Moufang sets of finite Morley rank

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Moufang sets of finite Morley rank

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We study proper Moufang sets of finite Morley rank for which either the root groups are abelian or the roots groups have no involutions and the Hua subgroup is nilpotent. We give conditions ensuring that the little projective group of such a Moufang set is isomorphic to $\mathrm{PSL}_2(F)$ for F an algebraically closed field. In particular, we show that any infinite quasisimple L^*-group of finite Morley rank of odd type for which (B, N, U) is a split BN-pair of Tits rank 1 is isomorphic to $\mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$ provided that U is abelian. Additionally, we show that same conclusion can reached by replacing the hypothesis that U be abelian with the hypotheses that $B \cap N$ is nilpotent and U is definable and without involutions. As such, we make progress on the open problem of determining the simple groups of finite Morley rank with a split BN-pair of Tits rank 1, a problem tied to the current attempt to classify all simple groups of finite Morley rank.

Dedication

To Papa.

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Contents

Chapter 1

Introduction

The theory of groups of finite Morley rank has its roots in the study of uncountably categorical first-order theories, i.e. theories that posses exactly one model (up to isomorphism) of each uncountable cardinality. Michael Morley introduced and utilized a model-theoretic notion of dimension, now called Morley rank, when proving his Categoricity Theorem: a first-order theory in a countable language which possesses exactly one model of **some** uncountable cardinality is in fact uncountably categorical. Further investigation of uncountable categorical theories is intertwined with groups of finite Morley rank, see [17, Section 2.5], indicating that the study of these groups is inevitable, at least for the model theorist.

We will approach the study of groups of finite Morley rank via the Borovik-Poizat axioms for ranked structures. These axioms define a notion of rank that need not be the same as the Morley rank of a structure, but it is shown in [17] that for groups these notions do in fact coincide. The groups of finite Morley rank include both the finite groups and the linear algebraic groups over algebraically closed fields. In general, groups of finite Morley rank share a wealth of similarities with algebraic groups, and the similarities are expected to run quite deep as witnessed by the following conjecture due to Gregory Cherlin and Boris Zil'ber.

Algebraicity Conjecture. An infinite simple group of finite Morley rank is an algebraic group over an algebraically closed field.

Using the fact that the Sylow 2-subgroups of a group of finite Morley rank are conjugate, see Fact 2.3.5, the analysis of groups of finite Morley rank splits into four cases based on the structure of the connected component of a Sylow 2-subgroup, S. These cases, whose names correspond to the characteristic of a possible interpretable field, are

Degenerate type: S° is trivial,

Odd type: S° is nontrivial, divisible and abelian (S° is a 2-torus),

Even type: S° is nontrivial, nilpotent, and of bounded exponent (S° is 2-unipotent), and

Mixed type: S° contains a nontrivial 2-torus and a nontrivial 2-unipotent subgroup.

In [1], Tuna Altınel, Alexandre Borovik, and Gregory Cherlin show that mixed type simple groups of finite Morley rank do not exist and that those of even type are indeed algebraic, see Fact 2.3.28. If the Algebraicity Conjecture is true, degenerate type simple groups will not exist, but this is far from being established. Therefore, it is desirable to approach the classification of the odd type simple groups in a way that decouples them from those of degenerate type. This leads to the study of L and L^*-groups. A group of finite Morley rank is an *L-group* if every infinite definable simple section of odd type is isomorphic to a Chevalley group over an algebraically closed field; where as, the group is an *L^*-group* if every **proper** infinite definable simple section of odd type is isomorphic to a Chevalley group over an algebraically closed field.

One approach to the Algebraicity Conjecture utilizes the theory of BN-pairs developed by Jacques Tits. Every simple algebraic group over an algebraically closed field has a BN-pair that arises naturally from a Borel subgroup and the normalizer of any maximal torus contained in the Borel. BN-pairs possess their own notion of rank called Tits rank, and there is theory in place for identifying groups of finite Morley rank with a BN-pair of Tits rank at least 3, with partial results for Tits rank 2 [15].

We address the Tits rank 1 situation. One difficulty in Tits rank 1 is that the geometry associated to the BN-pair degenerates, and one is left simply with a group acting 2-transitively on a set. Nevertheless, progress has been made by investigating those groups in which $B \cap N$ has a normal complement in B, i.e. groups with a **split** BN-pair of Tits rank 1. Our approach is through the theory of Moufang sets. Moufang sets were introduced by Jacques Tits in [24] and capture the essence of a split BN-pair of Tits rank 1. Definitions and basic results for Moufang sets and split BN-pair of Tits rank 1 will be given in Chapter 3.

1.1 Results

This thesis is targeted at the following problem which is motivated by the fact that PSL_2 is the only simple algebraic group over an algebraically closed field whose standard BN-pair has Tits rank 1 [13, Corollary 32.3].

Conjecture 1.1.1 ([4, Question B.17])**.** If G is an infinite simple group of finite Morley rank with a split BN-pair of Tits rank 1, then $G \cong \mathrm{PSL}_2(K)$ for some algebraically closed field K.

The study of groups with a split BN-pair of Tits rank 1 more-or-less reduces to the study of Moufang sets, and we will work for the most part in the latter language. Additionally, we will avoid those Moufang sets which are associated to sharply 2-transitive actions (a separate and challenging issue) and work only with Moufang sets for which the 2-point stabilizers are nontrivial, i.e. *proper* Moufang sets. As such, the real target of our work is the following conjecture.

Conjecture 1.1.2. If $\mathbb{M}(U,\tau)$ is an infinite proper Moufang set of finite Morley rank, then $\mathbb{M}(U,\tau) \cong \mathbb{M}(F)$ for F an algebraically closed field.

The results of this thesis address two special cases of the previous conjecture. The first two theorems address (special) abelian Moufang sets of finite Morley rank. By recent result of Yoav Segev [20], a proper Moufang set that is abelian must also be special. The third, and final, theorem deals with Moufang sets for which the 2-point stabilizers are nilpotent.

Theorem 1.1.3. *Let $\mathbb{M}(U,\tau)$ be an infinite special abelian Moufang set of finite Morley rank with Hua subgroup H. Assume that* $\mathrm{char}(U) = 2$ *and $K := \mathrm{End}_H(U)$ is infinite. Then $\mathbb{M}(U,\tau) \cong \mathbb{M}(K)$ with K an algebraically closed field.*

This first theorem is somewhat recreational. We provide a fairly elementary proof of this result from within the theory of special Moufang sets, but a much shorter proof, which does not require the hypothesis that $\mathrm{End}_H(U)$ be infinite, can be obtained by appealing to the previously mentioned deep theorem of [1], see Fact 2.3.28. Theorem 1.1.3 complements the result of [11] which addresses the $\mathrm{char}(U) \neq 2$ situation, see Fact 3.2.4.

An obvious goal is to remove the hypothesis that $\mathrm{End}_H(U)$ be infinite from the $\mathrm{char}(U) \neq 2$ theorem of [11]. When $\mathrm{char}(U) = 0$, the condition that $\mathrm{End}_H(U)$ be infinite is always true, and our next result makes progress on the $\mathrm{char}(U) > 2$ case.

Theorem 1.1.4. *Let $\mathbb{M}(U, \tau)$ be an infinite special abelian Moufang set of finite Morley rank with Hua subgroup H. Assume that $p := \mathrm{char}(U) > 2$. Then either of the following conditions ensure that $\mathbb{M}(U, \tau) \cong \mathbb{M}(F)$ for some algebraically closed field F:*

(1) H is without infinite elementary abelian p-subgroups, or

(2) H is an L-group.

It is worth mentioning that sharply 2-transitive groups of finite Morley rank with abelian point stabilizers are known to be isomorphic to the 1-dimensional affine group $\mathrm{Aff}_1(F)$ for F an algebraically closed field, see [4, Proposition 11.61]. In particular, any abelian Moufang set of finite Morley rank with a **simple** little projective group must in fact be proper, hence special. As such, the previous theorem may be combined with the "absolute" results in characteristics 0 and 2, the latter relying on [1], to obtain the following result, see Corollary 3.4.2.

Corollary 1.1.5. *Let (B, N, U) be a split BN-pair of Tits rank 1 for an infinite quasisimple L^*-group of finite Morley rank G with U infinite and abelian. Then $G \cong \mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$ for F an algebraically closed field.*

Our final result arises from a study of not necessarily abelian Moufang sets of finite Morley rank for which we focus on those whose little projective group has odd type. The goal is to relax the restriction that U be abelian from our previous results. We accomplish this at the expense of extra restrictions on the Hua subgroup.

Theorem 1.1.6. *Let $\mathbb{M}(U,\tau)$ be an infinite proper Moufang set of finite Morley rank whose little projective group has odd type. Further assume that U_∞ is a definable $2^\perp$-group and that the Hua subgroup is nilpotent. Then $\mathbb{M}(U,\tau) \cong \mathbb{M}(F)$ for some algebraically closed field F.*

As before, we may use Corollary 3.4.2 to see that Theorem 1.1.6 yields the following corollary.

Corollary 1.1.7. *Let (B,N,U) be a split BN-pair of Tits rank 1 for an infinite odd type quasisimple group of finite Morley rank G. If U is a definable $2^\perp$-subgroup and $B \cap N$ is nontrivial and nilpotent, then $G \cong \mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$ for F an algebraically closed field.*

1.2 Outline

Chapter 2 provides a brief introduction to groups of finite Morley rank. The chapter includes definitions and examples as well as numerous background results which are usually given without proof. The last section of the chapter collects few facts specific to algebraic groups.

In Chapter 3 we lay out background on Moufang sets and split BN-pairs of Tits rank 1. The latter portion of the chapter specializes to groups of finite Morley rank. Corollary 3.2.3 will be used frequently and usually without reference. Corollary 3.4.2 illustrates how we may lift results for Moufang sets to groups with a split BN-pair of Tits rank 1.

Our study of Moufang sets of finite Morley rank really begins in Chapter 4. Here we define P^*-Moufang sets which attempt to capture the nature of minimal counterexamples to Conjecture 1.1.2. We initiate the study of P^*-Moufang sets and a serious attempt is made to keep the results as general as possible so that they may find use in a continued assault on Conjecture 1.1.2 that goes beyond what we accomplish here. Proposition 4.1.1 is the focal point of the chapter.

The final three chapters contain the proofs of the main theorems, in order. Each of the chapters prove results that are slightly stronger than what is needed to establish their respective theorems, and there is a feeling that the extra work may find future use. Additionally, the final section of Chapter 5 shows how the classification of the even type simple groups of finite Morley rank can be utilized to give a short proof of Theorem 1.1.3 which does not require the hypothesis that $\mathrm{End}_H(U)$ be infinite.

Chapter 2

Groups and groups of finite Morley rank

This chapter contains a brief introduction to groups of finite Morley rank via the Borovik-Poizat axioms for ranked structures. Further, it lays out the background required for later chapters. With few exceptions, these results will be extracted, without proof, from [4] and [1]. The final section addresses a handful of points about algebraic groups.

2.1 Group theoretic notation

We begin by establishing some notation, which is for the most part standard. Let G be a group acting on a set X; we usually will work with right actions. For any $Y \subseteq X$ and any $H \subseteq G$, we make the follow definitions.

- H_Y and $C_H(Y)$ both denote the set $\{h \in H : \forall y \in Y(yh = y)\}$. We often replace Y with a list of its elements.

- $H_{\{Y\}}$ and $N_H(Y)$ both denote the set $\{h \in H : Yh = Y\}$. Again, we often replace Y with a list of its elements.

- $C_Y(H) := \{y \in Y : \forall h \in H(yh = y)\}$. We will write $C_Y(h)$ for $C_Y(\{h\})$.

- In the case that X is a group and $g \in G$ acts as a group automorphism of X, $C_X^-(g)$ is the set of elements of X inverted by g, and $C_X^{\pm}(g) := C_X(g) \cup C_X^-(g)$.

- $\mathrm{Aut}(G)$ is the automorphism group of G. If G is abelian, $\mathrm{End}(G)$ denotes the endomorphism ring of G.

- When H is a subgroup of G, $H\backslash G$ stands for the set of right cosets of H in G.

- $\mathrm{Core}_G(H) := \bigcap_{g\in G} H^g$, and $H^* := H - \{1\}$.

In general, our group theoretic terminology and notation should be consistent with [4].

2.2 Borovik-Poizat axioms for ranked structures

Fix a first-order language $\mathcal{L}$, and let $\mathbb{M}$ be an $\mathcal{L}$-structure. Let $\mathcal{D}$ denote the $\mathbb{M}$-interpretable sets; that is, $\mathcal{D}$ is the collection of definable sets (the subsets of M^n that are solutions to $\mathcal{L}$-formulas with parameters from M) taken modulo definable equivalence relations. $\mathbb{M}$ is called a *ranked structure* if there is a function $\mathrm{rk} : (\mathcal{D} - \{\emptyset\}) \to \mathbb{N}$ satisfying the following four axioms.

Monotonicity of rank: For each interpretable set A, $\mathrm{rk}(A) \geq n+1$ if and only if A contains infinitely many pairwise disjoint nonempty interpretable subsets A_i with $\mathrm{rk}(A_i) \geq n$.

Definability of rank: If A, B, and $f : A \to B$ are interpretable, then for each integer n the set $\{b \in B : \mathrm{rk}(f^{-1}(b)) = n\}$ is interpretable.

Additivity of rank: If A, B, and $f : A \to B$ are interpretable with f surjective and $\mathrm{rk}(f^{-1}(b)) = n$ for every $b \in B$, then $\mathrm{rk}(A) = \mathrm{rk}(B) + n$.

Elimination of infinite quantifiers: If A, B, and $f : A \to B$ are interpretable, then there is an integer n such that for every $b \in B$, $|f^{-1}(b)| \geq n$ implies that $f^{-1}(b)$ is infinite.

If $\mathbb{M}$ is a ranked structure, then the rank of $\mathbb{M}$ is defined to be the rank of its underlying set M. Note that any structure interpretable in a ranked structure will again be ranked (in the language of the original structure).

We will be concerned with *ranked groups*, by which we mean a structure $\mathbb{G}$ in a language $\mathcal{L} = \{\cdot, ^{-1}, 1, \ldots\}$ possibly extending the group language such that $(G; \cdot, ^{-1}, 1)$ is a group. Similarly, when considering a *ranked ring*, we allow ourselves to work in a language that possibly extends the language of rings. It is shown in [17] that a ranked group is in fact a group of finite Morley rank where the rank of each interpretable set coincides with its Morley rank. The converse holds as well, a proof of which may again be found in [17]. As such, will refer to ranked groups as groups of finite Morley rank. Additionally, we will refer to a ranked field as a field of finite Morley rank.

Example 2.2.1. Finite structures have rank 0. By a result of Angus Macintyre, an infinite field has finite Morley rank if and only if it is an algebraically closed field, see [4, Theorem 8.1]. Examples of infinite groups of finite Morley rank include

(1) the affine algebraic groups over algebraically closed fields,

(2) the Prüfer p-group $\mathbb{Z}(p^\infty) := \{x \in \mathbb{C} : x^{p^n} = 1 \text{ for some } n\}$ where p is a prime, and

(3) any torsion-free divisible abelian group.

Now, an interpretable set of rank n can not be broken into **infinitely** many pairwise disjoint nonempty interpretable subsets each of rank n, but it may be possible to break it into **finitely** many, say d. It can be shown that there is a maximal such d, called the *degree* of the set. It is important to note that rank n and degree d interpretable sets usually contain proper interpretable subsets also of rank n and degree d. However, when passing from an interpretable group to a definable subgroup, this can not happen. As such, we obtain the following fundamental fact.

Fact 2.2.2 (see [4, Section 5.2])**.** *Let G be a group of finite Morley rank. Then G has no infinite descending chain of proper definable subgroups.*

Example 2.2.3. The group $(\mathbb{Z}; +, -, 0)$ does **not** have finite Morley rank because it has the following infinite descending chain of definable subgroups:

$$\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset 8\mathbb{Z} \supset \cdots .$$

The previous fact allows us to define the *connected component* of a group of finite Morley rank, G, to be its minimal definable subgroup of finite index, which is denoted G°. A group of finite Morley rank is said to be *connected* if it is equal to its connected component, and it is a theorem that a group of finite Morley rank is connected if and only if it has degree 1. Note that a connected group of finite Morley rank acting interpretably on a finite set must in fact fix the set pointwise since the kernel of the action will be a definable subgroup of finite index.

Example 2.2.4. Infinite simple groups of finite Morley rank and divisible groups of finite Morley rank are necessarily connected since neither have subgroups of finite index. By the latter observation, the multiplicative group of an algebraically closed field is connected. Since the degree of an infinite set remains unchanged by adding or removing finitely many elements, the additive part of an algebraically closed field is also connected.

The following fact, which is one of several incarnations of Zil'ber's Indecomposability Theorem, is often used to show a group (or subgroup) is connected.

Fact 2.2.5 (see [4, Theorem 5.26, Corollary 5.28])**.** *In a group of finite Morley rank, the subgroup generated by a set of definable connected subgroups is definable and connected.*

Another consequence of Fact 2.2.2 is that there is a minimal definable subgroup containing any subset of a group of finite Morley rank. This subgroup will be called the *definable closure* (or *definable hull*) of the set. Many properties a subgroup will pass to its definable closure such as being abelian, nilpotent, or solvable. For these examples and others, see [1, I Lemma 2.15]. We also extend our notion of connectedness to a not necessarily definable subgroup by defining it to be connected if its definable closure is connected.

2.3 General facts about groups of finite Morley rank

This section collects numerous results on groups of finite Morley rank. We begin with some definitions inspired by algebraic groups.

Definition 2.3.1. Let G be a group and π be a set of primes.

- G is called a π-*group* if every element is a π-element.
- G is called a π-*torus* if it is a divisible abelian π-group.

- G is called a $\pi^{\perp}$-*group* if G has no nontrivial π-elements.

For the remaining definitions, we further insist that G has finite Morley rank.

- G is called a *torus* if it is divisible and abelian.
- G is called a *decent torus* if it is a torus that is the definable closure of its torsion.
- G is called a *good torus* if it is a torus in which every definable subgroup is a decent torus.
- G is called *unipotent* if it is connected, nilpotent, and of bounded exponent.
- G is called π-*unipotent* if it is a unipotent π-group.

If a π-torus is a subgroup of a group of finite Morley, it will usually not be definable and hence not a torus. However, its definable closure will be a decent torus.

2.3.1 Core properties

Fact 2.3.2 (see [1, I Lemma 2.16]). *The definable closure of a cyclic subgroup of a group of finite Morley rank is the direct sum of a torus and a finite cyclic group.*

Fact 2.3.3 (see [1, I Lemma 2.17]). *A group of finite Morley rank with no elements of order dividing n is uniquely n-divisible.*

Fact 2.3.4 (see [1, I Lemma 2.18]). *Let G be a group of finite Morley rank, H a normal definable subgroup, x any element of G, and π a set of primes. If the image of x in G/H is a π-element, then Hx contains a π-element.*

The next two facts give conjugacy and structure results for *Sylow* 2-*subgroups*, i.e. maximal 2-subgroups. We will refer to the connected components of the Sylow 2-subgroups as the *Sylow*$^{\circ}$ 2-*subgroups.*

Fact 2.3.5 (see [1, I Proposition 6.11]). *The Sylow 2-subgroups of a group of finite Morley rank are conjugate.*

Fact 2.3.6 (see [1, I Proposition 6.4]). *Every Sylow$^\circ$ 2-subgroup of a group of finite Morley rank is a central product of a 2-unipotent group and a 2-torus.*

We briefly address involutive automorphisms, i.e. automorphisms of order 2.

Fact 2.3.7 (see [1, I Lemma 10.3]). *Let G be a connected group of finite Morley rank and α a definable involutive automorphism of G with finitely many fixed points. Then G is abelian and inverted by α.*

Fact 2.3.8 (see [1, I Lemma 10.4]). *Let G be a group of finite Morley rank without involutions and α a definable involutive automorphism of G. Then $G = C_G(\alpha)C_G^-(\alpha)$ with the multiplication map $C_G(\alpha) \times C_G^-(\alpha) \to G$ a bijection.*

2.3.2 Fields

The following results on fields lie at the heart of everything that we do.

Fact 2.3.9 (see [1, I Proposition 4.2]). *Let F be an infinite field of finite Morley rank. Then F is algebraically closed.*

Fact 2.3.10 (see [1, I Proposition 4.5]). *Let F be an infinite field of finite Morley rank and G a definable group of automorphisms of F. Then $G = 1$.*

Fact 2.3.11 (see [1, I Proposition 4.11]). *Let G be a connected group of finite Morley rank acting definably, faithfully, and irreducibly on an abelian group V, and let T be a normal infinite abelian subgroup of G. Then the subring, K, of $\mathrm{End}(V)$ generated by T is an algebraically closed field definable over G, and under the action of K, V is a finite dimensional vector space on which G acts K-linearly.*

2.3.3 The solvable radical and the generalized Fitting subgroup

Recall that the *Fitting subgroup* of a group G, denoted $F(G)$, is the subgroup generated by all normal nilpotent subgroups of G, while the *solvable radical*, denoted $\sigma(G)$, is the subgroup generated by all normal solvable subgroups of G.

Fact 2.3.12 (see [4, Theorem 7.3]). *Let G be a group of finite Morley rank. Then $F(G)$ and $\sigma(G)$ are definable subgroups which are respectively nilpotent and solvable.*

Fact 2.3.13 (see [1, I Lemma 8.3]). *Let G be a connected solvable group of finite Morley rank. Then $G/F^{\circ}(G)$ is a torus.*

Recall that a group is called *quasisimple* if it is perfect and modulo its center it is nonabelian simple. The *components* of a group are defined to be the subnormal quasisimple subgroups. Then the *generalized Fitting subgroup* of a group G, denoted $F^{*}(G)$, is defined to be the subgroup generated by $F(G)$ together with the components of G.

Fact 2.3.14 (see [4, Lemma 7.10]). *Let G be a group of finite Morley rank. Then G has finitely many components each of which are definable. If G is connected, then each component is normal.*

Fact 2.3.15 (see [4, Corollary 7.14]). *Let G be a group of finite Morley rank. Then $G/Z(F(G)) \leq$* $\mathrm{Aut}(F^{*}(G))$.

2.3.4 Lifting centralizers and generation by centralizers

The first fact addresses a situation when we may "lift centralizers".

Fact 2.3.16 (see [1, I Proposition 9.15]). *Let $H \rtimes T$ be a group of finite Morley rank with T a π-group of bounded exponent and $Q \lhd H$ a T-invariant definable π-divisible subgroup. Suppose that Q and T are solvable and H is connected. Then*

$$C_H(T)Q/Q = C_{H/Q}(T).$$

We now give two facts about generation by centralizers.

Fact 2.3.17 (see [1, I Proposition 9.16]). *Let $Q \rtimes V$ be a group of finite Morley rank with Q a definable connected solvable group with no nontrivial p-unipotent subgroups and V a finite abelian p-group. Then $Q = \langle C_Q(V_0) : V/V_0$ is cyclic$\rangle$.*

Fact 2.3.18 (see [2, Proposition 9.1]). *Let V be a group generated by distinct commuting involutions, and assume that V acts definably on a group of finite Morley rank, G, that is without involutions. Then $G = \langle C_G(v) : v \in V^* \rangle$.*

2.3.5 Generosity and decent tori

This section addresses so-called generous subgroups. We begin by defining generic subsets. Let G be a group of finite Morley rank and A an interpretable set. A subset X of A is said to be *generic* in A if X contains a definable subset of rank equal to the rank of A. We think of generic subsets as being "large" (relative to A). Now, a definable subgroup $H \subseteq G$ is called *generous* (in G) if the union of the G-conjugates of H is generic in G.

It turns out that generosity results are closely tied to conjugacy results. A necessary condition for being generous is to have "enough" conjugates, a point which gives rise to the following definition. We call a definable subgroup, H, of a group of finite Morley rank, G, *almost self-normalizing* (in G) if H is of finite index in $N_G(H)$, i.e. $N_G^\circ(H) \subseteq H$.

Fact 2.3.19 (see [1, IV Lemma 1.25]). *Let G be a connected group of finite Morley rank and H a definable, connected, and almost self-normalizing subgroup of G. Let $\mathcal{F}$ be the family of conjugates of H in G. Then the following are equivalent.*

(1) H is generous in G.

(2) The definable set

$$H_0 = \{h \in H : \{X \in \mathcal{F} : h \in X\} \text{ is finite}\}$$

is generic in H.

(3) The definable set

$$G_0 = \left\{g \in \bigcup H^G : \{X \in \mathcal{F} : g \in X\} \text{ is finite}\right\}$$

is generic in G.

As a divisible abelian (or nilpotent) group of finite Morley rank will have finitely many elements of each finite order, any connected group of finite Morley rank that normalizes a decent torus must centralize it, a fact which will be referred to as the *rigidity of decent tori.* This also shows that the centralizer of a decent torus is almost self-normalizing since the center of the centralizer will contain a unique maximal decent torus.

Fact 2.3.20 (see [1, IV Lemma 1.14])**.** *Let G be a connected group of finite Morley rank and T a decent torus. Then, $C_G^\circ(T)$ is generous in G.*

Fact 2.3.21 (see [1, IV Proposition 1.15])**.** *Let G be a group of finite Morley rank. Then any two maximal decent tori of G are conjugate, and the same holds for maximal good tori.*

2.3.6 Actions on p-unipotent groups

We now arrive at a situation that we will frequently encounter in our work: actions on p-unipotent groups. This next lemma is little more than a collection of results from [6]. Define the *p-unipotent radical* of a group G, denoted $U_p(G)$, to be the subgroup generated by all p-unipotent subgroups of G. When G is solvable, $U_p(G)$ will be p-unipotent as well. We refer the reader to [6] for the definitions of the $U_{0,r}(-)$ and $U_0(-)$ operators.

Lemma 2.3.22. *Let H be a connected nilpotent group of finite Morley rank acting faithfully and interpretably on an interpretable p-unipotent group U. For T the maximal decent torus of H, we have the following:*

(1) $H = T \times U_p(H)$ with T a $p^\perp$ good torus,

(2) $C_U(U_p(H))$ is infinite, and

(3) $C_U(T)$ is connected.

Proof. We work in the group $G := U \rtimes H$. By [6, Corollary 3.6],

$$H = T * U_{0,1}(H) * U_{0,2}(H) * \cdots * U_{0,r_{\max}}(H) * U_2(H) * U_3(H) * U_5(H) * \cdots .$$

Now, [6, Proposition 4.1] implies that $UU_q(H)$ is nilpotent for each prime q. If $q \neq p$, $[U, U_q(H)] = 1$. If $q = p$, U has an infinite intersection with the center of $UU_p(H)$, so $U_p(H)$ centralizes infinitely many elements of U. Additionally, [6, Lemma 4.5] shows that $[U, U_{0,r}(H)] = 1$ for all $r \geq 1$. This establishes the second item, and to finish the first item it only remains to show that T is a $p^\perp$ good torus.

We have that $H = T \times U_p(H)$. T is a good torus since we have seen that $U_{0,r_{\max}}(H) = \{1\}$ implying that T has no torsion-free definable section. We now show that T is $p^\perp$. Let S be the Sylow p-subgroup of T, which is connected since T is connected abelian. Then US is a connected solvable p-subgroup of UT. By [1, I Proposition 5.28], US is nilpotent, as solvable plus torsion implies that US is locally finite. We conclude that $[U, S] = 1$ by applying [6, Corollary 3.6] to the definable closure of US. Thus, $S = 1$.

Finally we address the third point. For this, we follow the proof of [1, I Lemma 10.6]. We proceed by induction on the rank of U. Set $Z = Z^\circ(U)$, and inductively assume that $C_{U/Z}(T)$ is connected. Let W be the full preimage in U of $C_{U/Z}(T)$, so W is connected as well. By [1, I Corollary 9.10], $W = [W, T]C_W(T) = ZC_U(T)$, and in fact, $W = ZC_U^\circ(T)$ since W is connected. Thus, $C_U(T) = C_W(T) = C_Z(T)C_U^\circ(T)$, and as $C_Z(T)$ is connected by [1, I Lemma 11.8], we see that $C_U(T)$ is connected. We note that [1, I Lemma 11.8] was not used at the outset because the lemma should state that the group being acted on is abelian instead of only nilpotent. □

A *Borel subgroup* of a group of finite Morley rank is defined to be a maximal definable connected solvable subgroup.

Lemma 2.3.23. *Let G be a group of finite Morley rank acting interpretably on an interpretable p-unipotent group. Additionally assume that G has no nontrivial p-unipotent subgroups. Then the Borel subgroups of G are $p^\perp$ good tori and are conjugate. Further, the same is true of any quotient of G by a normal solvable subgroup.*

Proof. Let B be a Borel subgroup of G. From the previous lemma, $F^\circ(B)$ is a $p^\perp$ good torus, so B centralizes $F^\circ(B)$ by the rigidity of good tori. Further, $B/F^\circ(B)$ is abelian by Fact 2.3.13, so

B is nilpotent. Hence, B is a good torus, and the conjugacy of the Borels of G now follows from Fact 2.3.21.

Now suppose that N is a normal solvable subgroup of G. Then every Borel of G/N will be the image of a Borel of G, so the Borels of G/N will be $p^\perp$ good tori as well. Their conjugacy follows as before. $\square$

The previous lemma leads us to a configuration dealt with by the following fact.

Fact 2.3.24 (see [5, Proposition 3.3 and Theorem 3.8])**.** *Let G be a connected group of finite Morley rank without nontrivial unipotent torsion whose Borel subgroups are nilpotent and whose definable subgroups all satisfy conjugacy of their Borel subgroups.*

(1) If H is a definable normal subgroup of G that is without involutions, then H is centralized by every 2-torus in G.

(2) If G is simple, then G has no involutions.

2.3.7 Algebraic subgroups

When examining L^*-groups of finite Morley rank, we hope to encounter definable subgroups that are algebraic. We now give a few facts regarding this situation.

The first result refers to graph automorphisms of a quasisimple algebraic group. We will not elaborate on this topic except to say that the result will only be applied when the group in question has no graph automorphisms. Indeed, we will be concerned only with quotients of SL_2.

Fact 2.3.25 (see [1, II Proposition 2.26])**.** *Let G be a group of finite Morley rank and L a normal subgroup isomorphic to a quasisimple algebraic group over an algebraically closed field. Suppose that G is connected or L has no graph automorphisms. Then, $G = L * C_G(L)$.*

Fact 2.3.26 (see [1, II Proposition 3.1])**.** *Let G be a perfect group of finite Morley rank such that $G/Z(G)$ is a quasisimple Chevalley group. Then G is a Chevalley group over the same field. In particular, $Z(G)$ is finite.*

Lemma 2.3.27. *Let G be a group of finite Morley rank that is a perfect central extension of $\mathrm{PSL}_2(F)$ for F an algebraically closed field. Then G is isomorphic to $\mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$. Thus, if G contains more than one involution, $G \cong \mathrm{PSL}_2(F)$, and $Z(G) = 1$.*

Proof. By Fact 2.3.26, we see that G is a Chevalley group over F with $Z(G)$ finite. The dimension of a maximal algebraic torus of G will be the same as the dimension of a maximal algebraic torus of $G/Z(G)$, so [13, Corollary 32.3] shows that $G \cong \mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$. □

2.3.8 Heavy hitters

This section contains some of the more powerful theorems that we have about groups of finite Morley rank, the first being the deepest.

Fact 2.3.28 (see [1])**.** *There are no infinite simple groups of finite Morley rank of mixed type and those of even type are algebraic.*

Fact 2.3.29 (see [2])**.** *Let G be a connected group of finite Morley rank whose Sylow 2-subgroups are finite. Then the Sylow 2-subgroups are trivial, i.e. G has no involutions.*

Fact 2.3.30 (see [2, Proposition 1.1])**.** *Let G be a connected and nontrivial group of finite Morley rank. Then the centralizer of any element of G is infinite.*

Fact 2.3.31 (see [7, Theorem 3])**.** *Let G be a connected group of finite Morley rank, π a set of primes, and g any π-element of G such that $C_G^\circ(g)$ is without nontrivial π-unipotent subgroups. Then g belongs to a π-torus.*

2.4 Algebraic groups

This final section contains a handful of facts on algebraic groups over algebraically closed fields. These facts will be applied to algebraic subgroups found in L^*-groups of finite Morley rank. Further information on algebraic groups, including the relevant definitions, may be found in [13].

Fact 2.4.1 ([13, Theorem 19.3])**.** *Let G be a connected solvable affine algebraic group, and let G_u be the set of unipotent elements of G. Then, G_u is a closed connected normal subgroup of G, and for T any maximal (algebraic) torus of G, $G = G_u \rtimes T$.*

Fact 2.4.2 ([13, Corollary 21.4])**.** *Let G be a connected affine algebraic group and B a Borel subgroup (in the algebraic sense). Then, $C_G(B) = Z(G)$.*

Fact 2.4.3 ([13, Proposition 21.4 B(a)])**.** *Let G be a connected affine algebraic group. If some Borel subgroup (in the algebraic sense) is nilpotent, then G is nilpotent.*

Fact 2.4.4 ([13, Corollary 23 A])**.** *Let G be a connected affine algebraic group. Each Borel subgroup (in the algebraic sense) of G is a maximal solvable subgroup of G.*

Chapter 3

BN-pairs of Tits rank 1 and Moufang sets

The present chapter is devoted to background material on groups with a split BN-pair of Tits rank 1, Moufang sets, and their connection. Much of the information presented here is free from the restriction of finite Morley rank, but the later sections do indeed impose this requirement. The most important portions of this chapter are the facts about special abelian Moufang sets as well as Corollary 3.2.3 which deals with the definability and connectedness of certain groups for Moufang sets of finite Morley rank. We begin with the essential definitions.

Definition 3.0.5. A *split BN-pair of Tits rank* 1 for a group G is a triple of subgroups (B, N, U) such that for $H := B \cap N$ the following conditions are met.

(1) $G = \langle B, N \rangle$.

(2) $|N : H| = 2$.

(3) For any $\omega \in N - H$, we have that

  (a) $H = B \cap B^{\omega}$,

  (b) $G = B \sqcup B\omega B$, and

  (c) $B^{\omega} \neq B$.

(4) $B = U \rtimes H$.

Notice that item (3c) implies that $U \neq 1$. Also, (3b) is equivalent to the condition that $G = B \sqcup B\omega U$, in the presence of (4).

When studying a split BN-pair of Tits rank 1, we usually work modulo the core of B, and this gives rise to a split 2-transitive permutation group acting on $B\backslash G$. Recall that a *split 2-transitive* permutation group on a set X is a group G acting faithfully and 2-transitively on X such that for any distinct $x, y \in X$, G_x splits as $G_x = U_x \rtimes G_{x,y}$ for some normal subgroup U_x of G_x. This condition forces U_x to act regularly on $X - \{x\}$, and $(G_x, G_{\{x,y\}}, U_x)$ is a split BN-pair of Tits rank 1 for G. We often further narrow our focus to the group generated by the set $\{U_x : x \in X\}$. This leads to the idea of a Moufang set, which we now introduce.

Definition 3.0.6. For a set X, $|X| \geq 3$, and a collection of groups $\{U_x : x \in X\}$ with each $U_x \leq \mathrm{Sym}(X)$, $(X, \{U_x : x \in X\})$ is called a *Moufang set* if for $G := \langle U_x : x \in X \rangle$ the following conditions hold:

(1) Each U_x fixes x and acts regularly on $X - \{x\}$,

(2) $\{U_x : x \in X\}$ is a conjugacy class of subgroups in G.

G is called the *little projective group* of the Moufang set, and the collection of U_x for $x \in X$ are called the *root groups*.

A Moufang set is called *proper* if its little projective group is **not** sharply 2-transitive on X. Two Moufang sets $(X, \{U_x : x \in X\})$ and $(Y, \{V_y : y \in Y\})$ are said to be *isomorphic* if there is a bijection $\varphi : X \to Y$ such that induced map $\mathrm{Sym}(X) \to \mathrm{Sym}(Y) : g \mapsto g^\varphi := \varphi^{-1} g \varphi$ resticts to an isomorphism from U_x to $V_{x\varphi}$ for each $x \in X$. The following well-known fact connects Moufang sets and split BN-pairs of Tits rank 1.

Fact 3.0.7. *Let* (B, N, U) *be a split* BN*-pair of Tits rank* 1 *for a group* G*. Set* $K := \mathrm{Core}_G(B)$*,* $M := \langle U^g : g \in G \rangle K$*, and* $X := B\backslash G$*. Then,* $(X, \{(UK/K)^x : x \in X\})$ *is a Moufang set with little projective group* M/K *and root groups isomorphic to* U*. Further,* G/K *embeds into* $\mathrm{Aut}(M/K)$*.*

3.1 Moufang sets

When studying split BN-pairs of Tits rank 1, Fact 3.0.7 implies that we should start by studying Moufang sets. Here we take a moment to recall some ideas from the theory of Moufang sets; more information can be found in [9], [10], [12], [19], and [21]. We begin with the $\mathbb{M}(U, \tau)$ construction.

As described in [12], every Moufang set can be constructed as an $\mathbb{M}(U, \tau)$ as follows. For a, not necessarily abelian, group $(U; +, -, 0)$, construct a set $X = U \cup \{\infty\}$ where ∞ is a symbol not appearing in U. Additionally, choose a $\tau \in \mathrm{Sym}(X)$ such that τ interchanges 0 and ∞. Define $\mathbb{M}(U, \tau)$ to be $(X, \{U_x : x \in X\})$ where each U_x is a subgroup of $\mathrm{Sym}(X)$ defined as follows.

(1) For each $u \in U$, α_u is the permutation of X that fixes ∞ and sends each $v \in U$ to $v + u$.

(2) $U_\infty = \{\alpha_u : u \in U\}$.

(3) $U_0 = U_\infty^\tau$.

(4) $U_u = U_0^{\alpha_u}$ for each $u \in U^* := U - \{1\}$.

Notice that a Moufang set constructed this way is *abelian*, i.e. has abelian root groups, if and only if U is abelian. There are some important distinguished elements arising from this construction. It is shown in [9] that there is associated to each $u \in U^*$ a unique element of $U_0 \alpha_u U_0$ interchanging 0 and ∞, and it is referred to as μ_u. It is a fact that $\mu_u^{-1} = \mu_{-u}$. For $h_u := \tau\mu_u$, the collection $\{h_u : u \in U^*\}$ will be called the *Hua maps*, and it is a theorem of [12] that $\mathbb{M}(U, \tau)$ is a Moufang set precisely when the Hua maps are in $\mathrm{Aut}(U)$.

When $\mathbb{M}(U, \tau)$ is a Moufang set, the pointwise stabilizer of 0 and ∞ in the little projective group is called the *Hua subgroup*, and it is generated by the set $\{\mu_a\mu_b : a, b \in U^*\}$, see [12]. Notice that two different choices of τ will give rise to identical Moufang sets exactly when they conjugate U_∞ to the same subgroup of $\mathrm{Sym}(X)$. Now if $\mathbb{M}(U, \tau) = \mathbb{M}(U, \rho)$, both constructions have the same μ-maps, but their Hua maps will certainly differ when $\tau \neq \rho$.

Finally we mention two frequently used identities in Moufang sets. For any $a \in U^*$, $h \in G_{0,\infty}$, and $n \in G_{\{0,\infty\}} - G_{0,\infty}$, we have that $\mu_a^h = \mu_{ah}$ and $\mu_{-a}^n = \mu_{an}$, see [9, Proposition 3.9(2)].

Example 3.1.1. Let F be a field. We define $\mathbb{M}(F)$ to be $\mathbb{M}(U, \tau)$ where $U := F^+$ and τ is the permutation of $X := U \cup \{\infty\}$ swapping 0 and ∞ and sending each $x \in F^*$ to $-x^{-1}$. Then $\mathbb{M}(F)$

is a Moufang set, see [12, Example 3.1]. Let G be the little projective group of $\mathbb{M}(F)$ and H the Hua subgroup.

- $G = \mathrm{PSL}_2(F)$, $X = \mathbb{P}_1(F)$, and the action of G on X is the natural one.
- μ_a sends $x \in F^*$ to $-a^2x^{-1}$, and μ_a is an involution.
- h_a sends $x \in F^*$ to a^2x.
- $\tau = \mu_1$ for 1 the identity of $F^\times$.
- $U_\infty \cong F^+$ and $H = \{h_a : a \in F^*\}$ with H isomorphic to the subgroup of squares of $F^\times$.

If φ is an isomorphism from $\mathbb{M}(U,\tau)$ to $\mathbb{M}(V,\rho)$, it need not be true that φ maps U to V. However, φ may be composed with a suitable element of the little projective group of $\mathbb{M}(V,\rho)$ to yield an isomorphism ψ from $\mathbb{M}(U,\tau)$ to $\mathbb{M}(V,\rho)$ that does indeed send U to V. In fact, this can be arranged so that ψ restricts to a group isomorphism from U to V, and in this case, ψ will respect α and μ-maps, i.e. $\alpha_u^\psi = \alpha_{u\psi}$ and $\mu_u^\psi = \mu_{u\psi}$ for all $u \in U^*$. This observation can be combined with the above example to obtain the following fact.

Fact 3.1.2. *Let $\mathbb{M}(U,\tau)$ be a Moufang set with little projective group G and Hua subgroup H. Suppose that $\mathbb{M}(U,\tau) \cong \mathbb{M}(F)$ for F a field. Then for any $a \in U^*$, we have that $H = \{\mu_a\mu_u : u \in U^*\}$ and $G_{\{0,\infty\}} - H = \{\mu_u : u \in U^*\}$.*

3.1.1 Special Moufang sets

A Moufang set $\mathbb{M}(U,\tau)$ is called *special* if the action of τ on U^* commutes with inversion. We now collect some facts about special abelian Moufang sets some of which do not require that the Moufang set is both special and abelian. All of these properties are reflected in the above example.

Fact 3.1.3. *Let $\mathbb{M}(U,\tau)$ be a special abelian Moufang set.*

(1) U has the structure of a vector space, see [23, Theorem 5.2(a)].

(2) U is an elementary abelian 2-group, or the Hua subgroup acts irreducibly on U, see [21, Theorem 1.2].

(3) For $a \in U^*$, μ_a is an involution, see [9, Lemma 5.1].

(4) For $a, b \in U^*$, $b\mu_a = -b$ if and only if $a = \pm b$, see [9, Lemma 4.3(2), Proposition 4.9(1)].

(5) For $a, b \in U^*$, $\mu_a = \mu_b$ if and only if $a = \pm b$, see [9, Proposition 5.2(5)].

To connect the notions of special and abelian, we have the following important theorem of Yoav Segev.

Fact 3.1.4 ([20, Main Theorem])**.** *If $\mathbb{M}(U, \tau)$ is an abelian Moufang set, then $\mathbb{M}(U, \tau)$ is either special or has a sharply 2-transitive little projective group.*

3.1.2 Root subgroups

We will make much use of so-called root subgroups.

Definition 3.1.5. Let $\mathbb{M}(U, \tau)$ be a Moufang set. A *root subgroup* of U is a subgroup $V \leq U$ such that there exists some $v \in V^*$ with $V^*\mu_v = V^*$.

Root subgroups are important because they give rise to new Moufang sets. Indeed, if V is a root subgroup of U and v is any $v \in V^*$, then $\mathbb{M}(V, \rho)$ is a Moufang set where ρ is the restriction of μ_v to $V \cup \{\infty\}$, see [10, Corollary 1.8]. This Moufang set will be the same for each $v \in V^*$ and will be called the *Moufang set induced by V*. A source of root subgroups is the collection of subgroups of the form $C_U(h)$ for h an element of the Hua subgroup, see [10, Corollary 1.9]. For special abelian Moufang sets, the following fact yields another collection of root subgroups.

Fact 3.1.6 ([19, Lemma 3.5])**.** *Assume that $\mathbb{M}(U, \tau)$ is a special abelian Moufang set with Hua subgroup H. Set F to be* $\mathrm{GF}(p)$ *if U has characteristic $p > 0$ and $\mathbb{Q}$ otherwise. Let $h \in H^*$, and assume that $\lambda \in F^*$ is an eigenvalue of h. Let $V_{h,\lambda}$ be the λ-eigenspace of h. Then*

(1) $V_{h,\lambda}$ is a root subgroup of U, and

(2) if $-\lambda$ is also an eigenvalue of h, then for each $x \in V_{h,-\lambda}^*$ we have $V_{h,\lambda}^* \mu_x = V_{h,\lambda}^*$

We often need to pull information from induced Moufang sets back to the original Moufang set. The following definition introduces some subgroups useful in this process.

Definition 3.1.7. Let $\mathbb{M}(U, \tau)$ be a Moufang set with little projective group G and Hua subgroup H. For V a root subgroup of U and $a \in V^*$, we make the following definitions:

- $G(V) := \langle \alpha_v, \mu_v : v \in V^* \rangle \leq G$,
- $H(V) := \langle \mu_v \mu_w : v, w \in V^* \rangle \leq H$,
- $U_\infty(V) := \{\alpha_v : v \in V\} \leq U_\infty$, and
- $U_0(V) := U_\infty(V)^{\mu_a} \leq U_0$.

$U_\infty(V)$ and $U_0(V)$ are often referred to as V_∞ and V_0, respectively, but we will avoid this. Each of the above subgroups may be thought of as the V-part of the corresponding subgroup, and they all act on $V \cup \{\infty\}$. The α-maps and μ-maps for the induced Moufang set are just the restrictions to $V \cup \{\infty\}$ of the corresponding maps in $G(V)$. In particular, the induced little projective group can be identified with $G(V)/C_{G(V)}(V)$.

Fact 3.1.8 ([19, Lemma 3.2])**.** *Let $\mathbb{M}(U, \tau)$ be a Moufang set with little projective group G and Hua subgroup H. Suppose that V is a nontrivial root subgroup of U, and set $Y := V \cup \{\infty\}$.*

(1) $G(V)$ is generated by $U_\infty(V)$ and $U_0(V)$.

(2) $G(V) \trianglelefteq N_G(Y)$ and $N_G(Y) = G(V)N_H(V)$.

(3) $C_G(Y) = C_H(Y)$ and $[C_G(Y), G(V)] = 1$; in particular, $C_{G(V)}(Y) = Z(G(V))$.

(4) If $|V| \geq 3$ and $\mathbb{M}(U, \tau)$ is special, then $G(V)$ is a perfect group.

3.1.3 Moufang sets of finite Morley rank

We will say that a Moufang set $(X, \{U_x : x \in X\})$ with little projective group G is *interpretable* in a ranked structure if X, G, and the action of G on X are interpretable in the structure. An important point is that we do **not** require the root groups to be interpretable. Now define a *Moufang set of finite Morley rank* to be a Moufang set interpretable in a ranked structure. We choose the name "Moufang set of finite Morley rank" over "ranked Moufang set" since the little projective group will be a group of finite Morley rank.

Lemma 3.1.9. *Let $\mathbb{M}(U, \tau)$ be a Moufang set with little projective group G. Suppose that U_∞ is a definable subgroup of G. Then the following hold.*

(1) $X := U \cup \{\infty\}$ and the action of G on X are interpretable.

(2) U is interpretable as a group.

(3) The map $U \to G : a \mapsto \alpha_a$ is interpretable.

(4) The map $U^* \to G : a \mapsto \mu_a$ is interpretable.

(5) If τ is definable, the map $U^* \to G : a \mapsto h_a$ is interpretable.

(6) If V is a definable root subgroup of U and $G(V)$ is definable, then the induced little projective group and its action on $V \cup \{\infty\}$ are interpretable.

Proof. Recall that G acts 2-transitively on X. As ∞ is the unique fixed-point of U_∞, we see that $N_G(U_\infty) = G_\infty$, so G_∞ is definable. Thus, X and the action of G on X are interpretable. Additionally, U is interpretable (as a set). The interpretability of $a \mapsto \alpha_a$ is immediate from the fact that U_∞ is definable and acts regularly on U. This map is a group isomorphism, so the group structure on U can be interpreted by pulling back the (definable) group structure on U_∞.

The interpretability of $a \mapsto \mu_a$ and $a \mapsto h_a$ is given by

$$a \mapsto \mu_a = \{(a, g) \in U^* \times G : g \in G_{\{0,\infty\}} \cap U_0 \alpha_a U_0\} \text{ and}$$
$$a \mapsto h_a = \{(a, g) \in U^* \times G : g = \tau\mu_u\}.$$

The final point follows from the fact that the induced little projective group can be identified with $G(V)/C_{G(V)}(V)$. $\square$

3.2 Split BN-pairs of Tits rank 1 for groups of finite Morley rank

Here we collect some facts concerning the definability and connectivity of certain subgroups of groups of finite Morley rank with a split BN-pair of Tits rank 1.

Proposition 3.2.1. *Let (B, N, U) be a split BN-pair of Tits rank 1 for an infinite group of finite Morley rank G. Set $H := B \cap N$, $K := \mathrm{Core}_G(B)$, $M := \langle U^g : g \in G\rangle K$, and $X := B\backslash G$. If U is infinite and UK is definable, then the following statements are true.*

(1) B, N, H, K, and M are definable.

(2) X and the action of G on X are interpretable with X of degree 1.

(3) If $K = 1$, then U and M are connected.

(4) If $K = 1$ and $M = G$, then B and H are connected.

Proof. Assume that U is infinite and UK is definable. Set $x := B$. As x is the unique common fixed point of UK, we see that $N_G(UK) = B$, so B and K are definable. We also get that X and the action of G on X are interpretable. Now, X is infinite, so the primitivity of the action of G on X ensures that G° acts transitively on X. Thus, X has degree 1. H is definable as $H = B \cap B^\omega$, and N is definable since H is of finite index in it.

The definability of M will follow from the definability of M/K, so we move to the case when $K = 1$. Our assumption is now that U is definable. U acts regularly on $X - \{x\}$, so U is in interpretable bijection with the degree 1 set $X - \{x\}$. Thus, U is connected, so $M = \langle U^g : g \in G\rangle$ is definable and connected by Zil'ber's Indecomposability Theorem, see Fact 2.2.5.

Now in addition to assuming that $K = 1$, we further assume that $M = G$. Thus, G is connected. Pick $\omega \in N - H$, and recall that $G = B \sqcup B\omega U$. In fact, the regularity of U on $X - \{x\}$ easily yields that every element of $B\omega U$ has a unique representation as $b\omega a$ for $b \in B$ and

$a \in U$. Denoting the degree of G by $\deg(G)$, we have $1 = \deg(G) = \deg(B\omega U) = \deg(B)\deg(U)$, so $\deg(B) = 1$. Further, $B = U \rtimes H$, so $\deg(H) = 1$. □

The previous proposition shows that the definability of $U \cdot \mathrm{Core}_G(B)$ has nice consequences. When U is abelian, the following well-known lemma shows that $U \cdot \mathrm{Core}_G(B) = C_G(U)$. In a group of finite Morley rank $C_G(U) = C_G(u_1, \ldots u_n)$ for some $u_1, \ldots, u_n \in U$, so $C_G(U)$ is definable.

Lemma 3.2.2. *Let (B, N, U) be a split BN-pair of Tits rank 1 for a group G. If U is abelian, then $U \cdot \mathrm{Core}_G(B) = C_G(U)$.*

Proof. Set $K := \mathrm{Core}_G(B)$, $X := B\backslash G$, and $x := B$. Being a subgroup of B, K certainly normalizes U. As K and U intersect trivially, K centralizes U, and $UK \leq C_G(U)$. Additionally, the transitivity of U on $X - \{x\}$ yields that $C_G(U)/K$ is regular on $X - \{x\}$. Hence, $UK = C_G(U)$. □

We will make frequent use of the following corollary rephrasing the above proposition for Moufang sets with definable root groups. By the previous lemma, we can omit the definability requirement when assuming that the root groups are abelian.

Corollary 3.2.3. *Let G be a group of finite Morley rank that is the little projective group of an infinite Moufang set $(X, \{U_x : x \in X\})$ with definable root groups. Then X and the action of G on X are interpretable with all 1-point stabilizers, all 2-point stabilizers, and all root groups definable and connected. Further, G is connected.*

We end this section with a result on abelian Moufang sets of finite Morley rank due to Tom De Medts and Katrin Tent. Recall from Fact 3.1.3, that U has the structure of a vector space whenever $\mathbb{M}(U, \tau)$ is a special abelian Moufang set, and notice that the condition that $\mathrm{End}_H(U)$ be infinite is always satisfied when U has characteristic 0.

Fact 3.2.4 ([11, Theorem 2.1])**.** *Let $\mathbb{M}(U, \tau)$ be an infinite special abelian Moufang set of finite Morley rank with Hua subgroup H. Assume that $\mathrm{char}(U) \neq 2$ and $K := \mathrm{End}_H(U)$ is infinite. Then $\mathbb{M}(U, \tau) \cong \mathbb{M}(K)$ with K an algebraically closed field.*

3.3 The affine group

This section collects some well-known lemmas regarding Aff_1 and concludes by specializing these results to groups of finite Morley rank. Given a field F, recall that $\mathrm{Aff}_1(F)$ is the subgroup of $\mathrm{Sym}(F)$ consisting of the invertible linear functions $\{x \mapsto mx + b : m \in F^* \text{ and } b \in F\}$.

Lemma 3.3.1. *Let $A := \mathrm{Aff}_1(F)$ for F a field, and let N be the socle of A. The following statements are true.*

(1) All complements to N in A are N-conjugate.

(2) Every complement to N in A is self-normalizing.

(3) $N = C_A(n)$ for any $n \in N^*$.

(4) If H is any complement to N in A, then $H = C_A(h)$ for any $h \in H^*$.

Proof. We know that N is an abelian regular (under the natural action of A on F) normal subgroup of A, and N^* is the set of fixed-point free elements of A. This implies that $N = C_G(n)$ for any $n \in N^*$. Also, for H any complement to N in A, each $h \in H^*$ fixes exactly one point. Fix an $h \in H^*$. Note that H is abelian, so $H \leq C_A(h)$. However, h is contained in a point-stabilizer, say A_f for some $f \in F$. We know that $A_f = C_A(h)$, so $H \leq A_f$. As H and A_f are both complements to N, they must be equal. We conclude that every complement to N in A is a point-stabilizer, and these are known to be N-conjugate and self-normalizing. □

Corollary 3.3.2. *Let G be a group isomorphic to $A := \mathrm{Aff}_1(F)$ for F a field. If H is any complement to the socle of G, there is an isomorphism $\varphi : A \to G$ mapping A_0 (the stabilizer of the additive identity of F) to H. Hence, the action of G on $H\backslash G$ is isomorphic to the natural action of A on F.*

We now consider groups with Aff_1 as a normal subgroup.

Lemma 3.3.3. *Let G be a group with a normal subgroup M such that $M \cong \mathrm{Aff}_1(F)$ for F a field. If N is the socle of M and H is any complement to N in M, then*

(1) $G = N \rtimes N_G(H)$.

(2) The image of H *in* $\mathrm{End}(N)$ *is* $E - \{0_E\}$ *for* E *an* M*-definable field isomorphic to* F.

(3) The image of $N_G(H)$ *in* $\mathrm{End}(N)$ *acts by conjugation on* E *as field automorphisms.*

Proof. By Lemma 3.3.1, N acts regularly on the G-conjugates of H, so $G = N \rtimes N_G(H)$. For the remaining two points, we first observe that the action of H on N (by conjugation) is isomorphic to the action of H on $H\backslash M$, the latter being isomorphic to the natural action of $\mathrm{Aff}_1(F)$ on F by the previous corollary. We now show that the ring generated by H in $\mathrm{End}(N)$ is M-definable; we will use additive notation for N . Introduce a symbol 0_E for the zero endomorphism and then addition on $E := H \cup \{0_E\}$ can be defined using N and H as follows. For a fixed $n \in N^*$ and any $h_1, h_2 \in H$, either

(1) $n^{h_1} + n^{h_2} = 0$, or

(2) there is a unique $h_3 \in H$ such that $n^{h_1} + n^{h_2} = n^{h_3}$.

In either case, one can see from the action of H on N that the conclusion does not depend on our choice of n. Thus, the first case defines $h_1 + h_2 = 0_E$, and the second case defines $h_1 + h_2 = h_3$. The defining formulas for adding two elements of H can be extended to $H \cup \{0_E\}$ in the obvious way. The multiplication for E is clearly definable using H. Lemma 3.3.1 shows that H an N are M-definable, so E is an M-definable field. Finally, note that $N_G(H)$ acts on H by conjugation, and H generates E in $\mathrm{End}(N)$. Thus, the image of $N_G(H)$ in $\mathrm{End}(N)$ acts on E as field automorphisms. □

Corollary 3.3.4. *Let* G *be a group of finite Morley rank with a definable normal subgroup* M *such that* $M \cong \mathrm{Aff}_1(F)$ *for* F *an infinite field. If* N *is the socle of* M *and* H *is any complement to* N *in* M*, then* $G = M \rtimes C_{N_G(H)}(N)$.

Proof. By the previous lemma, the image of H in $\mathrm{End}(N)$ is $E - \{0_E\}$ for E an infinite definable field on which $N_G(H)$ acts by field automorphisms. By Fact 2.3.10, $N_G(H)$ acts trivially on the E. Thus, $N_G(H)$ acts E-linearly on N which is 1-dimensional over E. We conclude that $N_G(H) = H \rtimes (N_G(H) \cap C_G(N))$. The corollary now follows from the fact that $G = N \rtimes N_G(H)$. □

3.4 Lifting results from Moufang sets

Fact 3.0.7 describes how to "descend" from split BN-pairs of Tits rank 1 to Moufang sets. This section addresses the reverse process in a context of finite Morley rank.

Lemma 3.4.1. *Let (B, N, U) be a split BN-pair of Tits rank 1 for a group of finite Morley rank G such that U is definable and $\mathrm{Core}_G(B) = 1$. Set $M := \langle U^g : g \in G\rangle$.*

(1) If $M \cong \mathrm{PSL}_2(F)$ for F an algebraically closed field, then $G = M$.

(2) If $M \cong \mathrm{Aff}_1(F)$ for F an algebraically closed field, then $G = M$.

Proof. First assume that $M \cong \mathrm{PSL}_2(F)$. By Fact 2.3.25, $G = M * C_G(M)$. When acting on $B\backslash G$, each U^g has Bg as its unique fixed point. Thus, $C_G(M)$ fixes every $x \in B\backslash G$, and lies in the kernel of the action. We conclude that $C_G(M) \le \mathrm{Core}_G(B) = 1$, so $G = M$.

Next assume that $M \cong \mathrm{Aff}_1(F)$. Let S be the socle of M. By Corollary 3.3.4, $G = MC_G(S)$. Recall that G acts (faithfully and) 2-transitively on $B\backslash G$. As S is a normal abelian subgroup of G, S acts regularly on $B\backslash G$, so $S = C_G(S)$. We conclude that $G = M$. □

Corollary 3.4.2. *Let (B, N, U) be a split BN-pair of Tits rank 1 for a group of finite Morley rank G. Set $K := \mathrm{Core}_G(B)$, and let $\mathbb{M}$ be the associated Moufang set $(B\backslash G, \{(UK/K)^x : x \in B\backslash G\})$. If $\mathbb{M} \cong \mathbb{M}(F)$ for F an algebraically closed field and UK is definable, then G has a definable normal subgroup Q such that $G = Q * K$ and $Q \cong \mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$.*

Proof. Set $M := \langle U^g : g \in G\rangle K$. Since $\mathbb{M} \cong \mathbb{M}(F)$, $M/K \cong \mathrm{PSL}_2(F)$, and M/K acts on $B\backslash G$ as $\mathrm{PSL}_2(F)$ acts on the projective line. By the previous lemma, $G = M$. Set $Q = \langle U^g : g \in G\rangle$. For each $g \in G$, K normalizes U^g and $K \cap U^g = \{1\}$. Thus, $G = Q * K$, and it remains to show that Q is definable and identify the isomorphism type of Q. As $Q/(Q \cap K) \cong \mathrm{PSL}_2(F)$ with $Q \cap K$ central in Q, we will show that Q is definable and perfect and then appeal to Lemma 2.3.27.

Set $H := B \cap N$. By the structure of $\mathrm{PSL}_2(F)$, we have that $[UK/K, H/K] = UK/K$. We translate this to Q. Set $H_Q := H \cap Q$ and $K_Q = K \cap Q$. Then, $[UK_Q/K_Q, H_Q/K_Q] = UK_Q/K_Q$, so $[U, H_Q]K_Q = UK_Q$. As $[U, H_Q] \leq U$ and $U \cap K = 1$, we see that $[U, H_Q] = U$. We conclude that Q is perfect. Additionally, we have that Q is a normal quasisimple subgroup of G, so Q is definable by [4, Lemma 7.10]. $\square$

Chapter 4

Projective root subgroups

In this chapter we introduce projective root subgroups. These are root subgroups that induce PSL_2, and we expect to encounter such root subgroups when considering minimal counterexamples to Conjecture 1.1.2. The analysis of projective root subgroups contained in this chapter will be extended in later chapters and utilized in the proofs of Theorems 1.1.4 and 1.1.6.

It should be noted that our definition of a projective root subgroup, as given below, is tailored for Moufang sets of finite Morley rank. In particular, we only deal with root subgroups that induce $\mathrm{PSL}_2(F)$ for F a **field**. For the definition, recall that $\mathbb{M}(F)$ was introduced in Example 3.1.1.

Definition 4.0.3. A Moufang set is *projective* if it is isomorphic to $\mathbb{M}(F)$ for F a field. If $\mathbb{M}(U, \tau)$ is a Moufang set and $V \leq U$ is a root subgroup, V is called *projective* if it induces a projective Moufang set.

Minimal counterexamples to Theorem 1.1.4 have the property that **every** infinite proper definable root subgroup is projective. This situation gives rise to the following definition, which could also be extended to a more general setting.

Definition 4.0.4. An infinite Moufang set of finite Morley rank $\mathbb{M}(U, \tau)$ will be called a P^*-*Moufang set* if each infinite proper definable root subgroup of U induces an interpretable projective Moufang set.

The "P" in the definition is for "projective". It is not immediate that a definable root subgroup will induce an interpretable Moufang set, but we will see that this is indeed the case provide the original Moufang set has definable root groups. Our proof of this fact makes use of the following well-known lemma.

Lemma 4.0.5. *Let G be a group of finite Morley rank acting interpretably and nontrivially on an interpretable set X. If X is infinite and the action is primitive, then X has degree 1.*

Proof. Since the action is primitive and G° is normal in G, G° acts transitively or trivially on X. As X is infinite and G is transitive, G modulo the kernel of the action must be infinite. Hence, G° acts transitively. The orbits of a connected group have degree 1, so X has degree 1. □

Lemma 4.0.6. *Let $\mathbb{M}(U, \tau)$ be an infinite Moufang set of finite Morley rank with little projective group G. If U_∞ is definable and V is an infinite definable root subgroup of U, then $G(V)$ is definable and connected, and V induces an interpretable Moufang set. If $\mathbb{M}(U, \tau)$ is special, then the induced Moufang set is also special.*

Proof. We show that $G(V)$ is definable and connected as the interpretability of the induced Moufang follows from this by Lemma 3.1.9. The fact that the induced Moufang set is special whenever $\mathbb{M}(U, \tau)$ is special holds for all Moufang sets and is given in [10, Lemma 1.8].

Now, $G(V)$ acts 2-transitively on $Y := V \cup \{\infty\}$, so the definable group $N_G(Y)$, which contains $G(V)$, also acts 2-transitively on Y. This implies that Y has degree 1 by the previous lemma, so V has degree 1 as well. As V is in interpretable bijection with $U_\infty(V)$ using $v \mapsto \alpha_v$, $U_\infty(V)$ is connected. Hence, $U_0(V)$ is also connected. From Fact 3.1.8, we know that $G(V) = \langle U_\infty(V), U_0(V)\rangle$, so $G(V)$ is definable and connected by Zil'ber's Indecomposability Theorem. □

4.1 The structure of $G(V)$ for projective root subgroups

We fix some notation for the present section.

Setup. $\mathbb{M}(U, \tau)$ is an infinite proper Moufang set of finite Morley rank with little projective group G and Hua subgroup H. X denotes $U \cup \{\infty\}$. Further, assume that G has definable root groups.

Proposition 4.1.1. *Let V be an infinite definable projective root subgroup of U. Then V induces a Moufang set with little projective group isomorphic to $\mathrm{PSL}_2(F)$ for F an algebraically closed field, and for $Y := V \cup \{\infty\}$, the following hold:*

(1) $C_{G(V)}(Y) = C_{G(V)}(V) = Z(G(V))$,

(2) $G(V)/Z(G(V))$ *acts on* Y *as* $\mathrm{PSL}_2(F)$ *acts on* $\mathbb{P}_1(F)$,

(3) $G(V) \cong \mathrm{SL}_2(F)$ *or* $\mathrm{PSL}_2(F)$,

(4) $H(V)$ *is a torus isomorphic to* $F^\times$ *that contains* $Z(G(V))$,

(5) the image of $H(V)$ *in* $\mathrm{End}(V)$ *generates an interpretable field isomorphic to* F,

(6) $N_H(V) = H(V) * C_H(V)$ *with* $H(V) \cap C_H(V) = Z(G(V))$, *and*

(7) $N_G(Y) = G(V) * C_H(V)$ *with* $G(V) \cap C_H(V) = Z(G(V))$.

Proof. We identify the induced little projective group with $G(V)/C_{G(V)}(V)$. As V is projective, there is an infinite field F such that $G(V)/C_{G(V)}(V)$ acting on Y is isomorphic to $\mathrm{PSL}_2(F)$ acting naturally on $\mathbb{P}_1(F)$. Since V is definable, Lemma 4.0.6 ensures that $G(V)/C_{G(V)}(V)$ is interpretable. Note that the stabilizer of ∞ in $G(V)/C_{G(V)}(V)$ is interpretable and isomorphic to $\mathrm{Aff}_1(F)$, so the image of $H(V)C_{G(V)}$ in $\mathrm{End}(V)$ generates an interpretable field isomorphic to F, see Lemma 3.3.3. In particular, F is algebraically closed.

We now verify that Lemma 2.3.27 applies to $G(V)$. By Lemma 4.0.6, $G(V)$ is definable. Fact 3.1.8 says that $G(V)$ has center $C_{G(V)}(V)$. Additionally, the proof of Fact 3.1.8(4) only requires that the **induced** Moufang set be special, a hypothesis we certainly meet, so $G(V)$ is perfect. Lemma 2.3.27 applies and $G(V) \cong \mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$. This completes the first three items.

For the next two items, we first note that the pointwise stabilizer of 0 and ∞ in $G(V)$, namely $H(V)C_{G(V)}(V)$, is definable and isomorphic to the stabilizer of 0 and ∞ in $\mathrm{SL}_2(F)$ or $\mathrm{PSL}_2(F)$. We claim that $C_{G(V)}(V) \leq H(V)$. The nontrivial case is when $G(V) \cong \mathrm{SL}_2(F)$. Choose a $v \in V^*$.

Then, μ_v is in $G(V)_{\{0,\infty\}} - G(V)_{0,\infty}$. When $\mathrm{SL}_2(F)$ acts on $\mathbb{P}_1(F)$, each element that swaps 0 and ∞ squares to the central involution. Thus, $\langle \mu_v^2 \rangle = Z(G(V)) = C_{G(V)}(V)$, and the fourth item is complete as μ_v^2 is in $H(V)$. The fifth item follows, as mentioned earlier, from Lemma 3.3.3.

We now give the structure of $N_H(V)$. By construction of $H(V)$, $H(V)$ is normal in $N_H(V)$ and is centralized by $C_H(V)$. We have already seen that $H(V) \cap C_H(V) = Z(G(V))$, so we need only show that $N_H(V) = H(V)C_H(V)$. Let $A := V \rtimes (N_H(V)/C_H(V))$ and $B := V \rtimes (H(V)C_H(V)/C_H(V))$. Since B is a normal subgroup of A isomorphic to $\mathrm{Aff}_1(F)$, Corollary 3.3.4 tells us that $A = B \cdot C_A(V) = BV = B$. We conclude that $N_H(V)/C_H(V) = H(V)C_H(V)/C_H(V)$, so $N_H(V) = H(V)C_H(V)$. This is the sixth item. Additionally, Fact 3.1.8 tells us that $N_G(Y) = G(V)N_H(V)$, so the structure of $N_G(Y)$ follows from the structure of $N_H(V)$. $\square$

This proposition has a pair of useful corollaries.

Corollary 4.1.2. *If V is an infinite definable projective root subgroup of U and A is an $H(V)$-invariant subgroup of U, then either $V \leq A$ or $V \cap A = 1$.*

Proof. $H(V)$ acts transitively on V^* and normalizes $V \cap A$, so $V \cap A = 1$ or V. $\square$

Corollary 4.1.3. *If $V \leq W$ are two infinite definable projective root subgroups of U, then $V = W$.*

Proof. Clearly, we have that $H(V) \leq H(W)$, and $H(V)$ acts on W. By the Proposition 4.1.1, the image of $H(W)$ in $\mathrm{End}(W)$ generates an interpretable algebraically closed field F_W that acts transitively on W^*. Now use Fact 2.3.11 modulo $C_{H(W)}(W)$ to see that the image of $H(V)$ in $\mathrm{End}(W)$ generates another interpretable algebraically closed field F_V. As F_V is a definable subfield of F_W, the finiteness of the Morley rank forces $F_V = F_W$. Notice that F_V normalizes V since $H(V)$ does. Since F_V, which is equal to F_W, acts transitively on W, we must have $V = W$. $\square$

If U is p-unipotent, we obtain an additional corollary of Proposition 4.1.1.

Corollary 4.1.4. *Assume that $\mathbb{M}(U, \tau)$ is a P^*-Moufang set and that U is p-unipotent. For every $u \in U^*$ and every subgroup $A \leq C_H(u)$, $C_U(A)$ is infinite provided that $C_{C_H(u)}(A)$ is infinite. In particular, every h in $C_H(u)$ has an infinite fixed-point space.*

Proof. Let $u \in U^*$, and let A be a subgroup of $C_H(u)$ such that $C := C_{C_H(u)}(A)$ is infinite. Then C contains a nontrivial connected definable abelian subgroup, so Lemma 2.3.22 implies that C contains an element c with an infinite fixed-point space V. Since A centralizes c, $A \leq N_H(V)$. By Proposition 4.1.1, $N_H(V)/C_H(V)$ acts regularly on V, and A fixes $u \in V$. Thus, $A \leq C_H(V)$, and we see that A has an infinite fixed-point space.

Now, let $h \in C_H(u)$, and suppose that $W := C_U(h)$ is finite. By Fact 2.3.30, $C_H^\circ(h)$ is infinite. As $C_H^\circ(h)$ is connected and acts on the finite set W, $C_H^\circ(h)$ centralizes W. Thus $C_H^\circ(h) \leq C_H(u)$, so $C_{C_H(u)}(h)$ is infinite. Thus, we may apply our result to the case when $A = \langle h \rangle$ to see that h, in fact, fixes infinitely many points. $\square$

4.2 H-invariant projective root subgroups

We continue with the setup from the previous section.

Setup. $\mathbb{M}(U,\tau)$ is an infinite proper Moufang set of finite Morley rank with little projective group G and Hua subgroup H. X denotes $U \cup \{\infty\}$. Further, assume that G has definable root groups.

We begin with a lemma whose third point is slightly technical. The idea is to approximate the property of $\mathbb{M}(F)$ that $G_{\{0,\infty\}} - H = \{\mu_u : u \in U^*\}$.

Lemma 4.2.1. *If V is an infinite definable H-invariant projective root subgroup of U, then*

(1) $H = H(V) * C_H^\circ(V)$ with $H(V) \leq Z(H)$,

(2) $N_G(Y) = G(V) * C_H^\circ(V)$, and

(3) $G_{\{0,\infty\}} - H = \bigcup_{v \in V^*} (\mu_v C_H^\circ(V))$.

Proof. Let $Y := V \cup \{\infty\}$. The first two points follow quickly from Proposition 4.1.1 since $H = N_H(V)$ and H is connected. Now let $n \in G_{\{0,\infty\}} - H$. As $N_G(Y)$ contains H and μ_v for any $v \in V^*$, $N_G(Y)$ contains $G_{\{0,\infty\}}$. Thus, $n = gc$ for $g \in G(V)$ and $c \in C_H^\circ(V)$. As $n \in G_{\{0,\infty\}} - H$, $g \in G(V)_{\{0,\infty\}} - H(V)$. Since $G(V)/C_{G(V)}(V) \cong \mathrm{PSL}_2(F)$, Fact 3.1.2 implies that g is a μ-map modulo $C_{G(V)}(V)$. Thus if $C_{G(V)}(V) = 1$, we are done. Otherwise, $G(V)$ has a central involution i and $g \in \mu_v\langle i\rangle$ for some $v \in V^*$. If $g = \mu_v i$, we must deal with the possibility that $i \in C_H(V) - C_H^\circ(V)$. We are assuming that $G(V)$ has a central involution, so the μ-maps have order 4. Hence, in the case that $g = \mu_v i$, we see that $g = \mu_v i = \mu_v \mu_v^2 = \mu_v^{-1} = \mu_{-v}$, so g is a μ-map. □

The next lemma says that in a P^*-setting either H is isomorphic to the multiplicative group of a field or H is close to acting irreducibly on U, in some weak sense. Both conclusions are nice approximations to the situation for $\mathbb{M}(F)$.

Lemma 4.2.2. *Assume that $\mathbb{M}(U,\tau)$ is a P^*-Moufang set. If U contains distinct infinite proper definable H-invariant root subgroups V and W, then $H = H(V) = H(W)$.*

Proof. We begin by showing that H is abelian. By Lemma 4.2.1, we have that $H = H(V) * C_H^\circ(V) = H(W) * C_H^\circ(W)$ with both $H(V)$ and $H(W)$ abelian. We claim that $C_H(V) \cap C_H(W) = \{1\}$. If not, there is an $h \in H^*$ such that $C_U(h)$ properly contains V. By assumption, $C_U(h)$ is projective, and this contradicts Corollary 4.1.3. We conclude that $C_H(V) \cap C_H(W) = \{1\}$, so H embeds into $H/C_H^\circ(V) \times H/C_H^\circ(W)$. Thus, H is abelian.

We will be done if we can show that $C_H^\circ(V)$ and $C_H^\circ(W)$ are both trivial. Suppose not. Without loss of generality, we assume that $C_H^\circ(V) \neq \{1\}$. Since $C_H(V) \cap C_H(W) = \{1\}$, the image of $C_H(V)$ in $H/C_H(W)$ is infinite, and we now work to contradict this fact. Note that $G_{\{0,\infty\}}$ is generated by H and any μ-map, so $C_H(V)$ is central in $G_{\{0,\infty\}}$ since H is abelian and $C_H(V)$ is centralized by every μ_v with $v \in V^*$. Fix a $w \in W^*$, and let $a \in C_H(V)$ be arbitrary. We study the image of a in $H/C_H(W)$. Now, there is an $h \in H(W)$ such that $aC_H(W) = hC_H(W)$. Observe that

$$aC_H(W) = (aC_H(W))^{\mu_w} = (hC_H(W))^{\mu_w} = h^{-1}C_H(W),$$

since $G(V)/C_{G(V)}(V) \cong \mathrm{PSL}_2(F)$ and μ-maps invert the Hua subgroup in PSL_2. We conclude that the coset $aC_H(W)$ is of the form $hC_H(W)$ where $h^2 \in C_H(W)$. This means that h^2 is either 1 or the unique involution of $H(W)$ (if $H(W)$ even has an involution), so h is an element of order dividing 4 in $H(W)$. Thus, there are at most four possibilities for the image of a in $H/C_H(W)$. As

a was an arbitrary element of $C_H(V)$, we have that $H/C_H(W)$ is finite, contradicting our earlier observation. $\square$

Chapter 5

Abelian Moufang sets of finite Morley rank in characteristic 2

The focus of this chapter is a proof of Theorem 1.1.3.

Theorem 1.1.3. *Let* $\mathbb{M}(U, \tau)$ *be an infinite special abelian Moufang set of finite Morley rank with Hua subgroup* H*. Assume that* $\operatorname{char}(U) = 2$ *and* $K := \operatorname{End}_H(U)$ *is infinite. Then* $\mathbb{M}(U, \tau) \cong \mathbb{M}(K)$ *with* K *an algebraically closed field.*

We pursue a fairly elementary approach from within the theory of special Moufang sets. However, the final section of this chapter gives an alternative proof of Theorem 1.1.3, utilizing the deep result of [1] (see Fact 2.3.28), that is free from the hypothesis that $\operatorname{End}_H(U) := C_{\operatorname{End}(U)}(H)$ be infinite. For the most part, this chapter is a reproduction of [25]. We adopt the following setup for the rest of the chapter.

Setup. $\mathbb{M}(U, \tau)$ is an infinite special abelian Moufang set of finite Morley rank with little projective group G and Hua subgroup H. $\operatorname{char}(U)$ is assumed to be 2. X denotes $U \cup \{\infty\}$, and $K := \operatorname{End}_H(U)$.

5.1 H is transitive on U^*

The main result of this section is that H is transitive on U^*. By [16, Thm. 1.2(b)], K is an interpretable field that is either finite or algebraically closed. We follow the proof of [8, Proposition 2.1].

Definition 5.1.1. Let $A := \bigcup_{x \in X} U_x^*$ and $\overline{A} := A - U_\infty^*$.

Lemma 5.1.2. *We have that*

(1) A consists of involutions,

(2) $A^G = A$,

(3) $A \cap G_\infty = U_\infty^*$, and

(4) for $t \in U_\infty^*$, $C_A(t) \subseteq U_\infty$ and hence $C_A(t)t \subseteq U_\infty$.

Proof. By construction of $\mathbb{M}(U, \tau)$, U_∞ is isomorphic to U. As the U_x are conjugate, each U_x is isomorphic to U, so (1) holds since U has characteristic two. Item (2) follows from the fact that $\{U_x : x \in X\}$ is a conjugacy class of subgroups of G. For (3), it is clear that $U_\infty^* \subseteq A \cap G_\infty$. Now $U_\infty^* \supseteq A \cap G_\infty$ because each U_x is regular on $X - \{x\}$. Finally, $C_A(t)$ acts on the fixed points of t, which consists of only ∞. Thus $C_A(t) \subseteq G_\infty$. By (3), $C_A(t) \subseteq U_\infty$, so (4) holds. □

Proposition 5.1.3. *A is a conjugacy class of involutions in G.*

Proof. By Lemma 5.1.2, we need only show that all elements of A are conjugate. To prove this fact, it suffices to show that for all $s \in \overline{A}$ and $t \in U_\infty^*$, s and t are conjugate. As we will be working with involutions, it will be useful to recall that two involutions i and j generate a dihedral group $\langle ij \rangle \rtimes \langle i \rangle$. We will exploit the fact that if $|ij|$ is odd then i and j are conjugate. Regardless of the order of ij, the subgroup generated by ij is inverted by both i and j. In the case that $|ij|$ is even, this subgroup contains a unique involution, and that involution is in the center of $\langle ij \rangle \rtimes \langle i \rangle$.

Claim. If there exists $r \in \overline{A}$ such that r is conjugate to s and $|rt|$ is finite, then $|rt|$ is odd. Hence, r is $\langle r, t \rangle$-conjugate to t, and most importantly s and t are conjugate.

Proof. If $|rt|$ is even, there is a unique involution w in $\langle rt \rangle$, and w is centralized by r and t. Also, wt is conjugate (in $\langle r, t \rangle$) to r or t, so by Lemma 5.1.2(2) $wt \in A$. Similarly, $wr = rw \in A$. By Lemma 5.1.2(4), $t \in U_\infty^*$ implies that $(wt)t = w \in U_\infty^*$. But since $w \in U_\infty^*$, we can apply Lemma 5.1.2(4) again to get $(rw)w = r \in U_\infty^*$. This is a contradiction, so $|rt|$ is odd. □

In light of the claim, it suffices to directly show s is conjugate to t or to produce $r \in \overline{A}$ such that r is conjugate to s and $|rt|$ is finite. Set $g = st$. We are done if g has finite order, so assume that it does not. We now proceed in a fashion similar to the proof of [11, Proposition 3.4]. Let $d(g)$ be the definable hull of $\langle g\rangle$. Note that s and t invert g. As the set of elements of $d(g)$ inverted by s and t is a definable subgroup containing g (using that $d(g)$ is abelian), we get that $d(g)$ is inverted by s and t. Also, $d(g) = D \times C$ with D a divisible group and C a finite cyclic group. Using divisibility, write $g = d^2c$ with $d \in D$ and $c \in C$. As s inverts d, we have

$$s^d = d^{-1}sdss = d^{-2}s = cg^{-1}s = ct\,.$$

Hence, $s^dt = c$ is of finite order. If $c = 1$, then s is conjugate to t, and we are done.

Otherwise $c \neq 1$. We claim that this forces s^d to be in $\overline{A}$ so that the claim applies with $r = s^d$. Since s^d is conjugate to s, $s^d \in A$. If $s^d \notin \overline{A}$, s^d must be in U_∞^*. Thus $s^dt = c$ is in U_∞^*, so c is an involution. As s inverts $c \in d(g)$, s fixes c. By Lemma 5.1.2(4), s is in U_∞, which is a contradiction. We conclude that s^d is in $\overline{A}$. The claim applies, and we are done. $\square$

As the action of H on U is isomorphic to H acting on U_∞ by conjugation, we obtain the following corollary.

Corollary 5.1.4. *H is transitive on U^*. Hence, H acts irreducibly on U, and K is an interpretable field that is either finite or algebraically closed.*

Proof. For any $s, t \in U_\infty^*$, Proposition 5.1.3 tells us that s and t are G-conjugate, say $s^g = t$. Since ∞ is the unique fixed point of both s ant t, g must fix ∞. Further, $G_\infty = U_\infty H$, and since U_∞ acts trivially on itself, we may assume $g \in H$. Thus for any $s, t \in U_\infty^*$, s and t are H-conjugate.

Let $u, v \in U^*$. Now, $\alpha_u, \alpha_v \in U_\infty^*$ are conjugate in H, say $\alpha_u^h = \alpha_v$. Since $\alpha_{uh} = \alpha_u^h = \alpha_v$, we have $uh = v$, so H is transitive on U^*. Then H acts irreducibly on U, so K is a division ring. By [16, Thm. 1.2(b)], K is definable, so K is either a finite field or an algebraically closed field. $\square$

By Fact 3.1.3(2), an arbitrary special Moufang set $\mathbb{M}(V, \rho)$, that is not necessarily of finite Morley rank, has the property that either V is an elementary abelian 2-group or the Hua subgroup acts irreducibly on V. Combining this result with the previous corollary and the fact that special

Moufang sets with V finite and of characteristic two are isomorphic to $\mathrm{PSL}_2(\mathbb{F}_{2^\alpha})$, see [8], we obtain the following.

Corollary 5.1.5. *Suppose that the Hua subgroup of a special Moufang set $\mathbb{M}(V,\rho)$ acts reducibly on V. Then V is an infinite elementary abelian two-group, and the Moufang set is not of finite Morley rank.*

We now return our focus to $\mathbb{M}(U,\tau)$. The next lemma, which is really a corollary of H being transitive on U^*, establishes some properties of the μ-maps and Hua-maps. The substance of the lemma is the final item which we extract from [8, Proposition 2.1].

Lemma 5.1.6. *For $a,b \in U^*$ with $a \neq b$, we have that*

(1) a is the unique fixed point of μ_a,

(2) $\mu_a \neq \mu_b$,

(3) $h_a \neq h_b$, and

(4) $\{\mu_a : a \in U^*\}$ is a conjugacy class of non-commuting involutions in $G_{\{0,\infty\}}$ that are in fact H-conjugate.

Proof. For the first two points see Fact 3.1.3. Item (3) now follows as $h_x = \tau\mu_x$ for all $x \in U^*$. We now address (4). First, the μ-maps are involutions since $\mu_x^{-1} = \mu_{-x}$ and we are in characteristic 2. Now, we already know that the set of μ-maps is $G_{\{0,\infty\}}$-normal, so the transitivity of H on U^* shows that the set of μ-maps is a $G_{\{0,\infty\}}$-conjugacy class of involutions that are H-conjugate. Finally, let $a,b \in U^*$ such that $a \neq b$. If $[\mu_a,\mu_b] = 1$, then $\mu_{a\mu_b} = \mu_a^{\mu_b} = \mu_a$, and this forces $a\mu_b = a$. This contradicts the fact that b is the unique fixed point of μ_b, so $[\mu_a,\mu_b] \neq 1$. □

The final two points of this section give criteria for proving $\mathbb{M}(U,\tau) \cong \mathbb{M}(K)$.

Fact 5.1.7 ([11, Proposition 2.2])**.** *If H acts regularly on U^*, then $\mathbb{M}(U,\tau) \cong \mathbb{M}(K)$.*

Corollary 5.1.8. *If H is abelian, then $\mathbb{M}(U,\tau) \cong \mathbb{M}(K)$. In particular, if $\dim_K(U) = 1$, then $\mathbb{M}(U,\tau) \cong \mathbb{M}(K)$.*

Proof. We have seen that H acts transitively on U^*. If in addition H is abelian, then H will act regularly on U^*, so the previous fact applies. If $\dim_K(U) = 1$, then $H \leq \mathrm{GL}(U) = K^\times$, and H will be abelian. □

5.2 When $\mathrm{End}_H(U)$ is infinite

We now work to prove Theorem 1.1.3. For the remainder of this section we **assume that** K **is infinite**. Hence, K is an algebraically closed field, by Corollary 5.1.4.

Lemma 5.2.1. *U is an n-dimensional vector space over K for some $n \in \mathbb{N}$, and $H \leq \mathrm{GL}(U)$.*

Proof. Clearly, U is a vector space over K. Since U and K have finite Morley rank and K is infinite, U must be n-dimensional over K for some finite n. By definition of K, we see that $H \leq \mathrm{GL}(U)$. □

We now proceed by contradiction. Assume that the $\mathbb{M}(U, \tau)$ that we are working with is not isomorphic to $\mathbb{M}(K)$ and further that among all counterexamples to Theorem 1.1.3, U is of minimal rank. We now extract some results of [11] about our minimal counterexample.

Lemma 5.2.2. *If $V < U$ is a proper nontrivial definable root subgroup of U that is K-invariant, then V is 1-dimensional over K, and V induces a Moufang set isomorphic to $\mathbb{M}(K)$. In particular, we have that for all $h \in H^*$ either $C_U(h)$ is trivial or $C_U(h)$ is 1-dimensional over K and induces a Moufang set isomorphic to $\mathbb{M}(K)$.*

Proof. Let $V < U$ be a proper nontrivial definable root subgroup of U that is K-invariant. Since V is a definable root subgroup, V induces an interpretable Moufang set $\mathbb{M}' = \mathbb{M}(V, \mu_v|_{V \cup \{\infty\}})$ for any $v \in V^*$. Additionally, V has rank strictly less than the rank of U since U is connected. As V is a K-subspace of U, K restricts faithfully to a field of automorphisms of V, which we also call K. Let H' be the Hua subgroup of $\mathbb{M}'$, and note that $L := \mathrm{End}_{H'}(V) \supseteq K$. This is because a generating set for H' can be obtained by restricting some of the generators of H, i.e. certain products of pairs of μ-maps. Thus L is infinite, hence an algebraically closed field, so the finiteness of the Morley rank forces $L = K$. By the minimality of our counterexample $\mathbb{M}' \cong \mathbb{M}(K)$, and V is 1-dimensional over K. □

Lemma 5.2.3. *For each $a, b \in U^*$, we have the following:*

(1) $aK \leq U$ induces a Moufang set isomorphic to $\mathbb{M}(K)$,

(2) $(a \cdot t)\mu_b = a\mu_b \cdot t^{-1}$ for all $t \in K^*$, and

(3) $a\mu_{b \cdot t} = a\mu_b \cdot t^2$ for all $t \in K^*$.

Proof. Because we are assuming that $\mathbb{M}(U, \tau) \ncong \mathbb{M}(K)$, Fact 5.1.7 implies that there is some $h \in H^*$ that has fixed points in U^*. Since H acts transitively on U^*, we may replace h with a suitable conjugate and assume that h fixes a. By the previous lemma, aK induces a Moufang set isomorphic to $\mathbb{M}(K)$. This is the first point. Using that aK induces a Moufang set isomorphic to $\mathbb{M}(K)$, we have that $(a \cdot t)\mu_a = a \cdot t^{-1}$ for all $t \in K^*$. The following calculation proves (2):

$$(a \cdot t)\mu_b = (a\mu_a \cdot t)\mu_b = (a \cdot t^{-1})\mu_a\mu_b = a\mu_a\mu_b \cdot t^{-1} = a\mu_b \cdot t^{-1}.$$

Finally, we show (3). Replacing a in (2) with $a + b$, we have

$$(a \cdot t + b \cdot t)\mu_b = (a + b)\mu_b \cdot t^{-1}$$

By [10, Lemma 5.2(4)], we can rewrite the above equation as

$$((a \cdot t)\mu_{b \cdot t} + b \cdot t)\mu_b + (b \cdot t)\mu_b = (a\mu_b + b)\mu_b \cdot t^{-1} + b\mu_b \cdot t^{-1}.$$

Using (2), we get

$$((a \cdot t)\mu_{b \cdot t} + b \cdot t)\mu_b + (b \cdot t)\mu_b = (a\mu_b \cdot t + b \cdot t)\mu_b + (b \cdot t)\mu_b,$$

which simplifies to $(a \cdot t)\mu_{b \cdot t} = a\mu_b \cdot t$. Apply (2) one last time to get $a\mu_{b \cdot t} = a\mu_b \cdot t^2$. □

Corollary 5.2.4. *We have that $K^* \leq Z(H)$.*

Proof. By the definition of K it suffices to show that $K^* \leq H$. Let $t \in K^*$, and write $t = s^2$ for $s \in K^*$. By Lemma 5.2.3(3), $\mu_b\mu_{b \cdot s} = \mu_b^2 \cdot s^2 = t \cdot \mathrm{id}$, so $t \cdot \mathrm{id} \in H$. □

With Lemma 5.2.3 in hand, we are now able to extend Lemma 5.2.2 from fixed-point spaces to eigenspaces corresponding to eigenvalues in K by simply repeating the brief calculation in [19, Lemma 3.5].

Proposition 5.2.5. *For all $h \in H$, $h = \lambda \cdot id$ for some $\lambda \in K^*$, or each eigenspace of h is 1-dimensional and induces a Moufang set isomorphic to $\mathbb{M}(K)$.*

Proof. Assume that $h \notin K \cdot \mathrm{id}$. Let $\lambda \in K$ be an eigenvalue of h, and let V be the eigenspace of h corresponding to λ. We need only show that V is a root subgroup of U as Lemma 5.2.2 then shows that V is 1-dimensional and induces a Moufang set isomorphic to $\mathbb{M}(K)$.

Let $w \in V^*$. It suffices to show that V^* is μ_w invariant. Observe that for all $v \in V^*$, we have

$$v\mu_w h = vh\mu_w^h = vh\mu_{wh} = (v \cdot \lambda)\mu_{w\cdot\lambda} = v\mu_w \cdot \lambda^{-1}\lambda^2 = v\mu_w \cdot \lambda,$$

so $v\mu_w$ is indeed in V^*. □

For $h \in H$, we define $\mathrm{Spec}(h)$ to be the *spectrum* of h, i.e. the set of eigenvalues of h.

Lemma 5.2.6. *For all $h \in H$, $\alpha, \beta \in \mathrm{Spec}(h)$ implies that $\alpha^{-1}\beta^2 \in \mathrm{Spec}(h)$.*

Proof. Let $a, b \in U$ be eigenvectors of h corresponding to α and β respectively. Note that h is invertible, so neither α nor β is zero. We now show that $a\mu_b$ is an eigenvector of h corresponding to $\alpha^{-1}\beta^2$. Indeed,

$$a\mu_b h = ah\mu_{bh} = (a \cdot \alpha)\mu_{b\cdot\beta} = a\mu_b \cdot \alpha^{-1}\beta^2,$$

where the final equality comes from Lemma 5.2.3. □

The next lemma is a main ingredient of the proof of Theorem 1.1.3. We have seen that in our minimal counterexample Fact 5.1.7 implies that some element of H fixes a point in U^*. The next lemma says that in fact we can choose such an element of the form $\mu_a\mu_b$ for $a \neq b$. This will be the last result extracted from [11].

Lemma 5.2.7. *There exists $a, b \in U^*$ such that $a \neq b$ and $\mu_a\mu_b$ fixes a point in U^*.*

Proof. Pick $a, c \in U^*$ that are K-linearly independent; this is possible by Corollary 5.1.8. Let λ be an eigenvalue of $\mu_a\mu_c$, and write $\lambda^{-1} = t^2$ for some $t \in K^*$. Then setting $b = c \cdot t$, we have that

$$\mu_a\mu_b = \mu_a\mu_{c\cdot t} = \mu_a\mu_c \cdot t^2 = \mu_a\mu_c\lambda^{-1}.$$

Hence 1 is an eigenvalue of $\mu_a\mu_b$, so $\mu_a\mu_b$ has nontrivial fixed points in U^*. As a and b are linearly independent, we have that $a \neq b$. □

The next proposition sets the stage for the final contradiction. The proof of Theorem 1.1.3 will follow.

Proposition 5.2.8. *H does not contain involutions.*

Proof. Let $\mathrm{Inv}(H)$ denote the set of involutions in H. By way of contradiction, assume that $\mathrm{Inv}(H) \neq \emptyset$.

Claim. $\dim_K(U) = 2$.

Proof. Pick $h \in \mathrm{Inv}(H)$. The minimal polynomial for h is $x^2 - 1 = (x-1)^2$, and 1 is the only eigenvalue of h. Thus, U decomposes into a sum of 1 and 2-dimensional h-invariant subspaces. Now, h restricts to a linear transformation of each of these subspaces with minimal polynomials that divide x^2-1. Thus, each of these subspaces have a nontrivial fixed point space. By Proposition 5.2.2, we must have only one such subspace, and it must be 2-dimensional since h is nontrivial. □

We next show that $\mathrm{Inv}(H)$ can be characterized as the nontrivial elements of H that fix points in U^*. As each involution is unipotent, every involution will fix a point in U^*. Now suppose that $h \in H^*$ fixes a point in U^*. As $\dim_K(U) = 2$, we must have that $\mathrm{Spec}(h) = \{1, \lambda\}$. By Lemma 5.2.6 (with $\alpha = \lambda$ and $\beta = 1$), we see that $\lambda^{-1} \in \mathrm{Spec}(h)$. Because we are in characteristic 2, we must have $\lambda = 1$. As h is nontrivial, the minimal polynomial of h must be $(x-1)^2 = x^2 - 1$. We conclude that h is an involution, and we have established that

$$\mathrm{Inv}(H) = \{h \in H^* : \ h \text{ fixes a point in } U^*\}.$$

Now, for all $a, b \in U^*$ with $a \neq b$, we must have $(\mu_a \mu_b)^2 \neq 1$. Indeed, $(\mu_a \mu_b)^2 = 1$ implies that μ_a and μ_b commute, and this contradicts Lemma 5.1.6. Thus $\mu_a \mu_b$ is a nontrivial element of H that is not an involution. We have just seen that this forces $\mu_a \mu_b$ to act freely on U^*, but this contradicts Lemma 5.2.7. $\square$

We now expose the final contradiction and prove Theorem 1.1.3.

Proof of Theorem 1.1.3. By Fact 2.3.3, the previous proposition implies that H is uniquely 2-divisible. By Lemma 5.2.7, there exists $a, b \in U^*$ with $a \neq b$, such that $\mu_a \mu_b$ fixes some point c in U^*. Thus,

$$\mu_c^{\mu_a \mu_b} = \mu_{c\mu_a \mu_b} = \mu_c,$$

so $\mu_a \mu_c \mu_a = \mu_b \mu_c \mu_b$. Hence, $(\mu_a \mu_c)^2 = (\mu_b \mu_c)^2$. The unique 2-divisibility of H shows that $\mu_a \mu_c = \mu_b \mu_c$, so $\mu_a = \mu_b$. However, this implies that $a = b$, which is a contradiction. $\square$

5.3 An alternative approach

We now give an alternative proof of Theorem 1.1.3 that does not require the theory of Moufang sets nor does it require the hypothesis that $\mathrm{End}_H(U)$ be infinite. We instead appeal to the classification of the simple groups of finite Morley rank of even type, see Fact 2.3.28.

As before, let $\mathbb{M}(U, \tau)$ be an infinite special abelian Moufang set of finite Morley rank with little projective group G and Hua subgroup H. The characteristic of U is assumed to be 2, and X denotes $U \cup \{\infty\}$. As U_∞ is isomorphic to U, G possesses an infinite elementary abelian two-subgroup. Further, G is perfect by [10, Theorem 1.12] and hence simple by a well-known lemma of Iwasawa (see [18, Theorem 9.27]). Thus, we may appeal to Fact 2.3.28 and take G to be an algebraic group over an algebraically closed field of characteristic 2.

We now show that the action of G on X can be taken to be algebraic. As G is transitive on X, it suffices to show that G acts algebraically on the right cosets of a point stabilizer, and for that, it is enough to show that G_∞ is a closed subgroup. By Lemma 3.2.2, $U_\infty = C_G(U_\infty)$, so U_∞ is closed. It then follows that G_∞ is closed since $G_\infty = N_G(U_\infty)$.

We now have a simple algebraic group G acting algebraically and 2-transitively on an algebraic variety X. By [14], there are not many choices for this action, and it must be that the action of G on X is isomorphic to $\mathrm{PGL}_n(F)$ acting naturally on $\mathbb{P}_{n-1}(F)$ with F an algebraically closed field. It only remains to show that $n = 2$, as it will follow that F is isomorphic to $\mathrm{End}_H(U)$. The following lemma, which surely has seen many proofs, concludes our alternative proof of Theorem 1.1.3.

Lemma 5.3.1. *Let $G := PGL_n(L)$ act naturally on $X := \mathbb{P}_{n-1}(L)$ with L an algebraically closed field and $n \geq 2$. A point stabilizer of G contains an abelian subgroup transitive on the remaining points if and only if $n = 2$.*

Proof. Recall that G acts 2-transitively on X. If $n = 2$ and $p \in X$, then G_p contains the abelian subgroup whose nontrivial elements are induced by the transvections from $\mathrm{GL}_2(L)$ that fix p. This subgroup is transitive on $X - \{p\}$.

Now choose $p \in X$, and let U be an abelian subgroup of G_p that is transitive on $X - \{p\}$. We show that $n = 2$. Note that U is regular on $X - \{p\}$, so U is closed since $U = C_G(U)$. As U acts regularly on the degree 1 set $X - \{p\}$, it must be that U is connected. Now, there is a closed and connected $A \leq \mathrm{GL}_n(L)$ such that A contains $Z := Z(\mathrm{GL}_n(L))$ and $A/Z = U$. A is certainly solvable. Hence, A is contained in a Borel subgroup of $\mathrm{GL}_n(L)$, and the group, U, that A induces on projective space fixes a maximal flag. If U is to be transitive on $X - \{p\}$, we must have $n = 2$. $\square$

Chapter 6

Abelian Moufang sets of finite Morley rank in characteristic larger than 2

This chapter is devoted to the proof of Theorem 1.1.4.

Theorem 1.1.4. *Let $\mathbb{M}(U, \tau)$ be an infinite special abelian Moufang set of finite Morley rank with Hua subgroup H. Assume that $p := \mathrm{char}(U) > 2$. Then either of the following conditions ensure that $\mathbb{M}(U, \tau) \cong \mathbb{M}(F)$ for some algebraically closed field F:*

(1) H is without infinite elementary abelian p-subgroups, or

(2) H is an L-group.

In light of the "absolute" results for abelian Moufang sets of finite Morley rank in characteristic 0 and 2 (see Fact 3.2.4 and Section 5.3), Corollary 1.1.5 follows readily from Theorem 1.1.4 by using Corollary 3.4.2 in conjunction with Fact 3.1.4 and the fact that sharply 2-transitive groups of finite Morley rank with abelian point stabilizers are of the form Aff_1, see [4, Proposition 11.61]. This chapter is a slightly modified version of [26]. We adopt the following setup throughout the present chapter.

Setup. $\mathbb{M}(U, \tau)$ is an infinite special abelian Moufang set of finite Morley rank with little projective group G and Hua subgroup H. Let $p := \mathrm{char}(U)$, and assume that $p > 2$. X denotes $U \cup \{\infty\}$.

In this setting, U will be an elementary abelian p-group, and Lemma 2.3.22 applies to all definable connected nilpotent subgroups of the Hua subgroup. In particular, H has no nontrivial q-unipotent subgroups for q a prime different from p. H is also without definable torsion-free subgroups.

6.1 Preliminary analysis

The starting point for our study of $\mathbb{M}(U, \tau)$ will be Fact 3.2.4. We begin by refocusing the condition that the Hua subgroup has an infinite centralizer in $\mathrm{End}(U)$.

Lemma 6.1.1. *We have that $\mathbb{M}(U, \tau) \cong \mathbb{M}(F)$ for some algebraically closed field F provided that*

(1) the solvable radical of H, $\sigma(H)$, is infinite, or

(2) $C_H(a)$ is finite for some $a \in U$.

Proof. Suppose that $\sigma(H)$ is infinite, so H has an infinite abelian definable normal subgroup A. As H acts irreducibly on U (see Fact 3.1.3(2)), Fact 2.3.11 applies, and we get that A generates a field in $\mathrm{End}(U)$ over which H acts linearly. In particular, $C_{\mathrm{End}(U)}(H)$ is infinite, so point (1) follows from Fact 3.2.4.

For the second item, assume that there is an $a \in U^*$ such that $C_H(a)$ is finite. We will show that H is abelian and conclude by way of the first item (or [12, Theorem 6.1]). As $|C_H^{\pm}(a) : C_H(a)| \leq 2$, $C_H^{\pm}(a)$ is finite as well. Now, μ_a is an involutive automorphism of H, so let us calculate its fixed-point space. We have that $\mu_a^h = \mu_a$ if and only if $\mu_{ah} = \mu_a$, and the latter occures if and only if $ah = \pm a$, see Fact 3.1.3(5). We conclude that when μ_a acts on H, it leaves fixed exactly $C_H^{\pm}(a)$. As this subgroup is finite, μ_a inverts H by Fact 2.3.7. $\square$

We now deal with the case when the elements of G have few fixed points.

Lemma 6.1.2. *If $C_U(h)$ is finite for all $h \in H^*$, then $\mathbb{M}(U, \tau) \cong \mathbb{M}(F)$ for some algebraically closed field F.*

Proof. Suppose that every $h \in H^*$ fixes finitely many elements of U. For each $u \in U^*$, Lemma 2.3.22 implies that $C_H(u)$ has no nontrivial connected definable abelian subgroups, so $C_H(u)$ must be finite. Now the previous lemma applies applies. $\square$

6.2 The P^*-setting

In this section, we assume that $\mathbb{M}(U, \tau)$ has the P^*-property.

Setup. $\mathbb{M}(U, \tau)$ is an infinite special abelian P^*-Moufang set of finite Morley rank with little projective group G and Hua subgroup H. Let $p := \mathrm{char}(U)$, and assume that $p > 2$. X denotes $U \cup \{\infty\}$.

Recall that P^*-Moufang sets were defined in Chapter 4. Our present goal is to show the following.

(1) H contains a unique infinite subnormal quasisimple subgroup Q,

(2) Q contains nontrivial p-unipotent torsion, and

(3) $Q/Z(Q)$ is of odd type.

We begin by utilizing the fact that U is abelian to strengthen Proposition 4.1.1.

Proposition 6.2.1. *Let V be an infinite proper definable root subgroup of U. Then V induces a Moufang set with little projective group isomorphic to* $\mathrm{PSL}_2(F)$ *for F an algebraically closed field, and for $Y := V \cup \{\infty\}$, the following hold:*

(1) $G(V) \cong \mathrm{PSL}_2(F)$ and acts faithfully on Y as $\mathrm{PSL}_2(F)$ acts on $\mathbb{P}_1(F)$,

(2) $H(V)$ is a torus isomorphic to $F^\times$,

(3) $H(V)$ generates an interpretable field in $\mathrm{End}(V)$ that is isomorphic to F,

(4) $N_H(V) = H(V) \times C_H(V)$, and

(5) $N_G(Y) = G(V) \times C_H(V)$.

Proof. Note that $G(V)$ contains μ_v for each $v \in V^*$. Since the μ-maps are involutions by Fact 3.1.3, $G(V)$ has too many involutions to be $\mathrm{SL}_2(F)$, and everything follows from Proposition 4.1.1. $\square$

The next proposition is an observation that we do not need for our proof of Theorem 1.1.4, but we hope that it may be helpful in strengthening the theorem.

Proposition 6.2.2. *The following hold.*

(1) Every definable connected nilpotent subgroup of H is a direct product of a $p^{\perp}$ good torus and an elementary abelian p-group.

(2) For every nontrivial, proper $A < H$ such that $V := C_U(A)$ is infinite and $C_{U/V}(A)$ is nontrivial, A is an elementary abelian p-group.

Proof. If $\mathbb{M}(U, \tau) \cong \mathbb{M}(F)$ for some algebraically closed field F, the first item is clear, and there is nothing to show for the second item. Thus, we assume that $\mathbb{M}(U, \tau) \not\cong \mathbb{M}(F)$ for any algebraically closed field F.

By Lemma 2.3.22, every definable connected nilpotent subgroup of H is a direct product of a $p^{\perp}$ good torus and a p-unipotent subgroup. Thus, we must show that every p-unipotent subgroup of H is in fact elementary abelian. Now, every nontrivial p-unipotent subgroup of H centralizes an infinite subgroup of U, say V. Similarly, the p-unipotent subgroup will fix an infinite subgroup of U/V, so the first item will follow from the second.

Let A be a nontrivial, proper subgroup of H such that $V := C_U(A)$ is infinite and $C_{U/V}(A)$ is nontrivial. Suppose that A fixes $w + V$ for $w \notin V$. Let W be the pre-image of $C_{U/V}(A)$ in U . Then W is a definable subgroup of U containing $w+V$. As W properly contains V, Corollary 4.1.3 implies that A acts faithfully on W. Further, A acts quadratically on W, i.e. $[W, A, A] = 0$. As we will see (and is well-known), this implies that A is an elementary abelian p-group.

Let $a, b \in A$. Then for all $w \in W$, $[w, a, b] = 0 = [w, b, a]$. Thus $wab - wb - wa + w = wba - wa - wb + w$, so $wab = wba$. Hence, $[a, b]$ fixes every element of W. By the faithfulness of the action, A is abelian. Since A centralizes $[W, A]$, we get that for each $w \in W$ $[w, a^p] = p[w, a] = 0$, so a^p fixes every element of W. Again by faithfulness of the action, A has exponent p. $\square$

6.2.1 The generalized Fitting subgroup

We now begin to analyze the generalized Fitting subgroup of H. Recall that the components of a group and the generalized Fitting subgroup were defined in Section 2.3.3. The next lemma will be sharpened below.

Lemma 6.2.3. *If $\mathbb{M}(U,\tau) \not\cong \mathbb{M}(F)$ for any algebraically closed field F, then $F(H) = Z(H)$ is finite, H has a component, and either:*

(1) H has exactly one component, or

(2) every component of H acts freely on U^*.

Proof. By Lemma 6.1.1(1), $F(H)$ is finite and hence central. Let $E(H)$ be the layer of H, i.e. the (central) product of all of the components. Then $F^*(H) = E(H) * Z(H)$. By Fact 2.3.15, $H/Z(H)$ embeds into $\mathrm{Aut}(F^*(H))$. Since H is not finite, we must have that $E(H)$ is infinite, and H has a component. Further, as H is connected, each component is infinite.

Now assume that H has more than one component and that Q is a component of H that does not act freely on U^*. Then there is some $u \in U^*$ such that $Q \cap C_H(u)$ is nontrivial. By Corollary 4.1.4, we see that there is an element of Q with an infinite fixed-point space, V. Let $\widehat{Q}$ be the product of the remaining components. Recall that $\widehat{Q}$ centralizes Q, so $\widehat{Q} \leq N_H(V)$. As the Moufang set is P^*, $N_H(V)/C_H(V)$ is abelian. Since each component is a perfect group, we see that $\widehat{Q} \leq C_H(V)$, so by Corollary 4.1.3, $V = C_U(\widehat{Q})$. Now reverse the argument. Q centralizes $\widehat{Q}$, so $Q \leq N_H(V)$. As before, $Q \leq C_H(V)$. We now have that the layer of H is contained in $C_H(V)$. As the layer is a normal subgroup, $H \leq N_H(V)$ contradicting the fact that H acts irreducibly on U. We conclude that H has exactly one component or every component of H acts freely on U^*. □

Note that the second case in the previous lemma will force the components of H to be of *$p^\perp$ type*, i.e. they have no nontrivial p-unipotent subgroups, since p-unipotent subgroups of H fix many points of U. This observation allows us to use a recent result of Alexandre Borovik and Jeffrey Burdges to clarify the situation in the preceding lemma.

Proposition 6.2.4. *If $\mathbb{M}(U,\tau) \ncong \mathbb{M}(F)$ for any algebraically closed field F, then H has exactly one component, and the component contains nontrivial p-unipotent torsion.*

Proof. Suppose that $\mathbb{M}(U,\tau) \ncong \mathbb{M}(F)$ for any algebraically closed field F. Let Q be a component of H that is of $p^{\perp}$ type. As the center of Q lies in $F(H)$, $Z(Q)$ is finite. We claim that Q has no involutions. If Q does have a nontrivial Sylow 2-subgroup, it must be infinite by Fact 2.3.29. This forces the simple group $Q/Z(Q)$ to have involutions. More is true. As Q has $p^{\perp}$ type, Lemma 2.3.23 shows that the Borel subgroups of $Q/Z(Q)$ are good tori and are conjugate. By Fact 2.3.24, we arrive at a contradiction, so Q has no involutions.

Lemma 6.1.2 implies that there is an $h \in H$ such that $V := C_U(h)$ is infinite. From the P^*-hypothesis, $H(V) \cong F^{\times}$ for some algebraically closed field F of characteristic larger than 2. Let S be the definable closure of the Sylow 2-subgroup of $H(V)$. We show that $[Q,S] = 1$. As Q and S are of $p^{\perp}$ type, QS has $p^{\perp}$ type, so the Borel subgroups of QS are good tori and are conjugate. Now Fact 2.3.24 applies to show that S centralizes Q. We conclude that S centralizes every component of $p^{\perp}$ type.

If H has more than one component, we have noted that every component is of $p^{\perp}$ type, so S centralizes the entire layer of H. As $F(H)$ is central, S centralizes $F^*(H)$. This implies that S is actually contained in $F(H)$, contradicting that $F(H)$ is finite. We conclude that H has a single component, and the same argument shows that it is not of $p^{\perp}$ type. □

6.2.2 Involutions

The goal of this subsection is to show that $Q/Z(Q)$ is of odd type where Q is the unique component of H. We begin by reworking a proposition of Tom De Medts and Katrin Tent that locates the element of H inverting U. Let ι be the permutation of X that fixes ∞ and inverts U.

Proposition 6.2.5. *If $a, b \in U^*$ are such that $a\mu_b = a$, then $\mu_a\mu_b = \iota$.*

Proof. We follow [11, Proposition 3.4] and [19, Proposition 6.2]. Let $a, b \in U^*$ such that $a\mu_b = a$, and set $h := \mu_a\mu_b$. We have that $h^2 = \mu_a\mu_b\mu_a\mu_b = \mu_a\mu_{a\mu_b} = \mu_a\mu_a = 1$. Also, $h \neq 1$ as this would force $\mu_a = \mu_b$ and $a = a\mu_b = a\mu_a = -a$; recall that $a\mu_a = -a$ by Fact 3.1.3.

Assume that $h \neq \iota$. Set $V_+ = C_U(h)$, and $V_- = C_U^-(h)$. By Fact 3.1.6, V_+ and V_- are (nontrivial) root subgroups of U, and $U = V_+ \oplus V_-$ by Fact 2.3.8. Note that $a, b \in V_-$.

Claim. There is a $c \in V_+^*$ such that $\mu_c\mu_a$ has finite order.

Proof of claim. Choose any $c \in V_+^*$ and set $g := \mu_c\mu_a$. Assume that g has infinite order. For M the definable closure of $\langle g \rangle$, M is a direct product of a divisible group D and a finite cyclic group C. As μ_c inverts g, μ_c inverts M. Write $g = d^2c$ for $d \in D$ and $c \in C$. Now compute

$$\mu_{cd} = \mu_c^d = d^{-1}\mu_c d = d^{-2}\mu_c = cg^{-1}\mu_c = c\mu_a.$$

Thus $\mu_{cd}\mu_a = c$, and provided that $cd \in V_+$ we are done. We see from Fact 3.1.6 that μ_c and μ_a both normalize V_+^*, so g and M do as well. As $d \in M$, cd is indeed in V_+. □

Fix $c \in V_+^*$ such that $g := \mu_c\mu_a$ has finite order, and set $K := \langle \mu_c, \mu_a \rangle \leq G_{\{0,\infty\}}$. As mentioned above, V_+^* and V_-^* are both K-normal.

We first consider when g has odd order. Then there is some $n \in K$ such that $\mu_a = \mu_c^n = \mu_{cn}$. Thus $cn = \pm a$ contradicting the fact that c, hence cn, is in V_+^* while a is in V_-^*. We conclude that g must have even order.

Now assume that g has order $4t - 2$ for some t. Set $d := cg^t$, and note that $d \in V_+$. Then $\mu_d = \mu_c^{g^t}$ commutes with μ_a, so $\mu_{d\mu_a} = \mu_d^{\mu_a} = \mu_d$. Thus, $d\mu_a = \pm d$. If $d\mu_a = -d$, then $a = \pm d$ contradicting the fact that $d \in V_+^*$ and $a \in V_-^*$. We conclude that $d\mu_a = d$. As $d \in V_+$, $d = dh = d\mu_a\mu_b = d\mu_b$, and we see that μ_b fixes d. If $p \equiv 1 \pmod 4$, then there is an $s \in \mathbb{F}_p$, the field with p elements, that squares to -1. By [10, Proposition 7.7(4)], $d \in \{\pm bs\}$ contradicting the fact that $d \in V_+^*$ and $b \in V_-^*$. Hence, $p \equiv 3 \pmod 4$, and $-1 = s_1^2 + s_2^2$ for $s_1, s_2 \in \mathbb{F}_p^*$. Since μ_b fixes a and d, [19, Proposition 4.1(3)] implies that $(as_1 + ds_2)\mu_b = -(as_1 + ds_2)$, so $b = \pm(as_1 + ds_2)$. Thus $\pm ds_2$, hence d, is in V_-, which is a contradiction.

Thus, g has order $4t$ for some t. Observe that $g^2 = [\mu_c, \mu_a] = \mu_c\mu_{c\mu_a} = \mu_{a\mu_c}\mu_a \in H(V_+) \cap H(V_-)$ by again using that V_+^* and V_-^* are K-normal. Thus $g^{2t} \in H(V_+) \cap H(V_-)$. Now, g^{2t} is an involution, and each of $H(V_+)$ and $H(V_-)$ contain a unique involution that inverts V_+ and V_-, respectively. Thus g^{2t} inverts both V_+ and V_-, so $g^{2t} = \iota$. Additionally, h is an involution in $H(V_-)$, so in fact $h = g^{2t} = \iota$. □

We obtain the following two corollaries addressing fixed points of the μ-maps.

Corollary 6.2.6. *If $a \in U^*$, then μ_a has exactly two fixed points. Additionally, $-\mu_a := \iota\mu_a$ is a μ-map fixing $\pm a$, and $-\mu_a = \mu_b$ for b a fixed point of μ_a.*

Proof. By Example 3.1.1, we may assume that $\mathbb{M}(U, \tau) \ncong \mathbb{M}(F)$ for any algebraically closed field F. We first show that μ_a has at most two fixed points. Assume that μ_a fixes b and c in U^*. Then $\mu_b\mu_a = \iota = \mu_c\mu_a$, so $\mu_b = \mu_c$. This forces $b = \pm c$, so μ_a fixes at most two points.

By Lemma 6.1.1, a is fixed by some $h \in H^*$, and $V := C_U(h)$ is infinite by Corollary 4.1.4. Then, $\mu_a \in G(V)$, and $G(V)$ acting on V is isomorphic to $\mathrm{PSL}_2(F)$ acting on $\mathbb{P}_1(F)$ with F an algebraically closed field. Thus, μ_a fixes $\pm a\gamma$ for $\gamma \in F$ a square root of -1, so μ_a fixes exactly two points.

Finally, let b be a fixed point of μ_a, so $\mu_b\mu_a = \iota$. Then $-\mu_a = \iota\mu_a = \mu_b$. Further, [10, Proposition 7.7(1)] shows that μ_b fixes a, hence fixes $-a$ as well. □

Corollary 6.2.7. *If $a, b \in U^*$ are such that $b\mu_a = b$, then any infinite definable root subgroup containing a or b contains them both.*

Proof. Let V be an infinite proper definable root subgroup containing a. Then, $G(V)$ acting on V is isomorphic to $\mathrm{PSL}_2(F)$ acting on $\mathbb{P}_1(F)$ with F an algebraically closed field. As $\mu_a \in G(V)$, μ_a fixes two points of V^*, so the fixed points of μ_a lie in V. [10, Proposition 7.7(1)] shows that our hypotheses are symmetric in a and b, so we are done. □

We now work to show that H has a unique involution. We begin with a proposition that characterizes when two μ-maps commute.

Proposition 6.2.8. *Let $a, b \in U^*$. The following are equivalent:*

(1) $[\mu_a, \mu_b] = 1$,

(2) $\mu_a = \pm\mu_b$,

(3) $a \in \{\pm b, \pm c\}$ *for* c *a fixed point of* μ_b.

In particular if $[\mu_a, \mu_b] = 1$*, then any infinite definable root subgroup containing* a *or* b *contains them both.*

Proof. Assume that $[\mu_a, \mu_b] = 1$. Then $\mu_a = \mu_a^{\mu_b} = \mu_{a\mu_b}$, so $a\mu_b = \pm a$. If $a\mu_b = -a$, then $\mu_a = \mu_b$. If $a\mu_b = a$, then $\mu_a\mu_b = \iota$. In either case, $[\mu_a, \mu_b] = 1$ implies that $\mu_a = \pm\mu_b$. It is clear that the second item implies the first, so the first two items are equivalent.

Now assume that $\mu_a = \pm\mu_b$. If $\mu_a = \mu_b$, $a = \pm b$. Otherwise, $\mu_a = -\mu_b = \mu_c$ for c a fixed point of μ_b, and $a = \pm c$. We conclude that (2) implies (3). It is clear from Corollary 6.2.6 that (3) implies (2).

The final statement now follows from the previous corollary. □

The proof that H contains a unique involution will follow from the next lemma.

Lemma 6.2.9. *Let* V *be an infinite proper definable root subgroup of* U*. If* μ_x *normalizes* V^**, then* $x \in V$*.*

Proof. Choose $a \in V^*$. By the structure of $N_H(V)$, there exists $h \in H(V)$ and $c \in C_H(V)$ such that $\mu_a\mu_x = hc$. We may use Fact 3.1.2 to find a $b \in V^*$ such that $h = \mu_a\mu_b$, so $\mu_x = \mu_b c$. As μ_b centralizes $C_H(V)$, μ_b centralizes μ_x. By Proposition 6.2.8, $x \in V$. □

Proposition 6.2.10. H *contains a unique involution, all involutions of* G *are conjugate, and all involutions of* G *fix exactly 2 elements of* X*.*

Proof. We have already seen that H contains the central involution ι. Any other central involution would also invert U as H acts irreducibly on U.

Assume that H contains a non-central involution j. Let $V_+ := C_U(j)$ and $V_- := C_U^-(j)$. As in Proposition 6.2.5, V_+ and V_- are nontrivial root subgroups of U, and $U = V_+ \oplus V_-$. Choose

$x \in V_-^*$. By Fact 3.1.6, μ_x normalizes V_+^*, so the previous lemma shows that $x \in V_+$. This is a contradiction, and we conclude that H has a unique involution.

We now show that all involutions of G are conjugate. We may assume that $\mathbb{M}(U, \tau) \not\cong \mathbb{M}(F)$ for any algebraically closed field F. Then U contains an infinite proper definable root subgroup V, and ι is contained in the 2-torus of $H(V)$. Let S be a Sylow° 2-subgroup of G containing ι, and notice that ι, being toral, is central in S. S acts on the fixed points of ι, so $S \leq G_{\{0,\infty\}}$. S is connected, so in fact $S \leq H$. We conclude that S is a 2-torus with a unique involution, so G has odd type and Prüfer 2-rank equal to 1. The conjugacy of the involutions of G now follows from Fact 2.3.31 and the conjugacy of maximal 2-tori. □

Lemma 6.2.9 can also be used show that H acts transitively on U^*. The transitivity of H on U^* will not be used in our proof of Theorem 1.1.4, but as with Proposition 6.2.2, we hope that it may enlighten any effort remove the extra restrictions of the theorem.

Lemma 6.2.11. *For $a, b \in U^*$ with $\mu_a \neq \mu_b$, $\mu_a\mu_b$ acts freely on U^*.*

Proof. We may assume that $\mathbb{M}(U, \tau) \not\cong \mathbb{M}(F)$ for any algebraically closed field F. Let $a, b \in U^*$ such that $\mu_a \neq \mu_b$, and set $h = \mu_a\mu_b$. Towards a contradiction, suppose that $V := C_U(h)$ is nontrivial. Then V is infinite by Corollary 4.1.4. Since μ_a inverts h, μ_a acts on V^*, so $a \in V$ by Lemma 6.2.9. Similarly, $b \in V$. Thus $h = \mu_a\mu_b \in H(V)$. As $H(V)$ acts freely on V^*, it must be that $h = 1$, contradicting the fact that $\mu_a \neq \mu_b$. □

Proposition 6.2.12. *The set $\{\mu_a\mu_b : a, b \in U^*\}$, hence H, acts transitively on U^*.*

Proof. Choose $v, w \in U^*$. We will map v to w by an element of $\{\mu_a\mu_b : a, b \in U^*\}$. For each $x \in U^*$, let $A_x := \{x\mu_a : a \in U^*\}$, and note that A_x is a definable subset of U since the map $a \mapsto \mu_a$ is interpretable. We now show that A_x is a generic subset of U. Consider the definable surjection $\varphi : U^* \to A_x : a \mapsto x\mu_a$. Now, a and b are in the same fiber of φ precisely when $\mu_a\mu_b$ fixes x. By the previous lemma and Fact 3.1.3, this occurs only when $a = \pm b$. We conclude that φ has finite fibers, so A_x is indeed a generic subset of U. Thus, A_v and A_w have a nontrivial intersection. Suppose that $v\mu_a = w\mu_b$. Then, $v\mu_a\mu_b = w$. $\square$

We conclude our analysis of $\mathbb{M}(U,\tau)$ by showing that if $\mathbb{M}(U,\tau) \not\cong \mathbb{M}(F)$ for any algebraically closed field F then the sole component of H contains all Sylow$^\circ$ 2-subgroups of H. The idea is to show that the sole component contains some nontrivial 2-element, for then it must contain an infinite 2-subgroup. Now, to show that the component contains a 2-element, we observe that otherwise the component is of degenerate type with nontrivial involutive automorphisms. To deal with this configuration, we follow the approach taken in [5, Proposition 3.3]; however, our situation requires us to work around p-unipotent subgroups. We begin with a general lemma.

Lemma 6.2.13. *Let A be a group of finite Morley rank with a normal, definable $2^\perp$-subgroup B. Additionally, assume that there is an $i \in A$ such that i is an involution modulo $C_A(B)$. Then for every $b \in B$, we have that $C_B(i^b) \cap C_B^-(i) = \{1\}$.*

Proof. First observe that no $x \in B^*$ is B-conjugate to its inverse. Otherwise, the conjugating element would be in $C_B^\pm(x) - C_B(x)$, and this would force B to contain 2-torsion by Fact 2.3.4.

As B is normal, $i^b = [b, i^{-1}]i \in Bi$, so we may write $i^b = ci$ for $c \in B$. Now suppose that there is some nontrivial $x \in C_B(i^b) \cap C_B^-(i) = C_B(ci) \cap C_B^-(i)$. Then, $x^{-1} = x^i = (x^{ci})^i = (x^c)^{i^2} = x^c$, which contradicts our initial observation. $\square$

Proposition 6.2.14. *Suppose that $\mathbb{M}(U,\tau) \not\cong \mathbb{M}(F)$ for any algebraically closed field F and Q is the sole component of H. Then Q contains all Sylow$^\circ$ 2-subgroups of H. In particular, $Q/Z(Q)$ has odd type.*

Proof. First suppose that Q contains some nontrivial 2-torsion. As Q is connected, Fact 2.3.29 ensures that Q contains an infinite Sylow 2-subgroup. Since H is of odd type and has Prüfer 2-rank equal to 1, it must be that any Sylow$^\circ$ 2-subgroup of Q is in fact a Sylow$^\circ$ 2-subgroup of H. By the conjugacy of Sylow$^\circ$ 2-subgroups, Q contains all Sylow$^\circ$ 2-subgroups of H. Now, our assumption that $\mathbb{M}(U,\tau) \ncong \mathbb{M}(F)$ forces $F(H)$ to be finite, so $Z(Q) \leq F(H)$ is finite as well. Hence, $Q/Z(Q)$ contains an infinite Sylow 2-subgroup.

It remains to show that Q contains some nontrivial 2-torsion. Assume not, so Q is a normal $2^{\perp}$-subgroup of H. Let S be a Sylow$^{\circ}$ 2-subgroup of H. If S centralizes Q, hence $F^*(H)$, then S is contained in the finite group $F(H)$. This is a contradiction, so there is a $t \in S$ such that t is an involution modulo $C_H(Q)$. By Fact 2.3.8, $Q = C_Q(t)C_Q^-(t)$ where the multiplication map $C_Q(t) \times C_Q^-(t) \to Q$ is a bijection.

Claim. $C_Q^-(t)$ intersects some torus of Q nontrivially.

Proof of claim. By Lemma 2.3.22, Q is of $q^{\perp}$ type for every prime q different from p. In light of Fact 2.3.31, is suffices to show that $C_Q^-(t)$ has a nontrivial torsion element of order coprime to p.

We first show that $C_Q^-(t)$ has nontrivial torsion. Assume that $h \in C_Q^-(t)$ has infinite order, and let M be the definable closure of $\langle h \rangle$. Then M is infinite, abelian, and inverted by t. By Lemma 2.3.22, M° certainly contains nontrivial torsion.

We now show that $C_Q^-(t)$ has no elements of order divisible by p. Otherwise, $C_Q^-(t)$ has an element h of order p. Then, $V := C_U(h)$ is infinite; see [1, Lemma 1.23] for example. Note that $t \in N_H(V)$. By Proposition 6.2.10, $C_H(V)$ is $2^{\perp}$, so it must be that $t \in H(V)$. However, $C_H(V)$ centralizes $H(V)$, so h and t commute. This contradicts our assumption that $h \in C_Q^-(t)$. □

Choose h to be a nontrivial toral element of $C_Q^-(t)$. Let T be a maximal good torus of H containing t, and let R be a maximal good torus of QT containing h. By the conjugacy of maximal good tori, there is an $a \in Q$ and $b \in T$ such that $R = T^{ba} = T^a$. Then t^a centralizes h, so $h \in C_Q(t^a) \cap C_Q^-(t)$. This contradicts the previous lemma, so Q does indeed contain nontrivial 2-torsion. □

6.3 Proof of Theorem 1.1.4

We argue by contradiction. Let $\mathbb{M}(U,\tau)$ be a counterexample to Theorem 1.1.4 such that U has minimal rank among all counterexamples. Let G be the little projective group of $\mathbb{M}(U,\tau)$, H

the Hua subgroup, and $p := \mathrm{char}(U) > 2$. By Lemma 6.1.2, U has infinite proper definable root subgroups.

We wish to show that $\mathbb{M}(U, \tau)$ has the P^*-property. Let V be any infinite proper definable root subgroup of U. By Lemma 4.0.6 and the minimality of our counterexample, we need only show that the Hua subgroup of the induced Moufang set, namely $(H(V)C_{G(V)}(V))/C_{G(V)}(V)$, is either an L-group or is without infinite elementary abelian p-subgroups. Set $A := C_{G(V)}(V)$. As the action of $G(V)$ on V is interpretable, $G(V)_{0,\infty} = H(V)A$ is definable. If H is an L-group, then the section $H(V)A/A$ is certainly an L-group as well. Now suppose that H has no infinite elementary abelian p-subgroups. By Fact 3.1.8, A is the center of $G(V)$, so Lemma 2.3.23 ensures that $H(V)A/A$ has $p^\perp$ type as well. We conclude that $\mathbb{M}(U, \tau)$ is a P^*-Moufang set.

By Proposition 6.2.4, $F^*(H) = QZ(H)$ where Q is the sole component of H and $Z(H)$ is finite. Further, the same proposition together with Proposition 6.2.14 show that Q has infinite Sylow 2-subgroups as well as some nontrivial p-unipotent torsion. Thus, H has nontrivial p-unipotent torsion, so it must be that H is an L-group. As $Q/Z(Q)$ has odd type, the L-hypothesis forces $Q/Z(Q)$ to be a Chevalley group over an algebraically closed field. It then follows from Fact 2.3.26 that Q is a Chevalley group over the same field. In fact, Proposition 6.2.10 and its proof showing that G has Prüfer 2-rank equal to 1 force Q to be of the form SL_2.

We next show that $H = Q$. By Fact 2.3.25, $H = Q * C_H(Q)$. As $F^*(H) = QZ(H)$, we have that $H = F^*(H)C_H(F^*(H))$. However, $C_H(F^*(H))$ is always contained in $F^*(H)$, so $H = F^*(H) = QZ(H)$. Additionally, H is connected with $Z(H)$ finite, so $H = Q$.

We now have that H is a quasisimple affine algebraic group over an algebraically closed field that must have characteristic p. Let B be a Borel subgroup of H, in the algebraic sense. Note that B is definable as it is a maximal solvable subgroup of H (see Fact 2.4.4) and the definable closure of B is solvable. Since H is not solvable, Fact 2.4.1 and Fact 2.4.3 imply that $U_p(B)$ is nontrivial. Set $V := C_U(U_p(B))$, which must be infinite. As B normalizes $U_p(B)$, $B \leq N_H(V)$. Since $H(V)$ is central in $N_H(V)$, $H(V) \leq C_H(B)$. By Fact 2.4.2, $H(V) \leq Z(H)$. As we are in a counterexample, $Z(H)$ is finite, and we have a contradiction.

Chapter 7

Not necessarily abelian Moufang sets of finite Morley

In this final chapter, we work to extend our previous results to the setting where U is no longer assumed to be abelian. We carry out our analysis exclusively for Moufang sets of finite Morley rank whose little projective group has odd type. By [3, Lemma 5.8], we are only excluding the even type case, a case for which the classification of the even type simple groups should be helpful.

When considering proper but not necessarily abelian Moufang sets, one no longer has immediate use of the theory of special Moufang sets. This is a considerable restriction, and it gives us cause to call heavily on the theory of groups of finite Morley rank. One specific issue arising in proper, but not necessarily special, Moufang sets is that there is a possibility of encountering root subgroups for which the induced Moufang set is no longer proper. To highlight this situation, we make the following definition.

Definition 7.0.1. Let $\mathbb{M}(U,\tau)$ be an infinite proper Moufang set of finite Morley rank. Then $\mathbb{M}(U,\tau)$ is said to be *hereditarily proper* if every infinite definable root subgroup of U induces a Moufang set that is also proper.

It is a fact that special Moufang sets are necessarily proper provided the root groups have at least 3 elements. As the property of being special easily passes to root subgroups, we see that infinite special Moufang sets of finite Morley rank are hereditarily proper. In the course of our work, we will show that any infinite Moufang set of finite Morley rank that has definable $2^{\perp}$ root subgroups and a little projective group of odd type is hereditarily proper. The proof of Theorem 1.1.6, which we now recall, will conclude the chapter.

Theorem 1.1.6. *Let $\mathbb{M}(U, \tau)$ be an infinite proper Moufang set of finite Morley rank whose little projective group has odd type. Further assume that U_∞ is a definable $2^\perp$-group and that the Hua subgroup is nilpotent. Then $\mathbb{M}(U, \tau) \cong \mathbb{M}(F)$ for some algebraically closed field F.*

7.1 Few fixed-points

We adopt the following setup throughout this section.

Setup. $\mathbb{M}(U, \tau)$ is an infinite Moufang set of finite Morley rank with little projective group G and Hua subgroup H. X denotes $U \cup \{\infty\}$. Further, assume that G has odd type and definable root groups.

The first two results of this section address the situation when G is nearly a Zassenhaus group. Note that the following lemma includes the sharply 2-transitive case.

Lemma 7.1.1. *If $C_X(g)$ is finite for all $g \in G^*$, then G_∞ has odd type. If additionally H is nontrivial, then H has odd type as well.*

Proof. Define A to be H if H is nontrivial and G_∞ otherwise. We show that A has odd type.

Let T be the definable hull of a nontrivial 2-torus of G. Then, T is a decent torus, and $C^\circ_G(T)$ is generous in G by Fact 2.3.20. Recall that generosity was defined in Section 2.3.5.

We now work to establish that A is generous in G, and we begin by showing that A is almost self-normalizing. Let $N := N^\circ_G(A)$. Then N acts on $C_X(A)$, which is a nonempty finite set. As N is connected, so N fixes $C_X(A)$. Thus, $N \leq A$, so A is almost self-normalizing. Now our assumption that $C_U(a)$ is finite for all $a \in A^*$ implies the condition of Fact 2.3.19(2), so A is indeed generous.

We conclude that $\bigcup_{g\in G} C^\circ_G(T^g)$ and $\bigcup_{g\in G} A^g$ have a nontrivial intersection, so $C^\circ_G(T^g) \cap A \neq \{1\}$ for some $g \in G$. Choose a nontrivial $a \in C^\circ_G(T^g) \cap A$. Now, T^g is a connected group acting on the finite set $C_X(a)$. As before, T^g fixes $C_X(a)$, so $T^g \leq A$. $\square$

The previous lemma allows us to extend Lemma 6.1.2 to Moufang sets for which U need not be abelian.

Corollary 7.1.2. *If $\mathbb{M}(U,\tau)$ is proper and $C_U(h)$ is finite for all $h \in H^*$, then $\mathbb{M}(U,\tau) \cong \mathbb{M}(F)$ for some algebraically closed field F.*

Proof. It suffices to show that U is abelian and then appeal to Lemma 6.1.2. By Lemma 7.1.1, H contains an involution, and this involution acts on U with finitely many fixed-points. Hence, U is abelian by Fact 2.3.7. $\square$

Lemma 7.1.1 also yields a new class of hereditarily proper Moufang sets.

Corollary 7.1.3. *If U has no involutions, then $\mathbb{M}(U,\tau)$ is hereditarily proper.*

Proof. Let V be an infinite definable root subgroup of U. As G has odd type, the little projective group of the induced Moufang set has odd or degenerate type, but the latter case is ruled out by Fact 2.3.29 and the observation that 2-transitive groups of finite Morley rank contain involutions. Now assume that the induced Moufang set is not proper. Then the root groups of the induced Moufang set coincide with the 1-point stabilizers in the induced little projective group. By Lemma 7.1.1, the roots groups of the induced Moufang set have odd type. However, the root groups of the induced Moufang set are isomorphic to subgroups of the original Moufang set's root groups, and we have a contradiction. $\square$

7.2 The P^*-setting

We continue with the previous setup but further insist that $\mathbb{M}(U,\tau)$ be proper and have the P^*-property.

Setup. $\mathbb{M}(U,\tau)$ is an infinite proper P^*-Moufang set of finite Morley rank with little projective group G and Hua subgroup H. X denotes $U \cup \{\infty\}$. Further, assume that G has odd type and definable root groups.

Lemma 7.2.1. *H has odd type.*

Proof. By Lemma 7.1.1, we may assume that there exists some $h \in H^*$ that fixes infinitely many points of U. Then for $V := C_U(h)$, H contains the 2-torus from $H(V)$. $\square$

We now continue the analysis of H-invariant projective root subgroups that was started in Chapter 4.

Proposition 7.2.2. *If U contains an infinite proper definable H-invariant root subgroup, then H has Prüfer 2-rank at least 2.*

Proof. We must show that H contains distinct commuting Prüfer 2-groups. Notice that our assumption implies that $\mathbb{M}(U, \tau) \ncong \mathbb{M}(F)$ for any algebraically closed field F.

Let W be an infinite proper definable H-invariant root subgroup of U. Then the involution $i \in H(W)$ is central in $N_H(W) = H$. Set $V := C_U(i)$. We claim that V is again infinite, proper, definable, and H-invariant. Certainly V is definable, and the faithfulness of the action of H on U forces V to be proper. Further, i is central in H, so V is H-invariant. Finally, we show that V is infinite. If not, i inverts U, and U is abelian. However, this implies H acts irreducibly on U by Facts 3.1.4 and 3.1.3(2). This is a contradiction, so V is infinite.

Now $H(V)$ is central in H and contains a 2-torus, so we need only produce a 2-torus not contained in $H(V)$. We will find such a torus in $C^\circ_H(V)$. In fact, we only need to show that $C^\circ_H(V)$ contains some nontrivial 2-element and appeal to Fact 2.3.29.

Using that G acts 2-transitively on X, we get that $G_{\{0,\infty\}} - H$ contains an involution ω. Use Lemma 4.2.1 to write $\omega = \mu_v c$ for some $v \in V^*$ and $c \in C^\circ_H(V)$. Since $G(V)/C_{G(V)}(V) \cong \mathrm{PSL}_2(F)$, we see that μ_v is an element of order dividing 4 which centralizes c, so $1 = \omega^4 = \mu_v^4 c^4 = c^4$. If $c \neq 1$, we are done, so assume that $c = 1$. In this case, μ_v is an involution, so it must be that $G(V)$ is of the form PSL_2. Hence, $Z(G(V)) = 1$, and $H = H(V) \times C_H(V)$. As H is connected, $C_H(V)$ is connected as well, and $C_H(V)$ contains the involution i. We conclude that H has Prüfer 2-rank at least 2. $\square$

When V is an infinite proper definable root subgroup, the P^*-property tells us that $H(V)$ is isomorphic to the multiplicative group of a field. Thus, the previous proposition can be combined with Lemma 4.2.2 to obtain the following corollary.

Corollary 7.2.3. *U contains at most one infinite proper definable H-invariant root subgroup.*

When H is nilpotent, we get an additional corollary. This result will easily yield Theorem 1.1.6.

Corollary 7.2.4. *If H is nilpotent and either*

(1) U is $2^{\perp}$ or

(2) $\sigma(U)$ is nontrivial,

then $\mathbb{M}(U, \tau) \cong \mathbb{M}(F)$ for F an algebraically closed field.

Proof. We work by contradiction; assume that $\mathbb{M}(U, \tau) \not\cong \mathbb{M}(F)$. By Lemma 6.1.1 together with Fact 3.1.4, we may assume that U is not abelian. Thus, any involution of H must fix infinitely many points of U.

By Lemma 7.2.1, H contains a 2-torus, and this 2-torus will be central since H is nilpotent and connected. Thus, U contains an infinite proper definable H-invariant root subgroup, namely the fixed-point space of a central involution of H. Now Proposition 7.2.2 applies, so H has distinct central involutions, say i and j. Set $A := \langle i, j\rangle$, and define $V_a := C_U(a)$ for each $a \in A$. As before, U is not abelian, so V_a is infinite for each $a \in A^*$. Further, each V_a is H-invariant because A is central. We conclude that $V_i = V_j = V_{ij}$ by the previous corollary. Set $V = V_i$.

If U is $2^{\perp}$, Fact 2.3.18 applies and contradicts the fact that V is proper. Now assume that $\sigma(U)$ is nontrivial. We argue that $\sigma(U)$ is infinite. Assume not. Since V is infinite, Corollary 4.1.2 ensures that $V \cap \sigma(U)$ is trivial. Thus, each $a \in A^*$ inverts $\sigma(U)$. This is a contradiction since it also implies that ij fixes $\sigma(U)$. Hence, $\sigma(U)$ is infinite.

Now, A acts on the infinite solvable group $\sigma^{\circ}(U)$. By Fact 2.3.17, $\sigma^{\circ}(U) \leq V$, and another application of Corollary 4.1.2 shows that $\sigma^{\circ}(U) = V$. Thus, V is normal in U. Additionally, V

is either a vector space over the rational numbers or an elementary abelian p-group. Since G has odd type, V can not be an elementary abelian 2-group, so V is 2-divisible. Applying Fact 2.3.16, we see that $C_{U/V}(i)$ is trivial, so U/V is abelian. Thus, U is solvable, and it must have been that $U = \sigma^\circ(U) = V$. This contradicts the fact that H acts faithfully on U. $\square$

7.3 Proof of Theorem 1.1.6

We argue by contradiction. Let $\mathbb{M}(U, \tau)$ be a counterexample to Theorem 1.1.6 such that U has minimal rank among all counterexamples. By Corollary 7.1.3, $\mathbb{M}(U, \tau)$ is hereditarily proper, so the minimality of our counterexample implies that $\mathbb{M}(U, \tau)$ is a P^*-Moufang set. Thus, we appeal to Corollary 7.2.4.

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