Quantum Zeno Effect with unitary description of measurements

by

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Quantum Zeno Effect with unitary description of measurements

Thesis directed by Prof. Graeme Smith

The Quantum Zeno Effect is the phenomenon where continuous measurements freeze the evolution of a quantum system. We explore this effect using unitary descriptions of measurements. By describing the measurements with CNOT gates, we can produce the Quantum Zeno Effect. Our results show that we need an infinite number of ancillary qubits, thus infinite spaces are required. We also look into the case where we only have a finite number of ancillary qubits and reuse each of the ancillary qubits. In this case, our results show that reusing the ancillary qubits more than once has the same impact as reusing them once. Therefore, to produce the Quantum Zeno Effect, we need an infinite number of ancillary qubits that have not been previously used.

Dedication

This thesis is dedicated to my family.

Acknowledgements

I would like to thank Professor Graeme Smith for being my advisor and giving me the opportunity to work on this project. Also, I would like to thank everyone in Professor Smith's group for their supportive discussions.

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Introduction

1.1 Background

In Quantum Mechanics, measurement is a process by which we can obtain information from a quantum system. The postulate about quantum measurement is saying that we can describe the process of measurement by some observables. After measurement, the wave function "collapses" into an eigenstate of the observable being measured, and the probability of getting its eigenvalue corresponds to the Born rule. The measurement itself can impact a quantum system, and this creates a phenomenon, which is known as Quantum Zeno Effect. The Quantum Zeno Effect (QZE) is the phenomenon that involves the evolution of a quantum system being frozen by repeated measurements. In 1977, Misra and Sudarshan noticed that, for a decaying atom, the survival probability of the atom in its undecayed state approaches one in the limit of continuous measurements. That is, the atom does not decay due to being frequently measured. Misra and Sudarshan named this phenomenon "Zeno's paradox in quantum theory", analogous to the Zeno's Paradoxes purposed by Zeno of Elea. [1] This "paradox" then was exhibited in the experiment proposed by Itano et al. with a multiple-level atom.[2] While the Quantum Zeno Effect is rarely considered a "paradox" now, there remains some discussion space for the interpretation of measurement, and we can attempt to look into this phenomenon again with unitary descriptions of measurements.

1.2 Zeno's paradox

In the fifth century B.C., Zeno proposed paradoxes to ask whether we can infinitely divide a magnitude or assume a magnitude is composed of parts. One of his paradoxes concerns "stopping" the motion of an arrow. The scenario is as follows: suppose we shoot an arrow, then at every instant, the arrow is in a fixed position. If time comprises instants, then the arrow will never move because, for every instant, the arrow is in a fixed position. [1] While this paradox had been solved by concepts in calculus and we know that the arrow's motion is position, a similar scenario will happen in Quantum Mechanics, which then be given the name the "Quantum Zeno paradox".

1.3 Measurement

According to the postulate of quantum measurement, after we measure a quantum state ψ with an observable M, we will get the eigenvalue of M, and the state becomes the eigenvector of M. In other words, we can describe the measurements by a set of operators M_i. Let ρ be the density operator of our quantum state. After we measure the state and get the result i, the state becomes

$$\frac{M_i \rho M_i^\dagger}{\mathrm{tr}\left(M_i^\dagger M_i \rho\right)}, \tag{1.1}$$

with probability [5]

$$P_i = \mathrm{tr}\left(M_i^\dagger M_i \rho\right). \tag{1.2}$$

If we had a qubit q_0 with density matrix ρ:

Figure 1.1: A qubit that we want to know about its state.

After we measure q_0, its state becomes

$$\rho \to \rho' = \alpha \left|0\right\rangle \left\langle 0\right| + \beta \left|1\right\rangle \left\langle 1\right| \tag{1.3}$$

That is, we get $\left|0\right\rangle$ with probability α and $\left|1\right\rangle$ with probability β.

1.3.1 Example 1

Consider we have the following circuit, where q_0 is a plus state $\left|+\right\rangle$ and q_1 is a zero state $\left|0\right\rangle$. Let ρ be the density operator of this circuit.

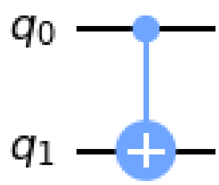

Figure 1.2: This figure is showing a quantum circuit where q_0 is a plus state $\left|+\right\rangle$ and q_1 is a zero state $\left|0\right\rangle$. These two states are entangled by a CNOT gate.

After we measure the circuit, by (1.1), if q_0 and q_1 are both measured to be $\left|0\right\rangle$, the state becomes

$$\frac{M_0 \rho M_0^\dagger}{\operatorname{tr}\left(M_0^\dagger M_0 \rho\right)}, \tag{1.4}$$

where $M_0 = \left|00\right\rangle \left\langle 00\right|$, with $P_0 = \frac{1}{2}$. If q_0 and q_1 are both measured to be $\left|1\right\rangle$, the state becomes

$$\frac{M_1 \rho M_1^\dagger}{\operatorname{tr}\left(M_1^\dagger M_1 \rho\right)}, \tag{1.5}$$

where $M_1 = \left|11\right\rangle \left\langle 11\right|$, with $P_1 = \frac{1}{2}$

So after the measurement, we have

$$\rho \to \rho' = \frac{1}{2}\left|00\right\rangle\left\langle 00\right| + \frac{1}{2}\left|11\right\rangle\left\langle 11\right|, \tag{1.6}$$

which is a classical state.

1.4 Unitary measurement

Furthermore, we can consider that measurements are just some interactions between the system and the "measurement devices", so we are able to describe the measurements by unitary operators. Indeed, we will later show that the unitary operator is the CNOT gate. Assume that we have a quantum system, q_0. To obtain information about this system, we add a "measurement device", q_1, to it. But since we are treating both our system q_0 and "measurement device" (q_1) as quantum systems, applying the "measurement device" is forming a larger quantum system where the measurement device entangles with our system. So, if we continue connecting more of these "measurement devices", all we are doing is forming a larger and larger quantum system that includes all the "measurement devices", and there should have some interaction Hamiltonian describing the "measurement devices" and q_0.

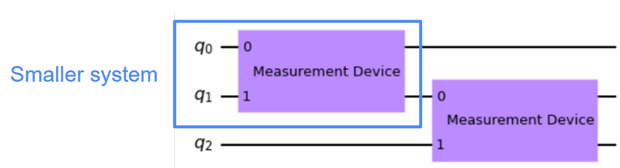

Figure 1.3: This figure is showing a system (q_0, q_1, q_2) that contains a smaller system (q_0, q_1). So the measurement device entangles q_0 and q_1 is contained in the system of (q_0, q_1, q_2).

Now consider only the smaller system in Figure 1.3, and let $|\psi\rangle$ be the wavefunction of q_0, and q_1 be initialized to a zero state $|0\rangle$ and will record the result of measurement of

q_0. Therefore, for q_0 (the system we interested in) and q_1 (the "measurement device"), we have the wave function

$$|\psi_{01}\rangle = M_0 |\psi\rangle |0\rangle + M_1 |\psi\rangle |1\rangle , \tag{1.7}$$

where $M_0 = |0\rangle \langle 0|$, $M_1 = |1\rangle \langle 1|$. Note that $|\psi_{01}\rangle$ can be simplified further so we have

$$|\psi_{01}\rangle = \alpha |0\rangle |0\rangle + \beta |0\rangle |0\rangle , \tag{1.8}$$

where α and β are some constants and satisfy $\alpha^2 + \beta^2 = 1$. Notice that the unitary which satisfies these conditions is the CNOT gate. Therefore, (1.8) is essentially the wave function of the following circuit,

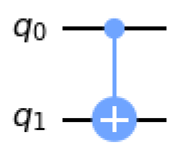

Figure 1.4: This figure shows the smaller system shown in Figure 1.2. q_0 and q_1 are entangled by a measurement device, which is essentially a CNOT gate.

where $|q_0\rangle = \alpha |0\rangle + \beta |1\rangle$ and $|q_1\rangle = |0\rangle$. To illustrate that the CNOT gate can be used as a measurement, we can consider the following example.

1.4.1 Example 2

Let us continue to use the circuit in Example 1 (Figure 1.2), but now we add a "measurement device" (a CNOT gate, as we proposed in the previous section) to the circuit. Note that q_0 is a plus state $|+\rangle$, q_1 is a zero state $|0\rangle$. Then let q_2 be a zero state $|0\rangle$.

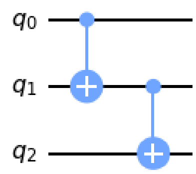

Figure 1.5: This figure is showing a circuit where q_0 is a plus state $|+\rangle$, q_1 and q_2 are zero states $|0\rangle$. There are two CNOT gates entangle (q_0, q_1) and (q_1, q_2).

We want to show that after we add q_2 to measure the circuit and then trace out q_2, we will get the classical state that is the same as the one we get after we measure the circuit in Figure 1.2 directly (1.6). For the circuit in Figure 1.5, its wave function is

$$|\psi\rangle = (I \otimes CNOT)(CNOT \otimes I) |+\rangle |0\rangle |0\rangle . \tag{1.9}$$

Tracing out q_2, we will get

$$\mathrm{Tr}_{q_2}(|\psi\rangle \langle\psi|) = \frac{1}{2} |00\rangle \langle 00| + \frac{1}{2} |11\rangle \langle 11| , \tag{1.10}$$

and we retrieve (1.6).

Chapter 2

Quantum Zeno Effect

2.1 Quantum Zeno Effect with projection opeartors

Recall that the Quantum Zeno Effect is saying that, a quantum system stops its evolution by being measured frequently. Suppose now we make the measurements with a projector, let $|0\rangle$ be our quantum system, and consider we have a Hamiltonian $H = \alpha\frac{\Delta t}{n}$, where α is a constant, n is the number of measurements, ΔT is the time interval between measurements. The system experiences unitary evolution, satisfying the Schrödinger equation. We can define the time evolution operator with the Hamiltonian defined above, so we have

$$U_{evolve} = \begin{pmatrix} \cos\frac{\alpha\Delta t}{n} & -\sin\frac{\alpha\Delta t}{n} \\ \sin\frac{\alpha\Delta t}{n} & \cos\frac{\alpha\Delta t}{n} \end{pmatrix}, \tag{2.1}$$

which is essentially a rotation matrix, and it changes our system from state $|0\rangle$ to state $|1\rangle$. Then we can use a projector $|0\rangle\langle 0|$ to measure the system:

$$|0\rangle\langle 0| U_{evolve} |0\rangle . \tag{2.2}$$

To produce the Quantum Zeno Effect, we need to measure the system repeatedly, so let us define U as

$$U = |0\rangle\langle 0| U_{evolve}, \tag{2.3}$$

then raise it to the power of n and apply it to our system:

$$|\psi_{sys}\rangle = U^n |0\rangle . \tag{2.4}$$

$$\begin{aligned} Prob &= |\langle 0|\psi_{sys}\rangle|^2 \\ &= \cos^{2n}\tfrac{\alpha T}{n} \end{aligned} \tag{2.5}$$

(2.5) approaches 1 as n goes to infinity. That is, the probability of getting back our original quantum state $|0\rangle$ approaches to 1 (i.e. the system does not evolve by the evolution operators) if we increase the number of measurements to infinity, thus showing the Zeno effect.

2.2 Measurement in Quantum Zeno Effect

The previous example assumes instantaneous "collapses" of wave function [4]. Look into our example again, since $|0\rangle$ is a pure state, let ρ be our density matrix so $\rho = |0\rangle\langle 0|$. It will become

$$U\rho U^\dagger \to |0\rangle\langle 0|\, U\rho U^\dagger\, |0\rangle\langle 0| \tag{2.6}$$

after we apply a projection operator to measure it. Note that if we let $Q = (|0\rangle\langle 0|\, U\, |0\rangle\langle 0|)^n$, that we apply measurements for n times, then the survival probability is

$$\mathrm{Tr}\big(Q\rho Q^\dagger\big) = \cos^{2n}\frac{\alpha T}{n} \tag{2.7}$$

and we retrieve (2.5) [3, 4]. This approach to deriving the Zeno effect is rather instructive, and based on the assumption that we have measurements that happen instantaneously. We would like to know if it is possible to produce the Zeno effect without the assumption of wave function collapse. To accomplish this, we could employ the unitary measurements described in Chapter 1.

Chapter 3

Unitary Quantum Zeno Effect

3.1 Quantum Zeno Effect with unitary measurements

Instead of interpreting in the way how the Zeno effect is caused by a lot of measurements instantaneously acting on a quantum system and the wave function collapses, as we purposed in Chapter 1, measurements can be described by unitary (in particular, the CNOT gate), we should be able to produce the Quantum Zeno Effect with unitary measurements.

Assume we have a qubit q_0 and we want to know about its state $|\phi\rangle$. If we add an ancilla and use a CNOT gate to obtain information of $|\phi\rangle$, the ancilla and the CNOT gate form a larger system with q_0. Adding more measurement devices could be seen as forming a larger system, which can be described unitarily. Therefore, if we use the CNOT gates to be the measurement unitary and measure the quantum state $|\phi\rangle$ continuously, we will see the Quantum Zeno Effect. The CNOT gate is defined as the following:

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.1}$$

and let our system q_0 has the state $|0\rangle$.

To use the CNOT gate for measuring our quantum system, we also need an ancillary qubit for each CNOT gate. Each CNOT gate connects our initial quantum state $|0\rangle$ to an ancillary qubit (which is prepared in zero state $|0\rangle$). If our initial quantum state is measured to be $|1\rangle$ (which means that our initial state evolved), the CNOT gate will switch the state of its corresponding ancillary qubit from $|0\rangle$ to $|1\rangle$. By using n CNOT gates and n ancillary qubits, we can implement n measurements which have the same effect as what the projection operators did in the previous example. Based on this idea, we have the following circuit (Fig. 3.1):

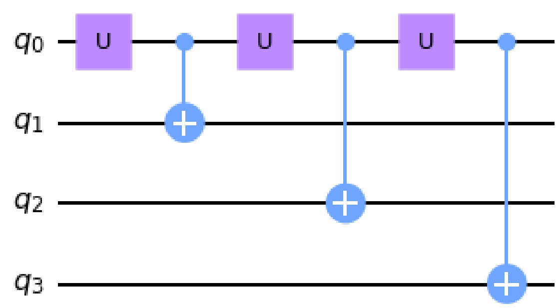

Figure 3.1: Example of a quantum circuit that produces the Quantum Zeno Effect, with three ancillary qubits. The first qubit (q_0) on the top line is our initial quantum state $|0\rangle$, and the rest of the qubits (q_1,q_2,q_3) are the ancillary qubits. The purple box labeled as R_x is our time evolution operator defined by equation (2.1), and the blue lines connecting two qubits are the CNOT gates defined by equation (3.1).

3.1.1 Qiskit simulation

The circuit in Fig. 3.1 can produce the Quantum Zeno Effect with n ancillary qubits. It can be simulated by using Qiskit with the code written in Appendix A.

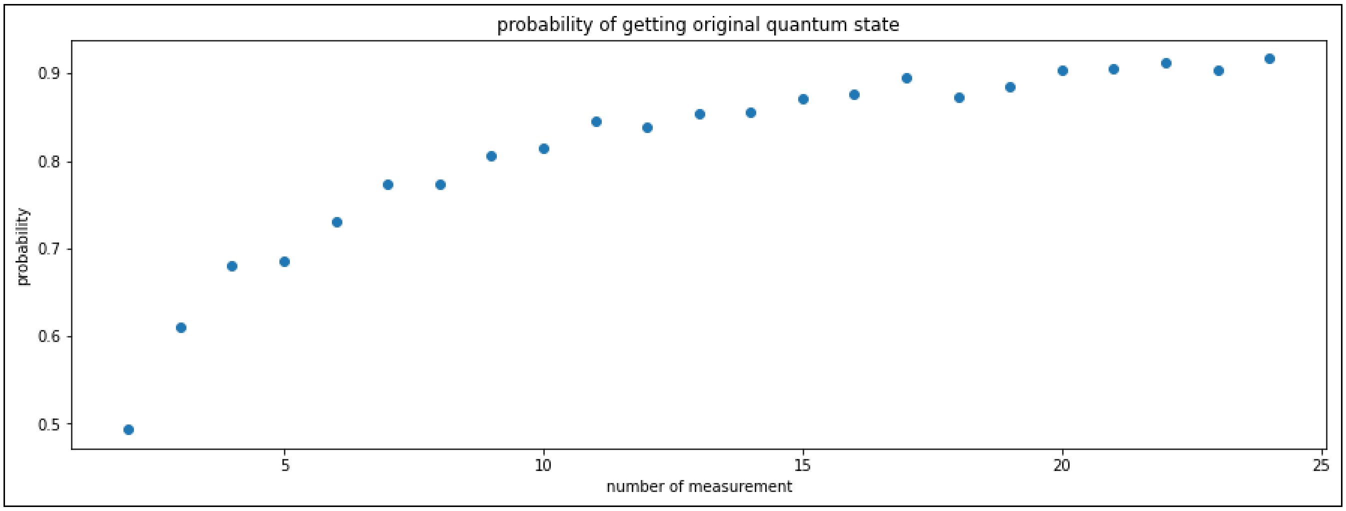

Figure 3.2: It shows that as we increase the number of measurements (the number of CNOT gates and ancillary qubits, as described above), the probability of getting our initial quantum state (the probability that our initial state does not evolve) approaches one. The Quantum Zeno Effect has been shown here.

3.2 Overlap

For the circuit in Fig 3.1, we observe the Quantum Zeno Effect by calculating the probability of q_0 being a $|0\rangle$ state. Let k be the number of ancillary qubits, and n be the number of times that we apply the CNOT gates on each ancillary qubit. Define the evolution unitary as

$$U = \begin{pmatrix} \cos \frac{T}{nk} & -\sin \frac{T}{nk} \\ \sin \frac{T}{nk} & \cos \frac{T}{nk} \end{pmatrix}, \tag{3.2}$$

where T is the total time of measurements. The evolution unitary is repetitively applied to our system. Thus, we adjust the dimension of U to align with the dimension of our whole system (q_0 plus k ancillary qubits):

$$U_{evolve} = U \otimes I^{\otimes k}. \tag{3.3}$$

Also, adjust the dimension of the CNOT gates defined at (3.1), so we have

$$U_{CNOT,i} = |0\rangle \langle 0| \otimes I^{\otimes i} \otimes I^{\otimes k-i} + |1\rangle \langle 1| \otimes I^{\otimes i-1} \otimes X \otimes I^{\otimes k-i}. \tag{3.4}$$

As we want to apply the evolution operator and unitary measurement repetitively, define U_{circ} where these two operators act repeatedly,

$$U_{circ} = \prod_{i=1}^{i=k} U_{CNOT,i} U_{evolve}, \tag{3.5}$$

then apply U_{circ} to our system to get $|\psi\rangle$,

$$|\phi\rangle = U_{circ}^{n} |0\rangle |0\rangle^{\otimes k}. \tag{3.6}$$

Tracing out the ancillary qubits, we can get the overlap, which is the probability of q_0 being in $|0\rangle$ state:

$$overlap = \langle 0| \operatorname{Tr}_{ancillas}(|\phi\rangle \langle \phi|) |0\rangle. \tag{3.7}$$

With the formula (3.7), we look into two cases:

(1) The case where we have an infinite number of ancillary qubits, and

(2) the case where we only have a limited number of ancillary qubits.

(1) is described in Fig. 3.1 and is tested with Qiskit in Fig. 3.2. In this case, each ancillary qubit interacts with q_0 with exactly one CNOT gate. So we are not reusing the ancillary qubits, and the number of ancillary qubits goes to infinity as we increase the number of CNOT gates to infinity. Therefore, the conditions for (3.7) in this case will be $n = 1$ and varying k. (2) is the case that we only change the number of CNOT gates while fixing the number of ancillary qubits. In this case, we reuse the ancillary qubits so each ancillary qubit will contain n CNOT gates. Thus the conditions for calculating (3.7) will be $n > 1$ and varying k. We will discuss (1) in Section 3.2, and (2) in Section 3.3.

3.3 Infinite number of ancillary qubits

First, we look into the case that we do not reuse the ancillary qubits. So we set $n = 1$, assign different integers to k, and calculate (3.7). By doing so, we get a bunch of formulas representing the probability of q_0 being at $|0\rangle$ state. The formulas for $n = 1$ to 8 are summarized in the following table.

	Expression of overlap for n=1
k=1	$\cos[T]^2$
k=2	$\cos[\frac{T}{2}]^4 + \sin[\frac{T}{2}]^4$
k=3	$\cos[\frac{T}{3}]^6 + 3\cos[\frac{T}{3}]^2 \sin[\frac{T}{3}]^4$
k=4	$\cos[\frac{T}{4}]^8 + 6\cos[\frac{T}{4}]^4 \sin[\frac{T}{4}]^4 + \sin[\frac{T}{4}]^8$
k=5	$\cos[\frac{T}{5}]^{10} + 10\cos[\frac{T}{5}]^6 \sin[\frac{T}{5}]^4 + 5\cos[\frac{T}{5}]^2 \sin[\frac{T}{5}]^8$
k=6	$\cos[\frac{T}{6}]^{12} + 15\cos[\frac{T}{6}]^8 \sin[\frac{T}{6}]^4 + 15\cos[\frac{T}{6}]^4 \sin[\frac{T}{6}]^8 + \sin[\frac{T}{6}]^{12}$
k=7	$\cos[\frac{T}{7}]^{14} + 21\cos[\frac{T}{7}]^{10} \sin[\frac{T}{7}]^4 + 35\cos[\frac{T}{7}]^6 \sin[\frac{T}{7}]^8 + 7\cos[\frac{T}{7}]^2 \sin[\frac{T}{7}]^{12}$
k=8	$\cos[\frac{T}{8}]^{16} + 28\cos[\frac{T}{8}]^{12} \sin[\frac{T}{8}]^4 + 70\cos[\frac{T}{8}]^8 \sin[\frac{T}{8}]^8 + 28\cos[\frac{T}{8}]^4 \sin[\frac{T}{8}]^{12} + \sin[\frac{T}{8}]^{16}$

Table 3.1: This table is showing the formulas of overlap for $n = 1$ when $k = 1, 2, 3, 4, 5, 6, 7, 8$. Recall that n is the number of CNOT gates on each ancillary qubit, and k is the number of the ancillary qubit. Setting $n = 1$ means we are not reusing the ancillary qubits.

We observe that the coefficients in these formulas are some particular binomial coefficients. Thus, with some algebraic simplification, we can rewrite all of them into a single formula as the following:

$$\frac{1}{2}((\cos^2\frac{T}{k} + \sin^2\frac{T}{k})^k + (\cos^2\frac{T}{k} - \sin^2\frac{T}{k})^k), \tag{3.8}$$

and it can be simplified further as:

$$\frac{1}{2}(1 + \cos^k\frac{2T}{k}). \tag{3.9}$$

(3.9) represents the probability of q_0 not evolving to another state when we are not reusing the ancillary qubits. If we increase the number of measurements with a fixed time interval (like what we did in Chapter 1), in other words, when k goes to infinity, (3.9) will approach one.

$$\lim_{k\to\infty} \frac{1}{2}(1+\cos^k \frac{2T}{k}) = 1 \tag{3.10}$$

We then observe the Quantum Zeno Effect. We can furthermore examine the relation between T and k. That is, give a value to the overlap, for example, let the cosine term equal $\frac{1}{2}$. So,

$$\begin{aligned} \cos^k(\frac{2T}{k}) &= \frac{1}{2} \\ T &= \frac{1}{2}k\cos^{-1}((\frac{1}{2})^{\frac{1}{k}}) \\ &= \frac{1}{2}\sqrt{k}(\sqrt{k}\cos^{-1}(\frac{1}{2})^{\frac{1}{k}}). \end{aligned} \tag{3.11}$$

For simplification, let $\epsilon = \frac{1}{k}$, so (3.10) becomes

$$T = \frac{1}{2}\frac{1}{\sqrt{\epsilon}}(\frac{1}{\sqrt{\epsilon}}\cos^{-1}(\frac{1}{2^\epsilon})), \tag{3.12}$$

where $\frac{1}{\sqrt{\epsilon}}\cos^{-1}(\frac{1}{2^\epsilon})$ is a constant when $k \to \infty, \epsilon \to 0$:

$$\lim_{\epsilon\to 0}(\frac{1}{\sqrt{\epsilon}}\cos^{-1}(\frac{1}{2^\epsilon})) = \frac{\sqrt{\log 4}}{2}. \tag{3.13}$$

Therefore, we can see that T and k have the following relations:

$$T = \alpha\sqrt{k}, \tag{3.14}$$

where α is a constant.

Before proceeding to the next section where we will discuss the case that only a limited number of ancillary qubits are available, we want to note that the formulas in Table 3.1 can be further simplified to the form of $\frac{1}{\alpha}(\beta + (\text{Cosine terms}))$, where α and β are some constant. That is, the first six formulas in Table 3.1 can become:

	Expression of overlap for n=1
k=1	$\cos[T]^2$
k=2	$\frac{1}{4}(3+\cos[2T])$
k=3	$\frac{1}{8}\left(4+3\cos\left[\frac{2T}{3}\right]+\cos[2T]\right)$
k=4	$\frac{1}{16}(11+4\cos[T]+\cos[2T])$
k=5	$\frac{1}{32}\left(16+10\cos\left[\frac{2T}{5}\right]+5\cos\left[\frac{6T}{5}\right]+\cos[2T]\right)$
k=6	$\frac{1}{64}\left(42+15\cos\left[\frac{2T}{3}\right]+6\cos\left[\frac{4T}{3}\right]+\cos[2T]\right)$

Table 3.2: The formulas in Table 3.1 are simplified and written in the form $\frac{1}{\alpha}(\beta + (\text{Cosine terms}))$.

This simplified version of formulas allows us to observe the similarity between different conditions of calculation.

3.4 Limiting the number of ancillary qubits

Thus far we are assuming that each ancillary qubit is only used once ($n = 1$). If we reuse it, say let $n = 2$ and calculate (3.7) again, then the probability of q_0 being a $|0\rangle$ state for different numbers of ancillary qubits will be:

	Expression of overlap for n=2
k= 1	$\frac{1}{4}(3 + \text{Cos}[2T])$
k= 2	$\frac{1}{8}(7 + \text{Cos}[2T])$
k= 3	$\frac{1}{16}\left(12 + 3\,\text{Cos}\left[\frac{2T}{3}\right] + \text{Cos}[2T]\right)$
k= 4	$\frac{1}{32}(27 + 4\,\text{Cos}[T] + \text{Cos}[2T])$
k= 5	$\frac{1}{64}\left(48 + 10\,\text{Cos}\left[\frac{2T}{5}\right] + 5\,\text{Cos}\left[\frac{6T}{5}\right] + \text{Cos}[2T]\right)$
k= 6	$\frac{1}{128}\left(106 + 15\,\text{Cos}\left[\frac{2T}{3}\right] + 6\,\text{Cos}\left[\frac{4T}{3}\right] + \text{Cos}[2T]\right)$

Table 3.3: These formulas are the overlap for $n = 2$, calculated with $k = 1, 2, 3, 4, 5, 6$. In this case, each ancillary qubit has two CNOT gates ($n = 2$) so we reuse each of them once.

If we consider the formulas with the form $\frac{1}{\alpha}(\beta + (\text{Cosine terms}))$ as we did in the previous section for $n = 1$, then, for $n = 2$, the formulas in Table 3.3 will have the form

$$\frac{1}{2\alpha}(\beta + 2^k + (\text{Cosine terms})). \tag{3.15}$$

We can see the difference between the overlap for $n = 1$ (not reusing each ancillary qubit) and $n = 2$ (reuse each ancillary qubit once) is that we have an extra $\frac{1}{2}$ factor outside the parentheses and an extra 2^k term inside the parentheses. Indeed, we will see that the formulas in Table 3.3 and (3.14), which seems to be an example made by a handy choice of conditions, non-trivially appear again later with different assumption on the setting of the whole system.

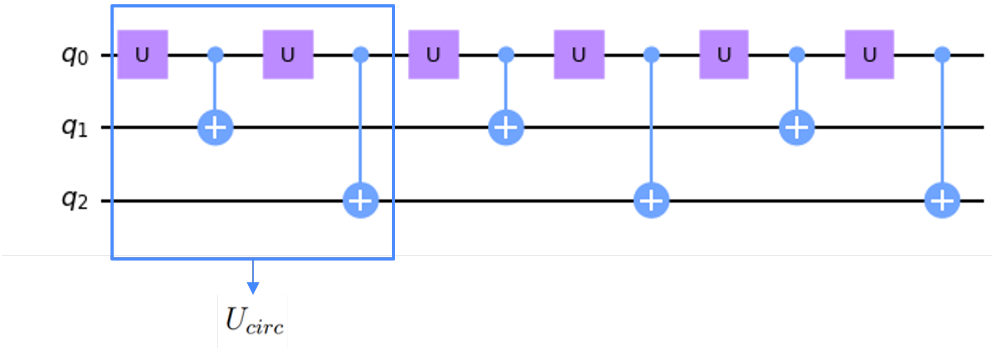

Figure 3.3: This is the circuit that we limited our number of ancillary qubits to two ancillary qubits, and reuse each ancillary qubit twice ($n = 3$).

While in the previous section, we showed that the Quantum Zeno Effect can be produced by using unitary measurements, infinitely many ancillary qubits are used in that case. This then brings up a question: Can we produce the Quantum Zeno Effect using the same condition of evolution and unitary measurements but with a limited number of ancillary qubits? To answer this question let us consider case (2) where we only have a limited number of ancillary qubits. In this case, to make as many measurements as possible, we inevitably need to reuse the ancillary qubits. That is, each ancillary qubit will have multiple CNOT gates entangled with our system (see Figure 3.3). Thus, by giving a fixed value of k (the number of ancillary qubits) and letting $n > 1$ (reusing the ancillary qubits), we calculate the overlap with (3.7). The result for $k = 1, 2, 3, 4$ is shown in Table 3.4 and Table 3.5.

We see that the results of this case show a different pattern from that of the case (1). Note that, for even numbers, we get the same formula of overlap, which is the same as the result we got from the case where $n = 2$ with the corresponding value of k. Moreover, if we look into the results from n equals to odd numbers (except $n = 1$), the only changing terms are the extra cosine terms that different with those of even number n, and as n increases these extra cosine terms go to one. That is, the results for odd values of n (excluding $n = 1$) are approaching those of even values of n as n increase. Therefore, in this case, we are not able to produce the Quantum Zeno Effect since the probability does not have the tendency to approach one.

Table 3.4: The formulas in these tables represent the overlap for $k = 1$ and $k = 2$, calculated with $n = 1$ to $n = 20$. Recall that when $n > 1$, we are reusing the ancillary qubits.

	Expression of overlap for k=1		Expression of overlap for k=2
n= 1	$\text{Cos}[T]^2$	n= 1	$\frac{1}{4}(3+\text{Cos}[2T])$
n= 2	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 2	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 3	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$	n= 3	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 4	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 4	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 5	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{5}\right]+\text{Cos}[2T]\right)$	n= 5	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{5}\right]+\text{Cos}[2T]\right)$
n= 6	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 6	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 7	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{7}\right]+\text{Cos}[2T]\right)$	n= 7	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{7}\right]+\text{Cos}[2T]\right)$
n= 8	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 8	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 9	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{9}\right]+\text{Cos}[2T]\right)$	n= 9	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{9}\right]+\text{Cos}[2T]\right)$
n= 10	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 10	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 11	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{11}\right]+\text{Cos}[2T]\right)$	n= 11	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{11}\right]+\text{Cos}[2T]\right)$
n= 12	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 12	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 13	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{13}\right]+\text{Cos}[2T]\right)$	n= 13	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{13}\right]+\text{Cos}[2T]\right)$
n= 14	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 14	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 15	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{15}\right]+\text{Cos}[2T]\right)$	n= 15	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{15}\right]+\text{Cos}[2T]\right)$
n= 16	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 16	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 17	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{17}\right]+\text{Cos}[2T]\right)$	n= 17	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{17}\right]+\text{Cos}[2T]\right)$
n= 18	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 18	$\frac{1}{8}(7+\text{Cos}[2T])$
n= 19	$\frac{1}{4}\left(2+\text{Cos}\left[\frac{2T}{19}\right]+\text{Cos}[2T]\right)$	n= 19	$\frac{1}{8}\left(6+\text{Cos}\left[\frac{2T}{19}\right]+\text{Cos}[2T]\right)$
n= 20	$\frac{1}{4}(3+\text{Cos}[2T])$	n= 20	$\frac{1}{8}(7+\text{Cos}[2T])$

Table 3.5: These formulas represent the overlap for $k = 3$ and $k = 4$.

	Expression of overlap for k=3
n= 1	$\frac{1}{8}\left(4+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 2	$\frac{1}{16}\left(12+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 3	$\frac{1}{16}\left(8+3\,\text{Cos}\left[\frac{2T}{9}\right]+4\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 4	$\frac{1}{16}\left(12+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 5	$\frac{1}{16}\left(8+3\,\text{Cos}\left[\frac{2T}{15}\right]+\text{Cos}\left[\frac{2T}{5}\right]+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 6	$\frac{1}{16}\left(12+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 7	$\frac{1}{16}\left(8+3\,\text{Cos}\left[\frac{2T}{21}\right]+\text{Cos}\left[\frac{2T}{7}\right]+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 8	$\frac{1}{16}\left(12+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 9	$\frac{1}{16}\left(8+3\,\text{Cos}\left[\frac{2T}{27}\right]+\text{Cos}\left[\frac{2T}{9}\right]+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$
n= 10	$\frac{1}{16}\left(12+3\,\text{Cos}\left[\frac{2T}{3}\right]+\text{Cos}[2T]\right)$

	Expression of overlap for k=4
n= 1	$\frac{1}{16}\left(11+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 2	$\frac{1}{32}\left(27+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 3	$\frac{1}{32}\left(22+4\,\text{Cos}\left[\frac{T}{3}\right]+\text{Cos}\left[\frac{2T}{3}\right]+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 4	$\frac{1}{32}\left(27+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 5	$\frac{1}{32}\left(22+4\,\text{Cos}\left[\frac{T}{5}\right]+\text{Cos}\left[\frac{2T}{5}\right]+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 6	$\frac{1}{32}\left(27+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 7	$\frac{1}{32}\left(22+4\,\text{Cos}\left[\frac{T}{7}\right]+\text{Cos}\left[\frac{2T}{7}\right]+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 8	$\frac{1}{32}\left(27+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 9	$\frac{1}{32}\left(22+4\,\text{Cos}\left[\frac{T}{9}\right]+\text{Cos}\left[\frac{2T}{9}\right]+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$
n= 10	$\frac{1}{32}\left(27+4\,\text{Cos}[T]+\text{Cos}[2T]\right)$

Chapter 4

Conclusion

We discussed the reason why we can describe measurement with unitary (CNOT gate) in Chapter 1, and looked into the Quantum Zeno Effect produced by projectors in Chapter 2. Then we examined how can we produce the Quantum Zeno Effect by unitary measurements with an unlimited/limited number of ancillary qubits. In summary, with unitary measurements, we can produce the Quantum Zeno Effect on a quantum circuit. However, it is required to use an infinite number of ancillary qubits that have not been used before. If we limited the number of ancillary qubits and reuse them, reusing each ancillary qubit once ($n = 2$) has the same impact as reusing them multiple times ($n > 2$). Thus, infinite space is required to produce the Quantum Zeno Effect.

Appendix A

Qiskit Simulation

```
import numpy as np
from qiskit import *
from qiskit.providers.aer import QasmSimulator
import matplotlib.pyplot as plt
import matplotlib as mpl
probabilities = []
n = []
backend = BasicAer.get_backend('qasm_simulator')
for i in range(2, 25):
  q = QuantumRegister(i)
  cs = ClassicalRegister(1)
  qc = QuantumCircuit(q)
  x = np.pi/(i)
  qc.add_register(cs)
  for j in range(0, i-1):
    qc.rx(x, 0)
    qc.cnot(0,j+1)
  qc.measure(0, 0)
  job = execute(qc, backend)
  result = job.result()
  p = result.get_counts()['0']/1024
  probabilities.append(p)
  n.append(i)
```

```
fig = plt.figure(figsize=(15,5))
plt.scatter(n, probabilities)
plt.xlabel("n")
plt.title("probability of getting original quantum state")
plt.ylabel("probability")
plt.show()
```

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