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NON-DEGENERATE SURFACE-WAVE MODE
COUPLING OF A SYSTEM OF DIELECTRIC WAVEGUIDES†

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ABSTRACT

A number of different methods in the literature for calculating the coupling between surface wave modes on parallel optical waveguides are examined and found to give identical results only for degenerate modes. A new variational derivation of the coupling coefficients is given, which is extended in a straightforward manner to include anisotropic guides and/or an arbitrary number of guides or modes. The character of the differences between the various methods in the non-degenerate case is discussed, and the approximations required are investigated numerically for the case of two parallel slab waveguides.
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<table>
<thead>
<tr>
<th>SECTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.   INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II.  COUPLING BETWEEN TWO ISOTROPIC GUIDES</td>
<td>2</td>
</tr>
<tr>
<td>III. COUPLED-MODE REPRESENTATION</td>
<td>8</td>
</tr>
<tr>
<td>IV.  RELATIONS TO OTHER METHODS</td>
<td>14</td>
</tr>
<tr>
<td>V.   NUMERICAL RESULTS FOR SLAB WAVEGUIDES</td>
<td>19</td>
</tr>
<tr>
<td>VI.  ANISOTROPIC WAVEGUIDES</td>
<td>29</td>
</tr>
<tr>
<td>VII. MULTIMODE COUPLING</td>
<td>31</td>
</tr>
<tr>
<td>VIII. CONCLUDING REMARKS</td>
<td>34</td>
</tr>
<tr>
<td>APPENDIX A</td>
<td>36</td>
</tr>
<tr>
<td>APPENDIX B</td>
<td>39</td>
</tr>
<tr>
<td>APPENDIX C</td>
<td>43</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>45</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

FIGURE                                      PAGE
1. Geometry of two parallel guides            3
2. Power transfer $|S_{12}|^2$ as a function of \( \Lambda \) ............... 12
3. \(|K| = |\Delta \Gamma|\) as a function of normalized guide separation \( \lambda k \) and relative non-degeneracy .............. 13
4. Geometry of two parallel slab guides ....... 20
5. Comparison of \( \delta^2_J \) and \( \delta^2_M \) for TE\(_0\) mode as a function of relative slab widths. \( \varepsilon_3 = 1.00, \varepsilon_1 = \varepsilon_2 = 1.04, k_o a_1 = 8.68 \) ................. 23
6. Comparison of \( \delta^2_J \) and \( \delta^2_A \) for TE\(_0\) mode as a function of relative slab widths. \( \varepsilon_3 = 1.00, \varepsilon_1 = \varepsilon_2 = 1.04, k_o a_1 = 8.68 \) ................. 24
7. Comparison of \( \delta^2_J \) and \( \delta^2_M \) for TE\(_0\) mode as a function of relative slab permittivities. \( \varepsilon_3 = 1.00, \varepsilon_1 = 1.04, k_o a_1 = k_o a_2 = 8.68 \) ................. 25
8. Comparison of \( \delta^2_J \) and \( \delta^2_A \) for TE\(_0\) mode as a function of relative slab permittivities. \( \varepsilon_3 = 1.00, \varepsilon_1 = 1.04, k_o a_1 = k_o a_2 = 8.68 \) ................. 26
9. Comparison of calculated exact value of \( \Delta \Gamma = \frac{1}{2}(\Gamma_+ - \Gamma_-)\) for two degenerate TE\(_0\) slab modes to Jones'\([8,9]\) value and the second-order value from equation (10). \( \varepsilon_3 = 1.00, \varepsilon_1 = \varepsilon_2 = 1.04, k_o a_1 = 8.68 \) ................. 27
10. Comparison of calculated exact value of \( \Delta \Gamma = \frac{1}{2}(\Gamma_+ - \Gamma_-)\) for two non-degenerate TE\(_0\) slab modes to Jones'\([8,9]\) value and the second-order value from equation (10). \( \varepsilon_3 = 1.00, \varepsilon_1 = \varepsilon_2 = 1.04, k_o a_1 = 8.68 \) ................. 28
NON-DEGENERATE SURFACE-WAVE MODE
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I. Introduction

With the recent surge of interest in optical waveguides as a viable economic communications medium, there has been a great deal of interest in the study of possible devices for the processing of signals on such guides, as well as in various means of isolating these signals. One topic which has applications in both these areas is the coupling of modes between parallel surface-wave lines. There exist in the literature many theoretical treatments of this subject: general theories for arbitrary lines [1-6] and specialized theories which depend to a greater or lesser extent on particular symmetries of the problem [7-23]. Further, there are treatments of open structures other than optical waveguides [24-26] which also relate to this problem, as well as general coupled mode theories [27-32] which, however, do not indicate a method for calculating the coupling coefficients in specific cases.

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It is the purpose of this paper to investigate the relationships between the various theories for calculating the coupling between two or more arbitrary parallel open dielectric waveguides which we allow to be lossy, inhomogeneous, and anisotropic. It is shown that, in a situation where significant power may be transferred between guide modes (that is to say for nearly degenerate modes), all of the general theories (and those others capable of generalization) give the same results. In this work the coupling results are derived by a variational method and are therefore in some sense the most accurate; the various approaches are compared in the areas where they differ: for nondegenerate modes and for very closely spaced waveguides.

II. Coupling between two isotropic guides

For simplicity we consider first the case of two parallel isotropic (but possibly lossy and inhomogeneous) waveguides as shown in Fig. 1. The relative permittivities \( \varepsilon_1 = n_1^2 \) and \( \varepsilon_2 = n_2^2 \) are "difference" permittivities which against the background of \( \varepsilon_3 = n_3^2 \) form the two dielectric surface waveguides. We consider a single surface-wave mode on each of the guides 1 and 2, with field distributions in the transverse (x-y) plane given by \( \vec{E}_{1,2}^+ \) and \( \vec{n}_{1,2}^+ \), and propagation constants \( \beta_1 \) and \( \beta_2 \). The t and z dependence is assumed to be \( \exp[j(\omega t - \beta z)] \) and is dropped when not required.
Fig. 1 Geometry of two parallel guides
If the guides are sufficiently far apart, the fact that the fields of each guide mode are evanescent outside the guide and thus small compared to the fields of the other guide in its vicinity leads us to search for system modes with propagation constant $\Gamma$ whose fields we represent approximately as $E^+ = m_1 E^+_1 + m_2 E^+_2$, $H^+ = m_1 H^+_1 + m_2 H^+_2$, with the relative components $m_1$ and $m_2$ of modes 1 and 2 as yet arbitrary.

Gabriel and Brodwin [33] have derived a variational formula for the propagation constant of a general uniform lossy inhomogeneous anisotropic waveguide. When specialized to the isotropic case, this becomes (see also [34, p. 347] and [35]):

$$\beta = \frac{\omega \int [\epsilon E^- \cdot E^- - \mu H^- \cdot H^-] dS + j \int [E^- \cdot \nabla_t \times H^- + H^- \cdot \nabla_t \times E^+] dS}{\int [E^- \times H^+_t - E^+_t \times H^-_t] \cdot \hat{a}_z dS} \tag{1}$$

Here the integrals are over the (infinite) cross section of the guide, $\nabla_t$ is the transverse dier operator, and $E^-$ and $H^-$ are the so-called "transpose" fields corresponding to a mode traveling with propagation constant $-\beta$, and are related in this case to $E^+$ and $H^+$ by

$$E^-_t = E^+_t; \quad E^-_z = -E^+_z; \quad H^-_t = -H^+_t; \quad H^-_z = H^+_z \tag{2}$$

These are sometimes characterized as "adjoint" fields, but this does not conform to the usage of the mathematical theory of linear operators, since (1) is a result obtained using a bilinear form which does not possess all the properties of a Hilbert space inner product. The transpose fields are used here because of their advantage of having a clear physical interpretation. See [36, 37] for details.
In our present case, since we have no conducting boundaries, we require only that the trial fields be suitably square integrable over the infinite cross section, and that \( \vec{E}_{\text{tan}} \) and \( \vec{H}_{\text{tan}} \) be continuous at any interfaces.

We note that \( \beta_1, \vec{E}_1 \) and \( \vec{H}_1 \) or \( \beta_2, \vec{E}_2 \) and \( \vec{H}_2 \) will individually satisfy (1) if \( \varepsilon_2 \) or \( \varepsilon_1 \) is set equal to zero, respectively. To find the system mode propagation constants \( \Gamma \), we insert our trial fields into (1) and find the stationary values. We obtain

\[
\Gamma = \frac{\frac{m_1^2 N_1 + 2m_1 m_2 K + m_2^2 N_2}{m_1 P_1 + 2m_1 m_2 L + m_2^2 P_2}}
\]  \hspace{1cm} (3)

where

\[
P_{1,2} = \int (\vec{E}_1 \times \vec{H}_1^+) - (\vec{E}_1^+ \times \vec{H}_1) \cdot \vec{a}_z \, dS \hspace{1cm} (4a)
\]

\[
= 2 \int (\vec{E}_1^+ \times \vec{H}_1^+) \cdot \vec{a}_z \, dS \hspace{1cm} (4b)
\]

and

\[
N_{1,2} = \beta_1,2 P_{1,2} + d_{1,2} ; \quad d_{1,2} = \omega \varepsilon_0 \int \varepsilon_2 \vec{E}_1 \cdot \vec{E}_1^+ \, dS
\]

\[
K = \frac{1}{2}(\omega \varepsilon_0 \int (\varepsilon_1 + \varepsilon_2) \vec{E}_1 \cdot \vec{E}_2 \, dS - \beta_1 \int (\vec{H}_2 \cdot \vec{a}_z \times \vec{E}_1^+ + \vec{E}_2 \cdot \vec{a}_z \times \vec{H}_1^+) \, dS
\]

\[
- \beta_2 \int (\vec{H}_1 \cdot \vec{a}_z \times \vec{E}_2 \cdot \vec{a}_z \times \vec{H}_2^+) \, dS)
\]

\[
L = \int (\vec{E}_1 \times \vec{H}_2^+ - \vec{E}_2 \times \vec{H}_1^-) \cdot \vec{a}_z \, dS
\]

All surface integrations are over the infinite cross section in the transverse plane; however since the integrands for \( d_{1,2} \) are zero outside \( S_{2,1} \), these are only finite surface integrals.
Using the vector identity [3]

\[
\int_A \nabla \cdot \mathbf{F} \, ds = \frac{3}{\partial z} \int_A \mathbf{F} \cdot \mathbf{a}_z \, ds + \oint_C \mathbf{F} \cdot \mathbf{n} \, dl
\]  

(5)

where \( A \) is an area in the transverse plane, \( C \) its boundary, and \( \mathbf{a}_z \) and \( \mathbf{a}_n \) unit vectors in the z-direction and the outward normal direction to \( C \) respectively, we let

\[
\mathbf{F} = [\mathbf{E}^+ - \mathbf{H}^+ \times \mathbf{H}^+] e^{j(\beta_1 - \beta_2)z}
\]  

(6)

and after some manipulation obtain

\[
L = \frac{c_1 - c_2}{\beta_1 - \beta_2} \quad \text{; } \quad K = \frac{\beta_1 c_1 - \beta_2 c_2}{\beta_1 - \beta_2}
\]  

(7)

where

\[
c_{1,2} = \omega \varepsilon_0 \int \mathbf{E}_1^+ \cdot \mathbf{E}_2^- \
\]  

(8)

Again, the infinite surface integrals for \( c_{1,2} \) reduce to finite ones since the integrands vanish outside \( S_{1,2} \).

To determine the stationary values of \( \Gamma \), and the corresponding values of the only other unknown, \( q = m_2/m_1 \), we apply the conditions \( \partial \Gamma / \partial m_1 = 0 \) and \( \partial \Gamma / \partial m_2 = 0 \) [38]. The result is

\[
q = \frac{(\beta_1' - \Gamma)P_1}{LT - K} = \frac{LT - K}{(\beta_2' - \Gamma)P_2}
\]  

(9a)

where

\[
\beta_{1,2}' = \beta_{1,2} + \frac{d_{1,2}}{P_{1,2}} \approx \beta_{1,2}
\]  

(9b)
The term \( d_{1,2}/P_{1,2} \) in (9b) is exponentially small to the second-order with respect to \( \beta_{1,2} \) for well-separated guides, because the exponentially-decayed fields \( \bar{E}_{1,2} \) are squared and only occur in the cross-section of the other guide.

Solving (9a) for the propagation constant \( \Gamma \), we obtain

\[
\Gamma = \frac{\beta_1' + \beta_2' - \frac{2LK}{P_1P_2} \pm \sqrt{(\beta_1' - \beta_2')^2 + \frac{4}{P_1P_2} (K - \beta_1'L)(K - \beta_2'L)}}{2(1 - \frac{L^2}{P_1P_2})}
\]

(10a)

for the two possible system modes. Now since our separation assumptions further imply

\( L \ll P_{1,2} \quad \text{and} \quad K \ll \beta_{1,2} P_{1,2} \)

(exponentially to the first-order), we may simplify the above expression to yield

\[
\Gamma = \beta_{av} \pm \Delta \Gamma ;
\]

(10b)

where \( \beta_{av} = \frac{1}{2}(\beta_1 + \beta_2) \) and \( \Delta \Gamma = \sqrt{\Delta^2 + \delta^2} \),

(11)

and

\[
\Delta = \frac{1}{2}(\beta_1 - \beta_2) \quad \text{and} \quad \delta^2 = c_1c_2/P_1P_2
\]

(12)

The ratio \( q = m_2/m_1 \) is also obtainable from (9) and (10a) or (10b) as

\[
q^2 = \frac{\beta_1' - \Gamma_\pm P_1}{(\beta_2' - \Gamma_\pm P_2} \approx \frac{P_1}{P_2} \left[ \frac{-\Delta \pm \sqrt{\Delta^2 + \delta^2}}{\delta} \right]^2
\]
In writing an expression for \( q_\pm \), we must take care to choose the proper sign of the square root. Examining the individual members of (9a) reveals that we have

\[
q_\pm \approx \sqrt{\frac{P_1}{P_2}} \left[ -\Delta \pm \sqrt{\frac{\Delta^2 + \delta^2}{\delta}} \right] \tag{13}
\]

only if we define \( \delta \) as \(^2\)

\[
\delta = \frac{c_2}{\sqrt{P_1 P_2}} \sqrt[4]{\frac{c_1}{c_2}} = \frac{c_1}{\sqrt{P_1 P_2}} \sqrt[4]{\frac{c_2}{c_1}} \tag{14}
\]

since a change in sign of the fields for one of the modes must result in a change in sign for \( q_\pm \). The system mode fields are thus

\[
\tilde{E}_{s\pm} = m_1 (\tilde{E}_{1\pm} + q_\pm \tilde{E}_{2\pm}) \tag{15}
\]

and similarly for \( \tilde{H}_{s\pm} \).

III. Coupled-mode representation

In order to express our results in terms of general coupled-mode theory [27-32], we now assume that each mode is normalized to unit power carried in the + z-direction:

\[
\frac{1}{2} \text{Re} \int \{ \tilde{E}_{t1}^+ \times \tilde{H}_{t1}^{+*} \} \cdot \tilde{a}_z \, ds = 1
\]

so that the most general field expressible in terms of these two modes is

\(^2\)In fact, since \( L \) as given by (7) must remain finite for \( \beta_1 = \beta_2, c_1 = c_2 \); even in the non-degenerate case we expect \( c_1 \approx c_2 \) and so could simplify (14). See also section III.
\[ \bar{E}^+ = A_1(z)\bar{E}_1^+ + A_2(z)\bar{E}_2^+ \]  

(16a)

and similarly for \( \bar{H}^+ \). Since this field must be uniquely expressible also in terms of the system modes:

\[ E^+ = A_+\bar{E}_s^+ e^{-j\Gamma_+ z} + A_-\bar{E}_s^- e^{-i\Gamma_- z} \]  

(16b)

we have by comparing these two expressions, and the use of (15), the following relationships:

\[ A_1(z) = A_+ e^{-j\Gamma_+ z} + A_- e^{-j\Gamma_- z} \]  

(17)

\[ A_2(z) = A_+ q_+ e^{-j\Gamma_+ z} + A_- q_- e^{-j\Gamma_- z} \]

Solving this pair of equations for \( A_+ e^{-j\Gamma_+ z} \) and differentiating, we can show that

\[ \frac{d}{dz} [q_+ A_1(z) - A_2(z)] = -j\Gamma_+ [q_+ A_1(z) - A_2(z)] \]

which, upon solving for \( \frac{d}{dz} A_1(z) \) and \( \frac{d}{dz} A_2(z) \), becomes

\[ j \frac{d}{dz} \begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix} = \begin{bmatrix} \beta_1 \delta \sqrt{P_1} \delta \sqrt{P_2} \\ \delta \sqrt{P_1} \beta_2 \end{bmatrix} \begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix} \]  

(18a)

where use has been made of (10b) and (13). But

\[ \delta \sqrt{\frac{P_2}{P_1}} = \frac{c_1}{P_1} \sqrt{\frac{c_2}{c_1}} ; \quad \delta \sqrt{\frac{P_1}{P_2}} = \frac{c_2}{P_2} \sqrt{\frac{c_1}{c_2}} \]

so that if \( \sqrt{c_2/c_1} = 1 \), (18a) reduces to the standard coupled mode equations\[ [1-3, 7-9, 27-32] \] for this pair of modes.
If (18a) is regarded as a description of a four-port network of differential length, mirror-image reciprocity will require the two off-diagonal elements to be equal, or \( P_1 \approx P_2 \) as long as the two-mode assumption is valid [46]. In fact, in the lossless case we will have \( P_1 = P_2 \) as a result of the power normalization at the beginning of the section (see Appendix C). Now additionally, from (7):

\[
c_1 = c_2 + (\beta_1 - \beta_2) L
\]

but as will be seen below, significant power transfer can occur only for

\[
|\beta_1 - \beta_2| \lesssim \delta = \left| \frac{c_1 c_2}{P_1 P_2} \right|
\]

and since \( \sqrt{\frac{L}{P_1 P_2}} \ll 1 \),

\[
|\frac{(\beta_1 - \beta_2)L}{c_2}| \lesssim \left| \frac{c_1}{c_2} \frac{L}{\sqrt{P_1 P_2}} \right| \ll \sqrt{\frac{c_1}{c_2}}
\]

so that in this situation, self-consistency dictates that \( c_1 / c_2 \approx 1 \). We can therefore also write (18a) in the form:

\[
j \frac{d}{dz} \begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix} = \begin{bmatrix} \beta_1 & c_{12} \\ c_{21} & \beta_2 \end{bmatrix} \begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix} \tag{18b}
\]

where \( c_{12} = c_1 / P_1 \) and \( c_{21} = c_2 / P_2 \) are the usual forms of the coupling coefficients [1-3]. Thus, our system-mode approach is indeed consistent with the commonly used coupled mode theory. It is clear from our derivation that the coupling
coefficients are directly related to parameters which characterize the two system modes.

Now, it is easily verified that the solution of (18a) for which \( A_1(0) = 1 \) and \( A_2(0) = 0 \) is

\[
\begin{bmatrix}
A_1(z) \\
A_2(z)
\end{bmatrix} = \begin{bmatrix}
(q_- e^{-j\Gamma z} - q_+ e^{-j\Gamma_+ z})/(q_- - q_+)
\\
(q_- q_+ (e^{-j\Gamma z} - e^{-j\Gamma_+ z})/(q_- - q_+)
\end{bmatrix}
\]

so that the power transferred into guide 2 is

\[
|A_2(z)|^2 = \frac{|P_1|}{|P_2|} \left| \frac{\delta}{\Delta^2 + \delta^2} \right|^2 \cdot e^{-2\alpha_{av} z} \left[ \sin^2(\Delta h z) + \sinh^2(\Delta \alpha z) \right]
\]

(19)

where we have written \( \Delta \Gamma = \Delta h - j\Delta \alpha \) and \( \beta_{av} = h_{av} - j\alpha_{av} \) from equation (11) in terms of their real and imaginary parts. Obviously, in the lossless case \( (\Delta \alpha = \alpha_{av} = 0) \), maximum power transfer occurs at \( z_M = (2n + 1)\pi/2\Delta \Gamma; \ n = 0, 1, 2, \ldots \) and the power transfer ratio is

\[
|S_{12}|^2 = \left| \frac{A_2(z_M)}{A_1(0)} \right|^2 = \frac{|P_1|}{|P_2|} \frac{|\delta|^2}{|\Delta^2 + \delta^2|} \approx \frac{|\delta|^2}{|\Delta^2 + \delta^2|}
\]

(20)

so that significant power transfer is possible only if \( |\Delta| < |\delta| \), and transfer is complete only if \( \Delta = 0 \). This is a well-known result [3; 15; 18-21; 27; 29, chapter 1; 31; 32]. In the general lossy situation \( |S_{12}|^2 \) will obviously never exceed the value in (20). The behavior of \( |S_{12}|^2 \) is shown in Fig. 2. The coupling constant \( |K| = |\Delta \Gamma| \) is plotted as a function of the normalized guide separation \( \lambda k_l \) for a typical pair of guides for several values of \( \Delta \) in Fig. 3.
Fig. 2 Power transfer $|S_{12}|^2$ as a function of $\Delta$
Fig. 3 \(|K| = |\Delta r|\) as a function of normalized guide separation \(\lambda k l\) and relative non-degeneracy.
In lossy situations it may be more important to determine the amount of isolation between the two guides. To do this, we consider the ratio at any point \( z \) between the power in guide 2 and the power in guide 1:

\[
|R_{21}|^2 = \left| \frac{A_2(z)}{A_1(z)} \right|^2 = \frac{\sin^2(\Delta h z) + \sinh^2(\Delta \alpha z)}{\cos^2(\Delta h z - \theta_r) + \sinh^2(\Delta \alpha z + \theta_i)} \tag{21}
\]

The angle \( \theta = \theta_r + j \theta_i \) is given by

\[
\theta = -j \sinh^{-1} \left( \frac{\Delta}{\delta} \right) = \sin^{-1}(-j \frac{\Delta}{\delta})
\]

If \( \Delta \alpha \neq 0 \), \( |R_{21}|^2 \) eventually approaches unity, indicating that the power has been equalized between the two modes [1], while the overall power has also been reduced exponentially. This occurs for \( z \gg |\theta_i/\Delta \alpha| \), which corresponds roughly to \( |\Delta h \theta_i/\pi \Delta \alpha| \) cycles of power exchange. (21) may be used to determine the relative isolation of the signal at any point along the pair of guides.

IV. Relations to other methods

The results derived here agree with those obtained by Marcuse [1,2], Snyder [3], and Andreyev [11] by various techniques and have the additional advantage of assuming second-order accuracy in the error fields since they were derived by a variational technique [38]. To investigate the relationships of this method to others in the literature, we consider the
auxiliary geometries shown in Fig. 1. It will be noted that to calculate $c_1$ or $c_2$ requires a surface integration over the finite cross-section $S_1$ or $S_2$, respectively. Arnaud [6] also obtains (11) and (12), with the exception that his coupling constants are given by contour integrals of the type

$$e_{a,b} = \oint [\vec{E}_1^- \times \vec{H}_2^+ - \vec{E}_2^+ \times \vec{H}_1^-] \cdot \vec{a}_n \, d\xi; \quad \text{for } j = a, b \tag{22}$$

and $\vec{a}_na'$, $\vec{a}_nb'$ are outward unit normal vectors on contours $C_a$ and $C_b$ as indicated in Fig. 1. To transform (8) into a similar integral, we apply again the identity (5) to the vector $\vec{F}$ of (6), with now $A$ and $C$ replaced by $S_a$ and $C_a$ or $S_b$ and $C_b$, and obtain

$$\begin{cases} 
    c_1 = (\beta_1 - \beta_2)D_a - je_a \\
    -c_2 = (\beta_1 - \beta_2)D_b - je_b
\end{cases} \tag{23}$$

where

$$D_{a,b} = \oint_{S_{a,b}} [\vec{E}_1^- \times \vec{H}_2^+ - \vec{E}_2^+ \times \vec{H}_1^-] \cdot a_z \, dS$$

Equation (23) holds for arbitrary contours $C_{a,b}$ and enclosed surfaces $S_{a,b}$ subject only to the condition that $S_1$ (resp. $S_2$) be completely contained in $S_a$ (resp. $S_b$). (See Fig. 1) In the degenerate case $(\beta_1 = \beta_2)$, and in particular when the finite portions of $C_a$ and $C_b$ are allowed to coincide we have $e_a = -e_b = jc_1 = jc_2$ and our result therefore coincides with that of Arnaud [6]. However, for
nondegenerate modes, his results correspond to ours only if the $D_{a,b}$ terms can be neglected. We will see in the next section that the error incurred by this neglect may in fact be quite large if $C_a$ is chosen close to one of the guides.

Alternatively, we may choose $C_a$ and $C_b$ to coincide with the boundary contours $C_1$ and $C_2$ of the individual guides. Now the $D_{a,b}$ become $D_{1,2}$, which are likely to be much smaller than in the previous case, and although we now have two contour integrals, $e_1$ and $e_2$, to evaluate, they are both along finite contours and therefore presumably easier to calculate numerically. These coupling constants are the ones obtained in the detailed treatment of the lossless fiber case by Vanclooster and Phariseau [7] whose connection with the Marcuse-Snyder-Andreyev form is mentioned by Snyder [3]. Again, $c_1 = c_2 = -je_1 = +je_2$ holds exactly in the degenerate case, all results are identical and the contour(s) may be taken anywhere outside the guides that suits the particular problem's geometry.

Jones [8,9] obtains the same form for the coupling constants as in [7], by a method which is noteworthy insofar as it is the only formally exact treatment, including the continuous mode spectrum of both guides. A dyadic Green's function for each individual guide resulting from a point

---

3 Snyder's method [3], while exact for the isolated perturbed fiber, can only be heuristically extended to the multiple fiber case.
source on the guide surface is constructed, and a Green's identity is applied to the "background region" (region 3 of Fig. 1). Since the Green's function can be expanded in terms of the modal eigenfunctions (which are both discrete and continuous) of each guide, the end result is the above mentioned contour integral of one modal field multiplied by another. Of course, this theory could give significantly different results from the inexact theories only when significant coupling from one of the modes of interest to the continuous spectrum occurred - in other words when the approximation of only two (or any finite number) modes broke down. But in that case, we would not be interested in using the structure to begin with, since a significant amount of power would be radiated away.

Variational techniques involving $\beta^2$ rather than $\beta$ have been used by Taylor [18, 19] (who only solves the problem in the scalar approximation and therefore can only rigorously treat TE waves on slab structures; there is no way, for instance, using this method, that polarization effects can be accounted for) and by Matsuhara and Kumagai [4,5]. As opposed to the mixed-field formulation used here, the latter have used E-and H-field variational principles (see [39]; these are related to the forms given by Harrington [34, p.347]). Since they yield expressions for $\beta^2$, these are not directly comparable to the present treatment; however, for $\beta_1$ not too different from $\beta_2$, we show in Appendix A that the
present formulas may be derived from those of either [4] or [5]. Since, as will be shown below, the present treatment may be very simply extended to anisotropic or even bianisotropic media, it would seem preferable to the E- or H- field methods, whose generalization is not at all apparent. The early work of Bracey et al. [10] is applicable to a pair of lossless reciprocal guides which are mirror images of one another, so that the system modes are even and odd and the modes are degenerate. Since in this case the problem is the same as that obtained by placing an electric or magnetic wall in the plane of symmetry between the guides, a perturbation solution of the latter problems is done. The method employs a concept due to Maclean known as the resonator action theorem [40-42] using the artifice of inserting perfectly conducting planes at two values of $z$ such that the guide forms a resonator at the operating frequency $\omega$, and determining what change in the $z$-dimension (hence in the guide wavelength) is necessary to maintain the same resonant frequency when the electric or magnetic wall is brought in from infinity. It is straightforward to show that this method too is equivalent to the Marcuse-Snyder-Andreyev results. This is hardly a surprise, since the resonator action theorem is simply a way of expressing a classical variational principle.
V. Numerical results for slab waveguides

In order to quantitatively compare the various results presented here, we consider two parallel slab waveguides of widths \( d_1 = 2a_1 \) and \( d_2 = 2a_2 \), separated by a width \( d \), as shown in Fig. 4. Guide 1, guide 2, and the substrates are taken to have dielectric constants \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \), respectively, which are assumed to be constant scalars, but may be complex.\(^4\) For simplicity we consider only the coupling between even TE modes of these structures; results will be similar for other cases.

As is well-known, we may write the fields of either guide in isolation as

\[
\begin{align*}
E_{y_i} &= A_i \cos p_i x_i \\
H_z &= -\frac{p_i}{\omega \mu_0} A_i \sin p_i x_i \quad |x_i| < a_i \\
H_{x_i} &= -\frac{\beta_i}{\omega \mu_0} A_i \cos p_i x_i
\end{align*}
\]

inside the slabs, where \( p_i^2 = k_o^2 \varepsilon_i - \beta_i^2 \), and

\[
\begin{align*}
E_{y_i} &= A_i \cos p_i a_i e^{-\gamma_i |x_i| - a_i} \\
H_z &= \frac{x_i}{|x_i|} + \frac{j\gamma_i}{\omega \mu_0} A_i \cos p_i a_i e^{-\gamma_i |x_i| - a_i} \quad |x_i| > a_i \\
H_{x_i} &= -\frac{\beta_i}{\omega \mu_0} A_i \cos p_i a_i e^{-\gamma_i |x_i| - a_i}
\end{align*}
\]

\(^4\)Note the redefinition of \( \varepsilon_1 \) and \( \varepsilon_3 \) in this section for convenience.
Fig. 4  Geometry of two parallel slab guides
outside the slabs, where \( \gamma_1^2 = \beta_1^2 - k_o^2 \varepsilon_3 \), and the \( \beta_i \) satisfy the characteristic equation

\[
\tan p_i a_i = \frac{\gamma_i}{p_i}
\]

or equivalently

\[
\cos p_i a_i = \frac{p_i}{k_o \sqrt{\varepsilon_i - \varepsilon_3}} ; \quad \sin p_i a_i = \frac{\gamma_i}{k_o \sqrt{\varepsilon_i - \varepsilon_3}}
\]

The above equations are used to calculate the various quantities relevant to the coupling problem in the Appendix.

To compare the coupling coefficients of Marcuse-Snyder-Andreyev, Arnaud, and Jones for various degrees of non-degeneracy (in the degenerate case all three are equivalent as discussed in section IV), we calculate \( \delta \) as used in eqns. (10) - (14) from the formulas derived in Appendix B, labeling them \( \delta_M, \delta_A, \) and \( \delta_J \) respectively.

\[
\delta_M^2 = Q \frac{[(\gamma_1 + \gamma_2) + (\gamma_1 - \gamma_2)e^{-2\gamma_2 a_1}][(\gamma_1 + \gamma_2) - (\gamma_2 - \gamma_1)e^{-2\gamma_1 a_2}]}{[k_o^2(\varepsilon_1 - \varepsilon_3) + \beta_2^2 - \beta_1^2][k_o^2(\varepsilon_2 - \varepsilon_3) + \beta_1^2 - \beta_2^2]}
\]

(24)

\[
\delta_A^2 = Q \frac{(\gamma_1 + \gamma_2)^2}{k_o^4(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)}
\]

(25)

\[
\delta_J^2 = Q \frac{[(\gamma_1 + \gamma_2) + (\gamma_1 - \gamma_2)e^{-2\gamma_2 a_1}][(\gamma_1 + \gamma_2) - (\gamma_2 - \gamma_1)e^{-2\gamma_1 a_2}]}{k_o^4(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)}
\]

(26)

where

\[
Q = \frac{p_1^2 p_2^2 e^{-(\gamma_1 + \gamma_2)d}}{4\beta_1 \beta_2 (a_1 + 1/\gamma_1)(a_2 + 1/\gamma_2)}
\]

(27)
In Figs. 5-8, the differences between $\delta^2_M$, $\delta^2_A$, and $\delta^2_J$ are plotted against the degree of degeneracy of the modes, due to differences between $a_1$ and $a_2$, or $\epsilon_1$ and $\epsilon_2$. These relative differences are independent of $d$, and become most pronounced when one of the modes comes close to cutoff, but are zero in the degenerate case.

In Figs. 9-10, $\Delta \Gamma = \frac{1}{2} (\Gamma_+ - \Gamma_-)$ is compared for a degenerate and a non-degenerate case for the result using Jone's coupling constant; the second-order variational result obtained by keeping all terms in equation (10); and the exact result, obtained by solving the exact modal characteristic equation for the system [21]. The variational result is quite superior close to the guide, when the other result becomes much more in error.

In Arnaud's paper [6] it was assumed that the finite part of the contour (= the point $\frac{d}{2} + w$ of evaluation of the fields in the slab case) could be chosen anywhere; however, in the non-degenerate case, it is apparent that this can result in a relative error of a factor as large as $e^{\pm (\gamma_2 - \gamma_1) \frac{d}{2}}$ in evaluating $\delta_A$. Now this is absorbed into a factor $e^{-(\gamma_1 + \gamma_2) \frac{d}{2}}$; to assure then, an error bound on $\log \delta$, we require

$$|\gamma_1 - \gamma_2| \ll |\gamma_1 + \gamma_2| \quad (28)$$

or equivalently . . .
Fig. 5 Comparison of $\delta_J^2$ and $\delta_M^2$ for $TE_0$ mode as a function of relative slab widths. $\varepsilon_3 = 1.00$, $\varepsilon_1 = \varepsilon_2 = 1.04$, $k_0 a_1 = 8.68$. 
Fig. 6 Comparison of $\delta_J^2$ and $\delta_A^2$ for TE$_0$ mode as a function of relative slab widths. 
$\varepsilon_3 = 1.00$, $\varepsilon_1 = \varepsilon_2 = 1.04$, $k_0a_1 = 8.68$. 
Fig. 7 Comparison of $\delta_J^2$ and $\delta_M^2$ for TE$_0$ mode as a function of relative slab permittivities. $\varepsilon_3 = 1.00$, $\varepsilon_1 = 1.04$, $k_o a_1 = k_o a_2 = 8.68$. 
Fig. 8 Comparison of $\delta_J^2$ and $\delta_A^2$ for TE mode as a function of relative slab permittivities. $\varepsilon_3 = 1.00$, $\varepsilon_1 = 1.04$, $k_0a_1 = k_0a_2 = 8.68$. 
Fig. 9 Comparison of calculated exact value of $\Delta \Gamma = \frac{1}{2} (\Gamma_+ - \Gamma_-)$ for two degenerate TE\(_0\) slab modes to Jones' [8,9] value and the second-order value from equation (10). $\varepsilon_3 = 1.00$, $\varepsilon_1 = \varepsilon_2 = 1.04$, $k_0 a_1 = 8.68$. 
Fig. 10 Comparison of calculated exact value of $\Delta \Gamma = \frac{1}{2}(\Gamma_+ - \Gamma_-)$ for two non-degenerate TE$_0$ slab modes to Jones' [8,9] value and the second-order value from equation (10). $\varepsilon_3 = 1.00$, $\varepsilon_1 = \varepsilon_2 = 1.04$, $k_0a_1 = 8.68$. 

\[ a_1/a_2 = 1.8 \]

- JONES
- 2nd ORDER SOL'N.
\[ |\beta_1 - \beta_2| \ll \frac{\beta_{1,2}^2 - k_o^2 \varepsilon_3}{\beta_1 + \beta_2} \] (29)

One consequence of (28) is that the closer one of the modes is to cutoff, the less non-degeneracy may be tolerated between the two modes to allow Arnaud's contour to be arbitrarily chosen.

Further, in order to be able to neglect the \( D_{a,b} \) or \( D_{1,2} \) terms which are the difference between Arnaud's, Jones' and Marcuse-Snyder's coupling coefficients, we must be able to say

\[ |\gamma_1 - \gamma_2|^{-2a_{1,2} \gamma_{2,1}} \ll |\gamma_1 + \gamma_2| \] (30)

as well as

\[ |\beta_1^2 - \beta_2^2| \ll k_o^2 |\varepsilon_{1,2} - \varepsilon_3| \] (31)

However, (28) clearly implies (30), and (29) similarly (31). Hence, the choice between any of these formulas is arbitrary so long as the approximate degeneracy condition (28)-(29) holds.

Moreover, if \( |\Delta| >> |\delta| \), the calculation of \( \Delta \Gamma \) is virtually independent of \( \delta \), and since no significant power transfer occurs, this case is academic anyway.

VI. Anisotropic waveguides

When one or more of the permittivites \( \varepsilon_1, \varepsilon_2 \) or \( \varepsilon_3 \) is a tensor, we may proceed from a more general variational formula in [33] in almost exactly the same manner as in
section II. The transpose fields required in this case are no longer given by (2) in general, but are solutions corresponding to negative-z-traveling waves in the "transpose" structures, which are defined by the transposes $\tilde{\varepsilon}^T$ of the original permittivity tensors [36]. Except in certain special cases [37,43], there is no general way to obtain the transpose fields from the original fields in the anisotropic case. This is not considered too great a problem, however, since the transpose fields will be of the same order of difficulty to calculate as were the original fields, which are assumed already obtained anyway. (We are assured of the existence of the transpose mode with propagation constant $-\beta$, which is necessary to generalize the method, by reciprocity [44]).

Following the analysis of section II, then, and keeping in mind that (2) no longer holds, we again obtain (10) - (14) in the standard approximations, with now

$$c_{1,2} = \frac{\omega \varepsilon_0}{2} \int \{ \bar{E}_2^- \cdot \tilde{\varepsilon}_{1,2} \cdot \bar{E}_1^+ + \bar{E}_1^- \cdot \tilde{\varepsilon}_{1,2} \cdot \bar{E}_2^+ \} \, dS$$

(32)

replacing (8). Interestingly enough, the anisotropy of the substrate affects the coupling only indirectly as it affects the modal fields $\bar{E}_{1,2}^\pm$. Further, by considering the vector

$$\bar{F} = \frac{1}{2} [\bar{E}_1^- \times \bar{H}_2^+ - \bar{E}_2^+ \times \bar{H}_1^-] e^{j(\beta_1 - \beta_2)z} - \frac{1}{2} [\bar{E}_2^- \times \bar{H}_1^+ - \bar{E}_1^+ \times \bar{H}_2^-] e^{-j(\beta_1 - \beta_2)z}$$

we may obtain the contour integral form with (22) and (23) now replaced by
\[ D_{a,b} = \frac{1}{2} \int_{S_{a,b}} [E_1^- \times H_2^+ - E_2^+ \times H_1^- + F_2 \times H_1^+ - E_1^+ \times H_2^-] \cdot a_z \, ds \]

(33)

\[ c_{a,b} = \frac{1}{2} \int_{C_{a,b}} [E_1^- \times H_2^+ - E_2^+ \times H_1^- - F_2 \times H_1^+ + E_1^+ \times H_2^-] \cdot n_{1,2} \, dl \]

This result can also be extended to bianisotropic media as discussed in [45].

VII. Multimode coupling

Let us now generalize our outlook to a finite number, \( N \), of nearly degenerate modes (some of which may be on the same structure). We now assume

\[ \overline{E}^+ = \sum_{i=1}^{N} m_i \overline{E}_i \]
\[ \overline{H} = \sum_{i=1}^{N} m_i \overline{H}_i \]

and obtain as in the two-mode case

\[ \Gamma = \frac{\sum_{i=1}^{N} m_i^2 N_i + \sum_{i \neq j} m_i m_j K_{ij}}{\sum_{i=1}^{N} m_i^2 P_i + \sum_{i \neq j} m_i m_j L_{ij}} \]

(34)

Here the notation generalization is obvious:

\[ L_{ij} = L_{ji} = \frac{c^i_{ij} - c^j_{ij}}{\beta_i - \beta_j} ; \quad K_{ij} = K_{ji} = \frac{\beta_i c^i_{ij} - \beta_j c^j_{ij}}{\beta_i - \beta_j} \]

where \( K_{ij} = \beta_i, j L_{ij} = c^j_{ij}, \omega \varepsilon_0 \int \varepsilon_j, i E_1^+ \cdot E_j^- \, ds \)
We now set $\partial \Gamma / \partial m_i = 0$ for each $i = 1, \ldots, N$ which gives a system of linear equations for the vector $\vec{m} = (m_1, \ldots, m_N)$ whose determinant we set equal to zero to obtain non-trivial solutions $\vec{m}$ and system mode eigenvalues $\Gamma$:

$$\|A - B\Gamma\| = 0 \quad (35)$$

where $A$ is a matrix with

$$A_{ii} = \beta_i' \; ; \; A_{ij} = \frac{c_{ij}}{P_i} + \frac{L_{ij}}{P_i} \beta_j \quad i \neq j \quad (36)$$

and $B$ is a matrix with

$$B_{ii} = 1 \; ; \; B_{ij} = \frac{L_{ij}}{P_i} \quad i \neq j \quad (37)$$

Evidently $B$ is "almost" the identity matrix since the off-diagonal terms are (as in section II) much smaller than one. If we consider the off-diagonal terms small to first order, then $B^{-1}$ is given to the second order approximately as

$$[B^{-1}]_{ii} = 1 \; ; \; [B^{-1}]_{ij} = - B_{ij} \quad i \neq j \quad (38)$$

Equation (40) is now equivalent to

$$\|B^{-1}A - \Gamma I\| = 0$$

and we can identify $B^{-1}A$ with the coupling matrix which is a generalization of the $2 \times 2$ case as in (18), and which are obtained in the general coupled-mode theory [3, 29, 30].
Calculating $C = B^{-1}A$ to the second order enables us to obtain

$$C_{ii} = \beta_i; \quad C_{ij} = \frac{c_{ij}^i}{p_i} + (\beta_j - \beta_i) \frac{L_{ij}}{p_i} \approx \frac{c_{ij}^i}{p_i}$$

(39)

as in [3]. The last approximation of (39) is valid since

$$|\beta_j - \beta_i| \leq |c_{ij}^i/p_i|$$

for appreciable power transfer (section III; see also [3, 15, 47]) whereas

$$|L_{ij}| \ll |p_i|$$

in the usual guide-separation approximations. Recalling the earlier question in section IV of whether to drop the terms $(\beta_1 - \beta_2)D_{a,b}$ from the contour integral forms, we note that $L = D_a + D_b$ and that the same approximation is applicable here; i.e., if the pair of modes is "not too non-degenerate" relative to their separation, we may indeed make such approximations. It should also be noted that the $C_{ij}$ will be of the second order (that is, of similar form to $d_{1,2}$ of section II) for a pair of orthogonal modes on the same guide, since their fields can only couple in the regions of the other guides where their own fields are already small.
VIII. Concluding remarks

We have presented here a fairly complete investigation of the relationships between the various coupled-mode theories for parallel dielectric waveguides to be found in the literature. It would seem appropriate here to conclude with a few remarks about the assumptions we must make in treating a problem such as this by coupled-mode theory. Now, in the first place, we must make the approximation that any field of the system of guides may be represented as a finite expansion over the two (or, as in section VII, N) modes being considered. The question of how this is to be done is of some interest.

Each guide, in the absence of the other, possesses a complete spectrum of discrete and continuous modes, in terms of which an arbitrary field of that structure may be expanded. In the region between the guides (where \( \epsilon = \epsilon_3 \) in Fig. 1), a mode of the other guide may in fact be so expanded. Faced with two complete sets of modes, we may obviously assign to a given field an arbitrary amount of a certain modal field on one guide, and represent the balance of the field in terms of the complete mode set of the other guide, so that such a representation is highly non-unique. In the guides themselves, however, (i.e., where \( \epsilon = \epsilon_3 + \epsilon_{1,2} \)), we may represent the field uniquely in terms of the mode set of the appropriate guide, arguing that a concentration of power within the guide may quite reasonably be assigned to a surface-wave mode of that
guide, rather than to a collection of (mostly) continuous modes of the other guide. It is apparent, then, that the approximation of a finite number of modes depends upon factors such as the degree of concentration of the modes involved within the guide itself (equivalently, the rate of field decay outside the guide), the separation between guides, and so on. A more quantitative statement of these conditions for the case of slab waveguides was given in section V.

What we have done up to this point of course, is to completely neglect all but a finite number of modes in our analysis. The justification for this is that if the $\beta_i$'s are sufficiently close together and all other modal values of $\beta$ sufficiently far away from them, then the only significant signal transfer occurs between the modes of interest. When this justification does not apply (and this is related to a more quantitative description of the word "sufficiently" above) we may still consider only the finite number of modes for purposes of determining a coupling length between them, but the loss of power to other modes is something which cannot be accounted for in this manner. The coupling length depends only on $\Gamma_+ - \Gamma_-$, which was calculated using a variational formula and so is relatively insensitive to the presence or absence of other modal fields in the trial forms. In section V, it was seen that $\Gamma_\pm$ are most accurately calculated using equation (10), and retaining the second-order terms. Thus, in cases where the first-order approximations break down, and a high degree of accuracy is required, this formula is the one to use.
Appendix A

In [4], an E-field variational principle is used to calculate the system mode propagation constants for the two-guide case. The result is

\[
\Gamma^2 = \frac{\frac{1}{2}(\beta_1^2 + \beta_2^2) + \frac{N_{12}D_{12}}{D_{11}D_{22}} + \sqrt{\left(\frac{\beta_1^2 - \beta_2^2}{\frac{1}{2}}\right)^2 + \frac{(N_{12} + \beta_1^2D_{12})(N_{12} + \beta_2^2D_{12})}{D_{11}D_{22}}} \right)}{1 - \frac{D_{12}^2}{D_{11}D_{22}}}
\]

(40)

where, after a good deal of manipulation of the expressions given in [4] using (5) as well as the identity

\[
\int_C [\mathbf{A} \times \mathbf{B}] \cdot \mathbf{a} d\mathbf{l} + \frac{\partial}{\partial z} \int_A [\mathbf{A} \times \mathbf{B}] \cdot \mathbf{a}_z dS
\]

\[
= \int_A \mathbf{A} \cdot \nabla \times \mathbf{B} dS - \int_A \mathbf{A} \cdot \mathbf{v} \times \mathbf{B} dS
\]

we can show that

\[
N_{12} = \omega^2 \beta_1^2 \varepsilon_0 \int \varepsilon_1 \mathbf{E}_t^+ \cdot \mathbf{E}_t^+ dS + \omega^2 \beta_2^2 \varepsilon_0 \int \varepsilon_2 \mathbf{E}_t^+ \cdot \mathbf{E}_t^+ dS
\]

\[
+ \beta_1 \beta_2 \{\omega_2 \int \mathbf{E}_t^+ \times \mathbf{H}_t^+ \cdot \mathbf{a}_z dS - \omega^2 \varepsilon_0 \int \varepsilon_1 \mathbf{E}_z^+ \cdot \mathbf{E}_z^+ dS\}
\]

\[
D_{12} = -\omega \beta_2 \mathbf{E}_t^+ \times \mathbf{H}_t^+ \cdot \mathbf{a}_z dS - \omega^2 \varepsilon_0 \int \varepsilon_1 \mathbf{E}_t^+ \cdot \mathbf{E}_t^+ dS
\]

\[
D_{11} = -\frac{1}{2} \omega \beta_1 p_1
\]

\[
D_{22} = -\frac{1}{2} \omega \beta_2 p_2
\]
Now for sufficiently small $\Delta$; more precisely, when
\[
\frac{1}{2}(\beta_1 + \beta_2)^2 = \beta_1^2 + \beta_2^2 = 2\beta_1\beta_2,
\]
we have
\[
N_{12} + \beta_1^2 D_{12} = \omega^2 \beta_2 \varepsilon_0 \int \varepsilon_2 \left[ \beta_2 E_t^+ \cdot E_t^+ - \beta_1 E_z^+ E_z^+ \right] dS = \omega^2 \beta_2 c_2
\]
\[
N_{12} + \beta_2^2 D_{12} = \omega^2 \beta_1 \varepsilon_0 \int \varepsilon_1 \left[ \beta_1 E_t^+ \cdot E_t^+ - \beta_2 E_z^+ E_z^+ \right] dS = \omega^2 \beta_1 c_1
\]
so that
\[
\Gamma_\pm = \frac{1}{2}(\beta_1 + \beta_2) \pm \sqrt{\frac{1}{4}(\beta_1 - \beta_2)^2 + \frac{c_1 c_2}{P_1 P_2}}
\]
as in (10), with our approximations.

A similar method is used in [5], employing an $H$-field form, and a consequent need for contour integrals where the dielectric constant is discontinuous in the transverse plane. Thus, for simplicity we assume that $\varepsilon_3$ varies continuously over the transverse plane and that $\varepsilon_1$ (or $\varepsilon_2$) approaches zero continuously as $C_1$ (or $C_2$) is approached. If we make similar manipulations on the results of [5], we obtain again (40), where now, due to some slight notational changes, $D_{12} \rightarrow -D_{12}$ and
\[
N_{12} - \beta_1^2 D_{12} = \omega^2 \beta_2 \varepsilon_0 \int \varepsilon_2 E_t^+ \cdot E_t^+ dS - \omega^2 \beta_2 \varepsilon_0 \int \frac{\varepsilon_3}{\varepsilon_2 + \varepsilon_3} E_z^+ E_z^+ dS
\]
\[
+ \omega^2 \beta_1 \varepsilon_0 \int \frac{\varepsilon_1}{\varepsilon_1 + \varepsilon_3} E_z^+ E_z^+ dS
\]
\[ N_{12} - \beta_{12}^2 D_{12} = \omega^2 \beta_1^2 \beta_2 \varepsilon_0 \int \frac{E_t^+}{\varepsilon_1} t_1 E_t^2 dS - \omega^2 \beta_1^2 \varepsilon_0 \int \frac{\varepsilon_1 \varepsilon_3}{\varepsilon_1 + \varepsilon_3} l_1 E_{z1}^+ E_{z2}^+ dS \]

\[ + \omega^2 \beta_2^2 \varepsilon_0 \int \frac{\varepsilon_2^2}{\varepsilon_2 + \varepsilon_3} z_1 E_{z1}^+ E_{z2}^+ dS \]

If we are dealing with low-permittivity contrast optical wa guides, we will have

\[ |\varepsilon_{1,2}| \ll |\varepsilon_3| \]

and so

\[ N_{12} - \beta_{12}^2 D_{12} \approx \omega \beta_2^2 c_2 \]

\[ N_{12} - \beta_{22}^2 D_{12} \approx \omega \beta_1^2 c_1 \]

and the same sequence of approximations as for [4] reduces the results again to (10), within our usual approximations.
Appendix B

We use the equations of section V to calculate the various quantities relevant to the coupling problem. Thus

\[ P_i = -2 \int_{-\infty}^{\infty} E_{y1} H_{x1} dx_1 \cdot A_1^2 \frac{2\beta_1}{\omega \mu_o} \left( a_i + \frac{1}{Y_i} \right) \]  

(41)

\[ c_{1,2} = \omega e \int_{-a_{1,2}}^{a_{1,2}} (\varepsilon_{1,2} - \varepsilon_3) E_{y1} E_{y2} dx_{1,2} \]

\[ = \frac{A_1 A_2}{\omega \mu_o} \sqrt{\frac{\varepsilon_{1,2} - \varepsilon_3}{\varepsilon_{1,2} - \varepsilon_3}} \frac{p_{1p2}}{k_0^2 (\varepsilon_{1,2} - \varepsilon_3) + \beta_{1,2}^2} . \]

\[ \cdot \left[ (\gamma_1 + \gamma_2) + (\gamma_{1,2} - \gamma_{2,1}) e^{-2\gamma_{2,1} a_{1,2}} \right] \]

(42)

\[ d_{1,2} = \omega e \int_{-a_{2,1}}^{a_{2,1}} (\varepsilon_{2,1} - \varepsilon_3) E_{y1}^2 dx_{1,2} \]

\[ = \frac{A_{1,2}^2}{\omega \mu_o} \frac{\varepsilon_{2,1} - \varepsilon_3}{\varepsilon_{1,2} - \varepsilon_3} \frac{p_{1,2}^2}{2\gamma_{1,2}} e^{-2\gamma_{1,2} d} [1 - e^{-4\gamma_{1,2} a_{2,1}}] \]

(43)

\[ D_a = \frac{1 + \beta_2}{\omega \mu_o} \int_{-\infty}^{w + \frac{d}{2}} E_{y1} E_{y2} dx_1 \]

\[ = (\beta_1 + \beta_2) \frac{A_1 A_2}{\omega \mu_o} \frac{p_{1p2}}{k_0^2 (\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)} . \]

(44)
\[
\left\{ \frac{(\gamma_1+\gamma_2)\left[1-e^{(\gamma_2-\gamma_1)(w+\frac{d}{2})/2}\right]}{\beta_1^2 - \beta_2^2} + \frac{(\gamma_1+\gamma_2)^{-2}\gamma_2 a_1}{k_o^2(\varepsilon_1-\varepsilon_3) + \beta_2^2 - \beta_1^2} \right\}
+ \frac{(\gamma_1+\gamma_2) + (\gamma_1-\gamma_2)e^{-2\gamma_2 a_1}}{k_o^2(\varepsilon_1-\varepsilon_3) + \beta_2^2 - \beta_1^2}
\]

\[
D_b = \frac{\beta_1 + \beta_2}{\omega \mu_o} \int_{w+\frac{d}{2}}^{\infty} E_1 E_2 \, dx_1
\]

\[
= (\beta_1 + \beta_2) \frac{A_1 A_2}{\omega \mu_o} \frac{p_1 p_2 e^{-\gamma_1 d}}{k_o^2(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)}
\]

\[
\left\{ \frac{(\gamma_1+\gamma_2)\left[1-e^{(\gamma_1-\gamma_2)(\frac{d-w}{2})/2}\right]}{\beta_2^2 - \beta_1^2} + \frac{(\gamma_2-\gamma_1)e^{-2\gamma_1 a_2}}{k_o^2(\varepsilon_2-\varepsilon_3) + \beta_1^2 - \beta_2^2} \right\}
+ \frac{(\gamma_1+\gamma_2) + (\gamma_2-\gamma_1)e^{-2\gamma_1 a_2}}{k_o^2(\varepsilon_2-\varepsilon_3) + \beta_1^2 - \beta_2^2}
\]

Here \( w \) is the distance from the midpoint between guides 1 and 2 to the position chosen for the integration "contour" (which has degenerated into a point in this two-dimensional problem). Further,

\[
L = D_a + D_b = \frac{c_1 - c_2}{\beta_1 - \beta_2}
\]

\[
= (\beta_1 + \beta_2) \frac{A_1 A_2}{\omega \mu_o} \frac{p_1 p_2 e^{-(\gamma_1+\gamma_2)\frac{d}{2}}}{\beta_1 - \beta_2}.
\]
\[
\left\{ \sqrt{\frac{\varepsilon_1 - \varepsilon_3}{\varepsilon_2 - \varepsilon_3}} \left[ \frac{1}{\gamma_1 - \gamma_2} + \frac{\text{e}^{-2\gamma_2a_1}}{\gamma_1 + \gamma_2} \right] \frac{\text{e}^{(\gamma_1 - \gamma_2) \frac{d}{2}}}{k_o^2(\varepsilon_1 - \varepsilon_3) + \beta_2^2 - \beta_1^2} + \\
\sqrt{\frac{\varepsilon_1 - \varepsilon_3}{\varepsilon_2 - \varepsilon_3}} \left[ \frac{1}{\gamma_2 - \gamma_1} + \frac{\text{e}^{-2\gamma_1a_2}}{\gamma_1 + \gamma_2} \right] \frac{\text{e}^{(\gamma_2 - \gamma_1) \frac{d}{2}}}{k_o^2(\varepsilon_2 - \varepsilon_3) + \beta_2^2 - \beta_1^2} \right\}
\]

(46)

and

\[
\kappa = \frac{1}{\beta_1 - \beta_2} \{ \beta_1 c_1 - \beta_2 c_2 \} = \frac{1}{2}(\beta_1 + \beta_2) L + \frac{1}{2}(c_1 + c_2)
\]

\[
= (\beta_1 + \beta_2) \frac{A_1 A_2}{\omega \mu_o} p_1 p_2 e^{-(\gamma_1 + \gamma_2) \frac{d}{2}} .
\]

(47)

In addition, we have the "contour integral" coupling constants of Arnaud [6]:

\[
e_{a,b} = \pm \left( E_{y_1} H_{z_2} - E_{y_2} H_{z_1} \right)_{x_1 = a_1 + d/2 + w}
\]

\[
= \pm j(\gamma_1 + \gamma_2) \frac{A_1 A_2}{\omega \mu_o} \frac{p_1 p_2}{k_o^2(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)} e^{-(\gamma_1 + \gamma_2) \frac{d}{2}} (\gamma_2 - \gamma_1) \frac{d}{2}
\]

(48)
or of Vanclooster and Phariseau [7] or Jones [8,9]:

\[
e_{1,2} = \pm \left[ E_{1} z_{2} - E_{2} z_{1} \right]^{a_{1,2}}
\]

\[
= \pm j \frac{A_{1} A_{2}}{\omega \mu_{0}} \frac{P_{1} P_{2}}{k_{0}^{2} \sqrt{(\varepsilon_{1} - \varepsilon_{3})(\varepsilon_{2} - \varepsilon_{3})}} e^{-\gamma_{2,1}^{d}}
\]

\[
\cdot \left[ (\gamma_{1} + \gamma_{2}) + (\gamma_{1,2} - \gamma_{2,1}) e^{-2\gamma_{2,1}^{a_{1,2}}} \right]
\]

(49)

In both (48) and (49), the plus sign is associated with "a" or "1" and the minus sign with "b" or "2". We will, in (48), take \( w = 0 \) so that the "contour" is midway between the guides, as suggested by Arnaud [6].
Appendix C

We demonstrate here that for lossless modes, normalization of the mode power to unity results in $P_1 = P_2$. Following Adler [48] we have that $E_z$ and $H_z$ satisfy

$$\nabla_t^2 E_z + (k^2 - \beta^2) E_z = \frac{1}{k^2 - \beta^2} \left\{ [\beta^2 \frac{\nabla_t e}{e} + k^2 \frac{\nabla_t \mu}{\mu}] \cdot \nabla_t E_z ight\}$$

$$- \omega \mu \beta \vec{a}_z \cdot \left[ \frac{\nabla_t k^2}{k^2} \times \nabla_t H_z \right]$$

and

$$\nabla_t^2 H_z + (k^2 - \beta^2) H_z = \frac{1}{k^2 - \beta^2} \left\{ [\beta^2 \frac{\nabla_t \mu}{\mu} + k^2 \frac{\nabla_t e}{e}] \cdot \nabla_t H_z \right\}$$

$$+ [\omega \epsilon \beta \vec{a}_z \cdot \left[ \frac{\nabla_t k^2}{k^2} \times \nabla_t E_z \right]$$

valid for arbitrary inhomogeneous isotropic structures where both $\epsilon$ and $\mu$ are allowed to be functions of position, and $k^2 = \omega^2 \mu \epsilon$. Since in the lossless case these equations have real coefficients, and since the boundary conditions

$$E_z, \ H_z \text{ continuous}$$

$$+ j(\vec{a}_n \times \vec{H}_t) = \frac{\beta}{k^2 - \beta^2} \vec{a}_n \times \nabla_t H_z + \frac{\omega \epsilon}{k^2 - \beta^2} \vec{a}_n \times (\vec{a}_z \times \nabla_t E_z)$$

and

$$+ j(\vec{a}_n \times \vec{E}_t) = \frac{\beta}{k^2 - \beta^2} \vec{a}_n \times \nabla_t E_z - \frac{\omega \mu}{k^2 - \beta^2} \vec{a}_n \times (\vec{a}_z \times \nabla_t H_z)$$

continuous, all have real coefficients, the solutions $E_z$ and $H_z$ can and will be chosen real. Further, $\vec{E}_t$ and $\vec{H}_t$ are given by
\[ \overline{H}_t = -j \left\{ \frac{\beta}{k^2 - \beta^2} \ \nabla_t H_z + \frac{\omega c}{k^2 - \beta^2} \ \overline{a}_z \times \nabla_t E_z \right\} \]

\[ \overline{E}_t = -j \left\{ \frac{\beta}{k^2 - \beta^2} \ \nabla_t E_z - \frac{\omega \mu}{k^2 - \beta^2} \ \overline{a}_z \times \nabla_t H_z \right\} \]

so that \( \overline{E}_t \times \overline{H}_t^* = -\overline{E}_t \times \overline{H}_t \) are both real, so that all lossless modes with unit power normalization have also \( P_1 = P_2 \).
BIBLIOGRAPHY


