Bridge Cohomology: A Generalization of Hochschild and Cyclic Cohomologies

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The connection between Hochschild and cyclic cohomologies with generalized De Rham homology and index theories for arbitrary algebras has long been established by the work of Connes, Karoubi, Loday, Feigin, Tsygan, et al. Here we generalize these cohomology theories even further, essentially creating a theory that establishes a step-wise bridge between the two. Motivation for this construction comes from trying to generalize the Hochschild-Kostant-Rosenberg-Connes theorem to manifolds with boundary, and applications in tracial constructions in certain classes of pseudodifferential operators. The situation that arises is a subcomplex of Hochschild functionals that are not cyclic, but rather descend to a subcomplex under the cyclic operator. The complexes that arise from this are developed at length, ending with Gysin-Connes sequences that relate bridge cohomology to Hochscild and cyclic cohomologies, as well as the theorem of excision. Further geometric and topological interests of this theory include extending Chern-Weil theory to manifolds with boundary via pairings between bridge cohomology and higher K-theories.

Dedication

This is dedicated to Dave Weaner and to all the teachers and instructors who have made a difference in my life.

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Chapter 1

Introduction

In this thesis we revisit Hochschild and cyclic cohomologies. With some extra motivation stemming from manifolds with boundary, we generalize these chain complexes to incorporate the information of an extra algebra analogous to the algebra of Whitney functions on a manifold with boundary. Recall that for a compact Riemannian manifold without boundary M, the space of de Rham k-currents, $\mathcal{D}_k(M)$, is the set of continuous linear functions on k-forms. For any current $C \in \mathcal{D}_k(M)$ and $\omega \in \Omega^k(M)$ we denote by an integral the pairing

$$\int_C \omega := \langle C, \omega \rangle.$$

Taken together, the collection of currents of all degrees form a complex $(\mathcal{D}_\bullet(M), \delta)$ where the differential δ is dual to the exterior derivative

$$\int_{\delta C} \omega = \int_C d\omega$$

for all $C \in \mathcal{D}_k(M)$ and $\omega \in \Omega^{k-1}(M)$. For example, if n is the dimension of the manifold, any $(n-k)$-form $\eta \in \Omega^{n-k}(M)$ defines a current $C_\eta \in \mathcal{D}_k(M)$, by

$$\int_{C_\eta} \omega = \int_M \omega \wedge \eta$$

for any $\omega \in \Omega^k(M)$. Moreover, if η is closed one checks by a simple application of the Leibnitz rule and Stokes Theorem that the current C_η is also closed.

In noncommutative language (cf. [Con94, pg. 187]) given an algebra A, one describes "currents" as cycles i.e. a triple $(\Omega^\bullet, d, \int)$, where $(\Omega^\bullet, d)$ is a differential graded algebra of degree k

over A for some homomorphism $\rho : A \to \Omega^0$, and $\int : \Omega^k \to \mathbb{C}$ is a closed graded trace. One recognizes that every cycle over an algebra A is given by a closed cyclic cocycle (cf. [Con94, Prop. 4 pg. 190],[GBVF13, pg. 350], [Lod13, pg. 74]). That is, there exists a multilinear functional $\tau : A^{\otimes k} \to \mathbb{C}$ such that

$$\tau(a_0, ..., a_k) = \int \rho(a_0)d\rho(a_1)...d\rho(a_k)$$

and $b\tau = 0$, $(1 - \lambda)\tau = 0$, where b is the Hochschild boundary map, and $\lambda\tau(a_0, ..., a_k) = (-1)^k\tau(a_k, a_0, ..., a_{k-1})$ is the cyclic operator (to be defined in chapter 1). When $A = \mathscr{C}^\infty(M)$ the closed graded trace $\int$ coincides with a closed current $\varphi \in \mathscr{D}_k(M)$. Thus, the goal of Hochschild and cyclic cohomology is to start by looking at a complex of multinear functionals, and with some added structure, recover the complex of de Rham currents without having to appeal to the commutativity of the algebra. The classical result of Hochschild-Kostant-Rosenberg [HKR62], later extended by Connes [Con85], tells us just that. For an algebra of smooth functions on a compact manifold $\mathscr{C}^\infty(M)$, the Hochschild cohomology is isomorphic to the complex of de Rham currents ([Con85, II.5 Lemma 45]),

$$HH^k(\mathscr{C}^\infty(M)) \cong \mathscr{D}(M).$$

Likewise, we have an isomorphism for cyclic cohomology (cf. [Con85, II.5 Thm. 46])

$$HC^k(\mathscr{C}^\infty(M)) \cong \ker b(\subset \mathscr{D}_k) \oplus H^{dR}_{k-2}(M) \oplus H^{dR}_{k-4}(M) \oplus ...$$

If we wish to consider this analogy for a manifold with boundary $(M, \partial M)$ however, we must be more careful. Let $f_0, ..., f_k \in \mathscr{C}^\infty(M)$ and define a k-multlinear functional $\tau \in C^k(\mathscr{C}^\infty(M))$ by

$$\tau(f_0, ..., f_k) = \int_M f_0 df_1 \wedge ... \wedge df_k.$$

By the tracial property of the integral we have

$$\begin{aligned}\int_M f_0 df_1 \wedge ... \wedge df_k &= (-1)^{k-1} \int_M df_k \wedge f_0 df_1 \wedge ... \wedge df_{k-1} \\ &= (-1)^{k-1} \int_M d(f_k f_0) \wedge df_1 \wedge ... \wedge df_{k-1} + (-1)^k \int_M f_k df_0 \wedge df_1 \wedge ... \wedge df_{k-1}.\end{aligned}$$

Hence

$$(1-\lambda)\tau(f_0,...,f_k) = (-1)^{k-1}\int_M d(f_k f_0)\wedge df_1\wedge ...\wedge df_{k-1} = (-1)^{k-1}\int_{\partial M} f_k f_0 \wedge df_1 \wedge ... \wedge df_{k-1}$$

And we see that τ is cyclic only when one of the f_i vanish on the boundary. However, since ∂M is a manifold itself with an integral $\int_{\partial M}$, we can again define a $(k-1)$-multilinear functional $\psi \in C^{k-1}(\mathscr{E}^\infty(\partial M))$ to coincide with this:

$$\psi(g_0,...,g_{k-1}) = \int_{\partial M} g_0 dg_1 \wedge ... \wedge dg_{k-1}.$$

The above line of equalities show that

$$(1-\lambda)\tau = (-1)^{k-1} b_k \sigma^* \psi$$

where $\sigma : \mathscr{C}^\infty(M) \to \mathscr{E}^\infty(\partial M)$ is the natural surjection from the algebra of smooth functions to the algebra of Whitney functions on M (cf. [Mal67], [Tou], [BP08],[PPT12]), and $b_k : C^k(\mathscr{E}^\infty(\partial M)) \to C^{k-1}(\mathscr{E}^\infty(\partial M))$ is the co-contraction on the k^{th} index i.e. $b_k\psi(f_0,...,f_k) = \psi(f_k f_0,...,f_{k-1})$. Up to relabeling, this means that for some $\psi \in C^k(\mathscr{E}^\infty(\partial M))$

$$(1-\lambda)\tau = \sigma^*\psi. \tag{$\circledast$}$$

The original idea for this thesis was introduced by Lesch, Moscovici, and Pflaum [LMP03], arising both in the aforementioned situation, as well as in certain classes of parameter dependent pseudo-differential operators [LP00]. Here we expand upon the above situation using noncommutative language. That is, given a surjection between two algebras $A \xrightarrow{\sigma} B$ we construct a subcomplex of the standard Hochschild complex, where the n^{th} degree module is given by

$$R^n(\sigma) = \{\varphi \in C^n(A) \mid (1-\lambda)\varphi \in \sigma^* C^n(B)\}. \tag{$\circledast\circledast$}$$

This will be called the **bridge complex** and its cohomology the **bridge cohomology of** σ.

1.1 Outline

In Chapter 2 will first provide background on the standard Hochschild, bar, and cyclic complexes, as well as their reduced and normalized versions. We also recall the Gysin-Connes sequence that relates Hochschild and cyclic cohomology. Such constructions can be found in [Lod13]. At the end of Chapter 2 we provide a proof of the theorem of excision in the cohomological setting, dualizing the paper of Guccione and Guccione [GG96].

In Chapter 3 we introduce the notion of the normalized, degenerate, and reduced bar complexes. These complexes are the respective versions of the complex $(C^\bullet(A), b')$ (cf. [Lod13, 1.1.11 pg. 12]) which until now hasn't been needed since $(C^\bullet(A), b')$ is contractible when A is unital. We parallel the development of Hochschild and cyclic complexes found in Loday [Lod13], now with this new framework from the bar complexes. After their construction we prove exactness for the (λ, Q)-sequences

$$... \to D^\bullet(A) \xrightarrow{1-\lambda} D^\bullet_{\text{bar}}(A) \xrightarrow{Q} D^\bullet(A) \xrightarrow{1-\lambda} D^\bullet_{\text{bar}}(A) \xrightarrow{Q} D^\bullet(A) \to ...$$

This allows us to then create cyclic bicomplexes, Connes' complexes, and provides another way to derive Gysin-Connes (SBI)-sequences for reduced and normalized cohomologies.

In Chapter 4 we construct our notion of bridge cohomology. From the motivation in the introduction we use equation ⊛⊛ to define a subcomplex of the Hochschild complex and set out to construct the appropriate normalized and reduced complexes, as well as define the appropriate constructions for non-unital algebras. The first step toward this is to realize the defining equation ⊛ as a pullback (Definition 4.1.1). After this we construct the category of surjective unital algebra homomorphisms, $\mathcal{S}_{\mathbb{1},\Bbbk}$, so as to define the bridge complex as a functor from this category to the category of complexes. This, combined with cohomology functor provides a functorial Definition of bridge cohomology. This functor can then be extended so non-unital morphisms by applying this functor to the inclusion of the identity into an augmented morphism (Definition 4.1.4). After constructing the normalized and reduced bridge complexes, we find that the reduced bridge cohomology of an augmented morphsim coincides with the functorial definition (Theorem 4.3.1). We end this Chapter by constructing a two bicomplex that gives us a long exact sequence relating bridge cohomology and "naive" bridge cohomology.

In Chapter 5 we construct bridge cohomology from a bicomplex analogous to that of the cyclic bicomplex $CC(A)$ (cf. [Lod13, Def. 2.4.1]). We construct normalized and reduced version of this complex as well as extend it to a triple complex, which can be used to compute the bridge cohomology of an algebra homomorphism whether it is unital or not.

In Chapter 6 we present the two main theorems, the existence of Gysin-Connes sequences (Theorem 6.1.1) and the theorem of excision (Theorem 6.3.2). We construct three Gysin-Connes sequences. The first comes from the bicomplex construction in Chapter 5 and considering the short exact sequence obtained by the inclusion of the relative cyclic bicomplex. This sequence provides a relationship between bridge cohomology and relative cyclic cohomology. The second sequence was proved by Lesch, Moscovici, and Pflaum [LMP03] though never published. Using the complexes developed in Chapter 4, we generalize the result to non-unital algebras. The last sequence provides a periodicity map $S : HR^n(\sigma) \to HR^{n+2}(\sigma)$ for bridge cohomology. To arrive at this sequence we first make another construction. Following the ideas of Lesch, Moscovici, and Pflaum modifying the cone complex of the morphism σ^*,

$$CC(B) \xrightarrow{\sigma^*} CC(A) \to \text{cone}(\sigma^*)$$

by removing the first column provides a complex to compute bridge cohomology. Modifying this using complexes constructed in Chapter 3, and then using similar techniques as the first Gysin-Connes sequence provides the third Gysin-Connes sequence.

In the last part of Chapter 6 we use the above sequences to prove the theorem of excision (Theorem 6.3.2) for bridge cohomology. Before doing so we show that the category of surjective algebra homomorphisms has kernels and cokernels, which ultimately leads to showing that short exact sequences in this category are given by a nine-diagram of algebras. We then provide the notion of a relative bridge complex. Given a short exact sequence of surjective algebra homomorphisms which is copure (the dual notion to that of Cohn [Coh59]) we say that the kernel is excisive (or excision holds) when the map from the relative bridge complex to the bridge complex of the kernel is a quasi-isomorphism. Generalizing the original result of Wodzicki [Wod89], we show that excision holds if and only if the source and target algebras of the kernel homomorphism are coH-unital.

In Appendix A we provide the details on the homological framework of the normalized and reduced bar complexes. We then parallel the development of Hochschild and cyclic cohomologies found in Loday [Lod13] and ultimate provide an alternate proof of the quasi-isomorphism between various cyclic complexes (cf. [Lod13, Prop. 2.2.16], Propositions A.3.9 and A.3.12).

Chapter 2

Preliminaries

In this chapter we introduce notation and background that will be used throughout. Most of this information can be found in Loday, **Cyclic Homology**[Lod13]. Throughout this paper all algebras are assumed to be associative and $\Bbbk$ will be a commutative ring. Where not explicitly written, all Hom sets are to be taken over $\Bbbk$ i.e. $\mathrm{Hom}(X, M) = \mathrm{Hom}_{\Bbbk}(X, M)$ for $\Bbbk$-modules X and M, and as usual we will let $\mathrm{Hom}(M)$ indicate $\mathrm{Hom}(M, \Bbbk)$ and often write this as M^*.

2.1 Basic Complexes

For any $\Bbbk$-algebra A and any A-bimodule M we define the cochain complexes $C^\bullet(A, M)$ and $C^\bullet_{\text{bar}}(A, M)$ by

$$C^\bullet(A, M) := \ M \xrightarrow{b} \mathrm{Hom}(A, M) \xrightarrow{b} \ldots \xrightarrow{b} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{b} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{b} \ldots$$

$$C^\bullet_{\text{bar}}(A, M) := \ M \xrightarrow{b'} \mathrm{Hom}(A, M) \xrightarrow{b'} \ldots \xrightarrow{b'} \mathrm{Hom}(A^{\otimes n}, M) \xrightarrow{b'} \mathrm{Hom}(A^{\otimes n+1}, M) \xrightarrow{b'} \ldots$$

where for any $\varphi \in \mathrm{Hom}(A^{\otimes n}, M)$,

$$\begin{aligned} b\varphi(a_1, \ldots, a_{n+1}) = {} & a_1\varphi(a_2, \ldots, a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i \varphi(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\ & + (-1)^{n+1} \varphi(a_1, \ldots, a_n) a_{n+1} \end{aligned}$$

and

$$b'\varphi(a_1, ..., a_{n+1}) = a_1\varphi(a_2, ..., a_{n+1}) + \sum_{i=1}^{n}(-1)^i\varphi(a_1, ..., a_ia_{i+1}, ..., a_{n+1})$$

Of special note is the case when $M = A^*$, where the A-bimodule structure of A^* is given by $(x\varphi y)(a) = \varphi(yax)$ for $a, x, y \in A$ and $\varphi \in A^*$. In such a case we will write $C^\bullet(A) := C^\bullet(A, A^*)$. The notation $C^\bullet_{\text{bar}}(A)$ will be used analogously for the bar complex. When A is unital, we have the extra degeneracy map $s : C^\bullet_{(\text{bar})}(A) \to C^{\bullet-1}_{(\text{bar})}(A)$ where $s\varphi(a_0, ..., a_{n-1}) = \varphi(1, a_0, ..., a_{n-1})$. This map gives us a contracting homotopy for $C^\bullet_{\text{bar}}(A)$, where $sb' + b's = 1$.

When A is a unital algebra, M an A-bimodule, the **Hochschild cohomology of A valued in M** is defined as the cohomology of the complex $C^\bullet(A, M)$ and will be denoted $H^\bullet(A, M)$. When $M = A^*$ we simply have the **Hochschild cohomology of A**, written $HH^\bullet(A)$. For any algebra A (not necessarily unital) the **bar cohomology of A** (with coefficients in M) is the cohomology of the complex $C^\bullet_{\text{bar}}(A)$ (or $C^\bullet_{\text{bar}}(A, M)$), and will be denoted $HB^\bullet(A)$ (or $HB^\bullet(A, M)$).

Acting on the modules $C^n_{(\text{bar})}(A)$ we have the cyclic operator $\lambda : C^n_{(\text{bar})}(A) \to C^n_{(\text{bar})}(A)$ where $\lambda\varphi(a_0, ..., a_n) = (-1)^n\varphi(a_n, a_0, ..., a_{n-1})$. From this we can also create the norm operator $Q : C^n_{(\text{bar})}(A) \to C^n_{(\text{bar})}(A)$ where $Q = \sum_{i=0}^n \lambda^i$. These operators have the following relationships with the differentials b and b' (c.f [Lod13, Lemma 2.1.1 pg. 53]):

$$(1 - \lambda)b = b'(1 - \lambda) \qquad Qb' = bQ.$$

Thus we get chain maps $(1 - \lambda) : C^\bullet(A) \to C^\bullet_{\text{bar}}(A)$ and $Q : C^\bullet_{\text{bar}}(A) \to C^\bullet(A)$.

2.2 Normalized and Reduced Complexes

In the homological setting (See Appendix A), for a unital algebra A the complex $C_\bullet(A)$ has a large degenerate complex $D_\bullet(A)$ where we have

$$D_n(A) = \{(a_0, ..., a_n) \,|\, a_i = 1 \text{ for some } 1 \leq i \leq n\}$$

Quotienting by this complex gives us the normalized complex $\overline{C}_\bullet(A) = C_\bullet(A)/D_\bullet(A)$.

Definition 2.2.1. The **normalized Hochschild cochain complex** of A is the kernel

$$0 \to \overline{C}^\bullet(A) \to C^\bullet(A) \to D^\bullet(A) \to 0$$

where $D^n(A)$ is the set of functionals in $\mathrm{Hom}(D_n(A))$ that extend to $C^n(A)$ i.e. $\iota^*(C^n(A))$ where ι^* is the dual to $\iota : D_n(A) \to C_n(A)$. In other words,

$$\overline{C}^n(A) = \{\varphi \in C^n(A) \,|\, \varphi(a_0, ..., a_n) = 0 \text{ if some } a_i \in \Bbbk,\ 1 \leq i \leq n\}.$$

We call $D^\bullet(A)$ the **degenerate cochain complex**. The fact that $D^\bullet(A)$ has zero cohomology follows from spectral arguments ([Lod13, 1.6.4 pg. 46] for the homological case).

For the reduced cochain complex, we need make a slight change from what might be expected. We start with the exact sequence

$$0 \longrightarrow \Bbbk \longrightarrow A \longrightarrow \overline{A} \longrightarrow 0. \tag{2.2.1}$$

Since the Hom(,$\Bbbk$) functor is left exact we don't necessarily have $\Bbbk$ as the cokernel. So, for the given algebra A, we define $\mathring{\mathbb{j}}_A$ as the cokernel of the diagram:

$$0 \longrightarrow \overline{A}^* \longrightarrow A^* \longrightarrow \mathring{\mathbb{j}}_A \longrightarrow 0$$

i.e. $\mathring{\mathbb{j}}_A = \{k \in \Bbbk \,|\, \varphi(1) = k \text{ for some } \varphi \in A^*\}$ is determined by the image of the evaluation map $\mathrm{ev}_1 : A^* \to \Bbbk$.

Under reasonable conditions $\mathring{\mathbb{j}}_A = \Bbbk$. Since $\mathring{\mathbb{j}}_A$ is an "ideal", i.e. a $\Bbbk$-submodule of $\Bbbk$, if $\Bbbk$ is a field then $\mathring{\mathbb{j}}_A = 0$ or $\Bbbk$, with $\mathring{\mathbb{j}}_A = \Bbbk$ only needing the extra condition that there exists some non-zero $\Bbbk$-homomorphism $\varphi : A \to \Bbbk$. Alternatively, if the sequence (2.2.1) is $\Bbbk$-split, such as when we have an augmented algebra A^+, the canonical algebra obtained from joining a unit to A, we get $\mathring{\mathbb{j}}_A = \Bbbk$.

Definition 2.2.2. The **Reduced Hochschild cochain complex** for a unital algebra A is defined as the kernel

$$0 \to C^\bullet(A)_{\mathrm{red}} \to \overline{C}^\bullet(A) \to \mathbb{j}_A[0] \to 0.$$

where $\mathbb{j}_A[0]$ is the cochain complex with $\mathbb{j}_A$ in degree 0. The **reduced Hochschild cohomology** is then $\overline{HH}^\bullet(A) := H^\bullet(C^\bullet(A)_{\mathrm{red}})$.

We will also use the following construction. Define the **reduced degenerate cochain complex** $D^\bullet(A)_{\mathrm{red}}$ as the cokernel

$$0 \to C^\bullet(A)_{\mathrm{red}} \to C^\bullet(A) \to D^\bullet(A)_{\mathrm{red}} \to 0.$$

Note that when comparing the pairs of complexes $\overline{C}^\bullet(A)$ with $C^\bullet(A)_{\mathrm{red}}$, and $D^\bullet(A)$ with $D^\bullet(A)_{\mathrm{red}}$ the only difference is the 0^{th} degree with $D^0(A) = 0$ and $D^0(A)_{\mathrm{red}} = \mathbb{j}_A$.

Definition 2.2.3. When A is non-unital, we extend the functor $HH^\bullet$ by defining

$$HH^\bullet(A) := \ker\left(HH^\bullet(A^+) \to HH^n(\Bbbk)\right).$$

Proposition 2.2.4. *From the above definitions one checks (cf. [Lod13, Prop. 1.4.5]) that for a non-unital algebra A, $HH^n(A) = H^n(C(A^+)_{\mathrm{red}})$.*

For an augmented algebra A^+ we have a bicomplex, $CC^\bullet(A)^{\{2\}}$, associated to $C^\bullet(A)_{\mathrm{red}}$. It can be shown that the Hochschild cohomology of A (whether unital or not) is the cohomology of the bicomplex $CC^\bullet(A)^{\{2\}}$ given below.

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} \\
C^2(A) & \xrightarrow{1-\lambda} & C^2_{\mathrm{bar}}(A) \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} \\
C^1(A) & \xrightarrow{1-\lambda} & C^1_{\mathrm{bar}}(A) \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle b'} \\
C^0(A) & \xrightarrow{1-\lambda} & C^0_{\mathrm{bar}}(A)
\end{array}$$

We will denote by $\mathscr{H}\mathscr{H}^\bullet(A)$ the naive Hochschild cohomology of A, which is the cohomology of $C^\bullet(A)$ whether A is unital or not. We get the following long exact sequence

$$... \to HB^{n-1}(A) \to HH^n(A) \to \mathscr{H}\mathscr{H}^n(A) \to HB^n(A) \to ... \tag{2.2.4}$$

2.3 Cyclic Cohomology

The last set of preliminary complexes are the cyclic complexes. The **Connes cochain complex** is defined as the kernel $C^\bullet_\lambda(A) := \ker(1-\lambda) = \{\varphi \in C^\bullet(A) \,|\, (1-\lambda)\varphi = 0\}$. It's cohomology, the **cyclic cohomology of** A will be denoted $HC^\bullet_\lambda(A) := H^\bullet(C^\bullet_\lambda(A))$.

We will call the following diagram the **cyclic cochain bicomplex** $CC^{\bullet\bullet}(A)$ and denote the cohomology of its total complex by $HC^\bullet(A)$, also called **the cyclic cohomology of** A ([Lod13, Def. 2.4.1 pg. 72]).

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \\
C^2(A) & \xrightarrow{1-\lambda} & C^2_{\text{bar}}(A) & \xrightarrow{Q} & C^2(A) & \xrightarrow{1-\lambda} & C^2_{\text{bar}}(A) & \xrightarrow{Q} & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \\
C^1(A) & \xrightarrow{1-\lambda} & C^1_{\text{bar}}(A) & \xrightarrow{Q} & C^1(A) & \xrightarrow{1-\lambda} & C^1_{\text{bar}}(A) & \xrightarrow{Q} & \cdots \\
\uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \uparrow{\scriptstyle b} & & \uparrow{\scriptstyle -b'} & & \\
C^0(A) & \xrightarrow{1-\lambda} & C^0_{\text{bar}}(A) & \xrightarrow{Q} & C^0(A) & \xrightarrow{1-\lambda} & C^0_{\text{bar}}(A) & \xrightarrow{Q} & \cdots
\end{array}$$

Proposition 2.3.1. *When* $\Bbbk$ *contains* $\mathbb{Q}$, *the sequence*

$$... \to C^\bullet(A) \xrightarrow{1-\lambda} C^\bullet_{\text{bar}}(A) \xrightarrow{Q} C^\bullet(A) \xrightarrow{1-\lambda} C^\bullet_{\text{bar}}(A) \xrightarrow{Q} C^\bullet(A) \to ...$$

is exact.

Proof. In the homological case (see A.2.1) we have a contracting homotopy

$$\dots \xrightarrow{Q} C_n^{\text{bar}}(A) \xrightarrow{1-\lambda} C_n(A) \xrightarrow{Q} C_n^{\text{bar}}(A) \xrightarrow{1-\lambda} C_n(A) \xrightarrow{Q} \dots$$

given by $h' : C_n^{(\text{bar})}(A) \to C_n^{(\text{bar})}(A)$, $h' = \frac{1}{n+1}$, along with $h : C_n^{(\text{bar})}(A) \to C_n^{(\text{bar})}(A)$, $h = -\frac{1}{n+1}\sum_{i=1}^{n} i\lambda^i$ (see Loday[Lod13, Thm. 2.1.5, pg. 55]). Since Hom is functorial, we have an analogous homotopy in the cohomological case. □

The last thing we mention is the Gysin-Connes sequence (also known as the SBI-sequence [Con94, Theorem 26, pg. 208]), which is a long exact sequence relating cyclic cohomology with Hochschild cohomology. It is given by

$$... \xrightarrow{I} HH^{n-1}(A) \xrightarrow{B} HC^{n-2}(A) \xrightarrow{S} HC^{n}(A) \xrightarrow{I} HH^{n}(A) \to ... \qquad \text{(Gysin-Connes)}$$

2.4 Excision

The original proof of excision was due to Wodzicki [Wod89], but was only proved for the homological case. The cohomological case has since been believed to be true, and if it has been proven is nonetheless difficult to find in the literature. Luckily, Guccione and Guccione [GG96] provided a much more direct proof again in the homological case, the arguments of which we will dualize to cohomology. Throughout this section, we will work with extensions of $\Bbbk$-algebras $I \to A \to B$. For any index, we will use lowercase a's to be elements of A, u's $\in I$, and s's $\in B$. We introduce the following notation: let V be a $\Bbbk$-bimodule, then we will write $C^\bullet(A, M; V)$ for the complex $C^\bullet(A, \text{Hom}(V, M))$ where the A-bimodule structure of $\text{Hom}(V, M)$ is inherited from M. Similarly we define $C^\bullet_{\text{bar}}(A, M; V)$ to be the complex $C^\bullet_{\text{bar}}(A, \text{Hom}(V, M))$. Note that we may identify $C^n(A, M; V)$ with $\text{Hom}(V \otimes A \otimes A^{\otimes n}, M)$, and along with the corresponding boundary map we see that $C^\bullet(A, M; V)$ is the dual to the complex $(C_\bullet(A, M) \otimes V, b \otimes 1)$.

Definition 2.4.1. An extension of algebras $I \to A \to B$ is said to be **copure** [Coh59, Sec. 2], or I is **copure** in A, if for any $\Bbbk$ module M the induced map $\text{Hom}(A, M) \to \text{Hom}(I, M)$ is surjective. For example, this occurs when the sequence is $\Bbbk$-split. Note that this implies that the maps $\mathrm{Hom}(A\otimes I, M) \to \mathrm{Hom}(I\otimes I, M)$ and $\mathrm{Hom}(A\otimes A, M) \to \mathrm{Hom}(A\otimes I, M)$ are surjective, the first being from the identification with $\mathrm{Hom}(A, \mathrm{Hom}(I, M)) \to \mathrm{Hom}(I, \mathrm{Hom}(I, M))$ and the second following similarly. Continuing inductively, we use this property to obtain proper filtrations and cofiltrations in the upcoming theorems.

Definition 2.4.2. Let I be a $\Bbbk$-algebra and M an I-bimodule. We say M is **coH-unitary** (read: co-H-unitary) if for every $\Bbbk$-module V, the complex $C^\bullet_{\text{bar}}(I, M; V)$ is exact, where the left I-module action of $\mathrm{Hom}(V, M)$ is induced from M. When $M = I^*$ we say I is **coH-unital**.

Note. that if I is coH-unital then $(M\otimes I)^*$ is coH-unitary. Indeed, since the left I-module structure is given by $(u_1\varphi)(m, u_0) = \varphi(m, u_0u_1)$, we see that the complex

$$\begin{aligned}\big(\mathrm{Hom}(I^{\otimes\bullet}, \mathrm{Hom}(V, \mathrm{Hom}(M\otimes I, \Bbbk))), b'\big) &= \big(\mathrm{Hom}(I^{\otimes\bullet}, \mathrm{Hom}(V\otimes M\otimes I, \Bbbk)), b'\big) \\ &= \big(\mathrm{Hom}(I^{\otimes\bullet}, \mathrm{Hom}(V\otimes M, I^*)), b'\big)\end{aligned}$$

is still exact.

Remark. We make the remark that in the literature the notion of coH-unitality doesn't appear. Instead, H-unitality is enough even in cohomological situations. We note that when A is free and $\Bbbk$ is a PID (e.g. when $\Bbbk$ is a field), the universal coefficient theorem holds for bar cohomology.

$$0 \to \mathrm{Ext}(HH_{n-1}(A), \Bbbk) \to HH^n(A) \to \mathrm{Hom}(HH_n(A), \Bbbk) \to 0$$

Hence H-unitality implies coH-unital. In a more general algebraic setting however, this may not be the case.

Theorem 2.4.3. *Let $0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$ be a copure extension of $\Bbbk$-algebras, M an A-bimodule and V a $\Bbbk$-module. If M is coH-unitary as an I-bimodule, then the canonical surjections $i^* : C^\bullet(A, M; V) \to C^\bullet(I, M; V)$ and $i^*_{\text{bar}} : C^\bullet_{\text{bar}}(A, M; V) \to C^\bullet_{\text{bar}}(I, M; V)$ are quasi-isomorphisms.*

Proof. Let's first consider i^*. We create a cofiltration of $C^\bullet(A,M;V) \to ... \to F_1^\bullet \to F_0^\bullet$ where

$$F_p^\bullet := \mathrm{Hom}(V,M) \xrightarrow{b} \mathrm{Hom}(A,\mathrm{Hom}(V,M)) \xrightarrow{b} ... \xrightarrow{b} \mathrm{Hom}(A^{\otimes p},\mathrm{Hom}(V,M)) \xrightarrow{b}$$
$$\mathrm{Hom}(I\otimes A^{\otimes p},\mathrm{Hom}(V,M)) \xrightarrow{b} \mathrm{Hom}(I^{\otimes 2}\otimes A^{\otimes p},\mathrm{Hom}(V,M)) \xrightarrow{b} ...$$

Note that the maps $i_p^* : F_{p+1}^\bullet \to F_p^\bullet$ are surjective by Definition 2.4.1. Let $J_{p+1}^\bullet$ be the kernel of i_p^*, which we compute:

$$\begin{array}{ccccccc}
F_p^\bullet : ... & \longrightarrow & \mathrm{Hom}(I^{\otimes n}\otimes A^{\otimes p},\mathrm{Hom}(V,M)) & \longrightarrow & \mathrm{Hom}(I^{\otimes n+1}\otimes A^{\otimes p},\mathrm{Hom}(V,M)) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle i_{p+1}^*} & & \uparrow{\scriptstyle i_{p+1}^*} & & \\
F_{p+1}^\bullet : ... & \longrightarrow & \mathrm{Hom}(I^{\otimes n-1}\otimes A^{\otimes p+1},\mathrm{Hom}(V,M)) & \longrightarrow & \mathrm{Hom}(I^{\otimes n}\otimes A^{\otimes p+1},\mathrm{Hom}(V,M)) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & J_{p+1}^{n+p} & \longrightarrow & J_{p+1}^{n+p+1} & \longrightarrow & \cdots
\end{array}$$

We see that

$$\begin{aligned} J_{p+1}^{n+p} &= \{\varphi \in F_{p+1}^{n+p} \mid \varphi(u_1,...,u_{n-1},a_n,...a_{n+p}) = 0 \text{ if } a_n \in I\} \\ &\cong \mathrm{Hom}(I^{\otimes n-1}\otimes B\otimes A^{\otimes p},\mathrm{Hom}(V,M)), \end{aligned}$$

and hence $J_{p+1}^\bullet \cong \mathrm{Hom}(I^{\otimes \bullet - p-1}\otimes B\otimes A^{\otimes p},\mathrm{Hom}(V,M))$ where the differential is induced by the differential on F_{p+1} and then applying the quotient map $A\to B$ on the n^{th} index. For $\varphi \in J_{p+1}^{n+p}$ we compute:

$$\begin{aligned}
(b\varphi)(u_1,...,u_n,a_{n+1},...,a_{n+p+1}) &= u_1\varphi(u_2,...,u_n,a_{n+1},...,a_{n+p+1}) \\
&+ \sum_{i=1}^{n-1}(-1)^i\varphi(u_1,...,u_iu_{i+1},...,u_n,a_{n+1},...,a_{n+p+1}) \\
&+ (-1)^n\varphi(u_1,...,u_na_{n+1},...,a_{n+p+1}) \qquad (1) \\
&+ \sum_{i=n+1}^{n+p}(-1)^i\varphi(u_1,...,u_n,a_{n+1},...,a_ia_{i+1},...,a_{n+p+1}) \qquad (2) \\
&+ (-1)^{n+p+1}\varphi(u_1,...,u_n,a_{n+1},...,a_{n+p})a_{n+p+1} \qquad (3)
\end{aligned}$$

From the definition of J_{p+1} we see that terms (1), (2), and (3) are all zero since this step results in the nth index being in I. Thus, under the isomorphism $J^\bullet_{p+1} \cong \mathrm{Hom}(I^{\otimes \bullet - p - 1} \otimes B \otimes A^{\otimes p}, \mathrm{Hom}(V, M))$ the differential becomes

$$\begin{aligned}(b\varphi)(u_1, ..., u_n, s_{n+1}, a_{n+2}, ..., a_{n+p+1}) &= u_1\varphi(u_2, ..., u_n, s_{n+1}, a_{n+2}, ..., a_{n+p+1}) \\ &+ \sum_{i=1}^{n-1}(-1)^i\varphi(u_1, ..., u_iu_{i+1}, ..., u_n, s_{n+1}, a_{n+2}, ..., a_{n+p+1})\end{aligned}$$

Now taking advantage of the Hom-Tensor adjunction, we conclude

$$(J^\bullet_{p+1}, b) \cong \mathrm{Hom}(I^{\otimes \bullet - p - 1}, \mathrm{Hom}(B \otimes A^{\otimes p} \otimes V, M)), b')$$

where the left A-module structure of $\mathrm{Hom}(B \otimes A^{\otimes p} \otimes V, M)$ is induced by M. By our assumptions on M, this is exact. Hence from the cohomological long exact sequence obtained from

$$0 \to J^\bullet_{p+1} \to F^\bullet_{p+1} \to F^\bullet_p \to 0$$

we see that each i^*_{p+1} is a quasi-isomorphism, thus so is i^*. The proof for i^*_{bar} follows similarly. $\square$

Corollary 2.4.4. *Let* $0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$ *be a copure extension of* $\Bbbk$*-algebras and* V *a* $\Bbbk$*-module. If* I *is coH-unital then the canonical maps* $q^* : C^\bullet(B, B^*; V) \to C^\bullet(A, B^*; V)$ *and* $q^*_{\mathrm{bar}} : C^\bullet_{\mathrm{bar}}(B, B^*; V) \to C^\bullet_{\mathrm{bar}}(A, B^*; V)$ *are quasi-isomorphisms.*

Proof. We construct a filtration $C^\bullet(B, B^*; V) \to ... \to \widetilde{F}^\bullet_{p+1} \to \widetilde{F}^\bullet_p \to ... \to \widetilde{F}^\bullet_1 \to C^\bullet(A, B^*; V)$ where $\widetilde{F}^\bullet_p$ is given by:

$$\begin{aligned}\widetilde{F}^\bullet_p := \mathrm{Hom}(V, B^*) \xrightarrow{b} \mathrm{Hom}(B, \mathrm{Hom}(V, B^*)) \xrightarrow{b} ... \xrightarrow{b} \mathrm{Hom}(B^{\otimes p}, \mathrm{Hom}(V, B^*)) \xrightarrow{b} \\ \mathrm{Hom}(B^{\otimes p} \otimes A, \mathrm{Hom}(V, B^*)) \xrightarrow{b} \mathrm{Hom}(B^{\otimes p} \otimes A^{\otimes 2}, \mathrm{Hom}(V, B^*)) \xrightarrow{b} ...\end{aligned}$$

The differential b is induced by the usual differential and then quotienting any necessary indexes. To make our work a bit more clear later on, we identify $\mathrm{Hom}(V, B^*)$ with $\mathrm{Hom}(V \otimes B, \Bbbk)$ again with the A-bimodule structure given by $(a_1\varphi a_2)(v, s) = \varphi(v, a_2sa_1)$. Note that the maps $\widetilde{F}^\bullet_{p+1} \xrightarrow{q^*_{p+1}} \widetilde{F}^\bullet_p$ are injective. We now compute the cokernel of q^*_{p+1}.

$$
\begin{array}{ccccc}
\widetilde{F}^{\bullet}_{p+1} : \ldots \to & \mathrm{Hom}(B^{\otimes p+1} \otimes A^{\otimes n-1}, \mathrm{Hom}(V \otimes B, \Bbbk)) & \longrightarrow & \mathrm{Hom}(B^{\otimes p+1} \otimes A^{\otimes n}, \mathrm{Hom}(V \otimes B, \Bbbk)) & \to \cdots \\
& \downarrow q^*_{p+1} & & \downarrow q^*_{p+1} & \\
\widetilde{F}^{\bullet}_{p} : \ldots \longrightarrow & \mathrm{Hom}(B^{\otimes p} \otimes A^{\otimes n}, \mathrm{Hom}(V \otimes B, \Bbbk)) & \longrightarrow & \mathrm{Hom}(B^{\otimes p} \otimes A^{\otimes n+1}, \mathrm{Hom}(V \otimes B, \Bbbk)) & \to \cdots \\
& \downarrow & & \downarrow & \\
Q^{\bullet}_{p} : \ldots \to & \mathrm{Hom}(B^{\otimes p} \otimes I \otimes A^{\otimes n-1}, \mathrm{Hom}(V \otimes B, \Bbbk)) & \to & \mathrm{Hom}(B^{\otimes p} \otimes I \otimes A^{\otimes n}, \mathrm{Hom}(V \otimes B, \Bbbk)) & \to \cdots
\end{array}
$$

The factor of I in the $(p+1)^{\text{st}}$ position is obtained by applying the Hom-Tensor adjunction along with the fact that the extension of algebras is copure. Let $\varphi \in Q_p^{n+p}$. We now compute the induced differential b:

$$
\begin{aligned}
(b\varphi)(s_1, ..., s_p, u_{p+1}, a_{p+2}, ..., a_{n+p+1}) &= s_1\varphi(s_2, ..., s_p, u_{p+1}, a_{p+2}, ..., a_{n+p+1}) && (4)\\
&+ \sum_{i=1}^{p-1} (-1)^i \varphi(s_1, ..., s_i s_{i+1}, ..., s_p, u_{p+1}, a_{p+2}, ..., a_{n+p+1}) && (5)\\
&+ (-1)^p \varphi(s_1, ..., s_p u_{p+1}, a_{p+2}, ..., a_{n+p+1}) && (6)\\
&+ (-1)^{p+1} \varphi(s_1, ..., s_p, u_{p+1} a_{p+2}, ..., a_{n+p+1})\\
&+ \sum_{i=p+2}^{n+p} (-1)^i \varphi(s_1, ..., s_p, u_{p+1}, a_{n+2}, ..., a_i a_{i+1}, ..., a_{n+p+1})\\
&+ (-1)^{n+p+1} \varphi(s_1, ..., s_p, u_{p+1}, a_{p+2}, ..., a_{n+p}) a_{n+p+1}
\end{aligned}
$$

Note that the right side of (4), (5), and (6) are all zero since here $u_{p+1} \equiv 0 \in B$. We thus see that $Q_p^{\bullet} \cong (\mathrm{Hom}(A^{\otimes \bullet - p}, B(p)), b)$. Where $B(p) = \mathrm{Hom}(V \otimes B \otimes B^{\otimes p} \otimes I, \Bbbk)$ and the left and right A-module actions of $\psi \in B(p)$ are given by:

$$
(a_{p+2}\psi a_{n+p+1})(v, s_0, s_1, ..., s_p, u_{p+1}) = \psi(v, a_{n+p+1}s_0, s_1, ..., s_p, u_{p+1}a_{p+2})
$$

Since I is coH-unital $B(p)$ is coH-unitary, and we can apply Theorem 2.4.3 to obtain an isomorphism $Q_p^{\bullet} \cong (\mathrm{Hom}(I^{\otimes \bullet - p}, B(p)), b)$, and we now note that the right action of I on $B(p)$ is trivial. Hence $Q_p^{\bullet} \cong (\mathrm{Hom}(I^{\otimes \bullet - p}, B(p)), b) \cong (\mathrm{Hom}(I^{\otimes \bullet - p}, B(p)), b')$ which is exact. Thus, the maps $q^*_{p+1} : \widetilde{F}^{\bullet}_{p+1} \to \widetilde{F}^{\bullet}_p$ are quasi-isomorphisms for each p, and so is $q^* : C^{\bullet}(B, B^*; V) \to C^{\bullet}(A, B^*; V)$. The same proof holds for q^*_{bar}. □

Theorem 2.4.5. *Let I be a $\Bbbk$-algebra. The Following statements are equivalent:*

(1) I is coH-unital,

(2) I satisfies excision for $\mathscr{H}\mathscr{H}$-cohomology,

(3) I satisfies excision for bar cohomology,

(4) I satisfies excision for Hochschild cohomology,

(5) I satisfies excision for cyclic cohomology

Proof. (1) $\implies$ (2); Let $0 \to I \xrightarrow{i} A \xrightarrow{q} B \to 0$ be a copure extension of $\Bbbk$-algebras, V a $\Bbbk$-module and $q^* : C^\bullet(B; V) \to C^\bullet(A; V)$ the canonical inclusion. Consider the following commutative diagram of short exact sequences:

$$
\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}(B^{\otimes \bullet}, \mathrm{Hom}(V, B^*)) & \longrightarrow & \mathrm{Hom}(A^{\otimes \bullet}, \mathrm{Hom}(V, A^*)) & \xrightarrow{q^*} & \mathrm{Coker}(q^*) & \longrightarrow & 0 \\
& \Big\downarrow j & & \Big\downarrow = & & \Big\downarrow \varrho_1 & & \\
0 \to & \mathrm{Hom}(A^{\otimes \bullet}, \mathrm{Hom}(V, B^*)) & \longrightarrow & \mathrm{Hom}(A^{\otimes \bullet}, \mathrm{Hom}(V, A^*)) & \longrightarrow & \mathrm{Hom}(A^{\otimes \bullet}, \mathrm{Hom}(V, I^*)) & \longrightarrow & 0 \\
& & & & & \Big\downarrow \varrho_2 & & \\
& & & & & \mathrm{Hom}(I^{\otimes \bullet}, \mathrm{Hom}(V, I^*)) & &
\end{array}
$$

By Proposition 2.4.4, j is a quasi-isomorphism, hence so is ϱ_1, and by Theorem 2.4.3, so is ϱ_2.

(1) $\Rightarrow$ (3); similar to (1) $\Rightarrow$ (2).

(2) and (3) $\Rightarrow$ (4) follows from the long exact sequence 2.2.4.

(4) $\Rightarrow$ (5); follows from Gysin-Connes.

(2) $\Rightarrow$ (1); Given a $\Bbbk$-module V let $A = I \oplus V$ be the algebra with multiplication defined by $(u_1, v_1)\dot{(}u_2, v_2) = (u_1 u_2, 0)$ and the canonical inclusion $\pi^* : (\mathrm{Hom}(V^{\otimes \bullet}, V^*), b) \to (\mathrm{Hom}(A^{\otimes \bullet}, A^*), b)$. Since the complex $(\mathrm{Hom}(I^{\otimes \bullet - 1}, \mathrm{Hom}(V \otimes I)), b') \oplus (\mathrm{Hom}(I^{\otimes \bullet}, I^*), b)$ is a direct summand of $\mathrm{Coker}(q^*)$

and I satisfies excision for $\mathscr{H}-$cohomology, we must have that $(\text{Hom}(I^{\otimes \bullet -1}, \text{Hom}(V \otimes I)), b')$ is exact.

(3) $\Rightarrow$ (1); Similar to (2) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1); Let V and A be as in the case (2) $\Rightarrow$ (1), $f : A \to V$ and $g : A \to I$ the canonical projection, $\tilde{f}^* : CC^\bullet(V)^{\{2\}} \to CC^\bullet(A)^{\{2\}}$ be the canonical inclusion and $\mathscr{L}$ the subcomplex of $\text{Coker}(\tilde{f}^*)$ generated by the elements $\{\varphi \in CC^\bullet(A)^{\{2\}} \mid \varphi(a_0, ..., a_n; a'_0, ..., a'_{n-1}) = 0$ if $a_i, a'_j \in V \ \forall\ i,j$ or $a_i, a'_j \in I \ \forall\ i,j\}$. Since $\text{Coker}(\tilde{f}^*) = CC^\bullet(I)^{\{2\}} \oplus \mathscr{L}$ and I satisfies excision for Hochschild cohomology, $\mathscr{L}$ is exact. Now suppose I is not coH-unital. Let $\varphi \in \text{Hom}(I^{\otimes n-1}, \text{Hom}(V, I^*)) = \text{Hom}(V \otimes I \otimes I^{\otimes n-1})$ be a cycle for b' that is not a boundary. Extend φ to $\text{Hom}(A \otimes A^{\otimes n})$ by $\varphi(a_0, ..., a_n) = \varphi(f(a_0), g(a_1), ..., g(a_n))$, and now we create the element $(0, \varphi) \in \mathscr{L}$ of degree $n+1$ which is seen to be a cycle but not a boundary, contradicting the exactness of $\mathscr{L}$. $\square$

Chapter 3

Normalized and Reduced Constructions

In this chapter we will further discuss the notions of Normalized and Reduced cochain complexes branching from the previous chapter. In the homological case, these have been thoroughly developed in Loday [Lod13], however for the purposes of this paper, we require an additional construction, that of a "Normalized bar cochain complex", as well as a strong understanding of how it fits in with the other complexes using the canonical maps of the previous chapter. We provide most of the details for the homological setting in Appendix A, which are a bit easier to start with, but a reader familiar with Hochschild and Cyclic (co)homology will have no trouble continuing here and using the appendix for comparison.

3.1 Normalized and Reduced Cochain complexes

Definition 3.1.1. For a unital algebra A we define the **degenerate bar complex** $D^{\text{bar}}_{\bullet}(A)$ to be the subcomplex of $C^{\text{bar}}_{\bullet}(A)$ where for $n \geq 2$ the n^{th} term is given by

$$D^{\text{bar}}_n(A) = \widetilde{D}_n(A) + Q(D_n(A))$$

where $\widetilde{D}_n(A) = \{(a_0, ..., a_n) \,|\, a_i = 1 \text{ for some } 0 < i < n\}$. For the low degrees, we define

$$D^{\text{bar}}_1(A) := \Bbbk(1,1) + Q(D_1(A))$$

$$D^{\text{bar}}_0(A) := \Bbbk$$

Indeed, for $n \geq 2$, and any $(a_0, ..., 1, ..., a_n) \in \widetilde{D}_n(A)$ where the 1 is in the k^{th} position, $1 \leq k \leq n-1$, we have

$$b'(a_0, ..., 1, ..., a_n) = \sum_{i=1}^{k-2} (-1)^i (a_0, ..., a_i a_{i+1}, ..., 1, ..., a_n)$$
$$+ (-1)^{k-1}(a_0, ..., a_{k-1} \cdot 1, a_{k+1}, ..., a_n) + (-1)^k (a_0, ..., a_{k-1}, 1 \cdot a_k, ..., a_n)$$
$$\sum_{i=k+1}^{n-1} (-1)^n (a_0, ..., 1, ..., a_i a_{i+1}, ..., a_n).$$

Hence $b'\widetilde{D}_n(A) \subset \widetilde{D}_{n-1}(A)$ and we also have $b'Q(D_n(A)) = Q(bD_n(A)) \subset Q(D_{n-1}(A))$. It's easy to check b' in low degrees, so we have a well defined complex.

Notice that $D_n^{\text{bar}}(A)$ can be thought of as the submodule of $C_n^{\text{bar}}(A)$ generated by elements of the form

- $(a_0, ..., a_n)$ where $a_i = 1$ for some $0 < i < n$
- $(1, a_1, ..., a_{n-1}, 1)$ (3.1.1)
- $(a_0, ..., a_{n-1}, 1) + (-1)^n (1, a_0, ..., a_{n-1})$.

It's easy to see from the above construction that the map $s : D_n^{\text{bar}}(A) \to D_{n+1}^{\text{bar}}(A)$ is well defined, and hence gives a contracting homotopy for $D_n^{\text{bar}}(A)$ (see Section 2.1). Lastly, we note that the reduced case will be the same as the normalized case, so the **reduced degernate bar complex $D_\bullet^{\text{bar}}(A)_{\text{red}} = D_\bullet^{\text{bar}}(A)$.**

Definition 3.1.2. For a unital algebra A the **normalized bar cochain complex** $\overline{C}^\bullet_{\text{bar}}(A)$, is the subcomplex of $C^\bullet_{\text{bar}}(A)$, where the n^{th} degree consists of all functionals φ such that $\varphi|_{D_n^{\text{bar}}(A)} = 0$. In other words, $\varphi(a_0, ..., a_n) = 0$ when $(a_0, ..., a_n)$ is of the form (3.1.1).

The **degenerate bar cochain complex** $D^\bullet_{\text{bar}}(A)$ is the cokernel

$$0 \to \overline{C}^\bullet_{\text{bar}}(A) \to C^\bullet_{\text{bar}}(A) \to D^\bullet_{\text{bar}}(A) \to 0$$

i.e. the subcomplex of $\text{Hom}(D_\bullet^{\text{bar}}(A))$ whose functionals extend to $C^\bullet_{\text{bar}}(A)$. Note that $D^0_{\text{bar}}(A) = \mathbb{j}_A$. As with the homological case, $D^\bullet_{\text{bar}}(A)$ comes with a contracting homotopy $s : D^n_{\text{bar}}(A) \to$

$D^{n-1}_{\text{bar}}(A)$ where $s\varphi(a_0, ..., a_{n-1}) = \varphi(1, a_0, ..., a_n)$, and $b's + sb' = 1$. Hence $H^n(\overline{C}^{\bullet}_{\text{bar}}(A)) = H^n(C^{\bullet}_{\text{bar}}(A)) = HB^n(A)$.

Just as in the homological setting the **reduced degenerate bar cochain complex** is the same as the degenerate bar cochain complex. $D^{\bullet}_{\text{bar}}(A)_{\text{red}} := D^{\bullet}_{\text{bar}}(A)$, likewise the **reduced bar complex** is the same as the normalized bar complex $C^{\bullet}_{\text{bar}}(A)_{\text{red}} := \overline{C}^{\bullet}_{\text{bar}}(A)$.

Corollary 3.1.3. *Since the sequence* $\Bbbk \to A^+ \to A$ *is* $\Bbbk$*-split, by identifying* $\overline{C}^{\bullet}_{\text{bar}}(A^+)$ *with* $\text{Hom}(\overline{C}^{\text{bar}}_{\bullet}(A^+))$*, we obtain the isomorphism*

$$\overline{C}^{\bullet}_{\text{bar}}(A^+) \cong \left(C^{\bullet}_{\text{bar}}(A) \oplus C^{\bullet-1}_{\text{bar}}(A), \begin{bmatrix} b' & 0 \\ 1 & -b' \end{bmatrix} \right)$$

(See Section A.3 for the homological case).

Definition 3.1.4. The complex $CB^{\bullet\bullet}(A)^{\{2\}}$ is given by

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
C^2_{\text{bar}}(A) & \longrightarrow & C^2_{\text{bar}}(A) \\
\uparrow & & \uparrow \\
C^1_{\text{bar}}(A) & \longrightarrow & C^1_{\text{bar}}(A) \\
\uparrow{\scriptstyle b'} & & \uparrow{\scriptstyle -b'} \\
C^0_{\text{bar}}(A) & \xrightarrow{1} & C^0_{\text{bar}}(A)
\end{array}$$

Note that the total complex of $CB(A)^{\{2\}}$ is isomorphic to $\overline{C}_{\text{bar}}(A^+)$. Hence $H^n(\text{Tot}\, CB^{\bullet\bullet}(A)^{\{2\}}) = 0$.

3.2 The (λ, Q)-sequence for cochain complexes

Here we develop the (λ, Q)-sequence for the above complexes. Throughout the rest of this section, if not stated, we will assume $\Bbbk$ contains $\mathbb{Q}$.

By associating $\overline{C}^{\bullet}(A)$ with $\mathrm{Hom}(\overline{C}_{\bullet}(A))$, and $\overline{C}^{\bullet}_{\mathrm{bar}}(A)$ with $\mathrm{Hom}(\overline{C}^{\mathrm{bar}}_{\bullet}(A))$ we see that the maps Q and $1-\lambda$ descend to the normalized complexes, giving us the (λ, Q)-sequences

$$... \to \overline{C}^{\bullet}(A) \xrightarrow{1-\lambda} \overline{C}^{\bullet}_{\mathrm{bar}}(A) \xrightarrow{Q} \overline{C}^{\bullet}(A) \xrightarrow{1-\lambda} \overline{C}^{\bullet}_{\mathrm{bar}}(A) \xrightarrow{Q} \overline{C}^{\bullet}(A) \to ...$$

And likewise, we have a similar sequence for the reduced cochain complexes

$$... \to C^{\bullet}(A)_{\mathrm{red}} \xrightarrow{1-\lambda} \overline{C}^{\bullet}_{\mathrm{bar}}(A) \xrightarrow{Q} C^{\bullet}(A)_{\mathrm{red}} \xrightarrow{1-\lambda} \overline{C}^{\bullet}_{\mathrm{bar}}(A) \xrightarrow{Q} C^{\bullet}(A)_{\mathrm{red}} \to ...$$

Theorem 3.2.1. *When $\Bbbk$ contains $\mathbb{Q}$, the reduced (λ, Q)-sequence*

$$... \to C^{\bullet}(A)_{\mathrm{red}} \xrightarrow{1-\lambda} \overline{C}^{\bullet}_{\mathrm{bar}}(A) \xrightarrow{Q} C^{\bullet}(A)_{\mathrm{red}} \xrightarrow{1-\lambda} \overline{C}^{\bullet}_{\mathrm{bar}}(A) \xrightarrow{Q} C^{\bullet}(A)_{\mathrm{red}} \to ...$$

is exact. By Definition 2.2.2, we have a similar result for the normalized cochain complexes, the only difference being that in degree 0 this sequence becomes

$$... \to \overline{C}^{0}(A) \xrightarrow{0} \overline{C}^{0}_{\mathrm{bar}}(A) \xrightarrow{1} \overline{C}^{0}(A) \xrightarrow{0} \overline{C}^{0}_{\mathrm{bar}}(A) \xrightarrow{1} \overline{C}^{0}(A) \to ...$$

which could be inexact.

Proof. By Definitions 3.1.2 and 2.2.1 above it suffices to show that the degenerate (λ, Q)-sequence

$$... \to D^{\bullet}(A) \xrightarrow{1-\lambda} D^{\bullet}_{\mathrm{bar}}(A) \xrightarrow{Q} D^{\bullet}(A) \xrightarrow{1-\lambda} D^{\bullet}_{\mathrm{bar}}(A) \xrightarrow{Q} D^{\bullet}(A) \to ...$$

is exact. We make use of the homotopy in Proposition 2.3.1. Let's extend the diagram to help better visualize the following arguments.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & D^n_{\mathrm{bar}}(A) & \xrightarrow{Q} & D^n(A) & \xrightarrow{1-\lambda} & D^n_{\mathrm{bar}}(A) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \iota^*_{\mathrm{bar}}} & & \uparrow{\scriptstyle \iota^*} & & \uparrow{\scriptstyle \iota^*_{\mathrm{bar}}} & & \\
\cdots & \longrightarrow & C^n_{\mathrm{bar}}(A) & \xrightarrow{Q} & C^n(A) & \xrightarrow{1-\lambda} & C^n_{\mathrm{bar}}(A) & \longrightarrow & \cdots
\end{array}$$

Fix an $n > 0$ and let $\widehat{\varphi} \in D^n(A)$ be such that $(1-\lambda)\widehat{\varphi} = 0$. By Definition 3.1.2 there exists $\varphi \in C^n(A)$ such that $\iota^*\varphi = \widehat{\varphi}$, and we may construct the element $\frac{1}{n+1}\varphi \in C^n_{\mathrm{bar}}(A)$. We may not have $Q\frac{1}{n+1}\varphi = \varphi$ but notice that $\iota^* Q\frac{1}{n+1}\varphi = \widehat{\varphi}$. Indeed, if $(a_0, ..., a_n) \in D_n(A)$ then $\iota^* Q\frac{1}{n+1}\varphi(a_0, ..., a_n) = Q\frac{1}{n+1}\widehat{\varphi}(a_0, ..., a_n) = \frac{1}{n+1}\widehat{\varphi}(Q(a_0, ..., a_n)) = \widehat{\varphi}(a_0, ..., a_n)$. Where the last equality holds since $(1-\lambda)\widehat{\varphi} = 0$ implies $\varphi(a_0, ..., a_n) = \varphi(\lambda(a_0, ..., a_n))$ for all $(a_0, ..., a_n) \in D_n(A)$. Since the above diagram commutes, we see that $\iota^*_{\text{bar}} \frac{1}{n+1}\varphi$ is such that $Q\iota^*_{\text{bar}} \frac{1}{n+1}\varphi = \widehat{\varphi}$. Hence, $Q(D^n_{\text{bar}}) = \ker(1-\lambda)$.

Let's shift the diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & D^n(A) & \xrightarrow{1-\lambda} & D^n_{\text{bar}}(A) & \xrightarrow{Q} & D^n(A) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \iota^*} & & \uparrow{\scriptstyle \iota^*_{\text{bar}}} & & \uparrow{\scriptstyle \iota^*} & & \\
\cdots & \longrightarrow & C^n(A) & \xrightarrow{1-\lambda} & C^n_{\text{bar}}(A) & \xrightarrow{Q} & C^n(A) & \longrightarrow & \cdots
\end{array}$$

Let $\widehat{\psi} \in D^n_{\text{bar}}(A)$ be such that $Q\widehat{\psi} = 0$, and let $\psi \in C^n_{\text{bar}}(A)$ be such that $\iota^*_{\text{bar}}\psi = \widehat{\psi}$. We can then construct the element $\psi := -\frac{1}{n+1}\sum_{i=1}^n i\lambda^i\psi \in C^n(A)$. Again, we have $\iota^*_{\text{bar}}(1-\lambda)\psi = \widehat{\psi}$ since for any $(a_0, ..., a_n) \in D^n_{\text{bar}}(A)$,

$$\begin{aligned}
\iota^*(1-\lambda)\left(-\frac{1}{n+1}\sum_{i=1}^n i\lambda^i\psi\right)(a_0, ..., a_n) &= (1-\lambda)\left(-\frac{1}{n+1}\sum_{i=1}^n i\lambda^i\widehat{\psi}\right)(a_0, ..., a_n) \\
&= -\frac{1}{n+1}\left[\left(\sum_{i=1}^n i\lambda^i - \sum_{i=1}^n i\lambda^{i+1}\right)\widehat{\psi}\right](a_0, ..., a_n) \\
&= -\frac{1}{n+1}\left[\left(-(n+1)+Q\right)\widehat{\psi}\right](a_0, ..., a_n) \\
&= \widehat{\psi}(a_0, ..., a_n)
\end{aligned}$$

By commutativity of the diagram we see that $(1-\lambda)\iota^*\psi = \widehat{\psi}$. Hence $(1-\lambda)(D^n(A)) = \ker Q$.

Note that for degree 0, the degenerate complex becomes

$$\cdots \to 0 \to \mathbb{j}_A \to 0 \to \mathbb{j}_A \to \ldots$$

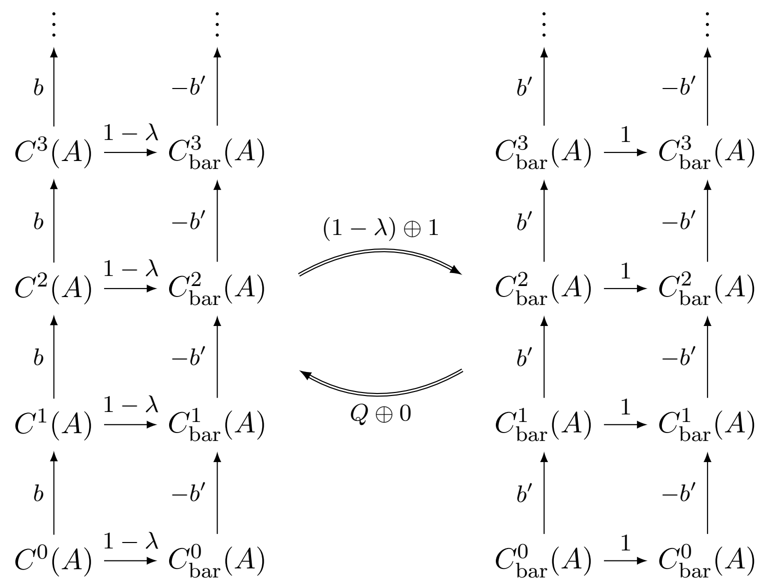
hence the inexactness for the normalized (λ, Q)-sequence in degree 0. However, it follows from Definition 2.2.3 that we re-obtain exactness in the reduced (λ, Q)-sequence. □

Note. As with the homological case, to avoid too much eye clutter we will often drop the double bullet indices from our notation when dealing with double cochain complexes. This does pose the slight problem of using the same notation in both situations. When it's not clear from the context which situation we are in, we will be sure to use the indices.

Corollary 3.2.2. *Given an augmented algebra* $\Bbbk \to A^+ \to A$, *and associating* $C^n(A^+)_{\text{red}}$ *and* $\overline{C}^n_{\text{bar}}(A^+)$ *with their bicomplexes (see Corollary 3.1.3) and Theorem 3.2.1 becomes*

$$\dots \xrightarrow{Q\oplus 0} CC(A)^{\{2\}} \xrightarrow{1-\lambda\oplus 1} CB(A)^{\{2\}} \xrightarrow{Q\oplus 0} CC(A)^{\{2\}} \xrightarrow{1-\lambda\oplus 1} CB(A)^{\{2\}} \xrightarrow{Q} \dots$$

We can summarize this in the following diagram:



(3.2.2)

Proof. We have a canonical association of $C^\bullet(A^+)_{\text{red}}$ with $\operatorname{Hom}(C_\bullet(A^+)_{\text{red}})$, so for $n \geq 1$, $\varphi \in C^n(A^+)_{\text{red}}$, and $(a_0 + t_0, ..., a_n) \in \overline{C}^{\text{bar}}_n(A^+)$ (see Proposition A.3.1) we compute

$$\begin{aligned}
(1-\lambda)\varphi(a_0 + t_0, a_1, ..., a_n) &= \varphi((1-\lambda)(a_0 + t_0, a_1, ..., a_n)) \\
&= \varphi(a_0 + t_0, a_1, ..., a_n) - (-1)^n \varphi(a_n, a_0 + t_0, a_1, ..., a_{n-1}) \\
&= \varphi(a_0, a_1, ..., a_n) + \varphi(t_0, a_1, ..., a_n) - (-1)^n \varphi(a_n, a_0, a_1, ..., a_{n-1}) \\
&= (1-\lambda)\varphi(a_0, ..., a_n) + \varphi(t_0, a_0, ..., a_n)
\end{aligned}$$

And now associating $C^n(A)_{\text{red}}$ with $C^n(A) \oplus C^{n-1}(A)$ we see that the map $1-\lambda$ becomes $(1-\lambda)\oplus 1$.

Similarly, we associate $\overline{C}^n_{\text{bar}}(A^+)$ with $\operatorname{Hom}(\overline{C}^{\text{bar}}_n(A^+))$. Let $\varphi \in \overline{C}^n_{\text{bar}}(A^+)$ and

$(a_0 + t_0, a_1, ..., a_n) \in C_n(A^+)_{\text{red}}$. We compute

$$\begin{aligned} Q\varphi(a_0 + t_0, a_1, ..., a_n) &= \varphi(Q(a_0 + t_0, a_1, ..., a_n)) \\ &= \varphi(Q(a_0, ..., a_n)) + \varphi(Q(t_0, a_1, ..., a_n)) \\ &= Q\varphi(a_0, ..., a_n). \end{aligned}$$

Note that the right had term in the second-to-last line above is zero since, whenever t_0 is in position 1 through $n-1$ the term is in $\overline{C}_n(A^+)$ where φ vanishes. The only terms left then are when t_0 is in position 0 and n, but the combination of these two is in $D_n(A^+)$ as well. Thus, the map Q becomes $Q \oplus 0$. □

3.3 Cyclic Cocomplexes

We can now define normalized and reduced versions of the cyclic cochain bicomplex (2.3), but of course, we must start with unital algebras.

Definition 3.3.1. For a unital algebra A, the **degenerate cochain bicomplex** $DD^{\bullet\bullet}(A)$ is the complex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
D^2(A) & \longrightarrow & D^2_{\text{bar}}(A) & \longrightarrow & D^2(A) & \longrightarrow & D^2_{\text{bar}}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
D^1(A) & \longrightarrow & D^1_{\text{bar}}(A) & \longrightarrow & D^1(A) & \longrightarrow & D^1_{\text{bar}}(A) & \longrightarrow & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow & & \uparrow & & \\
0 & \xrightarrow{1-\lambda} & \mathring{\mathbb{j}}_A & \xrightarrow{Q} & 0 & \longrightarrow & \mathring{\mathbb{j}}_A & \longrightarrow & \cdots
\end{array}$$

For notational simplicity later on, let's use $HD(A)$ to mean the homology of this complex. Since this complex has exact columns, for any n we have $HD^n(A) := H^n(\text{Tot}DD^{\bullet\bullet}(A)) = 0$.

Definition 3.3.2. For a unital algebra A, the **reduced degenerate cochain bicomplex** $DD^{\bullet\bullet}(A)_{\text{red}}$ is the complex

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
D^2(A) & \longrightarrow & D^2_{\text{bar}}(A) & \longrightarrow & D^2(A) & \longrightarrow & D^2_{\text{bar}}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
D^1(A) & \longrightarrow & D^1_{\text{bar}}(A) & \longrightarrow & D^1(A) & \longrightarrow & D^1_{\text{bar}}(A) & \longrightarrow & \cdots \\
{\scriptstyle b}\uparrow & & {\scriptstyle -b'}\uparrow & & \uparrow & & \uparrow & & \\
\mathring{\mathbb{j}}_A & \xrightarrow{1-\lambda} & \mathring{\mathbb{j}}_A & \xrightarrow{Q} & \mathring{\mathbb{j}}_A & \longrightarrow & \mathring{\mathbb{j}}_A & \longrightarrow & \cdots
\end{array}
$$

The **reduced degenerate cohomology of** A is then $\overline{HD}^{\bullet}(A) := H^{\bullet}(\text{Tot}(DD(A)_{\text{red}}))$.

With a slight abuse of notation let's use $\mathring{\mathbb{j}}_A[0,e]$ be the dual to our homological construction, that is, the bicomplex with $\mathring{\mathbb{j}}_A$'s in row zero and even columns with 0's everywhere else:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
{\scriptstyle b}\uparrow & & {\scriptstyle -b'}\uparrow & & \uparrow & & \uparrow & & \\
\mathring{\mathbb{j}}_A & \xrightarrow{1-\lambda} & 0 & \xrightarrow{Q} & \mathring{\mathbb{j}}_A & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

Thus, we have an exact sequence

$$0 \to DD(A) \to DD(A)_{\text{red}} \to \mathring{\mathbb{j}}_A[e,0] \to 0,$$

and from the induced cohomoligical long exact sequence, we see that $\overline{HD}^{2n}(A) = \mathring{\mathbb{j}}_A$ and $\overline{HD}^{2n+1}(A) = 0$ for $n \geq 0$.

Definition 3.3.3. For a unital algebra A, the **normalized cyclic complex** $\overline{CC}^{\bullet\bullet}(A)$ is given as the kernel

$$0 \to \overline{CC}(A) \to CC(A) \to DD(A) \to 0.$$

The full bicomplex is of course

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\overline{C}^2(A) & \longrightarrow & \overline{C}^2_{\text{bar}}(A) & \longrightarrow & \overline{C}^2(A) & \longrightarrow & \overline{C}^2_{\text{bar}}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\overline{C}^1(A) & \longrightarrow & \overline{C}^1_{\text{bar}}(A) & \longrightarrow & \overline{C}^1(A) & \longrightarrow & \overline{C}^1_{\text{bar}}(A) & \longrightarrow & \cdots \\
b\uparrow & & -b'\uparrow & & \uparrow & & \uparrow & & \\
\overline{C}^0(A) & \xrightarrow{1-\lambda} & \overline{C}^0_{\text{bar}}(A) & \xrightarrow{Q} & \overline{C}^0(A) & \longrightarrow & \overline{C}^0_{\text{bar}}(A) & \longrightarrow & \cdots
\end{array}$$

It follows that $H^\bullet(\operatorname{Tot}\overline{CC}(A)) = HC^\bullet(A)$.

Definition 3.3.4. For a unital algebra A the **reduced cyclic cochain bicomplex** $CC^{\bullet\bullet}(A)_{\text{red}}$ is the kernel

$$0 \to CC(A)_{\text{red}} \to \overline{CC}(A) \to \mathring{\mathbb{j}}_A[0,e] \to 0$$

or equivalently

$$0 \to CC(A)_{\text{red}} \to CC(A) \to DD(A)_{\text{red}} \to 0.$$

The diagram is then

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^2(A)_{\text{red}} & \longrightarrow & \overline{C}^2_{\text{bar}}(A) & \longrightarrow & C^2(A)_{\text{red}} & \longrightarrow & \overline{C}^2_{\text{bar}}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^1(A)_{\text{red}} & \longrightarrow & \overline{C}^1_{\text{bar}}(A) & \longrightarrow & C^1(A)_{\text{red}} & \longrightarrow & \overline{C}^1_{\text{bar}}(A) & \longrightarrow & \cdots \\
b\uparrow & & -b'\uparrow & & \uparrow & & \uparrow & & \\
C^0(A)_{\text{red}} & \xrightarrow{1-\lambda} & \overline{C}^0_{\text{bar}}(A) & \xrightarrow{Q} & C^0(A)_{\text{red}} & \longrightarrow & \overline{C}^0_{\text{bar}}(A) & \longrightarrow & \cdots
\end{array}$$

We are now in a position to develop the dual notions of the $\mathscr{B}(A)$ complex as well as all of the quasi-isomorphisms given in Section A.3, which we shall do briefly. Starting with the complex $CC(A)$ we may apply Lemma A.3.6 on the odd degree columns to obtain $\mathscr{B}^{\bullet\bullet}(A)$:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
b\uparrow & & b\uparrow & & b\uparrow & & \\
C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) & \longrightarrow & \cdots \\
b\uparrow & & b\uparrow & & & & \\
C^1(A) & \xrightarrow{B} & C^0(A) & & & & \\
b\uparrow & & & & & & \\
C^0(A) & & & & & &
\end{array}
$$

The horizontal differential is given by $B = Qs(1-\lambda)$. Normalizing each column gives the complex $\overline{\mathscr{B}}^{\bullet\bullet}(A)$, now with horizontal differential $\overline{B} = Qs$.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
b\uparrow & & b\uparrow & & b\uparrow & & \\
\overline{C}^2(A) & \xrightarrow{\overline{B}} & \overline{C}^1(A) & \xrightarrow{\overline{B}} & \overline{C}^0(A) & \longrightarrow & \cdots \\
b\uparrow & & b\uparrow & & & & \\
\overline{C}^1(A) & \xrightarrow{\overline{B}} & \overline{C}^0(A) & & & & \\
b\uparrow & & & & & & \\
\overline{C}^0(A) & & & & & &
\end{array}
$$

Alternatively, we can start with $CC(A)$, normalize first to get $\overline{CC}(A)$, and then apply Lemma A.3.6 to get $\overline{\mathscr{B}}(A)$. We have proved the following

Proposition 3.3.5. *The maps in the diagram induce quasi-isomorphisms on the total complexes.*

$$\begin{array}{ccc} \overline{CC}(A) & \longrightarrow & CC(A) \\ \downarrow & & \downarrow \\ \overline{\mathscr{B}}(A) & \longrightarrow & \mathscr{B}(A) \end{array}$$

For the reduced complexes, we can now take $\mathscr{B}(A)_{\text{red}}$ to be the kernel [Lod13, pg 65]

$$0 \to \mathscr{B}(A)_{\text{red}} \to \overline{\mathscr{B}}(A) \to \overline{\mathscr{B}}(\Bbbk) \to 0. \tag{3.3.5}$$

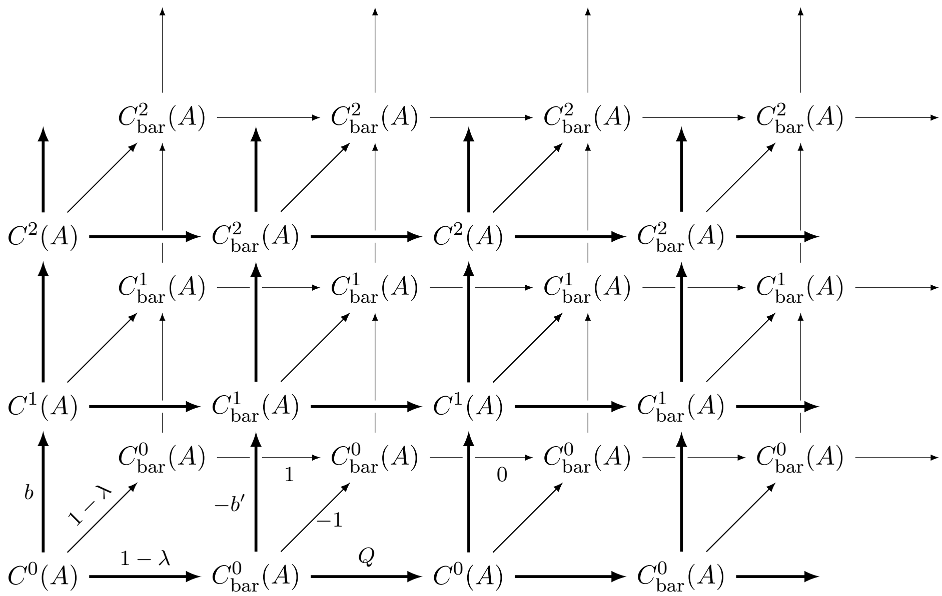
Alternatively, we can start with the complex $CC(A)_{\text{red}}$ and then apply the killing contractible complexes Lemma A.3.6 to get $\mathscr{B}(A)_{\text{red}}$.

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow b & & \uparrow b & & \uparrow b & & \\ C^2(A)_{\text{red}} & \xrightarrow{\overline{B}} & C^1(A)_{\text{red}} & \xrightarrow{\overline{B}} & C^0(A)_{\text{red}} & \longrightarrow & \cdots \\ \uparrow b & & \uparrow b & & & & \\ C^1(A)_{\text{red}} & \xrightarrow{\overline{B}} & C^0(A)_{\text{red}} & & & & \\ \uparrow b & & & & & & \\ C^0(A)_{\text{red}} & & & & & & \end{array}$$

We note that we can now obtain the reduced cyclic cohomology by $\overline{HC}^n(A) = H^n(\operatorname{Tot}\mathscr{B}(A)_{\text{red}}) = H^n(CC^n(A)_{\text{red}})$.

For a non-unital algebra A, the cyclic cohomology is typically defined by $HC^n(A) = H^n(\operatorname{Tot} CC(A))$ (cf. [Lod13, pg. 72]). With the above constructions this should match with our functorial definition of setting $HC^n(A) = \ker(HC(A^+) \to HC(\Bbbk))$, as well as with the reduced construction $HC^n(A) := \overline{HC}^n(A^+)$. Equivalence of the functorial definition and reduced construction follows from the homological long exact sequence obtained from (3.3.5) or from Definition 3.3.4 when you notice that $\mathbb{j}_{A^+} = \Bbbk$ and $\Bbbk[0,e] = \overline{CC}(\Bbbk)$. To match these with the standard definition we need a few more constructions.

Definition 3.3.6. For an augmented algebra A^+, each of the columns of the complex $CC(A^+)_{\mathrm{red}}$ can be expanded via Definition (3.1.4) and connected by (3.2.2) to obtain the tricomplex $CCB(A)$.



(3.3.6)

The back face of the above diagram we call the **cyclic bar complex of** A and denote this by $CB(A)$.

Proposition 3.3.7. *$HC^n(A) = H^n(\mathrm{Tot}\, CC(A)) = H^n(\mathrm{Tot}\, CC(A^+)_{\mathrm{red}})$.*

Proof. We have an exact sequence of complexes

$$0 \to CB(A) \to CCB(A) \to CC(A) \to 0.$$

Note that the rows of $CB(A)$ are exact. Hence this complex is quasi-isomorphic to the kernel of its first two columns, which is 0. By construction $CCB(A) \cong CC(A^+)_{\mathrm{red}}$. □

3.4 Connes Complexes

Definition 3.4.1. Recall that for an algebra A, the Connes cochain complex $C^\bullet_\lambda(A)$ is given as the kernel of the first two columns of $CC(A)$.

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^2_\lambda(A) & \dashrightarrow & C^2(A) & \longrightarrow & C^2_{\text{bar}}(A) & \longrightarrow & C^2(A) & \longrightarrow & C^2_{\text{bar}}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^1_\lambda(A) & \dashrightarrow & C^1(A) & \longrightarrow & C^1_{\text{bar}}(A) & \longrightarrow & C^1(A) & \longrightarrow & C^1_{\text{bar}}(A) & \longrightarrow & \cdots \\
b\uparrow & & b\uparrow & & -b'\uparrow & & \uparrow & & \uparrow & & \\
C^0_\lambda(A) & \dashrightarrow & C^0(A) & \xrightarrow{1-\lambda} & C^0_{\text{bar}}(A) & \xrightarrow{Q} & C^0(A) & \longrightarrow & C^0_{\text{bar}}(A) & \longrightarrow & \cdots
\end{array}
$$

When $\Bbbk$ contains $\mathbb{Q}$ the rows of $CC(A)$ are exact, so it follows that the inclusion $C_\lambda(A) \to CC(A)$ is a quasi-isomorphism.

Definition 3.4.2. The **degenerate Connes cochain complex $D^\bullet_\lambda(A)$,** is the kernel of the first two columns of $DD(A)$.

$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
D^2_\lambda(A) & \dashrightarrow & D^2(A) & \longrightarrow & D^2_{\text{bar}}(A) & \longrightarrow & D^2(A) & \longrightarrow & D^2_{\text{bar}}(A) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
D^1_\lambda(A) & \dashrightarrow & D^1(A) & \longrightarrow & D^1_{\text{bar}}(A) & \longrightarrow & D^1(A) & \longrightarrow & D^1_{\text{bar}}(A) & \longrightarrow & \cdots \\
b\uparrow & & b\uparrow & & -b'\uparrow & & \uparrow & & \uparrow & & \\
0 & \dashrightarrow & 0 & \xrightarrow{1-\lambda} & D^0_{\text{bar}}(A) & \xrightarrow{Q} & 0 & \longrightarrow & D^0_{\text{bar}}(A) & \longrightarrow & \cdots
\end{array}
$$

Now, while the total cohomology of $DD(A)$ is zero, the bottom row is not exact, hence the inclusion of $D_\lambda(A)$ into $DD(A)$ is not a quasi-isomorphism, and $D_\lambda(A)$ is not exact. However, the inexactness is quite manageable. Let $\widetilde{DD}(A)$ be the same bicomplex as $DD(A)$ except replace the zeroth row with all 0's. Then, when $\mathbb{k}$ contains $\mathbb{Q}$, the rows of $\widetilde{DD}(A)$ are exact, and thus the natural inclusion $D_\lambda(A) \to \widetilde{DD}(A)$ is a quasi-isomorphism.

$$
\begin{array}{ccccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
D^2_\lambda & & D^2 & \longrightarrow & D^2_{\text{bar}} & \longrightarrow & D^2 & \longrightarrow & D^2_{\text{bar}} & \longrightarrow & \\
\uparrow & \dashrightarrow & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
D^1_\lambda & & D^1 & \longrightarrow & D^1_{\text{bar}} & \longrightarrow & D^1 & \longrightarrow & D^1_{\text{bar}} & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow &
\end{array}
\tag{3.4.2}
$$

We will show that this implies that, similar to the homological case, $H^{2n}(D_\lambda(A)) = \mathring{\mathbb{j}}_A$ for $n > 0$, and $H^{2n+1}(D_\lambda(A)) = 0$. In the homological case since $(1, ..., 1) \in D^\lambda_{2n}(A)$, but $(1, ..., 1) \notin D^\lambda_{2n-1}$ it's very clear where the inexactness comes from. Conversely, we should suspect that in the cohomological case that there might be $\varphi \in D^{2n}_\lambda(A)$ such that $\varphi(1, ..., 1) \neq 0$ and $b\varphi = 0$. This would account for any inexactness, since for any $\psi \in D^{2n-1}_\lambda(A)$, $b\psi(1, ..., 1) = \psi(1, ..., 1) \equiv \psi(0, ..., 0) = 0$. Furthermore, the cohomology suggests that for every $t \in \mathring{\mathbb{j}}_A$ we have such a $\varphi \in D^{2n}_\lambda(A)$, with $b\varphi = 0$ and $\varphi(1, ..., 1) = t$.

Proposition 3.4.3. *Let $K^\bullet_{\mathbb{1}}(A)$ be the complex whose n^{th} degree consists of all $\varphi \in C^n(A)$ such that $\varphi(1, ..., 1) = 0$. Then we have the following cokernel*

$$0 \to K^\bullet_{\mathbb{1}}(A) \to C^\bullet(A) \to C^\bullet(\mathring{\mathbb{j}}_A) \to 0.$$

In other words, for any n and any $t \in \mathring{\mathbb{j}}_A$, there exists $\varphi \in C^n(A)$ such that $\varphi(1, ..., 1) = t$ and moreover these are the only such values $\varphi(1, ..., 1)$ can be.

Proof. For any $n \geq 0$ let $\varphi \in C^n(A)$ be such that $\varphi(1, ..., 1) = t \in \mathbb{k}$. Then we can construct $s^n\varphi \in C^0(A) = A^*$, such that $s^n\varphi(1) = \varphi(1, ..., 1) = t$. By definition we must have $t \in \mathring{\mathbb{j}}_A$.

Conversely, let $t \in \mathring{\mathbb{j}}_A$, and let $\varphi \in C^0(A)$ such that $\varphi(1) = t$. We have a map $d_0 : C^n(A) \to C^{n+1}(A)$ which is the dual to contracting on the zeroth position i.e. $d_0\varphi(a_0, a_1, ..., a_{n+1}) =$

$\varphi(a_0a_1, ..., a_{n+1})$. Thus for any $n > 0$ we can form the functional $d_0^n\varphi \in C^n(A)$ such that $d_0^n\varphi(1, ..., 1) = \varphi(1) = t$. □

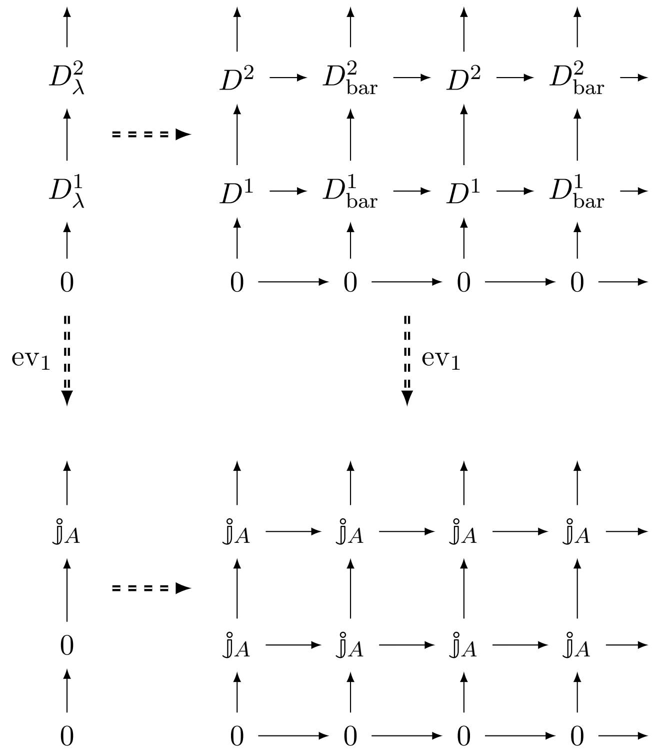
Proposition 3.4.4. *When* $\Bbbk$ *contains* $\mathbb{Q}$*, for any even* n *and any* $t \in \mathbb{j}_A$*, there exists* $\varphi \in C_\lambda^n(A)$ *such that* $\varphi(1, ..., 1) = t$*.*

Proof. Note that if n is even then $\lambda(1, ..., 1) = (-1)^n(1, ..., 1) = (1, ..., 1)$. By the previous proposition we have $\varphi \in C^n(A)$ such that $\varphi(1, ..., 1) = t$. Then $\frac{1}{n+1}Q\varphi$ will be such that $\frac{1}{n+1}Q\varphi(1, ..., 1) = t$ and of course $(1 - \lambda)\frac{1}{n+1}Q\varphi = 0$. □

In light of the above, when n is even we simply need to find some $\varphi \in C_\lambda^n(A)$ such that $\varphi(1, ..., 1) \neq 0$ and $b\varphi = 0$. Unfortunately, constructing such an element is not possible by conventional means, so we must turn to cohomological arguments to prove their existence. For any n let $\mathrm{ev}_1 : C^n(A) \to \mathbb{j}_A$ be the evaluation on $(1, ..., 1)$, $\varphi \mapsto \varphi(1, ..., 1)$. This map restricts to the degenerate complexes, and adjusting the diagram (3.4.2) gives the commutative square

$$\begin{array}{ccc} D_\lambda(A) & \longrightarrow & \widetilde{DD}(A) \\ \Big\downarrow{\scriptstyle \mathrm{ev}_1} & & \Big\downarrow{\scriptstyle \mathrm{ev}_1} \\ D_\lambda(\mathbb{j}_A) & \longrightarrow & \widetilde{DD}(\mathbb{j}_A) \end{array}$$

Viewing the expansion of the complexes, we see that on the right evaluation map is a quasi-isomorphism when restricted to columns, hence is a quasi-isomorphism on the total complexes. Likewise, since the rows of the bottom right complex are exact, the bottom horizontal inclusion is a quasi-isomorphism. Hence, the inclusion on the left is also a quasi-isomorphism.



We have shown the following:

Theorem 3.4.5. *When* $\Bbbk$ *contains* $\mathbb{Q}$*, for any unital algebra* A*, the evaluation map* $\mathrm{ev}_1 : D_\lambda(A) \to D_\lambda(\mathring{\mathbb{j}}_A)$ *is a quasi-isomorphism. Hence* $H^{2n}(D_\lambda(A)) = \mathring{\mathbb{j}}_A$ *for* $n > 0$*, and* $H^{2n+1}(D_\lambda(A)) = 0$ *for all* n*. Moreover the even cohomology groups are characterized by classes* $[\varphi]$ *such that* $\varphi(1, ..., 1) = t$ *for any* $t \in \mathring{\mathbb{j}}_A$*.*

Definition 3.4.6. We define the **reduced degenerate Connes complex** $D_\lambda(A)_{\mathrm{red}}$ to be the kernel of the first to columns of $DD(A)_{\mathrm{red}}$.

Corollary 3.4.7. *The maps in the following commutative square are quasi-isomorphisms. Hence,* $H^{2n}(D_\lambda(A)_{\mathrm{red}}) = \mathring{\mathbb{j}}_A$*,* $H^{2n+1}(D_\lambda(A)_{\mathrm{red}}) = 0$ *for all* $n \geq 0$*.*

$$\begin{array}{ccc} D_\lambda(A)_{\mathrm{red}} & \longrightarrow & DD(A)_{\mathrm{red}} \\ \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} \\ D_\lambda(\mathring{\mathbb{j}}_A)_{\mathrm{red}} & \longrightarrow & DD(\mathring{\mathbb{j}}_A)_{\mathrm{red}} \end{array}$$

Proof. Considering the top of the diagram, this follows analogously to the arguments made above.

$$\begin{array}{cccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
D^2_\lambda & & D^2 & \longrightarrow & D^2_{\mathrm{bar}} & \longrightarrow & D^2 & \longrightarrow & D^2_{\mathrm{bar}} & \longrightarrow \\
\uparrow & \dashrightarrow & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
D^1_\lambda & & D^1 & \longrightarrow & D^1_{\mathrm{bar}} & \longrightarrow & D^1 & \longrightarrow & D^1_{\mathrm{bar}} & \longrightarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\mathbb{j}A & & \mathbb{j}A & \longrightarrow & \mathbb{j}A & \longrightarrow & \mathbb{j}A & \longrightarrow & \mathbb{j}A & \longrightarrow
\end{array}$$

□

We would like to define the normalized and reduced Connes complexes, but beforehand we need to prove one small detail.

Proposition 3.4.8. *The map $C_\lambda(A) \to D_\lambda(A)_{\mathrm{red}}$ (and hence $C_\lambda(A) \to D_\lambda(A)$) is surjective.*

Proof. Let's name the maps of the short exact sequence

$$0 \to C(A)_{\mathrm{red}} \xrightarrow{i} C(A) \xrightarrow{q} D(A)_{\mathrm{red}} \to 0,$$

q of course restricts to $C_\lambda(A) \xrightarrow{q} D_\lambda(A)_{\mathrm{red}}$, which we need to show is surjective. The case for degree 0 is clear from Proposition 3.4.4. For $n > 0$ the proof comes from diagram chasing. Note that we have exact columns.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^n(A)_{\mathrm{red}} & \xrightarrow{i} & C^n(A) & \xrightarrow{q} & D^n(A)_{\mathrm{red}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle 1-\lambda} & & \downarrow & & \\
0 & \longrightarrow & \overline{C}^n_{\mathrm{bar}}(A) & \xrightarrow{i} & C^n_{\mathrm{bar}}(A) & \xrightarrow{q} & D^n_{\mathrm{bar}}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle Q} & & \downarrow & & \\
0 & \longrightarrow & C^n(A)_{\mathrm{red}} & \xrightarrow{i} & C^n(A) & \xrightarrow{q} & D^n(A)_{\mathrm{red}} & \longrightarrow & 0
\end{array}$$

Let $\widetilde{\varphi} \in D^n_\lambda(A)_{\mathrm{red}}$, hence we have $\widetilde{\varphi} \in D^n(A)_{\mathrm{red}}$ such that $(1-\lambda)\widetilde{\varphi} = 0$. Let $\varphi \in C^n(A)$ be in the preimage of $\widetilde{\varphi}$, i.e. $q\varphi = \widetilde{\varphi}$. By the commutativity of the diagram we have that $q(1-\lambda)\varphi = 0$, so there exists $\psi \in \overline{C}^n_{\mathrm{bar}}(A)$ with $i\psi = (1-\lambda)\varphi$. But by the injectivity of i we must have $Q\psi = 0$, so there exists $\tau \in C^n(A)_{\mathrm{red}}$ such that $(1-\lambda)\tau = \psi$. Thus, we can create the element $\varphi - \tau \in C^n(A)$ such that $(1-\lambda)(\varphi - \tau) = 0$, hence $\varphi - \tau \in C^n_\lambda(A)$, and $q(\varphi - \tau) = \widetilde{\varphi}$. □

Definition 3.4.9. The **normalized and reduced Connes Complexes** $\overline{C}_\lambda(A)$ and $C_\lambda(A)_{\text{red}}$ are the respective kernels

$$0 \to \overline{C}_\lambda(A) \to C_\lambda(A) \to D_\lambda(A) \to 0$$

and

$$0 \to C_\lambda(A)_{\text{red}} \to C_\lambda(A) \to D_\lambda(A)_{\text{red}} \to 0.$$

Note that the inclusion between $\overline{C}_\lambda(A) \to \overline{CC}(A)$ is not a quasi-isomorphism, but by corollary 3.4.7 the inclusion $C_\lambda(A)_{\text{red}} \to CC(A)_{\text{red}}$ is.

We made these last few definitions and theorems because we are tasked with finding information about the Hochschild and Cyclic cohomology groups, so it would be nice if we could extract some of the information from the degenerate complexes. Specifically, we want to extend each $[\varphi] \in H^{2n}(D_\lambda(A)_{\text{red}})$ to an element in $HC_\lambda^{2n}(A)$ (in geometric terms ([Con94, pg. 190]) this corresponds to finding a closed graded trace on A that doesn't vanish on the identity). Unfortunately, for arbitrary unital algebras this is wishful thinking. But we can at least get a relationship from the following 9-diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & CC(A)[2,0]_{\text{red}} & \longrightarrow & CC(A)_{\text{red}} & \longrightarrow & CC(A)^{\{2\}}_{\text{red}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \\
0 & \longrightarrow & CC(A)[2,0] & \longrightarrow & CC(A) & \longrightarrow & CC(A)^{\{2\}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \\
0 & \longrightarrow & DD(A)[2,0]_{\text{red}} & \longrightarrow & DD(A)_{\text{red}} & \longrightarrow & DD(A)^{\{2\}}_{\text{red}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Where $[2,0]$ indicates a shift to the right 2 columns, $(CC(A)[2,0])_{pq} = CC(A)_{p-2,q}$ (See [Lod13, pg. 61]), and of course the $\{2\}$ indicates only the first 2 columns of that respective complex. Applying the cohomology functor we get the following exact bicomplex:

$$
\begin{array}{ccccccccc}
 & & \overline{HD}^{2n-1}(A) & & \overline{HD}^{2n+1}(A) & & H^{2n+1}(DD(A)^{\{2\}}) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \delta} & & \downarrow & & \\
\overline{HH}^{2n+1}(A) & \longrightarrow & \overline{HC}^{2n}(A) & \longrightarrow & \overline{HC}^{2n+2}(A) & \longrightarrow & \overline{HH}^{2n+2}(A) & \longrightarrow & \overline{HC}^{2n+1}(A) \\
 & & \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow & & \\
HH^{2n+1}(A)^B & \longrightarrow & HC^{2n}(A) & \xrightarrow{S} & HC^{2n+2}(A) & \xrightarrow{I} & HH^{2n+2}(A)^B & \longrightarrow & HC^{2n+1}(A) \\
 & & \downarrow & & \downarrow{\scriptstyle q^*} & & \downarrow & & \\
H^{2n+1}(DD(A)^{\{2\}}) & \longrightarrow & \overline{HD}^{2n}(A) & \longrightarrow & \overline{HD}^{2n+2}(A)^{\delta} & \longrightarrow & H^{2n+2}(DD(A)^{\{2\}}) & \longrightarrow & \overline{HD}^{2n+1}(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \overline{HC}^{2n+1}(A) & & \overline{HC}^{2n+3}(A) & & \overline{HH}^{2n+3}(A) & &
\end{array}
$$

Now substituting in calculated values, and including 6 rows we get:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\overline{HH}^{2n+1}(A) & \longrightarrow & \overline{HC}^{2n}(A) & \longrightarrow & \overline{HC}^{2n+2}(A) & \longrightarrow & \overline{HH}^{2n+2}(A) & \longrightarrow & \overline{HC}^{2n+1}(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
HH^{2n+1}(A) & \xrightarrow{B} & HC^{2n}(A) & \xrightarrow{S} & HC^{2n+2}(A) & \xrightarrow{I} & HH^{2n+2}(A) & \longrightarrow & HC^{2n+1}(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathring{\mathbb{j}}_A & \longrightarrow & \mathring{\mathbb{j}}_A & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\overline{HH}^{2n+2}(A) & \longrightarrow & \overline{HC}^{2n+1} & \longrightarrow & \overline{HC}^{2n+3}(A) & \longrightarrow & \overline{HH}^{2n+3} & \longrightarrow & \overline{HC}^{2n+2}(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
HH^{2n+2}(A) & \xrightarrow{B} & HC^{2n+1}(A) & \xrightarrow{S} & HC^{2n+3}(A) & \xrightarrow{I} & HH^{2n+3}(A) & \longrightarrow & HC^{2n+2}(A) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathring{\mathbb{j}}_A \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \overline{HC}^{2n+2}(A) & & \overline{HC}^{2n+4}(A) & & \overline{HH}^{2n+4}(A) & &
\end{array}
\tag{3.4.9}
$$

Note that there is a slight deviation from the above in degree 0, which we leave to the reader to check. The map I is induced from the inclusion, $B = Qs(1-\lambda)$, and S is the periodicity map, which is rather technical to write out, so we reference the reader to [Lod13, pg. 63] or to [Con94, pg. 198]. The important thing to note is that for any $[\varphi] \in HC^n(A)$ the evaluation map ev_1 is well defined, $\mathrm{ev}_1([\varphi]) = \varphi(1,...,1)$, and by Corollary 3.4.7 we can take the map $HC(A) \to \mathbb{j}_A$ as this evaluation. Moreover $\mathrm{ev}_1(S[\varphi]) = S\varphi(1,...,1) = \varphi(1,...,1)$. Along with the quasi-isomorphisms between the Connes complexes and Cyclic complexes, we obtain the following results:

Theorem 3.4.10. *The induced periodicity map on the reduced degenerate cyclic (Connes) cohomology groups,* $S : \overline{HD}^n(A) \to \overline{HD}^{n+2}(A)$ $\left(S : \overline{H}^n(D_\lambda(A)) \to \overline{H}^{n+2}(D_\lambda(A))\right)$ *is an isomorphism.*

Theorem 3.4.11. *For n even the image of the map* $\mathrm{ev}_1 : HC^n(A) \to \mathbb{j}_A$ $(\mathrm{ev}_1 : H^n_\lambda(A) \to \mathbb{j}_A)$ *is non-decreasing i.e.* $\mathrm{ev}_1(HC^n(A)) \subseteq \mathrm{ev}_1(HC^{n+2}(A))$ $\left(\mathrm{ev}_1(H^n_\lambda(A)) \subseteq \mathrm{ev}_1(H^{n+2}_\lambda(A))\right)$.

Theorem 3.4.12. *Let* $\mathbb{h}^{2n}_A$ *be the image of* $\mathrm{ev}_1 : HC^{2n}(A) \to \mathbb{j}_A$ $\left(= \mathrm{ev}_1(H^{2n}_\lambda(A)) \subset \mathbb{j}_A\right)$. *Then* $HC^{2n}(A) = \overline{HC}^{2n}(A) \oplus \mathbb{h}^{2n}_A$ $\left(H^{2n}_\lambda(A) = \overline{H}^{2n}_\lambda(A) \oplus \mathbb{h}_A\right)$ *and* $HC^{2n+1}(A) = \overline{HC}^{2n+1}(A) \big/ (\mathbb{j}_A/\mathbb{h}^{2n}_A)$ $\left(H^{2n+1}_\lambda(A) = \overline{H}^{2n+1}_\lambda(A) \big/ (\mathbb{j}_A/\mathbb{h}^{2n}_A)\right)$.

Proof. For the first assertion, let $\mathcal{J} \subset \mathbb{h}_A$ be a minimal set of generators, and for each $t \in \mathcal{J}$ choose one class $[\varphi_t] \in HC^{2n}$ such that $\varphi_t(1,...,1) = t$. Then the submodule generated by all such $[\varphi_t]$ will be isomorphic to $\mathbb{h}_A$. Let $[\tau] \in HC^{2n}$ be such that $\tau(1,...,1) = t_0$. Then we can write $[\tau] = [\tau - \varphi_{t_0}] + [\varphi_{t_0}]$.

Let $CK(A)$ be the kernel of the evaluation map: $CC(A) \xrightarrow{\mathrm{ev}_1} CC(\mathbb{j}_A)$, and similarly define $DK(A)$ for the degenerate complexes. We again have a 9-diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & CC(A)_{\mathrm{red}} & \longrightarrow & CK(A) & \longrightarrow & DK(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & CC(A)_{\mathrm{red}} & \longrightarrow & CC(A) & \longrightarrow & DD(A)_{\mathrm{red}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \\
0 & \longrightarrow & CC(0) & \longrightarrow & CC(\mathbb{j}_A) & \longrightarrow & CC(\mathbb{j}_A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where we get exactness in the top row from exactness of the bottom. We've also shown in Proposition 3.4.4 that the map $DD(A) \xrightarrow{\mathrm{ev}_1} CC(\mathbb{j}_A)$ is a quasi-isomorphism. Hence $H^n(\mathrm{Tot}(DK(A))) = 0$

for all n. So without having to consider the full cohomological bicomplex, we get $\overline{HC}^n(A) \cong HK^n(A)$ where $HK^n(A) := H^n(\mathrm{Tot}(CK(A)))$.

Now considering $[\tau] = [\tau - \varphi_{t_0}] + [\varphi_{t_0}]$ it's clear that $[\tau - \varphi_{t_0}]$ defines a cohomology class in $HK^{2n}(A)$. Hence via the isomorphism, there exists a unique class $[\psi] \in \overline{HC}^{2n}(A)$ corresponding to $[\tau - \varphi_{t_0}]$. The map $HC^{2n}(A) \to \overline{HC}^{2n}(A) \oplus \mathbb{j}_A$, $[\tau] \to ([\psi], \tau(1, ..., 1))$ is then clearly an isomorphism. The claim for the odd degrees follows from the exactness of the diagram 3.4.9. □

Corollary 3.4.13. *If $\mathbb{j}_A$ is finitely generated e.g. if $\Bbbk$ is Noetherian, then the image* $\mathrm{ev}_1 : HC^{2n}(A) \to \mathbb{j}_A$ *is eventually stable. If* $\mathrm{ev}_1(HC^{2n}(A)) = \mathbb{j}_A$ *for some n, then* $HC^{2k}(A) \cong \overline{HC}^{2k}(A) \oplus \mathbb{j}_A$ *and* $\overline{HC}^{2k+1}(A) \cong HC^{2k+1}(A)$ *for all $k \geq n$.*

In Hochschild cohomology we of course have that the reduced Hochschild and Hochschild cohomology groups are isomorphic for $n > 0$. This last theorem is our analogue in cyclic cohomology. Ideally, we'd like to be in the corollary situation where the isomorphisms hold for all n, which of course requires that for all $t \in \mathbb{j}_A$ we have a functional $\varphi_t \in A^*$ such that $\varphi_t(a_0a_1 - a_1a_0) = 0$ (i.e. a trace) and $\varphi_t(1) = t$. Since $\mathbb{j}_A$ is an ideal in $\Bbbk$, then this last condition is easily satisfied if we have some trace φ on A such that $\varphi(1) = 1$. This is, of course, the case when A is augmented $A = \Bbbk \oplus I$, as the projection map $A \to \Bbbk$ then defines an algebra homomorphism. This will play an mportant role in constructing the next cohomology theory for non-unital algebras.

Chapter 4

Bridge Cohomology

Following the set-up in the introduction, our aim is now to consider the subcomplex of Hochschild functionals such that $(1-\lambda)\varphi = \sigma^*\psi$ where $\varphi \in C^n(A)$, $\psi \in C^n(B)$, and $A \xrightarrow{\sigma} B$ is a surjective map of algebras. Such a situation arises naturally when considering traces of algebras on a manifold with boundary, as well as in certain classes of pseudo-differential operators [LP00], [LMP07]. This tool has potential to extend Chern-characters and pairings between K-theory and cyclic cohomology to manifolds with boundary [Con94, Ch. 3, Sec. 3], [LMP09], [LMP12].

As it turns out, our theory will be a full generalization of both cyclic and Hochschild cohomologies that, informally, encompasses all steps in between. More precisely, for any unital $\Bbbk$-algebra A, we can consider the set of ideals of A, $\mathcal{I}_A$, and for any $I \in \mathcal{I}_A$ we get a short exact sequence

$$0 \to I \to A \to A/I \to 0$$

where the map $A \to A/I$ is a surjective unital algebra morphism. When $I = A$ is the largest it can be, we obtain the cyclic complex, and when $I = 0$ we obtain the Hochschild complex. Thus, we have a "bridge" between the two, and to not overuse the letter B, we will denote each complex $R(\sigma)$, based off of our unital algebra surjection σ and "R" for "rung" (as in "notch" or "step").

4.1 Definitions and the category $\mathcal{S}_{\mathbb{1},\Bbbk}$

Definition 4.1.1. For a surjective unital algebra homomorphism of unital algebras $A \xrightarrow{\sigma} B$, the bridge complex $R^\bullet(\sigma)$ can be defined as the pullback in the following diagram:

$$\begin{array}{ccc} R^\bullet(\sigma) & \longrightarrow & C^\bullet(A) \\ \big\downarrow & \ulcorner & \big\downarrow {\scriptstyle 1-\lambda} \\ C^\bullet_{\text{bar}}(B) & \xrightarrow{\sigma^*} & C^\bullet_{\text{bar}}(A) \end{array} \tag{4.1.1}$$

In the category of cochain complexes, this means that

$$R^n(\sigma) = \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in C^n(A) \times C^n_{\text{bar}}(B) \,\middle|\, (1-\lambda)\varphi = \sigma^*\psi \right\},$$

with differential $\begin{pmatrix} b & 0 \\ 0 & b' \end{pmatrix}$, though since the map σ^* is injective, including the bottom entry ψ is not quite necessary, so we will often only write elements as $\varphi \in R^n(\sigma)$. Also note that $\varphi \in R^n(\sigma)$ iff it satisfies

$$((1-\lambda)\varphi)(a_0, ..., a_n) = 0 \text{ if } a_j \in K \text{ for some } j.$$

The cohomology of $R^\bullet(\sigma)$ will be denoted by $HR^\bullet(\sigma)$ and called the **bridge cohomology of** σ.

Proposition 4.1.2. *We have a non-direct sum* $R^\bullet(\sigma) = C^\bullet_\lambda(A) + \sigma^* C^\bullet(B)$.

Proof. Given any $\varphi \in R^n(\sigma)$, let $\psi \in C^n_{\text{bar}}(B)$ be such that $(1-\lambda)\varphi = \sigma^*\psi$. Notice that since $Q\sigma^*\psi = Q(1-\lambda)\varphi = 0$, we have $\sigma^* Q\psi = 0$, and hence $Q\psi = 0$ since σ^* is injective. Thus, by the exactness of the (λ, Q)-sequence 2.3.1 there exists $\psi' \in C^n(B)$ such that $(1-\lambda)\psi' = \psi$. Now we write $\varphi = (\varphi - \sigma^*\psi') + \sigma^*\psi'$.

Lastly, for $\varphi + \sigma^*\psi \in R^n(\sigma)$ in the desired form, we have $b(\varphi + \sigma^*\psi) = b\varphi + \sigma^* b\psi$, and $(1-\lambda)b\varphi = b'(1-\lambda)\varphi = 0$. □

Our next task is to define bridge cohomology for non-unital algebras. We know from previous work that to simply apply the above construction to non-unital algebras is not quite the appropriate direction to take, but rather we should look to extend a functor from a "unital" category to a "non-unital" one. While this works well diagrammatically, we also have notions of normalized an reduced complexes that are used to extend previous cohomologies to non-unital algebras, and provide for a much more explicit framework to work with. Following in the work of Loday [Lod13] we will develop all of those notions here for bridge complexes.

Definition 4.1.3. We define the category $\mathcal{S}_{\mathbb{1},\Bbbk}$ to have as objects surjective unital $\Bbbk$-algebra homomorphisms. Given two objects $\sigma, \tau \in \mathrm{Obj}(\mathcal{S}_{\mathbb{1},\Bbbk})$, a morphism from σ to τ is a pair of unital algebra homomorphisms (f_1, f_2) such that $\tau f_1 = f_2\sigma$ i.e. we have a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ X & \xrightarrow{\tau} & Y \end{array}$$

Clearly for any σ, we have the identity morphism $\sigma \xrightarrow{(\mathrm{id}_A, \mathrm{id}_B)} \sigma$, and for morphisms $\sigma \xrightarrow{(f_1,f_2)} \tau$ and $\tau \xrightarrow{(g_1,g_2)} \upsilon$ we get the composition $\sigma \xrightarrow{(g_1 f_1, g_2 f_2)} \upsilon$.

This is a monoidal category. Indeed, since the category of associative unital $\Bbbk$-algebras is a monoidal category, we have a bifunctor $\_ \otimes \_ : \mathcal{Alg}_{\mathbb{1},\Bbbk} \times \mathcal{Alg}_{\mathbb{1},\Bbbk} \to \mathcal{Alg}_{\mathbb{1},\Bbbk}$, which sends two unital $\Bbbk$-algebras A and B to the algebra $A \otimes B$. Multiplication in $A \otimes B$ is performed component-wise,

$$a_1 \otimes b_1 \cdot a_2 \otimes b_2 = a_1 a_2 \otimes b_1 b_2.$$

Given two unital algebra homomorphisms, $\sigma : A \to B$ and $\tau : X \to Y$ the tensor bifunctor gives us the map $\sigma \otimes \tau : A \otimes X \to B \otimes Y$ given by

$$(\sigma \otimes \tau)(a \otimes x) = \sigma(a) \otimes \tau(x). \tag{4.1.3}$$

Thus, the tensor product on unital $\Bbbk$-algebras induces a tensor product on $\mathcal{S}_{\mathbb{1},\Bbbk}$, with the tensor on objects defined by the previous equation (4.1.3). It's clear that if σ and τ are surjective in equation (4.1.3) then so is $\sigma \otimes \tau$. Given two morphisms of objects in $\mathcal{S}_{\mathbb{1},\Bbbk}$, say $(f_1, f_2) : \rho \to \sigma$ and $(g_1, g_2) : \tau \to \upsilon$, we have the commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ W & \xrightarrow{\sigma} & X \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\tau} & D \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ Y & \xrightarrow{\upsilon} & Z \end{array}$$

The tensor product $(f_1, f_2) \otimes (g_1, g_2)$ is again induced by the tensor product on $\Bbbk$-algebras, that is $(f_1, f_2) \otimes (g_1, g_2) = (f_1 \otimes g_1, f_2 \otimes g_2)$, which gives us the commutative diagram

$$\begin{array}{ccc} A \otimes C & \xrightarrow{\rho \otimes \tau} & B \otimes D \\ \downarrow{\scriptstyle f_1 \otimes g_1} & & \downarrow{\scriptstyle f_2 \otimes g_2} \\ W \otimes Y & \xrightarrow{\sigma \otimes \upsilon} & X \otimes Z \end{array}$$

Thus, $(f_1, f_2) \otimes (g_1, g_2)$ is a morphism from $\rho \otimes \tau$ to $\sigma \otimes \upsilon$.

The unital object in this category is $\mathrm{id}_{\Bbbk} : \Bbbk \to \Bbbk$. For any $\sigma : A \to B$, we have the canonical inclusion $\mathrm{id}_{\Bbbk} \xrightarrow{(\iota_A, \iota_B)} \sigma$. The object $\mathrm{id}_{\Bbbk} \otimes \sigma$ gives us a map from $\Bbbk \otimes A \to \Bbbk \otimes B$, and the canonical isomorphisms $\Bbbk \otimes A \cong A$ and $\Bbbk \otimes B \cong B$ provide the canonical isomorphism $\mathrm{id}_{\Bbbk} \otimes \sigma \cong \sigma$. Isomorphisms for right tensoring with the identity follow similarly. The associativity and coherence conditions of the tensor product $\_ \otimes \_ : \mathcal{S}_{\mathbb{1},\Bbbk} \times \mathcal{S}_{\mathbb{1},\Bbbk} \to \mathcal{S}_{\mathbb{1},\Bbbk}$ are induced from those of the tensor product on unital $\Bbbk$-algebras.

We can now define the bridge complex as a (contravariant) functor $R^\bullet(\cdot) : \mathcal{S}_{\mathbb{1},\Bbbk} \to \mathcal{C}$, from the category of surjective unital algebra homomorphisms to the category of chain complexes, where the map $\sigma \xrightarrow{(f_1, f_2)} \tau$ goes to the map $f_1^* : R^\bullet(\tau) \to R^\bullet(\sigma)$. We check that $(1-\lambda)(f_1^*)(\varphi) = f_1^*(1-\lambda)\varphi = f_1^*\tau^*\psi = \sigma^*(f_2^*\psi)$ as needed. It's clear that this functor preserves the identity map, and given a composition $\sigma \xrightarrow{(f_1, f_2)} \tau \xrightarrow{(g_1, g_2)} \upsilon$, we clearly have $(g_1 \circ f_1)^* = f_1^* \circ g_1^*$.

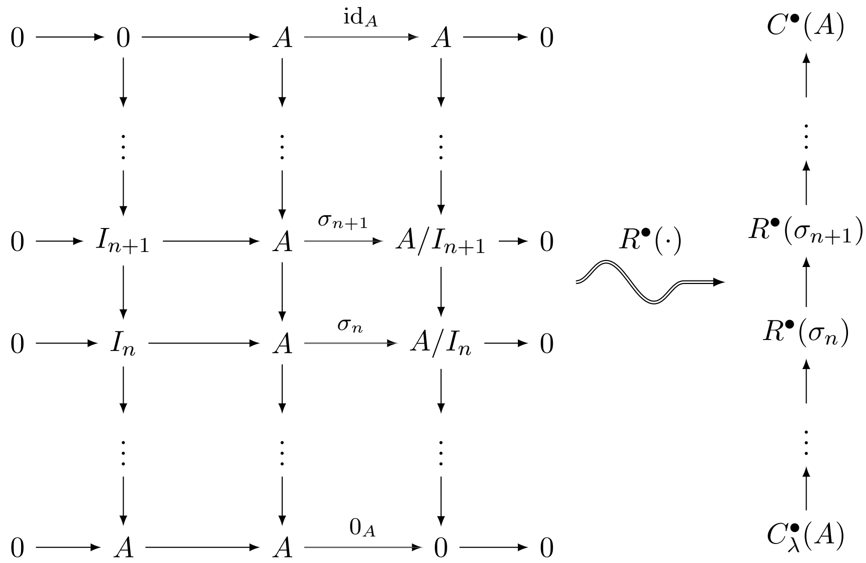
Note. One thing to note is that we could have just as well formed a category of short exact sequences of $\Bbbk$-algebras, where the middle and end algebras are unital, and then defined bridge cohomology as a functor from there to chain complexes. As we have it, it is much more simple to notate, though we should take note of a few important maps, namely the identity and zero map. For any unital algebra A and the identity map $\mathrm{id}_A : A \to A$, the bridge complex is found to be

$$R^\bullet(\mathrm{id}_A) = \{\varphi \in C^\bullet(A) \,|\, (1-\lambda)\varphi = \mathrm{id}_A^*\psi \text{ for some } \psi \in C^\bullet(A)\} = C^\bullet(A).$$

For the zero map we have

$$R^\bullet(0_A) = \{\varphi \in C^\bullet(A) \,|\, (1-\lambda)\varphi = 0\} = C_\lambda^\bullet(A).$$

So for any map σ with domain A, we have $C_\lambda^\bullet(A) \subset R^\bullet(\sigma) \subset C^\bullet(A)$, and the degree to how close $R^\bullet(\sigma)$ sits between either two is measured in some sense by the size of the kernel of σ. Likewise, for any ascending chain of ideals $0 \subset \ldots \subset I_{n+1} \subset I_n \subset \ldots \subset A$, we have the following picture



Combining the bridge complex functor with the cohomology functor, we have a functor $HR^n(\cdot) : \mathcal{S}_{\mathbb{1},\Bbbk} \to \mathcal{M}_{\Bbbk}$ from $\mathcal{S}_{\mathbb{1},\Bbbk}$ to $\Bbbk$-modules, and we can extend this functor in a natural way to the non-unital category $\mathcal{S}_{\Bbbk}$ of surjective $\Bbbk$-algebra homomorphisms.

Definition 4.1.4. Given any $\Bbbk$-algebras A and B (not necessarily unital) and a surjective algebra homomorphism $\sigma : A \to B$, let $\sigma^+ : A^+ \to B^+$ be the extension of σ to the augmented algebras A^+ and B^+. We define the n^{th} bridge cohomology module of σ as:

$$HR^n(\sigma) := \ker\left(HR^n(\sigma^+) \xrightarrow{\iota^*} HR^n(\mathrm{id}_{\Bbbk})\right).$$

4.2 Normalized and Reduced Bridge Cohomology (via Connes Complexes)

As with Hochschild and cyclic cohomologies, we would like to construct a chain complex that will give us the bridge cohomology of a map regardless of whether it is unital or not. To do so we make use of the degenerate complexes from Chapter 3 and construct equivalent notions for bridge complexes.

Definition 4.2.1. For unital $\Bbbk$-algebras A and B, and a surjective unital algebra homomorphism $\sigma : A \to B$, we define the **degenerate bridge complex of** σ, $D^\bullet(\sigma)$, to be the following pullback:

$$\begin{array}{ccc} D^\bullet(\sigma) & \longrightarrow & D^\bullet(A) \\ \big\downarrow & \ulcorner & \big\downarrow {\scriptstyle 1-\lambda} \\ D^\bullet_{\text{bar}}(B) & \xrightarrow{\sigma^*} & D^\bullet_{\text{bar}}(A) \end{array}$$

We will denote by $\mathcal{HD}^n(\sigma) := H^n(D(\sigma))$ to be the n^{th} cohomology of this complex. (The reason for the change in script will be clear in the next chapter.)

The natural maps $C^\bullet(A) \to D^\bullet(A)$, $C^\bullet_{\text{bar}}(A) \to D^\bullet_{\text{bar}}(A)$, and $C^\bullet(B) \to D^\bullet(B)$ induce a surjective map from $R^\bullet(\sigma) \to D^\bullet(\sigma)$. The collection of these maps we will draw as a double curved arrow:

$$\begin{array}{ccc} R^\bullet(\sigma) & \longrightarrow & C^\bullet(A) \\ \big\downarrow & \ulcorner & \big\downarrow {\scriptstyle 1-\lambda} \\ C^\bullet_{\text{bar}}(B) & \xrightarrow{\sigma^*} & C^\bullet_{\text{bar}}(A) \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} D^\bullet(\sigma) & \longrightarrow & D^\bullet(A) \\ \big\downarrow & \ulcorner & \big\downarrow {\scriptstyle 1-\lambda} \\ D^\bullet_{\text{bar}}(B) & \xrightarrow{\sigma^*} & D^\bullet_{\text{bar}}(A) \end{array} \tag{4.2.1}$$

Before defining a normalized complex for bridge cohomology, we first need two things: to show that $R(\sigma) \to D(\sigma)$ is surjective and to understand the cohomology of the complex $D^\bullet(\sigma)$. We will see that the inexactness in the degenerate Connes complexes (see Theorem 3.4.5) can still permeate into the degenerate bridge complex.

Proposition 4.2.2. *We have a non-direct sum $D(\sigma) = D_\lambda(A) + \sigma^* D(B)$. Moreover, if $\varphi + \sigma^*\tau \in D_\lambda(A) + \sigma^* D(B)$ is such that $b(\varphi + \sigma^*\tau) = 0$. Then we may find an element $\varphi' + \sigma^*\tau' = \varphi + \sigma^*\tau$ such that $b\varphi' = 0$ and $b\tau' = 0$.*

Proof. The first statement follows analogously to Proposition 4.1.2. Now let $\varphi + \sigma^*\tau \in D^n(\sigma)$ be such that $b(\varphi + \sigma^*\tau) = 0$. Let's first assume that n is odd. In this case since φ is cyclic we have $\varphi(1, ..., 1) = 0$, and hence $b\varphi(1, ..., 1) = 0$. Also note that $b\tau(1, ..., 1) = \tau(1, ..., 1)$, but since $b(\varphi + \sigma^*\tau) = 0$ this implies that we must have $\tau(1, ..., 1) = 0$.

Now since $b\sigma^*\tau = b\varphi \in D^{n+1}_\lambda(A)$ this implies that $b\tau \in D^{n+1}_\lambda(B)$. Thus by Theorem 3.4.5 there exists $\mu \in D^n_\lambda(B)$ such that $b\mu = b\tau$. Setting $\varphi' = (\varphi + \sigma^*\mu)$ and $\tau' = (\tau - \mu)$ gives us the desired form.

The even degree case follows similarly except we need not worry about evaluating on $(1, ..., 1)$.

□

Theorem 4.2.3. *$\mathcal{HD}^{2n-1}(\sigma) = 0$ and $\mathcal{HD}^{2n}(\sigma) = \mathring{\mathbb{j}}_A/\mathring{\mathbb{j}}_B$ $(:= \mathring{\mathbb{j}}_A/\sigma^*\mathring{\mathbb{j}}_B)$ for $n > 0$. $\mathcal{HD}^0(\sigma) = 0$.*

Proof. The zero case follows since $D^0(\sigma) = 0$, so let's start with the odd case. Let $\varphi + \sigma^*\tau \in D^{2n-1}(\sigma)$ be of the form in the previous proposition such that $b\varphi = b\sigma^*\tau = 0$. Then it's clear by the exactness of $D(B)$ and by theorem 3.4.5 that we have $\varphi' \in D_\lambda^{2n-2}(A)$ and $\tau' \in D^{2n-2}(B)$ such that $b\varphi' = \varphi$ and $b\tau' = \tau$. Then clearly $\varphi' + \sigma^*\tau' \in D^{2n-2}(\sigma)$ and $b(\varphi' + \sigma^*\tau') = \varphi + \sigma^*\tau$.

For the even case, let $\varphi + \sigma^*\tau \in D^{2n}(\sigma)$ be a cycle of the desired form from the previous proposition. Note that for any $t \in \mathring{\mathbb{j}}_B$ there exists $\mu \in D_\lambda^{2n}(B)$ such that $\mu(1, ..., 1) = t$ and $b\mu = 0$, so we may modify our element modulo cycles of $D_\lambda^{2n}(B)$, $\varphi + \sigma^*\tau = (\varphi - \sigma^*\mu) + \sigma^*(\tau + \mu)$ and still be of the desired form. It should also be clear that this is all such equivalent expressions in this form. Thus, if $\varphi(1, ..., 1) \equiv 0 \in \mathring{\mathbb{j}}_A/\mathring{\mathbb{j}}_B$ then we may rewite our element as $\widetilde{\varphi} + \sigma^*\widetilde{\tau}$ such that $\widetilde{\varphi}(1, ..., 1) = 0$. By Theorem 3.4.5 and the exactness of $D(B)$, there exists $\varphi' \in D_\lambda^{2n-1}(A)$ and $\tau' \in D^{2n-1}(B)$ such that $b\varphi' = \widetilde{\varphi}$ and $b\tau' = \widetilde{\tau}$. However, if $\varphi(1, ..., 1) \not\equiv 0 \in \mathring{\mathbb{j}}_A/\mathring{\mathbb{j}}_B$ then $\varphi + \sigma^*\tau$ cannot be a boundary, since for any $\varphi' + \sigma^*\tau' \in D^{2n-1}(\sigma)$ we must have $b(\varphi' + \sigma^*\tau')(1, ..., 1) = (\varphi' + \sigma^*\tau')(1, ..., 1) = \varphi'(1, ..., 1) + \tau'(1, ..., 1)$. But since φ' is cyclic and in an odd degree, $0 = (1 - \lambda)\varphi'(1, ..., 1) = 2\varphi'(1, ..., 1)$. Hence $b(\varphi' + \sigma^*\tau')(1, ..., 1) \in \mathring{\mathbb{j}}_B$. Recalling Theorem 3.4.5 gives us the existence of such a cycle $\varphi + \sigma^*\tau$ such that $(\varphi + \sigma^*\tau)(1, ..., 1) \not\equiv 0 \in \mathring{\mathbb{j}}_A/\mathring{\mathbb{j}}_B$. □

Corollary 4.2.4. *If $\mathring{\mathbb{j}}_B = \mathring{\mathbb{j}}_A$, such as when both $A = \mathbb{k} \oplus I$ and $B = \mathbb{k} \oplus J$ are augmented, then $D(\sigma)$ is exact.*

Proposition 4.2.5. *$R(\sigma) \to D(\sigma)$ is surjective.*

Proof. We have shown that $R(\sigma) = C_\lambda(A) + \sigma^*C(B)$ and $D(\sigma) = D_\lambda(A) + \sigma^*D(B)$. By Proposition 3.4.8 the map $C_\lambda(A) \to D_\lambda(A)$ is surjective, and by definition $C(B) \to D(B)$ is surjective. Hence $R(\sigma) \to D(\sigma)$ is a surjection. □

Definition 4.2.6. Given a unital algebra surjection $\sigma : A \to B$, the **normalized bridge complex** $\overline{R}^{\bullet}(\sigma)$ is the kernel

$$0 \to \overline{R}(\sigma) \to R(\sigma) \to D(\sigma) \to 0.$$

For $n > 1$, we will denote the cohomology of this complex by $\overline{HR}^{n}(\sigma) := H^n(\overline{R}(\sigma))$, For $n \in \{0, 1\}$, notice that the cohomology of the normalized complex is the same as the non-normalized version so we will continue to use $HR^0(\sigma)$ and $HR^1(\sigma)$. By the previous theorem we have the following exact sequence:

$$\ldots \to 0 \to \overline{HR}^{2n}(\sigma) \to HR^{2n}(\sigma) \to \mathring{\mathbb{j}}_A/\mathring{\mathbb{j}}_B \to \overline{HR}^{2n+1}(\sigma) \to HR^{2n+1}(\sigma) \to 0 \to \ldots \tag{4.2.6}$$

Proposition 4.2.7. *Similar to Definition 4.1.1, $\overline{R}(\sigma)$ is the pullback of the corresponding normalized complexes:*

$$\begin{array}{ccc} \overline{R}^{\bullet}(\sigma) & \longrightarrow & \overline{C}^{\bullet}(A) \\ \downarrow & \ulcorner & \downarrow{\scriptstyle 1-\lambda} \\ \overline{C}^{\bullet}_{\text{bar}}(B) & \xrightarrow{\sigma^*} & \overline{C}^{\bullet}_{\text{bar}}(A) \end{array}$$

Proof. Considering the diagram (4.2.1), the kernel is a commutative square.

$$\begin{array}{ccc} \overline{R}^{\bullet}(\sigma) & \longrightarrow & \overline{C}^{\bullet}(A) \\ \downarrow & & \downarrow{\scriptstyle 1-\lambda} \\ \overline{C}^{\bullet}_{\text{bar}}(B) & \xrightarrow{\sigma^*} & \overline{C}^{\bullet}_{\text{bar}}(A) \end{array} \rightsquigarrow \begin{array}{ccc} R^{\bullet}(\sigma) & \longrightarrow & C^{\bullet}(A) \\ \downarrow & \ulcorner & \downarrow{\scriptstyle 1-\lambda} \\ C^{\bullet}_{\text{bar}}(B) & \xrightarrow{\sigma^*} & C^{\bullet}_{\text{bar}}(A) \end{array} \rightsquigarrow \begin{array}{ccc} D^{\bullet}(\sigma) & \longrightarrow & D^{\bullet}(A) \\ \downarrow & \ulcorner & \downarrow{\scriptstyle 1-\lambda} \\ D^{\bullet}_{\text{bar}}(B) & \xrightarrow{\sigma^*} & D^{\bullet}_{\text{bar}}(A) \end{array}$$

The result follows categorically since limits and kernels commute.

Definition 4.2.8. Let $D(\sigma)_{\text{red}}$ be the **reduced degenerate bridge complex** defined by the following pullback

$$\begin{array}{ccc} D^{\bullet}(\sigma)_{\text{red}} & \longrightarrow & D^{\bullet}(A)_{\text{red}} \\ \downarrow & \ulcorner & \downarrow{\scriptstyle 1-\lambda} \\ D^{\bullet}_{\text{bar}}(B) & \xrightarrow{\sigma^*} & D^{\bullet}_{\text{bar}}(A) \end{array}$$

We leave off the "red" subscript from the bar complexes since these are the same. It should be clear that it's cohomology $\overline{HD}^n(\sigma) = \mathcal{HD}^n(\sigma)$ for $n > 0$ and $\overline{HD}^0(\sigma) = \mathring{\mathbb{j}}_A$.

Definition 4.2.9. We may define the **reduced bridge complex** $R(\sigma)_{\text{red}}$ as either of the following kernels:

$$0 \to R(\sigma)_{\text{red}} \to R(\sigma) \to D(\sigma)_{\text{red}} \to 0$$

or

$$0 \to R(\sigma)_{\text{red}} \to \overline{R}(\sigma) \to \mathring{\mathbb{j}}_A[0] \to 0$$

For lack of better notation, it's cohomology will be denoted $\overline{HR}^n(\sigma) = H^n(R(\sigma)_{\text{red}})$, we note that the reduced and normalized cohomologies are the same except in degree 0. Hence the long exact sequence (4.2.6) becomes:

$$\begin{gathered} 0 \to \overline{HR}^0(\sigma) \to HR^0(\sigma) \to \mathring{\mathbb{j}}_A \to \overline{HR}^1(\sigma) \to HR^1(\sigma) \to 0 \to \ldots \\ \ldots \to 0 \to \overline{HR}^{2n}(\sigma) \to HR^{2n}(\sigma) \to \mathring{\mathbb{j}}_A/\mathring{\mathbb{j}}_B \to \overline{HR}^{2n+1}(\sigma) \to HR^{2n+1}(\sigma) \to 0 \to \ldots \end{gathered} \tag{4.2.9}$$

Note: This sequence generalizes the first columns of (3.4.9), which is the dual of Loday [Lod13, 2.2.13.1 pg. 65].

Proposition 4.2.10. *Similar to Definition 4.1.1, $R(\sigma)_{\text{red}}$ is the pullback of the corresponding normalized complexes:*

$$\begin{array}{ccc} R^\bullet(\sigma)_{\text{red}} & \longrightarrow & C^\bullet(A)_{\text{red}} \\ \downarrow & \ulcorner & \downarrow {\scriptstyle 1-\lambda} \\ C^\bullet_{\text{bar}}(B)_{\text{red}} & \xrightarrow{\sigma^*} & C^\bullet_{\text{bar}}(A)_{\text{red}} \end{array}$$

Proof. This follows analogously to proposition 4.2.7. □

4.3 Bridge Cohomology for Non-unital Algebras (via Connes Complexes)

Recall that our functorial Definition, 4.1.4, of bridge cohomology for a surjective map of nonunital algebras $A \xrightarrow{\sigma} B$ is

$$HR^n(\sigma) := \ker\left(HR^n(\sigma^+) \xrightarrow{\iota^*} HR^n(\text{id}_{\Bbbk})\right).$$

where $\sigma^+ : A^+ \to B^+$ is the extension of σ to the augmented algebras A^+ and B^+, $\mathrm{id}_{\Bbbk}$ is the identity map on $\Bbbk$, and $\iota^* : \mathrm{id}_{\Bbbk} \to \sigma^+$ is the inclusion

$$\begin{array}{ccc} \Bbbk & \xrightarrow{\mathrm{id}_{\Bbbk}} & \Bbbk \\ \downarrow{\scriptstyle \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ A^+ & \xrightarrow{\sigma} & B^+ \end{array}$$

Theorem 4.3.1. *For a non-unital surjection $\sigma : A \to B$, $HR^n(\sigma) = H^n(R(\sigma^+)_{\mathrm{red}})$.*

Proof. It should be clear that $\mathbb{j}_{A^+} = \mathbb{j}_{B^+} = \Bbbk$, and from our discussion in Section 4.1, $R(\mathrm{id}_{\Bbbk}) = C(\Bbbk)$ and hence $HR^n(\mathrm{id}_{\Bbbk}) = HH^n(\Bbbk) = \begin{cases} 0 & n > 0 \\ \Bbbk & n = 0 \end{cases}$. The long exact sequence (4.2.9) then becomes

$$0 \to \overline{HR}^0(\sigma^+) \to HR^0(\sigma^+) \to \Bbbk \to \overline{HR}^1(\sigma^+) \to HR^1(\sigma^+) \to 0 \to ...$$
$$... \to 0 \to \overline{HR}^{2n}(\sigma^+) \to HR^{2n}(\sigma^+) \to 0 \to \overline{HR}^{2n+1}(\sigma^+) \to HR^{2n+1}(\sigma^+) \to 0 \to ...$$

Note that in degree 0, $R^0(\sigma^+) = (A^+)^*$ and that the map $HR^0(\sigma^+) \to \Bbbk$ is induced by the evaluation map $\mathrm{ev}_1 : (A^+)^* \to \Bbbk$. This is surjective since we can arbitrarily define $\varphi \in (A^+)^*$ such that $\varphi(1) = t$ for any $t \in \Bbbk$ and $\varphi(a) = 0$ for any $a \in A$, and in this case $b\varphi = 0$. □

Definition 4.3.2. The bicomplex $RB(\sigma)^{\{2\}}$ associated to $R(\sigma^+)_{\mathrm{red}}$ is given by the pullback

$$\begin{array}{ccc} RB(\sigma)^{\{2\}} & \longrightarrow & CC(A)^{\{2\}} \\ \downarrow & \ulcorner & \downarrow{\scriptstyle 1-\lambda \oplus 1} \\ CB(B)^{\{2\}} & \xrightarrow{\sigma^* \oplus \sigma^*} & CB(A)^{\{2\}} \end{array} \tag{4.3.2}$$

Corollary 4.3.3. $H^n(\mathrm{Tot}RB(\sigma)^{\{2\}}) = HR^n(\sigma)$.

Proof. The isomorphisms $\mathrm{Tot}CC(A)^{\{2\}} \cong C(A^+)_{\mathrm{red}}$ and $\mathrm{Tot}CB(A)^{\{2\}} \cong \overline{C}_{\mathrm{bar}}(A^+)$ are given in Corollary 3.1.3, with the analogues of $1 - \lambda$ and Q in Corollary 3.2.2. Hence $\mathrm{Tot}RB(\sigma)^{\{2\}} \cong R(\sigma^+)$. □

Explicitly, we get the following diagram for $RB(\sigma)^{\{2\}}$

$$
\begin{array}{ccc}
\uparrow {\scriptstyle b} & & \uparrow {\scriptstyle -b'} \\
R^2(\sigma) & \xrightarrow{(1-\lambda)^{\triangledown}} & C^2_{\text{bar}}(B) \\
\uparrow {\scriptstyle b} & & \uparrow {\scriptstyle -b'} \\
R^1(\sigma) & \xrightarrow{(1-\lambda)^{\triangledown}} & C^1_{\text{bar}}(B) \\
\uparrow {\scriptstyle b} & & \uparrow {\scriptstyle -b'} \\
R^0(\sigma) & \xrightarrow{(1-\lambda)^{\triangledown}} & C^0_{\text{bar}}(B)
\end{array}
$$

where $(1-\lambda)^{\triangledown} = (\sigma^*)^{-1}(1-\lambda)$ is well defined by our definition of $R(\sigma)$. The left map on diagram 4.3.2 is then $pr_2 \oplus 1$ while the top map can be written $pr_1 \oplus \sigma^*$, where pr_1 and pr_2 are the projection onto the first and second coordinate for an element $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in R(\sigma)$. Though, since we are suppressing the second coordinate from our notation, we may write the maps as follows:

$$
\begin{array}{ccc}
RB(\sigma)^{\{2\}} & \xrightarrow{1 \oplus \sigma^*} & CC(A)^{\{2\}} \\
{\scriptstyle (1-\lambda)^{\triangledown} \oplus 1} \downarrow & \ulcorner & \downarrow {\scriptstyle 1-\lambda \oplus 1} \\
CB(B)^{\{2\}} & \xrightarrow{\sigma^* \oplus \sigma^*} & CB(A)^{\{2\}}
\end{array}
$$

Thus, $\mathrm{Tot}^{\bullet\bullet} RB(\sigma)^{\{2\}} \cong R^{\bullet}(\sigma) \oplus C^{\bullet-1}_{\text{bar}}(B)$ with the differential being $\begin{pmatrix} b & 0 \\ (1-\lambda)^{\triangledown} & -b' \end{pmatrix}$. Letting $\mathcal{HR}^n(\sigma) := H^n(R(\sigma))$ represent the "naive" bridge cohomology of σ (whether unital or not) we obtain the following:

Theorem 4.3.4. *Let $A \xrightarrow{\sigma} B$ be a surjective (possibly non-unital) algebra homomorphism. The exact sequence relating "naive" bridge cohomology of σ with the bridge cohomology of σ is*

$$
\ldots \to HR^n(\sigma) \to \mathcal{HR}^n(\sigma) \to HB^n(B) \to HR^{n+1}(\sigma) \to \ldots \tag{4.3.4}
$$

Proof. This follows from the exact sequence of complexes:

$$
0 \to C^{\bullet-1}_{\text{bar}}(B) \to \mathrm{Tot}^{\bullet\bullet} RB(\sigma)^{\{2\}} \to R^{\bullet}(\sigma) \to 0.
$$

□

This is a rather interesting, though not too surprising, result. A priori we might expect that the difference between naive bridge cohomology and bridge cohomology to depend on more than just the bar cohomology of B, but recalling our discussion from Section 4.1, the bridge cohomology should give either the Hochschild cohomology of A or the cyclic cohomology of A when σ is trivial of either case. Suppose $\sigma : A \to A$ is the identity. Equation (4.3.4) then becomes

$$\begin{aligned} &... \to HR^n(\sigma) \to \mathcal{HR}^n(\sigma) \to HB^n(A) \to HR^{n+1}(\sigma) \to ... \\ =&... \to HH^n(A) \to \mathcal{HH}^n(A) \to HB^n(A) \to HH^{n+1}(A) \to ... \end{aligned}$$

exactly as we should expect, where $\mathcal{HH}(A)$ is the naive Hochschild cohomology of A (see [Lod13, pg. 31]). On the other end if $\sigma : A \to 0$ is the zero map then we get

$$\begin{aligned} &... \to HR^n(\sigma) \to \mathcal{HR}^n(\sigma) \to HB^n(0) \to HR^{n+1}(\sigma) \to ... \\ =&... \to HC^n_\lambda(A) \to HC^n_\lambda(A) \to 0 \to HC^{n+1}_\lambda(A) \to ... \end{aligned}$$

which, of course, looks like a generic result, but following the constructions it confirms that in fact “there is no difference between ‘naive’ cyclic [co]homology and cyclic [co]homology”([Lod13, pg. 67]).

Chapter 5

Bridge Cohomology via Cyclic Complexes

In this chapter we will see how relative complexes are related to our bridge complexes, which will come altering the complex $CC(A)$. We will change our notation slightly to emphasize this. Let $\sigma : A \to A/I$ be a surjective unital algebra homomorphism, where $I \subset A$ is an ideal. Then the relative Hochschild complex $C(A, I)$ is defined as the cokernel of $C(A/I) \to C(A)$. As such, by definition,

$$0 \to C(A/I) \to C(A) \to C(A, I) \to 0.$$

Its cohomology will be denoted $HH(A, I)$ called the relative Hochschild homology, and we of course have a long exact sequence relating the three

$$... \to HH^n(A, I) \to HH^n(A) \to HH^n(A/I) \to HH^{n+1}(A, I) \to ...$$

We also have the relative bar cohomology defined analogously, and for each of those we have degenerate, normalized, and reduced versions that all follow the analogous constructions used throughout chapter 3. Since the notation will be similar, we will not go over it here, and trust that it will be clear from the context. The last thing to note is that we have the (λ, Q)-sequence (in all its forms)

$$... \to C(A, I) \xrightarrow{1-\lambda} C_{\text{bar}}(A, I) \xrightarrow{Q} C(A, I) \to ...$$

which, when $\Bbbk$ contains $\mathbb{Q}$, will be exact. However, we will not necessarily assume that this is the case in this chapter.

5.1 Bridge and Relative Cohomologies

As with the previous chapter, our motivation comes from extending cyclic functionals $\varphi \in C^n_\lambda(A)$ to functionals $\varphi \in C(A)$ that descend to A/I under the cyclic operator i.e. $(1-\lambda)\varphi \in \sigma^* C(A/I)$. Though instead of using constructions that generalize the Connes complexes, we would like to use constructions that will generalize the full cyclic bicomplexes. Throughout this chapter we will denote the quotient map (on complexes) by $q : C(A) \to C(A, I)$. Our intuition comes from the (λ, Q)-sequence, which we extend from the short exact sequence of complexes:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C(A/I) & \xrightarrow{\sigma^*} & C(A) & \xrightarrow{q} & C(A,I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle 1-\lambda} & & \downarrow & & \\
0 & \longrightarrow & C_{\text{bar}}(A/I) & \xrightarrow{\sigma^*} & C_{\text{bar}}(A) & \xrightarrow{q} & C_{\text{bar}}(A,I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle Q} & & \downarrow & & \\
0 & \longrightarrow & C(A/I) & \xrightarrow{\sigma^*} & C(A) & \xrightarrow{q} & C(A,I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

With this in mind we see that this notion of $(1-\lambda)\varphi \in \sigma^* C(A/I)$ is similar to saying $\varphi \in \ker(q(1-\lambda))$.

Definition 5.1.1. Let A be a unital algebra and $0 \to I \to A \xrightarrow{\sigma} A/I \to 0$ be a short exact sequence of algebras. Define the **bridge bicomplex of σ, $RR^{\bullet\bullet}(\sigma)$**, as the bicomplex with the following columns

$$C^\bullet(A) \xrightarrow{q(1-\lambda)} C^\bullet_{\text{bar}}(A, I) \xrightarrow{Q} C^\bullet(A, I) \xrightarrow{1-\lambda} C^\bullet_{\text{bar}}(A, I) \xrightarrow{Q} \ldots \tag{5.1.1}$$

So as not to overdo naming conventions and notation, we will slightly abuse notation and denote the cohomology of the total complex by $HR^n(\sigma) := H^n(\operatorname{Tot} RR(\sigma))$ and call it the **bridge cohomology of σ**. It should be clear from the context which complex we are using to get this cohomology, and in many cases are interchangeable.

Proposition 5.1.2. *When $\Bbbk$ contains $\mathbb{Q}$, the total complex of $RR(\sigma)$ is quasi-isomorphic to the bridge complex,* $\mathrm{Tot}RR(\sigma) \overset{q}{\cong} R(\sigma)$.

Proof. When $\Bbbk$ contains $\mathbb{Q}$ note that the (λ, Q)-sequence for the relative complexes is exact, and since $q(1-\lambda) = (1-\lambda)q$ and q is surjective, we have that $\mathrm{Im}(q(1-\lambda)) = \ker(Q)$. Thus, the rows of (5.1.1) are exact, and by Proposition A.3.3 the cohomology of $\mathrm{Tot}\, RR(\sigma)$ is the same as the kernel of the first two columns, $\ker(q(1-\lambda)) = \{\varphi \in C(A) \,|\, (1-\lambda)\varphi \in \sigma^* C_{\mathrm{bar}}(A/I)\} = R(\sigma)$. □

Definition 5.1.3. The **degenerate bridge bicomplex of** σ, $DR(\sigma)$ is the bicomplex whose columns are given by

$$D(A) \xrightarrow{q(1-\lambda)} D_{\mathrm{bar}}(A, I) \xrightarrow{Q} D(A, I) \xrightarrow{1-\lambda} D_{\mathrm{bar}}(A, I) \xrightarrow{Q} \ldots \tag{5.1.3}$$

Its cohomology we will denote by $HD^n(\sigma) := H^n(\mathrm{Tot}\, DR(\sigma))$. Note that in this case we have a slight discrepency from the "Connes Complex" construction, so notation is different (with this being the standard). It will match once again in the reduced case.

Proposition 5.1.4. $HD^n(\sigma) = 0$.

Proof. Since $D(A)$, $D(A/I)$, $D_{\mathrm{bar}}(A)$, and $D_{\mathrm{bar}}(A/I)$ are exact, so are $D(A, I)$ and $D_{\mathrm{bar}}(A, I)$. Hence $DR(\sigma)$ has exact columns and $HD^n(\sigma) = 0$. □

Definition 5.1.5. The **normalized bridge bicomplex of** σ, $\overline{RR}(\sigma)$, is the kernel

$$0 \to \overline{RR}(\sigma) \to RR(\sigma) \to DR(\sigma) \to 0.$$

Alternatively, it is the bicomplex formed by using the following normalized complexes as columns

$$\overline{C}(A) \xrightarrow{q(1-\lambda)} \overline{C}_{\mathrm{bar}}(A, I) \xrightarrow{Q} \overline{C}(A, I) \xrightarrow{1-\lambda} \overline{C}_{\mathrm{bar}}(A, I) \xrightarrow{Q} \ldots$$

By the previous proposition, we have $H^n(\mathrm{Tot}\, \overline{RR}(\sigma)) = HR^n(\sigma)$.

Let's denote by $\mathring{\mathbb{j}}_{(A,I)} := \mathrm{ev}_1(C^0(A,I))$ the image of the "evaluation" map applied to the pair (A,I).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(A/I) & \longrightarrow & C^0(A) & \longrightarrow & C^0(A,I) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \\ 0 & \longrightarrow & \mathring{\mathbb{j}}_{A/I} & \longrightarrow & \mathring{\mathbb{j}}_{A} & \longrightarrow & \mathring{\mathbb{j}}_{(A,I)} & \longrightarrow & 0 \end{array}$$

By the above diagram it's clear that $\mathring{\mathbb{j}}_{(A,I)} \cong \mathring{\mathbb{j}}_A / \mathring{\mathbb{j}}_{A/I}$ ($= \mathring{\mathbb{j}}_A / \mathring{\mathbb{j}}_B$ by previous notation). Let $\mathring{\mathbb{j}}(\sigma)[0,e]$ be the bicomplex whose 0^{th} row is the sequence

$$\mathring{\mathbb{j}}_A \xrightarrow{q(1-\lambda)} 0 \xrightarrow{Q} \mathring{\mathbb{j}}_{(A,I)} \xrightarrow{1-\lambda} 0 \xrightarrow{Q} \mathring{\mathbb{j}}_{(A,I)} \to \ldots$$

with zeros elsewhere.

Definition 5.1.6. The **reduced bridge bicomplex** $RR(\sigma)_{\mathrm{red}}$ is the kernel

$$0 \to RR(\sigma)_{\mathrm{red}} \to \overline{RR}(\sigma) \to \mathring{\mathbb{j}}(\sigma)[0,e] \to 0.$$

Or alternatively,

$$0 \to RR(\sigma)_{\mathrm{red}} \to RR(\sigma) \to DR(\sigma)_{\mathrm{red}} \to 0$$

where we define $DR(\sigma)_{\mathrm{red}}$ to be the same as $DR(\sigma)$ except the bottom row is $\mathring{\mathbb{j}}_A \to \mathring{\mathbb{j}}_{(A,I)} \to \mathring{\mathbb{j}}_{(A,I)} \to \ldots$ We will denote the **reduced bridge cohomology of** σ by $\overline{HR}^n(\sigma) := H^n(\mathrm{Tot}\, RR(\sigma)_{\mathrm{red}})$, and the **reduced degenerate bridge cohomology** by $\overline{HD}^n(\sigma) := H^n(\mathrm{Tot}\, DR(\sigma)_{\mathrm{red}})$.

Proposition 5.1.7. *$\overline{HD}^n(\sigma)$, has the following values*

$$\overline{HD}^n(\sigma) := \begin{cases} \mathring{\mathbb{j}}_A & n = 0 \\ \mathring{\mathbb{j}}_{(A,I)} & n \ \mathit{even} \\ 0 & n \ \mathit{odd} \end{cases}$$

Hence, we have a long exact sequence

$$\begin{gathered} 0 \to \overline{HR}^0(\sigma) \to HR^0(\sigma) \to \mathring{\mathbb{j}}_A \to \overline{HR}^1(\sigma) \to HR^1(\sigma) \to 0 \to \ldots \\ \ldots \to 0 \to \overline{HR}^{2n}(\sigma) \to HR^{2n}(\sigma) \to \mathring{\mathbb{j}}_{(A,I)} \to \overline{HR}^{2n+1}(\sigma) \to HR^{2n+1}(\sigma) \to 0 \to \ldots \end{gathered} \tag{5.1.7}$$

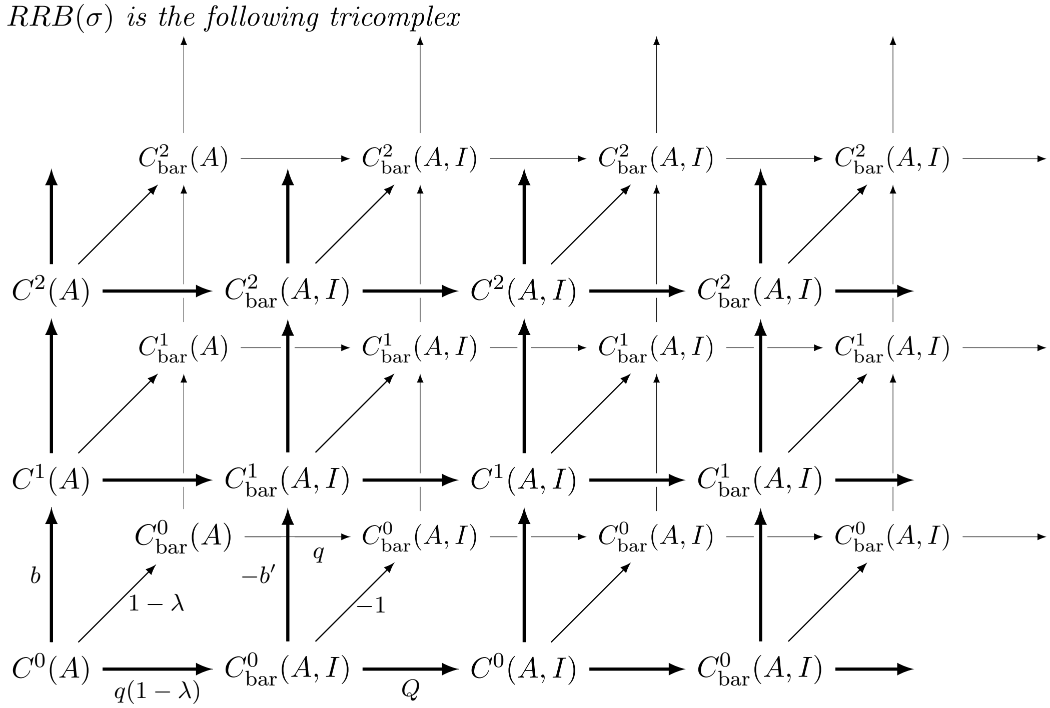
Proof. This hould be clear by construction since $\mathbb{j}(\sigma)[0,e] \to DR(\sigma)_{\text{red}}$ is a quasi-isomorphism, though moreover we should note that this quasi-isomorphism is also induced by the evaluation map (see Corollary 3.4.7). We have once again arrived at the same exact sequence as (4.2.9). □

Corollary 5.1.8. *For a non-unital surjection $A \xrightarrow{\sigma} B$, $\overline{HR}(\sigma_+) = HR(\sigma)$.*

Proof. $\mathbb{j}_{A^+} = \Bbbk$ and $\mathbb{j}_{(A,I)} = 0$. Comparing the long exact sequence above with Definition 4.1.4 gives the result. □

Our last task is to relate the bridge cohomology for a non-unital map with its naive version. Similar to how the results were obtained for Hochschild cohomology, we will need to add an extra dimension to our complex.

Lemma 5.1.9. *For an augmented morphism $\sigma_+ : A^+ \to A^+/I$, $HR^n(\sigma) = H^n(\operatorname{Tot} RRB(\sigma))$, where $RRB(\sigma)$ is the following tricomplex*



Proof. Note that by Corollary (3.1.3)

$$RR^{\bullet\bullet}(\sigma_+)_{\text{red}} = C^\bullet(A^+)_{\text{red}} \xrightarrow{q(1-\lambda)} \overline{C}^\bullet_{\text{bar}}(A^+, I) \to C^\bullet(A^+, I)_{\text{red}} \to$$

$$\cong C^\bullet(A) \oplus C^{\bullet-1}_{\text{bar}}(A) \to C^\bullet_{\text{bar}}(A,I) \oplus C^{\bullet-1}_{\text{bar}}(A,I) \to C^\bullet(A,I) \oplus C^{\bullet-1}_{\text{bar}}(A,I) \to$$

Comparing the total complex of $RR(\sigma_+)$ under this isomorphism with $\mathrm{Tot}\, RRB(\sigma)$ gives the result.

□

Definition 5.1.10. For any algebra surjection $\sigma : A \to B$, the **bar bicomplex of** σ, $RB(\sigma)$, is the back sheet of $RRB(\sigma)$. That is, $RB(\sigma)$ is the bicomplex with columns

$$RR_{\mathrm{bar}}(\sigma) := C_{\mathrm{bar}}(A) \xrightarrow{q} C_{\mathrm{bar}}(A, I) \xrightarrow{0} C_{\mathrm{bar}}(A, I) \xrightarrow{1} C_{\mathrm{bar}}(A, I) \xrightarrow{0} \ldots$$

The **bar cohomology of** σ, $HB^n(\sigma)$, is given as the total cohomology of this complex. $HB^n(\sigma) := H^n(\mathrm{Tot}\, RB(\sigma))$.

Definition 5.1.11. An algebra surjection $\sigma : A \to B$ is said to be **coH-unital** when $HB^n(\sigma) = 0$ for all n.

Theorem 5.1.12. *Let $\mathcal{HR}(\sigma)$ be the "naive" cohomology of σ, i.e. the cohomology of $RR(\sigma)$ where σ is unital or not. Then the following sequence relating bridge, naive, and bar cohomologies is exact:*

$$\ldots \to HB^{n-1}(\sigma) \to HR^n(\sigma) \to \mathcal{HR}^n(\sigma) \to HB^n(\sigma) \to \ldots$$

Proof. Considering the tricomplex $RRB(\sigma)$ it should be clear that $\mathrm{Tot}^\bullet(RRB(\sigma)) \cong \mathrm{Tot}^\bullet(RR(\sigma)) \oplus \mathrm{Tot}^{\bullet-1}(RB(\sigma))$. We then have a short exact sequence of bicomplexes:

$$0 \to \mathrm{Tot}^{\bullet-1}(RB(\sigma)) \to \mathrm{Tot}^\bullet(RR(\sigma)) \oplus \mathrm{Tot}^{\bullet-1}(RB(\sigma)) \to RR^\bullet(\sigma) \to 0.$$

□

This theorem is analogous to Theorem 4.3.4, and we can generalize the corollary that follows.

Corollary 5.1.13. *Let $\sigma : A \to B$ be an algebra surjection. $\mathcal{HR}(\sigma) = HR(\sigma)$ if and only if σ is coH-unital. If $\Bbbk$ contains $\mathbb{Q}$, this is equivalent to B being coH-unital.*

Proof. The first assertion is clear from the previous long exact sequence. Note that when $\Bbbk$ contains $\mathbb{Q}$ then the rows of $RR_{\mathrm{bar}}(\sigma)$ are exact, and hence $\mathrm{Tot}\, RB(\sigma)$ is quasi-isomorphic to the kernel of the first two rows. Computing, we see $\ker \left(C_{\mathrm{bar}}(A) \xrightarrow{q} C_{\mathrm{bar}}(A, I) \right) \cong \sigma^* C_{\mathrm{bar}}(B)$. □

Chapter 6

Excision and the Gysin-Connes Sequence

6.1 The Gysin-Connes Sequences

Throughout this chapter we will be working with a lot of bi- and tricomplexes. If we refer to the cohomology of such a complex, we mean the cohomology of its total complex.

Theorem 6.1.1. *For an exact sequence of* $\Bbbk$*-algebras* $I \to A \to A/I$*, there exists cohomological long exact sequences*

$$\cdots \to HC^n(A,I) \xrightarrow{S_1} HR^{n+2}(\sigma) \xrightarrow{I_1} HH^{n+2}(A) \xrightarrow{B_1} HC^{n+1}(A,I) \to \dots \qquad \text{(GC1)}$$

$$\cdots \to HC^n_\lambda(A) \xrightarrow{I_2} HR^n(\sigma) \xrightarrow{B_2} HC^{n-1}_\lambda(A/I) \xrightarrow{\sigma^*\circ S_2} HC^{n+1}_\lambda(A/I) \to \dots \qquad \text{(GC2)}$$

and

$$\cdots \to HR^{n-2}(\sigma) \xrightarrow{S_3} HR^n(\sigma) \xrightarrow{I_3} HH^n(A) \oplus HH^{n-1}(A/I) \xrightarrow{B_3} HR^{n-1}(\sigma) \to \dots \qquad \text{(GC3)}$$

We shall prove each sequence individually.

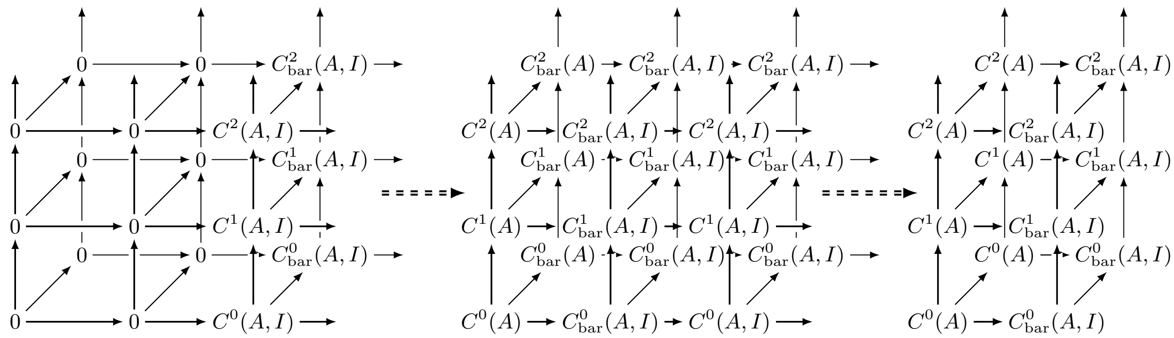
(GC1)

Proof. Beginning with the first sequence, when A is unital this follows from the short exact sequence

$$0 \to CC(A,I)[2,0] \to RR(\sigma) \to RR(\sigma)^{\{2\}} \to 0.$$

When A is non-unital, we have to appeal to the full tri-complex $RRB(\sigma)$:

$$0 \to CCB(A,I)[2,0,0] \to RRB(\sigma) \to RRB(\sigma)^{\{2\}} \to 0.$$



Let's first look at the cohomology of the right complex $RR(\sigma)^{\{2\}}$. By Corollary 3.1.3 we see that the first bicolumn is isomorphic to $C^\bullet(A^+)_{\text{red}}$, while the second bicolumn is isomorphic to $\overline{C}^\bullet_{\text{bar}}(A^+, I)_{\text{red}}$ which, along with $C^\bullet_{\text{bar}}(A^+)_{\text{red}}$ and $C^\bullet_{\text{bar}}(A^+/I)_{\text{red}}$, is contractible (see Definition 3.1.2). Hence, the right complex has the same cohomology as its first bicolumn i.e. $H(RR(\sigma)^{\{2\}}) \cong HH(A)$.

As for the left complex, starting with the cyclic bicomplexes $CC(A/I) \to CC(A)$ we may then form the relative bicomplex $CC(A, I)$, the relative complex for the augmented algebras $CC(A^+, I)$, and then the reduced relative bicomplex $CC(A^+, I)_{\text{red}}$ (see Definition 3.3.4). Splitting each column of this last complex into bicolumns (via 3.1.3), we obtain the complex $CCB(A, I)$. Hence, $CCB(A, I) \cong CC(A^+, I)_{\text{red}} \overset{q}{\cong} CC(A, I)$, where the latter quasi-isomorphism follows from the relative version of Theorem 3.3.7.

(GC2)

The second long exact sequence in Thereom 6.1.1 originates from Lesch, Moscovici, and Pflaum [LMP03]. We start with their construction, and then using complexes developed in this paper generalize the sequence to non-untal algeras. We start with the following lemma.

Lemma 6.1.2. *Let X, U, V be R-modules with U, V submodules of X, and set*

$$\pi_1 : U + V \to U/U \cap V \quad u + v \mapsto u \quad \mod U \cap V$$

$$\pi_2 : U + V \to V/U \cap V \quad u + v \mapsto v \quad \mod U \cap V$$

Then the following diagram is commutative with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U \cap V & \longrightarrow & U & \longrightarrow & U/U \cap V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \scriptstyle{\mathrm{id}} & & \\
0 & \longrightarrow & V & \longrightarrow & U + V & \xrightarrow{\pi_1} & U/U \cap V & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\pi_2} & & \downarrow & & \\
0 & \longrightarrow & V/U \cap V & \xrightarrow{\mathrm{id}} & V/U \cap V & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. The proof is straightforward though tedious to check.

When σ is unital, we recal Proposition 4.1.2 for the direct sum formulation of bridge cohomology and we obtain:

Proposition 6.1.3. *The following diagram is commutative with exact rows and columns both in the algebraic as well as the topological case.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^\bullet_\lambda(A/I) & \xrightarrow{\sigma^*} & C^\bullet_\lambda(A) & \longrightarrow & C^\bullet_\lambda(A)/\sigma^* C^\bullet_\lambda(A/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \scriptstyle{\mathrm{id}} & & \\
0 & \longrightarrow & C^\bullet(A/I) & \longrightarrow & R^\bullet(\sigma) & \xrightarrow{\pi_1} & C^\bullet_\lambda(A)/\sigma^* C^\bullet_\lambda(A/I) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \scriptstyle{\pi_2} & & \downarrow & & \\
0 & \longrightarrow & C^\bullet(A/I)/C_\lambda(A/I) & \xrightarrow{\mathrm{id}} & C^\bullet(A/I)/C^\bullet_\lambda(A/I) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

This induces long exact sequences on cohomology

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & HC^n_\lambda(A/I) & \xrightarrow{\sigma^*} & HC^n_\lambda(A) & \longrightarrow & HC^n_\lambda(A,I) & \xrightarrow{\delta} & HC^{n+1}_\lambda(A/I) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\
\cdots & \longrightarrow & HH^n(A/I) & \longrightarrow & HR^n(\sigma) & \longrightarrow & HC^n_\lambda(A,I) & \xrightarrow{I\circ\delta} & HH^{n+1}(A/I) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle B} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & HC^{n-1}_\lambda(A/I) & \xrightarrow{\mathrm{id}} & HC^{n-1}_\lambda(A/I) & \longrightarrow & 0 & \longrightarrow & HC^n_\lambda(A/I) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle \sigma^*\circ S} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & HC^{n+1}_\lambda(A/I) & \xrightarrow{\sigma^*} & HC^{n+1}_\lambda(A) & \longrightarrow & HC^{n+1}_\lambda(A,I) & \xrightarrow{\delta} & HC^{n+2}_\lambda(A/I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Remark. We remark that the topological setting follows from [Tre16, Ch. 43] and [BL01].

In order to extend the second Gysin-Connes sequence to non-unital surjections we would need a non-direct sum formulation for reduced bridge cohomology. Looking back at Section 4.3, in particular Corollary 4.3.3, we can use the total complex for $RB(\sigma)^{\{2\}}$,

$$
\mathrm{Tot}^\bullet(RB(\sigma)^{\{2\}}) = \left(R^\bullet(\sigma) \oplus C^{\bullet-1}_{\mathrm{bar}}(A/I), \begin{pmatrix} b & 0 \\ (1-\lambda)^\nabla & -b' \end{pmatrix} \right).
$$

Since the naive complex $R(\sigma)$ can still be written as a sum, we get the following bicomplex

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^\bullet_\lambda(A/I)\oplus 0 & \xrightarrow{\sigma^*\oplus\mathrm{id}} & C^\bullet_\lambda(A)\oplus 0 & \longrightarrow & C^\bullet_\lambda(A,I)\oplus 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i\oplus\mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}\oplus\mathrm{id}} & & \\
0 & \longrightarrow & C^\bullet(A/I)\oplus C^{\bullet-1}_{\mathrm{bar}}(A/I) & \xrightarrow{\sigma^*\oplus\mathrm{id}} & R^\bullet(\sigma)\oplus C^{\bullet-1}_{\mathrm{bar}}(A/I) & \xrightarrow{\pi_1\oplus 0} & C^\bullet_\lambda(A,I)\oplus 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi_2\oplus\mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & C^\bullet(A/I)/C^\bullet_\lambda(A/I)\oplus C^{\bullet-1}_{\mathrm{bar}}(A/I) & \xrightarrow{\mathrm{id}\oplus\mathrm{id}} & C^\bullet(A/I)/C^\bullet_\lambda(A/I)\oplus C^{\bullet-1}_{\mathrm{bar}}(A/I) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

This gives the same cohomological sequence as above. We note here that the map $B =$

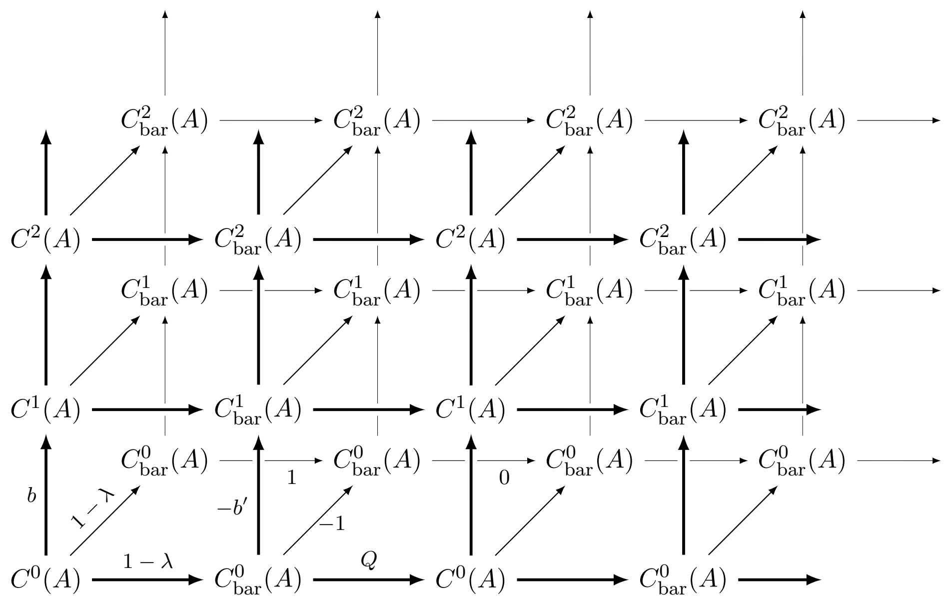
$Qs(1-\lambda)$ on the bicomplexes becomes

$$B = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}.$$

Moreover, the induced map on the quotient $B : C^\bullet(A/I)/C^\bullet_\lambda(A/I) \oplus C^{\bullet-1}_{\text{bar}}(A/I) \to C^{\bullet-1}_\lambda(A/I) \oplus 0$ is a quasi-isomorphism.

(GC3)

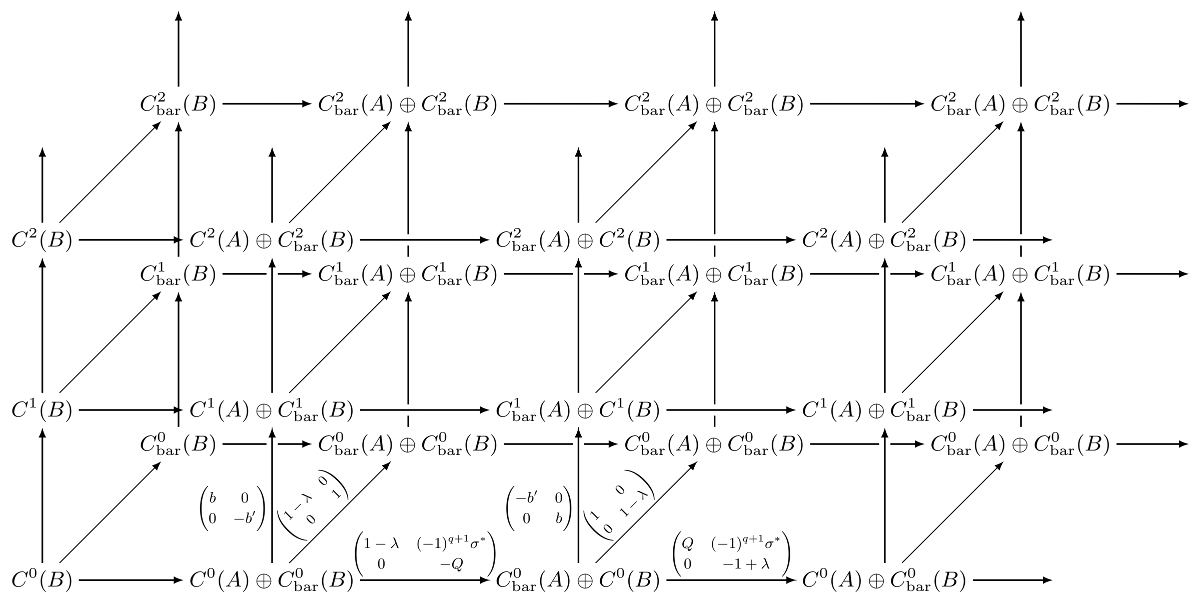
For the third Gysin-Connes sequence we first recall the tricomplex $CCB(A)$ from Definition 3.3.6.



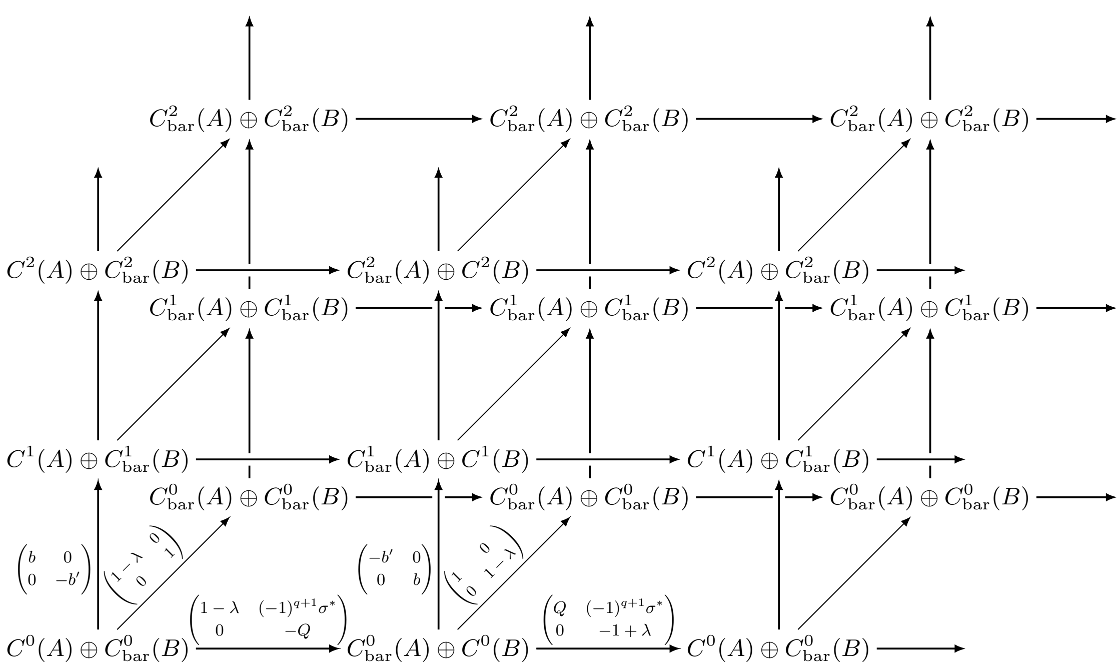
A surjective morphism $A \xrightarrow{\sigma} B$ (we once agiain use $B := A/I$ to reduce clutter in the diagrams that follow) now induces a cone complex

$$0 \to CCB(B) \xrightarrow{\sigma^*} CCB(A) \to \text{cone}(\sigma^*)$$

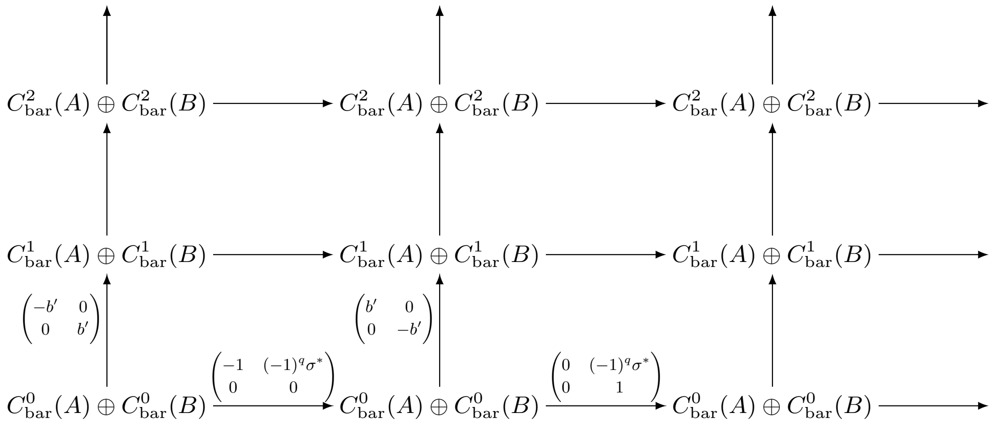
given by



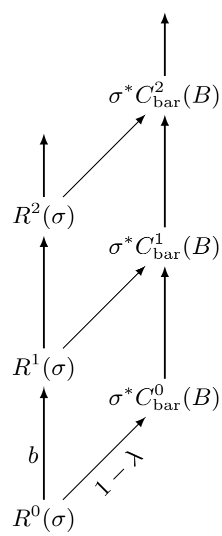
So as not to overclutter the diagram with maps, we left the factor of $(-1)^{q+1}\sigma^*$ in the maps above, which refers to the row height q. Following the ideas of Lesch, Moscovicci, and Pflaum [LMP03] we now remove the first column to get the complex $\widetilde{\mathrm{cone}}(\sigma^*)$



To see the maps on the back reference this diagram



When $\Bbbk$ contains $\mathbb{Q}$, one checks that the rows of $\widetilde{\text{cone}}(\sigma^*)$ are exact except at the first column. Hence this complex is quasi-isomorphic to the kernel of the first two columns, which one computes as



This as the same as 4.3.2, the cohomology of which is $HR(\sigma)$ whether σ is unital or not.

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
R^2(\sigma) & \longrightarrow & C^2_{\mathrm{bar}}(B) \\
\uparrow & & \uparrow \\
R^1(\sigma) & \longrightarrow & C^1_{\mathrm{bar}}(B) \\
b \uparrow & & \uparrow -b' \\
R^0(\sigma) & \xrightarrow{(1-\lambda)^{\triangledown}} & C^0_{\mathrm{bar}}(B)
\end{array}$$

Corollary 6.1.4. *For a short exact sequence of algebras $I \to A \to A/I$, there exists a long exact cohomological sequence*

$$... \to HR^n(\sigma) \to HC^{n+1}(A, I) \to HH^{n+1}(A/I) \to HR^{n+1}(\sigma) \to ...$$

Proof. From the above constructions we have a short exact sequence of complexes

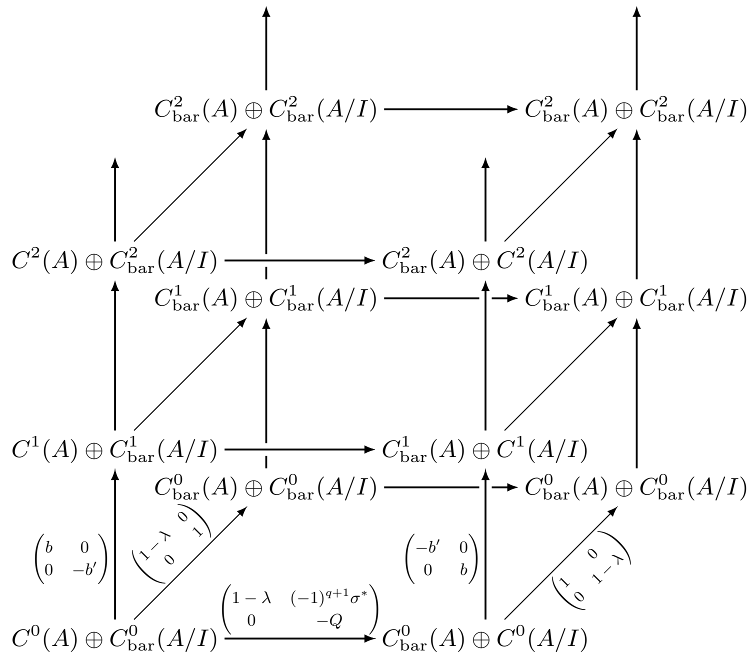
$$0 \to \widetilde{\mathrm{cone}}(\sigma^*) \xrightarrow{[1,0,0]} \mathrm{cone}(\sigma^*) \to C(A/I) \to 0$$

The lemma follows by taking cohomology and noting that $\mathrm{cone}(\sigma^*)$ is quasi-isomorphic to the relative cyclic bicomplex. □

Proof of Theorem. Now noticing that the columns of $\widetilde{\mathrm{cone}}(\sigma^*)$ are 2-periodic, we can use the same shift technique as in GC1 and get a short exact sequence

$$0 \to \widetilde{\mathrm{cone}}(\sigma^*)[2, 0, 0] \to \widetilde{\mathrm{cone}}(\sigma^*) \to \widetilde{\mathrm{cone}}(\sigma^*)^{\{2\}} \to 0.$$

We analyze the right complex, $\widetilde{\text{cone}}(\sigma)^{\{2\}}$:



(6.1.4)

We can collapse toward the total complex (in the z direction) and get

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
C^2(A) \oplus C^1_{\text{bar}}(A) \oplus C^2_{\text{bar}}(A/I) \oplus C^0_{\text{bar}}(A/I) & \longrightarrow & C^2_{\text{bar}}(A) \oplus C^1_{\text{bar}}(A) \oplus C^2(A/I) \oplus C^0_{\text{bar}}(A/I) \\
\begin{pmatrix} b & 0 & 0 & 0 \\ 1-\lambda & -b' & 0 & 0 \\ 0 & 0 & -b' & 0 \\ 0 & 0 & 1 & b' \end{pmatrix}\Big\uparrow & & \Big\uparrow\begin{pmatrix} -b' & 0 & 0 & 0 \\ 1 & b' & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 1-\lambda & -b' \end{pmatrix} \\
C^1(A) \oplus C^0_{\text{bar}}(A) \oplus C^1_{\text{bar}}(A/I) \oplus C^0_{\text{bar}}(A/I) & \longrightarrow & C^1_{\text{bar}}(A) \oplus C^0_{\text{bar}}(A) \oplus C^1(A/I) \oplus C^0_{\text{bar}}(A/I) \\
\uparrow & \begin{pmatrix} 1-\lambda & 0 & (-1)^{q+1}\sigma^* & 0 \\ 0 & -1 & 0 & (-1)^q\sigma^* \\ 0 & 0 & -Q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \uparrow \\
C^0(A) \oplus C^0_{\text{bar}}(A/I) & \longrightarrow & C^0_{\text{bar}}(A) \oplus C^0(A/I)
\end{array}$$

One should recognize that each column is now the direct sum of the reduced Hochschild and bridge complexes for their respective algebras (c.f. corollary3.1.3). We get

$$\begin{array}{ccc}
\uparrow & & \uparrow \\
C^2(A^+)_{\text{red}} \oplus C^2_{\text{bar}}(A^+/I)_{\text{red}} & \longrightarrow & C^2_{\text{bar}}(A^+)_{\text{red}} \oplus C^2(A^+/I)_{\text{red}} \\
\begin{pmatrix} b & 0 \\ 0 & -b' \end{pmatrix}\Big\uparrow & & \Big\uparrow\begin{pmatrix} -b' & 0 \\ 0 & b \end{pmatrix} \\
C^1(A^+)_{\text{red}} \oplus C^1_{\text{bar}}(A^+/I)_{\text{red}} & \longrightarrow & C^1_{\text{bar}}(A^+)_{\text{red}} \oplus C^1(A^+/I)_{\text{red}} \\
\uparrow & \begin{pmatrix} 1-\lambda & (-1)^{q+1}\sigma^* \\ 0 & -Q \end{pmatrix} & \uparrow \\
C^0(A^+)_{\text{red}} \oplus C^0_{\text{bar}}(A^+/I)_{\text{red}} & \longrightarrow & C^0_{\text{bar}}(A^+)_{\text{red}} \oplus C^0(A^+/I)_{\text{red}}
\end{array}$$

Note that the bar complexes are contractible and we may apply the killing contractible complexes lemma. Hence we get

$$\begin{array}{ccc} \uparrow & & \uparrow \\ C^2(A^+)_{\text{red}} & \longrightarrow & C^2(A^+/I)_{\text{red}} \\ b\uparrow & & \uparrow -b' \\ C^1(A^+)_{\text{red}} & \xrightarrow[0]{} & C^1(A^+/I)_{\text{red}} \\ \uparrow & & \uparrow \\ C^0(A^+)_{\text{red}} & \longrightarrow & C^0(A^+/I)_{\text{red}} \end{array}$$

We can now conclude that we have a long exact sequence

$$\cdots \to HR^{n-2}(\sigma) \xrightarrow{S} HR^n(\sigma) \xrightarrow{I} HH^n(A) \oplus HH^{n-1}(A/I) \xrightarrow{B} HR^{n-1}(\sigma) \to \dots \qquad \text{(GC3)}$$

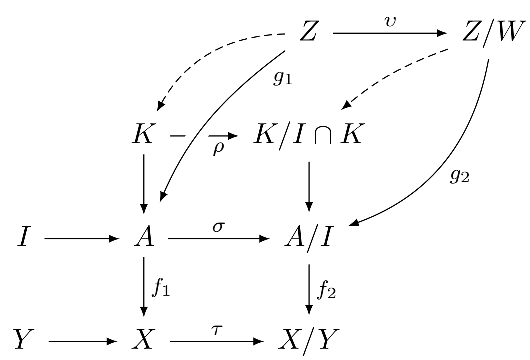
□

6.2 The Category $\mathcal{S}_{\Bbbk}$

We move on to prove excision for bridge cohomology, but first we provide some more info-mation about the category $\mathcal{S}_{\Bbbk}$. In particular, we will show that while $\mathcal{S}_{\Bbbk}$ is not abelian (since the category of $\Bbbk$-algebras is not additive, we cannot add two $\Bbbk$-algebra morphisms, hence we cannot add two objects in $\mathcal{S}_{\Bbbk}$), it does still have kernels and cokernels.

Proposition 6.2.1. *Let $\sigma : A \to A/I$ and $\tau : X \to X/Y$ be two surjective algebra morphisms with $(f_1, f_2) : \sigma \to \tau$ a map between them. Let $K := \ker(f_1)$. Then the kernel of (f_1, f_2) is the map $\rho : K \to K/I \cap K$.*

Proof. Let υ be the surjection $Z \xrightarrow{\upsilon} Z/W$, with a morphism $(g_1, g_2) : \upsilon \to \sigma$ such that $(f_1, f_2) \circ (g_1, g_2) : \upsilon \to \tau = (0, 0)$. Our task is to show that there exists a map from $\upsilon \to \rho$. We have the following diagram:

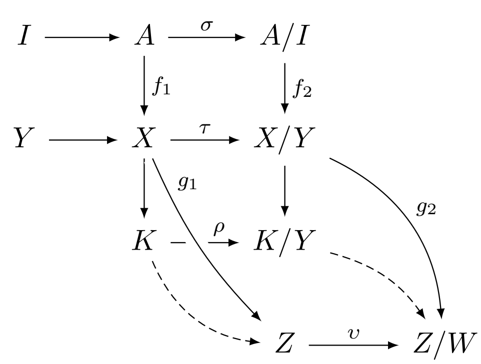


We need to show that the dashed arrows exist and make the diagram commute. The arrow $Z \dashrightarrow K$ exists by the universal property of the kernel K. Since υ is surjective, we have that $\sigma \circ g_1(Z) = g_2(Z/W)$, but $g_1(Z) \subset \mathrm{im}(K)$. Hence $g_2(Z/W) \subset \sigma(\mathrm{im}(K)) = \mathrm{im}(K/I \cap K)$. Thus we get the map $Z/W \dashrightarrow K/I \cap K$. Commutativity of the diagram should be clear by construction. □

Note. It should be noted that while the map $K/I \cap K \to A/I$ is injective and $K/I \cap K$ is an ideal in A/I, it is not in general the kernel of f_2 i.e. the right column is not exact.

Proposition 6.2.2. *Let $\sigma : A \to A/I$ and $\tau : X \to X/Y$ be two surjective algebra morphisms with $(f_1, f_2) : \sigma \to \tau$ a map between them. Let $K := \mathrm{coker}(f_1)$. Then the cokernel of (f_1, f_2) is the map $\rho : K \to K/Y$, where by K/Y we mean K modulo the image of Y in K.*

Proof. As before, let $\upsilon : Z \to Z/W$ be a surjection along with a morphism $(g_1, g_2) : \tau \to \upsilon$ such that $(g_1, g_2) \circ (f_1, f_2) = (0, 0)$. We need to construct the dashed arrows in the diagram:



The map $K \dashrightarrow Z$ is given by the univeral property of the cokernel - let's call this h. Then we get a map $K/Y \dashrightarrow Z/W$ by $k + Y \mapsto \upsilon \circ h(k)$. This is well defined since the image of Y in

Z/W is zero. □

Corollary 6.2.3. *In the category of* $\mathcal{S}_{\Bbbk}$ *a short exact sequence,* $0 \to \rho \to \sigma \xrightarrow{q} \tau \to 0$*, is given by a nine diagram of algebras.*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I \cap K & \to & K & \xrightarrow{\rho} & K/(I \cap K) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I & \to & A & \xrightarrow{\sigma} & A/I & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I/(I \cap K) & \to & A/K & \xrightarrow{\tau} & A/(I + K) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Note. It is important to note that a pair of surjections $(f_1, f_2) : \sigma \to \tau$ is not necessarily a cokernel, and likewise for injections.

Definition 6.2.4. Given a kernel $\rho \xrightarrow{(j_1,j_2)} \sigma$ with cokernel τ formally written as σ/ρ, the **relative bridge complex,** $R^{\bullet}(\sigma, \rho)$**,** is the cokernel

$$0 \to R^{\bullet}(\sigma/\rho) \to R^{\bullet}(\sigma) \to R^{\bullet}(\sigma, \rho) \to 0.$$

We leave it to the reader to check that the map $R^{\bullet}(\sigma/\rho) \to R^{\bullet}(\sigma)$ is injective. We of course now have a map $R^{\bullet}(\sigma, \rho) \to R^{\bullet}(\rho)$ and can make the following definition.

A kernel $\rho \to \sigma$ is said to satisfy **excision** if the map $R^{\bullet}(\sigma, \rho) \to R^{\bullet}(\rho)$ is a quasi-isomorphism.

6.3 Excision

Definition 6.3.1. The extension $0 \to \rho \to \sigma \to \tau \to 0$ of ideals is said to be **copure** if for any $\Bbbk$-module V, the bottom maps of the nine diagram of modules are surjective.

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(A/(I+K),V) & \xrightarrow{\tau^*} & \mathrm{Hom}(A/K,V) & \longrightarrow & \mathrm{Hom}(I/I\cap K,V) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(A/I,V) & \xrightarrow{\sigma^*} & \mathrm{Hom}(A,V) & \longrightarrow & \mathrm{Hom}(I,V) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}(K/I\cap K,V) & \xrightarrow{\rho^*} & \mathrm{Hom}(K,V) & \longrightarrow & \mathrm{Hom}(I\cap K,V) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
$$

Theorem 6.3.2. *(Excision) Let $0 \to \rho \to \sigma \to \tau \to 0$ be a copure short exact sequence in the category $\mathcal{S}_{\Bbbk}$, with associated nine diagram*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I\cap K & \longrightarrow & K & \xrightarrow{\rho} & K/(I\cap K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\sigma} & A/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I/(I\cap K) & \longrightarrow & A/K & \xrightarrow{\tau} & A/(I+K) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Then for any $\Bbbk$-module V, the map $R(\sigma,\rho;V) \to R(\rho;V)$ is a quasi-isomorphism if and only if K and $K/I\cap K$ are coH-unital.

Proof. Let V be a $\Bbbk$-module. The sequence GC2 for the maps σ and τ (given horizontally), along with the standard long exact cohomological sequences for the relative complexes (given vertically), give a relative version of GC2 (row 3 below).

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & HC^n_\lambda(A/K;V) & \xrightarrow{I} & HR^n(\tau;V) & \xrightarrow{B} & HC^{n-1}_\lambda(A/(I+K);V) & \xrightarrow{\tau^*\circ S} & HC^{n+1}_\lambda(A/K;V) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & HC^n_\lambda(A;V) & \xrightarrow{I} & HR^n(\sigma;V) & \xrightarrow{B} & HC^{n-1}_\lambda(A/I;V) & \xrightarrow{\sigma^*\circ S} & HC^{n+1}_\lambda(A;V) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & HC^n_\lambda(A,K;V) & \xrightarrow{I} & HR^n(\sigma,\rho;V) & \xrightarrow{B} & HC^{n-1}_\lambda(A/I,K/(I\cap K);V) & \xrightarrow{\rho^*_{\mathrm{rel}}\circ S} & HC^{n+1}_\lambda(A,K;V) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & HC^{n+1}_\lambda(A/K;V) & \to & HR^{n+1}(\tau;V) & \to & HC^n_\lambda(A/(I+K);V) & \to & HC^{n+2}_\lambda(A/K;V) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Comparing this with the same Gysin-Connes sequence for ρ gives us the diagram

$$
\begin{array}{ccccccccccc}
\cdots & \to & HC^n_\lambda(A,K;V) & \xrightarrow{I} & HR^n(\sigma,\rho;V) & \xrightarrow{B} & HC^{n-1}_\lambda(A/I,K/(I\cap K);V) & \xrightarrow{\rho^*_{\mathrm{rel}}\circ S} & HC^{n+1}_\lambda(A,K;V) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & HC^n_\lambda(K;V) & \xrightarrow{I} & HR^n(\rho;V) & \xrightarrow{B} & HC^{n-1}_\lambda(K/(I\cap K);V) & \xrightarrow{\rho^*\circ S} & HC^{n+1}_\lambda(K;V) & \to & \cdots
\end{array}
\tag{6.3.2}
$$

Suppose that K and $K/(I\cap K)$ are coH-unital. It follows from Theorem 2.4.5 that the vertical maps in the first, third, and fourth columns above are isomorphisms. Hence, so is the second.

Now suppose that ρ is excisive. We can once again create a relative version for the sequence GC3, and again using the map induced out of the cokernel gives us its relation with ρ

$$
\begin{array}{ccccccccccc}
\cdots & \to & HR^n(\sigma,\rho;V) & \xrightarrow{I} & HH^n(A,K;V)\oplus HH^{n-1}(A/I,K/(I\cap K);V) & \xrightarrow{B} & HR^{n-1}(\sigma,\rho;V) & \xrightarrow{S} & HR^{n+1}(\sigma,\rho;V) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & HR^n(\rho;V) & \xrightarrow{I} & HH^n(K;V)\oplus HH^{n-1}(K/(I\cap K);V) & \xrightarrow{B} & HR^{n-1}(\rho;V) & \xrightarrow{S} & HR^{n+1}(\rho;V) & \to & \cdots
\end{array}
\tag{6.3.2}
$$

Since ρ is excisive each of the maps for the bridge cohomologies $HR(\cdot)$ are isomorphisms. Hence, so are the maps

$$HH^n(A,K;V)\oplus HH^{n-1}(A/I,K/(I\cap K);V)\to HH^n(K;V)\oplus HH^{n-1}(K/(I\cap K);V)$$

This map is induced from the map on complexes $\widetilde{\mathrm{cone}}(\sigma,\rho)^{\{2\}}\to\widetilde{\mathrm{cone}}(\rho)^{\{2\}}$. Looking at this complex (6.1.4) and the steps following, demonstrate that there are no "cross maps" in the cohomological isomorphism above, i.e. both K and $K/(I \cap K)$ are excisive, hence coH-unital. $\square$

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Appendix A

Homological Constructions

A.1 Normalized and Reduced Complexes

Here we recall some of the homological constructions in Loday [Lod13] and then develop the notion of the normalized and reduced bar complexes.

Recall ([Lod13, 1.1 pg. 8]) that for a unital $\Bbbk$-algebra A, and A-bimodule M, the bar complex $C^{\text{bar}}_{\bullet}(A, M)$ and Hochschild complex $C_{\bullet}(A, M)$ are given by

$$C^{\text{bar}}_{\bullet}(A, M) := \; M \xleftarrow{b'} M \otimes A \xleftarrow{b'} ... \xleftarrow{b'} M \otimes A^{\otimes n} \xleftarrow{b'} M \otimes A^{\otimes n+1} \xleftarrow{b'} ...$$

$$C_{\bullet}(A, M) := \; M \xleftarrow{b} M \otimes A \xleftarrow{b} ... \xleftarrow{b} M \otimes A^{\otimes n} \xleftarrow{b} M \otimes A^{\otimes n+1} \xleftarrow{b} ...$$

We will use tuple notation for tensor products i.e. $(m, a_1, ..., a_n) = m \otimes a_1 \otimes ... \otimes a_n$. For any $(m, a_1, ..., a_n) \in M \otimes A^{\otimes n}$,

$$b'(m, a_1, ..., a_n) = \; (ma_1, ..., a_n) + \sum_{i=1}^{n-1} (-1)^i (m, a_1, ..., a_i a_{i+1}, ..., a_n)$$

and

$$b(m, a_1, ..., a_n) = \; b'(m, a_1, ..., a_n) + (-1)^n (a_n m, a_1, ..., a_{n-1})$$

Of special note is the case when $M = A$. In such a case we will write $C_{\bullet}(A) := C_{\bullet}(A, A)$. The notation $C^{\text{bar}}_{\bullet}(A)$ will be used analogously for the bar complex. As with the cohomological case, when A is unital, we have the extra degeneracy map $s : C^{(\text{bar})}_{\bullet}(A) \to C^{(\text{bar})}_{\bullet-1}(A)$ where $s(a_0, ..., a_{n-1}) = (1, a_0, ..., a_{n-1})$. This map gives us a contracting homotopy for $C^{\text{bar}}_{\bullet}(A)$ i.e. $sb' + b's = 1$.

From [Lod13, 1.1.14 pg. 13, 1.6.4 pg. 46] for a unital $\Bbbk$-algebra A, and any A-bimodule M, the Hochschild complex $C_\bullet(A, M)$ has a large degenerate subcomplex $D_\bullet(A, M)$, where

$$D_n(A, M) = \{(m, a_1, ..., a_n) \in C_n(A, M) \,|\, a_i = 1 \text{ for some } i\}.$$

The normalized Hochschild complex is then

$$\overline{C}_\bullet(A, M) = C_\bullet(A, M)/D_\bullet(A, M).$$

When $M = A$ we simply write $\overline{C}_\bullet(A)$ and $D_\bullet(A)$. We also have the **reduced Hochschild complex** ([Lod13, 1.4.2 pg. 30]) which is defined as the cokernel of

$$0 \to \Bbbk[0] \to \overline{C}_\bullet(A) \to C_\bullet(A)_{\text{red}} \to 0.$$

For cyclic homology ([Lod13, Sec. 2.1 pgs. 53-60]) we have the cyclic bicomplex $CC(A)$:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & \cdots
\end{array}
$$

Where the lower left position is of bidegree $(0, 0)$ with the first index counting to the right, the second counting upward. The homology of the total complex of $CC(A)$ is the cyclic homology of A, denoted $HC(A)$. Now, when A is unital the odd columns are contractible, and on the even columns we get Connes' boundary map $B : CC_{2,q}(A) \to CC_{0,q+1}(A)$ given by $B = (1-\lambda)sQ$, which gives a quasi-isomorphic bicomplex $\mathcal{B}(A)$:

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\
\downarrow b & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & \\
\downarrow b & & & & \\
A & & & &
\end{array}$$

Replacing each column with its normalization we get the normalized (b, B)-complex $\overline{\mathcal{B}}(A)$:

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
A \otimes \overline{A}^{\otimes 2} & \xleftarrow{\overline{B}} & A \otimes \overline{A} & \xleftarrow{\overline{B}} & A \\
\downarrow b & & \downarrow b & & \\
A \otimes \overline{A} & \xleftarrow{\overline{B}} & A & & \\
\downarrow b & & & & \\
A & & & &
\end{array}$$

The map $\overline{B}$ is induced by sQ in the above. Lastly, we define the reduced (b, B)-complex $\mathcal{B}_{\text{red}}(A)$ as the cokernel (Loday[Lod13, Sec. 2.2.13 pgs. 65-67]):

$$0 \to \overline{\mathcal{B}}(\Bbbk) \to \overline{\mathcal{B}}(A) \to \mathcal{B}_{\text{red}}(A) \to 0. \tag{A.1.1}$$

For the above mentioned complexes we have that for any unital $\Bbbk$-algebra the canonical maps are quasi-isomorphisms ([Lod13, pg. 59]):

$$\operatorname{Tot} \overline{\mathcal{B}}(A) \leftarrow \operatorname{Tot} \mathcal{B}(A) \hookrightarrow \operatorname{Tot} CC(A).$$

We also have

Proposition A.1.1. *For any k-algebra I, the complexes $CC(I)$ and $\mathcal{B}_{\text{red}}(I^+)$ are canonically isomorphic, hence $HC_\bullet(I) = \overline{HC}_\bullet(I^+)$ ([Lod13, Prop. 2.2.16 pg. 66]).*

A.2 Normalized and Reduced Bar complex

We now develop the notion of a normalized bar complex, which in view of the above, is the complex that the map $\overline{B}$ "factors through" i.e. if we were to start with the complex $CC(A)$ above and first normalize rather than removing the bar complex, what would result in the odd columns? Equivalently, our task is to create a normalized version of the exact sequence

$$\cdots \leftarrow C_\bullet(A) \xleftarrow{1-\lambda} C_\bullet^{\text{bar}}(A) \xleftarrow{Q} C_\bullet(A) \xleftarrow{1-\lambda} C_\bullet^{\text{bar}}(A) \leftarrow \dots \tag{A.2.0}$$

Recall from definition (3.1.1) for a unital algebra A, the degenerate bar complex $D_\bullet^{\text{bar}}(A)$ is the submodule of $C_n^{\text{bar}}(A)$ generated by elements of the form

- $(a_0, ..., a_n)$ where $a_i = 1$ for some $0 < i < n$
- $(1, a_1, ..., a_{n-1}, 1)$ (3.1.1)
- $(a_0, ..., a_{n-1}, 1) + (-1)^n(1, a_0, ..., a_{n-1})$.

We also have the contracting homotopy $s : D_n^{\text{bar}}(A) \to D_{n+1}^{\text{bar}}(A)$ from the usual bar complex. For a unital algebra A, we define the **normalized bar complex of** A to be the complex $\overline{C}_\bullet^{\text{bar}}(A) = C_\bullet^{\text{bar}}(A)/D_\bullet^{\text{bar}}(A)$. Note that $H_n\left(\overline{C}_\bullet^{\text{bar}}(A)\right) = HB_n(A) = 0$ for all n.

Theorem A.2.1. *When $\Bbbk$ contains $\mathbb{Q}$, for each $n \geq 1$ the sequences*

$$\cdots \leftarrow D_n(A) \xleftarrow{1-\lambda} D_n^{\text{bar}}(A) \xleftarrow{Q} D_n(A) \xleftarrow{1-\lambda} D_n^{\text{bar}}(A) \leftarrow \dots \tag{A.2.1}$$

and

$$\cdots \leftarrow \overline{C}_n(A) \xleftarrow{1-\lambda} \overline{C}_n^{\text{bar}}(A) \xleftarrow{Q} \overline{C}_n(A) \xleftarrow{1-\lambda} \overline{C}_n^{\text{bar}}(A) \leftarrow \dots \tag{A.2.2}$$

are exact, and for $n = 0$ we have

$$\cdots \leftarrow 0 \xleftarrow{1-\lambda} \Bbbk \xleftarrow{Q} 0 \xleftarrow{1-\lambda} \Bbbk \leftarrow \dots$$

and

$$\cdots \leftarrow A \xleftarrow{1-\lambda} \overline{A} \xleftarrow{Q} A \xleftarrow{1-\lambda} \overline{A} \leftarrow \dots$$

respectively.

Of course, proving exactness of (A.2.1) implies (A.2.2), which we will do in through the following lemmas. Note that the standard (λ, Q)-sequence comes with a contracting homotopy (see [Lod13, Thm. 2.1.5 pg. 55]) given by $h' = \frac{1}{n+1}$ and $h = -\frac{1}{n+1}\sum_{i=1}^{n} i\lambda^i$, satisfying the relations

$$h'Q + (1-\lambda)h = \text{id}, \quad Qh' + h(1-\lambda) = \text{id}$$

Unfortunately, these maps are not compatible with the degenerate complexes in general, but we can still put them to use. Exactness at $D_n^{\text{bar}}(A)$ follows easily from it's construction and making use of the map h'. Exactness at $D_n(A)$ is a bit more difficult primarily since both Q and h have a lot more terms as polynomials of λ. Because of this, the image of h from $D_n(A)$ may not land in $D_n^{\text{bar}}(A)$ and we must appeal more to the structure of elements involved. However, even in doing so, there is still a problem dealing with potential linear dependencies and torsion in A as a $\Bbbk$-module, which only exacerbate the rigor needed when dealing with tensor products. For this, we will have to appeal to the free module based on the generators of A.

Lemma A.2.2. *The sequence (A.2.1) is exact at* $D_n^{\text{bar}}(A)$.

Proof. Let $\alpha \in D_n^{\text{bar}}(A)$ be such that $(1-\lambda)\alpha = 0$. By Definition 3.1.1 we can write $\alpha = \alpha' + Q\beta$ where $\alpha' \in \widetilde{D}_n(A)$ and $\beta \in D_n(A)$. But then $h'(\alpha') = \frac{1}{n+1}\alpha' \in D_n(A)$ is such that $Q(h'(\alpha')+\beta) = \alpha$. Hence $\alpha \in \text{Im}(Q)$. □

Lemma A.2.3. *Any* $\alpha \in D_n(A)$ *can be written* $\alpha = \sum(a_{i0}, ..., a_{in-1}, 1) + (1-\lambda)\beta$ *for some* $\beta \in D_n^{\text{bar}}(A)$.

An example is sufficient to illustrate the idea of the proof:

$$\begin{aligned}(a_0, 1, a_2, a_3) &= (a_3, a_0, 1, a_2) - (a_3, a_0, 1, a_2) + (a_0, 1, a_2, a_3)\\ &= (a_3, a_0, 1, a_2) + (1-\lambda)(a_0, 1, a_2, a_3)\\ &= (a_2, a_3, a_0, 1) - (a_2, a_3, a_0, 1) + (a_3, a_0, 1, a_2) + (1-\lambda)(a_0, 1, a_2, a_3)\\ &= (a_2, a_3, a_0, 1) + (1-\lambda)(a_3, a_0, 1, a_2) + (1-\lambda)(a_0, 1, a_2, a_3)\\ &= (a_2, a_3, a_0, 1) + (1-\lambda)[(a_3, a_0, 1, a_2) + (a_0, 1, a_2, a_3)]\end{aligned}$$

Notice that the terms being affected by $(1-\lambda)$ will always be of the appropriate form to be in $D_n^{\text{bar}}(A)$.

Let $\mathscr{E} = \{e_i \mid i \in \mathscr{I}\} \subset A$ be a minimal set of generators that contains 1, indexed over a set $\mathscr{I}$. Let $\star \in I$ be the distinguished element such that $e_\star = 1$. Let E be the free $\Bbbk$-module with basis $\mathscr{E}$. Define $C_n(E)$ as the vector space with basis elements $(e_{i_0}, ..., e_{i_n})$ for $i_k \in \mathscr{I}$ i.e. using multi-index notation, elements of $C_n(E)$ are given by finite sums $\sum t_I e_I$ for $I = (i_0, ..., i_n) \in \mathscr{I}^{n+1}$ and $t_I \in \Bbbk$. For notational consistency, we'll define $C_n^{\text{bar}}(E) := C_n(E)$. It should be clear that $C_n(E) \cong E^{\otimes n+1}$ and that the projection map $\pi : E \to A$ lifts to $\pi_* : C_n(E) \to C_n(A)$ and $\pi_*^{\text{bar}} : C_n^{\text{bar}}(E) \to C_n^{\text{bar}}(A)$.

Define $D_n(E) \subset C_n(E)$ to be the submodule generated by elements of the form $(e_{i_0}, ..., e_{i_n})$ where $i_k = \star$ for some $1 \le k \le n$. Similarly define $D_n^{\text{bar}}(E) \subset C_n^{\text{bar}}(A)$ to be generated by elements of the form

- $(e_{i_0}, ..., e_{i_n})$ where $i_k = \star$ for some $1 \le k \le n-1$
- $(1, e_{i_1}, ..., e_{i_{n-1}}, 1)$
- $(1+\lambda)(e_{i_0}, ..., e_{i_{n-1}}, 1)$.

π_* descends to $\pi_* : D_n(E) \to D_n(A)$ and $\pi_* : D_n^{\text{bar}}(E) \to D_n^{\text{bar}}(A)$, and clearly we have (λ, Q)-sequences

$$\cdots \leftarrow D_n(E) \xleftarrow{1-\lambda} D_n^{\text{bar}}(E) \xleftarrow{Q} D_n(E) \xleftarrow{1-\lambda} D_n^{\text{bar}}(E) \leftarrow \dots \tag{A.2.3}$$

$$\cdots \leftarrow C_n(E) \xleftarrow{1-\lambda} C_n^{\text{bar}}(E) \xleftarrow{Q} C_n(E) \xleftarrow{1-\lambda} C_n^{\text{bar}}(E) \leftarrow \ldots \tag{A.2.4}$$

Note that the sequence (A.2.4) is exact by the contracting homotopy. Also notice that we can consider $D_n^{\text{bar}}(E)$ to be generated by elements in the first bullet plus $Q(D_n(E))$, hence exactness in (A.2.3) at $D_n^{\text{bar}}(E)$ follows as it did for $D_n^{\text{bar}}(A)$ in Lemma A.2.2. For $D_n(E)$ Lemma A.2.3 still holds, i.e. we can write any element $\alpha \in D_n(E)$ as $\alpha = \sum t_{I_\star} e_{I_\star} + (1-\lambda)\beta$ where $I_\star = (i_0, ..., i_{n-1}, \star) \in \mathscr{I}^{n+1}$ and $\beta \in D_n^{\text{bar}}(E)$. Let $J \in \mathscr{I}^{n+1}$ be an $(n+1)$-tuple of the form $(j_0, ..., j_{n-1}, \star)$ where $j_k \neq \star$ for any $0 \leq k \leq n-1$, and let $L \in \mathscr{I}^{n+1}$ be an $(n+1)$-tuple of the form $(l_0, ..., l_{n-1}, \star)$ where at least one other $l_k = \star$. For any $\alpha \in D_n(E)$ we can write

$$\alpha = \sum t_J e_J + \sum t_L e_L + (1-\lambda)\beta$$

for some $\beta \in D_n^{\text{bar}}(E)$.

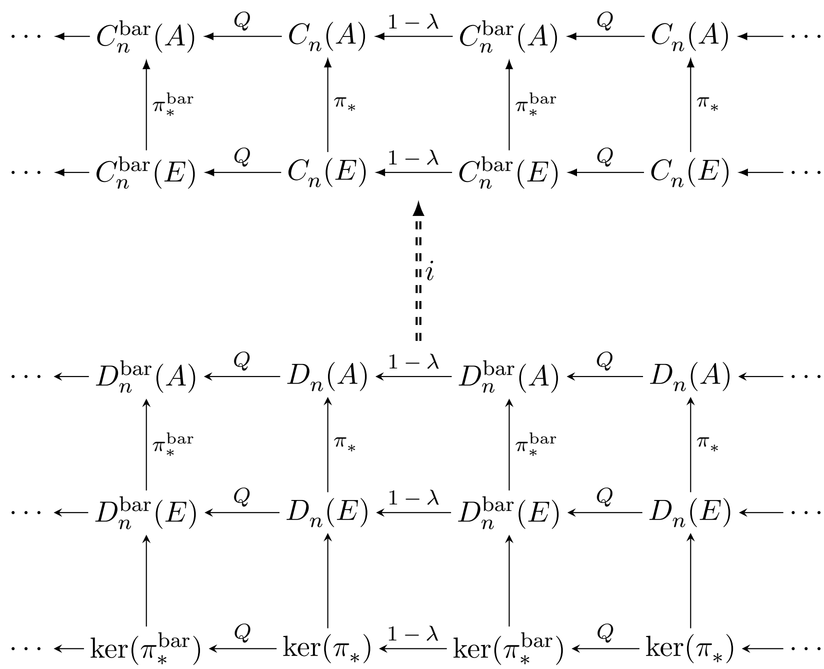
Lemma A.2.4. *The sequence (A.2.3) above is exact.*

Proof. As previously stated, exactness at $D_n^{\text{bar}}(E)$ follows as before. By construction, the collection $\{e_I \mid I \in \mathscr{I}\}$ are linearly independent, and note that $\lambda e_I = e_{\lambda I}$. Thus at $D_n(E)$, the collection $\{e_J, \lambda e_J, ..., \lambda^n e_J\}$ are linearly independent since there cannot be symmetry in the indexes. Thus for $\alpha \in D_n(E)$ such that $Q\alpha = 0$, we must have $t_J = 0$ for all J i.e. we can write α as

$$\alpha = \sum t_L e_L + (1-\lambda)\beta$$

with $Q(\sum t_L e_L) = 0$. Notice then that $h(\sum t_L e_L) \in D_n^{\text{bar}}(E)$ and we have $(1-\lambda)(h(\sum t_L e_L)) + \beta) = (1-h'Q)(\sum t_L e_L) + (1-\lambda)\beta = \sum t_J e_J + \sum t_L e_L + (1-\lambda)\beta = \alpha$. □

Proof of Theorem (A.2.1). Consider the diagrams



We will refer to the bottom set as the lower level or degenerate level, and likewise the upper level for the top set. All of the degenerate complexes can be included in the non-degenerate complexes, the collection of maps we have denoted by i (note that we have left out the kernel in the upper level, as it will not be needed). Our goal is to show that the (λ, Q)-sequence is exact for the kernels in the lower level.

Let $\alpha \in \ker(\pi_*^{\mathrm{bar}})$ be such that $(1-\lambda)\alpha = 0$. Then considering α as an element in $D_n^{\mathrm{bar}}(E)$ we have $h'(\alpha) = \frac{1}{n+1}\alpha \in D_n$ such that $Qh'(\alpha) = \alpha$, and we need to show $\pi_* Q h'(\alpha) = 0$. Notice that $ih'(\alpha) \in C_n(E)$ is such that $\pi_* i h'(\alpha) = 0$ since $C_n(A) = C_n^{\mathrm{bar}}(A)$ as $\Bbbk$-modules. Then by the commutativity of the diagram we have $i\pi_* h'(\alpha) = 0$ and by the injectivity of i, $\pi_* h'(\alpha) = 0$.

For exactness at $\ker(\pi_*)$ first note that for any $\alpha \in C_n(E)$ such that $\pi_*(\alpha) = 0$ we must also have $pi_* \lambda(\alpha) = 0$ by the injectivity of λ. Now let $\alpha \in \ker(\pi_*)$ be such that $Q\alpha = 0$. Then we can form $h(\alpha) \in D_n^{\mathrm{bar}}(E)$ such that $(1-\lambda)h(\alpha) = \alpha$. Considering $ih(\alpha) \in C_n^{\mathrm{bar}}(E)$ we claim that $\pi_* i^{\mathrm{bar}} h(\alpha) = 0$. Since h is just a polynomial in λ and considering $h\alpha \in C^n(E)$ we have $\pi_* h(\alpha) = 0$. Again since $C^n(A)$ and $C_n^{\mathrm{bar}}(A)$ are the same $\Bbbk$-modules, this implies that $\pi_*^{\mathrm{bar}} i h(\alpha) = 0$. Hence

$i\pi_*^{\text{bar}}h'(\alpha) = 0$ and by the injectivity of i, $\pi_*^{\text{bar}}h'(\alpha) = 0$.

To address the issue of this extra factor of $\Bbbk$ that we have placed in the 0^{th} degree of the degenerate bar complex, we note that it is an unfortunate (but manageable) consequence of trying to create such a construction. On the one hand, we like to have our normalized complexes have the same homology as their regular counterparts. In this sense we could either remove the $(1, 1)$ term from $D_1^{\text{bar}}(A)$ and leave $D_0(A) = 0$, or make the construction as we did. However, if we removed the $(1, 1)$ term from $D_1^{\text{bar}}((A)$ then the homology of the $(1 - \lambda, Q)$-degenerate sequence would have extra alternating factors of $\Bbbk$ in the 1^{st} degree that we would have to manage. On the other hand, one might think to perhaps allow the normalized bar complex to have slightly different homology than it's original counterpart and simply change the above construction to $D_0^{\text{bar}}(A) = 0$. This would work just as well, of course now the 0^{th} order homologies of the normalized bar and bar complex are off by this factor of $\Bbbk$, and then perhaps we may get rid of this in the "reduced" bar complex. The problem with this, is that it doesn't quite make sense functorially. Just as with the Hochschild homology functor, if we were to "properly" extend the bar homology functor to non-unital algebras we would define it as the cokernel of

$$\text{coker}(HB_n(\Bbbk) \to HB_n(I^+))$$

where I is some non-unital algebra, and I^+ its canonical extension. But the above definition is trivial since $HB_n(\Bbbk) = HB_n(I^+) = 0$ for all n. So it doesn't quite make sense to rename what we've been calling the "bar homology" to the "naive bar homology", if then the new "bar homology" would always be 0 for any algebra! Furthermore, we define the reduced Hochschild complex as such to take care of the philosophical conundrum of wanting a complex whose homology of the base ring is zero. But since the bar homology of $\Bbbk$ is already 0, there is not anything we really want to change.

So in light of the above, we keep the construction as it is, and with the functorial construction in mind we must then make the last definition.

Definition A.2.5. For a unital algebra A, the **reduced bar complex** $C_\bullet^{\text{bar}}(A)_{\text{red}}$ is the same as the normalized complex. $C^{\text{bar}}_{\bullet}(A)_{\text{red}} = \overline{C}^{\text{bar}}_{\bullet}(A)$. The reduced bar homology, which we may denote by $\overline{HB}_n(A)$, is 0 for all n when A is unital.

Corollary A.2.6. *When $\Bbbk$ contains $\mathbb{Q}$, for all n the sequences*

$$\cdots \leftarrow D_n(A)_{\text{red}} \xleftarrow{1-\lambda} D^{\text{bar}}_n(A) \xleftarrow{Q} D_n(A)_{\text{red}} \xleftarrow{1-\lambda} D^{\text{bar}}_n(A) \leftarrow \dots$$

and

$$\cdots \leftarrow C_n(A)_{\text{red}} \xleftarrow{1-\lambda} \overline{C}^{\text{bar}}_n(A) \xleftarrow{Q} C_n(A)_{\text{red}} \xleftarrow{1-\lambda} \overline{C}^{\text{bar}}_n(A) \leftarrow \dots$$

are exact.

A.3 Double Complexes

Here we provide the details for constructing the corresponding double complexes for reduced bar homology (see [Lod13, Sec. 1.4 pg. 30]). We also construct the full double complex for cyclic homology ([Lod13, Sec. 2.1 pgs. 53 - 59]).

Proposition A.3.1. *Let A^+ be an augmented algebra from a non-unital algebra A, then $\overline{C}^{\text{bar}}_{\bullet}(A^+) \cong C^{\text{bar}}_{\bullet}(A) \oplus C^{\text{bar}}_{\bullet-1}(A)$ with the differential given by* $\begin{bmatrix} b' & 1 \\ 0 & -b' \end{bmatrix}$.

Proof. It's clear that for any n-chain $\vec{a} \in \overline{C}^{\text{bar}}_n(A^+) = C^{\text{bar}}_n(A^+)/D^{\text{bar}}_n(A^+)$ we may begin by taking the equivalence class of the form $\vec{a} = (a_0 + t_0, a_1, ..., a_{n-1}, a_n + t_n) + D^{\text{bar}}_n(A^+)$ where $a_i \in A$ and

$t_i \in \Bbbk$. We then make adjustments modulo $D_n^{\text{bar}}(A^+)$:

$$\begin{aligned}(a_0+t_0, a_1, ..., a_{n-1}, a_n+t_n) =&(a_0, a_1, ..., a_{n-1}, a_n) + (t_0, a_1, ..., a_{n-1}, a_n)\\ &+ (a_0, a_1, ..., a_{n-1}, t_n) + (t_0, a_1, ..., a_{n-1}, t_n)\\ =&(a_0, a_1, ..., a_{n-1}, a_n) + (t_0, a_1, ..., a_{n-1}, a_n)\\ &+ (a_0, a_1, ..., a_{n-1}, t_n)\\ =&(a_0, a_1, ..., a_{n-1}, a_n) + (t_0, a_1, ..., a_{n-1}, a_n)\\ &+ (a_0, a_1, ..., a_{n-1}, t_n) - [(a_0, a_1, ..., a_{n-1}, t_n) + (-1)^n(t_n, a_0, ..., a_{n-1})]\\ =&(a_0, a_1, ..., a_{n-1}, a_n) + (t_0, a_1, ..., a_{n-1}, a_n)\\ &- (-1)^n(t_n, a_0, ..., a_{n-1})\end{aligned}$$

Thus for any $\alpha \in \overline{C}_n^{\text{bar}}(A^+)$ we can write $\alpha = \sum(a_{i0}, ..., a_{in}) + \sum(t_{j0}, a_{j1}, ..., a_{jn})$. Applying the canonical isomorphism we get $\overline{C}_n^{\text{bar}}(A^+) \cong C(A)_n^{\text{bar}} \oplus C_{n-1}^{\text{bar}}(A)$ where $(a_0, ..., a_n) + (t_0, x_1, ..., x_n) \mapsto (a_0, ..., a_n) \oplus (t_0x_1, ..., x_n)$. The differential on $\overline{C}_n^{\text{bar}}(A^+)$ is induced by b' in which we compute, taking an element in the form of the last equation above

$$\begin{aligned}&b'_n[(a_0, ..., a_n) + (t_0, x_1, ..., x_n)]\\ =&b'_n(a_0, ..., a_n) + b'_n(t_0, x_1, ..., x_n)]\\ =&b'_n(a_0, ..., a_n) + (t_0x_1, ..., x_n) - t_0 \otimes b'_{n-1}(x_1, ..., x_n)\\ \cong&b'_n(a_0, ..., a_n) + (t_0x_1, ..., x_n) \oplus -b'_{n-1}((t_0x_1, ..., x_n))).\end{aligned}$$

After dropping the subscripts on b', we see that this can be written

$$\begin{bmatrix} b' & 1 \\ 0 & -b' \end{bmatrix} \begin{bmatrix} (a_0, ..., a_n) \\ (t_0x_1, ..., x_n) \end{bmatrix}$$

□

Definition A.3.2. The bicomplex $CB(A)^{\{2\}}$ associated to $\overline{C}^{\text{bar}}(A^+)$ is given by

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle -b'} \\
A^{\otimes 3} & \xleftarrow{\text{id}} & A^{\otimes 3} \\
\downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle -b'} \\
A^{\otimes 2} & \xleftarrow{\text{id}} & A^{\otimes 2} \\
\downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle -b'} \\
A & \xleftarrow{\text{id}} & A
\end{array}
$$

We can now rederive the complexes mentioned in the previous Section (A.1). As previously mentioned we have the bicomplex $CC(A)$:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & \cdots
\end{array}
$$

And we can now form the degenerate bicomplex $DD(A)$:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
D_2(A) & \xleftarrow{1-\lambda} & D_2^{\text{bar}}(A) & \xleftarrow{Q} & D_2(A) & \xleftarrow{1-\lambda} & D_2^{\text{bar}}(A) & \xleftarrow{Q} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
D_1(A) & \xleftarrow{1-\lambda} & D_1^{\text{bar}}(A) & \xleftarrow{Q} & D_1(A) & \xleftarrow{1-\lambda} & D_1^{\text{bar}}(A) & \xleftarrow{Q} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
0 & \xleftarrow{1-\lambda} & \Bbbk & \xleftarrow{Q} & 0 & \xleftarrow{1-\lambda} & \Bbbk & \xleftarrow{Q} & \cdots
\end{array}
$$

There is a well known proposition ([Lod13, Prop. 1.0.12 pg. 6]):

Proposition A.3.3. *Let $C_{\bullet\bullet} \to C'_{\bullet\bullet}$ be a map of bicomplexes which is a quasi-isomorphism when restricted to each column. Then the induced map on the total complexes is a quasi-isomorphism. In particular, suppose that for all q the (horizontal) homology groups $H_p(C_{\bullet q})$ are 0 for $p > 0$ and put $K_n = H_0(C_{\bullet n})$. Then $H_n(\mathrm{Tot}C_{\bullet\bullet}) = H_n(K_\bullet, d^v)$. In other words, under the above hypothesis, the homology of the bicomplex is the homology of the cokernel of the first two columns.*

Since the columns of $DD(A)$ are exact, it follows from the above proposition that the total homology of $DD(A)$ is zero.

Definition A.3.4. The **normalized cyclic bicomplex** $\overline{CC}(A)$ is defined as the following cokernel

$$0 \to DD(A) \to CC(A) \to \overline{CC}(A) \to 0.$$

Corollary A.3.5. $H_n(\mathrm{Tot}(\overline{CC}(A)) = H_n(\mathrm{Tot}(CC(A)) := HC_n(A)$.

We may write $\overline{CC}(A)$ as

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\overline{C}_2(A) & \xleftarrow{1-\lambda} & \overline{C}_2^{\mathrm{bar}}(A) & \xleftarrow{Q} & \overline{C}_2(A) & \xleftarrow{1-\lambda} & \overline{C}_2^{\mathrm{bar}}(A) & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\overline{C}_1(A) & \xleftarrow{1-\lambda} & \overline{C}_1^{\mathrm{bar}}(A) & \xleftarrow{Q} & \overline{C}_1(A) & \xleftarrow{1-\lambda} & \overline{C}_1^{\mathrm{bar}}(A) & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\overline{C}_0(A) & \xleftarrow{1-\lambda} & \overline{C}_0^{\mathrm{bar}}(A) & \xleftarrow{Q} & \overline{C}_0(A) & \xleftarrow{1-\lambda} & \overline{C}_0^{\mathrm{bar}}(A) & \xleftarrow{Q} & \cdots
\end{array}$$

We slightly abuse the notation for $1 - \lambda$ and Q, which here are modulo our quotient (In degree zero they reduce to $1 - \lambda = 0$ and Q becomes the quotient map). In all degrees we may construct the normalized Connes boundary map $\overline{B} = (1 - \lambda)s\overline{Q}$, which reduces to $\overline{B} = sQ$.

Lemma A.3.6. *[Killing contractible complexes ([Lod13, Lemma 2.1.6 pg. 56])] Let*

$$\dots \to A_n \oplus A'_n \xrightarrow{d=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} A_{n-1} \oplus A'_{n-1} \to \dots$$

be a complex of $\Bbbk$*-modules such that* $(A'_\bullet, \delta)$ *is a complex and is contractible with contracting homotopy* $h : A'_n \to A'_{n+1}$*. Then the following inclusion of complexes is a quasi-isomorphism:*

$$(\mathrm{id}, -h\gamma) : (A_\bullet, \alpha - \beta h\gamma) \hookrightarrow (A_\bullet \oplus A'_\bullet, d)$$

Applying this lemma to the odd degree columns of $\overline{CC}(A)$ we arrive once again at the complex $\overline{\mathcal{B}}(A)$ ([Lod13, 2.1.9 pg. 58])

$$\begin{array}{ccccc}
\downarrow & & \downarrow & & \downarrow \\
A \otimes \overline{A}^{\otimes 2} & \xleftarrow{\overline{B}} & A \otimes \overline{A} & \xleftarrow{\overline{B}} & A \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
A \otimes \overline{A} & \xleftarrow{\overline{B}} & A & & \\
\downarrow{\scriptstyle b} & & & & \\
A & & & &
\end{array}$$

And we have again arrived at the following summary (with an extra addition [Lod13, 2.1.11 pg. 59]): For a unital $\Bbbk$-algebra A there are defined canonical morphisms of complexes

$$\begin{array}{ccccccc}
\mathrm{Tot}\,\overline{\mathcal{B}}(A) & \longleftarrow & \mathrm{Tot}\,\mathcal{B}(A) & \hookrightarrow & \mathrm{Tot}\,CC(A) & \longrightarrow & C^\lambda(A) \\
 & & & & \downarrow & & \\
 & & & & \mathrm{Tot}\,\overline{CC}(A) & &
\end{array} \tag{A.3.6}$$

The first two horizontal maps and vertical map are quasi-isomorphisms. We will define $C^\lambda(A)$ in the next section where we will see that the last horizontal map is also a quasi-isomorphism when the ground ring $\Bbbk$ contains $\mathbb{Q}$.

Definition A.3.7. For a unital algebra A, the **reduced cyclic bicomplex**, $CC(A)_{\text{red}}$ is given as the following cokernel

$$0 \to \Bbbk[0,e] \to \overline{CC}(A) \to CC(A)_{\text{red}} \to 0.$$

Where $\Bbbk[0,e]$ is the bicomplex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
0 & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & 0 & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
0 & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & 0 & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\Bbbk & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & \Bbbk & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & \cdots
\end{array}$$

The full complex is then

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\overline{C}_2(A) & \xleftarrow{1-\lambda} & \overline{C}_2^{\text{bar}}(A) & \xleftarrow{Q} & \overline{C}_2(A) & \xleftarrow{1-\lambda} & \overline{C}_2^{\text{bar}}(A) & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\overline{C}_1(A) & \xleftarrow{1-\lambda} & \overline{C}_1^{\text{bar}}(A) & \xleftarrow{Q} & \overline{C}_1(A) & \xleftarrow{1-\lambda} & \overline{C}_1^{\text{bar}}(A) & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
\overline{A} & \xleftarrow{1-\lambda} & \overline{A} & \xleftarrow{Q} & \overline{A} & \xleftarrow{1-\lambda} & \overline{A} & \xleftarrow{Q} & \cdots
\end{array}$$

That is, the reduced cyclic bicomplex is simply the bicomplex $CC(A)$ but with the columns replaced with their corresponding reduced complexes. The **reduced cyclic homology of** A is defined as $\overline{HC}_\bullet(A) := H_\bullet(CC(A)_{\text{red}})$.

From this we may apply the killing contractible complexes lemma to once again form the complex $\mathcal{B}(A)_{\text{red}}$.

Definition A.3.8. For a unital algebra A the bicomplex $\mathcal{B}(A)_{\text{red}}$ is given by

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ A\otimes \overline{A}^{\otimes 2} & \xleftarrow{\overline{B}} & A\otimes\overline{A} & \xleftarrow{\overline{B}} & \overline{A} \\ \downarrow b & & \downarrow b & & \\ A\otimes\overline{A} & \xleftarrow{\overline{B}} & \overline{A} & & \\ \downarrow b & & & & \\ \overline{A} & & & & \end{array}$$

The last thing we need to reconcile is the definition of cyclic homology for non-unital algebras. One the one hand it is standard to define the cyclic homology as the homology of the total complex $CC(A)$. On the other hand, our functorial procedure claims we should have $HC(A) := \text{coker}(HC(\Bbbk) \to HC(A^+))$, which should also match the homology of the reduced complex. This last statement follows directly from the procedures above which we can summarize as follows:

Proposition A.3.9. *The vertical maps in the following diagram are quasi-isomorphisms and* $\overline{HC}_n(A^+) = \text{coker}(HC(\Bbbk) \to HC(A^+))$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\mathcal{B}}(\Bbbk) & \longrightarrow & \overline{\mathcal{B}}(A^+) & \longrightarrow & \mathcal{B}(A^+)_{\text{red}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Bbbk[0,e] & \longrightarrow & \overline{CC}(A^+) & \longrightarrow & CC(A^+)_{\text{red}} & \longrightarrow & 0 \end{array} \tag{A.3.9}$$

To reconcile the standard defnition we make a few more constructions.

Proposition A.3.10. *For an augmented algebra* $\Bbbk \to A^+ \to A$ *the exact sequence in (A.2.6) becomes*

$$\dots \xrightarrow{Q\oplus 0} CB_n(A)^{\{2\}} \xrightarrow{1-\lambda\oplus 1} CC_n(A)^{\{2\}} \xrightarrow{Q\oplus 0} CB_n(A)^{\{2\}} \xrightarrow{1-\lambda\oplus 1} CC_n(A)^{\{2\}} \xrightarrow{Q} \dots$$

on the associated bicomplexes.

Proof. The first assertion follows immediately from Definition A.3.7. For the second, we showed in Proposition A.3.1 that for $\alpha \in C^{\text{bar}}(A)_{\text{red}}$, we can write $\alpha = \sum(a_{i0}, ..., a_{in}) + \sum(t_{j0}, a_{j1}, ..., a_{jn})$. Let's ignore the sums because of linearity and write $\alpha = (a_0, ..., a_n) + (t_0, a_1, ..., a_n)$. We compute

$$\begin{aligned}(1-\lambda)[(a_0, ..., a_n) + (t_0, a_1, ..., a_n)] &= (1-\lambda)(a_0, ..., a_n) + (1-\lambda)(t_0, a_1, ..., a_n) \\ &= (1-\lambda)(a_0, ..., a_n) + (t_0, a_1, ..., a_n) + (-1)^n(a_n, t_0, a_1, ..., a_{n-1}) \\ &= (1-\lambda)(a_0, ..., a_n) + (t_0, a_1, ..., a_n).\end{aligned}$$

The last equality holds since $(a_n, t_0, a_1, ..., a_{n-1}) = 0$ in $C_n(A)_{\text{red}}$. Hence, under the isomorphism of Proposition A.3.1 $C_n^{\text{bar}}(A)_{\text{red}} \cong C_n^{\text{bar}}(\overline{A}) \oplus C_{n-1}^{\text{bar}}(\overline{A})$ and the analogous isomorphism $C_n(A)_{\text{red}} \cong C_n(\overline{A}) \oplus C_{n-1}^{\text{bar}}(\overline{A})$, we see that the map $1-\lambda$ becomes

$$C_n^{\text{bar}}(\overline{A}) \oplus C_{n-1}^{\text{bar}}(\overline{A}) \xrightarrow{\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix}} C_n(\overline{A}) \oplus C_{n-1}^{\text{bar}}(\overline{A})$$

which corresponds with the above notation as a map between bicomplexes. For Q we compute similarly, letting $\alpha = (a_0, ..., a_n) + (t_0, a_1, ..., a_n) \in C_n(A)_{\text{red}}$, we compute

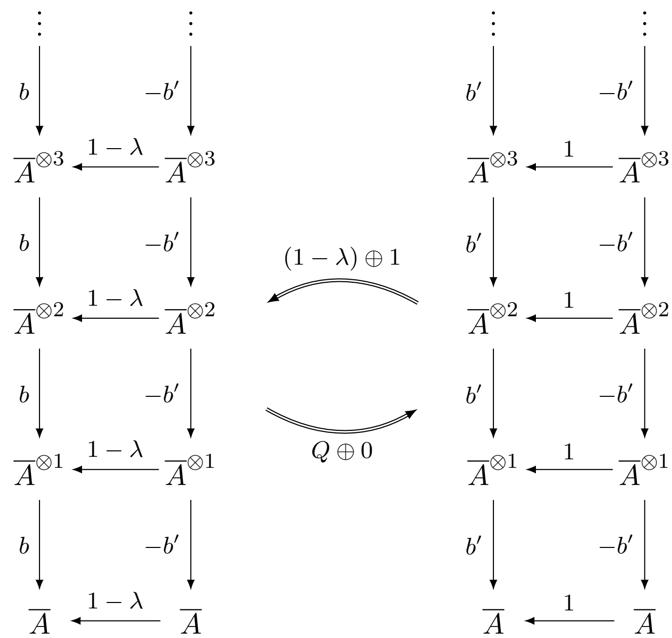
$$\begin{aligned}Q[(a_0, ..., a_n) + (t_0, a_1, ..., a_n)] &= Q(a_0, ..., a_n) + Q(t_0, a_1, ..., a_n) \\ &= Q(a_0, ..., a_n)\end{aligned}$$

where we recall that $Q(t_0, a_1, ..., a_n) = Q(-1)^n(a_n, t_0, a_1, ..., a_{n-1}) \in D_n^{\text{bar}}(A)$. Hence $Q(t_0, a_1, ..., a_n) = 0$, and under the canonical isomorphisms Q becomes

$$C_n(\overline{A}) \oplus C_{n-1}^{\text{bar}}(\overline{A}) \xrightarrow{\begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}} C_n^{\text{bar}}(\overline{A}) \oplus C_{n-1}^{\text{bar}}(\overline{A}).$$

□

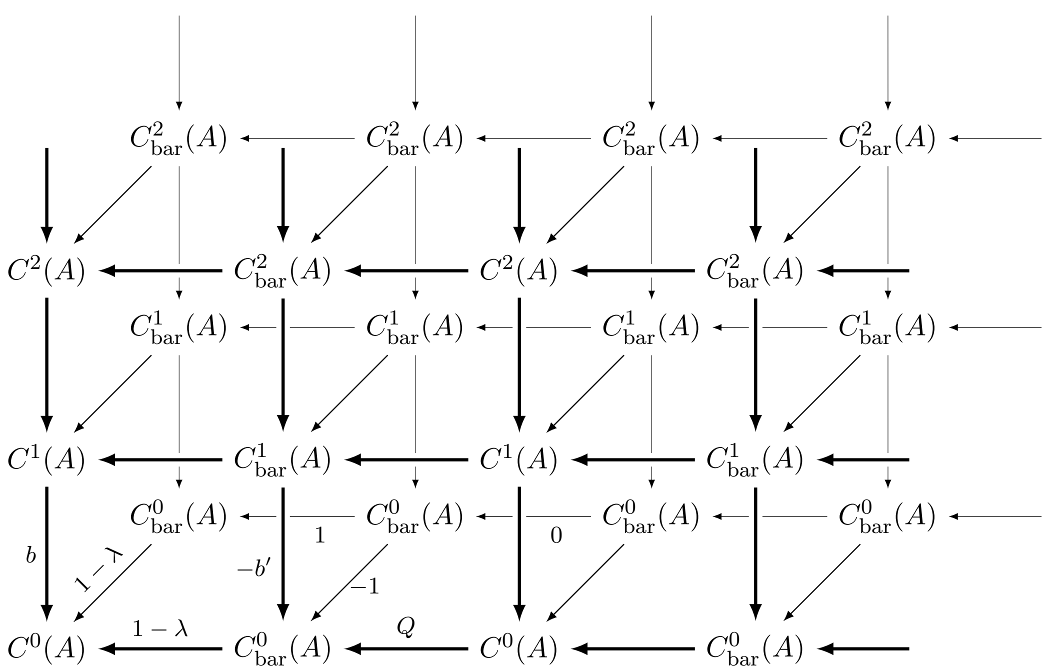
Let's summarize this in the following diagram



(A.3.10)

With the above complete we can make the following construction: for a non-unital algebra A start with the complex $CC(A^+)_{\text{red}}$ and expand each column via Definition A.3.2. Along with the above (λ, Q)-sequence this gives us the following complex.

Definition A.3.11. The tricomplex $CCB(A)$ is given by



The back face of $CCB(A)$ is called the **cyclic bar complex of** A and is denoted $CB(A)$.

Proposition A.3.12. $HC(A) = HC(A^+)_{\text{red}}$.

Proof. Note that $CCB(A)$ and $CC(A^+)_{\text{red}}$ are isomorphic as complexes. Moreover, the rows of $CB(A)$ are exact, hence the homology of the total complex is zero. The result then follows from the homological long exact sequence

$$0 \to CB(A) \to CCB(A) \to CC(A) \to 0$$

□

A.4 Connes' Complexes

The last thing we need to tie in is the Connes' complex, for which we will create an equivalent notion of degenerate and reduced complexes.

Definition A.4.1. Recall that the **Connes Complex** $C^\lambda(A)$, is defined as the cokernel of the first two rows of $CC(A)$:

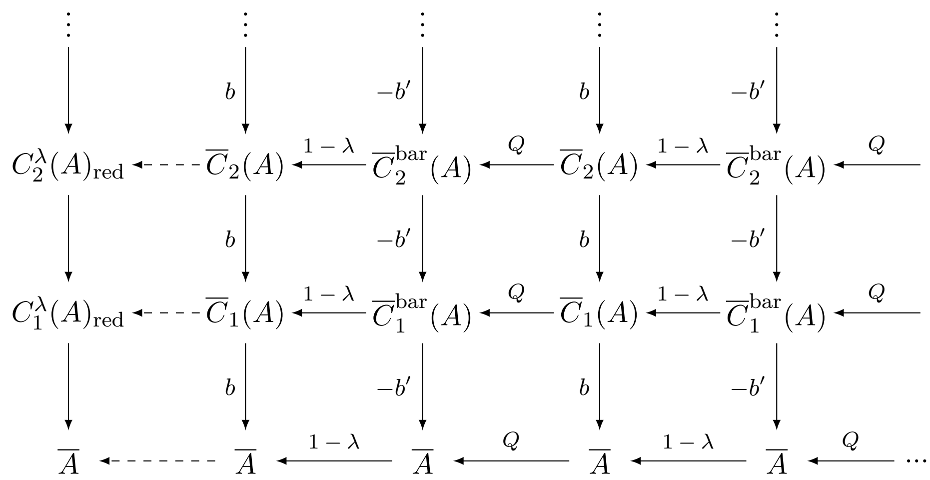
$$
\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_2^\lambda(A) & \leftarrow\!\text{-}\,\text{-} & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{Q} & \cdots \\
\downarrow & & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_1^\lambda(A) & \leftarrow\!\text{-}\,\text{-} & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{Q} & \cdots \\
\downarrow & & \downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A & \leftarrow\!\text{-}\,\text{-} & A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & A & \xleftarrow{1-\lambda} & A & \xleftarrow{Q} & \cdots
\end{array}
$$

The homology of this complex is denoted $HC^\lambda(A) := H_n(C^\lambda(A))$.

As noted previously, when $\Bbbk$ contains $\mathbb{Q}$ the rows of $CC(A)$ are exact, so by Proposition A.3.3 $HC^\lambda(A) = HC(A)$.

Definition A.4.2.

In light of our newer constructions, for a unital algebra A, we can define the **reduced Connes Complex** $C^{\lambda}(A)_{\text{red}}$, as the cokernel of the first two rows of the reduced cyclic bicomplex:



Loday defines "$\overline{C}^{\lambda}(A)$ as the quotient of $C^{\lambda}(A)$ by the sub-$\Bbbk$-module generated by $(a_0, ..., a_n)$ such that $a_i = 1$ for at least one index i, $0 \leq i \leq n$"([Lod13] pg. 65), which because the rows of $DD(A)_{\text{red}}$ are exact, matches our definition above. We note here that there is a slight misuse of notation as $\overline{C}^{\lambda}(A)$ should be used for the "normalized Connes complex" and not the "reduced Connes complex". It is, of course, a confusing convention since the overline in homology is used for the reduced homology, and the two complexes are the same except for the 0^{th} position (this can be seen in the higher vertical degrees of the diagram for $CC(A)_{\text{red}}$). In any case, we will stick with our notation to keep with the consistency set beforehand. The homology of the reduced complex is then notated by $\overline{HC}_n^{\lambda}(A) := H_n(C^{\lambda}(A)_{\text{red}})$ and as an immediate consequence (see Prop. A.3.3) by our construction we have arrived at another proposition by Loday ([Lod13, Prop. 2.2.14 pg. 65]).

Proposition A.4.3. *Assume that $\Bbbk$ is a direct summand of A as a $\Bbbk$-module and that $\Bbbk$ contains $\mathbb{Q}$. Then there is a canonical isomorphism $\overline{HC}_{\bullet}(A) \cong \overline{HC}^{\lambda}(A)$.*

We finish this section by noting that the "normalized Connes complex" should be defined by using the complex $\overline{CC}(A)$, however we cannot simply take the cokernel of the first two columns like the other definitions because the bottom row is not exact. Instead we can try to create a "degenerate Connes complex" from from the complex $DD(A)$. Letting $D_n^\lambda(A) := \operatorname{coker}(D_n^{\text{bar}}(A) \xrightarrow{1-\lambda} D_n(A))$, we define a complex $D\Lambda_{\bullet\bullet}(A)$ as

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
D_2^\lambda(A) & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & 0 & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
D_1^\lambda(A) & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & 0 & \xleftarrow{1-\lambda} & 0 & \xleftarrow{Q} & \cdots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
0 & \xleftarrow{1-\lambda} & \Bbbk & \xleftarrow{Q} & 0 & \xleftarrow{1-\lambda} & \Bbbk & \xleftarrow{Q} & \cdots
\end{array}
$$

We have a map $f : DD(A) \to D\Lambda(A)$ where f is the quotient map on the first column, the identity on the bottom row, and zero elsewhere. When $\Bbbk$ contains $\mathbb{Q}$, it's easy to see that f is an isomorphism when restricted to rows, so by Proposition $A.3.3$ f is a quasi-isomorphism on the total complexes. But, we also know that $H_n(\operatorname{Tot}(DD(A))) = 0$. Computing $\operatorname{Tot}(D\Lambda(A))$ gives us the following

Proposition A.4.4. *When $\Bbbk$ contains $\mathbb{Q}$, the sequence*

$$
... \xrightarrow{b} D_{2n+1}^\lambda(A) \oplus \Bbbk \xrightarrow{b} D_{2n}^\lambda(A) \xrightarrow{b} D_{2n-1}^\lambda \oplus \Bbbk \xrightarrow{b} D_{2n-2}^\lambda(A) \to ... \tag{A.4.4}
$$

is exact.

Note that the sequence

$$
... \xrightarrow{b} D_{2n+1}^\lambda(A) \xrightarrow{b} D_{2n}^\lambda(A) \xrightarrow{b} D_{2n-1}^\lambda \xrightarrow{b} D_{2n-2}^\lambda(A) \to ...
$$

is simply the cokernel of the first two columns of the degenerate cyclic bicomplex $DD(A)$, and may be used to obtain the reduced Connes complex. To explain the $\Bbbk$ factors that appear we note that this is the submodule $\Bbbk(1, ..., 1)$ for an even number of 1's (which is in an odd degree since we count indices from 0), and we'll look at what's happening in degree 1 to provide an example. As mentioned previously, $(1,1) \in D_1^{\text{bar}}(A)$ and $(1-\lambda)(\frac{1}{2})(1,1) = (1,1)$ hence $(1,1) \notin D_1^{\lambda}(A)$. However in degree 2 we have that $(1-\lambda)(1,1,1) = 0$ so $(1,1,1) \in D_2^{\lambda}(A)$, but $b(1,1,1) = (1,1)$. So we have to attach $(1,1)$ to $D_1^{\lambda}(A)$ to get exactness. The other odd degrees follow similarly.

Proposition A.4.4 will play a small but important roll in this paper. From this one could construct a "normalized Connes complex" which is dual to the construction in Defnition 3.4.1.