Some Applications of a Duality in Cyclic Homology

by

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Thesis directed by Prof. Alexander Gorokhovsky

In this thesis we define even and odd p-summable almost Fredholm modules, which may be viewed as a generalization of even and odd p-summable Fredholm modules. To define a character of an odd almost Fredholm module, we use a duality map, which was developed by A. Gorokhovsky in his thesis. The duality map is defined by using the character of an almost representation which was developed by A. Connes. To do something similar in the even case, we extend the duality map to $\mathbb{Z}/2$ graded algebras, and using this duality map we define a character for a pair of even almost Fredholm modules, and for a subset of the even almost Fredholm modules. Using this character, we are able to extend a result by J. Avron, R. Seiler, and B. Simon.

In particular, they show that if P is a projection and U is a unitary such that $P - UPU^{-1}$ is in the $(2n+1)$st Schatten Class, then $\text{Trace}((P - UPU^{-1})^{2n+1}) \in \mathbb{Z}$. One can recover this result by using Connes' character of an even almost Fredholm module. Our character allows us to give an extension of their result to almost projections, which are self adjoint maps such that $P - P^2$ is in the pth Schatten class.

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Chapter 1

Introduction

We will begin with a brief historical discussion of noncommutative geometry. For more information on K-theory, one may find Weibel's "The Development of Algebraic K-Theory before 1980" interesting, for more on the history of Fredholm modules and K-homology the introduction to "Analytic K-Homology" by Higson and Roe [8] is nicely detailed, and for more information on the history of cyclic cohomology and the Chern character of Fredholm modules, Connes discusses his motivations in both his seminal paper "Noncommutative Differential Geometry" [3] and his book "Noncommutative Geometry" [4]. We have taken some information from each of these sources for this section.

Inspired by Grothendieck's work on the Riemann-Roch theorem, Atiyah and Hirzebruch defined the K-theory of a topological space M as equivalence classes of vector bundles over M. When M is a compact manifold, Atiyah noticed that classes of linear elliptic operators can be paired with the K-theory of M and the pairing is an integer. In particular if $[V] \in K^0(M)$ then $\langle [V], [D] \rangle = \text{Index } D_V$ where the index on the right is the Fredholm index. This led Atiyah to abstractly define elliptic operators, which are now known as Fredholm Modules. K-theory is a (co)homology theory, and so it has a dual homology theory called K-homology. It has been shown that K-homology is a group of equivalence classes of these Fredholm modules.

Now, one can also define the K-theory for C^*-algebras, or more generally, rings. The K-theory of a C^*-algebra is a homology theory, which one may expect, as compact Hausdorff spaces and commutative unital C^*-algebras are contravariantly equivalent categories. If we let $C(M)$

denote the continuous functions on a compact manifold, then $C(M)$ is a commutative unital C^*-algebra.

Now, the topological K-theory of M is isomorphic to the K-theory of $C(M)$, $K^*(M) \cong K_*(C(M))$, and their K-homology is similarly isomorphic, so $K_*(M) \cong K^*(C(M))$. Further, there are maps called Chern characters from $ch : K^n(C(M)) \to H_n(M;\mathbb{R})$ and $ch : K_n(C(M)) \to H^n(M;\mathbb{R})$ such that the following diagram commutes

$$\begin{tikzcd}
K_n(C(M)) \times K^n(C(M)) \arrow[r] \arrow[d, "(ch_*, ch^*)"] & \mathbb{Z} \arrow[d] \\
H^n(M, \mathbb{R}) \times H_n(M, \mathbb{R}) \arrow[r] & \mathbb{R}
\end{tikzcd}$$

where the bottom map is the usual pairing of de Rham cohomology with homology. Connes wanted to find a similar statement for arbitrary algebras, as the first row of the above diagram makes sense when $\mathcal{A}$ is any k-algebra. For this reason, Connes began looking for an appropriate homology theory which would serve as the receptacle of the Chern character in the noncommutative case. This led him to develop cyclic cohomology and periodic cyclic cohomology. To define a Chern character analogous for the one on K-homology in the commutative case, Connes defined a subset of the Fredholm modules over an algebra, the finitely summable Fredholm modules. For each homotopy class of finitely summable Fredholm modules, he was able to define a class in the Cyclic cohomology of the algebra. These classes are called the Chern characters of the Fredholm modules in cyclic cohomology. Connes went on to define a pairing between the K-theory of an algebra and the cyclic cohomology of an algebra, which is integral when paired with a Chern character.

An almost representation is a linear map that is multiplicative modulo the pth Schatten classes, denoted $\mathcal{L}^p$, (sometimes called the Schatten ideals). In his development of the Chern character for an odd p summable Fredholm module, Connes found that for each almost representation there is a class in cyclic cohomology associated to the almost representation. For an appropriate almost representation, he showed that the character of the almost representation and the Chern character of an odd Fredholm module agree.

In chapter 5 of his thesis, Gorokhovsky developed a duality map between the negative (periodic) cyclic homology of an algebra $\mathcal{A}$ and the (periodic) cyclic cohomology of another algebra using Connes' character for an almost representation mentioned above. When Connes defined his character of even and odd Fredholm modules, he required that the operator F square to the identity. While every K-homology class has a representative with such a Fredholm module, it is a more strict condition than is commonly used, which is that $\rho(a)(1 - F^2)$ is compact. Using his duality map, Gorokhovsky was able to define a character of a more general notion of a Fredholm module, here requiring $(1 - F^2) \in \mathcal{L}^p$, which agrees with Connes' character when restricting to the case that $F^2 = 1$. While this result is not in this thesis, we find it interesting and worthwhile to mention here.

The first result of this thesis is a second application of the the duality map, which is in chapter 4. We will define an odd almost Fredholm module, which can be thought of as a generalization of an odd Fredholm module in the context of Connes, where the representation is replaced by an almost representation. Using the duality map, we then define an appropriate character, which agrees with Connes' character of an odd Fredholm module in that case. To do this, we apply the duality map to a specific cycle in the periodic cyclic homology of the degree 1 complex Clifford algebra. We then define a pairing between the invertible elements of an algebra with an almost odd Fredholm module. We do this by evaluating the character of the almost Fredholm module on the Chern character of the invertible element. We show that this results in an integer, as it is the index of a Fredholm operator.

To extend the duality map further, we define an appropriate character of an almost representation for $\mathbb{Z}/2$ graded algebras, which put simply, we applied the Kazoul sign convention to Connes' character of an almost representation. We check that this remains a cocycle in the $\mathbb{Z}/2$ graded cyclic cohomology for the algebra, along with other desirable properties that the original character has. After defining the duality map as Gorokhovsky did, we similarly apply it to a chain in the periodic cyclic homology of the $\mathbb{Z}/2$ graded degree 1 complex Clifford algebra, but this most desirable chain is not a cycle. Regardless, we are able to define a character for a special subset of (even) almost Fredholm modules, which we call weakly balanced almost Fredholm modules, and of which even Fredholm modules are a subset. In this case the character we define agrees with Connes. We also define a character of a pair of almost Fredholm modules.

In [2], Avron, Seiler, and Simon show that for certain projections P and Q and a unitary U, that $\mathrm{Tr}((P-Q)^{2n+1}) \in \mathbb{Z}$ and that $\mathrm{Tr}((P-UPU^{-1})^{2n+1}) \in \mathbb{Z}$. Using Connes' character of an even Fredholm module, one can easily come to this result. Using our character of weakly balanced Fredholm modules, we obtain a generalization of this result for almost projections, which are projections module $\mathcal{L}^p$.

1.1 Thesis Outline

An outline of the thesis is as follows. Chapter 2 is devoted to providing background material for cyclic (co)homology. We have primarily followed "Cyclic Homology" by Loday [11] for sections 2.1 through 2.6, when other sources have been used for a section we will mention them at the beginning of the section. Beyond the definitions of cyclic homology and its variations, information about the shuffle product map and cyclic shuffle product map will be provided as they are necessary to define the duality map. Lastly, we will discuss how the earlier sections of this chapter extend to the $\mathbb{Z}/2$ graded setting, for which we mainly follow the paper "A Künneth Formula for the Cyclic Cohomology of $\mathbb{Z}/2$-Graded Algebras" by C. Kassel [9].

Chapter 3 is devoted to (finitely summable) Fredholm modules, their characters, and the duality map. We define (finitely summable) Fredholm modules in section 3.1, the characters of even Fredholm modules in section 3.2, the characters of odd Fredholm modules and almost representations in 3.3, and the construction of the duality map in section 3.4. For sections 3.1 through 3.3 we will mainly be following Connes' seminal paper,"Noncommutative Differential Geometry" [3]. For section 3.4 we will follow chapter 5 of Gorokhovsky's thesis, "Explicit Formulae for Characteristic Classes in Noncommutative Geometry" [7].

Chapter 4 is the first original material in the thesis. In this chapter, we define a character for an (odd) almost Fredholm module using the duality map, and we compute the pairing of this character with an invertible element.

In chapter 5 section 1 we extend Connes' character of an almost representation to the $\mathbb{Z}/2$ graded setting and show that it retains periodicity and independence up to $\mathcal{L}^p$ perturbations of the almost representation. In section 5.2 we define the duality map in the $\mathbb{Z}/2$ graded setting, which follows from the material in sections 3.4 and 5.1. In section 5.3 we will define characters for (even) weakly balanced almost Fredholm modules (defined on page 91), and the character of a pair of almost Fredholm modules, of which the character of a weakly balanced almost Fredholm module is a special case. We then compute the paring of these characters with an idempotent, which in turn allows us to generalize a result in [2].

The appendices are devoted to proofs of results which are known, or likely known, for which we did not find a reference, and for which we felt the proof did not fit in a specific section or chapter.

Chapter 2

Cyclic Homology

The purpose of this chapter is to define cyclic (co)homology and its variants, and to define all of the tools needed to define the product first discussed in chapter 3 section 4. The definitions and theorems from sections 1 through 6 are mostly taken from [11], with some occasional alterations in notation. If a second source had a major influence on a section, we will note it at the beginning of the section. The definitions and conventions defined in section 7 are taken from [9].

Before we begin, we find it necessary to make the following remark about notation.

Remark 2.1. (About the letter B and Other Notational Choices). We have found that the letter B is a convenient choice to indicate various mathematical maps, objects, ideas and more, and so we will use it to denote various things which will be defined in chapter 2. Fortunately latex has enough stylizations of the letter so that they all may look different. $\mathcal{B}C$ and $\mathcal{B}(\mathcal{A})$ will either mean the (b, B)-bicomplex or the total complex of the (b, B)-bicomplex which we will try to differential with a subscript. So, for example $\mathcal{B}_*(\mathcal{A})$ is the total complex, while $\mathcal{B}(A)$ is the the bicomplex. Hopefully this will not be confusing with context. $\mathscr{B}$ will be an algebra. If $\mathcal{A}$ is an algebra, we thought $\mathscr{B}$ should be too. B will always be Connes' differential and b will be the Hochschild differential. If we have more than one complex, we might differentiate these with a subscript, which will never be a number. If we have b_k where $k \in \mathbb{N}$, then this will be an element of the algebra $\mathscr{B}$, and $(b_0, b_1, ..., b_n)$ will be an element of the k-module $\mathscr{B} \otimes \mathscr{B} \otimes .... \otimes \mathscr{B} = \mathscr{B}^{\otimes n+1}$. We will also write $\beta_n = (b_0, b_1, ..., b_n)$ and $\beta = \sum \beta_n$.

2.1 Hochschild Homology

Let k be a field of characteristic 0, and $\mathcal{A}$ a unital associative k-algebra. Suppose that M is an $\mathcal{A}$-bimodule, that is M is a left and right $\mathcal{A}$-module and $(a_1m)a_2 = a_1(ma_2)$. We note that this is equivalent to M being a right (or left) module over the algebra $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$ under the action $m \cdot (a_1 \otimes a_2) = a_2ma_1$. As a note, all tensor products are taken over k unless otherwise specified.

Definition 2.1. (Hochschild Homology). Define the k modules $C_n(\mathcal{A}, M)$ by

$$C_n(\mathcal{A}, M) = M \otimes \mathcal{A} \otimes ... \otimes \mathcal{A} = M \otimes \mathcal{A}^{\otimes n}.$$

We define the Hochschild boundary map, $b : C_n(\mathcal{A}, M) \to C_{n-1}(\mathcal{A}, M)$, as $b = \sum_{i=0}^{n}(-1)^i d_i$ where

$$\begin{aligned} d_0(m, a_1, ..., a_n) =& (ma_1, a_2, ..., a_n) \\ d_i(m, a_1, ..., a_n) =& (m, a_1, ..., a_ia_{i-1}, a_{i+1}, ..., a_n) \quad 1 \leq i \leq n-1 \\ d_n(m, a_1, ..., a_n) =& (a_nm, a_1, ..., a_{n-1}). \end{aligned}$$

The Chain complex $(C_n(\mathcal{A}, M), b)$ is called the Hochschild complex. The homology of this complex is called the Hochschild homology of $\mathcal{A}$ with coefficients in M, and is denoted $H_n(\mathcal{A}, M)$. If $M = \mathcal{A}$, the homology of this complex is called the Hochschild homology of $\mathcal{A}$, and is denoted $HH_n(\mathcal{A})$. As a note, we may write the elements of $C_n(\mathcal{A})$ as $\alpha_n = (a_0, a_1, ..., a_n) = a_0 \otimes a_1 \otimes ... \otimes a_n$.

Proposition 2.1. Define $\overline{\mathcal{A}} = \mathcal{A}/k$, and $\overline{C}_n(\mathcal{A}, M) = M \otimes \overline{\mathcal{A}}^{\otimes n}$. Then the canonical projection map $C_n(\mathcal{A}, M) \to \overline{C}_n(\mathcal{A}, M)$ is a quasi-isomorphism. The complex $\overline{C}_n(\mathcal{A}, M)$ is called the normalized Hochschild complex.

Definition 2.2. (Bar Complex). Consider the complex

$$... \xrightarrow{b'} \mathcal{A}^{\otimes n} \xrightarrow{b'} \mathcal{A}^{\otimes n-1} \xrightarrow{b'} ... \xrightarrow{b'} \mathcal{A}^{\otimes 3} \xrightarrow{b'} \mathcal{A}^{\otimes 2}$$

where $b' = \sum_{i=1}^{n-1}(-1)^i d_i$. The complex above is called the bar complex, and under the augmentation map $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, (a, a') \mapsto aa'$, it is called the bar resolution for the $\mathcal{A}^e$ module $\mathcal{A}$. We note that $s : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes n+1}$, $(a_1, ..., a_n) \to (1, a_1, ..., a_n)$ is a contracting homotopy called the extra degeneracy (referring to the simplical structure of this complex).

2.2 Cyclic Homology

2.2.1 Definitions

Definition 2.3. (The Cyclic Bicomplex). For each $n \in \mathbb{N}$ we define an action of $\mathbb{Z}/(n+1)\mathbb{Z}$ on $C_n(\mathcal{A}) = C_n(\mathcal{A}, \mathcal{A}) = \mathcal{A} \otimes \mathcal{A}^{\otimes n} = \mathcal{A}^{\otimes n+1}$ by

$$t_n \cdot (a_0, a_1, ..., a_n) = (-1)^n (a_n, a_0, a_1, ..., a_{n-1}),$$

where t_n is the standard generator of $\mathbb{Z}/(n+1)\mathbb{Z}$. We will omit the subscript n from t_n when confusion will not arise. Define $N : C_n(\mathcal{A}) \to C_n(\mathcal{A})$ by $N = \sum_{i=0}^{n} t^i$. Now, $(1-t)N = N - tN = N - N = 0$, $(1-t)b' = b(1-t)$, and $b'N = Nb$, so we have the following bicomplex, denoted $CC(\mathcal{A})$, called the cyclic bicomplex for $\mathcal{A}$.

$$\begin{tikzcd}
\vdots \arrow[d, "b"] & \vdots \arrow[d, "b'"] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "b'"] & \\
\mathcal{A}^{\otimes 3} \arrow[d, "b"] & \mathcal{A}^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "b'"] & \mathcal{A}^{\otimes 3} \arrow[l, "N"'] \arrow[d, "b"] & \mathcal{A}^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "b'"] & \cdots \arrow[l, "N"'] \\
\mathcal{A}^{\otimes 2} \arrow[d, "b"] & \mathcal{A}^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "b'"] & \mathcal{A}^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & \mathcal{A}^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "b'"] & \cdots \arrow[l, "N"'] \\
\mathcal{A} & \mathcal{A} \arrow[l, "1-t"'] & \mathcal{A} \arrow[l, "N"'] & \mathcal{A} \arrow[l, "1-t"'] & \cdots \arrow[l, "N"']
\end{tikzcd}$$

Definition 2.4. (Cyclic Homology). The cyclic homology groups of $\mathcal{A}$, $HC_n(\mathcal{A})$, are defined to be the homology groups of the total complex $\mathrm{Tot}_* CC(\mathcal{A})$,

$$HC_n(\mathcal{A}) := H_n(\mathrm{Tot}_* CC(\mathcal{A}))$$

Definition 2.5. (Connes' Complex). Denote by $C_n^\lambda(\mathcal{A}) := \mathcal{A}^{\otimes n+1}/(1-t)$ where $\mathcal{A}^{\otimes n+1}/(1-t)$ is the cokernel of the endomorphism $(1-t)$ of $\mathcal{A}^{\otimes n+1}$. Then

$$... \xrightarrow{b} C_n^\lambda(\mathcal{A}) \xrightarrow{b} C_{n-1}^\lambda \xrightarrow{b} ... \xrightarrow{b} C_0^\lambda(\mathcal{A})$$

is a well defined complex called Connes' Complex. Its nth homology group is denoted $H_n^\lambda(\mathcal{A})$.

Define $p : \mathrm{Tot}_* CC(\mathcal{A}) \to C^\lambda(\mathcal{A})$ to be the quotient map $\mathcal{A}^{\otimes n+1} \to \mathcal{A}^{\otimes n+1}/(1-t)$ on the first column and 0 on the others. Then $p_* : HC_*(\mathcal{A}) \to H_*^\lambda(\mathcal{A})$ is an isomorphism.

We note that in the general case, this map is not an isomorphism, it is required that k is a ring containing $\mathbb{Q}$.

We also note that in the previous two definitions, it was not required that $\mathcal{A}$ be unital, and so one can define cyclic homology for non-unital algebras if desired. In the following definition, which is the definition we will mainly use, it is required that $\mathcal{A}$ is unital.

Definition 2.6. (Connes' boundary map B and the (b, B)-bicomplex, $\mathcal{B}(\mathcal{A})$). Define $B : \mathcal{A}^{\otimes n+1} \to \mathcal{A}^{\otimes n+2}$ by $B = (1-t)sN$. Explicitly we have that

$$\begin{aligned} B(a_0, ..., a_n) = \sum_{i=0}^{n} (-1)^{ni}(1, a_i, ..., a_n, a_0, ..., a_{i-1}) \\ + (-1)^{ni}(a_i, 1, a_{i+1}, ..., a_n, a_0, ..., a_{i-1}) \end{aligned}$$

Then $bB + Bb = 0$ and $B^2 = 0$. Indeed,

$$bB + Bb = b(1-t)sN + (1-t)sNb = (1-t)b'sN + (1-t)sb'N = (1-t)(b's + sb')N = (1-t)N = 0,$$

and

$$B^2 = (1-t)sN(1-t)sN = (1-t)s(0)s(1-t) = 0.$$

Then we can define a bicomplex $\mathcal{B}(\mathcal{A})$, called the (b, B)-bicomplex of the algebra $\mathcal{A}$.

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
\mathcal{A}^{\otimes 3} \arrow[d, "b"] & \mathcal{A}^{\otimes 2} \arrow[l, "B"] \arrow[d, "b"] & \mathcal{A} \arrow[l, "B"] \\
\mathcal{A}^{\otimes 2} \arrow[d, "b"] & \mathcal{A} \arrow[l, "B"] & \\
\mathcal{A} & &
\end{tikzcd}$$

Then $\mathcal{B}(\mathcal{A})_{pq} = \mathcal{A}^{\otimes q-p+1}$ if $q \geq p$ and 0 otherwise. We will often write $\mathcal{B}_*(\mathcal{A})$ in place of $\mathrm{Tot}_*(\mathcal{B}(\mathcal{A}))$.

Now, the inclusion map $\mathcal{B}_*(\mathcal{A}) = \mathrm{Tot}_*\mathcal{B}(\mathcal{A}) \to \mathrm{Tot}_*CC(\mathcal{A})$ is a quasi-isomorphism, and we have $H_n(\mathrm{Tot}_*\mathcal{B}(\mathcal{A})) = HC_n(\mathcal{A})$.

We can define a normalized version of the $\mathcal{B}(\mathcal{A})$ complex by replacing the Hochschild complexes with their normalizations. That is, replace $\mathcal{A}^{\otimes n+1}$ with $\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes n}$ where $\overline{\mathcal{A}} = \mathcal{A}/k$. Then we have the following bicomplex, denoted $\overline{\mathcal{B}}(\mathcal{A})$,

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes 2} \arrow[d, "b"] & \mathcal{A} \otimes \overline{\mathcal{A}} \arrow[l, "\overline{B}"'] \arrow[d, "b"] & \mathcal{A} \arrow[l, "\overline{B}"'] \\
\mathcal{A} \otimes \overline{\mathcal{A}} \arrow[d, "b"] & \mathcal{A} \arrow[l, "\overline{B}"'] & \\
\mathcal{A} & &
\end{tikzcd}$$

where $\overline{B} = (1-t)sN = sN$, since $ts = 0$. Explicitly,

$$\overline{B}(a_0, ..., a_n) = \sum_{i=0}^{n} (-1)^{ni}(1, a_i, ..., a_n, a_0, ..., a_{i-1}).$$

We will almost always work in the normalized setting, and so we will often omit the bar from our notation. We will often denote elements of $\alpha \in \mathrm{Tot}_n\overline{\mathcal{B}}(\mathcal{A}) = \mathcal{B}_*(\mathcal{A})$ differently depending on context. These notations are

$$\alpha = \alpha_n + \alpha_{n-2} + ... = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \alpha_{2k+(n \bmod 2)} \text{ where } \alpha_k \in \overline{C}_k(A)$$

or

$$\alpha = (\alpha_{0+\delta}, \alpha_{2+\delta}, \alpha_{4+\delta}, ...) = \{\alpha_{2k+\delta}\},$$

where $\delta = n \bmod 2$.

Example 2.1. Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}F = \mathbb{C}[F]/(F^2 - 1)$ be the complex Clifford algebra of degree 1. Then

$$\overline{C_n}(\mathcal{A}) = A \otimes \overline{A}^{\otimes n} = (\mathbb{C} \oplus \mathbb{C}F) \otimes (\mathbb{C}F)^{\otimes n} = \mathbb{C} \otimes (\mathbb{C}F)^{\otimes n} \oplus (\mathbb{C}F)^{\otimes n+1}.$$

Thus, the chains in $\mathcal{B}_{2n}(\mathcal{A})$ and $\mathcal{B}_{2n-1}(\mathcal{A})$ are of the form

$$\sum_{k=0}^{n}(c_{2k}\cdot 1+c_{2k}^{F}F)\otimes F^{\otimes 2k}=\sum_{k=0}^{n}c_{2k}\otimes F^{2k}+c_{2k}^{F}F^{\otimes 2k+1}$$

$$\sum_{k=1}^{n}(c_{2k-1}+c_{2k-1}^{F}F)\otimes F^{2k-1}=\sum_{k=1}^{n}c_{2k-1}\otimes F^{2k-1}+c_{2k-1}^{F}F^{\otimes 2k-1}$$

respectively where $c_k, c_k^F \in \mathbb{C}$. So to find the cycles of $\mathcal{B}_{2n}(\mathcal{A})$ we compute

$$\begin{aligned}
b(F^{\otimes 2k+1}) =& 1\otimes F^{\otimes 2k-1}+\sum_{i=1}^{2k}(-1)^{i}(F\otimes F\otimes ...\otimes F\otimes 1\otimes ...\otimes F)+1\otimes F^{\otimes 2k-1}\\
=& 2\otimes F^{\otimes 2k-1}\\
B(F^{\otimes 2k+1}) =& \sum_{i=0}^{2k}(-1)^{(2k)i}1\otimes F\otimes ...\otimes F=\sum_{i=0}^{2k}1\otimes F^{\otimes 2k+1}\\
=& (2k+1)\otimes F^{\otimes 2k+1}\\
b(1\otimes F^{\otimes 2k}) =& F^{\otimes 2k}+\sum_{i=1}^{2k}(-1)^{i}(1\otimes F\otimes ...\otimes F\otimes 1\otimes ...\otimes F)+F^{\otimes 2k}\\
=& 2c_k^F F^{\otimes 2k}\\
B(1\otimes F^{\otimes 2k}) =& \sum_{i=0}^{2k}(-1)^{(2k)i}1\otimes F\otimes ...\otimes F\otimes c_k^F\otimes ...\otimes F)=0\\
(b+B)(1) =& 0.
\end{aligned}$$

So a cycle in $\mathcal{B}_{2n}(\mathcal{A})$ is of the form

$$c\cdot 1+c_0^F F+\sum_{k=1}^{n}c_{2k}^F F\otimes F^{\otimes 2k}$$

$$\text{where } c_{2k}^F=\frac{(-1)^k c_0^F}{2^k}\prod_{j=1}^{k}(2j-1)=\frac{(-1)^k(2k)!c_0^F}{4^k k!}.$$

Now, every chain in $\mathcal{B}_{2n-1}(\mathcal{A})$ is a boundary. Indeed, for any odd chain of the form $F^{\otimes 2k}$, we have that

$$(b+B)\left(\frac{1}{2}\otimes F^{\otimes 2k}\right)=F^{\otimes 2k},$$

and for any odd chain of the form $1\otimes F^{\otimes 2k-1}$, we have that

$$(b+B)\left(\frac{1}{2}F^{\otimes 2k+1}+\sum_{j=k+1}^{n}(-1)^{j-k}\cdot\frac{1}{2}\cdot\left(\prod_{i=k}^{j}\frac{2i+1}{2}\right)F^{\otimes 2j+1}\right)=1\otimes F^{\otimes 2k-1}.$$

So we have $HC_{2n+1}(\mathcal{A}) = 0$ and $HC_{2n}(\mathcal{A}) = \mathbb{C} \oplus \mathbb{C}$

Theorem 2.1. ***(Connes' Periodicity Exact Sequence).*** *For an associative but not necessarily unital k-algebra $\mathcal{A}$ there is a natural long exact sequence*

$$... \to HH_n(\mathcal{A}) \to HC_n(\mathcal{A}) \xrightarrow{S} HC_{n-2}(\mathcal{A}) \to HH_{n-1}(\mathcal{A}) \to ...$$

The map S is called the periodicity map. For $n = 2m+\delta$, $\delta = 0, 1$, S is defined on $\mathrm{Tot}_n\overline{\mathcal{B}}(\mathcal{A})$ by

$$S\left(\sum_{k=0}^{m} \alpha_{2k+\delta}\right) = \sum_{k=0}^{m-1} \alpha_{2k+\delta}$$

where $\alpha_k \in A \otimes \overline{A}^{\otimes k}$. Note that S can be defined on $C_n^\lambda(\mathcal{A})$ or $CC_n(\mathcal{A})$ if $\mathcal{A}$ is not unital.

2.2.2 Cohomology

If we wish to consider cohomology we construct the following cochain complexes. Let

$$CC^{pq}(\mathcal{A}) = C^q(\mathcal{A}) = \mathrm{Hom}_k(CC_{pq}(\mathcal{A}), k) = \mathrm{Hom}_k(C_q(\mathcal{A}), k) = \mathrm{Hom}_k(\mathcal{A}^{\otimes q+1}, k).$$

Then the cyclic cohomology of $\mathcal{A}$ is $H^n(\mathrm{Tot}\, CC^{**}(\mathcal{A}))$ denoted by $HC^*(\mathcal{A})$. We note that since k is a field we have $HC^*(\mathcal{A}) \cong \mathrm{Hom}_k(HC_*(\mathcal{A}), k)$.

We may also define cyclic cohomology as the homology of a subcomplex of the Hochschild cocomplex. This is the way that Connes originally defined cyclic cohomology. We call $f \in C^n(\mathcal{A})$ cyclic if

$$f(a_0, ..., a_n) = (-1)^n f(a_n, a_0, ..., a_{n-1}).$$

Then the set of cyclic cochains, denoted $C_\lambda^n(\mathcal{A})$, is a sub k-module of $C^n(\mathcal{A})$, and further $(C_\lambda^*(\mathcal{A}), b)$ is a sub-complex of $(C^*(\mathcal{A}), b)$. As in the homological case, if $\mathbb{Q} \subset k$, then the inclusion map $C_\lambda^*(\mathcal{A}) \hookrightarrow C^*(\mathcal{A})$ induces an isomorphism $H_\lambda^n(\mathcal{A}) \to HC^n(\mathcal{A})$.

Lastly, if $\mathcal{A}$ is unital we may use $\mathcal{B}(A)$, or rather, $\overline{\mathcal{B}}(A)$, to define the cyclic cohomology of $\mathcal{A}$, taking $\overline{\mathcal{B}}^*(\mathcal{A}) = \mathrm{Hom}_k(\overline{\mathcal{B}}_*(\mathcal{A}), k)$ and $HC^n(\mathcal{A}) = H^n(\overline{\mathcal{B}}^*(\mathcal{A}))$. We will also consider the periodicity map, S, defined on cyclic cohomology by $S\phi = \phi \circ S$, where that latter S is the periodicity map on cyclic homology.

2.2.2.1 Pairing

The evaluation map of cochains on chains

$$ev : \mathcal{B}^n(\mathcal{A}) \times \mathcal{B}_n(\mathcal{A}) \to k$$

by definition satisfies

$$ev(b\phi, \alpha) = b\phi(\alpha) = \phi(b\alpha) = ev(\phi, b\alpha) \text{ and } ev(B\phi, \alpha) = ev(\phi, B\alpha),$$

so there is a pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{HC}} = \langle \cdot, \cdot \rangle : HC^n(\mathcal{A}) \times HC_n(\mathcal{A}) \to k,$$

defined by

$$\langle [\phi], [\alpha] \rangle = \phi(\alpha)$$

where ϕ and α are (co)cycle representatives of $[\phi]$ and $[\alpha]$. Since k is a field, this product induces an isomorphism $HC^n(\mathcal{A}) \to \mathrm{Hom}_k(HC_n(\mathcal{A}), k)$.

2.2.3 Cyclic Modules

In this subsection, we will deviate from [11] slightly, as we wish to be a bit more general. We will take some definitions from [15] which originated in [6].

Let $t_n \in \mathbb{Z}/(n+1)\mathbb{Z}$ be the standard generator.

Definition 2.7. (Paracyclic k-Modules). A paracyclic k-module $C = (\{C_n\}_{n \geq 0}, d_i, s_i, t_n)$ is a simplicial k-module $(\{C_n\}_{n \geq 0}, d_i, s_i)$ with an automorphism for each n, $t_n : C_n \to C_n$, satisfying

$$\begin{aligned} d_i t_n &= t_{n-1} d_{i-1} \text{ and } s_i t_n = t_{n+1} s_{i-1} \text{ for } 1 \leq i \leq n, \\ d_0 t_n &= d_n \text{ and } s_0 t_n = t_{n+1}^2 s_n \end{aligned}$$

for $d_i : C_n \to C_{n-1}$ and $s_i : C_n \to C_{n+1}$. Each paracyclic module has maps

$$\begin{aligned}
&s = s_{-1} = t_{n+1}s_n : C_n \to C_{n+1}\\
&b = \sum_{i=0}^{n}(-1)^i d_i : C_n \to C_{n-1}\\
&T = t_n^{n+1} : C_n \to C_n\\
&N = \sum_{i=0}^{n}(-1)^{in} t_n^i : C_n \to C_n\\
&B = (1 + (-1)^n t_{n+1})sN : C_n \to C_n
\end{aligned}$$

such that $b^2 = B^2 = 0$ and $Bb + bB = 1 - T$.

Definition 2.8. A cyclic module is a paracyclic module such that $T = 1$.

A morphism of cyclic modules $f : C \to C'$ is a morphism of simplicial modules such that $f_n t_n \to t_n f_n$ for all n.

Definition 2.9. (The Extra Degeneracy). The operator $s = s_{n+1} = (-1)^{n+1} t_{n+1} s_n : C_n \to C_{n+1}$ satisfies all of the relations of the degeneracy operators except that in general $d_0 s_{n+1} \neq s_n d_0$. Thus it is called the extra degeneracy. Note that in the b'-complex we have $b's = sb' = 1$. In the case of $C = C(\mathcal{A})$, $s(a_0, ..., a_n) = (1, a_0, ..., a_n)$.

Proposition 2.2. Let $\mathcal{A}$ be an associative and unital k-algebra. The simplicial module $[n] \mapsto \mathcal{A}^{\otimes n+1}$ equipped with the action of the cyclic group $\mathbb{Z}/(n + 1)\mathbb{Z}$ given by

$$t_n(a_0, ..., a_n) = (a_n, a_0, ..., a_{n-1})$$

is a cyclic module denoted by $C(\mathcal{A})$.

To any cyclic module C there are associated cyclic bicomplexes CC, $\mathcal{B}C$, and $\overline{\mathcal{B}}C$ and a complex C^λ, which are all quasi-isomorphic and defined as for $CC(\mathcal{A})$, $\mathcal{B}C(\mathcal{A})$, $\overline{\mathcal{B}}C(\mathcal{A})$, and $C^\lambda(\mathcal{A})$. There is also a similarly defined periodicity map S which gives rise to a long exact sequence in homology.

Definition 2.10. ((Homology of a) Mixed Complex) A mixed complex (C, b, B) is a family of k-modules C_n, $n \geq 0$ and maps $b : C_n \to C_{n-1}$, $B : C_n \to C_{n+1}$ such that $b^2 = B^2 = Bb + bB = 0$. As a note, any cyclic module induces a mixed complex. The ordinary homology of a mixed complex, $HH_*(C)$ is the homology of the complex (C, b). The cyclic homology of a mixed complex, $HC_*(C)$ is the homology of the complex $(\mathcal{B}_*(C) = \mathrm{Tot}_*(\mathcal{B}C), b+B)$ where $\mathcal{B}_n(C) = \mathrm{Tot}_n(\mathcal{B}C) = C_n \oplus C_{n-2} \oplus ...$

Definition 2.11. (Bi-paracyclic module). A bi-paracyclic module is a sequence of k-modules $(\{C_{n,m}\}, d_i^h, s_i^h, t_n^h, d_j^v, s_j^v, t_m^v)$ where

$$d_i^h : C_{n,m} \to C_{n-1,m},\ s_i^h : C_{n+1,m},\ t_n^h : C_{n,m} \to t_{n,m}^h,\ \forall 0 \leq i \leq n$$

$$d_j^v : C_{n,m} \to C_{n,m-1},\ s_j^v : C_{n,m+1},\ t_m^v : C_{n,m} \to t_{n,m}^h,\ \forall 0 \leq j \leq m$$

such that for each $n_0, m_0 \geq 0$, $(\{C_{n,m_0}\}, d_i^h, s_i^h, t_n^h)$ and $(\{C_{n_0,m}\}, d_j^v, s_j^v, t_m^v)$ are paracyclic k-modules and the operators d_i^h, s_i^h, t_n^h commute with the operators d_j^v, s_j^v, t_m^v.

Definition 2.12. (Cylindrical Module). A cylindrical module is a bi-paracyclic module such that $(t_n^h)^{n+1}(t_m^v)^{m+1} = (t_m^v)^{m+1}(t_n^h)^{n+1} = 1$.

Remark 2.2. A bi-paracyclic module such that for each $n_0, m_0 \geq 0$, $(\{C_{n,m_0}\}, d_i^h, s_i^h, t_n^h)$ and $(\{C_{n_0,m}\}, d_j^v, s_j^v, t_m^v)$ are cyclic modules is a cylindrical module.

Remark 2.3. Let C and C' be cyclic modules. Then $(C \otimes C') = (\{C_n \otimes C'_m\}_{n,m}, d_i, s_i, t_n, d'_j, s'_j, t'_m)$ is a cylindrical module. From this we may obtain a mixed complex in two ways.

We may take $\Delta(C \otimes C') = C \times C'$ where

$$\begin{aligned} (C \times C')_n =& C_n \otimes C'_n, \\ d_i^\times = d_i \otimes d'_i,&\ s_j^\times = s_j \otimes s'_j,\ t_n^\times = (-1)^n t_n \otimes t'_n. \end{aligned}$$

Then $C \times C'$ is a cyclic module, and it induces a mixed complex $(C \times C', b_\times, B_\times)$.

We may also take $(\mathrm{Tot}(C \otimes C'), b_{\otimes}, B_{\otimes}) = (C \otimes C', b_{\otimes}, B_{\otimes})$ where

$$(C \otimes C')_n = \bigoplus_{p+q=n} C_p \otimes C'_q,$$

$$b_{\otimes} = b_h + (-1)^p b_v,\ B_{\otimes} = B_h 1 + (-1)^p B_v.$$

$$b_h = b \otimes 1,\ B_h = B \otimes 1$$

$$b_v = 1 \otimes b'\ B_v = 1 \otimes B'$$

The following two sections will provide theorems which show the cyclic homology of these mixed complexes are quasi-isomorphic.

2.3 Shuffles and Shuffle Product

All of this section except for the remark at the end is taken from [11]. The remark at the end is taken from a combination of [14] and [15].

Definition 2.13. ((p, q)-shuffles). Let S_n be the symmetric group acting on the set $\{1, 2, ..., n\}$. A (p, q)-shuffle is an element $\sigma \in S_{p+q}$ such that

$$\sigma(1) < \sigma(2) < ... < \sigma(p) \text{ and } \sigma(p+1) < \sigma(p+2) < ... < \sigma(p+q).$$

Example 2.2. Consider the set $X = \{1, 2, 3\}$. Then, using cycle notation, id, $(2\,3)$, $(1\,2\,3)$ are the $(2, 1)$-shuffles in S_{2+1}. $(1\,3\,2)$ is not a $(2, 1)$-shuffle since $1 < 2$ and $\sigma(1) = 3 > \sigma(2) = 1$. Similarly $(1\,2)$ and $(1\,3)$ are not $(2, 1)$-shuffles.

For any k-algebra $\mathcal{A}$, we let S_n act on $C_n(\mathcal{A})$ on the left by

$$\sigma \cdot (a_0, a_1, ..., a_n) = (a_0, a_{\sigma^{-1}(1)}, ..., a_{\sigma^{-1}(n)}).$$

Thus, if σ is a (p, q)-shuffle, then $\{a_1, .., a_p\}$ show up in the same order in $\sigma \cdot (a_0, a_1, ..., a_n)$, as do $\{a_{p+1}, ..., a_{p+q}\}$. For example,

$$(x_0, x_1, x_2, x_3, x_4) = (a_0, a_1, a_2, b_3, b_4)$$

under the $(2,2)$-shuffles $\sigma_1 = (2\,3)$ and $\sigma_2 = (1\,2\,4\,3)$ is

$$(x_0, x_{\sigma_1^{-1}(1)}, x_{\sigma_1^{-1}(2)}, x_{\sigma_1^{-1}(3)}, x_{\sigma_1^{-1}(4)}) = (x_0, x_1, x_3, x_2, x_4) = (a_0, a_1, b_3, a_2, b_4)$$

and

$$(x_0, x_{\sigma_2^{-1}(1)}, x_{\sigma_2^{-1}(2)}, x_{\sigma_2^{-1}(3)}, x_{\sigma_2^{-1}(4)}) = (x_0, x_3, x_1, x_4, x_2) = (a_0, b_3, a_1, b_4, a_2)$$

respectively. In each case we see that a_1 comes before a_2 and b_3 before b_4.

Definition 2.14. (Shuffle Product). Let $\mathcal{A}$ and $\mathcal{B}$ be two k-algebras. The shuffle product,

$$- \times - = sh_{pq} : C_p(\mathcal{A}) \otimes C_q(\mathcal{B}) \to C_{p+q}(\mathcal{A} \otimes \mathcal{B})$$

is defined by

$$sh_{pq}((a_0, a_1, .., a_p), (b_0, b_1, ..., b_q)) = \sum_{\sigma} (-1)^{\sigma} \sigma \cdot (a_0 \otimes b_0, a_1 \otimes 1, ..., a_p \otimes 1, 1 \otimes b_1, ..., 1 \otimes b_q)$$

where the sum is taken over all (p,q)-shuffles, and $(-1)^{\sigma} = \operatorname{sgn}(\sigma)$.

We note that the Hochschild boundary, b, is a graded derivation for the shuffle product,

$$b(sh_{pq}(x, y)) = sh_{(p-1)q}(b(x), y) + (-1)^{|p|} sh_{p(q-1)}(x, b(y)).$$

Definition 2.15. (Shuffle Product Map). We define the shuffle product map by

$$sh : (C_*(\mathcal{A}) \otimes C_*(\mathcal{B}))_n = \bigoplus_{p+q=n} C_p(\mathcal{A}) \otimes C_q(\mathcal{B}) \to C_n(\mathcal{A} \otimes \mathcal{B}),$$

$$sh = \sum_{p+q=n} sh_{pq}.$$

Since we have assumed k is a field, the shuffle product map induces an isomorphism

$$sh_* : HH_*(\mathcal{A}) \otimes HH_*(\mathcal{B}) \to HH_*(\mathcal{A} \otimes \mathcal{B}).$$

Remark 2.4. Let $C = (C_{p,q}, d_i^h, s_i^h, d_j^v, s_j^v)$ be a bi-simplical module. If it is desirable (and it will be), one can define a shuffle product map between the total complex of the bi-simplical module, $(\operatorname{Tot} C, b_{\otimes})$ and its diagonal $(\Delta C, b_{\times})$ in the following way, first we define

$$sh_{pq} : C_{p,q} \to C_{n,n},\ p + q = n$$

$$sh_{pq} = \sum_{\sigma} (-1)^{\sigma} s^h_{\sigma(n-1)} \cdots s^h_{\sigma(p)} s^v_{\sigma(p-1)} \cdots s^v_{\sigma(0)}$$

where the sum is taken over all (p,q)-shuffles of the set $\{0,...,p-1,p,...,p+q-1\}$, and we define the shuffle product map by

$$sh : \operatorname{Tot}_n C = \sum_{p+q=n} C_{p,q} \to C_{n,n}$$

$$sh = \sum_{p+q=n} sh_{pq}.$$

Then the above map induces a quasi-isomorphism in homology, and we have $shb_{\otimes} = b_{\times} sh$. This extends to the discussion above by taking

$$C_{p,q} = \mathcal{A}^{\otimes p+1} \otimes \mathcal{B}^{\otimes q+1}.$$

As we have $\mathcal{A} \otimes \mathcal{B} \cong \mathcal{B} \otimes \mathcal{A}$, we will have

$$(\Delta C)_n = \mathcal{A}^{\otimes n+1} \otimes \mathcal{B}^{\otimes n+1} \cong (\mathcal{A} \otimes \mathcal{B})^{\otimes n+1} = C_n(\mathcal{A} \otimes \mathcal{B}),$$

and so via the composition of these isomorphisms we have

$$H_n(C(\mathcal{A}) \otimes C(\mathcal{B})) = H_n(\operatorname{Tot}(C(\mathcal{A}) \otimes C(\mathcal{B})) \cong H_n(\Delta C) \cong H_n(C(\mathcal{A} \otimes \mathcal{B})).$$

This lens will be desirable when we work with $\mathbb{Z}/2$-graded algebras.

2.4 Cyclic Shuffles and Cyclic Shuffle Product

Like the previous section, most of this section is taken from [11]. The remark at the end is taken from [15], as well as a minor wording correction to the definition of a cyclic shuffle.

We recall the notation from the remark at the end of section 2.3, Let C and C' be cyclic modules, $(C \otimes C') = (\{C_n \otimes C'_m\}_{n,m}, d_i, s_i, t_n, d'_j, s'_j, t'_m)$ the associated cylindrical module, and let

$$(\Delta(C \otimes C') = C \times C', b_{\times}, B_{\times}), \; (\operatorname{Tot} C \otimes C' = C \otimes C', b^{\otimes}, B^{\otimes})$$

the induced mixed complexes.

Definition 2.16. ((p,q)-cyclic shuffle). A (p,q)-cyclic shuffle is a permutation $\sigma \in S_{p+q}$ defined in the following way: Let $t_1 \in \mathbb{Z}_p$ and $t_2 \in \mathbb{Z}_q$ act on the sets $\{1,..,p\}$ and $\{p+1,...,p+q\}$

respectively, and let σ be a (p,q)-shuffle acting on $\{t_1(1),...,t_1(p),t_2(p+1),..,t_2(p+q)\}$, and say $\sigma(k)=\sigma(t_i(k))$. We call σ a cyclic shuffle if $\sigma(1)<\sigma(p+1)$.

Example 2.3. Let $p=3$ and $q=2$ then $\{1,4,2,3,5\}$ and $\{3,5,1,4,2\}$ are examples of cyclic shuffles, as $\sigma(1)<\sigma(4)$ in either case and $1,2,3$ and $4,5$ appear in cyclic order (in order up to a cyclic permutation). $\{4,1,2,3,5\}$ is not a cyclic shuffle since $\omega(4)=3<\omega(1)=4$, and $\{1,4,3,5,2\}$ is not a cyclic shuffle since $\{1,3,2\}$ is not a cyclic permutation of $\{1,2,3\}$.

Definition 2.17. ((p,q)-Cyclic Shuffle Map). We define a map $\perp$: $C_p(\mathcal{A})\otimes C_q(\mathcal{B})\to C_{p+q}(\mathcal{A}\otimes\mathcal{B})$ given by

$$(a_0,...,a_p)\perp(b_0,...,b_q)=\sum_{\sigma}(-1)^{\sigma}\sigma\cdot(a_0\otimes b_0,a_1\otimes 1,...,a_p\otimes 1,1\otimes b_1,...,1\otimes b_q)$$

where the sum is extended over all (p,q)-cyclic shuffles. As will be discussed at the end of the section, one can define the following map for any cylindrical module, and so for a product of cyclic modules. As such, we will state the remainder of the section in this context. Let C and C' be cyclic modules. Using $\perp$ we define a map of homological degree 2, called the cyclic shuffle map,

$$-\times'-=sh'_{pq}:C_p\otimes C'_q\to C_{p+q+2}\otimes C'_{p+q+2},\text{ by }x\times' y=sh'_{pq}(x,y)=s(x)\perp s(y)$$

where $s:C_n\to C_{n+1}$ is the extra degeneracy. Thus, for any $x\in C_p$ and $y\in C'_q$

$$B(x\times' y)=Bx\times' y=x\times' By=0$$

since $Bx\times' y=s(Bx)\perp s(y)$ and $sB=0$ in the normalized setting. In the other notation this is written as

$$B_{\times}sh'_{pq}(x,y)=sh'_{(p+1)q}(Bx,y)=sh'_{p(q+1)}(x,By)=0.$$

Proposition 2.3. For $x\in C_p$ and $y\in C'_q$ we have

$$B_{\times}sh_{pq}-\left(sh_{(p+1)q}B_h+(-1)^p sh_{p(q+1)}B_v\right)=-b_{\times}sh'_{pq}+sh'_{(p-1)q}b_h+(-1)^p sh'_{p(q-1)}b_v.$$

Definition 2.18. (Cyclic Shuffle Map). The cyclic shuffle product, denoted by

$$sh':(C\otimes C')_n=\bigoplus_{p+q=n}C_p\otimes C_q\to(C\times C')_{n+2},$$

is the sum of all (p,q)-cyclic shuffle maps sh'_{pq} with $p+q=n$.

Proposition 2.4. The maps b, B, sh, and sh' satisfy the following equities in the normalized setting

$$[b, sh] = 0,$$

$$[B, sh] + [b, sh'] = 0,$$

$$[B, sh'] = 0.$$

Theorem 2.2. ***(Eilenberg - Zilber Theorem for Cyclic Homology).*** *Let C and C' be two cyclic modules and let $C \otimes C'$ be the tensor product of their associated mixed complexes. Then there is a canonical isomorphism*

$$Sh : HC_*(C \otimes C') \cong HC_*(C \times C')$$

induced by the shuffle product and cyclic shuffle product. It commutes with the morphisms B, I, and S of the Connes' exact sequence.

Remark 2.5. We can state something similar for any cylindrical module, which is more general than the product of two cyclic modules. In particular, if (C) is a cylindrical module, then one can define a generalized cyclic shuffle product, denoted

$$sh'_{pq} : C_{p,q} \to C_{p+q+2,p+q+2},$$

by

$$sh'_{pq} = \sum_{\substack{0 \le i \le p \\ 0 \le j \le q}} \sum_{\sigma} (-1)^{pi+qj+p+\sigma} s^h_{\sigma(n+1)} \cdots s^h_{\sigma(p+1)} s^h_{-1} (t^h_p)^i s^v_{\sigma(p)} \cdots s^v_{\sigma(0)} s^v_{-1} (t^v_q)^j$$

where the latter sum is taken over all (p, q)-shuffles over $\{0, ..., p+q-1\}$ such that $\sigma(i) < \sigma(p+j)$. The above maps induce the generalized cyclic shuffle map,

$$sh' : \mathrm{Tot}_n\, C \to \Delta_n(C)$$

$$sh' = \sum_{p+q=n} sh'_{pq}.$$

As with the product of cyclic modules,the map $Sh = sh + sh'$ is a quasi-isomorphism of the cyclic homology of the total complex and its diagonal. To bring this back to our discussion of algebras, since $\mathcal{A} \otimes \mathcal{B} \cong \mathcal{B} \otimes \mathcal{A}$, we have

$$\mathcal{B}_n(\mathcal{A} \otimes \mathcal{B}) = (\mathcal{A} \otimes \mathcal{B})^{\otimes n+1} \oplus (\mathcal{A} \otimes \mathcal{B})^{\otimes n-1} \oplus ... \cong \mathcal{A}^{\otimes n+1} \otimes \mathcal{B}^{\otimes n+1} \oplus ... = \mathcal{B}_n(\Delta(C(\mathcal{A}) \otimes C(\mathcal{B})))$$

and

$$\begin{aligned} \mathcal{B}_n(C(A) \otimes C(\mathcal{B})) &= B_n(\operatorname{Tot} C(\mathcal{A}) \otimes C(\mathcal{B})) \\ &= \bigoplus_{p+q=n} \mathcal{A}^{\otimes p+1} \otimes \mathcal{B}^{\otimes q+1} \oplus \bigoplus_{p+q=n-2} \mathcal{A}^{\otimes p+1} \otimes \mathcal{B}^{\otimes q+1} \oplus ... \end{aligned}$$

the above says that

$$HC_*(\mathcal{A} \otimes \mathcal{B}) \cong HC_*(C(\mathcal{A}) \otimes C(\mathcal{B})).$$

2.5 Negative and Periodic Theories

Let $C = (C_n)_{n \geq 0}$ be a cyclic module with face maps d_i, degeneracy maps s_j, and cyclic operator t, and $s = ts_n$, $b = \sum_{i=0}^{n} (-1)^i d_i$, $b' = \sum_{i=0}^{n-1} (-1)^i d_i$, $N = \sum (-1)^{in} t_n^i$, $B = (1 + (-1)^n t_{n+1}) sN$ the induced maps.

Definition 2.19. (Periodic and Negative Cyclic Bicomplexes). The following double complex, indexed by $\mathbb{Z} \times \mathbb{N}$ is called the periodic cyclic bicomplex:

$$\begin{tikzcd}
 & \vdots \arrow[d, "-b'"] & \vdots \arrow[d, "b"] & \vdots \arrow[d, "-b'"] & \\
\cdots & C_2 \arrow[l, "N"'] \arrow[d, "-b'"] & C_2 \arrow[l, "1-t"'] \arrow[d, "b"] & C_2 \arrow[l, "N"'] \arrow[d, "-b'"] & \cdots \arrow[l, "1-t"'] \\
\cdots & C_1 \arrow[l, "N"'] \arrow[d, "-b'"] & C_1 \arrow[l, "1-t"'] \arrow[d, "b"] & C_1 \arrow[l, "N"'] \arrow[d, "-b'"] & \cdots \arrow[l, "1-t"'] \\
\cdots & C_0 \arrow[l, "N"'] & C_0 \arrow[l, "1-t"'] & C_0 \arrow[l, "N"'] & \cdots \arrow[l, "1-t"'] \\
 & & & & \\
\text{col num} & -1 & 0 & 1 &
\end{tikzcd}$$

By deleting all of the negatively numbered columns from this complex we obtain CC, and by deleting all of the columns with numbers ≥ 2 we obtain the negative cyclic bicomplex that we denote by CC^-.

Definition 2.20. (Periodic and Negative Cyclic Homology).

Define $\mathrm{ToT}\, CC^{per}$ and $\mathrm{ToT}\, CC^{-}$ by

$$(\mathrm{ToT}\, CC^{per})_n = \prod_{p+q=n} CC^{per}_{pq} \text{ and } \left(\mathrm{ToT}\, CC^{-}\right)_n = \prod_{p+q=n} CC^{-}_{pq}.$$

The periodic and and negative cyclic homology of C is the homology of the above complexes,

$$HC^{per}(C) = H_n(\mathrm{ToT}\, CC^{per}) \text{ and } HC^{-}_n = H_n(\mathrm{ToT}\, CC^{-}).$$

If no confusion will arise, we will denote $HC^{per}_n(C)$ as HC^{per}_n. We note that for any integer n, $HC^{per}_{2n} = HC^{per}_0$ and $HC^{per}_{2n+1} = HC^{per}_1$, and for any integer less than or equal to 1, $HC^{-}_n = HC^{per}_n$.

Remark 2.6. We are using a direct product for the periodic and negative total complexes since the homology of the direct sum is trivial. This follows as the rows of CC^{per} and CC^{-} have trivial homology.

Proposition 2.5. For any cyclic module C there is an exact sequence

$$0 \to \varprojlim{}^1 HC_{n+2r+1} \to HC^{per}_n \to \varprojlim HC_{n+2r} \to 0$$

where $\lim_r HC_{n+2r}$ is the limit of

$$... \to HC_{n+2r} \xrightarrow{S} HC_{n+2(r-1)} \to ...$$

Remark 2.7. If S is surjective, or more generally if the above projective system satisfies the Mittag-Leffler condition, then

$$HC^{per}_n = \lim_r HC_{n+2r}.$$

By the definition of the above complexes there are maps

$$CC^{-} \xrightarrow{I} CC^{per} \xrightarrow{p} CC$$

which induce maps

$$I : HC^{-}_n \to HC^{per}_n \text{ and } p : HC^{per}_n \to HC.$$

As such, there are maps such that

$$HC^{-}_n \xrightarrow{I} HC^{per}_n \to \varprojlim HC_{n+2r} \to ... \to HC_{n+2r} \xrightarrow{S} HC_{n+2r-2} \to ...$$

Definition 2.21. (Periodic and Negative (b, B)-Bicomplexes).

We can also define the periodic and negative cyclic homology using the (b, B)-bicomplex, which is given below in the periodic case, and it is denoted $\mathcal{B}C^{per}$.

$$\begin{tikzcd}
 & & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \\
 & & \cdots \arrow[d] & C_1 \arrow[l] \arrow[d] & C_0 \arrow[l] \\
 & \cdots \arrow[d] & C_1 \arrow[l] \arrow[d] & C_0 \arrow[l] & \\
 & C_1 \arrow[d] & C_0 \arrow[l] & & \\
 & \cdots & & & \\
\text{col num} & -2 & -1 & 0 & 1
\end{tikzcd}$$

By deleting the positive numbered columns one obtains the negative (b, B)-complex, denoted $\mathcal{B}C^{-}$. Unless otherwise stated, we will be working with the normalized versions of these complexes. Then

$$HC_n^{-} = H_n(\mathrm{ToT}\,\mathcal{B}C^{-}) \text{ and } HC_n^{per} = H_n(\mathrm{ToT}\,\mathcal{B}C^{per})$$

We will use the notation $\alpha = \sum \alpha_{2i+\delta} = (\alpha_\delta, \alpha_{2+\delta}, ...)$ where $\alpha_{2i+\delta} \in C_{2i+\delta}$ for $\alpha \in \mathrm{ToT}_\delta \mathcal{B}C^{per}$ and similarly for elements of $\mathrm{ToT}\,\mathcal{B}C^{-}$. If $C = C(\mathcal{A})$, We will write $\mathcal{B}_*^{per}(\mathcal{A})$ in place of $\mathrm{ToT}_*\mathcal{B}^{per}(\mathcal{A})$ and $\mathcal{B}_*^{-}(\mathcal{A})$ in place of $\mathrm{ToT}_*\mathcal{B}^{-}(\mathcal{A})$.

Definition 2.22. (Periodic Cyclic Cohomology). We may similarly define a periodic and negative cyclic cocomplex by reversing arrows. We will denote this complex as $\mathcal{B}C_{per}$, and we denote the periodic cyclic cohomology as $HC^n_{per} = H^n(\mathrm{Tot}_*\mathcal{B}C_{per})$. As with periodic cyclic homology, there are only two groups up to isomorphism.

When using Connes' definition of periodic cyclic cohomology for a unital algebra $\mathcal{A}$, we will denote the periodic groups as $H^{ev}_\lambda(\mathcal{A})$ and $H^{odd}_\lambda(\mathcal{A})$ in place of $HC^0_{per}(\mathcal{A})$ and $HC^1_{per}(\mathcal{A})$. Connes' original definition was as a limit, which is in the following proposition replacing HC^n with H^n_λ.

Proposition 2.6. The periodic cyclic cohomology of C is

$$HC^n_{per}(C) = \varinjlim HC^n$$

where $\varinjlim HC^n$ is the limit of

$$\ldots \to HC^n \xrightarrow{S} HC^{n+2} \to \ldots$$

Example 2.4. Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}F = \mathbb{C}[F]/(F^2 - 1)$ be the complex Clifford algebra of degree 1. Then

$$\overline{C}_k(\mathcal{A}) = \mathcal{A} \otimes (\mathcal{A}/\mathbb{C})^{\otimes k} = (\mathbb{C} \oplus \mathbb{C}F) \otimes (\mathbb{C}F)^{\otimes k} = \mathbb{C} \otimes (\mathbb{C}F)^{\otimes k} \oplus (\mathbb{C}F)^{\otimes k+1}.$$

we will find the elements in $HC^{per}_i(\mathcal{A})$. Note that

$$\overline{\mathcal{B}}^{\text{per}}_0(\mathcal{A}) = \prod_{k\geq 0} \overline{C}_{2k}(\mathcal{A}) = \prod_{k\geq 0} C \otimes (\mathbb{C}F)^{\otimes 2k} \oplus (\mathbb{C}F)^{\otimes 2k+1},$$

$$\overline{\mathcal{B}}^{\text{per}}_1(\mathcal{A}) = \prod_{k\geq 0} \overline{C}_{2k+1}(\mathcal{A}) = \prod_{k\geq 0} C \otimes (\mathbb{C}F)^{\otimes 2k+1} \oplus (\mathbb{C}F)^{\otimes 2k+2}$$

Thus, even and odd chains are of the following respective form, were c^1_k and c^F_k are constants

$$\sum_{k\geq 0}(c^1_k \cdot 1 + c^F_k F) \otimes F^{\otimes 2k} = \sum_{k\geq 0} c^1_k \otimes F^{\otimes 2k} + c^F_k F^{\otimes 2k+1}$$

$$\sum_{k\geq 1}(c^1_k \cdot 1 + c^F_k F) \otimes F^{\otimes 2k-1} = \sum_{k\geq 1} c^1_k \otimes F^{\otimes 2k-1} + c^F_k F^{\otimes 2k}.$$

Even cycles are of the form

$$c^1_0 \cdot 1 + c^F_0 F + \sum_{k\geq 1} c^F_k F^{\otimes 2k+1}$$

$$\text{Where } c^F_k = \frac{(-1)^k c^F_0}{2^k} \prod_{j=1}^{k} (2j-1) = \frac{(-1)^k (2k)! c^F_0}{4^k k!}.$$

Indeed, consider that for $k > 0$

$$\begin{aligned}
(b+B)(c_0^1 \cdot 1) =& 0 \\
b\Big(c_k^F F^{\otimes 2k+1}\Big) =& c_k^F \otimes F^{\otimes 2k-1} + \sum_{i=1}^{2k} (-1)^i (c_k^F F \otimes F \otimes ... \otimes F \otimes 1 \otimes ... \otimes F) + c_k^F \otimes F^{\otimes 2k-1} \\
&= 2c_k^F \otimes F^{\otimes 2k-1} \\
B\Big(c_k^F F^{\otimes 2k+1}\Big) &= \sum_{i=0}^{2k} (-1)^{(2k)i} 1 \otimes F \otimes ... \otimes F \otimes c_k^F F \otimes ... \otimes F = \sum_{i=0}^{2k} c_k^F \otimes F^{\otimes 2k+1} \\
&= (2k+1) c_k^F \otimes F^{\otimes 2k+1} \\
b\Big(c_k^1 \otimes F^{\otimes 2k}\Big) =& c_k^F F^{\otimes 2k} + \sum_{i=1}^{2k} (-1)^i (c_k^F \otimes F \otimes ... \otimes F \otimes 1 \otimes ... \otimes F) + c_k^F F^{\otimes 2k} \\
&= 2c_k^F F^{\otimes 2k} \\
B\Big(c_k^1 \otimes F^{\otimes 2k}\Big) &= \sum_{i=0}^{2k} (-1)^{(2k)i} 1 \otimes F \otimes ... \otimes F \otimes c_k^F \otimes ... \otimes F) = 0.
\end{aligned}$$

Hence,

$$b\Big(c_{k+1}^F F^{\otimes 2(k+1)+1}\Big) + B\Big(c_k^F F^{\otimes 2k+1}\Big) = 0,$$

and so the above chain is a cycle.

Now, every odd chain is a boundary. Indeed, for any odd chain of the form $c_k^F F^{\otimes 2k}$, we have that

$$(b+B)\left(\frac{c_k^F}{2} \otimes F^{\otimes 2k}\right) = c_k^F F^{\otimes 2k},$$

and for any odd chain of the form $c_k^1 \otimes F^{\otimes 2k-1}$, we have that

$$(b+B)\left(\frac{c_k^1}{2} F^{\otimes 2k+1} + \sum_{j \geq k+1} (-1)^{j-k} \cdot \frac{c_k^1}{2} \cdot \left(\prod_{i=k}^{j} \frac{2i+1}{2}\right) F^{\otimes 2j+1}\right) = c_k^1 \otimes F^{\otimes 2k-1}$$

Thus, every odd chain is a boundary, and so $HC_1^{per}(\mathcal{A}) = 0$.

Example 2.5. Let $e \in \mathcal{A}$ be an idempotent. Then

$$ch(e) = \sum_{m \geq 0} (-1)^m \frac{(2m)!}{m!} e \otimes e^{\otimes 2m} \in HC_0^{per}(\mathcal{A})$$

is a cycle. We note that while we have only defined this for idempotents in $\mathcal{A}$, one can extend it to idempotents in $M_n(\mathcal{A})$. The cycle $ch(e)$ is the appropriate representation of an element of $[e] \in K_0(\mathcal{A})$ for the (b, B)-complex. A proof that $ch(e)$ is a cycle, and the appropriate representation of the element $[e] \in K_0(\mathcal{A})$ for the (b, B)-complex can be found in chapter 8 of [11].

Example 2.6. Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}F = \mathbb{C}[F]/(F^2 - 1)$ be the complex Clifford algebra of degree 1, and

$$\alpha = \frac{1+F}{2} + \sum_{k \geq 1} c_k^F F^{\otimes 2k+1}$$

$$\text{Where } c_k^F = \frac{(-1)^k}{2^{k=1}} \prod_{j=1}^{k} (2j-1) = \frac{(-1)^k (2k)!}{2^{2k+1} k!}.$$

If we let $e = \frac{1+F}{2}$, then e is an idempotent, and as we are working in the normalized setting we have

$$\begin{aligned}
ch(e) =& \sum_{k \geq 0} (-1)^k \frac{(2k)!}{k!} e \otimes e^{\otimes 2k} \\
=& e + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} e^{\otimes 2k+1} \\
=& \frac{1+F}{2} + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} \left(\frac{1}{2} + \frac{F}{2} \right)^{\otimes 2k+1} \\
=& \frac{1+F}{2} + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{k!} \left(\frac{1}{2} + \frac{F}{2} \right)^{\otimes 2k+1} \\
=& \frac{1+F}{2} + \sum_{k \geq 1} (-1)^k \frac{(2k)!}{2^{2k+1} k!} F^{\otimes 2k+1} \\
\\
=& \alpha.
\end{aligned}$$

In chapter 4 we will need to choose a cycle in the periodic cyclic homology for $\mathcal{A}$, and because of the above equality we will use the cycle α.

Example 2.7. Let $\mathcal{B} = \mathbb{C}[u, u^{-1}] = \langle 1, u, u^{-1} \rangle$. We will show that

$$ch(u) = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} (k-1)! \big(u^{-1} \otimes u \big)^k.$$

is a periodic cyclic cycle. Indeed, consider that

$$\begin{aligned}
b\Big((-1)^{k-1}(k-1)!\big(u^{-1}\otimes u\big)^{k}\Big) =&(-1)^{k-1}(k-1)!b\big(u^{-1}\otimes u\big)^{k}\\
=&(-1)^{k-1}(k-1)!\big(1\otimes(u^{-1}\otimes u)^{k-1}-1\otimes(u^{-1}\otimes u^{-1})^{k-1}\big)\\
B\Big((-1)^{k-2}(k-2)!\big(u^{-1}\otimes u\big)^{k-1}\Big) =&(-1)^{k-2}(k-2)!B\big(u^{-1}\otimes u\big)^{k-1}\\
=&(-1)^{k-2}(k-2)!\big(1(k-1)\otimes(u^{-1}\otimes u)^{k-1}\\
&\qquad -1(k-1)\otimes(u\otimes u^{-1})^{k-1}\big)\\
=&(-1)^{k-2}(k-1)!\big(1\otimes(u^{-1}\otimes u)^{k-1}-1\otimes(u\otimes u^{-1})^{k-1}\big)
\end{aligned}$$

so

$$b\Big((-1)^{k-1}(k-1)!\big(u^{-1}\otimes u^{-1})\big)^{k}\Big)+B\Big((-1)^{k-1}(k-2)!\big(u^{-1}\otimes u\big)^{k-1}\Big)=0.$$

In chapter 8 of [11], Loday shows that $ch(u)$ is a cycle in $\mathcal{B}_1^-(\mathcal{B})$, and goes on to show that it is the appropriate receptacle of the K-theory class of u in $K^1(\mathcal{B})$. As we always have a map $HC_n^-(\mathcal{B})\to HC_n^{per}(\mathcal{B})$ which is an isomorphism when $n=0,1$ so it can be viewed similarly in periodic cyclic. If there was a desire to differentiate these elements, one could write them as, $ch_1^-(u)$ and $ch_1^{per}(u)$, but we will not, as we will only every consider this cycle in the periodic cyclic homology.

2.6 Bivariant Cyclic Cohomology

Let $\mathcal{A}$ and $\mathcal{B}$ be unital k-algebras.

Definition 2.23. (Hom complex). Let (C,d) and (C',d') be chain complexes. The Hom complex, $\operatorname{Hom}(C,D)$, is the chain complex, which in degree n is given by

$$\prod_p \operatorname{Hom}(C_p,C'_{p+n}),$$

and whose differential, ∂, is given by

$$\partial(f)=d'f-(-1)^{|f|}fd$$

for a homogeneous element f.

Definition 2.24. (Bivariant Hochschild Cohomology). Let $C(\mathcal{A})$, $C(\mathcal{B})$ be the Hochschild complex of $\mathcal{A}$, $\mathcal{B}$. The bivariant Hochschild cohomology of $(\mathcal{A}, \mathcal{B})$ is defined to be

$$HH^n(\mathcal{A}, \mathcal{B}) := H_{-n}(\ \mathrm{Hom}(C(\mathcal{A}), C(\mathcal{B})), \partial), \quad n \in \mathbb{Z}.$$

Definition 2.25. (Bivariant Cyclic Cohomology). Since $\mathcal{A}$ and $\mathcal{B}$ are unital, one can use the bicomplex $\mathcal{B}(-)$ or $\overline{\mathcal{B}}(-)$ in the following definition. To simplify notation we will write $\mathcal{B}(\mathcal{A})$ in place of $\mathrm{Tot}_*\overline{\mathcal{B}}(\mathcal{A})$. This complex comes with a degree -2 endomorphism $S : \mathcal{B}(\mathcal{A}) \to \mathcal{B}(\mathcal{A})$ which is obtained by factoring out by the first column at the bicomplex level. In particular this endomorphism is surjective and its kernel is $C(\mathcal{A})$.

Let $\mathrm{Hom}^S(\mathcal{B}(\mathcal{A}), \mathcal{B}(\mathcal{B}))$ denote the submodule of $\mathrm{Hom}(\mathcal{B}(\mathcal{A}), \mathcal{B}(\mathcal{B}))$ of elements which commute with S. The bivariant cyclic cohomology of $(\mathcal{A}, \mathcal{B})$ is defined by

$$HC^n(\mathcal{A}, \mathcal{B}) := H_{-n}(\ \mathrm{Hom}^S(\mathcal{B}(\mathcal{A}), \mathcal{B}(\mathcal{B})), \partial).$$

This definition makes sense as $\partial(f)$ commutes with S if f does. Note that $HC^n(-,-)$ is contravariant in the first variable and covariant in the second. Note that a map $F \in \mathrm{Hom}^S(\mathcal{B}(\mathcal{A}), \mathcal{B}(\mathcal{B}))$ is a cocycle if and only if it is a map of complexes.

Proposition 2.7. By letting $\mathcal{B} = k$, one has

$$HC^n(\mathcal{A}, k) = HC^n(\mathcal{A}).$$

Further we have that

$$HC^n(k, \mathcal{A}) = HC^-_{-n}(\mathcal{A}).$$

To see the former, let F be a representative of a cycle $HC^n(\mathcal{A}, k)$. So F is a degree $-n$ chain map which commutes with S, ie $\mathcal{B}_*(\mathcal{A}) \to \mathcal{B}_{*-n}(k)$, $F_*S = SF_{*+2}$,and $F_*(b + B) = (b + B)F_{*+1}$. Thus, $F_n : \mathcal{B}_n(\mathcal{A}) \to \mathcal{B}_0(k) = k$, $F_n(b + B)_{\mathcal{A}} = (b + B)_k F_{n+1} = 0$ since for all $\lambda \in k$ the boundary is trivial, $(b + B)_k(\lambda) = 0$, so $F \in \mathrm{Hom}(\mathcal{B}_n(\mathcal{A}), k)$. The condition $F_nS = SF_{n+2}$ implies that $S(F_n) = F_{n+2} \in \mathrm{Hom}(\mathcal{B}_{n+2}(\mathcal{A}), k)$. Now, if $F_n \in \mathrm{Hom}(\mathcal{B}_n(\mathcal{A}), k)$ is a representative of a cocycle, let $F \in \operatorname{Hom}^S(\mathcal{B}(\mathcal{A}), \mathcal{B}(k))$ be defined by $F_j = 0$ for $j < n$ or if $n - j = 1 \mod 2$, and define $F_{n+2j} = S^j(F_n)$. Then F is a representative of a cycle for $H_{-n}(\operatorname{Hom}^S(\mathcal{B}(\mathcal{A}), \mathcal{B}(k)))$.

We will discuss one direction of the latter. In particular, if $\alpha = (0, ..., 0, \alpha_i, \alpha_{i+2}, ...) \in HC_i^-(\mathcal{A})$, then identify the cycle α as the element of $F \in \operatorname{Hom}^S(\mathcal{B}(k), \mathcal{B}(\mathcal{A}))$ defined by

$$F_{2n} : 1 \in \mathcal{B}_{2n}(k) \mapsto (0, ..., 0, \alpha_i, ..., \alpha_{i+2n})$$

assuming $i > 1$. If $i \leq 1$, then we take

$$F_{2n} : 1 \in \mathcal{B}_{2n}(k) \mapsto (\alpha_\delta, ..., \alpha_{\delta+2(n-i)}) \in \mathcal{B}_{\delta+2(n-i)}$$

where $\delta = 0, 1$ depending on if i is even or odd respectively. Now,

$$(b + B)\alpha = (0, ..., b\alpha_i, b\alpha_{i+2} + B\alpha_i, b\alpha_{i+4} + B\alpha_{i+2}, ...) = 0,$$

where $b\alpha_i$ is in the $i - 1$st position, so

$$(b + B)(0, ..., 0, \alpha_i, ..., \alpha_{i+2n}) = (0, ..., 0, b\alpha_i, b\alpha_{i+2} + B\alpha_i, ..., b\alpha_{i+2n} + B\alpha_{i+2(n-1)}) = 0,$$

and we have F is a chain map.

$$F((b + B)_k\lambda) + (-1)^n(b + B)_{\mathcal{A}}F(\lambda) = F(0) + (-1)^n(b + B)_{\mathcal{A}}\lambda(0, ..., 0, \alpha_i, ..., \alpha_{i+2n}) = 0.$$

Further,

$$F(S1_{2n+2}) = F(1_{2n}) = (0, ..., 0, \alpha_i, ..., \alpha_{i+2n}) = S(0, ..., 0, \alpha_i, ..., \alpha_{i+2n}, \alpha_{i+2n+2}) = SF(1_{2n+2}),$$

where the above subscripts mean that $1_{2k} \in \mathcal{B}_{2k}(k)$. So $F \in \operatorname{Hom}^S(\mathcal{B}(k), \mathcal{B}(\mathcal{A}))$ is a degree i map which is a representative of an element in $HC^{-i}(k, \mathcal{A})$. We will write $F = \alpha$.

Proposition 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be as above and let $\mathcal{C}$ be another k-algebra. Then there is a map*

$$\otimes_{\mathcal{C}} : HC^n(\mathcal{A}, \mathcal{B}) \to HC^n(\mathcal{A} \otimes \mathcal{C}, \mathcal{B} \otimes \mathcal{C}),$$

$$F \mapsto F_{\mathcal{C}}.$$

where $F_{\mathcal{C}}$ is a map which makes the following diagram commute and is induced by the quasi isomorphism, Sh,

$$\begin{tikzcd}
\mathcal{B}(\mathcal{A}) \arrow[r, "-\otimes 1"] \arrow[d, "F"] & \mathcal{B}(C(\mathcal{A}) \otimes C(\mathcal{C})) \arrow[r, "Sh"] \arrow[d, "F\otimes^S id"] & \mathcal{B}(\mathcal{A} \otimes \mathcal{C}) \arrow[d, "F_{\mathcal{C}}"] \\
\mathcal{B}(\mathcal{B}) \arrow[r, "-\otimes 1"] & \mathcal{B}(C(\mathcal{B}) \otimes C(\mathcal{C})) \arrow[r, "Sh"] & \mathcal{B}(\mathcal{B} \otimes \mathcal{C})
\end{tikzcd}$$

where $F \otimes^S id$ is the map making

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{B}(C(\mathcal{A}) \otimes C(\mathcal{C})) \arrow[r] \arrow[d, "F\otimes^S id"] & \mathcal{B}(\mathcal{A}) \otimes \mathcal{B}(\mathcal{C}) \arrow[r, "S"] \arrow[d, "F\otimes id"] & (\mathcal{B}(\mathcal{A}) \otimes \mathcal{B}\mathcal{C}))[2] \arrow[r] \arrow[d, "F\otimes id"] & 0 \\
0 \arrow[r] & \mathcal{B}(C(\mathcal{B}) \otimes C(\mathcal{C})) \arrow[r] & \mathcal{B}(\mathcal{B}) \otimes \mathcal{B}(\mathcal{C}) \arrow[r, "S"] & (\mathcal{B}(\mathcal{B}) \otimes \mathcal{B}(\mathcal{C}))[2] \arrow[r] & 0
\end{tikzcd}$$

commute.

Example 2.8. To be consistent with when we use this in Chapter 3, we will change notation from the above proposition, and we will let $k = \mathbb{C}$. We wish to make explicit,

$$\otimes_{\mathcal{B}} : HC_i^-(\mathcal{A}) = HC^{-i}(\mathbb{C}, \mathcal{A}) \to HC^{-i}(\mathcal{B}, \mathcal{A} \otimes \mathcal{B}).$$

Let $\alpha \in HC_i^-(\mathcal{A})$, and identify it in $HC^{-i}(\mathbb{C}, \mathcal{A})$ as in the discussion after proposition 2.7. Then the above diagram is as follows

$$\begin{tikzcd}
\mathcal{B}(\mathbb{C}) \arrow[r, "-\otimes 1"] \arrow[d, "\alpha"] & \mathcal{B}(C(\mathbb{C}) \otimes C(\mathcal{B})) = \mathcal{B}(\mathcal{B}) \arrow[r, "Sh=id"] \arrow[d, "\alpha\otimes^S id"] & \mathcal{B}(\mathbb{C} \otimes \mathcal{B}) = \mathcal{B}(\mathcal{B}) \arrow[d, "Sh(\alpha\otimes^S id)"] \\
\mathcal{B}(\mathcal{A}) \arrow[r, "-\otimes 1"] & \mathcal{B}(C(\mathcal{A}) \otimes C(\mathcal{B})) \arrow[r, "Sh"] & \mathcal{B}(\mathcal{A} \otimes \mathcal{B}).
\end{tikzcd}$$

So we have,

$$\otimes_{\mathcal{B}} : HC_i^-(\mathcal{A}) \to HC^{-i}(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$$

$$\alpha \mapsto Sh(\alpha \otimes^S id) : \mathcal{B}_*(\mathcal{B}) \to \mathcal{B}_{*+i}(\mathcal{A} \otimes \mathcal{B}),$$

$$\beta \mapsto Sh(\alpha \otimes \beta)$$

where if $\alpha \in \mathcal{B}_i^-(\mathcal{A})$ and $\beta \in \mathcal{B}_n(\mathcal{B})$, then

$$Sh(\alpha \otimes \beta) = \left\{ sh(\alpha \otimes \beta) + sh'(\alpha \otimes \beta) \right\}$$

and in degree j

$$sh(\alpha \otimes \beta) + sh'(\alpha \otimes \beta) = \sum_{p+q=j} sh_{pq}(\alpha_p, \beta_q) + \sum_{p+q=j-2} sh'_{pq}(\alpha_p, \beta_q)$$

noting that $\alpha_p = 0$ if $p < i$ or if i and p have different parity. As a more specific example, let $\alpha = (0, \alpha_2, \alpha_4, ...) \in \mathcal{B}_2^-(\mathcal{A})$ and $\beta = (\beta_1, \beta_3) \in \mathcal{B}_3(\mathcal{B})$. Then

$$Sh(\alpha \otimes \beta) = (0, sh_{21}(\alpha_2, \beta_1), sh'_{21}(\alpha_2, \beta_1) + sh_{41}(\alpha_4, \beta_1) + sh_{23}(\alpha_2, \beta_3)).$$

Definition 2.26. (Composition Product). Since the elements of bivariant Hochschild and Cyclic Homology are represented by maps of simplical and cyclic modules, one can compose them. These compositions pass to to the homology, and so for unital k-algebras, the composition of maps gives rise to functorial products, called composition products,

$$HH^p(\mathcal{A}, \mathcal{B}) \otimes HH^q(\mathcal{B}, \mathcal{C}) \to HH^{p+q}(\mathcal{A}, \mathcal{C})$$

$$HC^p(\mathcal{A}, \mathcal{B}) \otimes HC^q(\mathcal{B}, \mathcal{C}) \to HC^{p+q}(\mathcal{A}, \mathcal{C})$$

which satisfy the obvious associative condition.

2.7 Cyclic Homology of $\mathbb{Z}/2$ Graded Algebras

Definition 2.27. ($\mathbb{Z}/2$-Graded Vector Spaces). A $\mathbb{Z}/2$ graded vector space is a k-vector space V with a decomposition $V = V^+ \oplus V^-$. Elements of $V^\pm$ are called homogeneous of degree $|v| = 0$ if $v \in V^+$ and $|v| = 1$ if $v \in V^-$. There exists an involution $\gamma : V^\pm \to V^\pm$ such that $\gamma(v) = \pm v$ for homogeneous elements $v \in V^\pm$. We may take a $\mathbb{Z}/2$ graded vector space to be a $k[\mathbb{Z}/2]$ module where the generator of $\mathbb{Z}/2$ acts on V by γ. In this case, we might as well denote the generator of $\mathbb{Z}/2$ to be γ.

A $\mathbb{Z}/2$ graded vector space is said to be trivially graded if $V^+ = V$ and $V^- = 0$.

The $\mathbb{Z}/2$ grading on the tensor product of two $\mathbb{Z}/2$ graded vector spaces is given by

$$(V \otimes W)^\pm = (V^+ \otimes W^\pm) \oplus (V^- \otimes W^\mp),$$

the involution is given by $\gamma_V \otimes \gamma_W$.

We have that $V \otimes W \cong W \otimes V$ as $k[\mathbb{Z}/2]$ modules under the map $v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v$, which can be extended to the tensor product of n $k[\mathbb{Z}/2]$ modules.

We note that if V and W are $\mathbb{Z}/2$ graded, then $\operatorname{Hom}_k(V,W)$ has a $\mathbb{Z}/2$ grading. In particular, if $f \in \operatorname{Hom}_k(V,W)$, then f can be written as $f = f^+ + f^-$ where $f^+\gamma_V = \gamma_W f^+$ is called even and $f^-\gamma_V = -\gamma_E f^-$ is called odd. We will say even maps have degree zero and write $|f^+| = 0$, and odd maps have degree one and write $|f^-| = 1$. We will write $\operatorname{Hom}_k^+(V,W)$ for the even elements and $\operatorname{Hom}_k^-(V,W)$ for the odd, and we will have $\operatorname{Hom}_k(V,W) = \operatorname{Hom}_k^+(V,W) \oplus \operatorname{Hom}_k^-(V,W)$. As a further remark, elements of $\operatorname{Hom}(V,W)$ can be viewed as 2x2 matrices, even elements will be diagonal matrices, and odd elements will be anti-diagonal matrices. If $f : V \to V'$ and $g : W \to W'$ then we obtain a map,$f \otimes g$, from $V \otimes W \to V' \otimes W'$ by $(f \otimes g)(v \otimes w) = (-1)^{|g|\cdot|w|} f(v) \otimes g(w)$

Definition 2.28. ($\mathbb{Z}/2$-Graded Complex). A positively graded complex of vector spaces, $C_* = (V_*, d)$ is a $\mathbb{Z}/2$ graded complex if all vector spaces, V_n, are $\mathbb{Z}/2$ graded and the differentials, $d : V_n \to V_{n-1}$ are even. We may take this to be a complex of $k[\mathbb{Z}/2]$ modules. We note that the degree of an element $v \in V_p$ can be a confusing term as it has a chain complex degree and a $\mathbb{Z}/2$ graded degree. If refer to the degree of an element in this section, we will always mean the $\mathbb{Z}/2$ graded degree. The tensor product of two $\mathbb{Z}/2$ graded complexes $C_* = (V_*, d)$ and $C'_* = (W_*, d)$ is given by

$$(C \otimes C')_n = \bigoplus_{p+q=n} V_p \otimes W_q$$

with the $\mathbb{Z}/2$ grading given as above and with differential

$$d(v \otimes w) = d(v) \otimes w + (-1)^p v \otimes d(w)$$

for $v \in V_p$. As a note, we do not obtain a copy of $|v|$ in the exponent as d is even with respect to the $\mathbb{Z}/2$ grading.

Definition 2.29. ($\mathbb{Z}/2$-Graded Algebra). A $\mathbb{Z}/2$ graded algebra, $\mathcal{A}$, is an associative unital k-algebra such that the multiplication is a map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ of $\mathbb{Z}/2$ graded vector spaces. Equivalently, γ is a homomorphism of algebras with $\gamma(aa') = \gamma(a)\gamma(a')$ for all $a, a' \in \mathcal{A}$, or $\mathcal{A}^\pm \cdot \mathcal{A}^\pm \subset \mathcal{A}^+$ and $\mathcal{A}^\pm \cdot \mathcal{A}^\mp \subset \mathcal{A}^-$. The tensor product of $\mathbb{Z}/2$ graded algebras $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \otimes \mathcal{B}$ has an algebra structure given by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|a'|\cdot|b|}(a \cdot a' \otimes b \cdot b'),$$

where $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. If we are keeping score, I broke my rule as b, b' are used here as algebra elements, not differentials. This was entirely unnecessary as we could have used b_0 and b_1 in their place, but the temptation in this moment was too great. We hope the reader has found this minor digression humorous.

Definition 2.30. ($k[\mathbb{Z}/2]$-cyclic module structure and the induced homology). Let $\mathcal{A}$ be a $\mathbb{Z}/2$ graded algebra. Then $(C(\mathcal{A}), d_i, s_i, t_n)$ where

$$C_n(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A} \otimes ... \otimes \mathcal{A} = \mathcal{A} \otimes \mathcal{A}^{\otimes n}$$

$$d_i(a_0, a_1, ..., a_n) = (a_0, ..., a_{i-1}, a_i a_{i+1}, a_{i+2}, ..., a_n),\ i < n$$

$$d_n(a_0, a_1, ..., a_n) = (-1)^{|a_n|(|a_0|+...+|a_{n-1}|)}(a_n a_0, a_1, ..., a_{n-1})$$

$$s_i(a_0, ..., a_n) = (a_0, ..., a_i, 1, a_{i+1}, ..., a_n)$$

$$s(a_0, ..., a_n) = (1, a_0, ..., a_n)$$

$$t_n = (-1)^{|a_n|(|a_0|+...+|a_{n-1}|)}(a_n, a_0, ..., a_{n-1})$$

is a $k[\mathbb{Z}/2]$ cyclic module, were we have denoted the extra degeneracy above by s. Recall

$$N = \sum_{i=0}^{n} (-1)^{ni} t_n^i.$$

In the normalized setting, the induced differentials are then given by

$$\begin{aligned} b(a_0, ..., a_n) &= \sum_{i=0}^{n} (-1)^i d_i(a_0, ..., a_n) \\ &= \sum_{i=0}^{n-1} (-1)^i (a_0, ..., a_{i-1}, a_i a_{i+1}, a_{i+2}, ..., a_n) \\ &\qquad + (-1)^{n+|a_n|(|a_0|+...+|a_{n-1}|)}(a_n a_0, a_1, ..., a_{n-1}) \end{aligned}$$

$$B(a_0, ..., a_n) = sN(a_0, ..., a_n) = \sum_{i=0}^{n-1} (-1)^{ni+(|a_i|+...+|a_n|)(|a_0|+..+|a_{i-1}|)}(1, a_i, ..., a_n, a_0, ..., a_{i-1}).$$

As such we obtain the Hochschild homology, cyclic homology, periodic cyclic homology, and negative cyclic homology in the $\mathbb{Z}/2$ graded setting in this way.

Definition 2.31. (The Cyclic Cohomology of a $\mathbb{Z}/2$-Graded Algebra). Let $\mathcal{A}$ be a $\mathbb{Z}/2$ graded algebra, $(C_*(\mathcal{A}), b, B)$ the normalized mixed complex for $\mathcal{A}$. Define

$$C^n(\mathcal{A}) = \operatorname{Hom}_k(C_n(\mathcal{A}), k),$$

and for $f \in C^n(\mathcal{A})$ define $bf = f \circ b$ and $Bf = f \circ B$. The cyclic cohomology of $\mathcal{A}$ is the cohomology groups of $(C^*(\mathcal{A}), b, B)$. Since k is a field, we have $HC^n(\mathcal{A}) = \operatorname{Hom}_k(HC_n(\mathcal{A}), k)$, and when $\mathcal{A}$ is trivially graded, we have $HC^*(\mathcal{A}) = H^*_\lambda(\mathcal{A})$.

Theorem 2.3. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{Z}/2$-graded algebras. Then*

$$Sh : HC_*(\mathcal{A} \otimes \mathcal{B}) \cong HC_*(C_*(\mathcal{A}) \otimes C_*(\mathcal{B}))$$

where $HC_*(\mathcal{A} \otimes \mathcal{B})$ is the homology of cyclic module $C_*(\mathcal{A} \otimes \mathcal{B})$.

Remark 2.8. This follows from theorem 2.2. We note that in [9], the cyclic shuffle product was not immediately used, but one can obtain a proof with the cyclic shuffle product by replacing Kassel's map G with Sh. To dig a bit deeper, the above follows as $\mathcal{A} \otimes \mathcal{B} \cong \mathcal{B} \otimes \mathcal{A}$ as $k[\mathbb{Z}/2]$-modules, which will induce a quasi-isomorphism of the $k[\mathbb{Z}/2]$ cyclic modules $\Delta(C(\mathcal{A}) \otimes C(\mathcal{B}))$ and $C(\mathcal{A} \otimes \mathcal{B})$. The shuffle product map and the cyclic shuffle product map in the $\mathbb{Z}/2$ graded setting obtain signs of the form $(-1)^{|a_i| \cdot |b_k|}$ from when $a_i \otimes 1$ passes through $1 \otimes b_k$, which are induced by the repeated application of the above isomorphism. We note that the cyclic shuffle product will also have a sign induced by the cyclic actions applied to the elements of $\mathcal{A}^{\otimes p+1}$ and $\mathcal{B}^{\otimes q+1}$. As a final note, all of the formulas from sections 2.3 and 2.4 will apply to this setting, which brings us much joy.

Example 2.9. Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}F = \mathbb{C}[F]/(F^2 - 1)$ be the Clifford algebra viewed as a $\mathbb{Z}/2$-graded algebra with grading given by $\mathcal{A}^+ = \mathbb{C}$ and $\mathcal{A}^- = \mathbb{C}F$. We will compute the cycles for the periodic $\mathbb{Z}/2$-graded cyclic homology of $\mathcal{A}$. Consider that even and odd chains are of the form

$$\sum_{n=0}^{\infty} (c_n + c_n^F F) \otimes F^{\otimes 2n} = \sum_{n=0}^{\infty} c_n \otimes F^{\otimes 2n} + c_n^F F^{\otimes 2n+1}$$

$$\sum_{n=0}^{\infty} (c_n + c_n^F F) \otimes F^{\otimes 2n+1} = \sum_{n=0}^{\infty} c_n \otimes F^{\otimes 2n+1} + c_n^F F^{\otimes 2n+2},$$

where $c_n, c_n^F \in \mathbb{C}$. Now, applying the differential's to even chains we have

$$b(1 \otimes F^{\otimes 2n}) = F^{\otimes 2n} + \sum_{i=0}^{2n-1} (-1)^i 1 \otimes F^{\otimes i-1} \otimes 1 \otimes F^{\otimes 2n-1-i} + (-1)^{2n+|F|\cdot(2n-1)|F|} F^{\otimes 2n} = 0$$

$$B(1 \otimes F^{\otimes 2n}) = \sum_{i=0}^{2n} (-1)^{2ni+(2n-i-1)i} 1 \otimes F^{\otimes 2n-i+1} \otimes 1 \otimes F^{\otimes i-1} = 0$$

$$b(F \otimes F^{\otimes 2n}) = 1 \otimes F^{\otimes 2n-1} + 0 + (-1)^{2n+2n} 1 \otimes F^{\otimes 2n-1} = 2 \otimes F^{\otimes 2n-1}$$

$$B(F \otimes F^{\otimes 2n}) = \sum_{i=0}^{2n} (-1)^{2ni+(2n-i-1)i} 1 \otimes F^{\otimes 2n+1} = (2n+1) \otimes F^{\otimes 2n+1}$$

and applying to odd chains we have

$$b(1 \otimes F^{\otimes 2n+1}) = F^{\otimes 2n+1} + 0 + (-1)^{2n+1+2n} F^{\otimes 2n+1} = 0$$

$$B(1 \otimes F^{\otimes 2n+1}) = 0$$

$$b(F \otimes F^{\otimes 2n+1}) = 1 \otimes F^{\otimes 2n} + 0 + (-1)^{2n+1+2n+1} 1 \otimes F^{\otimes 2n} = 2 \otimes F^{\otimes 2n}$$

$$B(F \otimes F^{\otimes 2n+1}) = \sum_{i=0}^{2n+1} (-1)^{(2n+1)i+(2n-i)i} 1 \otimes F^{\otimes 2n+2} = (2n+2) \otimes F^{\otimes 2n+2}$$

So we have even cycles of the form

$$c_0^F F + \sum_{k \geq 1} c_{2k}^F F \otimes F^{\otimes 2k}$$

$$\text{where } c_{2k}^F = \frac{(-1)^k c_0^F}{2^k} \prod_{j=1}^{k} (2j-1) = \frac{(-1)^k (2k)! c_0^F}{4^k k!}.$$

We note that this differs from the non-graded case as 1 is now a boundary. Indeed we have

$$(b+B)\left(\frac{1}{2}\sum_{k=1}^{\infty} (-1)^{k-1}(k-1)! F \otimes F^{\otimes 2k-1}\right) = 1.$$

The chain whose boundary is 1 will be of particular interest in chapter 5.

Chapter 3

p-Summable Fredholm Modules and the Duality Map

From this point onward, $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$.

We will follow [3] for sections 1, 2, and 3. We note that Connes uses H_λ^n as his definition for cyclic cohomology. In Section 4, we will follow [7], which primarily uses the normalized (b, B)-bicomplex $\mathcal{B}(\mathcal{A})$ (with total complex $\mathcal{B}^*(\mathcal{A})$) for its definition of cyclic cohomology. Since $\mathcal{A}$ will always be unital algebra over $\mathbb{C}$, the inclusion mappings $C_\lambda^n(\mathcal{A}) \to \mathrm{Tot}_n CC(\mathcal{A}) \to \mathcal{B}^*(\mathcal{A})$ induce quasi-isomorphisms in cyclic cohomology.

In section 3.1, we will define (p-summable) Fredholm modules. In sections 3.2 and 3.3 we will define the Chern character for a Fredholm module, Ch_F^n, which will be an element of $H_\lambda^n(\mathcal{A})$. In 3.3 we will define a character map, Ch_ρ^n, for any almost representation ρ. Using Ch_ρ^n one can obtain Ch_F^n for an odd Fredholm module, and so we find the use of this notation appropriate.

Now if Ch_*^n is either Ch_F^n or Ch_ρ^n, we will have $Ch_*^n(a_0, ..., a_n) = 0$ if $a_i = 1$ for some i, so under the inclusion mapping Ch_*^n is an element of $\mathcal{B}^*(\mathcal{A})$). Lastly, there will be a change of normalization, as we would like for what we call Ch_*^n to be periodic, ie $S[Ch_*^n] = [Ch_*^{n+2}]$ in $HC^{n+2}(\mathcal{A})$, and the necessarily coefficients for this to occur are different in the two complexes.

3.1 p-summable Fredholm Modules

Definition 3.1. (Even p-summable Fredholm Module). Let $\mathcal{A}$ be a unital $\mathbb{Z}/2$ graded algebra over $\mathbb{C}$. An (even) p-summable Fredholm Module over $\mathcal{A}$ is a triple $(\rho, \mathcal{H}, F)$ where,

(1) $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is a $\mathbb{Z}/2$ graded Hilbert space with a self adjoint grading operator γ,

$\gamma h = (-1)^{\deg h} h$ for all $h \in \mathcal{H}^{\pm}$,

(2) $\mathcal{H}$ is a $\mathbb{Z}/2$ graded left $\mathcal{A}$-module, i.e. one has a graded homomorphism ρ of $\mathcal{A}$ in the algebra $\mathcal{L}(\mathcal{H})$ of bounded operators in $\mathcal{H}$,

(3) $F \in \mathcal{L}(\mathcal{H})$, $F^2 = 1$, $F = F^*$, $F\gamma = -\gamma F$ and for any $a \in \mathcal{A}$ one has

$$Fa - (-1)^{\deg a} aF = [F, a] \in \mathcal{L}^p(\mathcal{H}),$$

where $\mathcal{L}^p(\mathcal{H})$ is the pth Schatten ideal. That is $\mathcal{L}^p(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : \text{Trace}\,(|T|^p) < \infty\}$.

Here we are using the notation a in place of $\rho(a)$, and we will do this when the notation is unlikely to be confused.

Definition 3.2. (Odd p-summable Fredholm Module). An odd p-summable Fredholm Module over $\mathcal{A}$ is a triple $(\rho, \mathcal{H}, F)$ where,

(1) $\mathcal{H}$ is a left $\mathcal{A}$-module

(2) $F \in \mathcal{L}(\mathcal{H})$, $F^2 = 1$, $F = F^*$, and $[F, a] \in \mathcal{L}^p(\mathcal{H})$.

We note that in both of these definitions one can relax the condition that $F^2 = 1$ to $1 - F^2 \in \mathcal{L}^p(\mathcal{H})$, in [3] Connes shows that every Fredholm Module with that latter condition induces a Fredholm module with the former condition, and they represent the same K-homology class.

The following definition is taken from [8].

Definition 3.3. An even Fredholm module $(\rho, \mathcal{H}, F)$ is called balanced if $\mathcal{H}^+ = \mathcal{H}^- = \mathcal{H}'$ so $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'$, $\mathcal{H}$ is graded by this direct sum decomposition, and $\rho^+ = \rho^-$.

In chapter 5 we will define a weakly balanced almost Fredholm module. Yes, balanced Fredholm modules will be weakly balanced almost Fredholm modules.

3.2 Characters of Even p-Summable Fredholm Modules

As we are following [3], we will begin this section by defining cycles of dimension n and their characters over a graded algebra, we will define the universal differential graded algebra, provide Connes' proposition which shows that cycles of dimension n induce a cocycle in $H^n_\lambda(\mathcal{A})$, and finally define a cycle of dimension n for each even $(n+1)$-summable Fredholm module and give its character. We note that much of the material in this section is only used in this section, we have included it for the completeness of the exposition.

Definition 3.4. (Cycle of Dimension n). A cycle of dimension n is a triple $\left(\Omega, d, \int\right)$ where $\Omega = \bigoplus_{j=0}^n \Omega_j$ is a graded algebra, d is a graded derivation of degree 1 with $d^2 = 0$, and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace. That is

(1) $\Omega_i \times \Omega_j \subset \Omega_{i+j}$ for all $i, j \in \{0, 1, ..., n\}$, $i + j \leq n$;

(2) $d\Omega_i \subset \Omega_{i+1}$, $d(\omega_1\omega_2) = d(\omega_1)\omega_2 - (-1)^{|\omega_1|\cdot|\omega_2|}\omega_1 d(\omega_2)$

(3) $\int d\omega = 0$ for all $\omega \in \Omega^{n-1}$; $\int \omega_1\omega_2 = (-1)^{|\omega_1|\cdot|\omega_2|} \int \omega_2\omega_1$

where $|\omega|$ is the degree of ω.

Definition 3.5. (Character of a cycle). Given an n-dimensional cycle $\left(\Omega, d, \int\right)$ and a homomorphism $\rho : \mathcal{A} \to \Omega^0$, we define the character of the cycle by

$$\tau(a_0, ..., a_n) = \int \rho(a_0) d(\rho(a_1)) \cdots d(\rho(a_n)).$$

Definition 3.6. (Universal DGA associated to $\mathcal{A}$). Let $\mathcal{A}$ be an algebra. The universal differential graded algebra (DGA) associated to $\mathcal{A}$, denoted $\Omega(\mathcal{A})$, is given by

$$\Omega^j(\mathcal{A}) = \widetilde{\mathcal{A}} \otimes \mathcal{A}^{\otimes j}$$

where $\widetilde{\mathcal{A}}$ is the unitalization of $\mathcal{A}$, (and we do this even though $\mathcal{A}$ is unital), and

$$d((a_0 + \lambda 1) \otimes a_1 \otimes ... \otimes a_j) = (1 \otimes a_0 \otimes a_1 \otimes ... \otimes a_j).$$

Proposition 3.1. Let $\mathcal{A}$ be an algebra, and $\Omega(\mathcal{A})$ the universal DGA associated to $\mathcal{A}$. Then, for an $(n+1)$-linear functional τ on $\mathcal{A}$, the following are equivalent

(1) There exists an n-dimensional cycle $\left(\Omega, d, \int\right)$ and a homomorphism $\rho : \mathcal{A} \to \Omega^0$ such that

$$\tau(a_0, ..., a_n) = \int \rho(a_0)d(\rho(a_1)) \cdots d(\rho(a_n))$$

for all $a_0, .., a_n \in \mathcal{A}$.

(2) There exists a closed graded trace T of dimension n on $\Omega(\mathcal{A})$ such that

$$\tau(a_0, ..., a_n) = T(a_0 d(a_1) \cdots d(a_n))$$

for all $a_0, ..., a_n \in \mathcal{A}$.

(3) One has $\tau(a_1, ..., a_n, a_0) = (-1)^n \tau(a_0, ..., a_n)$ for all $a_0, ..., a_n \in \mathcal{A}$ and

$$\sum_{j=0}^{n} (-1)^j \tau(a_0, ..., a_j a_{j+1}, ..., a_{n+1}) + (-1)^{n+1} \tau(a_{n+1} a_0, a_1, ..., a_n) = 0$$

for all $a_0, ..., a_{n+1} \in \mathcal{A}$. Alternatively stated, $\tau \in Z^n_\lambda(\mathcal{A})$

Definition 3.7. (DGA induced by a Fredholm Module). Let $(\rho, \mathcal{H}, F)$ be an even $n+1$-summable Fredholm module over $\mathcal{A}$. Let $\widetilde{\mathcal{A}}$ be obtained from $\mathcal{A}$ by adjoining a unit which acts by the identity operator in $\mathcal{H}$. For any $T \in \mathcal{L}(\mathcal{H})$ we let $dT = i[F, T]$ where the commutator is a graded commutator. For each $j \in \mathbb{N}$ we let Ω^j be the linear span in $\mathcal{L}(\mathcal{H})$ of the operators of the form

$$a^0 da^1 \cdots da^j, a^k \in \widetilde{A}$$

Then $\Omega = \bigoplus_{j=1}^{n} \Omega^j$ is a differential graded algebra, with $d^2 = 0$.

Proposition 3.2. For $T \in \mathcal{L}(\mathcal{H})$ such that $[F, T] \in \mathcal{L}^1(\mathcal{H})$ let

$$\mathrm{Tr}_s(T) = \frac{1}{2}\mathrm{Trace}(\gamma F[F, T]).$$

Then,

(1) If T is homogeneous of odd degree, then $\mathrm{Tr}_s(T) = 0$.

(2) If $T \in \mathcal{L}^1(\mathcal{H})$, then $\mathrm{Tr}_s(T) = \mathrm{Trace}(\gamma T)$.

(3) One has $[F, \Omega^n] \subset \mathcal{L}^1(\mathcal{H})$ and the restriction of Tr_s to Ω^n defines a closed graded trace on the DGA Ω.

Definition 3.8. (Cycle of dimension $2m$ associated to an even $(2m+1)$-summable Fredholm Module). Let $n = 2m$, and $(\rho, \mathcal{H}, F)$ be an even $(2m+1)$-summable Fredholm Module over $\mathcal{A}$. The cycle of dimension $2m$ over $\mathcal{A}$ associated to $(\rho, \mathcal{H}, F)$ is given by the DGA induced by the Fredholm Module, the closed graded trace

$$\int \omega = m!\mathrm{Tr}_s(\omega) \text{ for all } \omega \in \Omega^n,$$

and the representation of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$.

Definition 3.9. (Character of an even $(2m+1)$-summable Fredholm Module, Chern character). The Chern character is the character of the cycle of dimension $2m$ associated to the $(2m+1)$-summable Fredholm Module $(\rho, \mathcal{H}, F)$ over $\mathcal{A}$, and it is

$$Ch_F^{2m}(a_0, .., a_{2m}) = m!\mathrm{Tr}_s(a_0 da_1 \cdots da_{2m}) = (-1)^m \frac{m!}{2}\mathrm{Trace}(\gamma F[F, a_0] \cdots [F, a_{2m}])$$

By proposition 3.1, given above, we have that $Ch_F^{2m} \in Z_\lambda^n(\mathcal{A})$. We note that if the Fredholm module is $2m$ summable, then we also have

$$Ch_F^{2m}(a_0, .., a_{2m}) = (-1)^m m!\mathrm{Trace}(\gamma a_0[F, a_1] \cdots [F, a_{2m}]).$$

Example 3.1. As a short example, if $n + 1 = 2m + 1 = 1$, we have $n = 2m = 0$, and $Ch_F^0(a) = \frac{1}{2}\mathrm{Trace}(\gamma F[F, a])$. As $[F, a]$ is trace class and $F^2 = 1$, one sees that

$$\begin{aligned}
Ch_F^0(a_0 a_1) =& \frac{1}{2}\mathrm{Trace}(\gamma F[F, a_0 a_1]) \\
=& \frac{1}{2}\mathrm{Trace}(\gamma(a_0 a_1 - F a_0 a_1 F) \\
=& \frac{1}{2}\mathrm{Trace}(\gamma(a_0 F^2 a_1 - a_0 F a_1 F + a_0 F a_1 F - F a_0 a_1 F) \\
=& \frac{1}{2}\mathrm{Trace}(\gamma(a_0 F[F, a_1] - [F, a_0] a_1 F) \\
=& \frac{1}{2}\mathrm{Trace}(\gamma(a_0 F[F, a_1] + a_1 F[F, a_0]) \qquad (3.1) \\
=& Ch_F^0(a_1 a_0) \qquad (3.2)
\end{aligned}$$

(3.1) follows as $[F, a_0]$ is trace class and $a_1 F\gamma = -\gamma a_1 F$. (3.2) follows by symmetry.

Proposition 3.3. The Chern character is periodic. That is, $SCh_F^n = Ch_F^{n+2}$ in $H_\lambda^{n+2}(\mathcal{A})$. Thus, it defines an element in $H_\lambda^{ev}(\mathcal{A}) \cong HC_{per}^0(\mathcal{A})$.

Proposition 3.4. There is a pairing $\langle \cdot, \cdot \rangle : K_0(\mathcal{A}) \times H_\lambda^n \to \mathbb{C}$, with

$$\langle [e], [\phi] \rangle = (m!)^{-1}(\phi\#\mathrm{Tr})(e, e, ..., e)$$

where if we identify $M_k(\mathcal{A})$ with $\mathcal{A} \otimes M_k(\mathbb{C})$ then

$$(\phi\#\mathrm{Tr})(a_0 \otimes m_0, ..., a_n \otimes m_n) = \phi(a_0, ..., a_n)\mathrm{Trace}(m_0 \cdots m_n).$$

Further, if $(\rho, \mathcal{H}, F)$ is a $(2m+1)$-summable Fredholm Module over $\mathcal{A}$, then $\langle e, [Ch_F^n] \rangle \in \mathbb{Z}$, and $\langle e, [Ch_F^n] \rangle = \text{Index } F_e^+$ where $F_e^+ = eF|_{e\mathcal{H}^+} : e\mathcal{H}^+ \to e\mathcal{H}^-$.

Example 3.2. Let $\mathcal{H}_1 = \mathcal{H} \oplus \mathcal{H}$ where $\mathcal{H}$ is a Hilbert Space, let P and Q be orthogonal projections on $\mathcal{H}$ such that $P - Q \in \mathcal{L}^{2m+1}(\mathcal{H})$, and $\mathcal{A} = \mathbb{C}[e]/(e^2 - e) = \langle 1_\mathcal{A}, e \rangle$ be an algebra generated by a multiplicative identity $1_\mathcal{A}$ and idempotent e. Let $1 \in \mathcal{L}(\mathcal{H})$ be the identity operator on $\mathcal{H}$.

Define $F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ so $F, \gamma \in \mathcal{L}(\mathcal{H}_1)$, $F^2 = 1$, $F = F^*$, and $F\gamma = -\gamma F$. Define $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H}_1)$ by

$$\rho(1_\mathcal{A}) = 1_{\mathcal{H}_1} \text{ and } \rho(e) = e = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix},$$

so

$$[F, e] = \begin{bmatrix} 0 & Q - P \\ P - Q & 0 \end{bmatrix}.$$

Since $P - Q \in \mathcal{L}^{2m+1}(\mathcal{H})$, we have $[F, e] \in \mathcal{L}^{2m+1}(\mathcal{H}_1)$ as

$$\Big|[F, e]\Big|^p = \begin{bmatrix} |P - Q|^p & 0 \\ 0 & |P - Q|^p \end{bmatrix},$$

and so

$$\mathrm{Trace}\Big(\Big|[F, e]\Big|^{2m+1}\Big) = 2\mathrm{Trace}\Big(|P - Q|^{2m+1}\Big) < \infty.$$

Thus, $(\rho, \mathcal{H}, F)$is a $(2m+1)$-summable Fredholm module over $\mathcal{A}$.

Now,

$$Ch_F^{2m}(e, e, ..., e) = (-1)^m \frac{m!}{2} \text{Trace}\Big(\gamma F[F, e]^{2m+1}\Big),$$

and

$$\gamma F[F, e]^{2n+1} = (-1)^m \begin{bmatrix} (P-Q)^{2n+1} & 0 \\ 0 & (P-Q)^{2n+1} \end{bmatrix}.$$

Thus,

$$Ch_F^{2m}(e, e, ..., e) = (-1)^m \frac{m!}{2} \text{Trace}\Big(\gamma F[F, e]^{2m+1}\Big) = (m!)\text{Trace}\Big((P-Q)^{2m+1}\Big),$$

and

$$\langle [e], [Ch_F^{2m}] \rangle = (m!)^{-1} Ch_F^{2m}(e, e, ..., e) = \text{Trace}\Big((P-Q)^{2m+1}\Big) = \text{Index}(QP)$$

where $QP : P(\mathcal{H}) \to Q(\mathcal{H})$.

Corollary 3.1. *If $P, Q \in \mathcal{L}(\mathcal{H})$ are projections such that $P - Q \in \mathcal{L}^{2m+1}$, then $Trace\Big((P-Q)^{2m+1}\Big) \in \mathbb{Z}$.*

Corollary 3.2. *If $P \in \mathcal{L}(\mathcal{H})$ is a projection, $U \in \mathcal{L}(\mathcal{H})$ is a unitary such that $[U, P] \in \mathcal{L}^{2m+1}$, and if we let $P_U = UPU^{-1}$, then*

$$Trace\Big((P - P_U)^{2m+1}\Big) = Trace\Big((P - UPU^{-1})^{2m+1}\Big) \in \mathbb{Z}.$$

As a note, $P - UPU^{-1} = [P, U]U^{-1} \in \mathcal{L}^{2m+1}$.

Remark 3.1. The result in the above example and subsequent corollaries appears in [2] where they use other methods to prove the result. In section 5.3.3 we will generalize this result.

3.3 Character of Odd p-Summable Fredholm Modules

Similarly to the previous section, much of the material is presented here as it is necessary to construct the Chern character for an odd n summable Fredholm module, but it will not be used in the later sections of this thesis.

Let $\mathcal{C}_1 = \mathbb{C} \oplus \alpha\mathbb{C} = \mathbb{C}[\alpha]/(\alpha^2 - 1)$ be the following $\mathbb{Z}/2$ graded Clifford algebra over $\mathbb{C}$,

$$\mathcal{C}_1^+ = \{\lambda 1, \lambda \in \mathbb{C}\},\ 1 \text{ the unit of } C_1$$

$$\mathcal{C}_1^- = \{\lambda\alpha, \lambda \in \mathbb{C}\},\ \alpha^2 = I.$$

Let $\mathcal{H}_1$ be the following $\mathbb{Z}/2$ graded Hilbert Space $\mathcal{H}_1^+ = \mathbb{C}$, $\mathcal{H}_1^- = \mathbb{C}$, and let $\mathcal{C}_1$ act on $\mathcal{H}_1$ by

$$\lambda + \mu\alpha \mapsto \begin{bmatrix} \lambda & \mu \\ \mu & \lambda \end{bmatrix} \in \mathcal{L}(\mathcal{H}_1).$$

We note that our notation is slightly inconsistent with the rest of the thesis, as in the other sections we have $\mathcal{A} = \mathbb{C}[F]/(F^2-1)$ as our Clifford algebra, but $\mathcal{A}$ and F are already taken, and we currently have no other use for the symbol α. This notation will only be used in this section, and briefly at that. We will abandon it after the following two lemmas.

Lemma 3.1. *Let $\mathcal{A}$ be a trivially $\mathbb{Z}/2$ graded algebra, and $(\rho, \mathcal{H}, F)$ an odd p-summable Fredholm module over $\mathcal{A}$.*

Then let $\hat{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_1$ be the obvious $\mathcal{A} \otimes \mathcal{C}_1$ module, whose representation we call π, and put $\hat{F} = i \begin{bmatrix} 0 & F \\ -F & 0 \end{bmatrix}$. Then $(\pi, \hat{\mathcal{H}}, \hat{F})$ is an even p-summable Fredholm module over the $\mathbb{Z}/2$ graded algebra $\mathcal{A} \otimes \mathcal{C}_1$.

Lemma 3.2. *Let τ_n be the n-dimensional character of $(\hat{\mathcal{H}}, \hat{F})$ for $n \geq p-1$. Then,*

a) if n is even one has $\tau_n = 0$;

b) if n is odd, one has $\tau_n = \tau_n' \otimes \gamma$, where γ is the graded trace on $\mathcal{C}_1$, $\gamma(\lambda + \mu\alpha) = \mu$ for all $\lambda + \mu\alpha \in \mathcal{C}_1$, and where

$$\tau_n'(a_0, ..., a_n) = (-1)^{\frac{n-1}{2}} c_n \, Trace(F[F, a_0][F, a_1] \cdots [F, a_n]), \text{ for all } a_i \in \mathcal{A};$$

$$c_{2m-1} = \left(m - \frac{1}{2}\right) \cdots \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)$$

c) One has $\tau_n' \in Z_\lambda^n(\mathcal{A})$

As it must be the case that n is odd, we take $n = 2m-1$ for the remainder of the section.

Definition 3.10. (Chern Character for an odd $(2m-1)$-summable Fredholm module). Let $(\rho, \mathcal{H}, F)$ be an odd $(2m-1)$-summable Fredholm module. Then the Chern character of $(\rho, \mathcal{H}, F)$ is $Ch_F^{2m-1} = \tau'_{2m-1}$.

As we have defined the Chern character, one might be inclined to call it a day for the odd case, however, we can form a more general formula.

Theorem 3.1. *Let Σ be an algebra, $J \subset \Sigma$ a two sided ideal, $m \in \mathbb{N}$, and τ a linear functional on J^m such that*

$$\tau(ab) = \tau(ba) \ \text{ for } a \in J^k,\ b \in J^q,\ k+q \geq m.$$

Let $\rho : \mathcal{A} \to \Sigma$ be a linear map which is multiplicative modulo J.

(1) *Let ϕ be the $2m$-linear functional on $\mathcal{A}$ given by*

$$\phi(a^0, ..., a^{2m-1}) = \tau(\omega_0 \varepsilon_2 \cdots \omega_{2m-2}) - \tau(\omega_1 \omega_3 \cdots \omega_{2m-1})$$

where $\omega_j = \rho(a^j a^{j+1}) - \rho(a^j)\rho(a^{j+1})$. Then $\phi \in Z_\lambda^{2m-1}(\mathcal{A})$.

(2) *Let $\rho' : \mathcal{A} \to \Sigma$ satisfy the same conditions as ρ, with $\rho(a) - \rho'(a) \in J$ for $a \in \mathcal{A}$; then one has $\phi' - \phi \in B_\lambda^{2m-1}(\mathcal{A})$.*

(3) *Let $\mathcal{A}$ and $(\rho, \mathcal{H}, F)$ be as above. Let $E_0 = \frac{1+F}{2} \in \mathcal{L}(\mathcal{H})$, $E = E_0(\mathcal{H})$, $\rho(a) = E_0 a E_0$ for all $a \in \mathcal{A}$, $\Sigma = \rho(\mathcal{A}) + \mathcal{L}^p(E)$, and $J = \mathcal{L}^p(E)$ for $m \geq p$. Then the corresponding $\phi \in Z_\lambda^{2m+1}(\mathcal{A})$ satisfies*

$$\phi = -(2^{-(2m+1)} c_{2m-1}^{-1}) \tau'_{2m-1} = -(2^{-(2m+1)} c_{2m-1}^{-1}) Ch_F^{2m-1}$$

Theorem 3.2. *Let $\mathcal{H}$ be a Hilbert Space, ρ a linear map of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ which is multiplicative modulo $\mathcal{L}^p(\mathcal{H})$.*

(a) *The following functional, denoted Ch_ρ^n where $n = 2m-1$ and $m \geq p$, belongs to $Z_\lambda^n(\mathcal{A})$:*

$$Ch_\rho^n(a_0, ..., a_n) = -2^{n+2} c_n \, \mathit{Trace}((\omega_0 \omega_2 \cdots \omega_{n-1}) - (\omega_n \omega_1 \cdots \omega_{n-2}))$$

where $\omega_j = \rho(a^j a^{j+1}) - \rho(a^j)\rho(a^{j+1})$.

(b) The class of Ch^n_ρ *in* $H^n_\lambda(\mathcal{A})$ *depends only on the quotient homomorphism* $\mathcal{A} \to \mathcal{L}(\mathcal{H})/\mathcal{L}^{p/2}(\mathcal{H})$.

(c) One has $S[Ch^n_\rho] = [Ch^{n+2}_\rho]$ *in* $H^{n+2}_\lambda(\mathcal{A})$. *Thus, it defines an element in* $H^{odd}_\lambda(\mathcal{A})$

3.4 Construction of the Duality Map

For this section we follow chapter 5 of [7]. As the duality map is the primary focus of Chapters 4 and 5, and as this specific work has yet to be published in a journal, this section will be done in significantly more detail than the presented background material to this point. Some additional proofs that do not appear in chapter 5 of [7] are in the appendix.

We note again that in this section, we will use the normalized (b, B)-bicomplex. A notational note, from here on we will use Tr = Trace.

3.4.1 The General Framework of the Duality Map

The purpose of this subsection is to give the general background for the following definition

Definition 3.11. (The Duality Map). Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras and $\mathcal{H}$ a Hilbert space such that $\mathcal{A}$ acts on $\mathcal{H}$, let $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ be a unital linear map which is multiplicative modulo $\mathcal{L}^p(\mathcal{H})$, $[a, \rho(b)] \in \mathcal{L}^p(\mathcal{H})$, and $s : \mathcal{A} \otimes \mathcal{B} \to \mathcal{L}(\mathcal{H})$ by $s(a \otimes b) = a\rho(b)$. Let n be an odd integer such that $n \geq 2p - 1$. The duality maps

$$\Psi^{i,n}_{\mathcal{A},\mathcal{B}} : HC^-_i(\mathcal{A}) \to HC^{n-i}(\mathcal{B})$$

$$P\Psi^{i,n}_{\mathcal{A},\mathcal{B}} : HC^{per}_i(\mathcal{A}) \to HC^{1-i}_{per}(\mathcal{B})$$

are given by

$$\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) = Ch^n_s\,(sh(\alpha \otimes \beta))$$

$$P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) = Ch^n_s\,(sh(\alpha \otimes \beta))$$

where

$$sh(\alpha \otimes \beta) = \sum_{k+l=n} sh_{kl}(\alpha_k, \beta_l).$$

Note that $\alpha_k = 0$ if k and i have different parity, and $\alpha_k = 0$ if $k < i$ in the negative case.

We will from now on maintain the assumptions from the previous definition. We note that ρ being multiplicative modulo $\mathcal{L}^p(\mathcal{H})$ means,

$$\omega(b_0, b_1) = \rho(b_0 b_1) - \rho(b_0)\rho(b_1) \in \mathcal{L}^p(\mathcal{H}).$$

ω is called the curvature of ρ.

Proposition 3.5. The linear map $s : \mathcal{A} \otimes \mathcal{B} \to \mathcal{L}(\mathcal{H})$, $s(a \otimes b) = a\rho(b)$ is multiplicative modulo $\mathcal{L}^p$. We will denote the curvature of s by ε.

Indeed,

$$\begin{aligned}\varepsilon(a_0 \otimes b_0, a_1 \otimes b_1) =& s((a_0 \otimes b_0)(a_1 \otimes b_1)) - s(a_0 \otimes b_0)s(a_1 \otimes b_1)\\ =& s(a_0 a_1 \otimes b_0 b_1) - s(a_0 \otimes b_0)s(a_1 \otimes b_1)\\ =& a_0 a_1 \rho(b_0 b_1) - a_0 \rho(b_0) a_1 \rho(b_1)\\ =& a_0 a_1 \rho(b_0 b_1) - a_0 a_1 \rho(b_0)\rho(b_1) + a_0 a_1 \rho(b_0)\rho(b_1) - a_0 \rho(b_0) a_1 \rho(b_1)\\ =& a_0 a_1 \omega(b_0, b_1) + a_0 [a_1, \rho(b_0)]\rho(b_1)\end{aligned}$$

and $\omega(b_0, b_1), [a_1, \rho(b_0)] \in \mathcal{L}^p$.

Proposition 3.6. The Chern Character, Ch_s^n, defined by

$$\begin{aligned}&Ch_s^n(x_0, ..., x_n) =\\ &\qquad c_n\Big(\mathrm{Tr}\big(\varepsilon(x_0, x_1)\varepsilon(x_2, x_3)\cdots\varepsilon(x_{n-1}, x_n)\big) - \mathrm{Tr}\big(\varepsilon(x_n, x_0)\varepsilon(x_1, x_2)\cdots\varepsilon(x_{n-2}, x_{n-1})\big)\Big)\\ &c_n = \frac{(-1)^{\frac{n-1}{2}}}{\left(\frac{n-1}{2}\right)!}\end{aligned}$$

is a cocycle in the cyclic cohomology of $\mathcal{A} \otimes \mathcal{B}$ using the normalized (b, B)-complex.

Remark 3.2. Theorem 3.1 from section 3.3, guarantees that this map is a cocycle in $H_\lambda^n(\mathcal{A} \otimes \mathcal{B})$. Since $Ch_s^n(x_0, ..., x_n) = 0$ if any $x_i = 1$, it is a cocycle in the the normalized (b, B)-bicomplex. We are using the notation from theorem 3.2 as this is the corresponding map in the normalized

(b, B)-bicomplex, the change in the coefficients is due to the change in the complex. The above coefficient is necessary for periodicity to hold in the normalized (b, B)-bicomplex, and so it is our choice, but it should be noted that we can scale this coefficient as desired. A discussion of why the properties of theorem 3.2 hold for this cocycle is in the appendix.

The requirement that $n \geq 2p-1$ is needed so that $\varepsilon(x_0, x_1) \cdots \varepsilon(x_{n-1}, x_n)$ is trace class.

In general, one can construct the following products using bivariant cyclic cohomology

$$HC_i^-(\mathcal{A}) \otimes_{\mathcal{A}} HC^n(\mathcal{A} \otimes \mathcal{B}) \to HC^{n-i}(\mathcal{B})$$

$$HC_i^-(\mathcal{B}) \otimes_{\mathcal{B}} HC^n(\mathcal{A} \otimes \mathcal{B}) \to HC^{n-i}(\mathcal{A})$$

$$HC_i^{per}(\mathcal{A}) \otimes_{\mathcal{A}} HC^n_{per}(\mathcal{A} \otimes \mathcal{B}) \to HC^{n-i}_{per}(\mathcal{B})$$

$$HC_i^{per}(\mathcal{B}) \otimes_{\mathcal{B}} HC^n_{per}(\mathcal{A} \otimes \mathcal{B}) \to HC^{n-i}_{per}(\mathcal{A}).$$

The products can be constructed by composing the natural map

$$HC_i^-(\mathcal{A}) = HC^{-i}(\mathbb{C}, \mathcal{A}) \overset{\otimes \mathcal{B}}{\to} HC^{-i}(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$$

and the composition product

$$HC^{-i}(\mathcal{B}, \mathcal{A} \otimes \mathcal{B}) \otimes HC^n(\mathcal{A} \otimes \mathcal{B}, \mathbb{C}) \to HC^{n-i}(\mathcal{B}, \mathbb{C}) = HC^{n-i}(\mathcal{B}).$$

These products are discussed in chapter 2 section 5. As a reminder, the map $\otimes_{\mathcal{B}}$ takes $\alpha = (0, ..., 0, \alpha_i, ...) \in HC_i^-(\mathcal{A})$ to the map $\Big(Sh(\alpha \otimes id) : \beta \mapsto Sh(\alpha \otimes \beta)\Big) \in \operatorname{Hom}^S(\mathcal{B}(\mathcal{B}), \mathcal{B}(\mathcal{A} \otimes \mathcal{B}))$, where

$$Sh(\alpha \otimes \beta) = \big\{ sh(\alpha \otimes \beta) + sh'(\alpha \otimes \beta) \big\}$$

and in degree $2r+1$ (which we consider since n is odd)

$$sh(\alpha \otimes \beta) + sh'(\alpha \otimes \beta) = \sum_{k+l=2r+1} sh_{kl}(\alpha_k, \beta_l) + \sum_{k+l=2r-1} sh'_{kl}(\alpha_k, \beta_l)$$

noting that $\alpha_k = 0$ if $k < i$ or if i and k have different parity. We note that this notation can be confusing since $sh + sh'$ doesn't have any indication of which chains it is being applied to. That said, we will from here on only consider when it is applied to degree n chains, since we will be evaluation chains under Ch_s^n, which is zero if the chain has degree other than n, and so from here on it should be less vague.

Now, using the above maps and fixing $Ch_s^n \in HC^n(\mathcal{A}\otimes\mathcal{B},\mathbb{C})$, we can define the duality maps

$$\Psi^{i,n}_{\mathcal{A},\mathcal{B}} : HC_i^-(\mathcal{A}) \to HC^{n-i}(\mathcal{B})$$
$$\Psi^{i,n}_{\mathcal{B},\mathcal{A}} : HC_i^-(\mathcal{B}) \to HC^{n-i}(\mathcal{A})$$

and

$$P\Psi^{i,n}_{\mathcal{A},\mathcal{B}} : HC_i^{per}(\mathcal{A}) \to HC^{1-i}_{per}(\mathcal{B})$$
$$P\Psi^{i,n}_{\mathcal{B},\mathcal{A}} : HC_i^{per}(\mathcal{B}) \to HC^{1-i}_{per}(\mathcal{A}).$$

where

$$\begin{aligned}\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) =& Ch_s^n\left(Sh(\alpha\otimes\beta)\right)\\ =& Ch_s^n\big(sh(\alpha\otimes\beta)+sh'(\alpha\otimes\beta)\big)\\ =& Ch_s^n\left(sh(\alpha\otimes\beta)\right).\end{aligned}$$

The middle equality holds because Ch_s^n is zero on chains of degree different from n, and the last equality is due to the fact that $sh'(\alpha\otimes\beta)$ always begin with $1\otimes 1$, and Ch_s^n vanishes on such elements. The maps in the periodic case are the maps in the negative case letting $i=0,1$ under the mapping $\mathcal{B}^{n-i}(\mathcal{B}) \to \mathcal{B}^{1-i}_{per}(\mathcal{B}), \phi\mapsto\phi$.

Proposition 3.7. The duality maps are periodic, and if ρ' is another almost representation of $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$, then the maps induced by Ch_s^n and $Ch_{s'}^n$ are chain homotopic. That is, $S\Psi^{i,n}_{\mathcal{A},\mathcal{B}} = \Psi^{i,n+2}_{\mathcal{A},\mathcal{B}}$ and $\Psi^{i,n+2}_{\mathcal{A},\mathcal{B}}$ is independent of the choice of ρ up to $\mathcal{L}^p$ perturbations.

We note that the periodicity of the maps Ch_s^n will imply periodicity of the duality maps, and that Ch_s^n and $Ch_{s'}^n$ are representatives of the same class in $HC^n(\mathcal{A}\otimes\mathcal{B})$ will imply the independence of $\Psi^{i,n}_{\mathcal{A},\mathcal{B}}$ on ρ. A more detailed argument is given in the appendix.

3.4.2 An Explicit Formula for the Duality Map

The goal of this subsection is to prove the following theorem

Theorem 3.3. *The duality map is*

$$\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) = Ch^n_s\left(sh(\alpha\otimes\beta)\right) = Ch^n_s\left(\sum_{k+l=n} sh_{kl}(\alpha_k,\beta_l)\right) = \sum_{k+l=n} Ch^n_s\left(sh_{kl}(\alpha_k,\beta_l)\right)$$

where for $k = 0$

$$\begin{aligned} Ch^n_s(sh_{0n}(\alpha_0,\beta_n)) =& c_{n,0}\Bigg(Tr\Big(a_0\omega(b_0,b_1)\omega(b_2,b_3)\cdots\omega(b_{n-1},b_n)\Big) \\ &- Tr\Big(\Big(a_0\omega(b_n,b_0) + [a_0,\rho(b_n)]\rho(b_0)\Big)\omega(b_1,b_2)\cdots\omega(b_{n-2},b_{n-1})\Big)\Bigg) \end{aligned}$$

and for $k > 0$

$$\begin{aligned} Ch^n_s(sh_{kl}(\alpha_k,\beta_l)) = c_{n,k}\Bigg(& \sum_{0\le i_1<i_2<\ldots<i_k\le\frac{n-1}{2}} Tr\Big(a_0[a_1,\rho(b_0)]\omega(b_1,b_2)\ldots[a_2,\rho(b_{2i_2-1})]\omega(b_{2i_2},b_{2i_2+1})\ldots \\ & \qquad + a_0\omega(b_0,b_1)\ldots[a_1,\rho(b_{2i_1})]\omega(b_{2i_1+1},b_{2i_1+2})\ldots\Big) \\ -(-1)^k & \sum_{0<i_1\le\ldots\le i_k\le\frac{n-1}{2}} Tr\Big(\Big(a_0\omega(b_n,b_0) + [a_0,\rho(b_n)]\rho(b_0)\Big)\omega(b_1,b_2)\cdots \\ & \qquad [a_1,\rho(b_{2i_1})]\omega(b_{2i_1+1},b_{2i_1+2})\ldots[a_j,\rho(b_{2i_j-j+1})]\ldots\Big)\Bigg) \end{aligned}$$

and

$$c_{n,k} = \frac{(-1)^{\frac{n-1}{2}+\frac{k(k-1)}{2}}}{(\frac{n-1}{2})!}.$$

To achieve this goal, we must understand

$$Ch^n_s(x_0,\ldots,x_n) = c_n\Bigg(\mathrm{Tr}\Big(\varepsilon(x_0,x_1)\cdots\varepsilon(x_{n-1},x_n)\Big) - \mathrm{Tr}\Big(\varepsilon(x_n,x_0)\cdots\varepsilon(x_{n-2},x_{n-1})\Big)\Bigg)$$

in this framework, so we first consider computations of the maps

$$\varepsilon(a_0\otimes b_0, a_1\otimes b_1) = a_0a_1\omega(b_0,b_1) + a_0[a_1,\rho(b_0)]\rho(b_1)$$

for the following cases.

$$\varepsilon(1\otimes b_0, a_1\otimes b_1) = a_1\omega(b_0,b_1) + [a_1,\rho(b_0)]\rho(b_1)$$

$$\varepsilon(a_0\otimes 1, a_1\otimes b_1) = a_0a_1\omega(1,b_1) + a_0[a_1,\rho(1)]\rho(b_1) = 0$$

$$\varepsilon(a_0\otimes b_0, 1\otimes b_1) = a_0\omega(b_0,b_1) + a_0[1,\rho(b_0)]\rho(b_1) = a_0\omega(b_0,b_1)$$

$$\varepsilon(a_0\otimes b_0, a_1\otimes 1) = a_0a_1\omega(b_0,1) + a_0[a_1,\rho(b_0)]\rho(1) = a_0[a_1,\rho(b_0)].$$

Next, we will compute

$$Ch_s^n(sh_{kl}(\alpha_k,\beta_l)) = c_n\left(\mathrm{Tr}\Big(\varepsilon(x_0,x_1)\cdots\varepsilon(x_{n-1},x_n)\Big) - \mathrm{Tr}\Big(\varepsilon(x_n,x_0)\cdots\varepsilon(x_{n-2},x_{n-1})\Big)\right)$$

where $\alpha = a_0\otimes ...\otimes a_k$, $\beta = b_0\otimes ...\otimes b_l$, and $x_r = 1\otimes b_i$ or $a_i\otimes 1$. To do this, we will first compute $\mathrm{Tr}\Big(\varepsilon(x_0,x_1)\varepsilon(x_2,x_3)\cdots\varepsilon(x_{n-1},x_n)\Big)$ using cases, and then we will compute $\mathrm{Tr}\Big(\varepsilon(x_n,x_0)\varepsilon(x_1,x_2)\cdots\varepsilon(x_{n-2},x_{n-1})\Big)$ using cases.

We first consider the case where $k = 0$, as a reminder n is odd such that $n \geq 2p-1$. Then $l = n$, $sh_{0n}(\alpha_0,\beta_n) = (a_0\otimes b_0, 1\otimes b_1, ..., 1\otimes b_n)$, and

$$\mathrm{Tr}\Big(\varepsilon(a_0\otimes b_0, 1\otimes b_1)\varepsilon(1\otimes b_2, 1\otimes b_3)\cdots\varepsilon(1\otimes b_{n-1}, 1\otimes b_n)\Big) = \mathrm{Tr}\Big(a_0\omega(b_0,b_1)\omega(b_2,b_3)\cdots\omega(b_{n-1},b_n)\Big).$$

We now consider the case where $k > 0$. We note that $x_0 = a_0\otimes b_0$, x_j is of the form $a\otimes 1$ or $1\otimes b$, and if $x_{2j} = a\otimes 1$, then $\varepsilon(x_{2j},x_{2j+1}) = 0$, so

$$\mathrm{Tr}\big(\varepsilon(x_0,x_1)\varepsilon(x_2,x_3)\cdots\varepsilon(x_{n-1},x_n)\big) = 0.$$

Thus, for $k > \frac{n+1}{2}$, $\mathrm{Tr}\big(\varepsilon(x_0,x_1)\varepsilon(x_2,x_3)\cdots\varepsilon(x_{n-1},x_n)\big) = 0$ since we must have $x_{2j} = a_i\otimes 1$ for some j. Therefore, we consider the case for which $k \leq \frac{n+1}{2}$.

Now, we will show that all of the (k,l)-shuffles that result in a possible non-zero value have the same sign. Suppose that we have a shuffle such that for some m,

$$\begin{aligned} x_{2m+1} =& 1\otimes b_i \\ x_{2m+2} =& 1\otimes b_{i+1} \\ x_{2m+3} =& a_j\otimes 1. \end{aligned}$$

Then if we apply the even permutation $(2m+1\ \ 2m+2\ \ 2m+3)$ to this shuffle, we obtain another shuffle that is equal to the first at all places except at

$$\begin{aligned} x'_{2m+1} =& x_{2m+3} = a_j \otimes 1 \\ x'_{2m+2} =& x_{2m+1} = 1 \otimes b_i \\ x'_{2m+3} =& x_{2m+2} = 1 \otimes b_{i+1}. \end{aligned}$$

These shuffles have the same sign since they differ by an even permutation. Thus, if we can get from one shuffle to another via even permutations, we will obtain shuffles of the same sign. Thus, for a fixed k, every permutation such that $a \otimes 1$ appear only in the odd places have the same sign.

Thus, the sign of the shuffle

$$(a_0 \otimes b_0, a_1 \otimes 1, 1 \otimes b_1, a_2 \otimes 1, 1 \otimes b_2, ..., a_k \otimes 1, 1 \otimes b_k, 1 \otimes b_{k+1}, ..., 1 \otimes b_l)$$

is the same as that for all shuffles with $(a \otimes 1)$ in odd places, and it is

$$(-1)^{\frac{k(k-1)}{2}}.$$

An argument for this is in the appendix.

Next we consider the case where $x_1 = a_1 \otimes 1$. Let $a_j \otimes 1$ be in position $2i_j + 1$, so $i_1 = 0$, and $i_j < i_{j'}$ if $j < j'$, and note that $i_k \leq \frac{n-1}{2}$. Then the contribution of such shuffles is

$$c_{n,k} \sum_{0=i_1<i_2<...<i_k\leq\frac{n-1}{2}} \mathrm{Tr}\Big(a_0[a_1, \rho(b_0)]\omega(b_1, b_2)...[a_2, \rho(b_{2i_2-1})]\omega(b_{2i_2}, b_{2i_2+1}) \cdots [a_j, \rho(b_{2i_j-j+1})]...\Big)$$

where

$$c_{n,k} = \frac{(-1)^{\frac{n-1}{2}+\frac{k(k-1)}{2}}}{(\frac{n-1}{2})!}.$$

Now consider if $x_1 = 1 \otimes b_1$. Similarly as before, let $a_j \otimes 1$ be in position $2i_j + 1$, noting that now $0 < i_1$. The contribution of such shuffles is

$$c_{n,k} \sum_{0<i_1<i_2<...<i_k\leq\frac{n-1}{2}} \mathrm{Tr}\Big(a_0\omega(b_1, b_2)...[a_1, \rho(b_{2i_1})]\omega(b_{2i_1+1}, b_{2i_1+2})...[a_j, \rho(b_{2i_j-j+1})]...\Big)$$

Then adding the above expressions, we obtain a formula for the first term:

$$c_{n,k} \sum_{0 \leq i_1 < i_2 < \ldots < i_k \leq \frac{n-1}{2}} \operatorname{Tr}\big(a_0[a_1, \rho(b_0)]\omega(b_0, b_1)\ldots[a_2, \rho(b_{2i_2-1})]\omega(b_{2i_2}, b_{2i_2+1})\ldots$$

$$+ a_0\omega(b_0, b_1)\ldots[a_1, \rho(b_{2i_1})]\omega(b_{2i_1+1}, b_{2i_1+2})\ldots\big)$$

We now compute the second term of Ch_s^n,

$$\operatorname{Tr}\big(\varepsilon(x_n, x_0)\varepsilon(x_1, x_2) \cdots \varepsilon(x_{n-2}, x_{n-1})\big).$$

As before, we first consider the case where $k = 0$. So we compute

$$\operatorname{Tr}\Big(\varepsilon(1 \otimes b_n, a_0 \otimes b_0)\varepsilon(1 \otimes b_1, 1 \otimes b_2) \cdots \varepsilon(1 \otimes b_{n-2}, 1 \otimes b_{n-1})\Big)$$

$$= \operatorname{Tr}\bigg(\Big(a_0\omega(b_n, b_0) + [a_0, \rho(b_n)]\rho(b_0)\Big)\omega(b_1, b_2) \cdots \omega(b_{n-2}, b_{n-1})\bigg)$$

We now let $k > 0$. Note that if $x_{2j+1} = a \otimes 1$, then $\varepsilon(x_{2j+1}, x_{2j+2}) = 0$, so

$$\operatorname{Tr}\big(\varepsilon(x_n, x_0)\varepsilon(x_1, x_2) \cdots \varepsilon(x_{n-2}, x_{n-1})\big) = 0,$$

and so it must be the case that $k \leq \frac{n-1}{2}$. Similarly as before, all of the shuffles that satisfy the condition that $x_{2i+1} \neq a \otimes 1$ have the same sign for a fixed k, and in particular that sign is $(-1)^{\frac{k(k+1)}{2}} = (-1)^k(-1)^{\frac{k(k-1)}{2}}$. As before, let i_j be such that $x_{2i_j} = a_j \otimes 1$. So for the second term we have

$$(-1)^k c_{n,k} \sum_{0 < i_1 \leq \ldots \leq i_k \leq \frac{n-1}{2}} \operatorname{Tr}\bigg(\Big(a_0\omega(b_n, b_0) + [a_0, \rho(b_n)]\rho(b_0)\Big)\omega(b_1, b_2) \cdots$$

$$[a_1, \rho(b_{2i_1})]\omega(b_{2i_1+1}, b_{2i_1+2}) \cdots [a_j, \rho(b_{2i_j-j+1})] \cdots\bigg).$$

Thus we have the following formulas. In the case that $k = 0$

$$\begin{aligned} Ch_s^n(sh_{0n}(\alpha_0, \beta_n)) =& c_n\Big(\operatorname{Tr}\big(\varepsilon(x_0, x_1) \cdots \varepsilon(x_{n-1}, x_n)\big) - \operatorname{Tr}\big(\varepsilon(x_n, x_0) \cdots \varepsilon(x_{n-2}, x_{n-1})\big)\Big) \\ =& c_{n,0}\Big(\operatorname{Tr}\big(a_0\omega(b_0, b_1)\omega(b_2, b_3) \cdots \omega(b_{n-1}, b_n)\big) \\ &- \operatorname{Tr}\big(\big(a_0\omega(b_n, b_0) + [a_0, \rho(b_n)]\rho(b_0)\big)\omega(b_1, b_2) \cdots \omega(b_{n-2}, b_{n-1})\big)\Big) \end{aligned}$$

and in the case that $k > 0$ we have

$$\begin{aligned}
Ch^n_s(sh_{kl}(\alpha_k, \beta_l)) &= c_n\Bigg(\mathrm{Tr}\big(\varepsilon(x_0, x_1)\cdots\varepsilon(x_{n-1}, x_n)\big) - \mathrm{Tr}\big(\varepsilon(x_n, x_0)\cdots\varepsilon(x_{n-2}, x_{n-1})\big)\Bigg)\\
&= c_{n,k}\Bigg(\sum_{0\leq i_1<i_2<\ldots<i_k\leq\frac{n-1}{2}} \mathrm{Tr}\big(a_0[a_1, \rho(b_0)]\omega(b_1, b_2)...[a_2, \rho(b_{2i_2-1})]\omega(b_{2i_2}, b_{2i_2+1})...\\
&\qquad + a_0\omega(b_0, b_1)...[a_1, \rho(b_{2i_1})]\omega(b_{2i_1+1}, b_{2i_1+2})...\big)\\
&\qquad -(-1)^k \sum_{0<i_1\leq\ldots\leq i_k\leq\frac{n-1}{2}} \mathrm{Tr}\Big(\big(a_0\omega(b_n, b_0) + [a_0, \rho(b_n)]\rho(b_0)\big)\omega(b_1, b_2)\cdots\\
&\qquad [a_1, \rho(b_{2i_1})]\omega(b_{2i_1+1}, b_{2i_1+2})...[a_j, \rho(b_{2i_j-j+1})]...\Big)\Bigg)
\end{aligned}$$

where

$$c_{n,k} = \frac{(-1)^{\frac{n-1}{2}+\frac{k(k-1)}{2}}}{(\frac{n-1}{2})!}.$$

3.4.3 Another Formula for the Duality Map

Note that in this section we will often be dealing with the differentials of various complexes. As such, we have decided to use a subscript to indicate which complex the differenital is for. Outside of this section we will not do this, but it is useful to see it this way once.

The purpose of this section is to prove the following theorem.

Theorem 3.4. *The duality map*

$$\begin{aligned}
\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) =& Ch^n_s(sh(\alpha\otimes\beta))\\
=& C^{n+1}_s\big(sh(B\,\alpha\otimes\beta)\big) + (-1)^i C^{n+1}_s\big(sh(\alpha\otimes B\,\beta)\big)
\end{aligned}$$

where for $k \leq \frac{n+1}{2}$

$$\begin{aligned}
&Ch^n_s(sh_{kl}(\alpha_k, \beta_l)) = \\
&(-1)c_{n+2,k}\Bigg(\sum_{\substack{1 \leq i_0 < i_1 < \ldots < i_k \leq \frac{n+1}{2} \\ \lambda-\mathit{cyclic\ permutation}}} (-1)^{\lambda}\, Tr\Big(\rho(b_0)\omega(b_1, b_2)\cdots[a_{\lambda(0)}, \rho(b_{2i_0-1})]\omega(b_{2i_0}, b_{2i_0+1})\cdots \\
&\qquad\qquad [a_{\lambda(j)}, \rho(b_{2i_j-j-1})]\omega(b_{2i_j-j}, b_{2i_j-j+1})\cdots\Big) \\
&+(-1)^{k+1} \sum_{\substack{1 \leq i'_1 < \ldots < i'_k \leq \frac{n+1}{2} \\ \sigma-\mathit{cyclic\ permutation}}} (-1)^{\sigma}\, Tr\Big(a_0\omega(b_{\sigma(0)}, b_{\sigma(1)})\cdots[a_1, \rho(b_{\sigma(2i'_1-2)})]\omega(b_{\sigma(2i'_1-1)}, b_{\sigma(2i'_1)})\cdots \\
&\qquad\qquad [a_j\rho(b_{\sigma(2i'_j-j-1)})]\omega(b_{\sigma(2i'_j-j)}, b_{\sigma(2i'_j-j+1)})\cdots\Big)\Bigg)
\end{aligned}$$

noting that the first sum is zero if $k = \frac{n+1}{2}$*, and*

$$c_{n+2,k} = \frac{(-1)^{\frac{n+1}{2} + \frac{k(k-1)}{2}}}{\left(\frac{n+1}{2}\right)!},$$

and $Ch^n_s(sh_{kl}(\alpha_k, \beta_l)) = 0$ *if* $k > \frac{n+1}{2}$.

We begin by noting that we have $Ch^n_s = B_{\mathcal{A}\otimes\mathcal{B}} C^{n+1}_s$ where

$$\begin{aligned}
C^{n+1}_s(x_0, x_1, ..., x_n, x_{n+1}) =& c_{n+2}\mathrm{Tr}\big(s(x_0)\varepsilon(x_1, x_2)\cdots\varepsilon(x_n, x_{n+1})\big) \\
c_{n+2} =& \frac{(-1)^{\frac{n+1}{2}}}{\left(\frac{n+1}{2}\right)!}
\end{aligned}$$

Note that C^{n+1}_s is the chain homotopy showing $[SCh^{n-2}_s] = [Ch^n_s]$ (see the appendix). The chains C^n_s satisfy

$$b_{\mathcal{A}\otimes\mathcal{B}} C^{n-1}_s + B_{\mathcal{A}\otimes\mathcal{B}} C^{n+1}_s = Ch^n_s - Ch^n_s = 0.$$

Further, we have the following identities for shuffles in the normalized (b, B)-cyclic complex:

$$\begin{aligned}
&B_{\mathcal{A}\otimes\mathcal{B}}\, sh(x \otimes y) - sh(B_{\mathcal{A}} x \otimes y) - (-1)^{\deg x} sh(x \otimes B_{\mathcal{B}} y) = \\
&\qquad - \big(b_{\mathcal{A}\otimes\mathcal{B}}\, sh'(x \otimes y) - sh'(b_{\mathcal{A}} x \otimes y) - (-1)^{\deg x} sh'(x \otimes b_{\mathcal{B}} y)\big) \\
&B_{\mathcal{A}\otimes\mathcal{B}}\, sh'(x \otimes y) = sh'(B_{\mathcal{A}} x \otimes y) = sh'(x \otimes B_{\mathcal{B}} y) = 0.
\end{aligned}$$

Then for $\alpha \in \mathcal{B}_i^-(\mathcal{A})$ we have that:

$$\begin{aligned}
Ch_s^n(Sh(\alpha\otimes\beta)) =& B_{\mathcal{A}\otimes\mathcal{B}}C_s^{n+1}(Sh(\alpha\otimes\beta)) = B_{\mathcal{A}\otimes\mathcal{B}}C_s^{n+1}(sh(\alpha\otimes\beta)+sh'(\alpha\otimes\beta))\\
=& C_s^{n+1}(B_{\mathcal{A}\otimes\mathcal{B}}\, sh(\alpha\otimes\beta)+B_{\mathcal{A}\otimes\mathcal{B}}\, sh'(\alpha\otimes\beta)) = C_s^{n+1}(B_{\mathcal{A}\otimes\mathcal{B}}\, sh(\alpha\otimes\beta))\\
=& C_s^{n+1}\big(sh(B_{\mathcal{A}}\,\alpha\otimes\beta)+(-1)^i sh(\alpha\otimes B_{\mathcal{B}}\,\beta)\\
&\qquad - b_{\mathcal{A}\otimes\mathcal{B}}\, sh'(\alpha\otimes\beta)+sh'(b_{\mathcal{A}}\,\alpha\otimes\beta)+(-1)^i sh'(\alpha\otimes b_{\mathcal{B}}\,\beta)\big)\\
=& C_s^{n+1}\big(sh(B_{\mathcal{A}}\,\alpha\otimes\beta)+(-1)^i sh(\alpha\otimes B_{\mathcal{B}}\,\beta)\\
&\qquad + sh'(b_{\mathcal{A}}\,\alpha\otimes\beta)+(-1)^i sh'(\alpha\otimes b_{\mathcal{B}}\,\beta)\big) - b_{\mathcal{A}\otimes\mathcal{B}}C^{n+1}(sh'(\alpha\otimes\beta))\\
=& C_s^{n+1}\big(sh(B_{\mathcal{A}}\,\alpha\otimes\beta)+(-1)^i sh(\alpha\otimes B_{\mathcal{B}}\,\beta)\\
&\qquad + sh'(b_{\mathcal{A}}\,\alpha\otimes\beta)+(-1)^i sh'(\alpha\otimes b_{\mathcal{B}}\,\beta)\big) - B_{\mathcal{A}\otimes\mathcal{B}}C^{n+3}(sh'(\alpha\otimes\beta))\\
=& C_s^{n+1}\big(sh(B_{\mathcal{A}}\,\alpha\otimes\beta)+(-1)^i sh(\alpha\otimes B_{\mathcal{B}}\,\beta)\\
&\qquad + sh'((b+B)_{\mathcal{A}}\,\alpha\otimes\beta)+(-1)^i sh'(\alpha\otimes(b+B)_{\mathcal{B}}\,\beta)\big)\\
=& C_s^{n+1}\big(sh(B_{\mathcal{A}}\,\alpha\otimes\beta)\big)+(-1)^i C_s^{n+1}\big(sh(\alpha\otimes B_{\mathcal{B}}\,\beta)\big)\\
&+ C_s^{n+1}\big(sh'((b+B)_{\mathcal{A}}\,\alpha\otimes\beta)\big)+(-1)^i C_s^{n+1}\big(sh'(\alpha\otimes(b+B)_{\mathcal{B}}\,\beta)\big).
\end{aligned}$$

So if we define

$$\psi(\alpha)(\beta) = C_s^{n+1}\big(sh(B\,\alpha\otimes\beta)\big)+(-1)^i C_s^{n+1}\big(sh(\alpha\otimes B\,\beta)\big)$$

and

$$\phi(\alpha)(\beta) = C_s^{n+1}\big(sh'(\alpha\otimes\beta)\big)$$

then

$$\Psi^{i,n}(\alpha)(\beta)-\psi(\alpha)(\beta) = \phi((b+B)_{\mathcal{A}}\alpha)(\beta)+(-1)^i(b+B)_{\mathcal{B}}\phi(\alpha)(\beta)$$

and so $\Psi^{i,n}$ and ψ are equivalent chain maps.

Further,

$$\begin{aligned}
&C_s^{n+1}\big(sh(B\,\alpha\otimes\beta)\big)+(-1)^i C_s^{n+1}\big(sh(\alpha\otimes B\,\beta)\big)\\
&\quad = \sum_{k+l=n} C_s^{n+1}\big(sh_{(k+1)l}(B_{\mathcal{A}}\,\alpha_k,\beta_l)\big)+(-1)^i C_s^{n+1}\big(sh_{k(l+1)}(\alpha_k,B_{\mathcal{B}}\,\beta_l)\big)
\end{aligned}$$

As before, we will compute these for a fixed k.

Suppose that $\alpha_k = a_0 \otimes a_1 \otimes ... \otimes a_k$ and $\beta_l = b_0 \otimes ... \otimes b_l$. Then in the reduced complex, we note that

$$B\,\alpha_k = \sum_{\lambda\text{-cyclic permutation}} (-1)^{\lambda} 1 \otimes a_{\lambda(0)} \otimes ... \otimes a_{\lambda(k)}$$

so we will first compute $C_s^{n+1}\Big(sh\big((1 \otimes \alpha_k) \otimes \beta_l\big)\Big)$.

Recall that

$$C_s^{n+1}(x_0, x_1, ..., x_n, x_{n+1}) = c_n \mathrm{Tr}\big(s(x_0)\varepsilon(x_1, x_2) \cdots \varepsilon(x_n, x_{n+1})\big).$$

First note that $x_0 = 1 \otimes b_0$. As before, if $x_{2j+1} = a \otimes 1$, then $C_s^{n+1}\Big(sh_{kl}\big((1 \otimes \alpha_k), \beta_l\big)\Big) = 0$. So we consider the shuffles with $x_{2i_j} = a_j \otimes 1$, and $k < \frac{n+1}{2}$. Now the sign of a shuffle is independent of the values of i_j, and is equal to $(-1) \cdot (-1)^{\frac{k(k-1)}{2}}$. So we have

$$C_s^{n+1}\Big(sh_{kl}\big((1 \otimes \alpha_k), \beta_l\big)\Big) = \\ (-1) \cdot c_{n+2,k} \sum_{1 \le i_0 < i_1 < ... < i_k \le \frac{n+1}{2}} \mathrm{Tr}\Big(\rho(b_0)\omega(b_1, b_2) \cdots [a_0, \rho(b_{2i_0-1})]\omega(b_{2i_0}, b_{2i_0+1}) \cdots \\ [a_j, \rho(b_{2i_j-j-1})]\omega(b_{2i_j-j}, b_{2i_j-j+1}) \cdots\Big)$$

where

$$c_{n+2,k} = \frac{(-1)^{\frac{n+1}{2} + \frac{k(k-1)}{2}}}{\big(\frac{n+1}{2}\big)!}.$$

Hence,

$$C_s^{n+1}(sh_{kl}(B\alpha_k \otimes \beta_l)) = \\ (-1) \cdot c_{n,k} \sum_{\substack{1 \le i_0 < i_1 < ... < i_k \le \frac{n+1}{2} \\ \lambda-\text{cyclic permutation}}} (-1)^{\lambda} \mathrm{Tr}\Big(\rho(b_0)\omega(b_1, b_2) \cdots [a_{\lambda(0)}, \rho(b_{2i_0-1})]\omega(b_{2i_0}, b_{2i_0+1}) \cdots \\ [a_{\lambda(j)}, \rho(b_{2i_j-j-1})]\omega(b_{2i_j-j}, b_{2i_j-j+1}) \cdots\Big)$$

Similarly, the second term can be computed as

$$\begin{aligned}
&C_s^{n+1}(sh_{kl}(\alpha_k, B\beta_l)) = \\
&c_{n+2,k} \sum_{\substack{1\leq i'_1<\ldots<i'_k\leq \frac{n+1}{2} \\ \sigma-\text{cyclic permutation}}} (-1)^{\sigma}\mathrm{Tr}\Big(a_0\omega(b_{\sigma(0)}, b_{\sigma(1)})\cdots[a_1, \rho(b_{\sigma(2i'_1-2)})]\omega(b_{\sigma(2i'_1-1)}, b_{\sigma(2i'_1)})\cdots \\
&\qquad\qquad [a_j\rho(b_{\sigma(2i'_j-j-1)})]\omega(b_{\sigma(2i'_j-j)}, b_{\sigma(2i'_j-j+1)})\cdots\Big)
\end{aligned}$$

Noting that this sum is possibly non-zero if $k = \frac{n+1}{2}$. Thus, we have that

$$\begin{aligned}
&Ch_s^{n}(sh_{kl}(\alpha_k, \beta_l)) = \\
&(-1)\cdot c_{n+2,k}\Bigg(\sum_{\substack{1\leq i_0<i_1<\ldots<i_k\leq \frac{n+1}{2} \\ \lambda-\text{cyclic permutation}}} (-1)^{\lambda}\mathrm{Tr}\Big(\rho(b_0)\omega(b_1, b_2)\cdots[a_{\lambda(0)}, \rho(b_{2i_0-1})]\omega(b_{2i_0}, b_{2i_0+1})\cdots \\
&\qquad\qquad [a_{\lambda(j)}, \rho(b_{2i_j-j-1})]\omega(b_{2i_j-j}, b_{2i_j-j+1})\cdots\Big) \\
&+(-1)^{k+1}\sum_{\substack{1\leq i'_1<\ldots<i'_k\leq \frac{n+1}{2} \\ \sigma-\text{cyclic permutation}}} (-1)^{\sigma}\mathrm{Tr}\Big(a_0\omega(b_{\sigma(0)}, b_{\sigma(1)})\cdots[a_1, \rho(b_{\sigma(2i'_1-2)})]\omega(b_{\sigma(2i'_1-1)}, b_{\sigma(2i'_1)})\cdots \\
&\qquad\qquad [a_j\rho(b_{\sigma(2i'_j-j-1)})]\omega(b_{\sigma(2i'_j-j)}, b_{\sigma(2i'_j-j+1)})\cdots\Big)\Bigg) \\
&= c_{n+2,k}\Bigg(\sum_{\substack{1\leq i'_1<\ldots<i'_k\leq \frac{n+1}{2} \\ \sigma-\text{cyclic permutation}}} (-1)^{\sigma+k}\mathrm{Tr}\Big(a_0\omega(b_{\sigma(0)}, b_{\sigma(1)})\cdots[a_1, \rho(b_{\sigma(2i'_1-2)})]\omega(b_{\sigma(2i'_1-1)}, b_{\sigma(2i'_1)})\cdots \\
&\qquad\qquad [a_j\rho(b_{\sigma(2i'_j-j-1)})]\omega(b_{\sigma(2i'_j-j)}, b_{\sigma(2i'_j-j+1)})\cdots\Big) \\
&+\sum_{\substack{1\leq i_0<i_1<\ldots<i_k\leq \frac{n+1}{2} \\ \lambda-\text{cyclic permutation}}} (-1)^{\lambda}\mathrm{Tr}\Big(\rho(b_0)\omega(b_1, b_2)\cdots[a_{\lambda(0)}, \rho(b_{2i_0-1})]\omega(b_{2i_0}, b_{2i_0+1})\cdots \\
&\qquad\qquad [a_{\lambda(j)}, \rho(b_{2i_j-j-1})]\omega(b_{2i_j-j}, b_{2i_j-j+1})\cdots\Big)\Bigg)
\end{aligned}$$

for $k \leq \frac{n+1}{2}$ and it is zero otherwise.

Chapter 4

An Application of the Duality Map

Definition 4.1. (Almost Representation). By an almost representation of a unital algebra $\mathcal{B}$ in a Hilbert space $\mathcal{H}$, we mean a unital linear map, ρ, such that $\omega(b_1, b_2) = \rho(b_1 b_2) - \rho(b_1)\rho(b_2) \in \mathcal{L}^p(\mathcal{H})$.

Definition 4.2. (An (Odd) (p-Summable) Almost Fredholm Module). By an (odd) (p-summable) almost Fredholm module over a unital algebra $\mathcal{B}$, we mean that we have a triple $(\rho, \mathcal{H}, F)$ where $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ is an almost representation, $F^* = F$, $F^2 = 1$, and $[F, \rho(b)] \in \mathcal{L}^p(\mathcal{H})$ for all $b \in \mathcal{B}$.

Definition 4.3. (The Character of an (Odd) (p-Summable) Almost Fredholm Module, χ_F) We define the character of an (odd) (p-summable) almost Fredholm module, denoted χ_F, by

$$\chi_F = P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)$$

where $P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)$ is defined in the previous chapter, $\mathcal{A} = \langle 1, F \rangle \subset \mathcal{L}(\mathcal{H})$, and α is the following cycle

$$\alpha = \frac{1+F}{2} + \sum_{k \geq 1} c^F_{2k} F \otimes F^{\otimes 2k} = \sum_{k \geq 0} \alpha_{2k}$$

$$\text{where } c^F_{2k} = \frac{(-1)^k}{2 \cdot 2^k} \prod_{j=1}^{k} (2j-1) = \frac{(-1)^k (2k)!}{2^{2k+1} k!}.$$

A proof that α is a cycle is given in example 2.4 near the end of section 2.5. As a note, if we let $e = \frac{1+F}{2}$, then e is an idempotent, and $\alpha = ch(e)$ which is shown in example 2.6.

We remind the reader that the Chern character of an invertible element $u \in \mathcal{B}$, $ch(u) \in HC_1^{per}(\mathcal{B})$, is defined by

$$ch(u) = \frac{1}{2}\sum_{m=1}^{\infty}(-1)^{m-1}(m-1)!(u^{-1}\otimes u)^m.$$

Note, as in this section we will always be requiring p-summability and that we have an odd almost Fredholm module, so we simply say an almost Fredholm module over a unital algebra $\mathcal{B}$. Also, since we are always working over a fixed Hilbert space, we will often write $\mathcal{L}^p$ in place of $\mathcal{L}^p(\mathcal{H})$.

The standing assumptions for this chapter are as follows, n is an odd integer such that $n \geq 2p-1$, $(\rho, \mathcal{H}, F)$ is an almost Fredholm module over the unital algebra $\mathcal{B}$, and $\mathcal{A}$ is the subalgebra of $\mathcal{L}(\mathcal{H})$ generated by 1 and F.

4.1 Character of an (Odd) Almost Fredholm Module

Our goal in this section is to state following theorem, which gives an explicit form of the character of an almost Fredholm module (defined above), χ_F, we will show that this character agrees with Connes' character when the almost Fredholm module is a Fredholm module, and as an example, we prove directly that we obtain a cocycle in the specific case $n = 1$.

Theorem 4.1. *Let $n \geq 2p-1$, and $(\rho, \mathcal{H}, F)$ an almost Fredholm module over $\mathcal{B}$. Then,*

$$\chi_F(\beta) = Ch_s^n(sh(\alpha\otimes\beta)) = \sum_{2k+2l+1=n} Ch_s^n(sh_{(2k)(2l+1)}(\alpha_{2k}, \beta_{2l+1}))$$

$$Ch_s^n(sh_{(2k)(2l+1)}(\alpha_{2k}, \beta_{2l+1})) = c_{2k}^F \cdot Ch_s^n\Big(sh_{(2k)(2l+1)}(F\otimes F^{2k}, b_0\otimes \ldots \otimes b_{2l+1}\Big)$$

where if $2k = 0$

$$\begin{aligned}
&\frac{1}{2} Ch_s^n(sh_{0(2l+1)}\,(1+F, b_0 \otimes ... \otimes b_{2l+1}) = \\
&\qquad \frac{c_{n,0}}{2} \Bigg(Tr\big(\omega(b_0,b_1)\omega(b_2,b_3)\cdots\omega(b_{n-1},b_n)\big) \\
&\qquad\qquad - Tr\big(\big(\omega(b_n,b_0)\omega(b_1,b_2)\cdots\omega(b_{n-2},b_{n-1})\big)\Bigg) \\
&\qquad + \frac{c_{n,0}}{2} \Bigg(Tr\big(F\omega(b_0,b_1)\omega(b_2,b_3)\cdots\omega(b_{n-1},b_n)\big) \\
&\qquad\qquad - Tr\big(\big(F\omega(b_n,b_0) + [F,\rho(b_n)]\rho(b_0)\big)\omega(b_1,b_2)\cdots\omega(b_{n-2},b_{n-1})\big)\Bigg)
\end{aligned}$$

and if $2k > 0$

$$\begin{aligned}
&c^F_{2k} Ch_s^n(sh_{(2k)(2l+1)}(F \otimes F^{2k}, b_0 \otimes ... \otimes b_{2l+1}) = \\
&\quad c^F_{2k} c_{n,2k} \cdot \Bigg(\sum_{0 \le i_1 < i_2 < ... < i_{2k} \le \frac{n-1}{2}} Tr\Big(F[F,\rho(b_0)]\omega(b_1,b_2)...[F,\rho(b_{2i_2-1})]\omega(b_{2i_2},b_{2i_2+1})... \\
&\qquad\qquad + F\omega(b_1,b_2)...[F,\rho(b_{2i_1})]\omega(b_{2i_1+1},b_{2i_1+2})...\Big) \\
&\qquad - \sum_{0 < i_1 \le ... \le i_{2k} \le \frac{n-1}{2}} Tr\Big(\big(F\omega(b_n,b_0) + [F,\rho(b_n)]\rho(b_0)\big)\omega(b_1,b_2)\cdots \\
&\qquad\qquad [F,\rho(b_{2i_1})]\omega(b_{2i_1+1},b_{2i_1+2})...[F,\rho(b_{2i_j-j+1})]...\Big)\Bigg)
\end{aligned}$$

$$c^F_{2k} c_{n,2k} = \frac{(-1)^{\frac{n-1}{2}}(2k)!}{2^{2k+1}\left(\frac{n-1}{2}\right)!k!}.$$

Alternatively, we may take

$$c^F_{2k}Ch^n_s(sh_{(2k)(2l+1)}(F\otimes F^{2k}, b_0\otimes ...\otimes b_{2l+1}) =$$

$$= -\, c^F_{2k}c_{n+2,2k}\cdot\Bigg(\sum_{\substack{1\leq i_0<i_1<...<i_{2k}\leq\frac{n+1}{2}\\ \lambda-\textit{cyclic permutation}}}(-1)^\lambda\, Tr\Big(\rho(b_0)\omega(b_1,b_2)\cdots[F,\rho(b_{2i_0-1})]\omega(b_{2i_0},b_{2i_0+1})\cdots$$

$$[F,\rho(b_{2i_j-j-1})]\omega(b_{2i_j-j},b_{2i_j-j+1})\cdots\Big)$$

$$-\sum_{\substack{1\leq i_0<i_1<...<i_{2k}\leq\frac{n+1}{2}\\ \sigma-\textit{cyclic permutation}}}(-1)^\sigma\, Tr\Big(F\omega(b_{\sigma(0)},b_{\sigma(1)})\cdots[F,\rho(b_{\sigma(2i_1-2)})]\omega(b_{\sigma(2i_1-1)},b_{\sigma(2i_1)})\cdots$$

$$[F,\rho(b_{\sigma(2i_j-j-1)})]\omega(b_{\sigma(2i_j-j)},b_{\sigma(2i_j-j+1)})\cdots\Big)\Bigg)$$

$$= -\, c^F_{2k}c_{n+2,2k}\cdot\Bigg(\sum_{1\leq i_0<i_1<...<i_{2k}\leq\frac{n+1}{2}}(2k+1)\, Tr\Big(\rho(b_0)\omega(b_1,b_2)\cdots[F,\rho(b_{2i_0-1})]\omega(b_{2i_0},b_{2i_0+1})\cdots$$

$$[F,\rho(b_{2i_j-j-1})]\omega(b_{2i_j-j},b_{2i_j-j+1})\cdots\Big)$$

$$-\sum_{\substack{1\leq i_0<i_1<...<i_{2k}\leq\frac{n+1}{2}\\ \sigma-\textit{cyclic permutation}}}(-1)^\sigma\, Tr\Big(F\omega(b_{\sigma(0)},b_{\sigma(1)})\cdots[F,\rho(b_{\sigma(2i_1-2)})]\omega(b_{\sigma(2i_1-1)},b_{\sigma(2i_1)})\cdots$$

$$[F,\rho(b_{\sigma(2i_j-j-1)})]\omega(b_{\sigma(2i_j-j)},b_{\sigma(2i_j-j+1)})\cdots\Big)\Bigg)$$

$$c^F_{2k}c_{n+2,2k} = \frac{(-1)^{\frac{n+1}{2}}(2k)!}{2^{2k+1}\left(\frac{n+1}{2}\right)!k!}.$$

Further, χ_F is independent of the choice of $n\geq 2p-1$, and if $(\rho',\mathcal{H},F)$ is a second almost Fredholm module so that $\rho'-\rho\in\mathcal{L}^p$, then their characters are the same.

This theorem follows near immediately from theorems 3.1, 3.3, and 3.4 of subsections 3.3, 3.4.2, and 3.4.3 respectively. For the first sum, we have $(-1)^\lambda = 1$ as λ is a cyclic permutation of the set $\{0,...,2k\}$, and as λ only permutes the letter F, with respect to λ each term of the first sum is the same.

Proposition 4.1. Let $n = 2m+1\geq 2p-1$, so $\frac{n-2k-1}{2} = m-k$, let $u\in\mathcal{B}$ an invertible element, and $(\rho, \mathcal{H}, F)$ an almost Fredholm module over $\mathcal{B}$. Then,

$$\chi_F(ch(u)) = Ch^n_s(sh(\alpha \otimes ch(u))) = \sum_{k=0}^{\lfloor\frac{n+1}{4}\rfloor} Ch^n_s(sh_{(2k)(n-2k)}(\alpha_{2k}, ch(u)_{n-2k}))$$

$$Ch^n_s(sh_{(2k)(n-2k)}(\alpha_{2k}, ch(u)_{n-2k})) = c^F_{2k}c^u_{n-2k} \cdot Ch^n_s\Big(sh_{(2k)(n-2k)}\Big(F \otimes F^{2k}, (u^{-1} \otimes u)^{m-k+1}\Big)\Big)$$

where if $2k = 0$

$$\begin{aligned}
&\frac{1}{2}c^u_n \cdot Ch^n_s(sh_{0n}\left(1+F, (u^{-1} \otimes u)^{m+1}\right) = \\
&\qquad \frac{1}{2}\Bigg(\mathrm{Tr}\Big(\omega(u^{-1}, u)^{m+1} - \omega(u, u^{-1})^{m+1}\Big) \\
&\qquad\qquad + \mathrm{Tr}\Big(F\omega(u^{-1}, u)^{m+1} - \Big(F\omega(u, u^{-1}) + [F, \rho(u)]\rho\big(u^{-1}\big)\Big)\omega(u, u^{-1})^m\Big)\Bigg) \\
&\qquad =\frac{1}{2}\Bigg(\mathrm{Tr}\Big((1+F)\Big(\omega(u^{-1}, u)^{m+1} - \omega(u, u^{-1})^{m+1}\Big) - [F, \rho(u)]\rho\big(u^{-1}\big)\omega(u, u^{-1})^m\Big)\Bigg)
\end{aligned}$$

and if $2k > 0$

$$\begin{aligned}
&c^F_{2k}c^u_{n-2k}Ch^n_s(sh_{(2k)(n-2k)}(F \otimes F^{2k}, (u^{-1} \otimes u)^{m-k+1}) = \\
&\qquad = -\, c^F_{2k}c^u_{n-2k}c_{n+2,2k} \cdot \Bigg(\sum_{\substack{2(j_0+j_1+\ldots+j_{2k}) \\ =n-2k-1}} (2k+1)\mathrm{Tr}\Big(\rho\big(u^{-1}\big)\omega(u, u^{-1})^{j_0}[F, \rho(u)] \cdots \\
&\qquad\qquad\qquad\qquad \cdots [F, \rho\big(u\big)]\omega(u^{-1}, u)^{j_{2k}}\Big) \\
&\qquad - \sum_{\substack{2(j_1+\ldots+j_{2k}) \\ =n-2k}} (m-k+1)\Bigg(\mathrm{Tr}\Big(F\omega(u^{-1}, u)^{j_1}[F, \rho\big(u^{-1}\big)] \cdots [F, \rho(u)]\omega(u^{-1}, u)^{j_{2k}}\Big) \\
&\qquad\qquad\qquad - \mathrm{Tr}\Big(F\omega(u, u^{-1})^{j_1}[F, \rho(u)] \cdots [F, \rho\big(u^{-1}\big)]\omega(u, u^{-1})^{j_{2k}}\Big)\Bigg)\Bigg)
\end{aligned}$$

$$-c^F_{2k}c^u_{n-2k}c_{n+2,2k} = \frac{(-1)^{k+1}(2k)!\left(\frac{n-2k-1}{2}\right)!}{2^{2k+1}k!\left(\frac{n-1}{2}\right)!} = \frac{(-1)^k(2k)!\,(m-k)!}{2^{2k+1}k!m!}$$

If one desires to expand the $\omega(u, u^{-1})$ terms, the formulas become a bit unruly due to the many parenthesis. For this reason we will rewrite the above proposition with some notational changes.

We let $U = \rho(u)$, $V = \rho(u^{-1})$, so $\omega(u, u^{-1}) = 1 - UV$. We further let $C_{n,2k} = c^F_{2k}c^u_{n-2k}c_{n+2,2k}$, and we will define terms $\Omega_{n,2k}$ and $\Omega'_{n,2k}$ in the proposition.

Proposition 4.2. Let $n = 2m + 1 \geq 2p - 1$, so $\frac{n-2k-1}{2} = m - k$, let $u \in \mathcal{B}$ an invertible element, and $(\rho, \mathcal{H}, F)$ an almost Fredholm module over $\mathcal{B}$. Then,

$$\chi_F(ch(u)) = \Omega_{n,0} + \sum_{k=1}^{\lfloor \frac{n+1}{4} \rfloor} C_{n,k}(\Omega_{n,k} - \Omega'_{n,k})$$

where

$$\Omega_{n,0} = \frac{1}{2}\mathrm{Tr}\Big((1+F)\Big((1-VU)^{m+1} - (1-UV)^{m+1}\Big) - [F,U]V(1-UV)^m\Big)$$

$$\Omega_{n,k} = \sum_{\substack{2(j_0+\ldots+j_{2k}) \\ =n-2k+1}} (m-k+1)\Bigg(\mathrm{Tr}\Big(F(1-VU)^{j_0}[F,V]\cdots[F,U](1-VU)^{j_{2k}}\Big)$$
$$- \mathrm{Tr}\Big(F(1-UV)^{j_1}[F,U]\cdots[F,V](1-UV)^{j_{2k}}\Big)\Bigg)$$

$$\Omega'_{n,k} = \sum_{\substack{2(j_0+j_1+\ldots+j_{2k+1}) \\ =n-2k-1}} (2k+1)\mathrm{Tr}\Big(V(1-UV)^{j_0}[F,U](1-VU)^{j_1}\cdots[F,V](1-UV)^{j_{2k+1}}\Big)$$

$$\Omega'_{n,\frac{n+1}{4}} = 0$$

$$C_{n,k} == \frac{(-1)^k(2k)!\left(\frac{n-2k-1}{2}\right)!}{2^{2k+1}k!\left(\frac{n-1}{2}\right)!} = \frac{(-1)^k(2k)!\,(m-k)!}{2^{2k+1}k!m!}$$

Example 4.1. We will show that the character of an almost odd Fredholm module agrees with the character of an odd Fredholm module up to a constant, and that evaluating χ_F on $ch(u)$ results in the same value as Connes' pairing for an element $[u] \in K_1(\mathcal{B})$ with $u \in \mathcal{B}$.

Fix $m = 2l + 1$. From the above formula, we see that in the instance ρ is a representation, then we only obtain a possible nonzero term if $2k = \frac{n+1}{2} = m + 1$ or $2k = \frac{n-1}{2} = m - 1$ as every other term contains a copy of $\omega(b_i, b_{i+1}) = 0$. Assuming $m + 1 = 2k = \frac{n+1}{2}$, then $n = 2m + 1$, and

$$\sum_\sigma (-1)^\sigma \mathrm{Tr}\Big(F[F, b_{\sigma(0)}]\cdots[F, b_{\sigma(m)}]\Big)$$
$$= \mathrm{Tr}\Big(F[F,b_0]\cdots[F,b_m]\Big) - \mathrm{Tr}\Big(F[F,b_1]\cdots[F,b_0]\Big) + \cdots - \mathrm{Tr}\Big(F[F,b_m]\cdots[F,b_{m-1}]\Big)$$
$$= Tr\Big(F[F,b_0]\cdots[F,b_m]\Big) + \mathrm{Tr}\Big(F[F,b_0]\cdots[F,b_m]\Big) + \cdots + \mathrm{Tr}\Big(F[F,b_0]\cdots[F,b_m]\Big)$$
$$= (m+1)\mathrm{Tr}\Big(F[F,b_0]\cdots[F,b_m]\Big)$$

since $F[F,b_i] = -[F,b_i]F$ and using the trace class property. Also,

$$\begin{aligned}-(m+1)c^F_{m+1}c_{2m+3,m+1} =&(m+1)\frac{(m+1)!}{2^{m+2}(m+1)!\left(\frac{m+1}{2}\right)!}\\ =&\frac{1}{2^{m+1}\left(\frac{m-1}{2}\right)!}\end{aligned}$$

Now, if $m-1 = 2k = \frac{n-1}{2}$, then $n = 2m-1$, and

$$\begin{aligned}m\mathrm{Tr}\Big(b_0[F,b_1]\cdots[F,b_m]\Big) =&\frac{m}{2}\mathrm{Tr}\Big((b_0+F^2b_0)[F,b_1]\cdots[F,b_m]\Big)\\ =&\frac{m}{2}\mathrm{Tr}\Big((b_0-Fb_0F)[F,b_1]\cdots[F,b_m]\Big)\\ =&\frac{m}{2}\mathrm{Tr}\Big(F[F,b_0][F,b_1]\cdots[F,b_m]\Big)\end{aligned}$$

and

$$\begin{aligned}\frac{m}{2}c^F_{m-1}c_{2m+1,m} =&\frac{m}{2}\frac{(m-1)!}{2^m m!\left(\frac{m-1}{2}\right)!}\\ =&\frac{1}{2^{m+1}\left(\frac{m-1}{2}\right)!}\end{aligned}$$

Thus, the above formulas agree regardless if $m = \frac{n-1}{2}$ or $m = \frac{n+1}{2}$, and result in

$$\chi_F(\beta) = \frac{1}{2^{m+1}\left(\frac{m-1}{2}\right)!}\mathrm{Tr}\Big(F[F,b_0][F,b_1]\cdots[F,b_m]\Big).$$

We note that the above formula is exactly the character of an odd Fredholm module in definition 3.10 (section 3.3) up to a change in constant. We note that this change in constant is due to our use of the normalized (b,B) complex.

Evaluating χ_F on $ch(u)$, we obtain

$$\chi_F(ch(u)) = c^F_{2k}Ch^n_s(sh_{(2k)m}(F\otimes F^{2k}, ch(u)) = \frac{(-1)^{\frac{m-1}{2}}}{2^{m+1}}\mathrm{Tr}(F[F,u^{-1}][F,u]\cdots[F,u]) = \langle[u], Ch^n_F\rangle.$$

where $\langle[u], Ch^n_F\rangle$ is Connes' pairing of a unitary element, $[u]\in K_1(\mathcal{A})$, with the Chern character of an odd Fredholm module.

Example 4.2. We will show as an example that χ_F is a cocycle in the case that $n=1$ by direct calculation. Note that in this case $[F,\rho(b)]$ and $\omega(b_1,b_2)$ are trace class (since $n=1$ and $n\geq 2p-1$, $1\geq p$.) Consider

$$c^F_0 = \frac{1}{2}.$$

We will do the computation without the constant by multiplying χ_F by -2. This will not change whether or not χ_F is a cocycle as b and B are linear, and it will save the writer a lot of time by fixing a careless sign error. We have,

$$\begin{aligned}
-2\chi_F(b_0,b_1) =& -Ch_s^1(sh((1+F)\otimes(b_0\otimes b_1))) \\
=& -Ch_s^1((1+F)\otimes b_0, 1\otimes b_1) \\
=& c_{1,0}\mathrm{Tr}\big(\varepsilon(1\otimes b_0, 1\otimes b_1) - \varepsilon(1\otimes b_1, 1\otimes b_0)\big) \\
&+ c_{1,0}\mathrm{Tr}\big(\varepsilon(F\otimes b_0, 1\otimes b_1) - \varepsilon(1\otimes b_1, F\otimes b_0)\big) \\
=& c_{1,0}\mathrm{Tr}(\rho(b_0)[1,\rho(b_1)] - \omega(b_0,b_1) + \omega(b_1,b_0)) \\
&+ c_{1,0}\mathrm{Tr}(\rho(b_0)[F,\rho(b_1)] - F\omega(b_0,b_1) + F\omega(b_1,b_0)) \\
=& c_{1,0}\mathrm{Tr}(\omega(b_1,b_0) - \omega(b_0,b_1)) \\
&+ c_{1,0}\mathrm{Tr}(\rho(b_0)[F,\rho(b_1)] - F\omega(b_0,b_1) + F\omega(b_1,b_0))
\end{aligned}$$

We will show that

$$\begin{aligned}
\psi_1(b_0,b_1) &= c_{1,0}\mathrm{Tr}(\omega(b_1,b_0) - \omega(b_0,b_1)) \\
\psi_F(b_0,b_1) &= c_{1,0}\mathrm{Tr}(\rho(b_0)[F,\rho(b_1)] - F\omega(b_0,b_1) + F\omega(b_1,b_0))
\end{aligned}$$

are each individually cocycles, and so their sum will be too. Since 1 has all of the necessary properties that F has for ψ_F to be a cocycle, we only really need to show that $\psi_F(b_0,b_1)$ is a cocycle, as ψ_1 is exactly ψ_F with 1 in place of F.

Then

$$\begin{aligned}
B\psi_F(b_0) =& \psi_F(1, b_0) \\
=& c_{1,0}\mathrm{Tr}(\rho(1)[F, \rho(b_0)] - F\omega(b_0, 1) + F\omega(1, b_0)) \\
=& c_{1,0}\mathrm{Tr}([F, \rho(b_0)]) - 0 + 0 \\
=& c_{1,0}\mathrm{Tr}(F^2[F, \rho(b_0)]) && \text{(since } F^2 = 1) \\
=& c_{1,0}\mathrm{Tr}(F[F, \rho(b_0)]F) && \text{(Trace class property)} \\
=& c_{1,0}\mathrm{Tr}(F(F\rho(b_0) - \rho(b_0)F)F) \\
=& c_{1,0}\mathrm{Tr}(F^2\rho(b_0)F - F\rho(b_0)F^2) \\
=& c_{1,0}\mathrm{Tr}(\rho(b_0)F - F\rho(b_0)) \\
=& -c_{1,0}\mathrm{Tr}(F\rho(b_0) - \rho(b_0)F) \\
=& -c_{1,0}\mathrm{Tr}([F, \rho(b_0)])
\end{aligned}$$

Thus,

$$B\psi_F(b_0) = \psi_F(1, b_0) = c_{1,0}\mathrm{Tr}([F, \rho(b_0)]) = -c_{1,0}\mathrm{Tr}([F, \rho(b_0)]),$$

so it must be the case that

$$B\psi_F(b_0) = \psi_F(1, b_0) = 0$$

Now,

$$\begin{aligned}
b\psi_F(b_0, b_1, b_2) =& \psi_F(b_0b_1, b_2) - \psi_F(b_0, b_1b_2) + \psi_F(b_2b_0, b_1) \\
=& c_{1,0}\mathrm{Tr}\Big(\rho(b_0b_1)[F, \rho(b_2)] - F\omega(b_0b_1, b_2) + F\omega(b_2, b_0b_1) \\
& - \rho(b_0)[F, \rho(b_1b_2)] + F\omega(b_0, b_1b_2) - F\omega(b_1b_2, b_0) \\
& + \rho(b_2b_0)[F, \rho(b_1)] - F\omega(b_2b_0, b_1) + F\omega(b_1, b_2b_0)\Big)
\end{aligned}$$

The following equalities are alternate forms of the terms in the last equality above.

$$\rho(b_0b_1)[F,\rho(b_2)] = \omega(b_0,b_1)F\rho(b_2) + \rho(b_0)\rho(b_1)[F,\rho(b_2)] - \omega(b_0,b_1)\rho(b_2)F \qquad (*)$$

$$\begin{aligned} -\rho(b_0)[F,\rho(b_1b_2)] &= -\rho(b_0)F\omega(b_1,b_2) + \rho(b_0)\omega(b_1,b_2)F \\ &\quad - \rho(b_0)[F,\rho(b_1)]\rho(b_2) - \rho(b_0)\rho(b_1)[F,\rho(b_2)] \end{aligned}$$

$$\rho(b_2b_0)[F,\rho(b_1)] = \omega(b_2,b_0)F\rho(b_1) - \omega(b_2,b_0)\rho(b_1)F + \rho(b_2)\rho(b_0)[F,\rho(b_1)]$$

$$F\omega(b_0,b_1b_2) - F\omega(b_0b_1,b_2) = F\omega(b_0,b_1)\rho(b_2) - F\rho(b_0)\omega(b_1,b_2) \qquad (**)$$

$$F\omega(b_2,b_0b_1) - F\omega(b_2b_0,b_1) = F\omega(b_2,b_0)\rho(b_1) - F\rho(b_2)\omega(b_0,b_1)$$

$$F\omega(b_1,b_2b_0) - F\omega(b_1b_2,b_0) = F\omega(b_1,b_2)\rho(b_0) - F\rho(b_1)\omega(b_2,b_0).$$

For a sample computation we will show $(*)$ and $(**)$.

Proof of $(*)$

$$\begin{aligned} \rho(b_0b_1)[F,\rho(b_2)] =& \rho(b_0b_1)F\rho(b_2) - \rho(b_0b_1)\rho(b_2)F \\ =& \rho(b_0b_1)F\rho(b_2) - \rho(b_0)\rho(b_1)F\rho(b_2) + \rho(b_0)\rho(b_1)F\rho(b_2) \\ & - \rho(b_0)\rho(b_1)\rho(b_2)F + \rho(b_0)\rho(b_1)\rho(b_2)F - \rho(b_0b_1)\rho(b_2)F \\ =& \omega(b_0,b_1)F\rho(b_2) + \rho(b_0)\rho(b_1)[F,\rho(b_2)] - \omega(b_0,b_1)\rho(b_2)F \end{aligned}$$

Proof of $(**)$

$$\begin{aligned} F\omega(b_0,b_1b_2) - F\omega(b_0b_1,b_2) &= F\rho(b_0b_1b_2) - F\rho(b_0)\rho(b_1b_2) - F\rho(b_0b_1b_2) + F\rho(b_0b_1)\rho(b_2) \\ &= F\rho(b_0b_1)\rho(b_2) - F\rho(b_0)\rho(b_1b_2) \\ &= F\rho(b_0b_1)\rho(b_2) - F\rho(b_0)\rho(b_1)\rho(b_2) \\ &\quad + F\rho(b_0)\rho(b_1)\rho(b_2) - F\rho(b_0)\rho(b_1b_2) \\ &= F\omega(b_0,b_1)\rho(b_2) - F\rho(b_0)\omega(b_1,b_2) \end{aligned}$$

Inserting these alternate forms into the previous equation, and using the fact that $\omega(b_i,b_j)$ and

$[F, \rho(b_i)]$ are trace class, we have

$$\begin{aligned}
b\psi_F(b_0, b_1, b_2) =& c_{1,0}\mathrm{Tr}\Big(\omega(b_0, b_1)F\rho(b_2) + \rho(b_0)\rho(b_1)[F, \rho(b_2)] - \omega(b_0, b_1)\rho(b_2)F \\
&- \rho(b_0)F\omega(b_1, b_2) + \rho(b_0)\omega(b_1, b_2)F \\
&- \rho(b_0)[F, \rho(b_1)]\rho(b_2) - \rho(b_0)\rho(b_1)[F, \rho(b_2)] \\
&+ \omega(b_2, b_0)F\rho(b_1) - \omega(b_2, b_0)\rho(b_1)F + \rho(b_2)\rho(b_0)[F, \rho(b_1)] \\
&+ F\omega(b_0, b_1)\rho(b_2) - F\rho(b_0)\omega(b_1, b_2) \\
&+ F\omega(b_2, b_0)\rho(b_1) - F\rho(b_2)\omega(b_0, b_1) \\
&+ F\omega(b_1, b_2)\rho(b_0) - F\rho(b_1)\omega(b_2, b_0)\Big) \\
=& c_{1,0}\mathrm{Tr}\Big(F\rho(b_2)\omega(b_0, b_1) + \rho(b_0)\rho(b_1)[F, \rho(b_2)] - F\omega(b_0, b_1)\rho(b_2) \\
&- F\omega(b_1, b_2)\rho(b_0) + F\rho(b_0)\omega(b_1, b_2) \\
&- \rho(b_2)\rho(b_0)[F, \rho(b_1)] - \rho(b_0)\rho(b_1)[F, \rho(b_2)] \\
&+ F\rho(b_1)\omega(b_2, b_0) - F\omega(b_2, b_0)\rho(b_1) + \rho(b_2)\rho(b_0)[F, \rho(b_1)] \\
&+ F\omega(b_0, b_1)\rho(b_2) - F\rho(b_0)\omega(b_1, b_2) \\
&+ F\omega(b_2, b_0)\rho(b_1) - F\rho(b_2)\omega(b_0, b_1) \\
&+ F\omega(b_1, b_2)\rho(b_0) - F\rho(b_1)\omega(b_2, b_0)\Big) \\
=& 0
\end{aligned}$$

Thus, $(b + B)\psi_F = 0$, and so $\chi_F = -\frac{1}{2}(\psi_1 + \psi_F)$ is indeed a cocycle in $HC^1_{per}(\mathcal{B})$.

4.2 Pairing with an Invertible Element

Our goal is to prove the following theorem.

Theorem 4.2. *Let $(\rho, \mathcal{H}, F)$ be an almost (odd) (p-summable) Fredholm module over the unital algebra $\mathcal{B}$, $u \in \mathcal{B}$ an invertible element, $ch(u)$ the periodic cyclic cycle associated to u in $HC_1^{per}(\mathcal{B})$, and $P = \frac{1+F}{2}$, then*

$$\langle ch(u), \chi_F\rangle = \chi_F(ch(u)) = Index_{P\mathcal{H}}\, P\rho(u)P.$$

Remark 4.1. Considering only the symbols, this is exactly the value of Connes' pairing of a Fredholm module with a unitary. This differs from his pairing as in his case $\rho(u)$ is a unitary, and in this case $\rho(u)$ is a Fredholm operator. Thus, the above formula immediately agrees with Connes' pairing when ρ is a representation.

To begin, we will define a second almost representation, ρ' such that $\rho' - \rho \in \mathcal{L}^p$ and $[F, \rho'(b)] = 0$ for all $b \in \mathcal{B}$. Define $\rho' : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ by $\rho'(b) = \rho(b) + \frac{1}{2}F[\rho(b), F]$.
Then $\rho'(b) - \rho(b) = \frac{1}{2}F[\rho(b), F] \in \mathcal{L}^p$, and

$$\begin{aligned}
F\rho'(b) =& F\Big(\rho(b) + \frac{1}{2}F[\rho(b), F]\Big) \\
=& F\rho(b) + \frac{1}{2}F^2[\rho(b), F] \\
=& F\rho(b) + \frac{1}{2}[\rho(b), F] \\
=& F\rho(b) + \frac{1}{2}\Big(\rho(b)F - F\rho(b)\Big) \\
=& \frac{1}{2}\Big(\rho(b)F + F\rho(b)\Big) \\
=& \rho(b)F + \frac{1}{2}\Big(F\rho(b) - \rho(b)F\Big) \\
=& \rho(b)F + \frac{1}{2}F\Big(\rho(b)F - F\rho(b)\Big)F \\
=& \rho(b)F + \frac{1}{2}F[\rho(b), F]F \\
=& \Big(\rho(b) + \frac{1}{2}F[\rho(b), F]\Big)F \\
=& \rho'(b)F
\end{aligned}$$

Let $s'(a \otimes b) = a\rho'(b)$. Then $s(a \otimes b) - s'(a \otimes b) = a(\rho(b) - \rho'(b)) \in \mathcal{L}^p$, so $Ch_s^n - Ch_{s'}^n$ induce the same element in the cyclic cohomology of $\mathcal{A} \otimes \mathcal{B}$ by theorem 3.1 in section 3.3, and so they both induce χ_F.

Let $\beta_l = b_0 \otimes ... \otimes b_l$. Consider that if $k > 0$, then

$$\begin{aligned}
Ch^n_{s'}(sh_{kl}(\alpha_k, \beta_l)) &= c_n\Big(\mathrm{Tr}\big(\varepsilon'(x_0, x_1)\cdots\varepsilon'(x_{n-1}, x_n)\big) - \mathrm{Tr}\big(\varepsilon'(x_n, x_0)\cdots\varepsilon'(x_{n-2}, x_{n-1})\big)\Big) \\
&= c^F_{2k} c_{n,k}\Bigg(\sum_{0 \le i_1 < i_2 < ... < i_k \le \frac{n-1}{2}} \mathrm{Tr}\big(F[F, \rho'(b_0)]\omega'(b_1, b_2)...[F, \rho'(b_{2i_2-1})]\omega'(b_{2i_2}, b_{2i_2+1})... \\
&\qquad + F\omega'(b_1, b_2)...[F, \rho'(b_{2i_1})]\omega'(b_{2i_1+1}, b_{2i_1+2})...\big) \\
&\qquad -(-1)^k \sum_{0 < i_1 \le ... \le i_k \le \frac{n-1}{2}} \mathrm{Tr}\Big(\big(F\omega'(b_n, b_0) + [F, \rho'(b_n)]\rho'(b_0)\big)\omega'(b_1, b_2)\cdots \\
&\qquad [F, \rho'(b_{2i_1})]\omega'(b_{2i_1+1}, b_{2i_1+2})...[F, \rho'(b_{2i_j-j+1})]...\Big)\Bigg) \\
&= 0
\end{aligned}$$

since each term contains a copy of $[F, \rho'(b)]$ and $[F, \rho'(b)] = 0$. Thus,

$$\begin{aligned}
\chi_F(ch(u)) &= Ch^n_{s'}(sh(\alpha \otimes ch(u))) \\
&= Ch^n_{s'}\left(\sum_{k+l=n} sh_{kl}(\alpha_k, ch(u)_l)\right) \\
&= Ch^n_{s'}(sh_{0n}(\alpha_0, ch(u)_n)).
\end{aligned}$$

So we compute

$$Ch^n_{s'}(sh_{0n}(\alpha_0, ch(u)_n)) = Ch^n_{s'}\left(sh_{0n}\left(\frac{1}{2}, ch(u)_n\right)\right) + Ch^n_{s'}\left(sh_{0n}\left(\frac{F}{2}, ch(u)_n\right)\right).$$

Let m be such that $n = 2m - 1$, then

$$c^u_{2m-1} c_{2m-1,0} = (-1)^{m-1}(m-1)!\frac{(-1)^{m-1}}{(m-1)!} = 1,$$

so

$$\begin{aligned}
Ch^n_{s'}\left(sh_{0n}\left(\frac{1}{2}, ch(u)_n\right)\right) =& \frac{1}{2}\Big(\mathrm{Tr}\big(\omega'(u^{-1},u)\cdots\omega'(u^{-1},u)\big) - \mathrm{Tr}\big(\omega'(u,u^{-1})\cdots\omega'(u,u^{-1})\big)\Big)\\
Ch^n_{s'}\left(sh_{0n}\left(\frac{F}{2}, ch(u)_n\right)\right) =& \frac{1}{2}\Bigg(\mathrm{Tr}\Big(F\omega'(u^{-1},u)\cdots\omega'(u^{-1},u)\Big)\\
&- \mathrm{Tr}\Big(\big(F\omega'(u,u^{-1}) + [F,\rho'(u)]\rho'(u^{-1})\big)\omega'(u,u^{-1})\cdots\omega'(u,u^{-1})\Big)\Bigg)\\
=& \frac{1}{2}\Bigg(\mathrm{Tr}\Big(F\omega'(u^{-1},u)\cdots\omega'(u^{-1},u)\Big) - \mathrm{Tr}\Big(F\omega'(u,u^{-1})\cdots\omega'(u,u^{-1})\Big)\Bigg)
\end{aligned}$$

We will compute each individually. Note that since $n \geq 2p-1$, we have $m \geq p$. Now,

$$\begin{aligned}
Ch^n_{s'}\left(sh_{0n}\left(\frac{1}{2}, ch(u)_n\right)\right) =& Ch^{2m-1}_{s'}\left(sh_{0(2m-1)}\left(\frac{1}{2}, ch(u)_{2m-1}\right)\right)\\
=& \frac{1}{2}\Bigg(\mathrm{Tr}\Big(\omega'(u^{-1},u)^m\Big) - \mathrm{Tr}\Big(\omega'(u,u^{-1})^m\Big)\Bigg)\\
=& \frac{1}{2}\Bigg(\mathrm{Tr}\bigg(\Big(1-\rho(u^{-1})\rho(u)\Big)^m\bigg) - \mathrm{Tr}\bigg(\Big(1-\rho(u^{-1})\rho(u)\Big)^m\bigg)\Bigg)\\
=& \frac{1}{2}\mathrm{Index}_{\mathcal{H}}\,\rho'(u)\\
=& \frac{1}{2}\mathrm{Index}_{\mathcal{H}}\,\rho(u)
\end{aligned}$$

with the last equality as $\rho' - \rho \in \mathcal{L}^p$ and the Fredholm index is invariant under compact perturbations.

Now, for the second term we have

$$\begin{aligned}
Ch^n_{s'}\left(sh_{0n}\left(\frac{F}{2}, ch(u)_n\right)\right) =& Ch^{2m-1}_{s'}\left(sh_{0(2m-1)}\left(\frac{F}{2}, ch(u)_{2m-1}\right)\right)\\
=& \frac{1}{2}\mathrm{Tr}\Big(F\omega'(u^{-1},u)^m\Big) - \frac{1}{2}\mathrm{Tr}\Big(F\omega'(u,u^{-1})^m\Big)\\
=& \frac{1}{2}\mathrm{Tr}\bigg(F\Big(1-\rho'(u^{-1})\rho'(u)\Big)^m\bigg) - \frac{1}{2}\mathrm{Tr}\bigg(F\Big(1-\rho'(u)\rho'(u^{-1})\Big)^m\bigg)
\end{aligned}$$

For now, we will do our computations without the $\frac{1}{2}$ and return it at the end.

Lemma 4.1. *(1)* $F = \frac{1+F}{2} - \frac{1-F}{2}$

(2) $\frac{1-F}{2}$ *and* $\frac{1+F}{2}$ *are orthogonal projections with* $\frac{1-F}{2} \perp \frac{1+F}{2}$

(3) For any $b \in \mathcal{B}$, $\rho'(b)$ *commutes with* $\frac{1-F}{2}$ *and* $\frac{1+F}{2}$

A proof of this lemma is in appendix B.

Let $P = \frac{1+F}{2}$, then $(1-P) = \frac{1-F}{2}$, and $F = 2P-1$. From the above lemma we have $\rho'(u)|_{P\mathcal{H}} = P\rho'(u)P = \rho'(u)P = P\rho'(u)$ and also for u^{-1}.

Lemma 4.2. *Assume P is an orthogonal projection and A is a trace class operator such that $PA = AP$. Then $Tr(PA) = Tr(A|_{P\mathcal{H}})$*

A proof of this lemma is in appendix B. Thus,

$$\begin{aligned}
\operatorname{Tr}&\Big(F\Big(1-\rho'(u^{-1})\rho'(u)\Big)^m\Big) - \operatorname{Tr}\Big(F\Big(1-\rho'(u)\rho'(u^{-1})\Big)^m\Big)\\
&=\operatorname{Tr}\Big((2P-1)\Big(1-\rho'(u^{-1})\rho'(u)\Big)^m\Big) - \operatorname{Tr}\Big((2P-1)\Big(1-\rho'(u)\rho'(u^{-1})\Big)^m\Big)\\
&=\operatorname{Tr}\Big(2P\Big(1-\rho'(u^{-1})\rho'(u)\Big)^m\Big) - \operatorname{Tr}\Big(2P\Big(1-\rho'(u)\rho'(u^{-1})\Big)^m\Big)\\
&\quad-\Big(\operatorname{Tr}\Big(\Big(1-\rho'(u^{-1})\rho'(u)\Big)^m\Big) - \operatorname{Tr}\Big(\Big(1-\rho'(u)\rho'(u^{-1})\Big)^m\Big)\Big)\\
&=2\operatorname{Tr}\Big(\Big(1_{P\mathcal{H}}-\rho'(u^{-1})|_{P\mathcal{H}}\,\rho'(u)|_{P\mathcal{H}}\Big)^m P\Big) - 2\operatorname{Tr}\Big(\Big(1_{P\mathcal{H}}-\rho'(u)|_{P\mathcal{H}}\,\rho'(u^{-1})|_{P\mathcal{H}}\Big)^m P\Big)\\
&\quad-\Big(\operatorname{Tr}\Big(\Big(1-\rho'(u^{-1})\rho'(u)\Big)^m\Big) - \operatorname{Tr}\Big(\Big(1-\rho'(u)\rho'(u^{-1})\Big)^m\Big)\Big)\\
&=2\mathrm{Index}_{P\mathcal{H}}\ \rho'(u)|_{P\mathcal{H}} - \mathrm{Index}_{\mathcal{H}}\ \rho'(u)\\
&=2\mathrm{Index}_{P\mathcal{H}}\ \rho'(u)|_{P\mathcal{H}} - \mathrm{Index}_{\mathcal{H}}\ \rho(u).
\end{aligned}$$

Now,

$$\begin{aligned}
\rho'(u) &=\rho(u) + \frac{1}{2}F[\rho(u), F]\\
&=\left(\frac{1+F}{2}\right)\rho(u)\left(\frac{1+F}{2}\right) + \left(\frac{1-F}{2}\right)\rho(u)\left(\frac{1-F}{2}\right)\\
&=P\rho(u)P + (1-P)\rho(u)(1-P)
\end{aligned}$$

Thus,

$$\rho'(u)|_{P\mathcal{H}} = \rho'(u)P = P\rho(u)P.$$

So we have that

$$\begin{aligned} Ch^n_{s'}\left(sh_{0n}\left(\frac{F}{2}, ch(u)_n\right)\right) =& \frac{1}{2}\Big(2\mathrm{Index}_{P\mathcal{H}}\ \rho'(u)|_{\mathcal{H}_1} - \mathrm{Index}_{\mathcal{H}}\ \rho(u)\Big) \\ =& \mathrm{Index}_{P\mathcal{H}}\ P\rho(u)P - \frac{1}{2}\mathrm{Index}_{\mathcal{H}}\ \rho(u) \end{aligned}$$

Putting this all together, we have our result,

$$\begin{aligned} \chi_F(ch(u)) =& Ch^n_{s'}\left(sh_{0n}\left(\frac{1}{2}, ch(u)_n\right)\right) + Ch^n_{s'}\left(sh_{0n}\left(\frac{F}{2}, ch(u)_n\right)\right) \\ =& \frac{1}{2}\mathrm{Index}_{\mathcal{H}}\ \rho(u) + \mathrm{Index}_{P\mathcal{H}}\ P\rho(u)P - \frac{1}{2}\mathrm{Index}_{\mathcal{H}}\ \rho(u) \\ =& \mathrm{Index}_{P\mathcal{H}}\ P\rho(u)P. \end{aligned}$$

Remark 4.2. The choice of α was so that it agreed with the Chern character of an idempotent. If we apply the duality map to

$$\hat{\alpha} = C^1 + C^F F + \sum_{k\geq 1} \frac{(-1)^k(2k)!C^F}{4^k k!} F \otimes F^{\otimes 2k},$$

and pair that with $ch(u)$, then we will have

$$P\Psi^{0,n}_{\mathcal{A},\mathcal{B}}(\hat{\alpha}, ch(u)) = 2C^F\mathrm{Index}_{P\mathcal{H}}\ P\rho(u)P + (C^1 - C^F)\mathrm{Index}_{\mathcal{H}}\ \rho(u).$$

Example 4.3. Since $\chi_F(ch(u)) = \mathrm{Index}_{P\mathcal{H}}\ P\rho(u)P$, we must have that $\chi_F(ch(u)) = 0$ if $\mathcal{H}$ is finite dimensional. We will check this with direct computation when $n = 1$ and $n = 3$. We choose to use the notation of proposition 4.2, so $\rho(u) = U$ and $\rho(U^{-1}) = V$.

We note that every operator is trace class for finite dimensional $\mathcal{H}$.

Now for $n = 1$, $\lfloor \frac{2}{4} \rfloor = 0$, so we only need to consider $k = 0$, $m = 0$, so

$$\begin{aligned} \chi_F(ch(u)) = \Omega_{1,0} = & \frac{1}{2}\mathrm{Tr}\Big((1+F)(1-VU) - (1+F)(1-UV) - [F,U]V\Big) \\ =& \frac{1}{2}\mathrm{Tr}\Big((1+F) - (1+F)VU - (1+F) + (1+F)UV - (FU - UF)V\Big) \\ =& \frac{1}{2}\mathrm{Tr}\Big((1+F)UV - U(1+F)V + UFV - FUV\Big) \\ =& \frac{1}{2}\mathrm{Tr}\Big(UV + FUV - UV - UFV + UFV - FUV\Big) \\ =& 0 \end{aligned}$$

For $n = 3$, so $m = 1$, we have $\lfloor \frac{4}{4} \rfloor = 1$, so we consider $k = 1$ and $k = 0$. We have

$$\begin{aligned}
\chi_F(ch(u)) =& \Omega_{3,1} + C_{3,1}(\Omega_{3,1} + \Omega'_{3,1}) \\
C_{3,1} =& -\frac{1}{4} \\
\Omega_{3,0} =& \frac{1}{2}\mathrm{Tr}\Big((1+F)\Big((1-VU)^2 - (1-UV)^2\Big) - [F,U]V(1-UV)\Big) \\
=& \frac{1}{2}\mathrm{Tr}\Big(\Big((1-VU)^2 - (1-UV)^2\Big) + F\Big((1-VU)^2 - (1-UV)^2\Big) - [F,U]V(1-UV)\Big) \\
=& \frac{1}{2}\mathrm{Tr}\Big(0 + -2FVU + FVUVU + 2FUV - FUVUV \\
& \qquad - FUV + FUVUV + UFV - UFVUV\Big) \\
=& \frac{1}{2}\mathrm{Tr}\Big(F[U,V]\Big) \\
\Omega_{3,1} =& \mathrm{Tr}\Big(F[F,V][F,U]\Big) - \mathrm{Tr}\Big(F[F,U][F,V]\Big) \\
=& \mathrm{Tr}\Big(F[F,V][F,U]\Big) + \mathrm{Tr}\Big([F,U]F[F,V]\Big) \\
=& 2\mathrm{Tr}\Big(F[F,V][F,U]\Big) \\
=& 2\mathrm{Tr}\Big(FU - UF - FVU + FVFUF\Big) \\
=& 2\mathrm{Tr}\Big(F[U,V]\Big) \\
\Omega'_{3,1} =& 0
\end{aligned}$$

so

$$\chi_F(ch(u)) = \frac{1}{2}\mathrm{Tr}\Big(F[U,V]\Big) - \frac{1}{4}\Big(2\mathrm{Tr}\Big(F[U,V]\Big) + 0\Big) = 0.$$

Note, in the case $n = 3$, Connes' pairing for odd Fredholm modules with a unitary will say that $\frac{1}{2}\mathrm{Tr}\Big(F[U,V]\Big) = 0$ which follows as in this case $V = U^{-1}$.

We next do an example with a non zero index.

Example 4.4. Let $\mathcal{H} = \ell^2(\mathbb{N})$, $\{e_i\}_{i=0}^{\infty}$ the standard basis, and E_i projection onto $\overline{\mathrm{Span}\{e_i\}}$, U the right shift operator, V the left shift operator, and $Fe_i = 1$ for $i \neq 1$ and $Fe_1 = -1$. Then

$P = \frac{1+F}{2} = 1 - E_1$, and $[F,U], [F,V], 1 - VU, 1 - UV$ are finite rank. Thus, χ_F is defined and $\chi_F(ch(u)) = \text{Index}_{P\mathcal{H}}\, PUP = -1$. In particular

$$VU = 1 \qquad\qquad [F,U] = 2U(E_1 - E_0)$$

$$UV = 1 - E_0 \qquad\qquad [F,V] = 2V(E_1 - E_2).$$

So For $n = 1$ we must compute

$$\begin{aligned}\chi_F(ch(u)) = \Omega_{1,0} &= \frac{1}{2}\text{Tr}\Big((1+F)(1-VU) - (1+F)(1-UV) - [F,U]V\Big)\\ &= \text{Tr}\Big((1+F)(0) - \frac{1+F}{2}(1-UV) - \frac{1}{2}[F,U]V\Big)\\ &= \text{Tr}\Big(-(1-E_1)(E_0) - \frac{1}{2}2U(E_1-E_0)V\Big)\\ &= \text{Tr}\Big(-E_0 - UE_1V + UE_0V\Big)\end{aligned}$$

and

$$\begin{aligned}\text{Tr}\Big(-E_0 - UE_1V + UE_0V\Big) &= \sum_{i=0}^{\infty}\langle(-E_0 - UE_1V + UE_0V)e_i, e_i\rangle\\ &= -\sum_{i=0}^{\infty}\langle E_0e_i, e_i\rangle - \sum_{i=0}^{\infty}\langle UE_1Ve_i, e_i\rangle + \sum_{i=0}^{\infty}\langle UE_0Ve_i, e_i\rangle\\ &= -\langle E_0e_0, e_0\rangle - \langle UE_1Ve_2, e_2\rangle + \langle UE_0Ve_1, e_1\rangle\\ &= -1\end{aligned}$$

Chapter 5

Duality Map for $\mathbb{Z}/2$ Graded Algebras

5.1 Ch^n_ρ for $\mathbb{Z}/2$ Graded Algebras

In this section we let $\mathcal{A}$ be a $\mathbb{Z}/2$ graded algebra, $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ an almost representation where $\mathcal{H}$ is a $\mathbb{Z}/2$-graded Hilbert space, and ρ respects the grading on $\mathcal{L}(\mathcal{H})$. Note that we will always take $a \in \mathcal{A}$ to be homogeneous since we may always reduce to this case by linearity. We let γ be the grading map

$$\gamma : \mathcal{H} \to \mathcal{H} \text{ by } \gamma(h) = (-1)^{|h|}h.$$

Our goal in this section is to prove the following theorem

Theorem 5.1. *Let n be an odd integer such that $n \geq 2p - 1$. Then,*

(1) the following map is a cocycle in the cyclic cohomology for $\mathcal{A}$,

$$Ch^n_\rho(a_0, .., a_n) = c_n\bigg(Tr\Big(\gamma\varepsilon(a_0, a_1)\cdots\varepsilon(a_{n-1}, a_n)\Big) - (-1)^{r_0}\, Tr\Big(\gamma\varepsilon(a_1, a_2)\cdots\varepsilon(a_n, a_0)\Big)\bigg)$$

where

$$r_0 = |a_0|\sum_{i=1}^{n}|a_i|, \text{ and } c_n = \frac{(-1)^{\frac{n-1}{2}}}{(\frac{n-1}{2})!}.$$

In particular, if $\sum_{i=0}^{n}|a_i|$ is odd, then

$$Ch^n_\rho(a_0, .., a_n) = 0,$$

and if $\sum_{i=0}^{n}|a_i|$ is even, then

$$Ch^n_\rho(a_0, .., a_n) = c_n\bigg(Tr\Big(\gamma\varepsilon(a_0, a_1)\cdots\varepsilon(a_{n-1}, a_n)\Big) - (-1)^{|a_0|}\, Tr\Big(\gamma\varepsilon(a_1, a_2)\cdots\varepsilon(a_n, a_0)\Big)\bigg).$$

(2) If ρ' is another almost representation such that $\rho' - \rho \in \mathcal{L}^p$, then Ch_ρ^n and $Ch_{\rho'}^n$ differ by a coboundary.

(3) SCh_ρ^n and Ch_ρ^{n+2} differ by a coboundary.

We will label and prove each of the above parts as separate propositions. We note that the proofs are all essentially the same as in the ungraded case, in that regard, the proofs of (1) and (2) can be found in [3] at the beginning of page 73.

Proposition 5.1. If $\sum_{i=0}^{n} |a_i|$ is odd, then

$$Ch_\rho^n(a_0, .., a_n) = 0,$$

and if $\sum_{i=0}^{n} |a_i|$ is even then

$$Ch_\rho^n(a_0, .., a_n) = c_n\Big(\mathrm{Tr}\Big(\gamma\varepsilon(a_0, a_1)\cdots\varepsilon(a_{n-1}, a_n)\Big) - (-1)^{|a_0|}\mathrm{Tr}\Big(\gamma\varepsilon(a_1, a_2)\cdots\varepsilon(a_n, a_0)\Big)\Big)$$

is a cocycle in $HC^n(\mathcal{A})$.

We note that the differentials for the cyclic homology of a $\mathbb{Z}/2$-graded algebra $\mathcal{A}$ are the following in the normalized setting

$$b(a_0, .., a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0, a_1, ..., a_i a_{i+1}, ..., a_n) + (-1)^{n+r_n}(a_n a_0, a_1, .., a_{n-1})$$

where $r_n = |a_n| \sum_{i=0}^{n-1} |a_i|$, and

$$B(a_0, ..., a_n) = \sum_{i=0}^{n}(-1)^{ni+\hat{r_i}}(1, a_i, ..., a_n, a_0, ..., a_{i-1})$$

where $\hat{r_i} = (|a_i| + ... + |a_n|)(|a_0| + ... + |a_{i-1}|)$.

Note that for homogeneous $T \in \mathcal{L}(\mathcal{H})$ we have $\gamma T = (-1)^{|T|}T\gamma$. So if $\omega \in \mathcal{L}(\mathcal{H})$ is trace class with odd degree, then

$$\mathrm{Tr}(\gamma\omega) = \mathrm{Tr}(\omega\gamma) = -\mathrm{Tr}(\gamma\omega),$$

and thus $\mathrm{Tr}(\gamma\omega) = 0$.

Also note that $|TS| = |T| + |S|$, so $|\varepsilon(a_0, a_1)| = |\rho(a_0a_1) - \rho(a_0)\rho(a_1)| = |a_0| + |a_1|$, and thus

$$|\varepsilon(a_0, a_1) \cdots \varepsilon(a_{n-1}, a_n)| = \sum_{i=0}^{n} |a_i|$$

with the above sum taken mod 2.

To make our differential calculations easier, we let $n-1$ be odd, and we do our analysis on

$$Ch_\rho^{n-1}(a_0, .., a_{n-1}) = c_{n-1}\Big(\mathrm{Tr}\Big(\gamma\varepsilon(a_0, a_1) \cdots \varepsilon(a_{n-2}, a_{n-1})\Big) - (-1)^{r_0}\mathrm{Tr}\Big(\gamma\varepsilon(a_1, a_2) \cdots \varepsilon(a_{n-1}, a_0)\Big)\Big)$$

From above, if $\sum_{i=0}^{n-1} |a_i|$ is odd, then

$$Ch_\rho^{n-1}(a_0, .., a_{n-1}) = 0.$$

So we let $\sum_{i=0}^{n-1} |a_i| = 0$ mod 2, which will imply

$$r_0 = |a_0| \sum_{i=1}^{n-1} |a_i| = |a_0| \cdot |a_0| = |a_0|.$$

Thus,

$$Ch_\rho^{n-1}(a_0, .., a_{n-1}) = c_{n-1}\Big(\mathrm{Tr}\big(\gamma\varepsilon(a_0, a_1) \cdots \varepsilon(a_{n-2}, a_{n-1})\big) - (-1)^{|a_0|}\mathrm{Tr}\big(\gamma\varepsilon(a_1, a_2) \cdots \varepsilon(a_{n-1}, a_0)\big)\Big).$$

Let

$$\phi^+(a_0, .., a_{n-1}) = \mathrm{Tr}\Big(\gamma\varepsilon(a_0, a_1) \cdots \varepsilon(a_{n-1}, a_{n-1})\Big)$$

and

$$\phi^-(a_0, .., a_{n-1}) = (-1)^{|a_0|}\mathrm{Tr}\Big(\gamma\varepsilon(a_1, a_2) \cdots \varepsilon(a_{n-1}, a_0)\Big)$$

so $Ch_\rho^{n-1} = c_{n-1}(\phi^+ - \phi^-)$.

Claim: $bCh_\rho^{n-1} = 0$.

First note that

$$\varepsilon(a_ia_{i+1}, a_{i+2}) - \varepsilon(a_i, a_{i+1}a_{i+2}) = \rho(a_i)\varepsilon(a_{i+1}, a_{i+2}) - \varepsilon(a_i, a_{i+1})\rho(a_{i+2}).$$

Indeed,

$$\begin{aligned}\varepsilon(a_ia_{i+1},a_{i+2})-\varepsilon(a_i,a_{i+1}a_{i+2}) =&\rho(a_ia_{i+1}a_{i+2})-\rho(a_ia_{i+1})\rho(a_{i+2})\\ &-\rho(a_ia_{i+1}a_{i+2})+\rho(a_i)\rho(a_{i+1}a_{i+2})\\ =&\rho(a_i)\rho(a_{i+1}a_{i+2})-\rho(a_ia_{i+1})\rho(a_{i+2})\\ =&\rho(a_i)\rho(a_{i+1}a_{i+2})-\rho(a_i)\rho(a_{i+1})\rho(a_{i+2})\\ &+\rho(a_i)\rho(a_{i+1})\rho(a_{i+2})-\rho(a_ia_{i+1})\rho(a_{i+2})\\ =&\rho(a_i)\varepsilon(a_{i+1},a_{i+2})-\varepsilon(a_i,a_{i+1})\rho(a_{i+2}).\end{aligned}$$

Then we have

$$\begin{aligned}b\phi^+(a_0,...,a_n) = &\sum_{i=0}^{n-1}(-1)^i\phi^+(a_0,a_1,...,a_ia_{i+1},...,a_n)+(-1)^{|a_n|}\phi^+(a_na_0,a_1,..,a_{n-1})\\ =&\sum_{i=0}^{n/2-1}\phi^+(a_0,a_1,...,a_{2i}a_{2i+1},a_{2i+2},...,a_n)-\sum_{i=0}^{n/2-1}\phi^+(a_0,a_1,...,a_{2i},a_{2i+1}a_{2i+2},...,a_n)\\ &+(-1)^{|a_n|}\phi^+(a_na_0,a_1,..,a_{n-1})\\ =&\sum_{i=0}^{n/2-1}\mathrm{Tr}\Big(\gamma\varepsilon(a_0,a_1)\cdots\Big(\varepsilon(a_{2i}a_{2i+1},a_{2i+2})-\varepsilon(a_{2i},a_{2i+1}a_{2i+2})\Big)\cdots\varepsilon(a_{n-1},a_n)\Big)\\ &+(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\varepsilon(a_na_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\Big)\\ =&\sum_{i=0}^{n/2-1}\mathrm{Tr}\Big(\gamma\varepsilon(a_0,a_1)\cdots\Big(\rho(a_{2i})\varepsilon(a_{2i+1},a_{2i+2})-\varepsilon(a_{2i},a_{2i+1})\rho(a_{2i+2})\Big)\cdots\varepsilon(a_{n-1},a_n)\Big)\\ &+(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\varepsilon(a_na_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\Big)\\ =&\mathrm{Tr}\Big(\gamma\rho(a_0)\varepsilon(a_1,a_2)\cdots\varepsilon(a_{n-1},a_{n-2})\Big)-\mathrm{Tr}\Big(\gamma\varepsilon(a_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\rho(a_n)\Big)\\ &+(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\varepsilon(a_na_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1}\Big),\end{aligned}$$

and

$$\begin{aligned}
&b\phi^-(a_0,...,a_n)\\
&\quad= \phi^-(a_0a_1,...,a_n) + (-1)^{|a_n|}\phi^-(a_na_0,a_1,..,a_{n-1}) + \sum_{i=1}^{n-1}(-1)^i\phi^-(a_0,a_1,...,a_ia_{i+1},...,a_n)\\
&\quad=(-1)^{|a_0|+|a_1|}\mathrm{Tr}\Big(\gamma\varepsilon(a_2,a_3)\cdots\varepsilon(a_n,a_0a_1)\Big)\\
&\qquad-\sum_{i=1}^{n/2}(-1)^{|a_0|}\mathrm{Tr}\Big(\gamma\varepsilon(a_1,a_2)\cdots\Big(\varepsilon(a_{2i-1}a_{2i},a_{2i+1})-\varepsilon(a_{2i-1},a_{2i}a_{2i+1})\Big)\cdots\varepsilon(a_n,a_0)\Big)\\
&\quad=(-1)^{|a_0|+|a_1|}\mathrm{Tr}\Big(\gamma\varepsilon(a_2,a_3)\cdots\varepsilon(a_n,a_0a_1)\Big)\\
&\qquad-(-1)^{|a_0|}\mathrm{Tr}\Big(\gamma\rho(a_1)\varepsilon(a_2,a_3)\cdots\varepsilon(a_n,a_0)\Big)+(-1)^{|a_0|}\mathrm{Tr}\Big(\varepsilon(a_1,a_2)\cdots\varepsilon(a_{n-1},a_n)\rho(a_0)\Big)
\end{aligned}$$

We note that the last term of the sum has degree $(-1)^{|a_n|+|a_0|+|a_n|}=(-1)^{|a_0|}$ and in the sum we take $a_{n+1}=a_0$.

Now,

$$\begin{aligned}
&(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\varepsilon(a_na_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\Big)-(-1)^{|a_0|+|a_1|}\mathrm{Tr}\Big(\gamma\varepsilon(a_2,a_3)\cdots\varepsilon(a_n,a_0a_1)\Big)\\
&\quad=(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\varepsilon(a_na_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\Big)-(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\varepsilon(a_n,a_0a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\Big)\\
&\quad=(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\rho(a_n)\varepsilon(a_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\Big)-(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\varepsilon(a_n,a_0)\rho(a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\Big)
\end{aligned}$$

Thus,

$$\begin{aligned}
&bCh^n(a_0,...,a_n)=b\phi^+(a_0,...,a_n)-b\phi^-(a_0,...,a_n)\\
&\quad=\mathrm{Tr}(\gamma\rho(a_0)\varepsilon(a_1,a_2)\cdots\varepsilon(a_{n-1},a_{n-2}))-(-1)^{|a_0|}\mathrm{Tr}(\gamma\varepsilon(a_1,a_2)\cdots\varepsilon(a_{n-1},a_n)\rho(a_0))\\
&\qquad+(-1)^{|a_n|}\mathrm{Tr}(\gamma\rho(a_n)\varepsilon(a_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1}))-\mathrm{Tr}(\gamma\varepsilon(a_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1})\rho(a_n))\\
&\qquad+(-1)^{|a_0|}\mathrm{Tr}(\gamma\rho(a_1)\varepsilon(a_2,a_3)\cdots\varepsilon(a_n,a_0))-(-1)^{|a_n|}\mathrm{Tr}(\gamma\varepsilon(a_n,a_0)\rho(a_1)\cdots\varepsilon(a_{n-2},a_{n-1}))\\
&\quad=\mathrm{Tr}(\gamma\rho(a_0)\varepsilon(a_1,a_2)\cdots\varepsilon(a_{n-1},a_{n-2}))-(-1)^{2|a_0|}\mathrm{Tr}(\gamma\rho(a_0)\varepsilon(a_1,a_2)\cdots\varepsilon(a_{n-1},a_n))\\
&\qquad+(-1)^{|a_n|}\mathrm{Tr}(\gamma\rho(a_n)\varepsilon(a_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1}))-(-1)^{|a_n|}\mathrm{Tr}(\gamma\rho(a_n)\varepsilon(a_0,a_1)\cdots\varepsilon(a_{n-2},a_{n-1}))\\
&\qquad+(-1)^{|a_0|}\mathrm{Tr}(\gamma\rho(a_1)\varepsilon(a_2,a_3)\cdots\varepsilon(a_n,a_0))-(-1)^{|a_0|+2|a_n|}\mathrm{Tr}(\gamma\rho(a_1)\varepsilon(a_2,a_3)\cdots\varepsilon(a_n,a_0))\\
&\quad=0
\end{aligned}$$

Further, $BCh^n_\rho(a_0, .., a_n) = 0$ since $\varepsilon(1, a) = 0$.

Thus, Ch^n_ρ is a cocycle in the cyclic cohomology of the $\mathbb{Z}/2$ graded algebra $\mathcal{A}$.

Proposition 5.2. If $\rho_0 - \rho_1 \in \mathcal{L}^p$ then $Ch^n_{\rho_0} - Ch^n_{\rho_1}$ is a coboundary.

Let $L(a) = \rho_0(a) - \rho_1(a)$, $\rho_t(a) = \rho_0(a) + tL(a)$, and $\varepsilon_t(a_1, a_2) = \rho_t(a_1a_2) - \rho_t(a_1)\rho_t(a_2)$. Then we will have that $\int_0^1 \frac{d}{dt}\left(Ch^n_{\rho_t}\right) dt = Ch^n_{\rho_1} - Ch^n_{\rho_0}$. So we will find a map ϕ_t, so that $(b + B)\psi_t = \frac{d}{dt}\left(Ch^n_{\rho_t}\right)$.

We will suppress the subscript t notation as it will be desirable to allow the subscripts to have a different meaning. So we write $\rho = \rho_t$, $\varepsilon = \varepsilon_t$, $\varepsilon_i = \varepsilon(a_i, a_{i+1})$ taken mod n and mod $(n+1)$ as necessary, and $(Ch^n)' = \frac{d}{dt}\left(Ch^n_{\rho_t}\right)$. Then,

$$(Ch^n)'(a_0, ..., a_n) = \mathrm{Tr}(\gamma(A_1 - (-1)^{|a_0|}A_2))$$

$$\text{where} \qquad A_1 = \varepsilon_0'\varepsilon_2 \cdots \varepsilon_{n-1} + \varepsilon_0\varepsilon_2' \cdots \varepsilon_{n-1} + ... + \varepsilon_0\varepsilon_2 \cdots \varepsilon_{n-1}'$$

$$A_2 = \varepsilon_1'\varepsilon_3 \cdots \varepsilon_n + \varepsilon_1\varepsilon_3' \cdots \varepsilon_n + ... + \varepsilon_1\varepsilon_3 \cdots \varepsilon_n'$$

$$\text{and} \qquad \varepsilon_j' = L(a_ja_{j+1}) - \rho(a_j)L(a_{j+1}) - L(a_j)\rho(a_{j+1}).$$

Recall that we must have $\sum_{i=0}^n |a_i| = 0$ mod 2 or the result follows trivially.

Define

$$\begin{aligned}
\psi_{2k}(a_0, ..., a_{n-1}) =& \mathrm{Tr}(\gamma\varepsilon_0 \cdots \varepsilon_{2k-2}L(a_{2k})\varepsilon_{2k+1} \cdots \varepsilon_{n-2}) \\
\psi_{2k-1}(a_0, ..., a_{n-1}) =& (-1)^{|a_0|}\mathrm{Tr}(\gamma\varepsilon_1 \cdots \varepsilon_{2k-3}L(a_{2k-1})\varepsilon_{2k} \cdots \varepsilon_{n-1}) \\
=& (-1)^{|a_{n-1}|}\mathrm{Tr}(\gamma\varepsilon_{n-1}\varepsilon_1 \cdots \varepsilon_{2k-3}L(a_{2k-1})\varepsilon_{2k} \cdots \varepsilon_{n-3}).
\end{aligned}$$

We will show that

$$(Ch^n)' = (b + B)\left(\sum_{j=0}^{n-1} \psi_j\right).$$

First note that $L(1) = \varepsilon(1, a) = \varepsilon(a, 1) = 0$, so $B\psi_j = 0$.

Then using

$$\varepsilon(a_ia_{i+1}, a_{i+2}) - \varepsilon(a_i, a_{i+1}a_{i+2}) = \rho(a_i)\varepsilon(a_{i+1}, a_{i+2}) - \varepsilon(a_i, a_{i+1})\rho(a_{i+2}),$$

we have for $k > 0$

$$\begin{aligned}
b\psi_{2k}(a_0, ..., a_n) =&\mathrm{Tr}\Big(\gamma\rho(a_0)\varepsilon_1\varepsilon_3\cdots\varepsilon_{2k-1}L(a_{2k+1})\varepsilon_{2k+2}\cdots\varepsilon_{n-1}\\
&-\gamma\varepsilon_0\cdots\varepsilon_{2k-2}\rho(a_{2k})L(a_{2k+1})\varepsilon_{2k+2}\cdots\varepsilon_{n-1}\\
&+\gamma\varepsilon_0\cdots\varepsilon_{2k-2}L(a_{2k}a_{2k+1})\varepsilon_{2k+2}\cdots\varepsilon_{n-1}\\
&-\gamma\varepsilon_0\cdots\varepsilon_{2k-2}L(a_{2k})\rho(a_{2k+1})\varepsilon_{2k+2}\cdots\varepsilon_{n-1}\\
&+\gamma\varepsilon_0\cdots\varepsilon_{2k-2}L(a_{2k})\varepsilon_{2k+1}\cdots\varepsilon_{n-2}\rho(a_n)\\
&-(-1)^{|a_n|}\gamma\varepsilon(a_na_0, a_1)\varepsilon_2\cdots\varepsilon_{2k-2}L(a_{2k})\varepsilon_{2k+1}\cdots\varepsilon_{n-2}\Big)\\
=&\mathrm{Tr}\Big(\gamma\rho(a_0)\varepsilon_1\varepsilon_3\cdots\varepsilon_{2k-1}L(a_{2k+1})\varepsilon_{2k+2}\cdots\varepsilon_{n-1}\\
&+\gamma\varepsilon_0\cdots\varepsilon_{2k-2}\varepsilon'_{2k}\varepsilon_{2k+2}\cdots\varepsilon_{n-1}\\
&+\gamma\varepsilon_0\cdots\varepsilon_{2k-2}L(a_{2k})\varepsilon_{2k+1}\cdots\varepsilon_{n-2}\rho(a_n)\\
&-(-1)^{|a_n|}\gamma\varepsilon(a_na_0, a_1)\varepsilon_2\cdots\varepsilon_{2k-2}L(a_{2k})\varepsilon_{2k+1}\cdots\varepsilon_{n-2}\Big)
\end{aligned}$$

and

$$\begin{aligned}
b\psi_{2k-1}(a_0, ..., a_n) =&(-1)^{|a_n|}\mathrm{Tr}\Big(\gamma\varepsilon(a_n, a_0a_1)\varepsilon_2\cdots\varepsilon_{2k-2}L(a_{2k})\varepsilon_{2k+1}\cdots\varepsilon_{n-2}\Big)\\
&-(-1)^{|a_0|}\mathrm{Tr}\Big(\gamma\rho(a_1)\varepsilon_2\cdots\varepsilon_{2k-2}L(a_{2k})\varepsilon_{2k+1}\cdots\varepsilon_n)\\
&\qquad+\gamma\varepsilon_1\cdots\varepsilon'_{2k-1}\varepsilon_{2k+1}\cdots\varepsilon_n\\
&\qquad+\gamma\varepsilon_1\cdots\varepsilon_{2k-3}L(a_{2k-1})\varepsilon_{2k}\cdots\varepsilon_{n-1}\rho(a_0)\Big)
\end{aligned}$$

noting that

$$\begin{aligned}
(-1)^{|a_n|}\psi_{2k-1}(a_na_0, a_1, .., a_{n-1}) =&(-1)^{|a_n|}(-1)^{|a_0|+|a_n|}\mathrm{Tr}(\gamma\varepsilon_1\cdots\varepsilon_{2k-3}L(a_{2k-1})\varepsilon_{2k}\cdots\varepsilon(a_{n-1}, a_na_0))\\
=&(-1)^{|a_0|}\mathrm{Tr}(\gamma\varepsilon_1\cdots\varepsilon_{2k-3}L(a_{2k-1})\varepsilon_{2k}\cdots\varepsilon(a_{n-1}, a_na_0)).
\end{aligned}$$

So, for $k > 0$,

$$\begin{aligned} b(\psi_{2k}+\psi_{2k-1})(a_0,...,a_n) =&\mathrm{Tr}\Big(\gamma\rho(a_0)\varepsilon_1\varepsilon_3\cdots\varepsilon_{2k-1}L(a_{2k+1})\varepsilon_{2k+2}\cdots\varepsilon_{n-1} \\ &+\gamma\varepsilon_0\cdots\varepsilon_{2k-2}\varepsilon_{2k}'\varepsilon_{2k+2}\cdots\varepsilon_{n-1} \\ &-(-1)^{|a_0|}\gamma\varepsilon_1\cdots\varepsilon_{2k-1}'\varepsilon_{2k+1}\cdots\varepsilon_n \\ &-\gamma\rho(a_0)\varepsilon_1\cdots\varepsilon_{2k-3}L(a_{2k-1})\varepsilon_{2k}\cdots\varepsilon_{n-1}\Big) \end{aligned}$$

Now as

$$\begin{aligned} b\psi_0(a_0,...,a_n) = \mathrm{Tr}\Big(&\gamma\rho(a_0)L(a_1)\varepsilon_2\cdots\varepsilon_{n-1}+\gamma\varepsilon_0'\varepsilon_2\cdots\varepsilon_{n-1} \\ &-(-1)^{|a_0|}\gamma\varepsilon_1\cdots\varepsilon_n' - \gamma\rho(a_0)\varepsilon_1\varepsilon_3\cdots\varepsilon_{n-2}L(a_n)\Big) \end{aligned}$$

we have that

$$(Ch^n)' = \sum_{j=0}^{n-1} b\psi_j = \sum_{j=0}^{n-1}(b+B)\psi_j = (b+B)\left(\sum_{j=0}^{n-1}\psi_j\right)$$

Proposition 5.3. $SCh_\rho^n - Ch_\rho^{n+2}$ is a coboundary.

This will follow similarly as in the ungraded case (proven in appendix A).

Let

$$\phi_n(a_0,...,a_n) = \mathrm{Tr}(\gamma\varepsilon_0\cdots\varepsilon_{n-1}) - (-1)^{|a_n|}\mathrm{Tr}(\gamma\varepsilon_n\varepsilon_1\cdots\varepsilon_{n-2})$$

and

$$\psi_{n+1}(a_0,...,a_{n+1}) = \mathrm{Tr}(\gamma\rho(a_0)\varepsilon_1\cdots\varepsilon_n).$$

Then,

$$\begin{aligned}
B\phi_{n+1}(a_0, ..., a_n) = &\sum_{i=0}^{n}(-1)^i\phi_{n+1}(1, a_i, ..., a_{i-1}) \\
= &\sum_{j=0}^{m}(-1)^{\sum_{k=2j}^{n}|a_k|}\mathrm{Tr}(\gamma\varepsilon_{2j}\cdots\varepsilon_{n-1}\varepsilon_0\cdots\varepsilon_{2j-2}) \\
&- (-1)^{\sum_{k=2j+1}^{n}|a_k|}\mathrm{Tr}(\gamma\varepsilon_{2j+1}\cdots\varepsilon_{n-2}\varepsilon_n\cdots\varepsilon_{2j-1}) \\
= &\sum_{j=0}^{m}(-1)^{\sum_{k=0}^{2j-1}|a_k|+\sum_{k=2j}^{n}|a_k|}\mathrm{Tr}(\gamma\varepsilon_0\cdots\varepsilon_{n-1}) \\
&- (-1)^{|a_n|+\sum_{k=0}^{2j}|a_k|+\sum_{k=2j+1}^{n}|a_k|}\mathrm{Tr}(\gamma\varepsilon_n\cdots\varepsilon_{n-2}) \\
= &\left(\frac{n+1}{2}\right)\Big(\mathrm{Tr}(\gamma\varepsilon_0\cdots\varepsilon_{n-1}) - (-1)^{|a_n|}\mathrm{Tr}(\gamma\varepsilon_n\cdots\varepsilon_{n-2})\Big) \\
= &\left(\frac{n+1}{2}\right)S(\phi_n)(a_0, ..., a_n).
\end{aligned}$$

and after doing the same reduction as in the ungraded case, we have

$$\begin{aligned}
b\phi_{n-1}(a_0, ..., a_n) = &\mathrm{Tr}(\gamma\rho(a_0)\varepsilon_1\cdots\varepsilon_{n-2}\rho(a_n)) - \mathrm{Tr}(\gamma\rho(a_0)\rho(a_1)\varepsilon_2\cdots\varepsilon_{n-1}) \\
&+ \mathrm{Tr}(\gamma\rho(a_0a_1)\varepsilon_2\cdots e_{n-1}) - (-1)^{|a_n|}\mathrm{Tr}(\gamma\rho(a_na_0)\varepsilon_1\cdots e_{n-2}) \\
= &(-1)^{|a_n|}\mathrm{Tr}(\gamma\rho(a_n)\rho(a_0)\varepsilon_1\cdots\varepsilon_{n-2}) - \mathrm{Tr}(\gamma\rho(a_0)\rho(a_1)\varepsilon_2\cdots\varepsilon_{n-1}) \\
&+ \mathrm{Tr}(\gamma\rho(a_0a_1)\varepsilon_2\cdots e_{n-1}) - (-1)^{|a_n|}\mathrm{Tr}(\gamma\rho(a_na_0)\varepsilon_1\cdots e_{n-2}) \\
= &\mathrm{Tr}(\gamma\rho(a_0a_1)\varepsilon_2\cdots e_{n-1}) - \mathrm{Tr}(\gamma\rho(a_0)\rho(a_1)\varepsilon_2\cdots\varepsilon_{n-1}) \\
&- (-1)^{|a_n|}(\mathrm{Tr}(\gamma\rho(a_na_0)\varepsilon_1\cdots e_{n-2}) - \mathrm{Tr}(\gamma\rho(a_n)\rho(a_0)\varepsilon_1\cdots\varepsilon_{n-2})) \\
= &\mathrm{Tr}(\gamma\varepsilon_0\cdots e_{n-1}) - (-1)^{|a_n|}\mathrm{Tr}(\gamma\varepsilon_n\varepsilon_1\cdots\varepsilon_{n-2}) \\
= &\psi_n(a_0, ..., a_n).
\end{aligned}$$

Hence, the proposition follows.

5.2 Duality Map in the $\mathbb{Z}/2$ Graded Setting

This sections will follow from section 5.1, section 2.7, and the construction of the duality map in the ungraded case.

We will begin this section by assuming that $\mathcal{A}$ and $\mathcal{B}$ are graded algebras, but we will very quickly change our assumption to $\mathcal{B}$ is trivially graded. We continue to assume that $n \geq 2p-1$ is an odd integer.

In the $\mathbb{Z}/2$ graded setting we may define the duality maps as in section 3.4,

$$\Psi^{i,n}_{\mathcal{A},\mathcal{B}} : HC^-_i(\mathcal{A}) \to HC^{n-i}(\mathcal{B})$$
$$P\Psi^{i,n}_{\mathcal{A},\mathcal{B}} : HC^{per}_i(\mathcal{A}) \to HC^{1-i}_{per}(\mathcal{B})$$

where

$$\begin{aligned}\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) =& Ch^n_s\left(Sh(\alpha\otimes\beta)\right)\\ =& Ch^n_s\left(sh(\alpha\otimes\beta)+sh'(\alpha\otimes\beta)\right)\\ =& Ch^n_s\left(sh(\alpha\otimes\beta)\right).\end{aligned}$$

These are still maps on homology. Indeed, for $\alpha \in \mathcal{B}^-_i(\mathcal{A})$ and $\beta \in \mathcal{B}_{n-i}(\mathcal{B})$ we have

$$(b+B)Sh(\alpha\otimes\beta) = Sh((b+B)\alpha\otimes\beta) + (-1)^i Sh(\alpha\otimes(b+B)\beta),$$

so

$$\begin{aligned}(b+B)_{\mathcal{B}}\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) =& Ch^n_s\left(Sh(\alpha\otimes(b+B)_{\mathcal{B}}\beta)\right)\\ =&(-1)^i Ch^n_s\left(Sh((b+B)_{\mathcal{A}}\alpha\otimes\beta)\right)\\ &-(b+B)_{\mathcal{A}\otimes\mathcal{B}}Ch^n_s\left(Sh(\alpha\otimes\beta)\right)\\ =&(-1)^i\Psi^{i,n}_{\mathcal{A},\mathcal{B}}((b+B)_{\mathcal{A}}\alpha)(\beta).\end{aligned}$$

Hence, $\Psi^{i,n}_{\mathcal{A},\mathcal{B}}$ sends cycles to cocycles and boundaries to coboundaries.

We now let $\mathcal{A}$ be a $\mathbb{Z}/2$ graded algebra such that $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a unital representation, for which we will write $\pi(a) = a$ if no confusion may occur. We let $\mathcal{B}$ be a trivially graded algebra such that $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ is an almost representation, and $[a, \rho(b)] \in \mathcal{L}^p$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Note that in this setting we are using the graded shuffle product, however in this case the shuffle product does not induce a sign as $\mathcal{B}$ is trivially graded. To see this let σ_{pq} be a (p,q)-shuffle. Then for $\alpha_p = (a_0, \cdots, a_p)$, $\beta_q = (b_0, \cdots, b_q)$, σ_{pq} does not change the relative ordering of the a_i with a_j or b_i with b_j. If it changes the relative order of a_i with b_j, then we obtain (-1) to a power of $|a_i| \cdot |b_j| = 0$. The cyclic shuffle product will induce a sign, but as before, $Ch^n_s\left(sh'(\alpha \otimes \beta)\right) = 0$. Since sh does not induce an additional sign, $\Psi^n_{\mathcal{A},\mathcal{B}}$ can be written exactly as it is in section 3.4.2,

Theorem 5.2. *The duality map is*

$$\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) = Ch^n_s\left(sh(\alpha \otimes \beta)\right) = Ch^n_s\left(\sum_{k+l=n} sh_{kl}(\alpha_k, \beta_l)\right) = \sum_{k+l=n} Ch^n_s\left(sh_{kl}(\alpha_k, \beta_l)\right)$$

where for $k = 0$

$$\begin{aligned} Ch^n_s(sh_{0n}(\alpha_0, \beta_n)) =& c_{n,0}\Big(Tr\Big(a_0\omega(b_0, b_1)\omega(b_2, b_3)\cdots\omega(b_{n-1}, b_n)\Big) \\ & - Tr\Big(\big(a_0\omega(b_n, b_0) + [a_0, \rho(b_n)]\rho(b_0)\big)\omega(b_1, b_2)\cdots\omega(b_{n-2}, b_{n-1})\Big)\Big) \end{aligned}$$

and for $k > 0$

$$\begin{aligned} Ch^n_s(sh_{kl}(\alpha_k, \beta_l)) =& c_{n,k}\Bigg(\sum_{0 \le i_1 < i_2 < \ldots < i_k \le \frac{n+1}{2}} Tr\Big(a_0[a_1, \rho(b_0)]\omega(b_1, b_2)\ldots[a_2, \rho(b_{2i_2-1})]\omega(b_{2i_2}, b_{2i_2+1})\ldots \\ & \qquad + a_0\omega(b_0, b_1)\ldots[a_1, \rho(b_{2i_1})]\omega(b_{2i_1+1}, b_{2i_1+2})\ldots\Big) \\ & - (-1)^k \sum_{0 < i_1 \le \ldots \le i_k \le \frac{n+1}{2}} Tr\Big(\big(a_0\omega(b_n, b_0) + [a_0, \rho(b_n)]\rho(b_0)\big)\omega(b_1, b_2)\cdots \\ & \qquad \cdots[a_1, \rho(b_{2i_1})]\omega(b_{2i_1+1}, b_{2i_1+2})\ldots[a_j, \rho(b_{2i_j-j+1})]\ldots\Big)\Bigg) \end{aligned}$$

and

$$c_{n,k} = \frac{(-1)^{\frac{n-1}{2} + \frac{k(k-1)}{2}}}{(\frac{n-1}{2})!}.$$

Now, similarly as in section 3.4.3, we have that

$$Ch^n_s = BC^{n+1}_s$$

where

$$C^{n+1}_s(x_0, \ldots, x_{n+1}) = c_{n+2}\mathrm{Tr}(\gamma\rho(x_0)\varepsilon_1\cdots\varepsilon_n),\ \varepsilon_i = \varepsilon(x_i, x_{i+1}),$$

and so as before, we will have

$$\begin{aligned} Ch_s^n(Sh(\alpha\otimes\beta)) =& C_s^{n+1}\big(sh(B_{\mathcal{A}}\,\alpha\otimes\beta)\big) + (-1)^i C_s^{n+1}\big(sh(\alpha\otimes B_{\mathcal{B}}\,\beta)\big) \\ & + C_s^{n+1}\big(sh'((b+B)_{\mathcal{A}}\,\alpha\otimes\beta)\big) + (-1)^i C_s^{n+1}\big(sh'(\alpha\otimes(b+B)_{\mathcal{B}}\,\beta)\big) \end{aligned}$$

and so we have the following, noting the additional sign induced by the permutations λ, and noting that there is no additional sign induced by the permutations σ since $\mathcal{B}$ is trivially graded.

Theorem 5.3.

$$\begin{aligned} \Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) =& Ch_s^n(sh(\alpha\otimes\beta)) \\ =& C_s^{n+1}\big(sh(B\,\alpha\otimes\beta)\big) + (-1)^i C_s^{n+1}\big(sh(\alpha\otimes B\,\beta)\big) \end{aligned}$$

where

$$Ch_s^n(sh_{kl}(\alpha_k,\beta_l)) = C_s^{n+1}\big(sh(B\,\alpha\otimes\beta)\big) + (-1)^i C_s^{n+1}\big(sh(\alpha\otimes B\,\beta)\big) =$$

$$\begin{aligned} = (-1)c_{n+2,k}\Bigg(& \sum_{\substack{1\le i_0<i_1<\ldots<i_k\le\frac{n+1}{2} \\ \lambda-cyclic\ permutation}} (-1)^{\lambda+|\lambda(\alpha_k)|}\, Tr\Big(\rho(b_0)\omega(b_1,b_2)\cdots[a_{\lambda(0)},\rho(b_{2i_0-1})]\omega(b_{2i_0},b_{2i_0+1})\cdots \\ & \qquad [a_{\lambda(j)},\rho(b_{2i_j-j-1})]\omega(b_{2i_j-j},b_{2i_j-j+1})\cdots\Big) \\ & +(-1)^{k+1} \sum_{\substack{1\le i'_1<\ldots<i'_k\le\frac{n+1}{2} \\ \sigma-cyclic\ permutation}} (-1)^{\sigma}\, Tr\Big(a_0\omega(b_{\sigma(0)},b_{\sigma(1)})\cdots[a_1,\rho(b_{\sigma(2i'_1-2)})]\omega(b_{\sigma(2i'_1-1)},b_{\sigma(2i'_1)})\cdots \\ & \qquad [a_j\rho(b_{\sigma(2i'_j-j-1)})]\omega(b_{\sigma(2i'_j-j)},b_{\sigma(2i'_j-j+1)})\cdots\Big)\Bigg) \end{aligned}$$

noting that the first sum is zero if $k=\frac{n+1}{2}$,

$$|\lambda(\alpha_k)| = \sum_{\substack{i<j \\ \lambda(j)<\lambda(i)}} |a_i|\cdot|a_j|,\quad c_{n+2,k} = \frac{(-1)^{\frac{n+1}{2}+\frac{k(k-1)}{2}}}{\big(\frac{n+1}{2}\big)!},$$

and $Ch_s^n(sh_{kl}(\alpha_k,\beta_l)) = 0$ *if* $k>\frac{n+1}{2}$.

Remark 5.1. In the following sections, we will be interested in applying the duality map to a chain α such that $(b+B)_{\mathcal{A}}\alpha = 1$, but since α is not a cycle, we can not guarantee that $P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)\in\mathcal{B}^0_{per}(\mathcal{B})$

is a cocycle. We will have,

$$\begin{aligned}(b+B)_{\mathcal{B}}P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)(\beta) =& P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}((b+B)_{\mathcal{A}}\alpha)(\beta)\\ =& P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(1)(\beta)\\ =& -Ch^n_s\left(sh(1\otimes\beta)\right)\\ =& -Ch^n_s(1\otimes b_0, 1\otimes b_1, ..., 1\otimes b_n).\end{aligned}$$

Now, if $\rho=\rho^+\oplus\rho^- : \mathcal{B}\to\mathcal{L}(\mathcal{H})$, then

$$\begin{aligned}&Ch^n_s(1\otimes b_0, 1\otimes b_1, ..., 1\otimes b_n) =\\ &\qquad c_n\operatorname{Tr}\big(\gamma\varepsilon(1\otimes b_0, 1\otimes b_1)\cdots\varepsilon(1\otimes b_{n-1}, 1\otimes b_n)\big)\\ &\qquad - c_n\operatorname{Tr}\big(\gamma\varepsilon(1\otimes b_1, 1\otimes b_2)\cdots\varepsilon(1\otimes b_n, 1\otimes b_0)\big)\\ &\quad = c_n\operatorname{Tr}\Big(\gamma\big(\omega(b_0,b_1)\cdots\omega(b_{n-1}b_n)-\omega(b_1,b_2)\cdots\omega(b_n,b_0)\big)\Big)\\ &\quad = c_n\Big(\operatorname{Tr}\big(\omega^+(b_0,b_1)\cdots\omega^+(b_{n-1},b_n)\big)-\operatorname{Tr}\big(\omega^-(b_0,b_1)\cdots\omega^-(b_{n-1},b_n)\big)\Big)\\ &\qquad - c_n\Big(\operatorname{Tr}\big(\omega^+(b_1,b_2)\cdots\omega^+(b_n,b_0)-\operatorname{Tr}\big(\omega^-(b_1,b_2)\cdots\omega^-(b_n,b_0)\big)\Big)\\ &\quad = c_n\Big(\operatorname{Tr}\big(\omega^+(b_0,b_1)\cdots\omega^+(b_{n-1},b_n)\big)-\operatorname{Tr}\big(\omega^+(b_1,b_2)\cdots\omega^+(b_n,b_0)\big)\Big)\\ &\qquad - c_n\Big(\operatorname{Tr}\big(\omega^-(b_0,b_1)\cdots\omega^-(b_{n-1},b_n)\big)-\operatorname{Tr}\big(\omega^-(b_1,b_2)\cdots\omega^-(b_n,b_0)\big)\Big)\\ &\quad = Ch^n_{\rho^+}(b_0,...,b_n)-Ch^n_{\rho^-}(b_0,...,b_n).\end{aligned}$$

So if $Ch^n_{\rho^+}=Ch^n_{\rho^-}$, then $P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)$ is a cocycle. For example, we see that in the instance $\rho^+=\rho^-$, then $P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)$ is a cocycle. We also note that the boundary of $P\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\alpha)$ does not depend on the representation of $\mathcal{A}$ in $\mathcal{H}$, which leads us to our next discussion and theorem.

Let π_1,π_2 be two representations of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ and let s_1,s_2 be the corresponding maps for $\mathcal{A}\otimes\mathcal{B}$. We note that $s_1(1\otimes b)=s_2(1\otimes b)$ for all $b\in\mathcal{B}$. Let $P\Psi^n_k : \mathcal{B}^{per}_1(\mathcal{A})\to\mathcal{B}^0_{per}(\mathcal{B}),\ k=1,2$ be the corresponding chain maps. Then for α such that $(b+B)(\alpha)=1$, we will have

$$(b+B)_{\mathcal{B}}P\Psi^n_2(\alpha)-(b+B)_{\mathcal{B}}P\Psi^n_1(\alpha)=0,$$

as both chains have the same boundary as mentioned above, and so $P\Psi^n_2(\alpha)-P\Psi^n_1(\alpha)$ is a cocycle.

Theorem 5.4. *Let $\pi_k : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, $k = 1, 2$ be unital representations, and $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ an almost representation such that $[\pi_k(a), \rho(b)] \in \mathcal{L}^p$. Let $s_k : \mathcal{A} \otimes \mathcal{B} \to \mathcal{L}(\mathcal{H})$ be $s_k(a \otimes b) = \pi_k(a)\rho(b)$, and let $P\Psi^n_k$ be the corresponding duality maps, viewed as maps on chains. Let $\alpha \in \mathcal{B}^{per}_1(\mathcal{A})$ such that $(b + B)(\alpha) = 1$. Then*

(1) $\mathcal{X}^n_{1,2} = P\Psi^n_2(\alpha) - P\Psi^n_1(\alpha)$ is a cocycle.

(2) $\mathcal{X}^{n+2}_{1,2} = S\mathcal{X}^n_{1,2}$ in cohomology.

(3) If ρ' is another almost representation of $\mathcal{B}$ with corresponding map $(\mathcal{X}'_{1,2})^n$, then $\mathcal{X}^n_{1,2} = (\mathcal{X}'_{1,2})^n$ in cohomology.

Proof. We have

$$(b + B)_{\mathcal{B}} P\Psi^n_k(\alpha)(\beta) = -\left(Ch^n_{\rho^+}(b_0, ..., b_n) - Ch^n_{\rho^-}(b_0, ..., b_n)\right) \tag{5.1}$$

and $Ch^n_{s_1}(1 \otimes \beta) = Ch^n_{s_2}(1 \otimes \beta)$. The negative appears in (5.1) because of the shuffle products. Thus we have $(b + B)_{\mathcal{B}}\mathcal{X}^n_{1,2} = 0$.

We note that by proposition 5.3 from the previous section, and mentioned above, we have

$$Ch^{n+2}_{s_k} - SCh^n_{s_1} = (b + B)_{\mathcal{A}\otimes\mathcal{B}} C^{n+1}_{s_k}.$$

Now,

$$\begin{aligned}(b + B)_{\mathcal{A}\otimes\mathcal{B}} C^{n+1}_{s_k}(Sh(\alpha \otimes \beta)) =& C^{n+1}_{s_k}(Sh((b + B)_{\mathcal{A}}\alpha \otimes \beta)) - C^{n+1}_{s_k}(Sh(\alpha \otimes (b + B)_{\mathcal{B}}\beta)) \\ =& C^{n+1}_{s_k}(1 \otimes \beta)) - C^{n+1}_{s_k}(Sh(\alpha \otimes (b + B)_{\mathcal{B}}\beta))\end{aligned}$$

and $C^{n+1}_{s_1}(1 \otimes \beta)) = C^{n+1}_{s_2}(1 \otimes \beta))$. Thus,

$$\begin{aligned}\mathcal{X}^{n+2}_{1,2} - S\mathcal{X}^n_{1,2} =& C^{n+1}_{s_2}(1 \otimes \beta)) - C^{n+1}_{s_2}(Sh(\alpha \otimes (b + B)_{\mathcal{B}}\beta)) \\ & - C^{n+1}_{s_1}(1 \otimes \beta)) + C^{n+1}_{s_1}(Sh(\alpha \otimes (b + B)_{\mathcal{B}}\beta)) \\ =& C^{n+1}_{s_1}(Sh(\alpha \otimes (b + B)_{\mathcal{B}}\beta)) - C^{n+1}_{s_2}(Sh(\alpha \otimes (b + B)_{\mathcal{B}}\beta)) \\ =& (b + B)_{\mathcal{B}}\Big(C^{n+1}_{s_1}(Sh(\alpha \otimes \beta)) - C^{n+1}_{s_2}(Sh(\alpha \otimes \beta))\Big).\end{aligned}$$

The 3rd part follows from proposition 5.2. In particular, let ψ_k be such that

$$Ch^n_{s_k} - Ch^n_{s'_k} = (b+B)_{\mathcal{A}\otimes\mathcal{B}}\psi_k(Sh(\alpha\otimes(b+B)_{\mathcal{B}}\beta)).$$

Then

$$(b+B)_{\mathcal{A}\otimes\mathcal{B}}\psi_k(Sh(\alpha\otimes\beta)) = \psi_k(1\otimes\beta) - \psi_k(Sh(\alpha\otimes(b+B)_{\mathcal{B}}\beta))$$

and we will similarly have $\psi_1(1\otimes\beta) = \psi_2(1\otimes\beta)$, and so

$$\begin{aligned}\chi^n_{1,2}(\beta) - (\chi'_{1,2})^n(\beta) = P\Psi^n_2(\alpha)(\beta) - P\Psi^n_1(\alpha)(\beta) - (P\Psi')^n_2(\alpha)(\beta) + (P\Psi')^n_1(\alpha)(\beta)\\ =&Ch^n_{s_2}(Sh(\alpha\otimes\beta)) - Ch^n_{s_1}(Sh(\alpha\otimes\beta)) - Ch^n_{s'_2}(Sh(\alpha\otimes\beta)) + Ch^n_{s'_1}(Sh(\alpha\otimes\beta))\\ =&\psi_2(1\otimes\beta) - \psi_2(Sh(\alpha\otimes(b+B)_{\mathcal{B}}\beta)) - \psi_1(1\otimes\beta) + \psi_1(Sh(\alpha\otimes(b+B)_{\mathcal{B}}\beta))\\ =&(b+B)_{\mathcal{B}}(\psi_1(Sh(\alpha\otimes\beta)) - \psi_2(Sh(\alpha\otimes\beta)))\end{aligned}$$

□

Finally, we will be able to compute these cycles using similar methods as above.

Theorem 5.5. *We have*

$$\begin{aligned}\chi^n_{1,2}(\beta) =&C^{n+1}_{s_2}\big(sh(B\,\alpha\otimes\beta)\big) - C^{n+1}_{s_2}\big(sh(\alpha\otimes B\,\beta)\big)\\ &- C^{n+1}_{s_1}\big(sh(B\,\alpha\otimes\beta)\big) + C^{n+1}_{s_1}\big(sh(\alpha\otimes B\,\beta)\big).\end{aligned}$$

5.3 Applications of the Duality Map for $\mathbb{Z}/2$-Graded Algebras.

5.3.1 The Character of a Weakly Balanced Almost Fredholm Module

Definition 5.1. (An (Even) (p-Summable) Almost Fredholm Module). By an (even) (p-summable) almost Fredholm module over a unital algebra $\mathcal{B}$, we mean a triple $(\rho,\mathcal{H},F)$ such that $\mathcal{H}=\mathcal{H}^+\oplus\mathcal{H}^-$ is $\mathbb{Z}/2$ graded Hilbert space, $\rho:\mathcal{B}\to\mathcal{L}(\mathcal{H})$ is a unital almost representation by even operators, so $\rho=\rho^+\oplus\rho^-$ where $\rho^\pm:\mathcal{H}^\pm\to\mathcal{H}^\pm$ and $\omega(b_0,b_1)=\rho(b_0b_1)-\rho(b_0)\rho(b_1)\in\mathcal{L}^p$, and F is an odd operator such that $F=F^*$, $F^2=1$, $\gamma F=-F\gamma$, and $[F,\rho(b)]\in\mathcal{L}^p$ for all $b\in\mathcal{B}$.

We will always be assuming p-summability and that the Hilbert space is graded in this section, so we will drop the words even and p-summable, and simply say an almost Fredholm module.

Definition 5.2. (A Weakly Balanced Almost Representation/ A Weakly Balanced Almost Fredholm Module). By a weakly balanced almost representation of $\mathcal{B}$, we mean an almost representation such that

$$Ch^n_{\rho^+}(b_0, ..., b_n) - Ch^n_{\rho^-}(b_0, ..., b_n) = 0.$$

If $(\rho, \mathcal{H}, F)$ is an almost Fredholm module over $\mathcal{B}$ such that ρ is a weakly balanced almost representation, then we call $(\rho, \mathcal{H}, F)$ a weakly balanced almost Fredholm module over $\mathcal{B}$.

Example 5.1. Suppose $\rho : \mathcal{B} \to \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is an almost representation and $(\rho, \mathcal{H}, F)$ an almost Fredholm module.

(1) If ρ is a representation, then ρ is a weakly balanced almost representation.

(2) If $(\rho, \mathcal{H}, F)$ is a balanced Fredholm module, then it is a weakly balanced almost Fredholm module.

(3) If $\mathcal{H}^+ = \mathcal{H}^-$ and ρ is an almost representation such that $\rho^+ = \rho^-$, then ρ is a weakly balanced almost representation. In this instance, it makes sense to call $(\rho, \mathcal{H}, F)$ a balanced almost Fredholm module.

Definition 5.3. (The Chain α). Let $\mathcal{A} = \mathbb{C}[F]/(F^2 - 1)$. Then

$$\alpha = \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}(k-1)!F \otimes F^{\otimes 2k-1} = \frac{1}{2}\sum_{k=1}^{\infty} c^F_{2k-1}\alpha_{2k-1},$$

is a chain such that, $(b + B)_{\mathcal{A}}\alpha = 1$ as was shown in example 2.9 at the end of section 2.7. From now on, if we refer to α, we mean this chain.

Definition 5.4. (Character of a Weakly Balanced Almost Fredholm Module). Let $(\rho, \mathcal{H}, F)$ be a weakly balanced almost Fredholm module over $\mathcal{B}$ and $\mathcal{A} = \langle 1, F\rangle \subset \mathcal{L}(\mathcal{H})$ the subalgebra generated by 1 and F. Then we call

$$\chi_F = P\Psi^{1,n}_{\mathcal{A},\mathcal{B}}(\alpha)$$

the character of the weakly balanced almost Fredholm module. We note that this will be a cocycle in the cyclic cohomology of $\mathcal{B}$ by the weakly balanced property of ρ.

Before the next example, we remind the reader of the following cycle. Let $e \in \mathcal{B}$. Then

$$ch(e) = \sum_{m=1}^{\infty} (-1)^m \frac{(2m)!}{m!} e \otimes e^{2m} \in HC_0^{per}(\mathcal{B}).$$

Example 5.2. If ρ is a representation, then we recover Connes' character of a Fredholm module up to a difference in constant due to the difference in complex, and so we view χ_F as an appropriate generalization.

To see this, consider that $\omega(b_0, b_1) = 0$ and so we have

$$Ch_s^n(sh_{rl}(\alpha_{2k-1}, \beta_{2m})) = 0$$

unless $2k-1 = \frac{n+1}{2}$ or $2k-1 = \frac{n-1}{2}$. If $2k-1 = \frac{n+1}{2}$, then $n = 4k-3$, and as $n = 2k-1+2m$, we have $k = m+1$ and $n = 4m+1$. Then we will have

$$\begin{aligned} c^F_{2k-1} c_{n,2k-1} =& (-1)^{k-1} \frac{(k-1)!}{2} \frac{(-1)^{\frac{n-1}{2} + \frac{(2k-1)(2k-2)}{2}}}{(\frac{n-1}{2})!} \\ =& (-1)^m \frac{m!}{2} \frac{(-1)^{2m+(2m-1)m}}{(2m)!} \\ =& \frac{m!}{2(2m)!} \end{aligned}$$

and we have

$$\chi_F(\beta) = \frac{m!}{2(2m)!} \mathrm{Tr}(\gamma F [F, b_0] \cdots [F, b_{2m}]).$$

Which is Connes' character of a Fredholm module in the (b, B)-complex. We check the paring for fun, Let $e \in \mathcal{B}$ be an idempotent, then we have

$$\langle ch(e), \chi_F \rangle = \chi_F(ch(e)) = (-1)^m \frac{1}{2} \mathrm{Tr}(\gamma F [F, \rho(e)]^{2m+1}) = \langle [e], Ch_F^{2m} \rangle = \text{Index } F^+_{\rho(e)}$$

where the pairing, Ch_F^{2m}, and $F^+_{\rho(e)} = \rho(e) F|_{\rho(e)\mathcal{H}^+} : \rho(e)\mathcal{H}^+ \to \rho(e)\mathcal{H}^-$ are as in section 3.1 (definition 3.9 and proposition 3.4).

If $2k-1 = \frac{n-1}{2}$, then we will obtain the alternate variation of Connes' cocycle.

5.3.2 The Character of a Pair of Almost Fredholm Modules

We now turn our attention to general almost Fredholm modules.

If $(\rho, \mathcal{H}, F_k)$, $k = 1, 2$ are almost Fredholm modules over an algebra $\mathcal{B}$, then we will have two representations for $\mathcal{A} = \mathbb{C}[F]/(F^2 - 1)$ in $\mathcal{H}$ which we will differentiate via a subscript of $k = 1, 2$. In the previous subsection we defined $\chi_{F_k} = (P\Psi^{1,n}_{\mathcal{A},\mathcal{B}})_k(\alpha)$, but in this general case, $(P\Psi^{1,n}_{\mathcal{A},\mathcal{B}})_k(\alpha)$ is not necessarily a cocycle, but it is close. Thus, we will use the notation $X_{F_k} = (P\Psi^{1,n}_{\mathcal{A},\mathcal{B}})_k(\alpha)$ as similarly X is not necessarily a χ, but it is close. If we have a weakly balanced almost Fredholm module, then $\chi_F = X_F$, and we will use the former notation.

As we will often have two almost representations of $\mathcal{B}$ we will indicate which almost representation we are using via a superscript, so for example if ρ and ρ' are almost representations for $\mathcal{B}$ we write $X^{\rho}_{F_k}$ and $X^{\rho'}_{F_k}$.

We define the following character as the map $\chi^n_{1,2}$ defined in theorem 5.4 of the previous section.

Definition 5.5. Let $(\rho, \mathcal{H}, F_k)$, $k = 1, 2$ be almost Fredholm modules over an algebra $\mathcal{B}$. We call

$$\chi_{F_1,F_2} = \chi^n_{1,2} = X_{F_2} - X_{F_1}$$

the character of the pair of almost Fredholm Modules. Note that by theorem 5.4, $\chi^n_{1,2}$ is a cocycle in the cyclic cohomology of $\mathcal{B}$.

Example 5.3. Suppose that ρ is a representation, so that $(\rho, \mathcal{H}, F_k)$, $k = 1, 2$ are even p-summable Fredholm Modules. Then we have that

$$\langle ch(e), \chi_{F_1,F_2}\rangle = \chi_{F_1,F_2}(ch(e)) = \chi_{F_2}(ch(e)) - \chi_{F_1}(ch(e)) = \text{Index } (F_2)^+_{\rho(e)} - \text{Index } (F_1)^+_{\rho(e)}.$$

5.3.3 Pairing With Idempotents

In this subsection, we will continue our investigation of the previously defined characters. We will do this by evaluating them on the cyclic cycle associated to an idempotent $e \in \mathcal{B}$. The material of this section will follow from theorems 5.2, 5.3, and 5.4, and the following two lemmas, whose proofs appear in Appendix C.

Lemma 5.1. *Let* $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. *Suppose* $T \in \mathcal{L}(\mathcal{H})$, *such that* $T - T^2 \in \mathcal{L}^p(\mathcal{H})$ *and suppose* $T = T^*$. *Then there exists* $P \in \mathcal{L}(\mathcal{H})$ *such that* $P = P^2 = P^*$ *and* $P - T \in \mathcal{L}^p(\mathcal{H})$.

Lemma 5.2. *If* $(\rho, \mathcal{H}, F)$ *is an almost Fredholm module and* ρ' *is an (even) almost representation such that* $\rho - \rho' \in \mathcal{L}^p$, *then* $(\rho', \mathcal{H}, F)$ *is an almost Fredholm module.*

Corollary 5.1. *(1) Suppose that* $(\rho, \mathcal{H}, F_k)$, $k = 1, 2$ *are even p-summable Fredholm Modules over* $\mathcal{B}$, *and that* ρ' *is another representation of* $\mathcal{B}$ *such that* $\rho - \rho' \in \mathcal{L}^p$. *Then* $(\rho', \mathcal{H}, F_k)$, $k = 1, 2$ *are even p-summable Fredholm Modules over* $\mathcal{B}$,

$$[Ch^n_{F_2,\rho} - Ch^n_{F_1,\rho}] = [Ch^n_{F_2,\rho'} - Ch^n_{F_1,\rho'}],$$

and

$$\textit{Index } (F_2^+)_{\rho(e)} - \textit{Index } (F_1^+)_{\rho(e)} = \textit{Index } (F_2^+)_{\rho'(e)} - \textit{Index } (F_1^+)_{\rho'(e)}.$$

(2) Suppose $(\rho, \mathcal{H}, F)$ *is a balanced Fredholm module, and that* ρ' *is another representation of* $\mathcal{B}$ *such that* $\rho - \rho' \in \mathcal{L}^p$, *then*

$$[Ch^n_{F,\rho}] = [Ch^n_{F,\rho'}]$$

and

$$\textit{Index } (F^+)_{\rho(e)} = \textit{Index } (F^+)_{\rho'(e)}$$

Remark 5.2. We suspect that this was known, however we have not found it referenced anywhere. On a different note, in general Index $F^+_{\rho(e)} \neq$ Index $F^+_{\rho'(e)}$ for $\rho - \rho' \in \mathcal{L}^p$. We will provide an example below.

Proof. That $(\rho', \mathcal{H}, F_k)$, $k = 1, 2$ are Fredholm modules follows from lemma 5.1, the rest follows from part (3) of theorem 5.4 and example 5.3. $\square$

Example 5.4. Suppose we have the Fredholm module of example 3.2 of section 3.2 , and for ease of computation assume $p = 1$. Then $\rho'(e) = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$ is another representation such that $\rho - \rho' \in \mathcal{L}^1$. Then

$$\text{Index } F^+_{\rho(e)} = \text{Trace}(P - Q) \neq \text{Index } F^+_{\rho'(e)} = \text{Trace}(P - P) = 0.$$

We now change our focus back to almost Fredholm modules. Since the pairing of idempotents and the character of a Fredholm module are integers, we will consider cases where $\rho(e)$ is an almost idempotent and E is an (even) idempotent such that $\rho(e) - E \in \mathcal{L}^p$, this will give us a nice avenue to study almost idempotents.

If $\rho(e)$ is a self adjoint almost idempotent, we will call it an almost projection. In this case, lemma 5.2 guarantees there is a projection P so that $\rho(e) - P \in \mathcal{L}^p$. Further, the projection can be chosen to be an even operator since each of $\rho^{\pm}(e)$ are almost projections which have associated nearby projections $P^{\pm}$, so we may take $P = P^+ \oplus P^-$.

We suspect that lemma 5.2 can be extended to the almost idempotent case, but we do not currently have a proof for this, though admittedly we have not put much effort into finding one.

Theorem 5.6. *If $(\rho, \mathcal{H}, F_k)$, $k = 1, 2$ are almost Fredholm modules, $e \in \mathcal{B}$ is an idempotent such that $\rho(e)$ is an almost idempotent, and $E \in \mathcal{L}(\mathcal{H})$ is an even idempotent such that $\rho(e) - E \in \mathcal{L}^p$, then*

$$\langle ch(e), \chi_{F_1,F_2} \rangle = \chi_{F_1,F_2}(ch(e)) = \mathit{Index}\ (F_2^+)_E - \mathit{Index}\ (F_1^+)_E,$$

where $(F_k^+)_E : E\mathcal{H}^+ \to E\mathcal{H}^-$.

Proof. If $\mathcal{B}' = \langle 1, e \rangle$, then $(\rho|_{\mathcal{B}'}, \mathcal{H}, F_k)$, $k = 1, 2$ are almost Fredholm modules over $\mathcal{B}'$ and $\chi^{\mathcal{B}}_{F_1,F_2}(ch(e)) = \chi^{\mathcal{B}'}_{F_1,F_2}(ch(e))$, so we may take $\mathcal{B} = \langle 1, e \rangle$ for our proof.

Define $\rho' : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ by $\rho'(1) = 1$ and $\rho'(e) = E$. Then extending by linearity, ρ' is a representation of $\mathcal{B}$ in $\mathcal{L}(\mathcal{H})$ such that $\rho' - \rho \in \mathcal{L}^p$, and by lemma 5.1, $(\rho', \mathcal{H}, F_k)$, $k = 1, 2$ are Fredholm modules. Thus, by theorem 5.4 and example 5.3, we have

$$\chi^{\rho}_{F_1,F_2}(ch(e)) = \chi^{\rho'}_{F_1,F_2}(ch(e)) = \text{Index } (F_2^+)_{\rho'(e)} - \text{Index } (F_1^+)_{\rho'(e)} = \text{Index } (F_2^+)_E - \text{Index } (F_1^+)_E.$$

□

We will use the following notation, and work in the specific setting that $\mathcal{H}^+ = \mathcal{H}^- = \mathcal{H}$. We assume $\rho(e)$ is an almost projection, and $\rho'(e)$ is a projection. We will write

$$\rho(e) = \begin{bmatrix} \rho^+(e) & 0 \\ 0 & \rho^-(e) \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} = P \oplus Q, \quad \rho'(e) = \begin{bmatrix} P' & 0 \\ 0 & P' \end{bmatrix} = P' \oplus P'$$

and

$$F_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F_U = \begin{bmatrix} 0 & U^{-1} \\ U & 0 \end{bmatrix}.$$

Now, as $\rho(e)$ is an almost projection, P and Q will be as well. We we will assume $P - Q \in \mathcal{L}^p$, and that $U \in \mathcal{H}$ is a unitary such that $[U, P] \in \mathcal{L}^p$, from which it will follow that

$$[U^{-1}, P], [U, Q], [U^{-1}, Q] \in \mathcal{L}^p.$$

We will assume P' a projection such that $P - P' \in \mathcal{L}^p$, and it will follow that $Q - P' \in \mathcal{L}^p$.

Corollary 5.2. *Let $(\rho, \mathcal{H} \oplus \mathcal{H}, F_1)$, $(\rho, \mathcal{H} \oplus \mathcal{H}, F_U)$ be almost Fredholm modules over $\mathcal{B}$. Then for an idempotent $e \in \mathcal{B}$ with $\rho(e), \rho'(e)$ defined as above, we have*

$$\chi_{F_1,F_U}(ch(e)) = \text{Index } (F_U^+)_{\rho'(e)} - \text{Index } (F_1^+)_{\rho'(e)} = \text{Index } (F_U^+)_{\rho'(e)} = \text{Index } P'UP' : P'\mathcal{H} \to P'\mathcal{H}.$$

If $\rho^+ = \rho^-$, then

$$\chi_{F_1,F_U}(ch(e)) = \chi_{F_U}(ch(e)) = \text{Index } P'UP' : P'\mathcal{H} \to P'\mathcal{H}.$$

We note that in this second case, $(\rho, \mathcal{H} \oplus \mathcal{H}, F_U)$ is a weakly balanced almost Fredholm module, and χ_{F_U} is its character.

Proof. Since $(\rho')^+(e) = (\rho')^-(e) = P'$, we have $[F_1, \rho'(e)] = 0$. Now, each term of $\chi^{\rho'}_{F_1}(ch(e))$ will have a copy of $[F_1, \rho'(e)]$ (see theorem 5.3), and so $\chi^{\rho'}_{F_1}(ch(e)) = 0$, and so the first equality follows.

If $\rho^+ = \rho^-$, then we will similarly have $\chi^{\rho}_{F_1}(ch(e)) = 0$, and the second equality will follow. □

Proposition 5.4. Using the notation from above, if $\rho(e) = E$ and $* = 1, U$

$$\begin{aligned}
&\chi_{F_1,F_U}(ch(e)) = X_{F_U} - X_{F_1} \\
&X_{F_*}(ch(e)) = \sum_{\substack{k=2m+1 \\ k\le \frac{n+1}{2}}} C_{n,k}\Big((n-k+1)\Omega_{n,k} + (k+1)\Omega'_{n,k}\Big) \\
&\Omega_{n,k} = (-1)^{\frac{k+1}{2}} \sum_{\substack{2(j_0+\ldots+j_k) \\ =n-2k+1}} \mathrm{Tr}(\gamma F_*(E-E^2)^{j_0}[F_*,E](E-E^2)^{j_1}\cdots[F_*,E](E-E^2)^{j_k}) \\
&\Omega'_{n,k} = (-1)^{\frac{k+1}{2}} \sum_{\substack{2(j'_0+\ldots+j'_{k+1}) \\ =n-2k-1}} \mathrm{Tr}(\gamma E(E-E^2)^{j'_0}[F_*,E](E-E^2)^{j'_1}\cdots[F_*,E](E-E^2)^{j'_{k+1}}) \\
&\Omega'_{n,\frac{n+1}{2}} = 0 \\
&C_{n,k} = -\frac{\left(\frac{k-1}{2}\right)!(n-k)!}{2\left(\frac{n+1}{2}\right)!\left(\frac{n-k}{2}\right)!}.
\end{aligned}$$

Further, we may reduce to the following

(1) If $\rho^+(e) = P$, $\rho^-(e) = Q$, $* = U$, and letting $G = U^{-1}QU$ then

$$\begin{aligned}
&\mathrm{Tr}(\gamma F_U(E-E^2)^{j_0}[F_U,E](E-E^2)^{j_1}\cdots[F_U,E](E-E^2)^{j_k}) = \\
&\qquad (-1)^{\frac{k+1}{2}}\mathrm{Tr}((G-G^2)^{j_0}(G-P)(P-P^2)^{j_1}(G-P)\cdots(G-P)(P-P^2)^{j_k}) \\
&\qquad +(-1)^{\frac{k+1}{2}}\mathrm{Tr}((P-P^2)^{j_0}(G-P)(G-G^2)^{j_1}(G-P)\cdots(G-P)(G-G^2)^{j_k}) \\
&\mathrm{Tr}(\gamma E(E-E^2)^{j'_0}[F_U,E](E-E^2)^{j'_1}\cdots[F_U,E](E-E^2)^{j'_{k+1}}) = \\
&\qquad (-1)^{\frac{k+1}{2}}\mathrm{Tr}((G-P)(G-G^2)^{j'_0}(G-P)(P-P^2)^{j'_1}\cdots(G-P)(P-P^2)^{j'_{k+1}})
\end{aligned}$$

(2) If $\rho^+(e) = P$, $\rho^-(e) = Q$, and $* = 1$ then

$$\begin{aligned}
&\mathrm{Tr}(\gamma F_1(E-E^2)^{j_0}[F_1,E](E-E^2)^{j_1}\cdots[F_1,E](E-E^2)^{j_k}) = \\
&\qquad (-1)^{\frac{k+1}{2}}\mathrm{Tr}((Q-Q^2)^{j_0}(Q-P)(P-P^2)^{j_1}(Q-P)\cdots(Q-P)(P-P^2)^{j_k}) \\
&\qquad +(-1)^{\frac{k+1}{2}}\mathrm{Tr}((P-P^2)^{j_0}(Q-P)(Q-Q^2)^{j_1}(Q-P)\cdots(Q-P)(Q-Q^2)^{j_k}) \\
&\mathrm{Tr}(\gamma F_1(E-E^2)^{j'_0}[F_1,E](E-E^2)^{j'_1}\cdots[F_1,E](E-E^2)^{j'_{k+1}}) = \\
&\qquad (-1)^{\frac{k+1}{2}}\mathrm{Tr}((Q-P)(Q-Q^2)^{j'_0}(Q-P)(P-P^2)^{j'_1}\cdots(Q-P)(P-P^2)^{j'_{k+1}})
\end{aligned}$$

Proof. We obtain this by plugging into theorem 5.3/ 5.4. We let $k+l=n$, noting k is odd and l is even. Let $c_k^F = (-1)^{\frac{k-1}{2}}\frac{1}{2}\left(\frac{k-1}{2}\right)!$ be the constant for α_k, and $c_l^E = (-1)^{\frac{l}{2}}\frac{l!}{\frac{l}{2}!}$ the constant term for $ch(e)_l$. We will write $\rho(e) = E$ and $\omega_E = \omega(e,e) = E - E^2$ to save space. We let $\operatorname{sgn}(\lambda) = (-1)^\lambda$. We are computing

$$
\begin{aligned}
Ch_s^n(sh_{kl}(\alpha_k,\beta_l)) &= C_s^{n+1}\big(sh(B\,\alpha\otimes\beta)\big) - C_s^{n+1}\big(sh(\alpha\otimes B\,\beta)\big) \\
&= (-1)c_{n+2,k}c_k^F c_l^E\Bigg(\sum_{\substack{1\leq i'_0<i'_1<\ldots<i'_k\leq\frac{n+1}{2}\\ \lambda-\text{cyclic permutation}}} (-1)^\lambda(-1)^{|\lambda(\alpha_k)|}\mathrm{Tr}\Big(E\omega_E\cdots[F,E]\omega_E\cdots\Big) \\
&\qquad + \sum_{\substack{1\leq i_0<i_1<\ldots<i_k\leq\frac{n+1}{2}\\ \sigma-\text{cyclic permutation}}} (-1)^\sigma\mathrm{Tr}\Big(F\omega_E\cdots[F,E]\omega_E\cdots\Big)\Bigg) \\
&= (-1)c_{n+2,k}c_k^F c_l^E\Bigg(\sum_{\substack{2(j'_0+\ldots+j'_{k+1})=n-2k-1\\ \lambda-\text{cyclic permutation}}} (-1)^\lambda(-1)^{|\lambda(\alpha_k)|}\mathrm{Tr}\Big(E\omega_E^{j'_0}[F,E]\omega_E^{j'_1}\cdots[F,E]\omega_E^{j'_{k+1}}\Big) \\
&\qquad + \sum_{\substack{2(j_0+\ldots+j_k)=n-2k+1\\ \sigma-\text{cyclic permutation}}} (-1)^\sigma\mathrm{Tr}\Big(F\omega_E^{j_0}[F,E]\omega_E^{j_1}\cdots[F,E]\omega_E^{j_k}\Big)\Bigg)
\end{aligned}
$$

As k is odd, we will have $\lambda\in S_{k+1}$ are possibly odd or even. We let $\lambda = (0\ 1\ \ldots\ k+1)$. Then $(-1)^{\lambda^i} = (-1)^{i \bmod 2}$. Now,

$$
|\lambda^i(\alpha_k)| = \sum_{j=0}^{i-1}|F|\sum_{j=i}^{k}|F| = i\cdot k = i \bmod 2,
$$

so $(-1)^{\lambda^i}(-1)^{|\lambda^i(\alpha_k)|} = 1$. As $2m$ is even, $\sigma\in S_{2m+1}$ is even, and so $(-1)^\sigma = 1$. Thus, the above sums reduce further to

$$
\begin{aligned}
(-1)c_{n+2,k}c_k^F c_l^E\Bigg(&\sum_{\substack{1\leq i'_0<i'_1<\ldots<i'_k\leq\frac{n+1}{2}\\ \lambda-\text{cyclic permutation}}} (k+1)\mathrm{Tr}\Big(E\omega\cdots[F,E]\omega_E\cdots\Big) \\
&+ \sum_{\substack{1\leq i_0<i_1<\ldots<i_k\leq\frac{n+1}{2}\\ \sigma-\text{cyclic permutation}}} (l+1)\mathrm{Tr}\Big(F\omega_E\cdots[F,E]\omega_E\cdots\Big)\Bigg)
\end{aligned}
$$

Finally, we compute $(-1)c_{n+2,k}c_k^F c_l^E(k+1), (-1)c_{n+2,k}c_k^F c_l^E(l+1)$ in terms of n and k. We have $l = n-k$. Thus,

$$\begin{align*}
(-1)c_{n+2,k}c_k^F c_l^E(k+1) =& (-1)c_{n+2,k}c_k^F c_{n-k}^E \\
=& (-1)(-1)^{\frac{n+1}{2}+\frac{k(k-1)}{2}}\frac{1}{(\frac{n+1}{2})!}(-1)^{\frac{k-1}{2}}\frac{1}{2}\left(\frac{k-1}{2}\right)!(-1)^{\frac{l}{2}}\frac{l!}{(\frac{l}{2})!} \\
=& (-1)^{1+\frac{n+1}{2}+\frac{k(k-1)}{2}+\frac{k-1}{2}}\frac{\left(\frac{k-1}{2}\right)!}{2(\frac{n+1}{2})!}(-1)^{\frac{n-k}{2}}\frac{(n-k)!}{(\frac{n-k}{2})!} \\
=& (-1)^{1+\frac{k(k+1)}{2}}\frac{(\frac{k-1}{2})!(n-k)!}{2(\frac{n+1}{2})!(\frac{n-k}{2})!} \\
=& (-1)^{1+\frac{k+1}{2}}\frac{(\frac{k-1}{2})!(n-k)!}{2(\frac{n+1}{2})!(\frac{n-k}{2})!} \tag{5.2} \\
=& (-1)^{\frac{k+1}{2}}C_{n,k}
\end{align*}$$

with (5.2) holding as k is odd.

Now, let $* = U$. As we are to compute

$$\mathrm{Tr}(\gamma F_U(E-E^2)^{j_0}[F_U,E](E-E^2)^{j_1}\cdots[F_U,E](E-E^2)^{j_k}),$$

we will write the above in matrix notation, and find the products

$$\gamma F_U(E-E^2)^{j_0}[F_U,E](E-E^2)^{j_1}$$

and

$$[F_U,E](E-E^2)^{j_{2r}}[F_U,E](E-E^2)^{j_{2r+1}}.$$

Recall, we let $G = U^{-1}QU$. We have

$$\begin{aligned}
(E - E^2)^j &= \left(\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}^2 \right)^j \\
&= \begin{bmatrix} (P - P^2)^j & 0 \\ 0 & (Q - Q^2)^j \end{bmatrix} \\
[F_U, E] &= \begin{bmatrix} 0 & U^{-1} \\ U & 0 \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} 0 & U^{-1} \\ U & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & U^{-1}Q - PU^{-1} \\ UP - QU & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & (G - P)U^{-1} \\ -U(G - P) & 0 \end{bmatrix} \\
U^{-1}(Q - Q^2)^j U &= U^{-1}(Q - Q^2)(Q - Q^2) \cdots (Q - Q^2)U \\
&= U^{-1}(Q - Q^2)UU^{-1}(Q - Q^2)UU^{-1} \cdots UU^{-1}(Q - Q^2)U \\
&= (U^{-1}QU - U^{-1}QUU^{-1}QU)^j \\
&= (G - G^2)^j
\end{aligned}$$

So,

$$\begin{aligned}
&\gamma F_U (E - E^2)^{j_0} [F_U, E] (E - E^2)^{j_1} = \\
&\gamma F_U \begin{bmatrix} (P - P^2)^{j_0} & 0 \\ 0 & (Q - Q^2)^{j_0} \end{bmatrix} \begin{bmatrix} 0 & (G - P)U^{-1} \\ -U(G - P) & 0 \end{bmatrix} \begin{bmatrix} (P - P^2)^{j_1} & 0 \\ 0 & (Q - Q^2)^{j_1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & U^{-1} \\ -U & 0 \end{bmatrix} \begin{bmatrix} -U^{-1}(Q - Q^2)^{j_0} U (G - P)(P - P^2)^{j_1} & 0 \\ 0 & U(P - P^2)^{j_0}(G - P)U^{-1}(Q - Q^2)^{j_1} \end{bmatrix} \\
&= \begin{bmatrix} -(G - G^2)^{j_0}(G - P)(P - P^2)^{j_1} & 0 \\ 0 & -U(P - P^2)^{j_0}(G - P)U^{-1}(Q - Q^2)^{j_1} \end{bmatrix}
\end{aligned}$$

and

$$[F_U, E](E - E^2)^{j_{2r}}[F_U, E](E - E^2)^{j_{2r+1}} =$$

$$\begin{bmatrix} -(G-P)(G-G^2)^{j_{2r}}(G-P)(P-P^2)^{j_{2r+1}} & 0 \\ 0 & -U(G-P)(P-P^2)^{j_{2r}}(G-P)U^{-1}(Q-Q^2)^{j_{2r+1}} \end{bmatrix}$$

Now, there are $\frac{k-1}{2}$ terms of the above form, so we have

$$\begin{aligned}
&\mathrm{Tr}\Big(\gamma F_U(E-E^2)^{j_0}[F_U,E](E-E^2)^{j_1}\cdots[F_U,E](E-E^2)^{j_k}\Big) = \\
&\qquad (-1)^{\frac{k+1}{2}}\mathrm{Tr}\Big((G-G^2)^{j_0}(G-P)(P-P^2)^{j_1}\cdots(G-P)(P-P^2)^{j_k}\Big) \\
&\qquad +(-1)^{\frac{k+1}{2}}\mathrm{Tr}\Big(U(P-P^2)^{j_0}(G-P)U^{-1}(Q-Q^2)^{j_1}U\cdots(G-P)U^{-1}(Q-Q^2)^{j_k}\Big) \\
&\qquad =(-1)^{\frac{k+1}{2}}\mathrm{Tr}\Big((G-G^2)^{j_0}(G-P)(P-P^2)^{j_1}\cdots(G-P)(P-P^2)^{j_k}) \\
&\qquad +(-1)^{\frac{k+1}{2}}\mathrm{Tr}\Big((P-P^2)^{j_0}(G-P)(G-G^2)^{j_1}\cdots(G-P)U^{-1}(Q-Q^2)^{j_k}U\Big) \\
&\qquad =(-1)^{\frac{k+1}{2}}\mathrm{Tr}\Big((G-G^2)^{j_0}(G-P)(P-P^2)^{j_1}\cdots(G-P)(P-P^2)^{j_k}\Big) \\
&\qquad +(-1)^{\frac{k+1}{2}}\mathrm{Tr}\Big((P-P^2)^{j_0}(G-P)(G-G^2)^{j_1}\cdots(G-P)(G-G^2)^{j_k}\Big)
\end{aligned}$$

That

$$\begin{aligned}
&\mathrm{Tr}(\gamma E(E-E^2)^{j'_0}[F_U,E](E-E^2)^{j'_1}\cdots[F_U,E](E-E^2)^{j'_{k+1}}) = \\
&\qquad (-1)^{\frac{k+1}{2}}\mathrm{Tr}(G(G-G^2)^{j'_0}(G-P)(P-P^2)^{j'_1}(G-P)\cdots(G-P)(P-P^2)^{j'_{k+1}}) \\
&\qquad -(-1)^{\frac{k+1}{2}}\mathrm{Tr}(P(P-P^2)^{j'_0}(G-P)(G-G^2)^{j'_1}(G-P)\cdots(G-P)(G-G^2)^{j'_{k+1}}) \\
&\qquad =(-1)^{\frac{k+1}{2}}\mathrm{Tr}((G-P)(G-G^2)^{j'_0}(G-P)(P-P^2)^{j'_1}\cdots(G-P)(P-P^2)^{j'_{k+1}})
\end{aligned}$$

follows similarly.

Lastly, (2) is a special case of (1) where $U = 1$. $\square$

Corollary 5.3. *If $\rho^+(e) = \rho^-(e) = P$, $G = U^{-1}PU$, and P' is a projection such that $P - P' \in \mathcal{L}^p$,* *then we have*

$$\begin{aligned}
\chi_{F_1,F_U}(ch(e)) =&\chi_{F_U}(ch(e)) = \mathit{Index}\ P'UP' : P'\mathcal{H} \to P'\mathcal{H} \\
=& \sum_{\substack{k=2m+1 \\ k\leq \frac{n+1}{2}}} C_{n,k}\Big((n-k+1)\Omega_{n,k} + (k+1)\Omega'_{n,k}\Big) \\
\Omega_{n,k} =& \sum_{\substack{2(j_0+\ldots+j_k) \\ =n-2k+1}} \Bigg(Tr\Big((G-G^2)^{j_0}(G-P)(P-P^2)^{j_1}\cdots(G-P)(P-P^2)^{j_k}\Big) \\
&+ Tr\Big((P-P^2)^{j_0}(G-P)(G-G^2)^{j_1}\cdots(G-P)(G-G^2)^{j_k}\Big)\Bigg) \\
\Omega'_{n,k} =& \sum_{\substack{2(j'_0+\ldots+j'_{k+1}) \\ =n-2k-1}} Tr((G-P)(G-G^2)^{j'_0}(G-P)(P-P^2)^{j'_1}\cdots(G-P)(P-P^2)^{j'_{k+1}}) \\
\Omega'_{n,\frac{n+1}{2}} =&0 \\
C_{n,k} =& -\frac{\left(\frac{k-1}{2}\right)!(n-k)!}{2\left(\frac{n+1}{2}\right)!\left(\frac{n-k}{2}\right)!}.
\end{aligned}$$

Remark 5.3. (About Notation). We will compare the above result to a result in [2]. They use the notation $P_U = UPU^{-1}$, whereas we have used $G = U^{-1}PU$ and so $G = P_{U^{-1}}$ in their notation. As it is undesirable to rewrite all of the preceding proofs and statements, we will simply write the previous corollary in their notation, and make the change from $P_{U^{-1}}$ to P_U.

Corollary 5.4. *If $\rho^+(e) = \rho^-(e) = P$, $P_U = UPU^{-1}$, and P' is a projection such that $P - P' \in \mathcal{L}^p$,* *then we have*

$$\begin{aligned}
\chi_{F_1,F_U}(ch(e)) &= \chi_{F_U}(ch(e)) = \mathit{Index}\ P'UP' : P'\mathcal{H} \to P'\mathcal{H} \\
&= \sum_{\substack{k=2m+1\\ k\leq \frac{n+1}{2}}} C_{n,k}\Big((n-k+1)\Omega_{n,k} + (k+1)\Omega'_{n,k}\Big) \\
\Omega_{n,k} &= \sum_{\substack{2(j_0+\ldots+j_k)\\ =n-2k+1}} \Bigg(Tr\Big((P-P^2)^{j_0}(P-P_U)(P_U-P_U^2)^{j_1}\cdots(P-P_U)(P_U-P_U^2)^{j_k}\Big) \\
&\qquad + Tr\Big((P_U-P_U^2)^{j_0}(P-P_U)(P-P^2)^{j_1}\cdots(P-P_U)(P-P^2)^{j_k}\Big)\Bigg) \\
\Omega'_{n,k} &= \sum_{\substack{2(j'_0+\ldots+j'_{k+1})\\ =n-2k-1}} Tr\Big((P-P_U)(P-P^2)^{j'_0}(P-P_U)(P_U-P_U^2)^{j'_1}\cdots(P-P_U)(P_U-P_U^2)^{j'_{k+1}}\Big) \\
\Omega'_{n,\frac{n+1}{2}} &= 0 \\
C_{n,k} &= -\frac{\left(\frac{k-1}{2}\right)!(n-k)!}{2\left(\frac{n+1}{2}\right)!\left(\frac{n-k}{2}\right)!}.
\end{aligned}$$

We may restate the above as follows,

Corollary 5.5. *If P is an almost projection, U is a unitary such that $[U,P] \in \mathcal{L}^p$, P' is a projection such that $P - P' \in \mathcal{L}^p$, and n is odd so that $n \geq 2p-1$, then we define Index (P,U) as*

$$\mathit{Index}\ (P,U) = \sum_{\substack{k=2m+1\\ k\leq \frac{n+1}{2}}} C_{n,k}\Big((n-k+1)\Omega_{n,k} + (k+1)\Omega'_{n,k}\Big) = \mathit{Index}\ P'UP' : P'\mathcal{H} \to P'\mathcal{H}.$$

Corollary 5.6. *If P is an almost projection, U is a unitary such that $[U,P] \in \mathcal{L}^p$, P' is a projection such that $P - P' \in \mathcal{L}^p$, and n is odd so that $n \geq 2p-1$, then*

$$\sum_{\substack{k=2m+1\\ k\leq \frac{n+1}{2}}} C_{n,k}\Big((n-k+1)\Omega_{n,k} + (k+1)\Omega'_{n,k}\Big)$$

is an integer.

Remark 5.4. This as a generalization of theorem 5.3 in [2], and so we have taken their notation. Part of their theorem states that if P is a projection and U is a unitary such that $[P,U] \in \mathcal{L}^{2m+1}$, then

$$\text{Trace}((P - UPU^{-1})^{2m+1}) = -\text{Index } PUP : P\mathcal{H} \to P\mathcal{H}.$$

We will show that we recover this result under these assumptions, ie P is a projection and U is a unitary such that $[P,U] \in \mathcal{L}^{2m+1}$.

Indeed, suppose first that $\frac{n+1}{2}$ is odd. Then $\Omega'_{n,k} = \Omega_{n,k} = 0$ if $k < \frac{n+1}{2}$ as each term will contain a copy of $P - P^2 = 0$ or $P_U - P_U^2 = 0$. Thus we consider $k = \frac{n+1}{2} = 2m+1$, and we will have $\Omega'_{n,\frac{n+1}{2}} = 0$, and as $j_0 = ... = j_{\frac{n+1}{2}} = 0$,

$$\Omega_{\frac{n+1}{2}} = 2\mathrm{Tr}((P - P_U)^k) = 2\mathrm{Tr}((P - P_U)^{2m+1}).$$

Now,

$$\begin{aligned}(n-k+1)C_k &= \left(\frac{n+1}{2}\right) C_{\frac{n+1}{2}} \\ &= -\left(\frac{n+1}{2}\right) \frac{\left(\frac{n-1}{4}\right)! \left(\frac{n+1}{2} - 1\right)!}{2\left(\frac{n+1}{2}\right)! \left(\frac{n-1}{4}\right)!} \\ &= -\frac{\left(\frac{n-1}{4}\right)! \left(\frac{n+1}{2}\right)!}{2\left(\frac{n+1}{2}\right)! \left(\frac{n-1}{4}\right)!} \\ &= -\frac{1}{2}\end{aligned}$$

As we are taking $P' = P$, we obtain the desired result,

$$-\mathrm{Tr}((P - P_U)^{2m+1}) = \text{Index } PUP.$$

If $\frac{n-1}{2}$ is odd, then similarly we will have $\Omega'_{n,k} = \Omega_{n,k} = 0$ if $k < \frac{n-1}{2}$ contain a copy of $P - P^2 = 0$ or $P_U - P_U^2 = 0$. In this case, $\Omega_{\frac{n-1}{2}} = 0$ as it too must have a copy of $P - P^2$ or $P_U - P_U^2$. Now, letting $2m - 1 = \frac{n-1}{2}$ we have

$$\begin{aligned}\Omega'_{\frac{n-1}{2}} &= \mathrm{Tr}((P - P_U)^{2m+1}) = \mathrm{Tr}(P(P - P_U)^{2m} - \mathrm{Tr}(P_U(P - P_U)^{2m}) \\ (k+1)C_{n,k} &= \left(\frac{n+1}{2}\right) C_{\frac{n-1}{2}} = -1.\end{aligned}$$

If in addition we have $\frac{n-1}{2} \geq p$, then $\Omega'_{\frac{n-1}{2}}$ is the map γ_n in [2]. To see this more explicitly, we recreate their proof,

$$\begin{aligned}
\mathrm{Tr}(P(P-P_U)^{2m} - \mathrm{Tr}(P_U(P-P_U)^{2m}) =& \mathrm{Tr}(P(P-UPU^{-1})^{2m}) - \mathrm{Tr}(UPU^{-1}(P-UPU^{-1})^{2m}) \\
=& \mathrm{Tr}(P(P-UPU^{-1})^{2m}) - \mathrm{Tr}(P(U^{-1}PU-P)^{2m}) \\
=& \mathrm{Tr}(P^m(P-PUPU^{-1}-UPU^{-1}P+UPU^{-1})^m) \\
& - \mathrm{Tr}(P^m(P-PU^{-1}PU-U^{-1}PUP+U^{-1}PU)^m) \\
=& \mathrm{Tr}((P-PUPU^{-1}-PUPU^{-1}P+PUPU^{-1})^m) \\
& - \mathrm{Tr}((P-PU^{-1}PU-PU^{-1}PUP+PU^{-1}PU)^m) \\
=& \mathrm{Tr}((P-PUPU^{-1}P)^m) - \mathrm{Tr}((P-PU^{-1}PUP)^m).
\end{aligned}$$

We will now do a few explicit examples to serve as sanity check, and for the fun of computation.

Example 5.5. If we are working over a finite dimensional Hilbert space, then every operator is almost the identity. Hence, it must be that Index $(P,U)=0$ for all matrices P and unitaries U. We will check our formula for $n=1$ and $n=3$.

If $n=1$, then we only must consider $k=1$, and so we compute

$$C_{1,1}\Omega_{1,1} + 2C_{1,1}\Omega'_{1,1}.$$

As $\frac{1+1}{2}=1$, $\Omega'_{1,1}=0$, $C_{1,1}=-\frac{1}{2}$ and

$$\Omega_{1,1} = 2\mathrm{Tr}(P-P_U) = \mathrm{Tr}(P-UPU^{-1}) = \mathrm{Tr}(P-P) = 0,$$

and so we obtain the expected result.

Now for $n=3$, we similarly need only consider $k=1$, and so we compute

$$3C_{3,1}\Omega_{3,1} + 2C_{3,2}\Omega'_{3,1}.$$

We have

$$\begin{aligned}
C_{3,1} =& -\frac{1}{2}\\
\Omega_{3,1} =& \sum_{2(j_0+j_1)=2} \mathrm{Tr}((P-P^2)^{j_0}(P-P_U)(P_U-P_U^2)^{j_1}) + \mathrm{Tr}((P_U-P_U^2)^{j_0}(P-P_U)(P-P^2)^{j_1})\\
=&\mathrm{Tr}((P-P^2)(P-P_U)) + \mathrm{Tr}((P_U-P_U^2)(P-P_U))\\
&+ \mathrm{Tr}((P-P_U)(P_U-P_U^2)) + \mathrm{Tr}((P-P_U)(P-P^2))\\
=&2\mathrm{Tr}((P-P_U)(P-P^2)) + 2\mathrm{Tr}((P-P_U)(P_U-P_U^2))\\
\Omega'_{3,1} =&\mathrm{Tr}((P-P_U)^3)
\end{aligned}$$

Keeping in mind that everything is trace class, we have,

$$\begin{aligned}
3C_{3,1}\Omega_{3,1} + 2C_{3,2}\Omega'_{3,1} =& -\mathrm{Tr}(3(P-P_U)(P-P^2) + 3(P-P_U)(P_U-P_U^2) + (P-P_U)^3)\\
=& -\mathrm{Tr}((P-P_U)(3P-3P^2+3P_U-3P_U^2+P^2-PP_U-P_UP+P_U^2))\\
=& -\mathrm{Tr}((P-P_U)(3P+3P_U-2P^2-2P_U^2-PP_U-P_UP))\\
=& -\mathrm{Tr}(3P^2+3PP_U-2P^3-2PP_U^2-P^2P_U-PP_UP\\
&\qquad -3P_UP-3P_U^2+2P_UP^2+2P_U^3+P_UPP_U+P_U^2P)\\
=& -\mathrm{Tr}(3P^2+3PP_U-2P^3-2PP_U^2-P^2P_U-P^2P_U\\
&\qquad -3PP_U-3P_U^2+2P^2P_U+2P_U^3+PP_U^2+PP_U^2)\\
=& -\mathrm{Tr}(3P^2-3P_U^2+2P_U^3-2P^3)\\
=& -\mathrm{Tr}(3P^2-3UP^2U^{-1}+2UP^3U^{-1}-2P^3)\\
=& -\mathrm{Tr}(3P^2-3P^2+2P^3-2P^3)\\
=&0
\end{aligned}$$

Next we will consider a case with a non-zero index.

Example 5.6. Let $\mathcal{H} = \ell^2(\mathbb{Z})$, U the right shift $P_{\mathbb{N}}$ the projection onto $\ell^2(\mathbb{N})$, P_0 the projection onto the zeroth component and $P = P_{\mathbb{N}} - \frac{1}{2}P_0$. Then $P - P_{\mathbb{N}} = \frac{1}{2}P_0 \in \mathcal{L}^1$, and according to our theorem Index $(P, U) =$ Index $P_{\mathbb{N}}UP_{\mathbb{N}} = -1$. We will check this for $n = 1$ and $n = 3$.

For $n = 1$ we only must consider $k = 1$, and so we compute

$$C_{1,1}\Omega_{1,1} + 2C_{1,1}\Omega'_{1,1}.$$

As $\frac{1+1}{2} = 1$, $\Omega'_{1,1} = 0$, $C_{1,1} = \frac{-1}{2}$ and $\Omega_{1,1} = 2\mathrm{Tr}(P - P_U)$. Now

$$P(e_i) = \begin{cases} 0, & i < 0 \\ \frac{1}{2}e_0, & i = 0 \\ e_i, & i > 0 \end{cases} \text{ and } P_U(e_i) = UPU^{-1}(e_i) = \begin{cases} 0, & i < 1 \\ \frac{1}{2}e_1, & i = 1 \\ e_i, & i > 1 \end{cases}$$

so

$$\begin{aligned} -\mathrm{Tr}(P - P_U) &= -\sum_{-\infty}^{\infty}\langle (P - P_U)(e_i), e_i\rangle \\ &= -\langle (P - P_U)(e_0), e_0\rangle - \langle (P - P_U)(e_1), e_1\rangle \\ &= -\frac{1}{2} - (1 - \frac{1}{2}) \\ &= -1. \end{aligned}$$

as desired.

For $n = 3$, we will similarly have

$$\begin{aligned} 3C_{3,1}\Omega_{3,1} + 2C_{3,2}\Omega'_{3,1} = &- \mathrm{Tr}(3P^2 + 3PP_U - 2P^3 - 2PP_U^2 - P^2P_U - PP_UP \\ &- 3P_UP - 3P_U^2 + 2P_UP^2 + 2P_U^3 + P_UPP_U + P_U^2P) \end{aligned}$$

We note that $P_UP = PP_U = P_U$ which leaves us with

$$\begin{aligned} 3C_{3,1}\Omega_{3,1} + 2C_{3,2}\Omega'_{3,1} &= -\mathrm{Tr}(3P^2 - 2P^3 - 3P_U^2 + 2P_U^3) \\ &= -\sum_{-\infty}^{\infty}\langle (3P^2 - 2P^3 - 3P_U^2 + 2P_U^3)(e_i), e_i\rangle \\ &= -\langle (3P^2 - 2P^3 - 3P_U^2 + 2P_U^3)(e_0), e_0\rangle \\ &\quad - \langle (3P^2 - 2P^3 - 3P_U^2 + 2P_U^3)(e_1), e_1\rangle \\ &= -\left(\frac{3}{4} - \frac{1}{4}\right) - \left(3 - 2 - \frac{3}{4} + \frac{1}{4}\right) \\ &= -1 \end{aligned}$$

As a final example, we will do a similar computation as above but with an $\mathcal{L}^3$ perturbation that is not $\mathcal{L}^2$.

Example 5.7. Let $\mathcal{H} = \ell^2(\mathbb{Z})$, U the right shift $P_{\mathbb{N}}$ the projection onto $\ell^2(\mathbb{N})$, define D by $D(e_i) = \frac{1}{\sqrt{i}}e_i$ for $i \geq 1$ and zero otherwise, and $P = P_{\mathbb{N}} - D$. Then $P - P_{\mathbb{N}} = -D \in \mathcal{L}^3$, and according to our theorem Index $(P, U) =$ Index $P_{\mathbb{N}}UP_{\mathbb{N}} = -1$. We will check this for $n = 3$. As before we have

$$\begin{aligned} 3C_{3,1}\Omega_{3,1} + 2C_{3,2}\Omega'_{3,1} = &- \operatorname{Tr}(3P^2 + 3PP_U - 2P^3 - 2PP_U^2 - P^2P_U - PP_UP \\ &\qquad - 3P_UP - 3P_U^2 + 2P_UP^2 + 2P_U^3 + P_UPP_U + P_U^2P) \\ = &- \operatorname{Tr}(3P^2 - 2P^3 - 3P_U^2 + 2P_U^3). \end{aligned}$$

as $P_UP = PP_U$. Now, P and P_U are diagonal operators, so their squares and cubes are as well, and they are the following for $i \geq 0$, noting they are zero for $i < 0$

$$\begin{aligned} &3P^2(e_0) = 3e_0,\ 3P^2(e_i) = 3\left(1 - \frac{1}{\sqrt{i}}\right)^2 e_i \\ &2P^3(e_0) = 2e_0,\ 2P^3(e_i) = 2\left(1 - \frac{1}{\sqrt{i}}\right)^3 e_i \\ &3P_U^2(e_0) = 3e_0,\ 3P_U^2(e_1) = 3e_1\, 3P_U^2(e_i) = 3\left(1 - \frac{1}{\sqrt{i-1}}\right)^2 e_i \\ &2P^2(e_0) = 3e_0,\ 2P_U^2(e_1) = 2e_1\, 2P_U^2(e_i) = 2\left(1 - \frac{1}{\sqrt{i-1}}\right)^2 e_i \end{aligned}$$

So we have

$$\sum_{i=0}^{m}\langle(3P^2 - 2P^3 - 3P_U^2 + 2P_U^3)e_i, e_i\rangle = 3\left(1 - \frac{1}{\sqrt{m}}\right)^2 - 2\left(1 - \frac{1}{\sqrt{m}}\right)^3 \overset{m\to\infty}{\longrightarrow} 1.$$

Hence,

$$-\operatorname{Tr}(3P^2 - 2P^3 - 3P_U^2 + 2P_U^3) = -1$$

as desired.

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Appendix A

Additional Proofs for Section 3.4

Proposition A.1. Let $\mathcal{A}$ be unital $\mathbb{C}$ algebra, $\mathcal{H}$ a Hilbert space, and $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ a unital linear map which is multiplicative modulo $\mathcal{L}^p(\mathcal{H})$, $\varepsilon(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$,and let $n \geq 2p - 1$ be odd. We use the notation $\varepsilon_i = \varepsilon(a_i, a_{i+1})$ modulo the appropriate index. Then,

(1) $\phi_n(a_0, ..., a_n) = \mathrm{Tr}(\varepsilon_0 \cdots \varepsilon_{n-1}) - \mathrm{Tr}(\varepsilon_n \varepsilon_1 \cdots \varepsilon_{n-2})$ is a cocycle in in the total normalized (b, B) cyclic bicomplex, $\mathrm{Tot}_n(\overline{\mathcal{B}}^*(\mathcal{A})) = \overline{\mathcal{B}}^n(\mathcal{A})$.

(2) If p' is another linear map with the same properties and $p - p' \in \mathcal{L}^p(\mathcal{H})$, then it induces the same element in cohomology.

(3) Let $\psi_{n+1}(a_0, ..., a_{n+1}) = \mathrm{Tr}(\rho(a_0)\varepsilon_1 \cdots \varepsilon_n)$. Then $\psi_{n+1} \in \overline{\mathcal{B}}^{n+1}(\mathcal{A})$, and

$$\psi_{n+2} - (-1)\left(\frac{n+1}{2}\right) S(\psi_n) = (b + B)\phi_{n+1}$$

Proof. (1) and (2) follow from theorem 3.1 from section 3.2 and the fact that

$$C^n_\lambda(\mathcal{A}) \hookrightarrow \mathrm{Tot}_n(CC^*(\mathcal{A})) \to \mathcal{B}^*(\mathcal{A})$$

are quasi isomorphisms, and $\phi \mapsto \phi$ under this composition. $\phi \in \overline{\mathcal{B}}^n(\mathcal{A})$ since if $a_i = 1$ for some i, then $\varepsilon_i = \varepsilon_{i-1} = 0$, so $\phi_n(a_0, ..., a_n) = 0$.

For (3), we note that $\psi_{n+1} \in \overline{\mathcal{B}}^{n+1}(\mathcal{A})$ since if $a_i = 1$ for some $i \geq 1$, then

$$\phi_{n+1}(a_0, ..., a_{n+1}) = 0.$$

Let $n = 2m+1$. Now,

$$\begin{aligned}
B\phi_{n+1}(a_0, ..., a_n) &= \sum_{i=0}^{n} (-1)^i \phi_{n+1}(1, a_i, ..., a_{i-1}) \\
&= \sum_{i=0}^{n} (-1)^i \mathrm{Tr}(\varepsilon_i \cdots \varepsilon_{i-2}) \\
&= \sum_{j=0}^{m} \mathrm{Tr}(\varepsilon_{2j} \cdots \varepsilon_{2(j-1)}) - \mathrm{Tr}(\varepsilon_{2j+1} \cdots \varepsilon_{2j-1}) \\
&= \sum_{j=0}^{m} \mathrm{Tr}(\varepsilon_0 \cdots \varepsilon_{n-1}) - \mathrm{Tr}(\varepsilon_n \cdots \varepsilon_{n-2}) \\
&= \left(\frac{n+1}{2}\right) \mathrm{Tr}(\varepsilon_0 \cdots \varepsilon_{n-1}) - \mathrm{Tr}(\varepsilon_n \cdots \varepsilon_{n-2}) \\
&= \left(\frac{n+1}{2}\right) S(\phi_n)(a_0, ..., a_n).
\end{aligned}$$

Now,

$$\varepsilon(a_i, a_{i+1}a_{i+2}) - \varepsilon(a_i a_{i+1}, a_{i+2}) = \varepsilon(a_i, a_{i+1})\rho(a_{i+2}) - \rho(a_i)\varepsilon(a_{i+1}, a_{i+2}) = \varepsilon_i \rho(a_{i+2}) - \rho(a_i)\varepsilon_{i+1}.$$

For convenience of indices, we will do the following computation for $n-1$ rather than $n+1$.

$$\begin{aligned}
b\phi_{n-1}(a_0,...,a_n) =& \sum_{i=1}^{n-1}(-1)^i\phi_{n-1}(a_0,...,a_ia_{i+1},...,a_n) \\
&+ \phi_{n-1}(a_0a_1,a_2,...,a_n) - \phi_{n-1}(a_na_0,...,a_{n-1}) \\
=& \sum_{j=1}^{m}\phi_{n-1}(a_0,...,a_{2j}a_{2j+1},...,a_n) - \phi_{n-1}(a_0,...,a_{2j-1}a_{2j},...,a_n) \\
&+ \phi_{n-1}(a_0a_1,a_2,...,a_n) - \phi_{n-1}(a_na_0,...,a_{n-1}) \\
=& \sum_{j=1}^{m}\Big(\mathrm{Tr}(\rho(a_0)\varepsilon_1\cdots\varepsilon(a_{2j-1},a_{2j}a_{2j+1})\varepsilon_{2j+2}\cdots\varepsilon_{n-1}) \\
&\qquad - \mathrm{Tr}(\rho(a_0)\varepsilon_1\cdots\varepsilon(a_{2j-1}a_{2j},a_{2j+1})\varepsilon_{2j+2}\cdots\varepsilon_{n-1})\Big) \\
&+ \mathrm{Tr}(\rho(a_0a_1)\varepsilon_2\cdots e_{n-1}) - \mathrm{Tr}(\rho(a_na_0)\varepsilon_1\cdots e_{n-2}) \\
=& \sum_{j=1}^{m}\Big(\mathrm{Tr}(\rho(a_0)\varepsilon_1\cdots\varepsilon_{2j-1}\rho(a_{2j+1})\varepsilon_{2j+2}\cdots\varepsilon_{n-1}) \\
&\qquad - \mathrm{Tr}(\rho(a_0)\varepsilon_1\cdots\rho(a_{2j-1})\varepsilon_{2j}\varepsilon_{2j+2}\cdots\varepsilon_{n-1})\Big) \\
&+ \mathrm{Tr}(\rho(a_0a_1)\varepsilon_2\cdots e_{n-1}) - \mathrm{Tr}(\rho(a_na_0)\varepsilon_1\cdots e_{n-2}) \\
=& \mathrm{Tr}(\rho(a_0)\varepsilon_1\cdots\varepsilon_{n-2}\rho(a_n)) - \mathrm{Tr}(\rho(a_0)\rho(a_1)\varepsilon_2\cdots\varepsilon_{n-1}) \\
&+ \mathrm{Tr}(\rho(a_0a_1)\varepsilon_2\cdots e_{n-1}) - \mathrm{Tr}(\rho(a_na_0)\varepsilon_1\cdots e_{n-2}) \\
=& \mathrm{Tr}(\rho(a_n)\rho(a_0)\varepsilon_1\cdots\varepsilon_{n-2}) - \mathrm{Tr}(\rho(a_0)\rho(a_1)\varepsilon_2\cdots\varepsilon_{n-1}) \\
&+ \mathrm{Tr}(\rho(a_0a_1)\varepsilon_2\cdots e_{n-1}) - \mathrm{Tr}(\rho(a_na_0)\varepsilon_1\cdots e_{n-2}) \\
=& \mathrm{Tr}(\rho(a_0a_1)\varepsilon_2\cdots e_{n-1}) - \mathrm{Tr}(\rho(a_0)\rho(a_1)\varepsilon_2\cdots\varepsilon_{n-1}) \\
&- (\mathrm{Tr}(\rho(a_na_0)\varepsilon_1\cdots e_{n-2}) - \mathrm{Tr}(\rho(a_n)\rho(a_0)\varepsilon_1\cdots\varepsilon_{n-2})) \\
=& \mathrm{Tr}(\varepsilon_0\cdots e_{n-1}) - \mathrm{Tr}(\varepsilon_n\varepsilon_1\cdots\varepsilon_{n-2}) \\
=& \psi_n(a_0,...,a_n)
\end{aligned}$$

Thus,

$$b\phi_{n+1} = \psi_{n+2}$$

and so we have

$$\psi_{n+2} - (-1)\left(\frac{n+1}{2}\right) S(\psi_n) = (b+B)\phi_{n+1}$$

□

Corollary A.1. *If* $C_\rho^{n+1} = c_{n+2}\phi_{n+1}$, *then*

$$S(Ch_\rho^n) - Ch_\rho^{n+2} = (B+b)C_\rho^{n+1}.$$

Proof. This follows as

$$(-1)\left(\frac{n+1}{2}\right) \cdot c_{n+2} = (-1)\left(\frac{n+1}{2}\right) \cdot \frac{(-1)^{\frac{n+3}{2}}}{\left(\frac{n+1}{2}\right)!} = \frac{(-1)^{\frac{n+1}{2}}}{\left(\frac{n-1}{2}\right)!} = c_n$$

□

This homotopy also works in the $\mathbb{Z}/2$ graded case.

In the following proposition we will use the notation $\{\alpha\}_i$ to represent $\alpha \in HC_i^-(\mathcal{A})$. We are doing this since we will be dealing with multiple negative cyclic homology degrees. We will also use the notation

$$sh : \mathcal{B}_i^-(\mathcal{A}) \otimes \mathcal{B}_{n-i}(\mathcal{B}) = \sum_{k=1}^{n} (\overline{C}(\mathcal{A}) \otimes \overline{C}(\mathcal{B}))_k \to \mathcal{B}_n(\mathcal{A} \otimes \mathcal{B}) = \sum_{k=1}^{n} \overline{C}_k(\mathcal{A} \otimes \mathcal{B}),\ sh = \sum_{k=1}^{n} sh^k$$

$$(\overline{C}(\mathcal{A}) \otimes \overline{C}(\mathcal{B}))_k = \sum_{j=i \bmod 2}^{k} \overline{C}_j(\mathcal{A}) \otimes \overline{C}_{k-j}$$

$$sh^k : (\overline{C}(\mathcal{A}) \otimes \overline{C}(\mathcal{B}))_k \to \overline{C}_k(\mathcal{A} \otimes \mathcal{B}),\ sh^k = \sum_{p+q=k} sh_{pq}$$

noting that if $\{\alpha\}_i = \sum_j \alpha_j$, then $\alpha_j = 0$ if $j < i$ or is of different parity than i. Thus, for

$$\{\alpha\}_i = \sum_{j=0}^{\infty} \alpha_{2j+i},\ \{\alpha\}_{i-2} = S\{\alpha\}_i = \sum_{j=0}^{\infty} \alpha_{2j+i}$$

$$\{\beta\}_{2(r+1)+\delta} = \sum_{k=0}^{r+1} \beta_{2k+\delta},\ \{\beta\}_{2r+\delta-2} = S\{\beta\}_{2r+\delta} = \sum_{k=0}^{r} \beta_{2k+\delta}$$

and $k < n+2 = (i+2r+\delta)+2$, we have

$$\begin{aligned} sh^k(\{\alpha\}_i \otimes S\{\beta\}_{2(r+1)+\delta}) =& sh^k(\{\alpha\}_i \otimes \{\beta\}_{2r+\delta}) \\ =& \sum_{p+q=k} sh_{pq}(\alpha_p, \beta_q) \\ =& sh^k(\{\alpha\}_i \otimes \{\beta\}_{2(r+1)+\delta}) \end{aligned}$$

as $p \geq i$ so $q < 2(r+1)+\delta$.

Further

$$sh^k(S\{\alpha\}_i \otimes \{\beta\}_{2(r+1)+\delta}) = sh^k(\{\alpha\}_i \otimes \{\beta\}_{2(r+1)+\delta})$$

holds trivially as

$$\{\alpha\}_i = S\{\alpha\}_i.$$

Thus,

$$\begin{aligned} S(sh(\{\alpha\}_i \otimes \{\beta\}_{2(r+1)+\delta})) =& S\left(\sum_{k=1}^{n+2} sh^k(\{\alpha\}_i \otimes \{\beta\}_{2(r+1)+\delta})\right) \\ =& \sum_{k=1}^{n} sh^k(\{\alpha\}_i \otimes \{\beta\}_{2(r+1)+\delta}) \\ =& \sum_{k=1}^{n} sh^k(\{\alpha\}_i \otimes \{\beta\}_{2r+\delta}) \\ =& \sum_{k=1}^{n} sh^k(\{\alpha\}_i \otimes S\{\beta\}_{2(r+1)+\delta}) \\ =& sh(\{\alpha\}_i \otimes S\{\beta\}_{2(r+1)+\delta}) \end{aligned}$$

Similarly,

$$S(sh(\{\alpha\}_i \otimes \{\beta\}_{2(r+1)+\delta})) = sh(S\{\alpha\}_i \otimes \{\beta\}_{2(r+1)+\delta}).$$

Proposition A.2. Let $\{\alpha\}_i \in HC_i^-(\mathcal{A})$. Then

$$S\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\{\alpha\}_i) = \Psi^{i,n+2}_{\mathcal{A},\mathcal{B}}(\{\alpha\}_i) = \Psi^{i-2,n}_{\mathcal{A},\mathcal{B}}(S\{\alpha\}_i).$$

Indeed, consider $\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\{\alpha\}_i) \in HC^{n-i}(\mathcal{B})$, $S\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\{\alpha\}_i) \in HC^{n-i+2}(\mathcal{B})$, so

$$\begin{aligned}
S\Psi^{i,n}_{\mathcal{A},\mathcal{B}}(\{\alpha\}_i)(\{\beta\}_{n-i+2}) =& \Psi^{i,n}(\{\alpha\}_i)(S\{\beta\}_{n-i+2}) \\
=& Ch^n\big(sh(\{\alpha\}_i \otimes S\{\beta\}_{n-i+2})\big) \\
=& Ch^n\big(S(sh(\{\alpha\}_i \otimes \{\beta\}_{n-i+2}))\big) \\
=& SCh^n\big((sh(\{\alpha\}_i \otimes \{\beta\}_{n-i+2}))\big) \\
=& Ch^{n+2}\big(sh(\{\alpha\}_i \otimes \{\beta\}_{n-i+2})\big) \\
=& \Psi^{i,n+2}_{\mathcal{A},\mathcal{B}}(\{\alpha\}_i)(\{\beta\}_{n-i+2})
\end{aligned}$$

Further, say $S\{\alpha\}_i = \{\alpha\}_{i-2}$

$$\begin{aligned}
Ch^n\big(sh(S\{\alpha\}_i \otimes \{\beta\}_{n-i+2})\big) =& Ch^n(sh(\{\alpha\}_{i-2} \otimes \{\beta\}_{n-i+2})) \\
=& \Psi^{i-2,n}_{\mathcal{A},\mathcal{B}}(\{\alpha\}_{i-2})(\{\beta\}_{n-i+2}) \\
=& \Psi^{i-2,n}_{\mathcal{A},\mathcal{B}}(S\{\alpha\}_i)(\{\beta\}_{n-i+2})
\end{aligned}$$

so

$$S\Psi^{i,n}_{\mathcal{A},\mathcal{B}} = \Psi^{i,n+2}_{\mathcal{A},\mathcal{B}}(\{\alpha\}_i) = \Psi^{i-2,n}_{\mathcal{A},\mathcal{B}}(S\{\alpha\}_i).$$

Remark A.1. Because the chain map respects S we can extend the definition to periodic cyclic homology.

Proposition A.3. If f and g differ by a boundary in $HC^n(\mathcal{A} \otimes \mathcal{B})$, then the chain maps induced on $HC^-_i(\mathcal{A}) \to HC^{n-i}(\mathcal{B})$ are equivalent in homology.

Proof. Let F and G be the induced chain maps, so

$$F(\alpha)(\beta) = f(Sh(\alpha \otimes \beta))$$

$$G(\alpha)(\beta) = g(Sh(\alpha \otimes \beta)).$$

Let h be the chain such that $f - g = (b+B)_{\mathcal{A}\otimes\mathcal{B}}h$. Then,

$$\begin{aligned} F(\alpha)(\beta) - G(\alpha)(\beta) =& f(Sh(\alpha\otimes\beta)) - g(Sh(\alpha\otimes\beta)) \\ =& (b+B)_{\mathcal{A}\otimes\mathcal{B}}h(Sh(\alpha\otimes\beta)) \\ =& h(Sh((b+B)_{\mathcal{A}}\alpha\otimes\beta)) + (-1)^{|\alpha|}h(Sh(\alpha\otimes(b+B)_{\mathcal{B}}\beta)) \\ =& H((b+B)_{\mathcal{A}}\alpha)(\beta) + (-1)^{|\alpha|}H(\alpha)((b+B)_{\mathcal{B}}\beta) \end{aligned}$$

where $H(\alpha)(\beta) = h(Sh(\alpha\otimes\beta))$. Thus, $F = G : HC_i^-(\mathcal{A}) \to HC^{n-i}(\mathcal{B})$ □

Proposition A.4. The sign of the (n,n)-shuffle taking the ordered set

$$(a_1, ..., a_n, b_1, .., b_n) \to (a_1, b_1, a_2, b_2, ..., a_n, b_n)$$

is $(-1)^{\frac{n(n-1)}{2}}$.

Proof. The inverse of this permutation in cycle notation is

$$(a_n\ b_1)\cdots(a_n\ b_{n-2})(a_n\ b_{n-1})\cdots(a_4\ b_1)(a_4\ b_2)(a_4\ b_3)(a_3\ b_1)(a_3\ b_2)(a_2\ b_1)$$

which is $\sum_{i=1}^{n-1} i$ transpositions. □

Appendix B

Additional Proofs for Section 4.2

Lemma B.1. *(1)* $F = \frac{1+F}{2} - \frac{1-F}{2}$

(2) $\frac{1-F}{2}$ *and* $\frac{1+F}{2}$ *are orthogonal projections with* $\frac{1-F}{2} \perp \frac{1+F}{2}$

(3) For any $b \in \mathcal{B}$, $\rho'(b)$ *commutes with* $\frac{1-F}{2}$ *and* $\frac{1+F}{2}$

Proof. (1) is obvious.

(2) $\left(\frac{1-F}{2}\right)^2 = \frac{1-2F+F^2}{4} = \frac{2-2F}{4} = \frac{1-F}{2}$ and $\left(\frac{1-F}{2}\right)^* = \frac{1^*-F^*}{2} = \frac{1-F}{2}$. The proof for $\frac{1+F}{2}$ is similar.

Now

$$\left(\frac{1-F}{2}\right)\left(\frac{1+F}{2}\right) = \frac{1-F^2}{4} = 0$$

(3) Indeed,

$$\rho'(b)\frac{1-F}{2} = \frac{\rho'(b)1 - \rho'(b)F}{2} = \frac{1\rho'(b) - F\rho'(b)}{2} = \frac{1-F}{2}\rho'(b).$$

The proof for $\frac{1+F}{2}$ is similar. $\square$

Lemma B.2. *Assume* P *is an orthogonal projection and* A *is a trace class operator such that* $PA = AP$. *Then* $Tr(PA) = Tr(A|_{P(\mathcal{H})})$

Let $\{e^1_k\}$ be an orthonormal basis for $P(\mathcal{H})$ and $\{e^2_k\}$ an orthonormal basis for $(1-P)\mathcal{H}$. So

$P(e_k^1) = e_k^1$ and $P(e_k^2) = 0$. Then

$$\begin{aligned}
\operatorname{Tr}_{\mathcal{H}}(PA) &= \sum_{\{e_k^1\}\cup\{e_k^2\}} \langle PAe_k^i, e_k^i\rangle \\
&= \sum_{\{e_k^1\}\cup\{e_k^2\}} \langle Ae_k^i, Pe_k^i\rangle \\
&= \sum_{\{e_k^1\}} \langle Ae_k^1, e_k^1\rangle \\
&= \sum_{\{e_k^1\}} \langle A_{P(\mathcal{H})}e_k^1, e_k^1\rangle \\
&= \operatorname{Tr}_{P(\mathcal{H})}(A|_{P(\mathcal{H})})
\end{aligned}$$

Appendix C

Additional Proofs for Chapter 5

Proposition C.1. If $\rho : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ is a unital map such that $\rho = \rho^+ \oplus \rho^-$, then ρ is an almost representation if and only if $\rho^\pm : \mathcal{B} \to \mathcal{L}(\mathcal{H}^\pm)$ are almost representations.

Proof. We extend $\rho^\pm$ to $\mathcal{H}$ by taking $\rho^\pm(b) = 0$ for all $b \in 0 \oplus \mathcal{H}^\mp$. Then $\rho = \rho^+ + \rho^-$, $\rho^+\rho^- = \rho^-\rho^+ = 0$, and $\omega(b_1, b_2) = \omega^+(b_0, b_1) + \omega^-(b_0, b_1)$. Now, $\omega(b_1, b_2)^* = \omega^+(b_0, b_1)^* + \omega^-(b_0, b_1)^*$, where $\omega^\pm(b_0, b_1)^*$ are the extensions of the adjoints as operators on $\mathcal{H}^\pm$ to $\mathcal{H}$ (this follows as $\omega^+\omega^- = \omega^-\omega^+ = 0$ and $\omega^\pm(b_0, b_1)h^\mp = 0$ for $h^\mp \in \mathcal{H}^\mp$.) Then,

$$\begin{aligned}
|\omega(b_0, b_1)|^2 =& \omega(b_0, b_1)^*\omega(b_0, b_1) \\
=& \omega^+(b_0, b_1)^*\omega^+(b_0, b_1) + \omega^-(b_0, b_1)^*\omega^-(b_0, b_1) \\
=& |\omega^+(b_0, b_1)|^2 + |\omega^-(b_0, b_1)|^2 \\
=& |\omega^+(b_0, b_1)|^2 + |\omega^+(b_0, b_1)| \cdot |\omega^-(b_0, b_1)| + |\omega^-(b_0, b_1)| \cdot |\omega^+(b_0, b_1)| + |\omega^-(b_0, b_1)|^2 \\
=& (|\omega^+(b_0, b_1)| + |\omega^-(b_0, b_1)|)^2.
\end{aligned}$$

and so $|\omega(b_0, b_1)| = |\omega^+(b_0, b_1)| + |\omega^-(b_0, b_1)|$, and more generally,

$$|\omega(b_0, b_1)|^p = |\omega^+(b_0, b_1)|^p + |\omega^-(b_0, b_1)|^p.$$

Thus, $\text{Trace}|\omega(b_0, b_1)|^p$ if and only if $\text{Trace}|\omega^\pm(b_0, b_1)|^p < \infty$. $\square$

Proposition C.2. Let $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{C}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Suppose $T \in \mathcal{L}(\mathcal{H})$, such that $T - T^2 \in \mathcal{L}^p(\mathcal{H})$ and suppose $T = T^*$. Then there exists $P \in \mathcal{L}(\mathcal{H})$ such that $P = P^2 = P^*$ and $P - T \in \mathcal{L}^p(\mathcal{H})$.

Proof. To begin, we will follow a proof from [1] which shows the existence of P so that $P-T \in \mathcal{K}(\mathcal{H})$. We first note that since $T-T^2 \in \mathcal{K}(\mathcal{H})$, the spectrum of T, $\sigma = \sigma(T)$, accumulates at 0 and 1. Let I be a closed interval containing 1 in its interior and $0 \notin I$, and $f : \mathbb{R} \to \mathbb{R}$ a continuous function such that $f(x) = 1$ on I and $f(x) = 0$ on $\sigma(T) \cap \mathbb{R} \backslash I$, noting that $\sigma(T)$ is countable. Let $P = f(T)$, so P is a self adjoint projection, and $P-T \in \mathcal{K}(\mathcal{H})$ by construction. Let $g(x) = x-x^2$ and $h(x) = f(x)-x$. By the spectral mapping theorem, we have $g(\sigma(T)) = \sigma(g(T))$ and $h(\sigma(T)) = \sigma(h(T))$, and since all of the operators considered are self adjoint, the singular values of each operator are the absolute values of the eigenvalues. Then by assumption, we have that

$$\sum_{g(\lambda_n) \in g(\sigma(T))} |g(\lambda)|^p = \sum_{\lambda_n \in \sigma(T)} |\lambda_n|^p |1-\lambda_n|^p < \infty.$$

We wish to show that

$$\sum_{h(\lambda_n) \in h(\sigma(T))} |h(\lambda_n)|^p = \sum_{\lambda_n \in \sigma(T)} |f(\lambda_n) - \lambda_n|^p < \infty.$$

Now, let $r \in (0,1)$ such that $r < \inf I$, let $U = \sigma(T) \cap (-\infty, \frac{1}{k}]$ and $V = \sigma(T) \cap (r, \infty)$. Then

$$S_1 = \sum_{\lambda_n \in U} |f(\lambda_n) - \lambda_n|^p = \sum_{\lambda_n \in U} |\lambda_n|^p$$
$$S_2 = \sum_{\lambda_n \in V} |f(\lambda_n) - \lambda_n|^p = \sum_{\lambda_n \in V} |1-\lambda_n|^p$$

Now,

$$S_1 \cdot |1-r|^p = \sum_{\lambda_n \in U} |\lambda_n|^p |1-r|^p \leq \sum_{\lambda_n \in U} |\lambda_n|^p |1-\lambda_n|^p < \infty$$
$$S_2 \cdot |r|^p = \sum_{\lambda_n \in V} |r|^p |1-\lambda_n|^p \leq \sum_{\lambda_n \in V} |\lambda_n|^p |1-\lambda_n|^p < \infty.$$

Thus,

$$\sum_{\lambda_n \in \sigma(T)} |f(\lambda_n) - \lambda_n|^p = S_1 + S_2 < \infty,$$

and so $P - T \in \mathcal{L}^p(\mathcal{H})$. □

Proposition C.3. If $(\rho, \mathcal{H}, F)$ is an almost Fredholm module and ρ' is an (even) almost representation such that $\rho - \rho' \in \mathcal{L}^p$, then $(\rho', \mathcal{H}, F)$ is an almost Fredholm module.

This follows as $[F, \rho'(b)] = [F, \rho'(b)] - [F, \rho(b)] + [F, \rho(b)] = [F, \rho'(b) - \rho(b)] + [F, \rho(b)]$.