + \xi_{01} h \sec^2 \xi_{01} h)]^{-1} \end{aligned} \quad (\text{A17})$$

The dependence on the coal seam parameters is now given explicitly in the expression for F_1 since $\xi_{j1} = (k_j^2 - \alpha_1^2)^{1/2}$; $j = 0, 1$ and $k_0 = k_1 \epsilon_1^{-1/2} = 2\pi f (\mu_0 \epsilon_0)^{1/2}$. Further simplification is still possible when both the thickness and height are small electrically, i.e., $k_0^2 h^2 \ll 1$ and $k_1^2 t^2 \ll 1$. In that case, a small argument expansion of the tangent functions in (A15) and (A17) immediately reduces to the

and

$$G \approx \pi^2 (\epsilon_0 S)^{-1} (\alpha_1 a) J_1^2(\alpha_1 a); \quad S = [(h+t)(h\epsilon_r^{-1}t)]^{1/2}/a \quad (A19)$$

where ϵ_r is the relative permittivity of coal defined as $\epsilon_r = \epsilon_1/\epsilon_0$.

Appendix B

Derivation for the Transverse Magnetic Field

It is well known that for a structure with rotational symmetry, the transverse magnetic field H_ϕ satisfies the following wave equation:

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + k_j^2 - \frac{1}{\rho^2} \right) H_\phi(\rho, z) = 0 \quad (B1)$$

in the j^{th} region specified by a wave number $k_j = \omega(\mu_0 \epsilon_j)^{1/2}$ where μ_0 is the permeability and ϵ_j the permittivity of that region. With the use of a Fourier-Bessel transform, one can therefore write down the spectrum representation of H_ϕ in each region as

$$H_\phi = \begin{cases} \int_0^\infty A(\alpha) \exp(-i\xi_2[z-h-t]) J_1(\alpha\rho) \alpha d\alpha, & j > h+t \\ \int_0^\infty B(\alpha) [\sin \xi_1(z-h) + Y_2(\alpha) \cos \xi_1(z-h)] J_1(\alpha\rho) \alpha d\alpha, & h < z < h+t \\ \int_0^\infty C(\alpha) [\sin \xi_0 z - R(\alpha) \cos \xi_0 z] J_1(\alpha\rho) \alpha d\alpha, & 0 < z < h \end{cases} \quad (B2)$$

where $\xi_j = (k_j^2 - \alpha^2)^{1/2}$, $\text{Im } \xi_j \leq 0$ and A, B, C, Y_2, R are some yet undetermined coefficients. The requirement that both H_ϕ and $E_\rho = -(i\omega\epsilon)^{-1} \partial H_\phi / \partial z$ be continuous at the interfaces $z=h$ and $z=h+t$ would then allow us to eliminate four of the five coefficients to yield

and

$$R(\alpha) = \frac{\epsilon_o \xi_1 \tan \xi_o h - Y_2(\alpha) \xi_o \epsilon_1}{\epsilon_o \xi_1 + \xi_o \epsilon_1 Y_2(\alpha) \tan \xi_o h} ; Y_2(\alpha) = \frac{i \xi_1 \epsilon_2 - \epsilon_1 \xi_2 \tan \xi_1 t}{\epsilon_1 \xi_2 + i \xi_1 \epsilon_2 \tan \xi_1 t} \quad (B4)$$

At $z=0$ plane, the tangential electric field is then given by

$$E_\rho(\rho, 0) = -(i\omega\epsilon_o)^{-1} \int_0^\infty C(\alpha) J_1(\alpha\rho) \alpha d\alpha \quad (B5)$$

so that the expression for $C(\alpha)$ can be determined from the inverse Fourier-Bessel transform, together with the boundary condition that $E_\rho(\rho, 0) = 0$ for $\rho > b$ or $\rho < a$ as

$$C(\rho) = -i\omega\epsilon_o \int_a^b E_\rho(\rho', 0) J_1(\alpha\rho') \rho' d\rho' \quad (B6)$$

Substitution of (B6) into (B3) would then yield the integral expression given in (A1).

Appendix C

Approximate Expression for the Kernel $G_0(\rho, \rho'; 0, 0)$

In order to derive the approximate expression given in (A10) under the assumption of a small-gap, we first note that G_0 has a logarithmic singularity at $\rho = \rho'$ and $z = z' = 0$ since the integrand in (A2) decays only as fast as $-i(\pi\alpha)^{-1} \cos \alpha(\rho - \rho')$ as $\rho \rightarrow \rho'$ and $z = 0$. Thus if we first subtract out the leading term, we can evaluate the remainder term approximately by setting $\rho = \rho' = a$. This means

$$\begin{aligned} G_0(\rho, \rho'; 0, 0) &= -i(\pi a)^{-1} \int_0^\infty \cos \alpha(\rho - \rho') \frac{d\alpha}{\xi_0} \\ &\quad + \int_0^\infty [\alpha R(\alpha) J_0(\alpha \rho) J_0(\alpha \rho') + i(\pi a)^{-1} \cos \alpha(\rho - \rho')] \frac{d\alpha}{\xi_0} \\ &= -i(\pi a)^{-1} \int_0^\infty \cos \alpha(\rho - \rho') \frac{d\alpha}{\xi_0} + \int_0^\infty [\alpha R(\alpha) J_0^2(\alpha a) + i(\pi a)^{-1}] \frac{d\alpha}{\xi_0} \end{aligned} \quad (C1)$$

The first integral is known exactly as $\frac{\pi}{2} H_0^{(2)}(k_0 |\rho - \rho'|)$, which because of the small gap assumption, i.e., $k_0^2 |\rho - \rho'|^2 \ll 1$ can be approximated by its small argument $\pi/2 - i(\ln k_0 |\rho - \rho'|/2 + \gamma)$ and $\gamma = 0.577216$ is the Euler's constant [Abramovitz and Stegun, 1965]. Consequently, we have from (C1)

$$G_0 = -(\pi a)^{-1} [\ln k_0 |\rho - \rho'|/2 + \gamma + iC_0] \quad (C2)$$

where C_0 is given in (A11).

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CHAPTER V

ON THE THEORY OF ELECTROMAGNETIC SCATTERING BY A NONUNIFORM COAL SEAM

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An analysis is given for scattering of plane waves by a conducting layer, the electrical properties of which vary in a periodic manner. The fields may be expressed in terms of Floquet functions or, alternatively, in terms of a related set of functions described below. The latter approach seems to have computational advantages. Both the transverse electric (TE) and the transverse magnetic (TM) cases are considered for arbitrary incident angle. Numerical examples for the TE case, relevant to remote probing of coal seams, are presented.

INTRODUCTION

As a first approximation in many remote sensing applications, the earth is assumed to be a uniformly stratified medium. The electromagnetic properties of such structures are well known¹. It is a problem of practical importance to understand how departures from the uniform layer model will affect any measurements made. As well as shedding some light on the validity of the layered approximation, such understanding should aid in the remote characterization of the anomalies themselves. In this report we consider layers in which there is a periodic lateral variation of the electrical properties. Similar problems have been attacked via perturbation theory^{2,3} as well as by direct modal expansion in terms of Floquet

functions^{4,5}. The present approach which, strictly speaking, is neither perturbational nor modal in nature, provides another way of attacking the problem and would seem to be a useful computational tool in a wide range of situations.

FIELDS IN A PERIODIC MEDIUM

We begin by considering the fields in an unbounded periodic medium such as that shown in Figure 1. Initially we will assume an x-dependent complex permittivity of the form

$$\epsilon(x) - i\sigma(x)/\omega = (\epsilon - i\sigma/\omega)(1 + M \cos 2\pi x/L) \quad (1)$$

for a time factor of $e^{+i\omega t}$. Uniformity is assumed in the "y" direction so the problem is two-dimensional ($\partial/\partial y \equiv 0$).

FIGURE 1: An unbounded periodic medium in two dimensions.

For the TE case (i.e., for the electric vector in the "y" direction) the electric field obeys the wave equation

$$[\partial^2/\partial x^2 + \partial^2/\partial z^2 + k^2(x)]E_y(x,z) = 0 \quad (2)$$

where

$$k^2(x) = [\epsilon(x) - i\sigma(x)/\omega]\mu_0\omega^2.$$

We assume a "Floquet" type of solution, namely

$$E_y(x,z) = \exp(-i\beta x) \sum_{n=-\infty}^{\infty} E_n(z) \exp(-i2\pi nx/L)$$

Substituting this expression into equation (2) and making use of the orthogonality of the functions $\exp(-i2\pi nx/L)$, gives

$$-\frac{d^2}{dz^2} E_n(z) = \xi_n^2 E_n(z) + \frac{Mk^2}{2} (E_{n+1}(z) + E_{n-1}(z))$$

$$k^2 = (\epsilon - i\sigma/\omega)\mu_0\omega^2$$

$$\xi_n^2 = k^2 - (\beta + 2\pi nx/L)^2$$

This may be written in the form

$$-\frac{d^2}{dz^2} E(z) = D^2 E(z) \quad (3)$$

Here, E is the vector $\{E_n\}$ and D^2 is a matrix operator with elements

$$D_{ij}^2 = \xi_i^2 \delta_{ij} + \frac{Mk^2}{2} (\delta_{i,j-1} + \delta_{i,j+1}) \quad (4)$$

(δ_{ij} is the Kronecker delta function which has the value "1" when $i=j$, and is "0" otherwise).

Equation (3) may be reduced to an eigenvalue problem with the substitution

$$E_n(z) = f_n \exp(-iuz)$$

Then

$$D^2 f_v = u_v^2 f_v \quad (5)$$

The subscript v serves to label any one of the possible eigenvalues, u_v^2 , and its corresponding eigenvector $f_v = \{f_{nv}\}$. In terms of the solutions of equation (5), the electric field in the periodic medium may be written as a superposition of functions of the type

$$E_{vy} = \exp(-i\beta x \pm iu_v z) \sum_{n=-\infty}^{\infty} f_{nv} \exp(-i2\pi nx/L) \quad (6)$$

For the TM polarization (i.e., for the magnetic vector in the "y" direction) the situation is slightly more complicated. The magnetic field obeys the equation

$$[\partial^2/\partial x^2 + \partial^2/\partial z^2 + k^2(x)]H_y - \frac{1}{k^2(x)} \frac{\partial k^2(x)}{\partial x} \frac{\partial H_y}{\partial x} = 0 \quad (7)$$

This problem, however, is still amenable to a "Floquet" substitution

$$H_y(x, z) = \exp(-i\beta x) \sum_{n=-\infty}^{\infty} H_n(z) \exp(-i2\pi nx/L)$$

Substitution of the above into equation (7) leads, after some algebra, to a relationship analogous to equation (3)

$$-\frac{d^2}{dz^2} H(z) = D^2 H(z) \quad (8)$$

where an element of the matrix D^2 now has the form

$$D_{ij}^2 = \xi_i^2 \delta_{ij} + \frac{Mk^2}{2} (\delta_{i,j-1} + \delta_{i,j+1}) + \frac{2\pi}{L} \operatorname{sgn}(j-i) \left[\frac{(1-M^2)^{1/2} - 1}{M} \right]^{|j-i|} \times \left[\beta + \frac{2\pi}{L} j \right] \quad (9)$$

$\text{sgn}(k)$ is the sign function

$$\text{sgn}(k) = \begin{cases} 1, & k > 0 \\ 0, & k = 0 \\ -1, & k < 0 \end{cases}$$

As in the TE case, the substitution of $H_n(z) = f_n \exp(-iuz)$ into (8) leads to an eigenvalue problem of the form of equation (5).

If we replace the complex permittivity of equation (1) by the more general form

$$\epsilon(x) - i\sigma(x)/\omega = (\epsilon - i\sigma/\omega)(1 + M \text{ per}(x)),$$

where $\text{per}(x)$ is an arbitrary function of period L and $|\text{per}(x)| < 1$, we obtain similar results. Let $F_y(x, z)$ represent the field, be it E_y for the TE polarization, or H_y for the TM polarization. A "Floquet" substitution

$$F_y(x, z) = \exp(-i\beta x) \sum_{n=-\infty}^{\infty} F_n(z) \exp(-i2\pi n x/L)$$

will reduce the field equation to the form

$$-\frac{d^2}{dz^2} F(z) = D^2 F(z) \quad (10)$$

where $F(z) = \{F_n(z)\}$. The elements of the matrix D^2 are related to the coefficients of the Fourier expansion of the complex permittivity and the choice of polarization (TE or TM).

A substitution

$$F_n(z) = f_n \exp(-iuz)$$

into equation (10) leads to the eigenvalue problem

$$D^2 f_v = u_v^2 f_v \quad (11)$$

In the general case then, the fields in the periodic layer of Figure 1 may be expressed in terms of the eigenfunctions

$$F_{vy} = \exp(-i\beta x \pm iu_v z) \sum_{n=-\infty}^{\infty} f_{nv} \exp(-i2\pi nx/L) \quad (12)$$

Central to this approach is the solution of the eigenvalue problem (11). When D^2 is of the form (4), it can be shown that u_v is a solution of a transcendental dispersion relation involving infinite continued fractions^{4,5}. In the more general case, one must truncate and diagonalize D^2 . These procedures, which involve numerical searches, may be somewhat awkward to carry out.

A way around this difficulty is suggested by equation (10). Consider the matrix functions

$$\begin{aligned} C(z) &= \cos(Dz) = 1 - D^2 \frac{z^2}{2!} + D^4 \frac{z^4}{4!} - \dots \\ S^{(1)}(z) &= D^{-1} \sin(Dz) = 1z - D^2 \frac{z^3}{3!} + D^4 \frac{z^5}{5!} - \dots \\ S^{(2)}(z) &= D \sin(Dz) = D^2 z - D^4 \frac{z^3}{3!} + D^6 \frac{z^5}{5!} - \dots \end{aligned} \quad (13)$$

$S^{(1)}$ and $S^{(2)}$ are not independent, of course, but it is useful to define them. The functions $F_v(z) = \{C_{nv}(z)\}$, $\{S_{nv}^{(1)}(z)\}$, or $\{S_{nv}^{(2)}(z)\}$ are easily seen to be formal solutions of (10). We propose to expand the fields in the periodic medium in terms of the quantities

$$F_{vy} = \exp(-i\beta x) \sum_n \begin{pmatrix} C_{nv}(z) \\ S_{nv}^{(1)}(z) \end{pmatrix} \exp(-i2\pi nx/L) \quad (14)$$

This is to be compared to the Floquet expansion of equation (12).

In order to discuss the properties of the functions (13), we will make the following hypothesis:

The matrix of coefficients, $\{f_{nv}\}$, of equation (12) exists and has an inverse, $\{f_{vm}^{-1}\}$, except, perhaps, at isolated values of the modulation parameter, M .

The author has not yet proved the above conjecture, but numerous numerical checks seem to bear it out.

Under the above hypothesis, it may be shown that $C(z)$, $S^{(1)}(z)$, and $S^{(2)}(z)$ exist and are differentiable for all values of the parameters. For example, from equation (11)

$$\sum_n C_{ln}(z) f_{nv} = \cos(u_v z) f_{lv}$$

which implies that the C_{ln} are all finite. Equation (11) also furnishes us with the relations

$$\begin{aligned} C_{nv}(z) &= \sum_k f_{nk} f_{kv}^{-1} \cos(u_k z) \\ S_{nv}^{(1)}(z) &= \sum_k f_{nk} f_{kv}^{-1} \frac{\sin(u_k z)}{u_k} \\ S_{nv}^{(2)}(z) &= \sum_k f_{nk} f_{kv}^{-1} u_k \sin(u_k z) \end{aligned} \tag{15}$$

AN EXAMPLE

Consider the situation shown in Figure 2. This is a four layer structure consisting of a half space of air, an overburden (depth d), a coal seam (depth s), and a semi-infinite substrate. All of these layers will be assumed homogeneous except for the coal, which will be periodically modulated as described in Section 2. A plane wave of amplitude E_0 is incident at angle θ ; we shall present the analysis for TE incidence only, the TM case being similar with only slight modifications. If the modulation is taken to be that of equation (1), this

model becomes identical to one we studied earlier via perturbation methods³.

FIGURE 2: A four layer model with a periodic anomaly.

The fields may be expanded as follows:

in air:

$$E_y = \exp(-i\beta x) \sum_{n=-\infty}^{\infty} [a_n \exp(iw_n z) + E_0 \delta_{n0} \exp(-iw_n z)] \exp(-i2\pi n x/L)$$

where $\beta = k_a \sin\theta$ and $w_n^2 = k_a^2 - (\beta + 2\pi n/L)^2$. In the overburden:

$$E_y = \exp(-i\beta x) \sum_{n=-\infty}^{\infty} [b_n \exp(iv_n z) + c_n \exp(-iv_n z)] \exp(-i2\pi n x/L)$$

where $v_n^2 = k_g^2 - (\beta + 2\pi n/L)^2$. In the coal:

$$E_y = \exp(-i\beta x) \sum_{\nu=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [g_{\nu} C_{n\nu}(z-d) + h_{\nu} S_{n\nu}^{(1)}(z-d)] \exp(-i2\pi n x/L)$$

where $C_{n\nu}$ and $S_{n\nu}^{(1)}$, defined in equation (13), are computed with the

matrix, D^2 , appropriate for the profile. In the substrate:

$$E_y = \exp(-i\beta x) \sum_{n=-\infty}^{\infty} d_n \exp(-it_n z) \exp(-i2\pi n x/L)$$

where $t_n^2 = k_s^2 - (\beta + 2\pi n/L)^2$.

These fields must satisfy boundary conditions. At each interface ($z=0, d, d+s$), we must assure the continuity of E_y and of $H_x \propto \partial E_y / \partial z$. Application of these conditions leads to three sets of equations for a_n, b_n, c_n, g_n, h_n and d_n :

$$\left. \begin{aligned} a_n + E_0 \delta_{no} &= b_n + c_n \\ w_n a_n - w_n E_0 \delta_{no} &= v_n b_n - v_n c_n \end{aligned} \right\} z = 0$$

$$\left. \begin{aligned} b_n \exp(iv_n d) + c_n \exp(-iv_n d) &= g_n \\ v_n b_n \exp(iv_n d) - v_n c_n \exp(-iv_n d) &= -ih_n \end{aligned} \right\} z = d \quad (16)$$

$$\left. \begin{aligned} \sum_v [g_v c_{nv}(s) + h_v S_{nv}^{(1)}(s)] &= d_n \exp(-it_n(d+s)) \\ \sum_v [ig_v S_{nv}^{(2)}(s) - ih_v c_{nv}(s)] &= -t_n d_n \exp(-it_n(d+s)) \end{aligned} \right\} z = d + s$$

Here, we have used the facts:

$$C(0) = 1; \quad S^{(1)}(0) = S^{(2)}(0) = 0$$

$$\frac{d}{dz} C(z) = -S^{(2)}(z); \quad \frac{d}{dz} S^{(1)}(z) = C(z)$$

Elementary operations allow us to reduce equations (16) to the following:

$$\sum_n T L_k^{-1} T P_{kn} T U_n \begin{pmatrix} E_0 \delta_{no} \\ a_n \end{pmatrix} = \begin{pmatrix} d_k \\ 0 \end{pmatrix} \quad (17)$$

where:

$$TL_k^{-1} = \frac{1}{2} \begin{pmatrix} \exp(it_k(d+s)) - \frac{1}{t_k} \exp(it_k(d+s)) \\ \exp(-it_k(d+s)) - \frac{1}{t_k} \exp(-it_k(d+s)) \end{pmatrix}$$

$$TP_{kn} = \begin{pmatrix} C_{kn}(s) & iS_{kn}^{(1)}(s) \\ iS_{kn}^{(2)}(s) & C_{kn}(s) \end{pmatrix}$$

$$TU_n = \frac{1}{2} \begin{pmatrix} \frac{1}{v_n} [(v_n + w_n) \exp(-iv_n d) + (v_n - w_n) \exp(iv_n d)] \\ - [(v_n + w_n) \exp(-iv_n d) - (v_n - w_n) \exp(iv_n d)] \\ \frac{1}{v_n} [(v_n - w_n) \exp(-iv_n d) + (v_n + w_n) \exp(iv_n d)] \\ - [(v_n - w_n) \exp(-iv_n d) - (v_n + w_n) \exp(iv_n d)] \end{pmatrix}$$

Equations (17) explicitly indicate the coupling of modes in the upper and lower regions via the periodic layer. Equations of the form (17) will arise generally whenever a periodic layer, of the type we have been discussing, is imbedded in any uniformly stratified medium illuminated by a plane wave of either TE or TM polarization.

One may also arrive at equations (17) if the fields in the periodic layer are expanded in the Floquet functions of equation (12). The connection is supplied by the relationships (15).

A NUMERICAL EXAMPLE

Equations (17) may be truncated and solved numerically for the coefficients of the scattered wave, $a_{-N}, \dots, a_N$. In the results presented below, N was chosen large enough so that $|a_N/a_0| < .001$. This generally has required an N of 2 or 3 for the range of parameters we have considered so far. The matrix expansions (13) have converged well.

There are several parameters which may be varied. We choose the following representative values as our standard set:

Overburden Layer Thickness	$d = .1\text{m}$
Overburden Conductivity	$\sigma_g = .001\text{mho/m}$
Overburden Permittivity	$\epsilon_g = 10\epsilon_0$
Coal Layer Thickness	$s = .1\text{m}$
Coal Conductivity	$\sigma = .01\text{mho/m}$
Coal Permittivity	$\epsilon = 20\epsilon_0$
Substrate Conductivity	$\sigma_s = .01\text{mho/m}$
Substrate Permittivity	$\epsilon_s = 20\epsilon_0$
Coal Modulation Index	$M = .3$
Operating Frequency	$\omega/2\pi = 100\text{Mhz}$
Angle of Incidence	$\theta = 45^\circ$
Spatial Period	$L = 2\text{m}$

These values are assumed in the following unless otherwise specified.

In addition, we will assume the complex permittivity of equation (1) in all our calculations in this report.

TE CASE

In discussing TE incidence, we will give the results in terms of field ratios rather than in terms of the field amplitudes. These ratios are directly accessible to the experimenter⁶ and are probably more reliably measured.

Figures 1a, 1b, and 2a, 2c show the results of calculations for the standard parameter set. Figures 1a and 1b give the magnitude and phase of the magnetic wave tilt at the air-ground interface, $MWT = H_z/H_x \Big|_{z=0}$. Figures 2a and 2b give the magnitude and phase of the surface admittance, $SA = -H_x/E_y \Big|_{z=0}$. The magnitude of the surface admittance is normalized in the plots by a factor of

$$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \approx 120 \text{ ohms,}$$

in order that it be unitless. In addition to the exact solution, a "local" approximation is also plotted. This approximation consists of replacing, for each value of x , the model of Figure 2 by a homogeneous structure with the electrical properties found locally at x . When the period of the modulation, L , becomes much larger than the wave length of the incident wave, one might expect the "local" approximation and the exact solution to agree. In Figures 1a,b and 2a,b, comparison of the exact and "local" solutions indicates that the surface admittance is a somewhat better indicator of local structure than is the wave tilt, at least for the standard parameters.

Figures 3 and 4 compare the present exact solutions with the perturbation solutions of a previous report³. As can be seen, the first order results agree reasonably well with the exact solution. The second

order calculations are so close to the exact theory that they cannot be resolved at the scale of the graphs. The general range of parameters for which such good agreement can be expected between the perturbation and "Floquet" approaches has yet to be established. Where one can use perturbation theory, it has the advantage that the field expressions may be expressed in closed form, and thus, the computation time is minimal.

Figures 5, 6, 7, and 8 form a series comparing the magnetic wave tilt with the "local" approximation for coal seam depths of .05, .2, .5, and 10 meters, respectively. The 10 meter depth is a good approximation of a semi-infinite seam thickness, as the average skin depth in the coal is only 2.4m at 100 Mhz. Glancing through the figures, one sees that the magnetic wave tilt is often not a good indicator of local structure. The exact solution and the local approximation differ in structure and tend to be shifted relative to one another.

Figures 9, 10, 11, and 12, showing the surface admittance for coal seam depths .05, .2, .5, and 10 meters, respectively, may be compared with Figures 5, 6, 7, and 8. The surface admittance seems to follow the "local" approximation quite well, having similar form and lacking any appreciable shift.

Figures 13 and 14 show the magnitudes of the magnetic wave tilt and surface admittance for the several coal seam depths which have been studied. Shifts are evident in the position of the maxima in the magnetic wave tilt with coal seam depth. The surface admittance data is fairly symmetric about the midpoint of the interval, as would be expected of a good indicator of local structure.

Figure 15 is a plot of the surface admittance when the period of the modulation L is .5m. In this case, the wavelength of the incidental beam in the coal ($\lambda=.67\text{m}$) is larger than the scale size of the modulation. Under these circumstances, measurements tend to average the disturbance. In Figure 15 it is apparent that the surface admittance is no longer indicating local structure, but is tending to the average value ($M=0$ case). This example underscores the importance of selecting wavelengths which are smaller than the objects one wishes to study; or, perhaps, wavelengths which are larger than the objects one wishes to ignore.

Figure 16 shows the magnetic wave tilt for several incidence angles. The unmodulated ($M=0$) case is shown for reference. Not only is the average value a strong function of angle, but the structure changes also. In contrast, the surface admittance is a very weak function of incidence angle. A plot of surface admittance corresponding to Figure 16 would barely distinguish the different cases.

Figures 17 and 18 show the magnetic wave tilt for overburden depths of .05 and .2 meters, respectively. These may be compared with Figures 19 and 20, which show corresponding plots of the surface admittance. Again, the surface admittance seems to be the better indicator of local structure.

Figures 21 and 22 show the magnitude of the magnetic wave tilt and surface admittance for the various overburden depths considered. The surface admittance curves are more or less symmetric about the midpoint of the interval, while the magnetic wave tilt curves show varying structure.

The numerical results varying the overburden thickness seem to be more erratic than those varying the coal seam thickness. This appears to be due, in part, to the fact that the skin depth in the overburden is quite large compared to its thickness (skin depth = 16.8m), the results thus being complicated by strong reflection-resonance effects.

CONCLUDING REMARKS

A method has been presented which allows the calculation of electromagnetic fields in periodic layers in a wide range of situations of interest. Theoretical work is proceeding to generalize this approach to incident beams (as opposed to plane waves) and arbitrary azimuthal angles (as opposed to strict 2-dimensionality).

With such a large number of parameters, the number of variations which can be considered is considerable. Further work needs to be done to clarify the useful range of perturbation theory. For the parameters considered thus far, it appears that the surface admittance is a better indicator of local structure than is the magnetic wave tilt. The surface admittance also has the advantage of being very insensitive to the incident angle of the probing beam. Finally, it would be of interest to consider some simple periodic profiles other than sinusoidal.

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FIGURE 1a - The magnitude of the magnetic wave tilt is compared with the "local" approximation. The standard parameter values are assumed.

FIGURE 1b - The phase of the magnetic wave tilt is compared with the "local" approximation. The standard parameter values are assumed.

FIGURE 2a - The magnitude of the surface admittance is compared to the "local" approximation. The standard parameter values are assumed.

FIGURE 2b - The phase of the surface admittance is compared to the "local" approximation. The standard parameter values are assumed.

FIGURE 3 - First and second order perturbation results³ are compared with the exact theory of this report. The second order perturbation results and the exact solution are barely distinguishable at this scale.

FIGURE 4 - First and second order perturbation results are compared with the exact theory of this report. The second order perturbation results and the exact solution are barely distinguishable at

FIGURE 5a - The magnitude of the magnetic wave tilt is compared with the "local" approximation. The depth of the coal seam is .05m.

FIGURE 5b - The phase of the magnetic wave tilt is compared with the local approximation. The depth of the coal seam is .05m.

FIGURE 6a - The magnitude of the magnetic wave tilt is compared with the "local" approximation. The depth of the coal seam is .2m.

FIGURE 6b - The phase of the magnetic wave tilt is compared with the "local" approximation. The depth of the coal seam is .2m.

FIGURE 7a - The magnitude of the magnetic wave tilt is compared with the "local" approximation. The depth of the coal seam is .5m.

FIGURE 7b - The phase of the magnetic wave tilt is compared with the "local" approximation. The depth of the coal seam is .5m.

FIGURE 8a - The magnitude of the magnetic wave tilt is compared with the "local" approximation. The depth of the coal seam is 10m.

FIGURE 8b - The phase of the magnetic wave tilt is compared with the "local" approximation. The depth of the coal seam is 10m.

FIGURE 9a - The magnitude of the surface admittance is compared with the "local" approximation. The depth of the coal seam is .05m.

FIGURE 9b - The phase of the surface admittance is compared with the "local" approximation. The depth of the coal seam is .05m.

FIGURE 10a - The magnitude of the surface admittance is compared with the "local" approximation. The depth of the coal seam is .2m.

FIGURE 10b - The phase of the surface admittance is compared with the "local" approximation. The coal depth is .2m.

FIGURE 11a - The magnitudes of the surface admittance is compared to the local approximation. The depth of the coal seam is .5m.

FIGURE 11b - The phase of the surface admittance is compared with the local approximation. The depth of the coal seam is .5m.

FIGURE 12a - The magnitude of the surface admittance is compared with the "local" approximation. The depth of the coal seam is 10m.

FIGURE 12b - The phase of the surface admittance is compared with the "local" approximation. The depth of the coal seam is 10m.

FIGURE 13 - The magnitude of the magnetic wave tilt is plotted for several coal depths. The curves are labeled by the corresponding seam thicknesses.

FIGURE 14 - The magnitude of the surface admittance is plotted for several coal depths. The curves are labeled by the corresponding seam thicknesses.

FIGURE 15a - The magnitude of the surface admittance is plotted for $L = .5m$. The wavelength in the unmodulated coal seam is $.67m$, slightly larger than the scale length of the disturbance. At this wavelength, and longer wavelengths, the measured quantity approaches an average value (i.e. the $M = 0.0$ case) and is no longer a good indicator of

FIGURE 15b - The phase of the surface admittance is plotted for $L = .5m$.

FIGURE 16 - The magnetic wave tilt is plotted for several incident angles. The unmodulated case ($M=0.0$) is given for reference. Not only does the average value vary with angle, but the structure changes also. In contrast, the surface admittance is a very weak function of incident angle.

FIGURE 17a - The magnitude of the magnetic wave tilt is compared with the "local" approximation. The depth of the overburden is .05m.

FIGURE 17b - The phase of the magnetic wave tilt is compared with the "local" approximation. The depth of the overburden is .05m.

FIGURE 18a - The magnitude of the magnetic wave tilt is compared with the "local" approximation. The depth of the overburden is .2m.

FIGURE 18b - The phase of the magnetic wave tilt is compared with the "local" approximation. The depth of the overburden is .2m.

FIGURE 19a - The magnitude of the surface admittance is compared with the "local" approximation. The depth of the overburden is .05m.

FIGURE 19b - The phase of the surface admittance is compared with the "local" approximation. The depth of the overburden is .05m.

FIGURE 20a - The magnitude of the surface admittance is compared with the "local" approximation. The depth of the overburden is .2m.

FIGURE 20b - The phase of the surface admittance is compared with the "local" approximation. The depth of the overburden is .2m.

FIGURE 21 - The magnitude of the magnetic wave tilt is plotted for several overburden depths. The curves are labeled by the corresponding overburden thicknesses.

FIGURE 22 - The magnitude of the surface admittance is plotted for several overburden depths. The curves are labeled by the corresponding overburden thicknesses.

SIMULATED PULSE TECHNIQUES FOR MEASURING COAL LAYER THICKNESS

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INTRODUCTION AND SUMMARY

This report is concerned with the use of broad-band electromagnetic measurements in the remote probing of geological structures.

An application might be the measurement of "header" thickness in a coal mine. Here a thin layer of coal must be left for safety reasons, as the overlying slate is unstable when exposed to air. Currently, mining operations must be stopped periodically while a hole is manually drilled into the roof. The ability to monitor "header" thickness remotely would be a definite advantage for use with automatic mining machinery.

Also useful in mining applications would be the capability to look ahead of the shaft to detect and identify hazards such as large water or gas pockets.

The techniques discussed herein make use of a large number of reflection measurements made at various microwave frequencies. The data are processed via computer to simulate the effect of an incident pulse. When this pulse encounters a discontinuity, part of it will be reflected. By examining the time of flight for the reflected pulse, it is possible to determine the distance to the interface. The amplitude and phase of the pulse give information on the electrical properties of the materials at the interface and just beyond.

Previous work which is directly related to this report should include that of Lundien [1] who used a swept frequency radar to measure stratified media. This work essentially used a simulated pulse, though this fact was not stated directly. Cook [2] has made effective use of an actual (not simulated) pulse. Ellerbruch and Adams [3] made reflection measurements at various microwave frequencies under actual mine conditions. They then tried to match this data with the frequency response of a mathematical model. Lytle [4] explores the idea of using the "natural" frequencies of a layered structure as a key to determining its properties. This approach is closely related to Lundien's but more general (and more complicated). Ellerbruch and Belsher [5] have successfully used an FM-CW system in actual coal mines.

SIMULATED PULSE

The function

$$\exp[i\omega(t-z/c)] \quad (1)$$

represents, mathematically, a plane wave of electromagnetic radiation traveling with the velocity, $c = 3.00 \times 10^8$ m/sec, in the positive z direction. A plane wave of this type might be present when an experimenter, making a steady-state measurement at a frequency, $f = \omega/2\pi$, points his antenna through the air toward a target. The wave that is reflected from the target has the form

$$R_o \exp[-i\omega 2h_o/c] \exp[i\omega(t+z/c)] \quad (2)$$

This is a plane wave traveling in the negative z direction. The factor $\exp[-i\omega^2 h_o/c]$ is due to the advance in phase as the wave travels from the antenna to the target (distance h_o) and back again. $R_o, (|R_o| \leq 1)$,

is the reflection coefficient (the ratio of the reflected to the incident field at the target surface), and it contains the information concerning the target. The result of a reflection measurement is commonly the quantity

$$R_o \exp[-i\omega 2h_o/c]$$

obtained by comparing expression 1) and 2) at $z = 0$ (the antenna-receiver location).

Let us assume we have made N measurements of the reflection coefficients, $R_o(\omega_n)$ over a spectrum of equally spaced frequencies $\omega_n = \omega_o + n\omega_s$. Using the computer, we may form quantities like:

$$\frac{1}{N} \sum_{n=0}^{N-1} \exp[i\omega_n(t-z/c)] \quad (3)$$

for the incident waves. And quantities like:

$$\frac{1}{N} \sum_{n=0}^{N-1} R_o(\omega_n) \exp[-i\omega 2h_o/c] \exp[i\omega(t+z/c)] \quad (4)$$

for the reflected wave where n is an integer.

Equations 3) and 4) are discrete Fourier transforms of the incident and reflected fields, respectively. By the superposition theorem, expression 3) is a valid incident field (albeit simulated) and expression 4) is the corresponding reflected field.

Assuming $N = 50$ and $\omega_s = 2\pi \times 100$ Mhz, we have plotted expression 3) as a function of z . Figures 1 and 2 show the real and imaginary parts, respectively. Physically, these represent the field strength of linearly polarized waves. Figure 3 shows the square of the magnitude which, physically, is proportional to the intensity of a circularly polarized wave.

As is evident, all three cases illustrate sharply defined pulses

Expression 3) is actually a periodic pulse, repeating itself every $2\pi/\omega_s$ seconds. This brings up the first major consideration for the experimenter, the "time window". If a pulse is reflected back from the target after a time greater than $2\pi/\omega_s$ seconds has elapsed following the first incident pulse, there is no way to tell whether the reflected pulse was due to the first incident pulse or to one of the subsequent periodically repeated incident pulses. In order to avoid this "wrap around" problem the experimenter should arrange his "time window", $2\pi/\omega_s$, so that all expected reflections are included. In the present example, $\omega = 2\pi \times 10^8 \text{ sec}^{-1}$, which gives a "time window" of 10^{-8} sec. In this time, a pulse will travel 150 cm down and back through air. Thus, if there is detectable structure below 150 cm optical depth, it would be wise to decrease ω_s .

The second main concern of the experimenter is resolution. If two layers are closely spaced, the reflected pulses may overlap so they appear to be one pulse. We may somewhat arbitrarily apply the criterion that when distance (down and back) between the two layers is greater than the width of the incident pulse at half maximum, then the layers are resolvable (Rayleigh criterion). Experimentally, this appears to be workable. One can show that the "smallest" resolvable distance is given by

$$\sim 181/\text{BW meters}$$

where BW = band width in Mhz. In the present case $\text{BW} = 50 \times 100 = 5000 \text{ Mhz}$, so the resolution is about 3.6 cm. Actually, in dense media there is a bonus since the speed of light is lower. The expression for resolution should be divided by the square root of the relative dielectric constant.

Finally, it should be noted that there are side lobes present on our synthesized pulses. If these become a problem, it is possible to weight the terms in the summation 3) so that they are much reduced [6]. This may be done only by increasing the width of the central peak, however.

REFLECTION OF A PULSE BY A LAYERED MEDIUM

Intuitively, we expect expression 4) for the reflected wave to consist (at least in part) of pulses reflected from discontinuities within the target. For the case of a layered medium, this may be shown explicitly.

Consider the situation shown in Figure 4 where the plane wave is assumed to be normally incident. According to theory [7] or [1], the reflection coefficient, $R_o = H_{ref}/H_{inc}$, is given by:

$$\begin{aligned}
 R_o &= r_o + \frac{t_o^2 R_1 \exp(-2ik_1 h_1)}{1 + r_o R_1 \exp(-2ik_1 h_1)} \\
 &\vdots \\
 R_m &= r_m + \frac{t_m^2 R_{m+1} \exp(-2ik_{m+1} h_{m+1})}{1 + r_m R_{m+1} \exp(-2ik_{m+1} h_{m+1})} \\
 &\vdots \\
 R_{M-1} &= r_{M-1}
 \end{aligned} \tag{5}$$

where

$$r_m = \frac{k_{m+1} - k_m}{k_{m+1} + k_m}$$

$$t_m^2 = 4k_{m+1}k_m / (k_{m+1} + k_m)^2$$

$$k_m = [\mu_o \omega (\epsilon_m \omega - i\sigma_m)]^{1/2}, \quad \text{Im}(k_m) < 0$$

r_m is the reflection coefficient calculated as if this interface were isolated from the rest of the system and t^2 is the transmission co-

efficient for a wave travelling down and back through the interface.

Equation (5) is a complicated expression which may be approximated easily. Assume that the reflection coefficients are relatively small so that multiple reflections may be ignored, then, from equation (5)

$$R_m \approx r_m + t_m^2 R_{m+1} \exp(-2ik_{m+1}h_{m+1})$$

(we have ignored the quadratic terms in the denominator). So we may write

$$\begin{aligned} R_0 = & r_0 + t_0^2 r_1 \exp[-2i(k_1 h_1)] + t_0^2 t_1^2 r_2 \exp[-2i(k_1 h_1 + k_2 h_2)] \\ & + \dots + t_0^2 \times \dots \times t_{M-2}^2 r_{M-1} \exp[-2i(k_1 h_1 + \dots + k_{M-1} h_{M-1})] \end{aligned} \quad (6)$$

This approximation amounts to keeping reflections corresponding to the solid line in Figure 4 and throwing away multiple reflections such as the one indicated by the dotted line in Figure 4. Higher order reflections will generally not be important, but if desired, they may be included by keeping as many terms as desired in the expansion of R_m . For our present purposes, the expansion (6) will suffice.

In making actual measurements, it is best to work in regions of the frequency spectrum where the loss tangent, $\sigma/\omega\epsilon$, is small. This makes the reflection coefficients, r (not R), insensitive to the frequency. The propagation constant, k , (defined below equation (5)), may be approximated as

$$k_m = (\omega/c)\epsilon_{mr}^{1/2} - i(\eta_0/2)\sigma_m/\epsilon_{mr}^{1/2}$$

Here η_0 is the impedance of free space

$$\eta_0 = (\mu_0/\epsilon_0)^{1/2} \sim 120\pi \text{ ohms}$$

and ϵ_{mr} is the relative dielectric constant of the m th layer. (Here we have tacitly assumed that ϵ_{mr} varies slowly with frequency).

Assuming a small loss tangent and making use of equation (6), expression (4) for the reflected wave becomes

$$\begin{aligned} & r_o \exp(-L_o) \times \frac{1}{N} \sum_{n=0}^{N-1} \exp i \omega_n [t + (y - 2D_o)/c] \\ & \vdots \\ & + t_o^2 \times \dots \times t_{m-1}^2 r_m \exp(-L_m) \times \frac{1}{N} \sum_{n=0}^{N-1} \exp i \omega_n [t + (y - 2D_m)/c] \\ & \vdots \end{aligned}$$

where,
$$L_m = \frac{1}{2} \eta_o \left(\frac{\sigma_o}{\epsilon_{or}^{1/2}} h_o + \dots + \frac{\sigma_m}{\epsilon_{mr}^{1/2}} h_m \right)$$

and

$$D_m = (\epsilon_{or}^{1/2} h_o + \dots + \epsilon_{mr}^{1/2} h_m)$$

The quantity D_m will hereafter be referred to as the optical depth. It is equivalent to the apparent depth of an interface if there were only air in the intervening space.

Comparing the summations in expression (7) with expression (3), indicates that expression (7) is a series of pulses that arrive at the receiver ($y=0$) at times $t = 2D/c$ (down to an optical distance D and back). The amplitude and phase of each reflected pulse is determined by the reflection coefficient, r , at the appropriate interface, the attenuation factor, $\exp(-L)$ due to travel through a lossy medium, and the transmission loss, $t_o^2 \times \dots$, due to reflections at the various interfaces.

Of course, these results are approximate. If multiple reflections were left in, one would find additional pulses with times of flight corresponding to the length of their optical paths. Since the loss tangent is usually finite, it will rarely be true that the r 's are independent of frequency, and so, the medium will be dispersive. Thus the reflected pulses will, in fact, be broadened and distorted.

For coal with $\epsilon_r \sim 9$ and $\sigma \sim .01$, the loss tangent is $\leq .02$ above 1 GHz so dispersive effects should be minimal in this range. Ellerbruch and Adams [3] have used frequencies between 1 and 2 GHz successfully in coal.

NUMERICAL CALCULATIONS AND EXPERIMENTAL RESULTS

In an attempt to estimate the usefulness of simulated time domain pulse techniques in remote probing of coal, several appropriate models were studied. Reflection coefficients were calculated for various frequencies using equation 5) unapproximated. These reflection coefficients were then fed as data into the pulse simulation program which basically performs a fast Fourier transform.

A typical model is shown in Figure 5. Here we are looking up through the roof of a coal mine. There are 20 cm of uncut coal followed by a 5 cm shale lense, then 10 cm of coal before the shale overburden is reached. The optical depth, calculated according to equation 7), is also shown for each interface. The electrical constants σ and ϵ have been chosen to be representative for coal and shale. With considerable hindsight, frequency "measurements" are made every 100 MHz. This gives a time window long enough for the pulse to penetrate 150 optical centimeters before wrap around difficulties are encountered. Using 50 frequencies gives a bandwidth of 5000 Mhz and a corresponding resolution of 3.6 cm. To minimize the effects of dispersion, the "measurements" were made in the band from 1 to 5.9 GHz.

Figure 6 shows the computed time domain response of the model. The ordinate is the magnitude squared of the circularly polarized pulse. The abscissa, which could have been labeled by time of flight, has been converted to optical depth for convenience. There are peaks at the expected optical depths and little sign of any multiple reflections. The pulse reflected at the air coal interface indicates that .75 of the energy of the incident pulse entered the roof. This is in keeping with the assignment of 9 to the relative dielectric constant of coal.

The real part of the reflected wave is shown in Figure 7. Though perhaps not quite as clear as in Figure 6, the response at the appropriate optical depths is evident. The pulse arising at the upper slate-coal interface, where there is a transition from a dense to less dense medium, is inverted. Distortion from the ideal pulse shape (see Figure 1) is due mainly to dispersion and multiple reflections.

An example of actual experimental results is shown in Figure 8. An automatic network analyzer (ANA) was used to make reflection coefficient measurements on a snow bank near Arlington, Wyoming. Measurements were made every 20 MHz from 240 to 2780 MHz. The synthesized time domain response is shown in the figure. The snow surface is all of the way to the left. The several sharp reflections indicated on the plot were correlated with actual physical layers in the pack, the ground shows up sharply. The physical depth of the drift was about 390 cm. Comparing this with an optical depth of 530 cm, gives an average relative dielectric constant of 1.85. The noise immediately to the right of the ground reflection is due to antenna cross-talk "wrapped around". The calculated resolution is about 7 cm, and this is observed in the widths of the peaks

The data used in this plot were normalized: a metal plate was placed over the snow and reflection coefficients were measured over the forementioned frequency range. Ideally, one should have measured a reflection coefficient of unity, but due to geometrical factors and frequency dependence of the antennas, this is not observed. The data are normalized by dividing, frequency by frequency, the reflection coefficient of the snow alone, by the reflection coefficient of the plate. It is easy to see that this procedure effectively levels the output of the antennas (thereby making the synthesized incident pulse sharper), corrects in part, for the geometry, and provides a reference so that the fraction of energy reflected may be read from the graph. This type of normalization also shifts the plate-snow surface to an optical depth of 0 cm.

(The information in the preceding two paragraphs was supplied courtesy of Doyle Ellerbruch and his group at the National Bureau of Standards, Boulder, Colorado).

CONCLUDING REMARKS

Over the past several years, there has been much improvement in instrumentation and it is presently possible to obtain an ANA and an accompanying controller-data processor capable of simulating pulse responses which will fit nicely on a desk top. Cycle time is still slow, requiring several minutes to take and process 100 data points. Reduction in size and improvement in speed are foreseeable in the near future, however. Advantages of simulated pulse over other currently available time domain techniques include (1) higher resolution--pulses of .1 nanosecond or less are easily attainable, (2) digital flexibility--can

satility--the ANA computer system is a flexible tool with many laboratory uses.

Of course, all time domain systems share the advantages of straight forward processing and easy interpretation of results.

At the present time, a simulated pulse capability would serve as a useful performance reference for more specialized devices designed for specific sensing tasks in the coal mine.

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FIG. 2 Imaginary part of the pulse (see expression (3)).

FIG. 3 Magnitude squared of the stimulated signal / $z=ct-10cm$ / $z=ct+1$

FIG. 4 Probing a layered structure.

FIG. 5 Model of a coal mine roof.

FIG. 6 Synthesized time domain response of the model of Fig. 5. Magnitude squared of the circularly polarized pulse (see expression (4)).

FIG. 7 Synthesized time domain response of the model of Fig. 5. Real part of the pulse (see expression (4)).

DATE: 770323. HORNS, SNOWBANK
CIRCULAR PULSE MAGNITUDE SQ

ground surface

ow surface

antenna
crosstalk

identifiable
features in snow

100 200 300 400 500 600 700
Optical Depth in cm

Synthesized time domain response of snowpack near Arlington, Wyoming (courtesy of Doyle Ellerbruch, NBS).

A REMOTE PROBING METHOD FOR DETERMINING THE THICKNESSES AND CONSTITUTIVE PROPERTIES OF PLANAR LAYERED MEDIA

by R. JEFFREY LYTLE

Abstract

A technique is given for determining the thicknesses and the electrical constitutive parameters of a planar layered medium, such as a coal seam in a mine environment. Time-domain experimental data are analyzed with Prony's method to determine the natural frequencies of the layered medium. Explicit relations are given (for dielectric layers) for determining the thicknesses and dielectric constants from the experimentally deter-

mined natural frequency results. Explicit expressions are also given (for conductive layers) for calculating the electrical thickness from natural frequency results. These natural frequency results are useful when using either a pulse excitation or a swept frequency excitation. Extensions of the technique to a non-planar medium and practical implications of the method are discussed. The basic concept is also applicable in acoustical probing.

Introduction

A problem of much interest is how to use remote probing to determine the electrical structure of planar layered media such as illustrated in Fig. 1. A more explicit statement of this problem in the context of subsurface probing would be, "How can one determine the thicknesses D_i , the relative dielectric constants ϵ_{ri} , and the conductivities σ_i of each of the N homogeneous subsurface layers by performing measurements on or above the surface $z = z_0$. A similar problem arises in connection with the determination of coal seam properties from measurements made within mine tunnels.¹

To address this general problem, we

layered media. This approach was motivated by the recent successes of the "natural frequency" concept in identifying the shape of metallic objects in free space.² This "natural frequency" concept should also be useful in identifying the electrical structure in the ground. The genesis of the idea is that an electrical structure has certain natural resonant frequencies that may be complex. Surprisingly, this classical phenomenon³ has seen very limited applications until quite recently in electromagnetics.⁴

The utility of the natural frequency idea has been dramatically illustrated for metallic object identification and an example is given below.

Fig. 1. A planar layered media can be represented by $N+1$ homogeneous layers. The m th layer has thickness D_m (for $1 \leq m \leq N$), electrical conductivity σ_m , relative dielectric constant ϵ_m , and permeability $\mu_m = \mu_0$.

prolate spheroids with various major axis-to-minor axis ratios (b/a). The shapes include an infinitesimally thin needle ($b/a = \infty$), a sphere ($b/a = 1$), and three other spheroids ($b/a = 100, 10, 5$). The natural frequencies of each one of these spheroids is unique to that object. The TM (transverse magnetic) natural frequencies⁵ for spheroids with a half-length in the long dimension of d are illustrated in Fig. 3. From Figs. 2 and 3, it is seen that if one knows the locations of the complex natural frequencies, then one can infer much about the shape of the object.

These natural frequencies can fortunately be determined experimentally. An

a metallic cylinder approximation of a Boeing 747 aircraft which scatters an incident electromagnetic pulse. The pulse induces currents which create a scattered field radiating at the aircraft's natural frequencies. An experimental result for the total scattered wave time signature of the 747 model is depicted in Fig. 4. It is known that for times greater than the width of the incident pulse (see Fig. 4), the scattered wave time signature is composed of the superposition of

signals radiating at the natural frequencies of the scattering object. By operating on the experimental time signature with a numerical algorithm known as Prony's method,⁶ one can extract the natural frequencies ($\omega_n = \omega_{nR} + j\omega_{nI}$) present in the data. Using this procedure, 50 natural frequencies present in the signature in Fig. 4 were determined. The time window indicated in Fig. 4 was used to determine these 50 natural frequencies. The superposition of the contribution of these 50 natural frequencies (see Fig. 5) gives a very adequate representation of the scattered field for times greater than the width of the incident pulse (compare Figs. 4 and 5). These results indicate

Fig. 3. The TM natural frequencies of the metallic objects in Fig. 2 are distinctive to the particular shape. Note: c is the speed of light in free space and d is the half-length

Fig. 4. The experimental response of a 747 metallic cylinder model excited by an incident pulse rings with many natural frequencies. The time window to which Prony's method was applied to find 50 natural frequencies is shown.

that by using Prony's method, one can determine the dominant natural frequencies present in even complicated experimental waveforms.

For metallic objects, the natural frequencies are independent of the angle of incidence for the incident pulse. The natural frequencies for metallic objects can be sensitive to the two orthogonal modes of polarization [TM(transverse magnetic) and TE(transverse electric)]. For planar layered media (such as some ground conditions), the natural frequencies are sensitive to the angle of incidence and the polarization. This will be shown below.

We propose using a finite width pulse incident upon a planar layered medium to excite the natural frequencies and to then

frequencies from the time-domain experimental data for the reflected signal. This paper discusses how to determine the values of ϵ_{ri} , σ_i , and D_i from the values of natural frequencies that are numerically determined from the experimental time signature.

Fig. 5. A reconstructed response of the scattered signal using 50 natural frequencies and extrapolations beyond the time window originally fit. (Compare with Fig. 4.)

Problem Formulation

A plane wave incident at angle θ_i on a planar layered medium excites a field in the upper medium (see Figs. 6a and 6b, respectively, for the TM and TE excitations). For a TM plane wave excitation, this field is

$$H_y = -H_0 \left\{ \exp[+jk_0(x\sin\theta_i + z\cos\theta_i)] + R^H \exp[(+jk_0(z\sin\theta_i - z\cos\theta_i))] \right\}. \quad (1)$$

For a TE plane wave excitation, this field is

$$E_y = E_0 \left\{ \exp[(+jk_0(z\sin\theta_i + z\cos\theta_i))] + R^E \exp[(+jk_0(x\sin\theta_i - z\cos\theta_i))] \right\}. \quad (2)$$

A time variation of $\exp(+j\omega t)$ is assumed, and the definitions of R^H and R^E in terms of the planar layer parameters are

$$R^E = \frac{N_0 - Y_1}{N_0 + Y_1}, \quad (4)$$

where Z_1 and Y_1 can be determined via a transmission-like procedure^{7,8} using

$$Z_m = K_m \frac{Z_{m+1} + jK_m \tan(k_m D_m \cos\theta_m)}{K_m + jZ_{m+1} \tan(k_m D_m \cos\theta_m)} \quad (5)$$

$$Y_m = N_m \frac{Y_{m+1} + jN_m \tan(k_m D_m \cos\theta_m)}{N_m + jY_{m+1} \tan(k_m D_m \cos\theta_m)} \quad (6)$$

with

$$k_m^2 = \omega^2 \mu_m \epsilon_m - j\omega \mu_m \sigma_m \quad (7)$$

$$\cos\theta_m = \sqrt{1 - \frac{k_0^2}{k_m^2} \sin^2\theta_i} \quad (8)$$

$$K_m = Z_{cm} \cos\theta_m \quad (9)$$

$$N_m = Y_{cm} \cos\theta_m \quad (10)$$

$$Y_{cm} = \sqrt{\frac{\epsilon_m - j\sigma_m/\omega}{\mu_m}} \quad (12)$$

In general, R^H and R^E are frequency dependent. However, for a single dielectric layer ($N = 1$) with a frequency independent relative dielectric constant ϵ_{r1} , R^H and R^E are frequency independent, and

$$R^H = \frac{\epsilon_{r1} \cos \theta_i - \sqrt{\epsilon_{r1} - \sin^2 \theta_i}}{\epsilon_{r1} \cos \theta_i + \sqrt{\epsilon_{r1} - \sin^2 \theta_i}} \quad (13)$$

$$R^E = \frac{\cos \theta_i - \sqrt{\epsilon_{r1} - \sin^2 \theta_i}}{\cos \theta_i + \sqrt{\epsilon_{r1} - \sin^2 \theta_i}} \quad (14)$$

Assuming an incident plane wave field with an impulse $[\delta(t)]$ time behavior, H_0 and E_0 are independent of frequency. Thus, for frequency independent values of R^H and R^E , the total field in the upper region is obtained by Fourier transforming Eqs. (1) and (2). This results in

$$2\pi h_y(x, z, t) = -H_0 \left\{ \delta \left(t + \frac{x \sin \theta_i + z \cos \theta_i}{c} \right) + R^H \delta \left(t + \frac{x \sin \theta_i - z \cos \theta_i}{c} \right) \right\} \quad (15)$$

$$2\pi e_y(x, z, t) = E_0 \left\{ \delta \left(t + \frac{x \sin \theta_i + z \cos \theta_i}{c} \right) + R^E \delta \left(t + \frac{x \sin \theta_i - z \cos \theta_i}{c} \right) \right\} \quad (16)$$

Thus the time response for a plane wave impulse for this model is the sum of the incident wave and a wave reflected from the surface with its amplitude de-

Fig. 6a. A TM plane wave incident upon a planar layered media excites a reflected field in the upper region.

of $2z\cos\theta_1/c$ relative to the incident pulse. This factor of $2z\cos\theta_1/c$ is obviously the time required for the plane wave impulse to travel from a height z , moving at an angle θ_1 , down to the surface and back to height z . These results are well known, but they help acquaint the reader with the general procedure.

We now consider a slightly more difficult problem, a single dielectric layer of thickness D_1 backed by a metallic conductor in region 2. Again, for a general angle of incidence, the total field in the upper region is expressed by Eqs. (1) and (2), where now via Eqs. (3) to (12),

$$R^H = \frac{Z_{co}\cos\theta_i - Z_{c1}\cos\theta_1 j \tan(k_1 D_1 \cos\theta_1)}{Y_{co}\cos\theta_i + Z_{c1}\cos\theta_1 j \tan(k_1 D_1 \cos\theta_1)} \quad (17)$$

$$R^E = \frac{Y_{co}\cos\theta_i - Y_{c1}\cos\theta_1 j \tan(k_1 D_1 \cos\theta_1)}{Y_{co}\cos\theta_i + Y_{c1}\cos\theta_1 j \tan(k_1 D_1 \cos\theta_1)} \quad (18)$$

For an impulsive incident plane wave field, the incident field time response is as in Eqs. (15) and (16), and the reflected field components are determined by

$$2\pi h y_{\text{reflected}} = - \int_{-\infty}^{\infty} H_0 R^H e^{+j\frac{\omega}{c}(x\sin\theta_1 - z\cos\theta_1) + j\omega t} d\omega \quad (19)$$

$$2\pi e y_{\text{reflected}} = + \int_{-\infty}^{\infty} E_0 R^E e^{+j\frac{\omega}{c}(x\sin\theta_1 - z\cos\theta_1) + j\omega t} d\omega \quad (20)$$

However, R^H and R^E are now frequency dependent [see Eqs. (17) and (18)]. By integration in the complex domain, the infinite integrals in Eqs. (19) and (20) can be evaluated in terms of the sums of the residues of the integrands (i. e., evaluated at the poles of R^H and R^E). These poles of R^H and R^E dictate the natural frequencies of the layered medium. For the present problem, the poles of R^H and R^E , respectively, are governed by the roots (ω) of the denominators of Eqs. (17) and (18), or

$$Z_{co}\cos\theta_i + Z_{c1}\cos\theta_1 j \tan(k_1 D_1 \cos\theta_1) = 0, \quad (21)$$

$$Y_{co}\cos\theta_i + Y_{c1}\cos\theta_1 j \tan(k_1 D_1 \cos\theta_1) = 0. \quad (22)$$

These equations could have alternatively been obtained by the transverse resonance procedure.⁹ For dielectric layers, there is no significant continuous spectrum of modes excited. This is also true for conducting layers. For layers wherein both conduction and displacement currents are significant, a continuous spectrum of modes may be important. This remains a subject for further investigation.

One can, in general, solve these equations for those complex values of ω which satisfy Eqs. (21) and (22). However, it is somewhat easier to visualize the natural resonant frequency phenomena by rewriting the expressions for R^H and R^E in terms of individual wave bounces, rather than the cumulative effect. For example, one

Fig. 7. The reflected wave component can be considered as being composed of a number of internally reflected waves (in region 1) which refract back into region 0. Note: R_0^L is the immediate reflection coefficient at the lower boundary of region 1. R_1^U is the immediate reflection coefficient at the upper boundary of region 1. $T_{01}T_{10}$ is the product of the immediate transmission coefficients for going from region 1 and then back to region 0.

$$\begin{aligned}
 R^{H,E} &= R_0^L \\
 &+ T_{01} \exp(-jk_1 D_1 \cos \theta_1) R_1^L \\
 &\times \exp(-jk_1 D_1 \cos \theta_1) T_{10} \\
 &\times \left\{ 1 + \left[R_1^U R_1^L \exp(-j2k_1 D_1 \cos \theta_1) \right]^1 \right. \\
 &\quad \left. + \left[R_1^U R_1^L \exp(-j2k_1 D_1 \cos \theta_1) \right]^2 + \dots \right\}.
 \end{aligned} \tag{23}$$

The superscript H and E on $R^{H,E}$ designate that the reflection and transmission coefficients on the right-hand side of Eq.(23) are to be, respectively, interpreted as TM and TE coefficients. By definition,

boundary of region 0 and 1. Hence, with a metallic conductor in region 2, $R_1^L = -1$ for a TE excitation and $R_1^L = +1$ for a TM excitation. R_1^U is the immediate reflection coefficient at the upper (U) boundary of region 1. T_{01} and T_{10} are, respectively, the immediate transmission coefficients in passing from region 0 into region 1 and in passing from region 1 into region 0. One can equivalently represent $R^{H,E}$, using geometric series arguments as⁷

$$R^{H,E} = R_0^L + \frac{T_{01} T_{10} R_1^L \exp(-2jk_1 D_1 \cos \theta_1)}{1 - R_1^U R_1^L \exp(-j2k_1 D_1 \cos \theta_1)}. \tag{24}$$

$$1 - R_1^U R_1^L \exp(-j2k_1 D_1 \cos\theta_1) = 0, \quad (25)$$

as well as Eqs. (21) and (22). This is the transverse resonance condition expressed in terms of the reflection coefficient formulation. It is the same form as the mode equation for idealized earth-ionosphere wave.⁷

With region 1 being a frequency independent dielectric medium, R_1^U and R_1^L are both frequency independent. Thus, the only frequency dependence in Eq. (25) occurs in $k_1 = \omega \sqrt{\mu_0 \epsilon_0 \epsilon_{r1}}$. Solving Eq. (25) for the natural frequencies ω_n results in

$$\omega_n = \frac{1}{j2\sqrt{\mu_0 \epsilon_0} D_1 \sqrt{\epsilon_{r1} - \sin^2 \theta_1}} \times \left[\ln |R_1^U R_1^L| + j |R_1^U R_1^L| \pm j2n\pi \right]. \quad (26)$$

These natural frequencies are located in the upper half of the ω plane, as will be illustrated later.

This natural frequency result could also have been obtained from the impedance formulations in Eqs. (21) and (22). It is, however, thought to be easier to visualize and determine the natural frequency results for more layers (discussed later) when using the multiple bounce formulation.

The behavior of $h_{y\text{reflected}}$ and $e_{y\text{reflected}}$ [see Eqs. (19) and (20)] can now be determined using the definition of $R^{H,E}$ as expressed in Eqs. (23) and (24). For an impulsive incident wave, H_0 and E_0 are frequency independent and thus

$$2\pi r(t) = \int R^{H,E} e^{j\frac{\omega}{c}(z\sin\theta_i - z\cos\theta_i) + j\omega t} d\omega. \quad (27)$$

Substituting Eq. (23) into Eq. (27) results in

$$\begin{aligned} 2\pi r(t) = & R_0^L \delta\left(t + \frac{x\sin\theta_i - z\cos\theta_i}{c}\right) \\ & + T_{01} T_{10} R_1^L \\ & \times \delta\left(t + \frac{x\sin\theta_i - z\cos\theta_i}{c} - \frac{2D_1 \sqrt{\epsilon_{r1} - \sin^2 \theta_1}}{c}\right) \\ & + T_{01} T_{10} R_1^L (R_1^U R_1^L)^1 \\ & \times \delta\left(t + \frac{x\sin\theta_i - z\cos\theta_i}{c} - \frac{4D_1 \sqrt{\epsilon_{r1} - \sin^2 \theta_1}}{c}\right) \\ & + T_{01} T_{10} R_1^L (R_1^U R_1^L)^2 \\ & \times \delta\left(t + \frac{x\sin\theta_i - z\cos\theta_i}{c} - \frac{6D_1 \sqrt{\epsilon_{r1} - \sin^2 \theta_1}}{c}\right) \\ & + \dots \quad (28) \end{aligned}$$

A pictorial example of the internally reflected portion of this result is given in Fig. 8.

Alternatively, substituting Eq. (24) into Eq. (27) results in

$$\begin{aligned} 2\pi r(t) = & R_0^L \delta\left(t + \frac{x\sin\theta_i - z\cos\theta_i}{c}\right) \\ & + \int_{-\infty}^{\infty} \frac{T_{01} T_{10} R_1^L e^{-j\frac{\omega^2}{c^2} D_1 \sqrt{\epsilon_{r1} - \sin^2 \theta_1} + j\frac{\omega}{c}(x\sin\theta_i - z\cos\theta_i) + j\omega t}}{1 - R_1^U R_1^L \exp(-j2\frac{\omega}{c} D_1 \sqrt{\epsilon_{r1} - \sin^2 \theta_1})} d\omega. \quad (29) \end{aligned}$$

The integral expression in Eq. (29) can

Fig. 8. The internally reflected portion of $r(t)$ creates a sequence of impulses, with the amplitude of each successive pulse being decreased by $R_1^U R_1^L$, and each succes-

the natural frequencies of the denominator [Eq. (26)]. That is, by extending the contour of integration to include the entire upper half of the ω plane,

$$2\pi r(t) = R_0^L \delta\left(t + \frac{x \sin \theta_i - z \cos \theta_i}{c}\right) + T_{01} T_{10} R_1^L (R_1^U R_1^L)^{-2} \sum_{n=-\infty}^{\infty} e^{+j\omega_n \left(t + \frac{x \sin \theta_i - z \cos \theta_i}{c}\right)}. \quad (30)$$

Equations (28) and (30) are mathematically equivalent. Equation (28) expresses the result in a physically intuitive form, whereas Eq. (30) does not. The influence of the natural frequencies ω_n on the solution is explicit in the Eq. (30) representation, whereas it is implicit in the Eq. (28) representation.

It should be noted from Eq. (30) that if one samples either in time or in position (x, z), the natural frequency influence is present and should be detectable when using Prony's method to extract the natural frequencies present in experimental data.

Illustration of the Use of the Natural Frequency Approach

For an incident plane wave which is an impulse, it is a simple matter to use the physically intuitive representation [Eq. (28)] to determine from the reflected field the values of R_0^L , $T_{01} T_{10} R_1^L$, $R_1^U R_1^L$, and $2D_1 \sqrt{\epsilon_{r1} - \sin^2 \theta_i} / c$. If two angles of incidence θ_i are used, then one can also uniquely determine D_1 and ϵ_{r1} . As $R_0^L = -R_1^U$, then R_1^L can also be determined. T_{01} and T_{10} can be determined using the relations $T_{01} = 1 + R_0^L$ and $T_{10} = 1 + R_1^U$. In this way, one can determine all the parameters of interest for the single dielectric layer.¹⁰

In a similar way, the parameters of interest can also be determined from the natural frequencies (evaluated by using Prony's method on the experimental data). From Eq. (26) it is seen that a number of natural frequencies ω_n will be determined. Let us denote them as ω_0 , ω_1 ,

Let us set

$$a_i = \left(\frac{2D_1}{c} \sqrt{\epsilon_{r1} - \sin^2 \theta_i} \right)^{-1}, \quad (31)$$

then from Eq. (26),

$$\text{Re}(\omega_0) = a_i \left| R_1^U R_1^L \right|, \quad (32)$$

$$\text{Re}(\omega_1) = a_i \left(\left| R_1^U R_1^L \right| + 2\pi \right), \quad (33)$$

and hence,

$$\left(\frac{2D_1}{c} \sqrt{\epsilon_{r1} - \sin^2 \theta_i} \right)^{-1} = \frac{\text{Re}(\omega_1) - \text{Re}(\omega_0)}{2\pi}, \quad (34)$$

$$\left| R_1^U R_1^L \right| = \frac{2\pi \text{Re}(\omega_0)}{\text{Re}(\omega_1) - \text{Re}(\omega_0)}. \quad (35)$$

It also follows from Eq. (26) that

The influence of the natural frequencies is quite evident in Eq. (40), but not as obvious (although present) in Eq. (39).

Equations (39) and (40) thus illustrate the advantage of looking at a problem from two different viewpoints.

Natural Frequency Modes for Multiple Layers and Their Relative Degree of Influence

The preceding results have been developed assuming there is only one finite thickness layer ($N = 2$ in Fig. 1). Many problems of interest have two or more finite thickness layers. The natural frequency modes for two finite thickness layers are discussed below, the extension of these results to multiple finite thickness layers follow by analogy.

Some of the possible ray paths for a two-layer problem are indicated in Fig. 9. The simplest modes excited are those characteristic of each layer (see Figs. 9a and 9b). The next simplest mode is that characteristic of the composite of the two layers (see Fig. 9c).

In addition to these ray paths, modes are excited which utilize reflections within both layers. Some of these paths are illustrated in Figs. 9d, 9e, and 9f.

In Figs. 9a-9f, the contribution from the various ray paths to the reflected signal in the upper region is indicated in equation form beside the ray path illustrations. It is thus seen that the natural frequencies for the two-layer problem are solutions of equations of the form

$$1 - R_1^U R_1^L \exp(-j2k_1 D_1 \cos\theta_1) = 0 \quad (41)$$

$$1 - R_2^U R_2^L \exp(-j2k_2 D_2 \cos\theta_2) = 0 \quad (42)$$

$$\times \exp[-j2(k_1 D_1 \cos\theta_1 + k_2 D_2 \cos\theta_2)] = 0 \quad (43)$$

$$1 - R_1^U R_2^L R_1^U R_1^L T_{12} T_{21}$$

$$\times \exp[-j2(2k_1 D_1 \cos\theta_1 + k_2 D_2 \cos\theta_2)] = 0 \quad (44)$$

$$1 - R_1^U R_2^L R_2^U R_2^L T_{12} T_{21}$$

$$\times \exp[-j2(k_1 D_1 \cos\theta_1 + 2k_2 D_2 \cos\theta_2)] = 0 \quad (45)$$

$$1 - R_1^U R_2^L R_1^U R_1^L T_{12} T_{21}$$

$$\times \exp[-j2(2k_1 D_1 \cos\theta_1 + k_2 D_2 \cos\theta_2)] = 0, \quad (46)$$

and the higher order forms of these equations.

The simplest equations to solve are those characteristic of each layer, or for layer ℓ ,

$$1 - R_\ell^U R_\ell^L \exp(-j2k_\ell D_\ell \cos\theta_\ell) = 0. \quad (47)$$

It is also of interest to determine what degree the various natural frequencies have an influence. By comparing the results in Figs. 9a-9f, it is seen that after deleting the factor of $T_{01} T_{10} \exp(-j2k_1 D_1 \cos\theta_1)$ common to all the modes, the relative influence (Fig. 9a) of the natural

Relative influence factor

3. 9. Possible ray paths for two finite thickness layers. The degrees of influence and the equations which dictate the natural frequencies of each mode are included in the equation for each characteristic ray path. The degree of influence for the various ray paths all include a common factor of $T_{01} T_{10} \exp(-j2k_1 D_1 \cos \theta_1)$.

for the second layer and also of the composite of the first and second layers is $R_2^L T_{12} T_{21} \exp(-j2k_2 D_2 \cos \theta_2)$, the relative influence of the modes in Figs. 9d and 9f is $R_1^U R_1^L R_2^L T_{12} T_{21} \exp(-j2k_1 D_1 \cos \theta_1 - j2k_2 D_2 \cos \theta_2)$, and the relative influence of the mode in Fig. 9e is $R_2^U R_2^L R_2^L T_{12} T_{21} \exp(-j2k_2 D_2 \cos \theta_2)$.

All higher order modes would have a relative influence which would include more products of immediate reflection coefficients, immediate transmission coefficients, and exponential decay factors. These results would, in most cases, indicate that the dominant behavior would be in the contribution of the natural frequencies of layer one, layer two, and the composite of layers one and two.

For three or more finite thickness layers, similar developments follow. It should be obvious that the relative degree of influence of the natural frequencies of the deeper layers becomes less and less. An indication of this is that the influence of the frequencies natural to a third layer is $R_3^L T_{12} T_{21} T_{23} T_{32} \exp(-j2k_2 D_2 \cos \theta_2 - j2k_3 D_3 \cos \theta_3)$.

As an example of the degree of influence effect, an example is considered such as might occur in a coal mine. We assume that the conductivity in each of the layers is zero. This eliminates any influence of the exponential factors on the results and would thus tend to overemphasize the relative influence of the lower lying layers. A medium with $N = 3$, with $\epsilon_{r0} = 1$, $\epsilon_{r1} = 9$, $\epsilon_{r2} = 25$, and $\epsilon_{r3} = 49$ is assumed. For normal incidence, the magnitudes of the various relative influence terms become: $|R_1^L| = 0.25$ (relative influence of the first layer, see Fig. 9a), $|R_1^L T_{12} T_{21}| = 0.17$ (relative influence of the second layer, see Fig. 9b), also the relative influence of the composite of layers one and two, see Fig. 9c), $|R_1^U R_1^L R_2^L T_{12} T_{21}| = 0.0196$ (relative influence of the modes depicted in Figs. 9d and 9f), and $|R_2^U R_2^L R_2^L T_{12} T_{21}| = 0.0065$ (relative influence of the mode illustrated in Fig. 9e). From these numbers, it is seen that the natural frequencies characteristic of the modes depicted in Figs. 9a-9c would have the largest influence on the reflected signal.

Locations of the Natural Frequencies

The natural frequencies characteristic of a single layer (i.e., not including transitions into other layers) are governed by the values of ω which satisfy

$$1 - R_\ell^U R_\ell^L \exp(-2jk_\ell D_\ell \cos \theta_\ell) = 0, \quad (48)$$

where the parameters are defined in the ℓ th layer. Various solutions of this

CASE 1: LAYERS $\ell-1$, ℓ , AND $\ell+1$ ARE DIELECTRICS

For $\epsilon_{r\ell} > \sin^2 \theta_i$ (the usual case of interest), the natural frequency solutions of Eq. (48) are

$$\omega_n = \frac{|R_\ell^U R_\ell^L \pm 2n\pi|}{2\sqrt{\mu_0 \epsilon_0} D_\ell \sqrt{\epsilon_{r\ell} - \sin^2 \theta_i}}$$

$$\ell n |R_\ell^U R_\ell^L|$$

For a sequence of dielectric layers, $|R_\ell^U R_\ell^L| \leq 1$, so $\ln |R_\ell^U R_\ell^L| \leq 0$. It is also true that $|R_\ell^U R_\ell^L| = 0$ or π . Thus, for a sequence of dielectric layers, the values of ω_n in the complex ω plane are in the upper half plane and located parallel to the real axis (see Fig. 10).

CASE 2: LAYERS $\ell-1$, ℓ , and $\ell+1$ ARE LAYERS WHERE CONDUCTION CURRENTS DOMINATE ($\sigma \gg \omega\epsilon$)

For this situation, $\cos\theta_\ell \approx \cos\theta_{\ell-1} \approx \cos\theta_{\ell+1} \approx 1$. Thus $|R_\ell^U R_\ell^L| \leq 1$ and $|R_\ell^U R_\ell^L| = 0$ or π . Using the relation

$k_\ell \approx \sqrt{-j\omega\mu_0\sigma_\ell}$, solutions of Eq. (48) are

$$\omega_n \approx \frac{(\pm 2n\pi + |R_\ell^U R_\ell^L|) \ln |R_\ell^U R_\ell^L|}{2\mu_0\sigma_\ell D_\ell^2} + j \frac{(|R_\ell^U R_\ell^L| \pm 2n\pi)^2 - (\ln |R_\ell^U R_\ell^L|)^2}{4\mu_0\sigma_\ell D_\ell^2} \quad (50)$$

Hence, for a sequence of conductive layers, the values of ω_n have a parabolic locus in the complex ω plane (see Fig. 11).

Fig.10. The natural frequencies for a dielectric layer have a characteristic locus from which one can determine (with data from two angles of incidence)

Case 1: $|R_\ell^U R_\ell^L| = 0$

Case 2: $|R_\ell^U R_\ell^L| = \pi$

Fig. 11. The natural frequencies for a conducting layer have a characteristic parabolic locus. From the locus of natural frequencies, one can determine $\sigma_\ell D_\ell^2$.

$|R_\ell^U R_\ell^L|$, and $|R_\ell^U R_\ell^L|$.

CASE 3: LAYERS $\ell-1$ AND $\ell+1$
ARE DIELECTRICS, AND LAYER ℓ
HAS A SMALL LOSS TANGENT

With the propagation constant in layer ℓ
being

$$k_\ell = \omega \sqrt{\mu_0 \epsilon_0 \epsilon_{r\ell}} - j \frac{\sigma_\ell}{2} \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_{r\ell}}}, \quad (51)$$

the natural frequency solutions of Eq. (48)
are (for $\epsilon_{r\ell} > \sin^2 \theta_i$)

$$\omega_n \cong \frac{|R_\ell^U R_\ell^L| + 2n\pi}{2\sqrt{\mu_0 \epsilon_0} D_\ell \sqrt{\epsilon_{r\ell} - \sin^2 \theta_i}} - j \frac{\ln |R_\ell^U R_\ell^L| - \sigma_\ell \sqrt{\frac{\mu_0}{\epsilon_0 (\epsilon_{r\ell} - \sin^2 \theta_i)}} D_\ell}{2\sqrt{\mu_0 \epsilon_0} D_\ell \sqrt{\epsilon_{r\ell} - \sin^2 \theta_i}}. \quad (52)$$

Hence, by comparing Eq. (49) and (52), it
is seen that a slightly lossy medium has
its natural frequency locations shifted to-
wards a slightly larger value (see Fig. 12)
of ω_{nI} (relative to the zero loss result).
This is, of course, physically intuitive as
the larger the loss in the medium, the
more rapid the natural frequency contri-
bution should decay with time ($e^{j\omega_n t} =$
 $e^{j\omega_{nR} t - \omega_{nI} t}$).

CASE 4: LAYERS ℓ AND $\ell+1$ ARE
REGIONS WHERE CONDUCTION
CURRENTS ARE DOMINATE AND
LAYER $\ell-1$ IS A DIELECTRIC

It can be shown that for a TE excitation

$$R_\ell^U \cong 1 \exp\left(-j \sqrt{\frac{2\omega \epsilon_0}{\sigma_\ell}} \cos \theta_{\ell-1}\right). \quad (53)$$

This result is valid for all values of $\theta_{\ell-1}$.

$$R_\ell^U \cong 1 \exp\left(j\pi - j \sqrt{\frac{2\omega \epsilon_0}{\sigma_\ell}} \frac{1}{\cos \theta_{\ell-1}}\right). \quad (54)$$

For $\theta_{\ell-1} > 45^\circ$, there is no simple
expression for R_ℓ^U for a TM excitation.
We thus restrict the results below to
 $\theta_{\ell-1} \leq 45^\circ$ for a TM excitation, there is
no restriction on the TE excitation.

We define

$$Q = \begin{cases} 0 & \text{for TE excitation} \\ \pi & \text{for TM excitation} \end{cases} \quad (55)$$

$$P = \begin{cases} \sqrt{\frac{2\epsilon_0}{\sigma_\ell}} \cos \theta_{\ell-1} & \text{for TE excitation} \\ \sqrt{\frac{2\epsilon_0}{\sigma_\ell}} \frac{1}{\cos \theta_{\ell-1}} & \text{for TM excitation} \end{cases} \quad (56)$$

and thus we can write

$$R_\ell^U \cong 1 \exp(jQ - j\sqrt{\omega} P). \quad (57)$$

By using this representation, the solu-
tion of Eq. (47) can be shown to be

$$\omega_n \cong \left[\frac{(\pm 2n\pi + Q + |R_\ell^L|) \ln |R_\ell^L|}{2\mu_0 \sigma_\ell D_\ell^2} + j \frac{(Q + |R_\ell^L| \pm 2n\pi)^2 - (\ln |R_\ell^L|)^2}{4\mu_0 \sigma_\ell D_\ell^2} \right] \times \left[1 - j \frac{P}{\sqrt{2\mu_0 \sigma_\ell} D_\ell} \right]. \quad (58)$$

Comparing this result with the result for
case 2 (three consecutive layers where
conduction currents dominate), it is seen
that the result in Eq. (58) is equal to the
result in Eq. (50) times the factor

Fig. 12. The natural frequencies for a layer with a small loss tangent are shifted upwards from the results of a layer with a loss tangent of zero. Compare these results with those of Fig. 10.

This means that the natural frequency results in Eq. (50) are each shifted by a constant amplitude and constant phase factor. Thus, the parabolic locus for three consecutive conducting layers (see Fig. 11) becomes a shifted parabolic locus which is not symmetric about the $\omega_{nR} = 0$ axis (see Fig. 13).

The preceding cases have been worked out to illustrate that the locus of natural frequencies indicates much information about the sequence of layers and the electrical parameters of the medium. The locus of the natural frequencies can also be worked out for combinations of layers other than those discussed in cases 1 to 4. If necessary, when analytical expressions cannot be easily derived, empirical relations can be established for the behavior of ω_n with the medium constants.

Fig. 13. The natural frequencies for a conducting layer between a conducting layer and a dielectric layer has a parabolic locus for the natural frequencies. The locus is, however, not symmetric about the ω_I axis. Compare these results with those of Fig. 11.

An Example of Natural Frequencies for a Three-Layer Subsurface

frequencies are indicated in Fig. 15. The natural frequencies having the next strongest influence are those of the second layer and those of the composite of layers one and two. These natural frequencies are also indicated in Fig. 15. There are also higher order natural frequencies excited, but they do not have as large an influence as these first three sets of natural frequencies. These other mode results are consequently not plotted in Fig. 15. It should be pointed out that for dielectric layers, with ξ_i being the spacing between two consecutive ω_{nR} values for layer one ($i = 1$) and layer two ($i = 2$), the spacing between two consecutive ω_{nR} for the composite of the two layers is ξ_c , where

$$\xi_c = \frac{1}{1/\xi_1 + 1/\xi_2} \quad (59)$$

No simple relationship holds for the relative spacings of ω_{nI} for layers one, two, and the composite layer. The relationship expressed by Eq. (59) can, of course, be used to detect which natural frequencies are those of composite layers, rather than individual layers.

$$\begin{array}{l} \epsilon_{r0} = 1, \sigma_0 = 0 \\ \hline \epsilon_{r1} = 9, \sigma_1 = 0, D_1 = 1 \text{ m} \\ \hline \epsilon_{r2} = 25, \sigma_2 = 0, D_2 = 2 \text{ m} \\ \hline \epsilon_{r3} = 9, \sigma_3 = 0 \end{array}$$

Fig. 14. A three-layer problem for which the dominant natural frequencies are desired.

Use of the Natural Frequency Results When Using a Swept Frequency Excitation

One possible excitation of the medium is a swept frequency excitation. By varying the frequency between the real (not complex) frequencies ω_1 and ω_2 , more information can be sometimes discerned about the unknown medium than is possible at a single frequency.^{1,11,12}

As an example of this, consider the case of probing a layered dielectric medium. It is known [Eq. (49)] that there are certain complex natural frequencies which cause true resonance within the layers. For layer ℓ [see Eq. (49)], the real part of the complex natural frequency is

$$\omega_{nR} = \frac{|R_{\ell}^U R_{\ell}^L| + 2n\pi}{2\sqrt{\mu_0 \epsilon_0} D_{\ell} \sqrt{\epsilon_{r\ell} - \sin^2 \theta_i}}. \quad (60)$$

If $|R_{\ell}^U R_{\ell}^L|$ is not much less than one [see the imaginary part of Eq. (49)], then the medium will exhibit partial (not true) resonance at those discrete values of ω_{nR} indicated in Eq. (60). An example of this effect is illustrated in Fig. 16, which depicts swept frequency wave tilt results over a two-layer subsurface. A variation of the wave tilt with real frequency radiation is noted.¹²

By noting the frequency shift between two successive partial resonances (say ω_{1R} and ω_{2R}) excited using a swept frequency excitation, one obtains

$$\omega_{1R} - \omega_{2R} = \frac{\pi}{\sqrt{\mu_0 \epsilon_0} D_{\ell} \sqrt{\epsilon_{r\ell} - \sin^2 \theta_i}}. \quad (61)$$

can then determine unique values of $\epsilon_{r\ell}$ and D_{ℓ} . It is noted that a small amount of loss in region ℓ does not distort this result [see Eq. (52)].

The same results as presented previously on the relative degree of influence of the natural frequencies of a medium with more than one finite thickness layer are still applicable for swept frequency excitation. The layer closest to the surface nominally has the largest influence, with the relative degree of influence of the partial resonant frequency being less the deeper the layer and the more complicated the mode structure.

It should also be noted from the imaginary part of Eq. (49) that if $|R_{\ell}^U R_{\ell}^L| \ll 1$, then it can be difficult using a real frequency excitation to detect the partial resonance. This is because the effect of the true resonance at the complex value of ω_n is then very small for frequencies on the real axis (as ω_n is far removed from the real ω axis). This can also be seen by the successive bounce formulation [Eq. (39)]..

If the regions ℓ and $\ell+1$ are conducting regions with a $\sigma \gg \omega\epsilon$, and region $\ell-1$ is either a dielectric or a conductor, then ω_{nR} becomes a complicated function [see Eq. (50) and (58)] of

$$|R_{\ell}^U R_{\ell}^L|, |R_{\ell}^U R_{\ell}^L| \text{ and } \sigma_{\ell} D_{\ell}^2.$$

If $|R_{\ell}^U R_{\ell}^L|$ and $|R_{\ell}^U R_{\ell}^L|$ do not change signi-

$$\omega_{2R} - \omega_{1R} = \frac{2\pi \ell n |R_{\ell}^U R_{\ell}^L|}{2\mu_0 \sigma_{\ell} D_{\ell}^2}, \quad (62)$$

$$\frac{|R_{\ell}^U R_{\ell}^L|}{\omega_{2R}/\omega_{1R}^{-1}} = \frac{2\pi(2 - \omega_{2R}/\omega_{1R})}{\omega_{2R}/\omega_{1R}^{-1}}, \quad (63)$$

Fig.16a. TM plane waves at grazing incidence ($\theta_i = 90^\circ$ in Fig. 6) to the ground create an electric field which tilts in the direction of propagation. The wave tilt is designed as $W = E_{\text{horizontal}}/E_{\text{vertical}}$, and W is indicative of the electrical properties of the ground.

(b)

$$\epsilon_{r0} = 1, \sigma_0 = 0$$

$$\epsilon_{r2} = 25, \sigma_2 = 10^{-2}$$

but a unique value of $\sigma_{\ell} D_{\ell}^2$ and $|R_{\ell}^U R_{\ell}^L|$ cannot be obtained. This is one advantage of using a pulse excitation (and determining both the natural frequencies ω_{nR} and ω_{nI}) rather than a swept frequency excitation which determines the values of ω_{nR} . For conducting layers, more diagnostic information is contained in ω_{nI} than ω_{nR} . It is noted [see Eq. (50)] that if one uses the values of ω_{nI} , one can obtain a unique value of $\sigma_{\ell} D_{\ell}^2$, and from that value obtain a unique result for $|R_{\ell}^U R_{\ell}^L|$ and $|R_{\ell}^U R_{\ell}^L|$. This would not be possible using a swept frequency excitation and observing ω_{nR} only.

Fig. 16c. The magnitude of the wave tilt varies with frequency and the thickness of the upper layer. For finite thicknesses D_1 , which are less than several skin depth (δ) in layer 1, the partial resonance effect is clearly observed. Note that the frequency width

General Remarks

The preceding development has been specifically applied to planar layered media. The same procedure can also be used with plane waves incident on spherically layered¹³ and cylindrically layered¹⁴ materials where the materials vary only in the r and ρ directions, respectively. The only modifications required are that the characteristic field variations between the layers become Bessel functions, rather than the exponential functions which are characteristic of the planar layer field behavior.

When the interfaces between layers are not planar, certain difficulties arise. Examples of non-planar interface problems could include: (1) surface roughness; (2) dipping interfaces; and (3) anomalies within the layers (see Fig. 17 for illustrations of these phenomena).

If the surface roughness is not a significant portion of a wavelength in the medium, then its effect should not be important. A surface roughness greater than about one-tenth the medium wavelength however, could lead to questionable results. This subject area merits further study. Possible approaches might utilize concepts used previously.^{15,16}

It is known from geometrical optics arguments¹⁷ that nonparallel interfaces can lead to "fundamental difficulties in trying to deduce layer thicknesses and subsurface conductivities." The degree of difficulty for dipping beds with measurements performed in a local region where there is only a small variation of local electrical thickness¹⁸ remains a subject worthy of further investigations

a small fraction ($\leq 10\%$) of the volume in the layer and their electrical character is not dramatically different than the host medium, their effect is minimal and is not significant.¹⁹ If the anomalies are electrically large and have a significantly different electrical character than the host medium, the anomaly can possess its own natural frequencies. One possible way of modeling this problem is to consider spherical anomalies within a host medium. The natural frequencies of the sphere are governed by those frequencies which satisfy $j_n(ka) = 0$, where a is the radius of the sphere, k is the propagation constant in the spherical anomaly, and j_n is the n th order spherical Bessel function. An interesting verification of this fact is the study²⁰ via alternate means of the quasi-static transient response of a conducting permeable sphere. Wait obtained a response composed of a superposition of terms varying with time as $\exp[-n^2\pi^2t/(\mu_0\sigma a^2)]$. The lowest order mode $j_0(ka) = 0$ implies $\sin(ka) = 0$, or $ka = n\pi$. For a conducting sphere, $ka = \sqrt{-j\omega\mu_0\sigma} a$. Hence, $\omega_n = jn^2\pi^2/(\mu_0\sigma a^2)$, and $\exp(j\omega_n t)$ is equivalent to Wait's expression. The problem of anomalies merits additional study.

Other problem areas needing investigation are the effects of a continuous electrical profile (i.e., not discrete homogeneous layers) and the effects of dispersion in the medium. All of the above mentioned possible difficulties are ones that have to be overcome (properly accounted for) in any successful remote probing technique

Fig. 17. Non-planar interface problems can cause difficulties. The degree of influence of these difficulties are items worthy of research.

tunnel, that doesn't occur for measurements performed over a planar surface, is the effect of the resonance of the tunnel. These effects can be properly accounted for. Another problem occurring in tunnels is the applicability of the planar layer interface theory to the geologic structures and interfaces naturally present in tunnels. It is suspected that for techniques that do not attempt to probe to depths greater than about one-tenth of the tunnel diameter, this effect would not be severe. By aligning the source and receiver in a longitudinal sense (rather than azimuthal), it is suspected that the use of the planar layer theory could be successfully extended to even greater depths of penetration. For short pulse lengths, horn antennas would be used for transmission and reception.

Another item worthy of study is the effect of non-plane wave excitations, e.g., a finite beam-width source and receiver. One way of handling this problem might be the number of significant Fresnel zones of interaction.¹⁵

The discussion presented herein has pertained to electromagnetic fields, but the general idea could also be used in acoustics.

Conclusions

Analytical results are given for the dependence of the frequencies naturally excited by a pulse incident on a planar layered medium. These natural frequencies can be determined from time domain experimental data by using Prony's method. Methods for determining the thicknesses and constitutive parameters from the natural frequency results are described.

The relative influence of each layer on the signal reflected from the layered medium is clearly described. The utility of these results for remote probing data obtained either using a pulse excitation or a swept frequency excitation is made evident. The natural frequency concepts presented herein can also be used in acoustical signal processing.

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natural frequency approach for metallic objects. The idea of using natural frequencies for layered media is a natural extension of this idea. The author also thanks J.R. Wait of CIRES for a number of helpful comments and insights regarding this work.

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THEORY RELATING TO REMOTE ELECTROMAGNETIC PROBING OF A NONUNIFORM COAL SEAM

by R. JEFFREY LYTLE

Abstract

The effect of lateral inhomogeneities on electromagnetic remote probing of layered structures is considered. The excitation is taken to be a plane wave incident at angle θ_0 . We determine the variation of the fields (E_y , H_x and H_z), the surface admittance Y_s , and the wave tilt W with angle of incidence θ_0 , with distance from the air-coal interface, with electrical contrast, and with the layer profile (e.g. sinusoidal, step, and slant profiles). Both low frequency and high frequency cases are illustrated and compared with corresponding results with no lateral inhomogeneities. We find that the surface admittance and the fields E_y

and H_x are "good" indicators of the local structure, whereas the field H_z and the wave tilt are not. However, the wave tilt is a good indication of subsurface anomalies.

As the observation point is moved away from the air-coal interface, the "information content" regarding the lateral inhomogeneities rapidly decreases. Thus, measurements performed on the surface, or measurements dependent upon the fields at the surface, e.g. reflection coefficients, are more sensitive to lateral inhomogeneities than measurements performed away from the surface.

Introduction

A problem of great practical importance¹ in geophysically interesting environments (such as coal seams), is the influence of lateral and vertical inhomogeneities on the behavior of electromagnetic fields used for remote probing. Figure 1 depicts examples of problems involving lateral and vertical inhomogeneities.*

In using a remote probing measurement method, it is prudent to choose a measurement method that determines those quantities that contain the most

significant information for the particular application. For example, in some problems, the approximate vertical thickness of layers (such as coal seams)

* In depicting such configurations, we imagine the source and observer (in the air) to be above the air-coal interface. In the situations of greatest interest to the U. S. Bureau of Mines sponsor, however, the figure geometry is inverted, i. e. we are looking at the ceiling of the mine haulage way.² Thus, the reader should bear in mind that the "sub-surface" could actually refer to the conductive medium above the surface in question. (Ed. note by J. R. W.)

Fig. 1. Lateral and vertical inhomogeneities in a coal environment can include rough surface effects, a spatially varying thickness, and anomalies contained within the coal deposit.

is the most significant parameter.² In other applications, a remote probing detection of lateral variations becomes more significant than determining the vertical thickness of individual layers. An example of this would occur in determining the presence of a previously undetected water pocket. In some cases determining both the presence and the thickness of a lateral inhomogeneity is important, e.g. in locating and assessing

the extent of a mineral deposit via remote probing means. Here we attempt to identify which electromagnetic field quantities or ratios contain the most information regarding the vertical and the lateral inhomogeneities. These results help determine the choice of measurement scheme for any specified purpose.

There are numerous analytical results and corresponding measurement procedures for determining the effect of vertical inhomogeneities (planar layered media) on electromagnetic fields. Until recently, however, there have been few results available for the effect of lateral inhomogeneities. Numerical results are presented here for a subsurface condition encompassing approximations of a rough surface and a spatially varying thickness. The approach used is an extension of the analytical procedures previously used by W. J. Hughes³ and Hughes and Wait.⁴

The basic procedure is to express the boundary between regions 1 and 2 (see Fig. 2) in terms of a Fourier series. If the deviations of this boundary from some mean value are electrically small, one can approximate the fields induced by a transverse electric (TE) plane wave incident at angle θ_0 upon the structure (see Fig. 3). The method is linked to the requirement for TE waves of having to simultaneously⁴ match, at the boundary between regions 1 and 2, the tangential and normal components of a magnetic field. The dual problem⁵ of a TM incident wave requires continuity of normal electric flux density and tangential electric field intensity. This prevents matching the equations for normal and tangential electric field intensity separately for a

Fig. 2. The problem to be solved consists of three regions, with the interface between regions 1 and 2 having a lateral spatial variation. The influence upon remote probing of the spatially varying thickness of region 1 is the subject of this report.

Fig. 3. A transverse electric (TE) plane wave is incident upon the air-coal interface. Due to the lateral variation in the thickness of region 1, the electromagnetic fields reflected from the coal/slate medium will also have a lateral variation. How the lateral variations of the electromagnetic fields in region 0 relate to the lateral variations in the spatial thickness of region 1, is the subject of this report.

TM wave (as could be done for normal and tangential magnetic field intensity for a TE wave). Thus, the problem of a

TM incident plane wave requires further study, although we do not expect the qualitative nature of the conclusions to be changed.

Previous considerations of lateral inhomogeneities have included scale model,^{6,7} analytical,^{8,9} and numerical studies.¹⁰⁻¹⁸ Much attention has been given to magneto-telluric applications.^{8,9,11,14,15,17} Other related investigations deal with the effect of lateral inhomogeneities on geomagnetic micro-pulsations,³ on surface wave propagation,^{4,19-24} on geophysical prospecting,^{6,7,12-14} and for basic geophysical investigations.^{10,16}

The analyses have commonly used Fourier series, while the numerical studies have been based primarily upon finite difference^{11,16} or finite element^{13,15} methods, and transmission-line analogies.^{10,14,17} With the numerical approaches, one can consider two orthogonal modes of polarization for the excitation. The numerical results obtained have been based typically upon an assumed plane wave at normal incidence, with displacement currents being negligible in the medium. Both oblique incident plane waves and displacement current effects can be included in these models. Hence these numerical procedures are general enough so that many problems of practical importance can be accurately modeled, if one is willing to pay the expense of computation.

The majority of the analytical, numerical, and scale modeling results to date have been concerned with low frequency phenomena. Few results have been presented for the dependence of the surface admittance^{5,21,25} and the wave

tilt^{5,21,26} on the subsurface profile. Most of the results have been concerned only with the behavior of the individual field components. Measurements that depend upon the ratio of two fields (e.g. the wave tilt and the surface admittance) are much less prone to error than absolute measurements of an individual field quantity. Thus, results for the dependence of these ratio quantities (wave tilt and surface admittance) are desired, not only for low frequencies, but also for high frequencies. It will be shown that, in many instances, the surface admittance is a better indicator of the local subsurface electrical profile than is the wave tilt. The dependence of these results upon the angle of incidence θ_0 for the TE incident plane wave will also be illustrated, as this can have a surprising effect.

The numerical results presented herein for a "rough surface" are for a model of an undulating "rough" surface which is definite, rather than the classi-

cal problem of a rough surface in a statistical sense. Other procedures can be used for the statistically "rough surface" problem.²⁷⁻²⁹

It should be mentioned that the models considered in this paper are frequency domain models, or models limited to a narrow range in frequency. If a pulse excitation or a wide range of frequencies are to be used in an experimental situation, the effects of dispersion (i.e., the frequency variation of the relative dielectric constant ϵ_r and conductivity σ in each geologic medium) have to properly be accounted for.³⁰

For those who are concerned about the depolarization brought about by scattering from a "rough surface," it has been noted by Fung²⁹ that "depolarization is a second order effect for scattering in the plane of incidence, in other directions it is a first order effect." Thus, proper alignment of transmitter and receiver can reduce the influence of depolarization.

Formulation of Problem

A TE plane wave is assumed incident at angle θ_0 upon a two-layered earth, where layers 1 and 2 are separated by an arbitrary profile (see Fig. 3). This arbitrary profile can be described by

$$z(x) = z_0 + \sum_{m=1}^M z_m \sin(m\nu x + \phi),$$

$$\text{where } \nu = 2\pi/L. \quad (1)$$

For this situation, it can be shown (using arguments similar to those previously used by Hughes³ and by Hughes and Wait,⁴

that for a time variation of $\exp(+j\omega t)$, the fields at the surface $z = 0$ are

$$E_y \approx \sum_{n=-M}^M (a_n + b_n) \exp(-jx(n\nu + \lambda)) \quad (2)$$

$$H_x \approx -\frac{1}{\omega\mu_0} \sum_{n=-M}^M (a_n - b_n) k_{n1} \times \exp(-jx(n\nu + \lambda)) \quad (3)$$

$$H_z \approx \frac{1}{\omega\mu_0} \sum_{n=-M}^M (a_n + b_n) \times (n\nu + \lambda) \exp(-jx(n\nu + \lambda)) \quad (4)$$

where for $n = 0$,

$$a_0 = \frac{k_{01} + k_{02}}{2k_{01}} \exp(-jk_{01}z_0)C_0, \quad (5)$$

$$b_0 = \frac{k_{01} - k_{02}}{2k_{01}} \exp(+jk_{01}z_0)C_0, \quad (6)$$

for $n = t$ and $1 \leq t \leq M$,

$$a_{\pm t} = \frac{\pm \frac{1}{2}(k_{02}^2 - k_{01}^2)z_t \exp(-jk_{\pm t}z_0 \pm j\omega)C_0}{(k_{\pm t1} - k_{\pm t2}) - g_{\pm t} \exp(-j2k_{\pm t}z_0)(k_{\pm t1} + k_{\pm t2})}, \quad (7)$$

$$b_{\pm t} = g_{\pm t}a_{\pm t}, \quad (8)$$

$$g_{\pm t} = \frac{k_{\pm t1} + j|\pm t\nu + \lambda|}{k_{\pm t1} - j|\pm t\nu + \lambda|}, \quad (9)$$

and for $n = t$ and $0 \leq t \leq M$,

$$k_{\pm t1} = \sqrt{\omega^2 \mu_0 \epsilon_0 \epsilon_{r1} - j\omega \mu_0 \sigma_1 - (\pm t\nu + \lambda)^2}, \quad (10)$$

$$k_{\pm t2} = \sqrt{\omega^2 \mu_0 \epsilon_0 \epsilon_{r2} - j\omega \mu_0 \sigma_2 - (\pm t\nu + \lambda)^2}. \quad (11)$$

The quantity λ is determined via $\lambda = k_0 \sin \theta_i = \omega \sqrt{\mu_0 \epsilon_0} \sin \theta_i$ where $i = 0, 1$, or 2 . The quantities μ_0 and ϵ_0 are respectively the permeability and the permittivity of free space (region 0). The quantities ϵ_{r1} and ϵ_{r2} are the relative dielectric constants (permittivities) of regions 1 and 2. The procedure used to determine these equations is based upon application of Fourier series. The method includes summations of terms including the factors $k_{\pm t1}$ and $k_{\pm t2}$. The 1 and 2 refer to the regions 1 and 2. The $\pm t$ notation includes terms from $-M \leq t \leq +M$. The lowest order term is $t = 0$, which leads to the terms k_{01} and k_{02} . These are the lowest-order propagation

constants in the regions 1 and 2 respectively.

The restrictions on Eqs. (2) - (4) being valid representations of the fields at $z = 0$ are

$$\omega^2 \mu_0 \epsilon_0 \ll \nu^2, \quad (12)$$

and for $1 \leq m \leq M$

$$k_{\pm t1} z_m \ll 1, \quad (13)$$

$$k_{\pm t2} z_m \ll 1. \quad (14)$$

These conditions can be satisfied for a large number of physically interesting situations as will be illustrated later. One can interpret Eqs. (13) and (14) as saying that the electrical variations of the spatially varying interface, Eq. (1), about its mean, z_0 , are small.

From the fields at $z = 0$ one can determine, at the surface $z = 0$, the wave tilt W and surface admittance Y_s for this TE incident wave. By definition,²⁶ the wave tilt is

$$W = \frac{H_z}{H_x}, \quad (\text{at } z = 0) \quad (15)$$

and the surface admittance is²⁵

$$Y_s = -\frac{H_x}{E_y} \quad (\text{at } z = 0). \quad (16)$$

We will be presenting numerical results for a number of spatially varying subsurface profiles. Examples of profiles of interest are shown in Fig. 4.

When numerical results are presented, insight into the importance of the lateral variations of $z(x)$ can be obtained by comparing these results with those where there are no lateral variations (i.e., when the subsurface is a true planar layered medium). For a two-layered

(a)

(b)

(c)

Fig. 4. Profiles of interest for their effects on the remote probing electromagnetic fields are: (a) a slanted interface, (b) a vertical discontinuity, (c) an undulating interface.

planar subsurface (see Fig. 5) with a finite layer thickness D_1 , the fields on the surface of a planar medium can be designated as E_{yp} , H_{xp} , and H_{zp} . For an incident TE electric field E_0 in region

Fig. 5. A planar layered medium consisting of three homogeneous regions is a first order approximation of the laterally varying profile depicted in Fig. 2.

0, the field components at $z = 0$, according to Wait,²⁴ are

$$E_{yp} = -E_0 \frac{2N_0}{N_0 + Y_1}, \quad (17)$$

$$H_{xp} = E_0 Y_{c0} \cos \theta_i \frac{2Y_1}{N_0 + Y_1} \quad (18)$$

$$H_{zp} = +E_0 Y_{c0} \sin \theta_i \frac{2N_0}{N_0 + Y_1}, \quad (19)$$

where

$$Y_1 = N_1 \frac{N_2 + jN_1 \tan(k_1 D_1 \cos \theta_1)}{N_1 + jN_2 \tan(k_1 D_1 \cos \theta_1)}, \quad (20)$$

and for $i = 0, 1, 2$

$$Y_{ci} = \sqrt{\frac{\epsilon_0 \epsilon_{ri} - j\sigma_i/\omega}{\mu_0}}, \quad (21)$$

$$N_{ci} = Y_{ci} \cos \theta_i, \quad (22)$$

$$\cos \theta_i = \sqrt{1 - \frac{k_0^2}{k_i^2} \sin^2 \theta_i}, \quad (23)$$

$$k_i = \sqrt{\omega^2 \mu_0 \epsilon_0 \epsilon_{ri} - j\omega \mu_0 \sigma_i}. \quad (24)$$

From Eqs. (17) - (24), the planar medium wavenumber W_p is

$$W_p = +\sin \theta_i \frac{Y_{c0}}{Y_1}, \quad (25)$$

and the planar medium surface admittance Y_{sp} is

$$Y_{sp} = +Y_1. \quad (26)$$

Requirements Imposed by the Mathematical Constraints

Equations (12) - (14) impose constraints upon the values of ν and z_m , for $1 \leq m \leq M$. The constraint on $\nu = 2\pi/L$ can be roughly interpreted, from Eq. (12), as

$$L \leq \frac{10^8}{f}, \quad (27)$$

where $\omega = 2\pi f$.

The subsurface profiles $z(x)$ that are of primary interest typically have $z_1 \geq z_m$, for $2 \leq m \leq M$. Thus, the best value of z_m to be used in Eqs. (13) and (14) is usually z_1 . The parameters $k_{\pm t1}$ and $k_{\pm t2}$ in Eqs. (13) and (14) have a conduction-current-dominant behavior when $\omega\mu_0\sigma_i \gg |\omega^2\mu_0\epsilon_0\epsilon_{ri} - (\pm t\nu + \lambda)^2|$, a displacement-current-dominant behavior when $\omega^2\mu_0\epsilon_0\epsilon_{ri} \gg |i\omega\mu_0\sigma_i + (\pm t\nu + \lambda)^2|$, and a periodic-function-dominant behavior when $|\pm t\nu + \lambda| \gg |\omega^2\mu_0\epsilon_0\epsilon_{ri} - i\omega\mu_0\sigma_i|$. In these inequalities, and in the relations that immediately follow, $i = 1$ or 2 .

The conduction-current-dominant-behavior result in Eqs. (13) and (14) is

$$z_1 \leq \frac{\delta_i}{10}, \quad (28)$$

where δ_i is the skin depth in medium i . This equation can also be written as

$$z_1 \leq \frac{1}{10\sqrt{\pi\mu_0\sigma_i f}}. \quad (29)$$

This constraint on z_1 depends upon σ_i and f . Lines indicating the maximum value of z_1 for various values of σ_i and f are indicated in Fig. 6.

The displacement-current-dominant-behavior result in Eqs. (13) and (14) is

$$z_1 \leq \frac{1}{10} \frac{\lambda_i}{2\pi}, \quad (30)$$

where λ_i is the wavelength in medium i . By definition, $\lambda_i = 3 \times 10^8 / (\sqrt{\epsilon_{ri}} f)$. A value of $\epsilon_{ri} = 25$ is chosen to represent nominal earth media. With this choice, then

$$z_1 \leq \frac{10^6}{f} \quad (31)$$

is the constraint imposed when displacement currents are dominant. In Fig. 6 the line $M = 1$ indicates the maximum value of z_1 allowed by this constraint.

The maximum contribution to $k_{\pm ti}$ of the periodic function component occurs when $\pm t = M$. When the term $|\pm t\nu + \lambda|$ is dominant in $k_{\pm ti}$, then the constraint on z_1 becomes

$$z_1 \leq \frac{1}{10} \frac{1}{M\nu}. \quad (32)$$

Using the fact $\nu = 2\pi/L$ and the constraint of Eq. (27), one obtains

$$z_1 \leq \frac{10^6}{fM}. \quad (33)$$

Fig. 6. The constraints that must be satisfied to make the mathematical representations of the electromagnetic fields valid. The requirements are that, at frequency f , the permissible values of z_1 fall to the left and below the governing constraint lines. The constraint lines are conduction current constraint (dependent upon the conductivities in regions 1 and 2), the periodic expansion constraint (dependent upon the number of terms M required to represent the boundary between regions 1 and 2), and the displacement current constraint (equivalent to the periodic expansion constraint for $M = 1$).

Lines indicating the maximum value of z_1 allowed by this constraint are illustrated in Fig. 6. M is the maximum number of terms used in the representation of the surface $z(x)$, see Eq. (1). For $M = 1$, i.e., a pure sinusoidal boundary, Eq. (33) is equivalent to the high frequency constraint, Eq. (31). It is evident that the more terms that are required to represent $z(x)$, the smaller the value of z_1 has to be.

All the constraint equations are indicated in Fig. 6, thus knowing M , σ_1 , σ_2 , and f , one can determine graphically the maximum value of z_1 , and all the other z_m (for $2 \leq m \leq M$). The maximum value of L permitted is determined from Eq. (27). These criteria are the only ones that have to be satisfied for the results in Eqs. (2) - (11) to be valid representations of the fields.

Variation of the Fields with Height

The effect of the lateral variation is generally most dramatic,⁷ in a remote probing sense, on the surface $z = 0$. That is, the influence of the subsurface lateral variation generally has a larger effect on the total field at $z = 0$ than it does at locations above the surface ($z < 0$).

This can be seen from the equations for the fields above the ground. In region 0 (see Fig. 3), the fields due to an incident TE plane wave of magnitude E_0 are

$$E_y \approx -E_0 \exp(+jk_0 x \sin \theta_i - jk_0 z \cos \theta_i) - R^E E_0 \exp(+jk_0 x \sin \theta_i + jk_0 z \cos \theta_i) + \left(\sum_{n=-M}^{-1} + \sum_{n=1}^M \right) \times \left[E_0 \alpha_n \exp(+|n\nu|z - jn\nu x - j\lambda x) \right], \quad (34)$$

$$H_x \approx + \frac{E_0 k_0 \cos \theta_i}{\omega \mu_0} \exp(+jk_0 x \sin \theta_i - jk_0 z \cos \theta_i) - R^E \frac{E_0 k_0 \cos \theta_i}{\omega \mu_0} \times \exp(+jk_0 x \sin \theta_i + jk_0 z \cos \theta_i) \times \frac{-jE_0}{\omega \mu_0} \left(\sum_{n=-M}^{-1} + \sum_{n=1}^M \right) \times \left[\alpha_n |n\nu| \exp(+|n\nu|z - jn\nu x - j\lambda x) \right], \quad (35)$$

and

$$H_z \approx + \frac{E_0 k_0 \sin \theta_i}{\omega \mu_0} \exp(+jk_0 x \sin \theta_i$$

$$\times \exp(+jk_0 x \sin \theta_i + jk_0 z \cos \theta_i) + \left(\sum_{n=-M}^{-1} + \sum_{n=1}^M \right) \left[E_0 \frac{\alpha_n (n\nu + \lambda)}{\omega \mu_0} \times \exp(+|n\nu|z - jn\nu x - j\lambda x) \right], \quad (36)$$

where

$$E_0 \alpha_n = a_n + b_n, \quad (37)$$

and

$$R^E = -1 - \frac{a_0}{E_0} \times \left(1 + \frac{k_{01} - k_{02}}{k_{01} + k_{02}} \exp(+j2k_{01}z_0) \right). \quad (38)$$

As the observation distance below the air-coal interface is increased, z is decreased (i. e., it becomes more negative as the positive z axis points into the coal). Thus, as the observation height is increased, the term $\exp(+|n\nu|z)$ becomes smaller, and the effect of the lateral variation influence, the third term in Eqs. (34) - (36), becomes smaller relative to the planar boundary terms, the first and second terms in Eqs. (34) - (36). Thus measurements performed on the surface $z = 0$ or measurements depending upon the field at the surface $z = 0$, e. g. reflection coefficients,² will contain more diagnostic information concerning the subsurface lateral profile than will measurements performed above the surface $z = 0$.

There are two special heights that should be noted. When

then only the lateral variation terms are important for E_y and H_z , see Eqs. (34) - (36). When

$$\exp(-jk_0 z \cos \theta_0) - R^E \exp(+jk_0 z \cos \theta_0) = 0, \quad (40)$$

then only the lateral variation terms are important for H_x .

It is difficult in general to predict those heights z above the air-coal surface that satisfy Eqs. (39) or (40) for a layered ground such as in Fig. 3. Some results are available for nonlayered ground,³¹ but the layered ground problem of Fig. 3 is more complex. Due to the general complexity of the results, it is not thought that measurements performed at those nonzero heights z satisfying Eqs. (39) or (40) would generally be that helpful. This is especially true at higher frequencies when, for example, horn antennas are used as transmitters and receivers. The directivity of high frequency antennas can have a large influence such that the direct wave, the first term in Eqs. (34) - (36), is negligible compared to the other terms. This is just another way of saying that in actual experimental situations, the directivity of the transmitting and receiving antennas has to be properly accounted for. These effects can be important, and thus it is not useful to belabor the fine points of the particular situation described by Eqs. (39) and (40).

A natural question is how rapidly the lateral inhomogeneity information present in Eqs. (34) - (36) decays with height. The height variation goes as $\exp(+|n\nu|z)$, or as $\exp(+|n|2\pi z/L)$. Using Eq. (27) and choosing the maximum value of L permissible, $L = 10^8/f$, the exponential term becomes $\exp(+|n|6\pi z/\lambda_0)$, where λ_0 is the wavelength in air at frequency f . The lowest order terms ($n = \pm 1$) would then decay with height as $\exp(+6\pi z/\lambda_0)$. The contribution of the lateral information terms at height $z = -\lambda_0/10$ would then be $\exp(-6\pi/10)$, or 0.15 relative to their contribution at the surface (height $z = 0$). The higher order terms $|n| > 1$ would decay even more rapidly. It is thus seen that the lateral information in the fields decays quite rapidly with height above the surface.

The rate of decay with height of the first order ($|n| = 1$) lateral information can be said to have a "skin depth" of at best $\lambda_0/6\pi$. The higher order contribution ($|n| > 1$) would have an even smaller "skin depth." Thus, measurements for determining the presence and effect of lateral variations should monitor the fields within a very small fraction of a wavelength of the surface. If the presence of lateral variations is unimportant, compared to the effect of vertical variations, the lateral variation information present in the signal can be "masked" by monitoring the fields at distances above the surface of $|z| \geq \lambda_0/4$.

Numerical Studies

$\omega\epsilon \gg \sigma$). The results have been calculated at particular frequencies, $f = 10^3$ Hz for the low frequency evaluations and $f = 10^9$ Hz for the high frequency evaluations. Specific dimensions (both laterally and vertically) have been chosen in the numerical studies. By the laws of electromagnetic scale modeling,⁶ the results calculated can be considered to be equivalent to results at other frequencies

$f' = sf$, as long as conduction currents (or displacement currents) remain dominant and the spatial dimensions d in the problems for $f = 10^3$ Hz (or 10^9 Hz) are scaled to become $d' = d/s$.

In all cases the parameters used in the numerical calculations have been chosen so that they satisfy the mathematical constraints, Eqs. (27) - (33) and Fig. 6.

Low Frequency Results

Three spatial profiles were chosen for study for the low frequency model. The models considered were a dipping profile, a step profile, and a sinusoidal profile, see Fig. 7. The dependence of the above-surface fields upon these profiles, upon the depth z_0 to the profile (see Fig. 7), upon the angle of incidence θ_0 of the incident TE plane wave, and upon the electrical contrast (σ_1/σ_2) of layers 1 and 2, has been studied. Depths z_0 of 0 and 50 m were chosen for the dipping and step profiles, and depths z_0 of 20 and 70 m were chosen for the sinusoidal profile. Angles of incidence of 0° (normal incidence) and 90° (grazing incidence) were evaluated. Electrical contrasts of $\sigma_1 = 10^{-4}$ mho/m, $\sigma_2 = 10^{-2}$ mho/m and $\sigma_1 = 10^{-2}$ mho/m, $\sigma_2 = 10^{-4}$ mho/m were evaluated. Due to the volumes of data generated, only an overview and representative samples of the data are presented below.

The calculations were based upon the first 99 terms in a Fourier expansion of the spatial profile separating regions 1 and 2. The dipping bed and sinusoidal

phenomenon, the true "step profile" of Fig. 7 was approximated by the "step profile" indicated in Fig. 8. Fortunately, the deviation of the approximation from the true "step profile" had no great influence on the field calculations.

In all cases, the electromagnetic fields were calculated. In general E_y and H_x varied smoothly, with the vertical inhomogeneity (rather than the lateral inhomogeneity) being the dominant information present in the spatial variation (i.e., the variation with x). The H_z component typically had a rapid and dramatic variation in the near vicinity of the lateral inhomogeneity. Thus, H_z seems to be the better indication of lateral anomalies whereas E_y and H_x seem to be better indicators of vertical anomalies. These general observations should not be construed to say that H_z information (or E_y and H_x information) cannot be used to determine vertical (lateral) inhomogeneity results. They, in fact, can. This is especially true for H_z (or the wave tilt H/H_z) for grazing incidence ($\theta_0 = 90^\circ$)

(a)

(b)

(c)

Fig. 7. The three profiles used in the low frequency studies (not to scale) include: (a) a dipping profile, (b) a step profile, (c) a sinusoidal profile. In all cases the functions were chosen as even functions about $x = 0$, and they were chosen with a spatial period, L , of 10,000 m. The portions of the profiles from 0 – 5000 m

Fig. 8. The Fourier expansion approximation (99 terms) substituted for the true step of Fig. 7b in the analysis.

a wider range of x than does H_z . This is because the effect of the lateral inhomogeneity in the spatial direction x is damp out much quicker for E_y and H_x than it is for H_z .

We do not present graphical results for E_y , H_x and H_z in this report, in order to condense the volume of results. We do present results for the ratios of these quantities as these are physically important. Reflection coefficient measurements² depend upon H_x/E_y , which is known as the surface admittance Y_s . Wave tilt measurements depend upon H_z/H_x , which is known as the wave tilt W . It is desirable to measure these quantities because they depend, not upon absolute field measurements, but upon the ratio of field measurements. Thus,

can be a difficult and crucial problem), is eliminated from the problem. If the reader is interested in the field behavior rather than the ratio behavior, please contact the author.

Results are given below for the ratio of the actual surface admittance $Y_s = H_x/E_y$ to the planar surface admittance Y_p evaluated at the same position x with depth z determined by Eq. (1). The planar surface admittance Y_p is concerned only with the vertical inhomogeneity at local position x and contains no influence of the actual lateral inhomogeneity. If

the actual surface admittance Y_s is not significantly influenced by the lateral inhomogeneities, then Y_s should be approximately equal to Y_p .

Results for $|Y_s/Y_p|$ and $\angle Y_s/Y_p$ are shown in Figs. 9 and 10 for electrical profiles of $\sigma_1 = 10^{-4}$ mho/m and $\sigma_2 = 10^{-2}$ mho/m, and in Figs. 11 and 12, for electrical profiles of $\sigma_1 = 10^{-2}$ mho/m and $\sigma_2 = 10^{-4}$ mho/m. Normal incidence ($\theta_0 = 0^\circ$) is assumed. When conduction currents are dominant, however, the results for Y_s/Y_p are independent of the angle of incidence. The depths z_0 (see

Fig. 9 A comparison of the magnitude of Y_s and Y_p for $\theta_0 = 0^\circ$ with $\sigma_1 = 10^{-4}$ mho/m

Fig. 10. A comparison of the phase angle of Y_s and Y_p for $\theta_0 = 0^\circ$ with $\sigma_1 = 10^{-4}$ mho/m and $\sigma_2 = 10^{-2}$ mho/m indicates almost no difference. See Fig. 7 for profile dimensions.

Fig. 7), are 50 m for the dipping and step profiles and $z_0 = 70$ m for the sinusoidal profile. It is seen from Figs. 9-12 that $Y_s \approx Y_p$ except very near the lateral inhomogeneity, and even then, the departure of Y_s from Y_p is not dramatic. The electrical profile of $\sigma_1 = 10^{-2}$ mho/m and $\sigma_2 = 10^{-4}$ mho/m seems to have a larger departure of Y_s from Y_p than the complement, although the departure is small in both cases.

for $|Y_s/Y_p|$ with $z_0 = 0$ for the dipping and slant profiles and $z_0 = 20$ m for the sinusoidal profile, are given in Figs. 13 and 14 for the two different electrical profiles. The profile of $\sigma_1 = 10^{-4}$ mho/m and $\sigma_2 = 10^{-2}$ mho/m, shows little dependence of $|Y_s/Y_p|$ on depth z_0 (compare Figs. 9 and 13). The profile of $\sigma_1 = 10^{-2}$ mho/m and $\sigma_2 = 10^{-4}$ mho/m shows some dependence of $|Y_s/Y_p|$ on depth z_0 (compare Figs. 11 and 14). The closer the lateral anomaly is to the surface, the

Comparisons of the actual wave tilt $W = H_z/H_x$ to the planar wave tilt W_p evaluated at the same position x , with depth z determined via Eq. (1), were also obtained. Comparisons of $|W/W_p|$ for $z_0 = 50$ m (for the dipping and step profiles) and $z_0 = 70$ m (for the sinusoidal profile) are given in Figs. 15 and 16 for the two different electrical profiles. An angle of incidence $\theta_0 = 90^\circ$ is assumed, so that actual wave tilt W tends towards the planar wave tilt W_p far from any lateral inhomogeneity. Thus, $|W/W_p|$

becomes one for the dipping and step inhomogeneities when x is far removed from the lateral inhomogeneity. The influence of the electrical profile of $\sigma_1 = 10^{-2}$ mho/m and $\sigma_2 = 10^{-4}$ mho/m (Fig. 15), is larger in magnitude and lasts farther in a lateral extent (x) than does the influence of the electrical profile of $\sigma_1 = 10^{-4}$ mho/m and $\sigma_2 = 10^{-2}$ mho/m (Fig. 16).

The closer the lateral inhomogeneity is to the surface, the larger its effect is on the wave tilt (compare Fig. 16 with Fig. 17).

It should be noted from Figs. 15-17 that the sharp electrical discontinuity (the step profile) has a more dramatic effect than the gradual electrical discontinuity (the dipping profile). The sinusoidal profile, which is constantly changing in a smooth manner, seems to lie

intermediately between the sharp and smooth profiles in its effect upon $|W/W_p|$. A wave tilt measurement over a sinusoidal profile is equivalent to the local planar wave tilt measurement only over the peaks and troughs of the sinusoid (compare Figs. 15-17 to Fig. 7c).

Fig. 13. A comparison of the magnitude of Y_s and Y_p for $\theta_0 = 0^\circ$ with $\sigma_1 = 10^{-4} \text{ mho/m}$ and $\sigma_2 = 10^{-2} \text{ mho/m}$, indicates almost no difference. See Fig. 7 for profile dimensions.

Fig. 14. A comparison of the magnitude of Y_S and Y_P for $\theta_0 = 0^\circ$ with $\sigma_1 = 10^{-2}$ mho/m and $\sigma_2 = 10^{-4}$ mho/m, indicates some small difference with the difference indicative of the subsurface profile. See Fig. 7 for profile dimensions.

Fig. 15. A comparison of the magnitude of W and W_p for $\theta_0 = 90^\circ$ with $\sigma_1 = 10^{-2}$ mho/m and $\sigma_2 = 10^{-4}$ mho/m, indicates a trend towards equality far from the anomaly and a peak about the anomaly. See Fig. 7 for profile dimensions.

Fig. 16. A comparison of the magnitude of W and W_p for $\theta_0 = 90^\circ$ with $\sigma_1 = 10^{-4}$ mho/m and $\sigma_2 = 10^{-2}$ mho/m, indicates a trend towards equality far from the anomaly and a peak about the anomaly. See Fig. 7 for profile dimensions.

ig. 17. A comparison of the magnitude of W and W_p for $\theta_0 = 90^\circ$ with $\sigma_1 = 10^{-4}$ mho/m and $\sigma_2 = 10^{-2}$ mho/m, indicates a trend towards equality far from the anomaly and a peak about the anomaly. See Fig. 7 for profile dimensions.

High Frequency Results

A frequency of $f = 10^9$ Hz was chosen for study purposes. This results in a free space wavelength of $\lambda_0 = 300$ mm.

number of terms representing $z(x)$ via Eq. (1), only a sinusoidal profile (see Fig. 18) has been considered for the high

Depths z_0 (see Fig. 18) of 2 mm and 12 mm were considered. Angles of incidence $\theta_i = 0^\circ$ and 90° were considered.

Figure 19 illustrates the magnitude of the actual surface admittance Y_s and the planar surface admittance Y_p for $\theta_0 = 0^\circ$ and $z_0 = 2$ mm. The results are not significantly different. Another way of showing this result is given in Fig. 20, which shows both $|Y_s/Y_p|$ and $\angle Y_s/Y_p$. It is seen that $Y_s \approx Y_p$ for normal incidence.

For grazing incidence ($\theta_0 = 90^\circ$), the quantities $|Y_s|$ and $|Y_p|$ are approximately equal in magnitude, but are shifted in location from each other (see Fig. 21). This is due to the fact that from ray-optic considerations, the actual wave tilt response is not due to the field interacting with the surface directly beneath it, but is due to the field interacting with the

surface at a smaller value of x (i. e., in the direction from which the wave is incident, see Fig. 22). An improvement in the planar surface admittance Y_p seen at the surface for high frequencies should include this shift in sensing position on the profile between regions 1 and 2. Even without this correction, from Fig. 23 it is seen that the magnitude and phase of Y_s are very well approximated by Y_p . From Fig. 24, it is seen that $|Y_s| \approx |Y_p|$ for both $z_0 = 2$ mm and $z_0 = 12$ mm.

A similar procedure can be used for the wave tilt. The phase angle of the actual wave tilt W and the planar wave tilt W_p are indicated in Fig. 25. A slight shift in location, again attributable to the effect noted in Fig. 22, is noted when comparing $\angle W$ and $\angle W_p$. Future studies should include the effect of the actual region sensed (Fig. 22).

Fig. 18. The profile used in the high frequency study. The sinusoidal function is an even function about $x = 0$. The portion of the profile between $x = 0$ and $x = 50$ mm is shown above. Note the disparity between the vertical and hori-

Fig. 19. A comparison of the magnitude of Y_s and Y_p for $\theta_0 = 0^\circ$ over a sinusoidal profile (Fig. 18) with $z_0 = 2 \text{ mm}$, indicates only a slight difference. The electrical profile is $\epsilon_{r1} = 9$ and $\epsilon_{r2} = 25$.

Fig. 20. Plots of $|Y_s/Y_p|$ and $\angle Y_s/Y_p$ versus x , indicate that for $\theta_0 = 0^\circ$, $Y_s \approx Y_p$ over a sinusoidal profile (Fig. 18) with $z_0 = 2 \text{ mm}$. The electrical profile is $\epsilon_{r1} = 9$ and $\epsilon_{r2} = 25$.

Fig. 21. A comparison of the magnitude of Y_s and Y_p for $\theta_0 = 90^\circ$ over a sinusoidal profile (Fig. 18) with $z_0 = 2 \text{ mm}$, indicates only a slight difference in amplitude, but a decided difference in the phase of $|Y_s|$ and $|Y_p|$. The electrical profile is $\epsilon_{r1} = 9$ and $\epsilon_{r2} = 25$.

Fig. 22. The above-surface field at position x is due to the interaction of the incident field and a field that leaks into region 1, is scattered at the boundary of regions 1 and 2, and passes back into region 0. Thus measurements at position x

Fig. 23. Even with the discrepancy in the actual sensed region of the profile between regions 1 and 2, the magnitudes and phase angles of Y_s and Y_p are almost equal for $\theta_0 = 90^\circ$ over a sinusoidal profile (Fig. 18) with $z_0 = 2 \text{ mm}$. The electrical profile is $\epsilon_{r1} = 9$ and $\epsilon_{r2} = 25$.

Fig. 24. For a dielectric subsurface with $\epsilon_{r1} = 9$ and $\epsilon_{r2} = 25$, the variation of $|Y_s/Y_p|$ with x for $z_0 = 2 \text{ mm}$, is almost the same as for $z_0 = 12 \text{ mm}$ with a sinusoidal profile (Fig. 18). The major difference is in the phase of $|Y_s/Y_p|$, which indicates the region sensed by the grazing incidence ($\theta_0 = 90^\circ$) plane wave (Fig. 22).

Fig. 25. A comparison of the phase angle of the actual wave tilt W and the planar wave tilt W_p for $\theta_0 = 90^\circ$ over a sinusoidal profile (Fig. 18) with $z_0 = 2$ mm, indicates only a slight difference in magnitude and a phase shift difference due to the region sensed by the plane wave (Fig. 22). The electrical profile is $\epsilon_{r1} = 9$ and $\epsilon_{r2} = 25$.

Conclusions

Results presented in this report indicate:

1. Surface admittance (and thus reflection coefficient measurements) and the field components E_y and H_x are good indicators of the subsurface vertical inhomogeneity.
2. Wave tilt and the field component H_z are good indicators of the subsurface lateral inhomogeneities that are near the surface.
3. The maximum "information con-

"information content" regarding lateral inhomogeneities rapidly decays with height above the air-coal interface. Measurements sampling fields above the surface would be good indicators of the vertical inhomogeneities.

4. Step, dipping, and sinusoidal subsurface profiles (for the parameters calculated) have little effect upon the surface admittance "tracking" the local subsurface

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