Least action approach to lumped element circuit mechanics

by

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Thesis directed by Professor Andrew Lucas

In the endeavor to design and implement useful quantum computers, various platforms have seen varying levels of success, and the route to the era of fully scalable quantum computing remains shrouded in mystery. One such platform for quantum computation is superconducting circuits, which are a fruitful environment for the observation of quantum physics on mesoscopic scales. In any effort to characterize the quantum behavior of superconducting circuits, a classical theory of circuits with a straightforward route to quantization is necessary. In this thesis, we approach the problem of establishing a maximally general theory of circuits. First, we formalize a framework wherein a classical Hamiltonian may be derived for an arbitrary nondissipative circuit, assuming only that Kirchhoff's laws admit a unique solution. We also provide an algorithm for deriving such Hamiltonians, and prove that it may always be executed successfully for nonsingular circuits. Secondly, we generalize the aforementioned framework in such a way that circuit duality is manifestly a relabeling transformation while providing novel insight into the ill–behaved properties of nonplanar circuit duals. Finally, we produce an even more general framework that captures the classical behavior of even dissipative circuits, including a formal argument that Johnson–Nyquist noise applies to all linear resistors and a derivation of Johnson–Nyquist noise for nonlinear resistors. We derive a principle of least action from which one may derive Langevin and Fokker–Planck equations describing the dynamics of circuits with linear and nonlinear dissipative elements alike.

Dedication

To my wife, April, and her bottomless affection.

Acknowledgements

I am seldom afforded a chance to wax poetic from a position beyond reproach. As such, I plan to take full advantage of this opportunity. I hope that my remarks here convey the depth of my gratitude and admiration.

To my wife, April, whose contribution to my life and goals is peerless, to my mother, Anna, who has urged me to hope for my improbable dreams earnestly, and to my father, Terry, who adoped me and taught me to look for my own happiness; I can offer in return only my love and gratitude. I will not outlive my profound debt to you three, who have given me myself.

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Chapter 1

Introduction

1.1 The structure of this thesis

This thesis should be regarded as a compendium of the very closely related works [90, 91]. The content of Chapter 2 first appeared in [90], and likewise Chapters 3 and 4 appeared originally in [91] and [**osborne2024stochastic**] respectively.

While the organization of chapters in this thesis is in correspondence with the chronological order of the related papers, it is also the most pedagogically sensible reading order. Chapter 2 builds a formalism[1] which is extended by the discussion in Chapter 3 and relied upon very heavily in Chapter 4. Chapter 2 begins with a gentle introduction to the theory of circuits, which is repeated to some extent in Chapters 3 and 4.

This introduction aims to put the contents of this thesis into context by motivating the problems addressed by the main body. We also provide what we hope is all of background necessary to understand the technical details of Chapter 2 with the material prerequisite for Chapter 3 would be provided by the first and so on.

1.2 Superconducting circuits, a platform for quantum computing

The study of quantum computers and quantum computation is a subject of great interest over recent years. Motivated in part by the power of quantum computers to perform tasks for which no

[1] Formal discussion of the contents of Chapter 2 is relegated to Appendix A.

efficient classical algorithm exists, many avenues toward quantum computation have been and are being explored both by academic institutions and private entities.

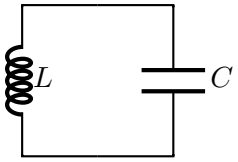
In our quest to develop scalable quantum computers, we find ourselves in the era of noisy intermediate scale quantum computation (NISQ) [61, 100]. Existing experimental implementations of quantum computers rely on a variety of platforms, but all face the challenge of correcting or preventing errors. Any kind of computation is subject to errors; for the task of classical computation, existing physical platforms and error correction techniques are sufficient to enable the production of even commercially available computers scaled to nearly arbitrary size. As a point of contrast, existing quantum computers are not capable of irrefutably demonstrating superiority over existing classical devices.

The greatest obstacle in demonstrating quantum supremacy is the problem of quantum error correction. Any quantum computer is subject to a number of sources of error, some of which do not have a classical analogue. Decoherence, gate infidelity, dephasing, and classical bit–flips all play a role in the host of problems for which a quantum computer must produce an answer.

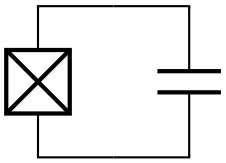
There are, in principle, many routes toward scalability. The first is so–called active error correction where errors are detected by the measurement of some quantity, and then corrected by applying some operation. Such schemes can make use of redundancy, or the structure of some set of stabilizers. Various schemes of active error correction have enjoyed some practical success. Another approach to protecting information from errors involves the encoding of information in some very nonlocal or entangled state

Yet a third approach is to use physical ingredients that somehow incorporate error protection in a way that is intrinsic to their hardware. This third approach is the primary motivation for the study of superconducting circuits as a platform for quantum computation [43, 44, 48].

As is probably clear from the nomenclature itself, the physical principles that underpin the physics of superconducting circuits is inseparable from the physics of superconductivity. Intuitively, a superconducting circuit is simply an electrical circuit fabricated out of a material which, below a critical temperature, exhibits superconductivity. Physically, the state of some superconducting



(a) A minimal one–loop conservative circuit.



(b) A minimal one–loop conservative circuit with a nonlinear element

Figure 1.1: Pictured above are a pair of circuits that mare minimal examples of conservative linear and nonlinear circuits respectively. The boxed and crossed circuit element is meant to denote the Josephson junction, which is a nonlinear inductor.

circuit can be understood as a wave function describing the configuration of a mesoscopic Cooper pair condensate. Despite the fact that there are, in principle, a macroscopic number of degrees of freedom to be specified by some such state, it happens to be the case that naively quantized classical circuit Hamiltonians agree with the observed behavior of superconducting circuits.

As a minimal example, consider the circuit drawn in Figure 1.1(a). From first–year undergraduate electricity and magnetism, we know well that

$$\begin{aligned} V_{\text{capacitor}} &= \frac{Q}{C}, \\ V_{\text{inductor}} &= -L\dot{I}_{\text{inductor}}. \end{aligned} \tag{1.1}$$

Using Kirchhoff's voltage law and the fact that $\dot{I} = \ddot{Q}$, we find that the dynamics of the circuit at hand are characterized by the second order linear ordinary differential equation

$$\ddot{Q} + \frac{Q}{LC} = 0. \tag{1.2}$$

Since we are interested in quantum mechanical behavior, it is desirable to write a Hamiltonian that produces (1.2); intuition guides us to

$$H = \frac{1}{2L}\Phi^2 + \frac{1}{2C}Q^2, \tag{1.3}$$

with

$$\Phi = L\dot{Q}. \tag{1.4}$$

It is not entirely surprising that we would discover that the variable conjugate to a charge is a flux. Happily, the superconducting circuit version of the circuit in Figure 1.1(a) is genuinely a quantum simple harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2L}\hat{\Phi}^2 + \frac{1}{2C}\hat{Q}^2 \tag{1.5}$$

and

$$[\Phi, Q] = \mathrm{i}\hbar. \tag{1.6}$$

One might correctly suspect that using linear elements to build a superconducting circuit as in (1.5) would produce most generally a system of coupled harmonic oscillators. However, in our pursuit of a useful qubit, it is necessary to incorporate more general ingredients in the form of nonlinearity. Nonlinear circuit elements are those for which the constitutive relations (1.1) is nonlinear. The most prevalent such element is the Josephson junction [1, 8, 27, 38, 57, 67, 74, 82, 111], with energy

$$E_{\text{Josephson}} = -E\cos(\Phi). \tag{1.7}$$

The (classical) Hamiltonian of the circuit drawn in Figure 1.1(b) is

$$H = \frac{1}{2C}Q^2 - E\cos(\Phi). \tag{1.8}$$

Why are nonlinear superconducting circuits desirable from an error–protection perspective? Heuristically, one can imagine constructing a potential of the form

$$V(\Phi) = \frac{1}{2L}(\Phi + \phi_0)^2 - E\cos(\Phi) \tag{1.9}$$

with ϕ_0 a constant tuned so that $V(\Phi)$ has two distinct wells. In each of these wells, there is a localized state, and parameters may be tuned such that these two states have small overlap. This is the mechanism of the “intrinsic error protection" that is a central virtue of superconducting circuits as a platform for qubits.

In effect, there is hope that superconducting circuits may be constructed that are safe from at least some sources of error simply by virtue of their construction. Then, active error correction may be used to correct errors from which the desired device is not safe. While the issue of error correction is not central to this thesis in a technical capacity, it is an underlying motivation to the study of superconducting circuits in general.

Superconducting circuits have been very successful experimentally [15, 17, 38, 62, 64, 78]. The transmon [62] and fluxonium qubits [78] in particular have been used as a key ingredient in state of the art quantum computers [5, 21, 83].There are circuit elements theorized but not yet implemented such as the quantum phase slip [6, 9, 63], which is thought to be the electromagnetic dual of the Josephson junction. The question of what might be accomplished with a broader suite nonlinear of circuit elements remains open. As is the case in many other settings, the proposition of experiments is intimately connected to the power of the theoretical predictions that can be made. In short, the search for novel and useful superconducting circuits demands first a broadly applicable theory of circuits.

The classical theory of conservative (i.e. without dissipation) circuits is very mature, and often the subject of textbooks and course materials. Nonetheless, there remain open problems in the theory of classical circuits, some of which are of importance to the theory of superconducting circuits.

While it is the case that Kirchhoff’s laws fully determine the dynamics of a classical circuit, it is not always clear how to derive a Hamiltonian (classical or quantum) that describing some circuit from Kirchhoff’s laws alone. Indeed, this problem is conventionally addressed by instead writing down a Lagrangian describing some circuit where Kirchhoff’s laws enter as Lagrange multipliers. Only after resolving constraints and eliminating degrees of freedom, or intuitively writing a Lagrangian such

that constraints are automatically satisfied can one derive a Hamiltonian via Legendre transform.

For a broad class of circuits, this procedure is satisfactory in the sense that it makes correct predictions. In another sense, however, such a formalism leaves much to be desired. One expects that conservative circuits generally admit a Hamiltonian description, but a rigorous demonstration that this expectation holds is not possible without a more general framework. Additionally, there are circuits that are not naively within the scope of ordinary Lagrangian mechanics; namely, dissipative and doubly nonlinear circuits. In particular the dualmon [69] is a doubly nonlinear circuit that is thought to have very desirable properties, with Hamiltonian

$$H_{\text{dualmon}} = -E_C \cos(q) - E_J \cos(\phi). \tag{1.10}$$

The form of the dualmon Hamiltonian (1.10) could only be postulated rather than derived prior to [90]. In particular, there is only one dynamical charge and one dynamical flux in the dualmon system due to the fact that the dualmon is a one–loop circuit. As such, (1.10) is intuitively the only Hamiltonian that could correctly describe it. However, a nontrivial network of dualmons, or simply including linear elements in addition to those present in the dualmon complicates such intuitive attempts at quantization prohibitively. This is not to say that there are no means by which one may endeavor to derive a quantum or classical Hamiltonian for such circuits (See e.g. [23, 88, 103, 104, 106, 117, 118, 122]). The issue is that there is not a sufficiently general theory of circuit mechanics; there exist circuits for whom the derivation of a Hamiltonian is not a rote problem of executing some algorithm but a problem requiring innovation, ingenuity, and creativity. A central result of Chapter 2 is a formalism wherein a circuit Lagrangian may be algorithmically derived for an arbitrary circuit. We also prove that every circuit that admits a unique solution to Kirchhoff's laws also admits a Hamiltonian description, and give an algorithm for deriving a Hamiltonian for circuits consistent with Kirchhoff's laws. Furthermore, we provide a number of examples of circuit Hamiltonians for which a physical derivation was unknown prior to the publication of [90].

In the theory of classical circuits, it has long been known that planar circuits with only linear

circuit elements have a number of so–called duals[105]. Circuit duality is a special case of duality in classical electromagnetism. Conveniently, there exists a simple, algorithmic procedure whereby a drawing of some circuit's dual may be derived using only a drawing of the original circuit and a set of rules. From an electrical engineering perspective, duality is a useful construction because a circuit and its dual have similar properties; the voltage across an element is related straightforwardly to the current across its dual element. When designing some device, it is generally possible to use either a circuit or one of its duals, and it can be the case that the dual of some circuit is easier to build, or requires less connectivity than the circuit itself.

Superconducting circuit qubits, however, use elements that are nonlinear. At the level of a quantum Hamiltonian, a simple change of basis? What is the dual element to the Josephson junction? At the level of a Hamiltonian, the transformation

$$\begin{aligned} p &\to -x' \\ x &\to p' \end{aligned} \tag{1.11}$$

is canonical in the sense that

$$\{x, p\} = \{x', p'\}. \tag{1.12}$$

Hence, this transformation is unitary when applied to a quantum Hamiltonian. Quantum mechanically, applying (1.11) leaves the eigenvalue spectrum of a Hamiltonian invariant. Is circuit duality the same transformation as (1.11)? Is the historical graph–theoretic circuit duality transformation applicable to circuits containing nonlinear elements? These questions were largely unanswered until Ulrich and Hassler[117], made a connection between the duality map (1.11), and the graph–theoretic circuit duality transformation. While the Hamiltonian transformation derived in [117] is precisely of the form (1.11), the Lagrangian version of the circuit duality transformation in [117] relied upon a nontrivial sequence of path integral operations. Furthermore, the validity of the transformation demands a number of conventions be chosen in regard to how charges and fluxes in a circuit are expressed.

Intuitively, one would hope for a Lagrangian duality transformation every bit as transparent as (1.11). Also, the circuit duality map on drawings does not require the specification of any conventions for which variables are used to express the state of the circuit. In chapter 3[91], we derive a Lagrangian theory where the duality transformation is manifestly a relabeling transformation. Using inspiration from the theory of algebraic topology, we also provide new insight on the reasons that nonplanar circuits need not have a well–defined dual.

The phenomenon of dissipation arises naturally when a system of interest is coupled to a "bath" with many degrees of freedom. If the interactions between the system and the bath conserve energy, keeping track of the bath degrees of freedom yields, in principle, a theory that is amenable to Lagrangian or Hamiltonian mechanics. In practice, however, this is often too cumbersome a request to fulfill. Especially when one is only interested in the dynamics of the smaller system, it is desirable to produce a model that captures the behavior of the smaller system without being burdened by the full complexity of the bath. Even when the system is coupled to the bath with only time reversal symmetric interactions, what naively arises is the phenomenon of dissipation, where the energy transfer from the system of interest to the bath is described as energy loss.

However, one expects that the bath would donate some energy to the system of interest at least occasionally even when the bath is at a higher temperature than the system. Moreover, it is conceptually appealing to derive an effective theory with the same symmetries as the microscopic interactions. At the classical level, both of these problems may be resolved through the use of stochastic differential equations in place of ordinary differential equations describing the dynamics of some system. From the perspective of stochastic differential equations, there arises a natural connection between the strength of dissipation and the variance of fluctuations. A so–called fluctuation–dissipation theorem [66, 101] is a mathematical relationship that relates the strength of dissipation to the variance of fluctuations.

In the context of circuit mechanics, there are various approaches to incorporating dissipation. The first and simplest is the ordinary linear resistor, equipped with a constitutive relation on equal footing with other circuit elements. Another approach is the Caldeira–Leggett model [18], where

bath degrees of freedom are explicitly tracked. We will focus on the former approach. Because ordinary Hamiltonian mechanics is intrinsically conservative, the dynamics of dissipative circuits may not be described by a Hamiltonian without further trickery.

Though it is possible to contrive techniques toward a general description of dissipative circuits (most often via Rayleigh dissipation function), such techniques are physically motivated only in the sense that they can be tuned to satisfy Kirchhoff's laws [79, 111]. Furthermore, such techniques generally only incorporate dissipation without fluctuations. In fact, the problem of deriving a theory wherein the Johnson–Nyquist [55, 89] fluctuation–dissipation theorem relating fluctuations in voltage δV across a resistor with resistance R at temperature T

$$\langle \delta V(t) \delta V(0) \rangle = 2TR\delta(t) \tag{1.13}$$

can be shown to hold (even only in the case of linear resistors) in general has remained open. Furthermore, the validity of (1.13) in the case of nonlinear resistors is a subject of some argument [57, 111].

In Chapter 4, we construct a physically motivated classical Lagrangian formalism describing arbitrary dissipative circuits that relies on the techniques of [90, 91]. Furthermore, we show that (1.13) holds for arbitrary linear resistors, while also demonstrating that fluctuation–dissipation theorems for nonlinear resistors may also be calculated within the scope of our formalism. We also show that LRC circuit duality is a simple relabeling transformation, which is another novel property of the Lagrangian we derive.

1.3 Preliminaries

In an effort to make this thesis a self–contained work, we will turn our attention toward a brief discussion of various technical prerequisites of the main body of this thesis. The expert reader would be forgiven for moving ahead to the start of chapter 2.

1.3.1 Newtonian, Lagrangian, and Hamiltonian mechanics

A first year classical mechanics course typically begins with a presentation of Newton's laws. For a particle of mass m in a one–dimensional potential $V(x)$, Newton's second law says that

$$m\ddot{x} = -\frac{\partial V}{\partial x}. \tag{1.14}$$

A second attempt at understanding classical mechanics would have you write

$$L = \frac{m}{2}\dot{x}^2 - V(x), \tag{1.15}$$

and observing that Lagrange's equations

$$0 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} \tag{1.16}$$

encode (1.14). It is possible to derive Lagrange's equations from Newton's. There is yet a third formulation of classical mechanics most often derived from Lagrangian mechanics by defining

$$p = \frac{\partial L}{\partial \dot{x}}, \tag{1.17}$$

and writing

$$H = p\dot{x} - L. \tag{1.18}$$

Provided that L contains the standard expression for kinetic energy, one finds that the Hamiltonian, H, *is* the total energy of the system. Hamilton's equations, which may be derived from (1.16), take the form

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial x} \\ \dot{x} &= \frac{\partial H}{\partial p}. \end{aligned} \tag{1.19}$$

Instead of the second order equation (1.16), we produce a coupled set of first order equations. Finally, we may define a bilinear, antisymmetric operation called a Poisson bracket by

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial p}. \tag{1.20}$$

In terms of the Poisson bracket, Hamilton's equations take the form

$$\begin{aligned} \dot{p} &= \{p, H\} \\ \dot{x} &= \{x, H\}. \end{aligned} \tag{1.21}$$

We say that x and p form a conjugate pair because

$$\{x, p\} = 1. \tag{1.22}$$

In matters pertaining a conservative system of a single classical point particle with kinetic energy

$$T = \frac{1}{2}m\dot{x}^2, \tag{1.23}$$

there is little more to say.

However, there are a great many systems of interest which fail to be conservative, or which consist of greater than one particle, or which are described not by a position and a momentum but by some other pair of observables. In preparation for the main body of this thesis, our aim in this introduction is to provide a description of Lagrangian and Hamiltonian mechanics that is both suitably self contained and sufficiently general that the main body is easily comprehensible.

1.3.2 Principle of least action

The principle of least actions postulates that one may use a Lagrangian L to define an *action functional*

$$S[x(t)] = \int \mathrm{d}t\, L(x(t), \dot{x}(t), \ddot{x}(t), \dots), \tag{1.24}$$

and that physical trajectories minimize S.

In order to find minima of S, we define $\delta x(t)$ be perturbatively small deviation from $x(t)$, and define $\frac{\delta S}{\delta x}$ to be the part of $S[x + \delta x]$ that is proportional to δx:

$$\delta S = S[x + \delta x] - S[x] = \int \mathrm{d}t \left[L(x + \delta x, \dot{x} + \dot{\delta x}, \dots) \right] - \int \mathrm{d}t L(x, \dot{x}, \dots). \tag{1.25}$$

Expanding about small δx,

$$\int \mathrm{d}t\, L(x + \delta x, \dot{x} + \dot{\delta x}, \dots) \approx \int \mathrm{d}t \left[\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \dot{\delta x} + \dots \right] = \int \mathrm{d}t \left[\frac{\partial L}{\partial x} \delta x - \partial_t \frac{\partial L}{\partial \dot{x}} \delta x + \cdots \right]. \tag{1.26}$$

Evidently a physical trajectory $x(t)$ satisfies,

$$0 = \frac{\delta S}{\delta x} = \sum_{i=0}^{\infty} (-1)^i \frac{\partial^i}{\partial t^i} \frac{\partial L}{\partial x^{(i)}}. \tag{1.27}$$

The textbook form of Lagrange's equations is recovered from (1.27) by demanding that L depend only upon $x(t)$ and $\dot{x}(t)$. Though the ordinary form of Lagrange's equations will suffice for this thesis, we remark that any departure from convention in defining or deriving some Lagrangian is on equal footing with textbook Lagrangian mechanics so long as the principle of least action is used to calculate equations of motion. We will shortly move on to some such departure from convention that is particularly useful.

1.3.3 Symplectic geometry

Hamiltonian mechanics, too, can be encoded by a principle of least action. Consider the action functional

$$S = \int \mathrm{d}t \left[p\dot{x} - H(x,p) \right]. \tag{1.28}$$

The equations of motion afforded by the principle of least action are

$$\begin{aligned} 0 &= \frac{\delta S}{\delta p} = \dot{x} - \frac{\partial H}{\partial p} \\ 0 &= \frac{\delta S}{\delta x} = -\dot{p} - \frac{\partial H}{\partial x}. \end{aligned} \tag{1.29}$$

At least in the case of the very simple (1.28), Hamilton's equations are recovered. One may safely regard (1.28) as an intrinsically Hamiltonian principle of least action. Nonetheless, we write

$$L = p\dot{x} - H \tag{1.30}$$

and refer to L as a Lagrangian simply because L is the argument of a principle of least action. The observation that a Hamiltonian principle of least action exists is often accredited to [35, 52].

Taking inspiration from Lagrangians of the form (1.28), the focus of this thesis is (approximately) Lagrangians of the form

$$L = \sum_{i,j} \xi_i M_{ij} \dot{\xi}_j - E(\xi). \tag{1.31}$$

When can a Lagrangian of the form (1.31) be used to derive a Hamiltonian system? Equivalently, when does (1.31) encode a Poisson bracket?

Since the equations of motion derived from (1.31) arise from a principle of least action, we are free to suppose that $M^\intercal = -M^\intercal$, since

$$\int \mathrm{d}t\, \xi_i M_{ij} \dot{\xi}_j + M_{ji} \xi_j \dot{\xi}_i = \frac{1}{2}(M_{ij} - M_{ji}) \int \mathrm{d}t\, \xi_i \dot{\xi}_j \tag{1.32}$$

by integration by parts. That is to say that any symmetric component of M is simply a total

derivative and does not effect equation of motion. Then,

$$0 = \frac{\delta S}{\delta \xi_i} = \sum_j M_{ij}\dot{\xi}_j - \frac{\partial E}{\partial \xi_i}. \tag{1.33}$$

While (1.33) has some cosmetic similarity to Hamilton's equations, the difference between (1.33) and (1.19) lies in the fact that $\dot{\xi}_j$ is not isolated in (1.33). Evidently, one can only put (1.33) in the form of Hamilton's equations if M is invertible, in which case,

$$0 = \dot{\xi}_j - \sum_i (M^{-1})_{ij}\frac{\partial E}{\partial \xi_i}. \tag{1.34}$$

Comparing (1.34) to (1.21) leads us to ascertain that the Poisson brackets determined by (1.31) may be defined by

$$\{f, g\} = \sum_{i,j} (M^{-1})_{ij}\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_j}. \tag{1.35}$$

Equivalently, one may define a so–called symplectic form

$$\omega = \sum_{i,j} M_{ij}\mathrm{d}\xi_i \wedge \mathrm{d}\xi_j. \tag{1.36}$$

It is straightforward to demonstrate that if M is both antisymmetric and invertible, then it must be a $2n$ by $2n$ matrix for some positive integer n. Even further, there exists some unitary matrix U and an n by n matrix Ω such that

$$U^\dagger M U = \frac{1}{2}\begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}. \tag{1.37}$$

Without loss of generality, we will suppose $U = \mathbb{I}$. With the notation

$$\xi_i = \begin{cases} p_i & 1 \leq i \leq n \\ x_i & n+1 \leq i \leq 2n, \end{cases} \tag{1.38}$$

we may rewrite

$$\begin{aligned} L &= \sum_{i,j=1}^{n} p_i \Omega_{ij} \dot{x}_j - E(p,x) \\ \{f,g\} &= \sum_{i,j=1}^{n} (\Omega^{-1})_{ij} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial p_j} \right). \end{aligned} \tag{1.39}$$

Now it is clear how the antisymmetry in M is encoded by the denomination of some variables as momenta and others as positions. Additionally, a Hamiltonian system must have an equal number of momenta and positions since the dimensions of an antisymmetric invertible matrix must be even.

This simple derivation is what underpins all Hamiltonian mechanics; classical and quantum. The mathematical study of manifolds with antisymmetric, invertible, biliear forms is called symplectic geometry. We discuss symplectic geometry in greater detail in Chapter 2.

1.3.4 Constraints

One of the virtues of Lagrangian formalisms is the particularly straightforward way that constraints appear and are solved. That is to say that each constraint to be satisfied corresponds to a term explicitly written in the Lagrangian, called a Lagrange multiplier. To impose some such constraint is often as simple as using Lagrange's equations to rewrite the Lagrangian itself. As a point of contrast, constraints enter into Hamiltonian mechanics by way of Dirac brackets [32, 42]. The constrained quantity affects the Poisson brackets, rather than the Hamiltonian function itself. In particular, the Hamiltonian function itself need not contain evidence of constraints to be satisfied.

As an example, suppose that the Lagrangian L describes the unconstrained dynamics of some

system, and one is interested in the dynamics of that same system constrained to satisfy

$$f(x) = 0. \tag{1.40}$$

The Lagrangian

$$L' = L + \lambda f(x) \tag{1.41}$$

has the equations of motion (via least action)

$$\begin{aligned} 0 &= \frac{\delta S'}{\delta \lambda} = f(x) \\ 0 &= \frac{\delta S'}{\delta x} = \frac{\delta}{\delta x} \int \mathrm{d}t L + \lambda \frac{\partial f}{\partial x}. \end{aligned} \tag{1.42}$$

The second line of (1.42) may be used to solve for λ in terms of dynamical variables in such a way that L' is a function of only x variables and not λ. Then, L' is precisely the sought after function: least action provides the equations of motion for a system with identical dynamics to L but constrained such that $f = 0$. Incorporating constraints in this manner is often simpler technically than imposing constraints at the Hamiltonian level, provided that both techniques are possible at all. Indeed, it is possible to impose constraints upon Lagrangians via Lagrange multipliers even for systems that do not admit a Hamiltonian description (such as those where there are overdamped degrees of freedom).

Such constraints also take on a natural role in Lagrangians of the form (1.31). Instead of supposing that Ω is a square, invertible matrix, suppose that ω is an n_1 by n_2 matrix of the form

$$\Omega = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix} \tag{1.43}$$

where ω is an m by m invertible, symmetric matrix with $m < \min(n_1, n_2)$. Then,

$$L = \sum_{i,j=1}^{m} p_i \omega_{ij} \dot{x}_j - E(x, p). \tag{1.44}$$

For $l > m$, equations of motion become constraints:

$$\begin{aligned} 0 &= \frac{\delta S}{\delta x_l} = -\frac{\partial E}{\partial x_l} \\ 0 &= \frac{\delta S}{\delta p_l} = -\frac{\partial E}{\partial p_l}. \end{aligned} \tag{1.45}$$

While it is certainly possible to build some Lagrangian for which (1.45) cannot be used eliminate x_l and p_l, we will ignore this possibility and suppose instead that there exist functions F_l and P_l from (1.45) such that

$$\begin{aligned} x_l &= F_l(x_1, x_2, \ldots, x_m) \\ p_l &= P_l(p_1, p_2, \ldots, p_m). \end{aligned} \tag{1.46}$$

Then, we may write

$$L = \sum_{i,j=1}^{m} p_i \omega_{ij} \dot{x}_j - E(x_1, x_2, \ldots, x_m, F_{m+1}, F_{m+2}, \ldots, F_{n_1}, p_1, p_2, \ldots, p_m, P_{m+1}, P_{m+2}, \ldots, P_{n_2}). \tag{1.47}$$

Since ω is invertible and symmetric, there exists some unitary matrix U such that

$$U \omega U^\dagger = \mathrm{diag}(w_1, w_2, \ldots, w_m). \tag{1.48}$$

Finally, define

$$\begin{aligned} X_i &= \sqrt{w_i} \sum_j U_{ij} x_j \\ P_i &= \sqrt{w_i} \sum_j (U^\dagger)_{ij} p_j, \end{aligned} \tag{1.49}$$

and rewrite

$$L = \sum_{i=1}^{m} P_i \dot{X}_i - E. \tag{1.50}$$

where, since ω is invertible, E may be regarded as a function of only X and P variables. After a Legendre transform, (1.50) yields a Hamiltonian and Poisson bracket where X_i and P_i form a conjugate pair.

Our ability to identify canonically conjugate pairs was no accident; Darboux's theorem [53, 72] guarantees that a one can always choose variables such that the Poisson brackets are locally of the form

$$\{X_i, P_j\} = \delta_{ij}. \tag{1.51}$$

The subject of the first two chapters of this thesis, approximately, is the means by which canonical conjugate pairs may be derived for as broad a class of circuits as possible.

1.3.5 Quantization

Equipped with Poisson brackets,

$$\{X_i, P_j\} = \delta_{ij}, \tag{1.52}$$

Dirac's canonical quantization prescription [33, 87, 120] dictates that the correct quantum mechanical analogue of a classical Hamiltonian system is given by the transformation

$$\begin{aligned} X_i &\to \hat{X}_i \\ P_i &\to \hat{P}_i \end{aligned} \tag{1.53}$$

such that

$$[\hat{X}_i, \hat{P}_i] = \mathrm{i}\hbar\{X_i, P_i\}. \tag{1.54}$$

There is, of course, much to be said about quantization in full generality (See e.g. [2, 7, 36, 46, 115]). For the purposes of this thesis, however, (1.53) and (1.54) will suffice unless explicitly stated otherwise.

1.3.6 Stochastic differential equations

Up to this point, we have concerned ourselves only with the mechanics of energy conserving systems. We now turn our attention toward systems that do not conserve energy. Prototypically,

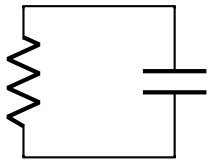


Figure 1.2: A one–loop circuit consisting of a resistor and a capacitor. It is not possible to construct a smaller circuit with nontrivial dynamics and dissipation, though one might also consider a circuit with only an inductor and a resistor as another minimal example.

dissipation enters into the theory of circuit mechanics in the form of a dissipative circuit element, the resistor.

There are two differing philosophies on how one might model a resistor. Firstly, one might model a resistor as a system with many degrees of freedom that interacts conservatively with the rest of the circuit. By keeping track of all of the “bath" degrees of freedom, the status of the model as energy preserving is kept intact. Ideally, we would simply proceed to solve Hamilton’s equations in order to produce simple, compact formulas that nicely describe the dynamics of all degrees of freedom. Unfortunately, this is too much to hope for in practice.

Happily, it is generally the case that bath degrees of freedom are not of particular interest due to the fact that they, as microscopic constituents in a resistive circuit element, are not easily manipulated. Thus, an alternative perspective naturally arises: how might one ignore the degrees of freedom in the bath, while still capturing the dynamics of the degrees of freedom of interest? At the level of first–year physics, the task at hand is most often accomplished by relying on Ohm’s law, the constitutive relation for resistors

$$V = IR. \tag{1.55}$$

Consider the circuit drawn in Figure 1.2. The equation of motion that governs the charge Q accumulated on the capacitor is

$$0 = \dot{Q} + \frac{1}{RC}Q. \tag{1.56}$$

For many purposes, (1.56) is sufficient. However, (1.56) is only approximately satisfied. More careful

observation would reveal that even with initial condition $Q(0) = 0$, there exist times t for which

$$\dot{Q}(t) \neq 0. \tag{1.57}$$

Of course, considering the fact that Ohm's law is an effective description of a much more complicated bath, this is hardly surprising. Without specifying the initial condition of the bath, it is generally possible that there is some energy transfer from the bath to the capacitor.

In order to proceed, we postulate that it is possible to correctly incorporate the dynamics of the bath into equations of motion via some *stochastic* variable $\xi(t)$. More precisely, we suppose that the equation

$$0 = \dot{Q} + \frac{1}{RC}Q + \xi(t) \tag{1.58}$$

correctly describes the time evolution of Q with

$$\langle \xi(t)\xi(0) \rangle = 2\sigma\delta(t - t'). \tag{1.59}$$

From a phenomenological perspective, the (generally temperature dependent) value of σ may be inferred from experiment. (1.58) is called a Langevin equation. One can take expectation values of physical variables via the path integral

$$\langle f(Q) \rangle = \int \mathcal{D}\xi\delta\left(\dot{Q} + \frac{1}{RC}Q + \xi(t)\right) f(Q) e^{-\frac{1}{4\sigma}\int \mathrm{d}t\xi^2}. \tag{1.60}$$

Suppose that the dynamics encoded by (1.58) can also be encoded by a probability distribution $\mathcal{P}(Q, t)$ such that expectation values may also be taken

$$\langle f(Q) \rangle = \int \mathrm{d}P\mathcal{P}(Q, t) f(Q). \tag{1.61}$$

Apparently,

$$\partial_t \langle f(Q) \rangle = \int \mathrm{d}P \, f(Q) \partial_t \mathcal{P}(Q,t). \tag{1.62}$$

On the other hand,

$$\partial_t \langle f(Q) \rangle \approx \langle f(\tilde{Q}(t+\mathrm{d}t)) \rangle - \langle f(\tilde{Q}(t)) \rangle = \left\langle f\left(\tilde{Q}(t) + \frac{1}{RC}\tilde{Q}(t)\mathrm{d}t + \mathrm{d}\xi\right) - f(\tilde{Q}(t)) \right\rangle \tag{1.63}$$

where $\tilde{Q}$ is a solution of (1.58). Further expanding (1.63) in small parameters, We find,

$$\partial_t \langle f(Q) \rangle \approx \left\langle f'\left(-\frac{1}{RC}Q\mathrm{d}t - \mathrm{d}\xi\right) + \frac{1}{2}f''\left(-\frac{1}{RC}Q\mathrm{d}t - \mathrm{d}\xi\right)^2 + \cdots \right\rangle. \tag{1.64}$$

Now, we neglect terms of quadratic order or higher in small parameters with the caveats that

$$\begin{aligned} \langle \mathrm{d}\xi \rangle &= 0 \\ \langle \mathrm{d}\xi^2 \rangle &\approx \sigma \mathrm{d}t \end{aligned} \tag{1.65}$$

to acquire

$$\partial_t \langle f(Q) \rangle \approx \left\langle -\frac{1}{RC}Qf' + \frac{1}{2}f'' \right\rangle = \int \mathrm{d}Q \, f\left(\partial_Q\left[\frac{1}{RC}Q\mathcal{P} + \frac{\sigma}{2}\partial_Q\mathcal{P}\right]\right). \tag{1.66}$$

If (1.62) and (1.66) are to be equal for arbitrary function f of Q, it must then be the case that

$$\partial_t \mathcal{P} = \partial_Q\left(\frac{1}{RC}Q\mathcal{P} + \frac{\sigma}{2}\partial_Q\mathcal{P}\right). \tag{1.67}$$

(1.67) is called a Fokker–Planck equation [37, 99].

The stationary state corresponding to (1.67) is a solution to

$$\partial_t \mathcal{P} = 0. \tag{1.68}$$

One such solution is

$$\mathcal{P}_{\text{ss}} = e^{-\frac{2}{R\sigma}E}. \tag{1.69}$$

with

$$E = \frac{1}{2C}Q^2. \tag{1.70}$$

Comparing $\mathcal{P}_{\text{ss}}$ with the Gibbs distribution, we find

$$\sigma = \frac{2T}{R}. \tag{1.71}$$

Interpreting ξ from (1.58) as fluctuations in current, we recover

$$\langle \xi(t)\xi(0)\rangle = \frac{2T}{R}\delta(t), \tag{1.72}$$

which is the famous Johnson–Nyquist noise formula [55, 89].

Evidently, we are faced with a choice; we can learn about the dynamics of the system of interest by working with (1.58) or with (1.67). Of course, in the derivation above, many subtleties have been neglected; greater care is necessary to produce a robust and complete theory of noisy systems. The most notable omission from the discussion above is that the manipulations above ought to be done in the framework of path integrals, where one may most naturally encounter and resolve issues that come from multiplicative noise and relate to ambiguities in defining Fokker–Planck equations.

Happily, [51] introduced the Martin–Siggia–Rose (MSR) formalism wherein a Lagrangian may be used as a "pneumonic" for the requesite path integral contingent upon the postulation of some steady state distribution, which we generally take to be the Gibbs distribution. For the circuit drawn in Figure 1.2, the corresponding MSR Lagrangian is

$$L = \Xi(\dot{Q} + \mathrm{i}T\Xi) + \frac{1}{RC}\Xi Q. \tag{1.73}$$

The Ξ variables should be thought of as purely imaginary, and they encode the role of noise. In

order to recover noiseless equations of motion (1.56), one need only calculate

$$0 = \frac{\delta}{\delta \Xi} \int \mathrm{d}t L \bigg|_{\Xi=0}. \tag{1.74}$$

On the other hand, one may derive a Fokker–Planck equation via

$$\partial_t \mathcal{P} = \mathrm{i} \left(L - \Xi \dot{Q} \right) \Big|_{\Xi=-\mathrm{i}\partial_Q} \mathcal{P}. \tag{1.75}$$

This formalism has a number of appealing properties. The first helpful property of MSR Lagrangians is that, since the formalism is Lagrangian in nature, constraints are easily imposed and interpreted. Perhaps most importantly for our purposes, however, since MSR Lagrangians are essentially a set of instructions for the evaluation of a path integral, it is possible to carefully resolve ambiguities that arise when noise is multiplicative.[2] Lastly, FDTs may be derived from MSR Lagrangians themselves. All of these points will be expounded upon in detail in Chapter 4.

In short, the MSR Lagrangian formalism is a principle of least action capable of producing both noiseless classical equations of motion and stochastic equations of motion for a class of systems with some generality. Indeed, MSR Lagrangians are related to their path integral counterparts in much the same way that quantum Hamiltonians are related to Feynman path integrals. As a point of contrast, techniques relying upon the Caldeira–Leggett [18] model or Rayleigh dissipation function [112] either require the full machinery of path integration or fail to be a principle of least action. In Chapter 4, we endeavor to use the MSR Lagrangian formalism to derive a theory of dissipative circuits that applies to circuits with reciprocal linear and nonlinear elements both conservative and dissipative alike.

[2] In the example at hand, noise being multiplicative if R is a nontrivial function of Q

Chapter 2

Symplectic geometry and circuit quantization

2.1 Introduction

The quantum mechanical description of superconducting circuits has paved the way for the rapid evolution of superconductor-based quantum computers [13, 31, 61, 65], enabled the discovery of the transmon [62], fluxonium [78], bosonic [44] and more complex circuits [48], facilitated the advancements of coupling between qubits [110], opened new avenues towards quantum simulation [3], and led to the theory of circuit quantum electrodynamics [10]. In this established formalism of circuit quantization [23, 30, 63, 88, 102–104, 106, 117, 118, 123], generalized branch or node fluxes, or charges, describe the energy of the elements. While these fluxes and charges are conjugates, in all but the simplest circuits, there are inevitable *constraints* that arise between variables, complicating a straightforward quantization prescription.

2.1.1 The standard approach to circuit quantization

The standard resolution in the literature is to begin by studying the classical Lagrangian mechanics of the circuit. In the Lagrangian formalism, we efficiently remove non-dynamical degrees of freedom by integrating them out; after this step, we perform a Legendre transformation to a classical Hamiltonian for the genuinely dynamical degrees of freedom. Since the Legendre transformation reveals the canonical momenta for each coordinate, we can quantize a Hamiltonian self-consistently.

As a simple example, consider an inductor, L, with two capacitors, C_1 and C_2, all in parallel [see Fig. 2.1(a)]. There is one degree of freedom, which can be identified as the flux ϕ across the

inductor. Since $\dot{\phi}$ is the voltage drop across the capacitor, one writes down a Lagrangian

$$L = \frac{1}{2}(C_1 + C_2)\dot{\phi}^2 - \frac{1}{2L}\phi^2. \tag{2.1}$$

This Lagrangian is interpreted as a “kinetic energy minus potential energy" term, and is mathematically equivalent to a simple pendulum. Note that we are also able to elegantly handle the two capacitors in parallel – the Lagrangian automatically adds them into a single effective capacitor for us. With a Lagrangian at hand, we find the conjugate momentum

$$q = \frac{\partial L}{\partial \dot{\phi}} = (C_1 + C_2)\dot{\phi}, \tag{2.2}$$

and finally the Hamiltonian reads

$$H = \frac{1}{2(C_1 + C_2)}q^2 + \frac{1}{2L}\phi^2, \tag{2.3}$$

where ϕ and q are conjugate variables, and their Poisson bracket is $\{\phi, q\} = 1$. We can quantize the circuit based on these conjugate pairs by imposing canonical commutation relations

$$\left[\hat{\phi}, \hat{q}\right] = \mathrm{i}\hbar, \tag{2.4}$$

where $\hbar$ is the reduced Planck constant, $\hat{\phi}$ is the flux operator, and $\hat{q}$ is the charge operator.

In the example above, we have linear capacitors and inductors. Circuits might also contain a combination of nonlinear and noninvertible capacitive and inductive elements, for example, Josephson junctions (JJs) [74, 75] and quantum phase slips (QPSs) [6, 58, 84, 108]. Treating both of these nonlinear and noninvertible elements in the same circuit is still an open problem: because the energy of the JJs depends on the flux ϕ as $E \sim \cos(2\pi\phi/\phi_0)$, while the energy of the QPSs on the charge q across the element as $E \sim \cos(2\pi q/2e)$, one can prove that no Lagrangian of the circuit with a single

type of variable (flux or charge) exists in general.[1] Here, $\phi_0 = h/2e$ is the fundamental flux quantum, h is the Planck constant, and e is the electron charge. Although for the minimal circuit of one JJ and one QPS included in a loop [see Fig. 2.1(b)], one can immediately write down a Hamiltonian [69]

$$H = -E_Q \cos\left(2\pi \frac{q}{2e}\right) - E_J \cos\left(2\pi \frac{\phi}{\phi_0}\right), \tag{2.5}$$

circuits involving even a few of these elements can involve non-trivial constraints [see Fig. 2.1(c)]. For example, the number of degrees of freedom is not equal to the number of JJs or QPSs; worse, the fluxes and charges across the different elements may not be conjugate pairs. Understanding how to even identify the dynamical degrees of freedom, let alone quantize the circuit, is an open problem. Moreover, modeling circuits that contain both JJs and QPSs is especially important for designing a next-generation of qubits which can realize Gottesman-Kitaev-Preskill-like states with advantageous error-correction properties [69].

2.1.2 Our quantization method

In this paper, we present an alternative approach to circuit quantization. Our approach is inspired by earlier work [17] that links graph theory to circuit quantization. However, rather than using Lagrangian mechanics as the starting point, we instead appeal to the mathematics of symplectic geometry [4, 56, 109], which generalizes textbook Hamiltonian mechanics to more abstract/general settings. We will show that this more abstract perspective elegantly resolves the puzzle of how to choose canonical coordinates, without relying on any assumptions about the constitutive relations of the inductive or capacitive elements in the circuit. Hence, our approach is universal for quantum circuits made out of nonlinear inductive and capacitive elements and capable of resolving the puzzle above.

The key insight of our theory is that the most natural quantization prescription involves building conjugate degrees of freedom out of charge variables on branches (q_e) and flux variables on

[1] A Lagrangian can be defined locally, but not globally, on configuration space. The obstruction to a global Lagrangian is that the Legendre transform is not defined because $\partial^2 L/\partial\dot{\phi}^2$ (or $\partial^2 L/\partial\dot{q}^2$) is not always positive definite.

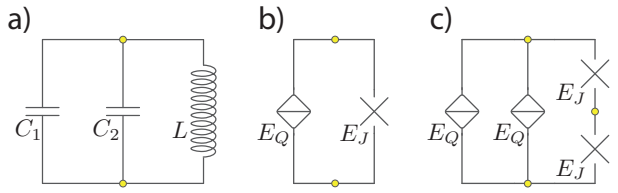


Figure 2.1: **Examples of quantum circuits with Lagrangians using one or two types of generalized coordinates.** (a) An inductor shunted by two capacitors: the Lagrangian of the circuit contains a single type of variable, the flux across the inductor, such that $L(\phi, \dot{\phi})$. The charge-flux conjugate pairs are defined at the Lagrangian level. (b) A Josephson junction and a quantum phase slip junction forming a loop. This quantum circuit can not be described by a Lagrangian using a single type of variable but only with a Lagrangian that contains both charge and flux variables, $L(\phi, \dot{\phi}, q, \dot{q})$. (c) A more complex circuit of multiple Josephson junctions and quantum phase slips, where writing down a Hamiltonian requires geometrical arguments.

nodes (ϕ_v) of the capacitive subgraph of the circuit (see Fig. 2.2). To understand our construction, let us remind the reader how one usually models circuits. The degrees of freedom are voltages V and currents I, which can be integrated in time to give flux ϕ and charge q. In circuits, ϕ and q are canonically conjugate variables, similar to position and momentum: recall Eq. (2.4). Of course, a typical circuit involves multiple elements and therefore multiple flux and charge variables. Due to Kirchhoff's voltage law (the sum of the voltage drops around a loop vanishes in the absence of external magnetic fields), it is natural to define voltages on nodes (or vertices) v of the circuit. On the other hand, currents I are naturally defined on branches (or edges) e in the circuit. In general, there is not a one-to-one mapping between the branches v and nodes e of a circuit. How, therefore, can we possibly find the conjugate pairs?

The mathematical puzzle above is not entirely semantic. In classical and quantum Hamiltonian mechanics, there must be an equal number of position and momentum coordinates. As we described above, the prior resolution in the literature has been to use Lagrangian mechanics to avoid tackling this issue head-on. With the notable exception of Ref. [117, 123], the Lagrangian is written using only one type of variable (for example, branch fluxes), and then the conjugate momenta is found at the Lagrangian level. Any excess in degrees of freedom is dealt with by integrating out non-dynamical variables. However, writing down a Lagrangian as a first step is not always possible. A simple

example is the circuit that we discussed above, the dualmon qubit [69] shown in Fig. 2.1(b). The Hamiltonian in Eq. (2.5) is only known because there is just one dynamical degree of freedom.

What this paper provides is a way of solving the constraints on ϕ and q variables, directly in a Hamiltonian formulation, such that we can find suitable linear combinations of charge and flux variables that are canonically conjugate (and equal in number). As stated above, the approach is inspired by symplectic geometry, together with the simple observation that the equations of motion for a circuit follow from the action

$$S = \int \mathrm{d}t \left[-E_{\mathrm{tot}}(q_e, \phi_v) + \sum_{e,v} q_e \Omega_{ev} \dot{\phi}_v \right]. \tag{2.6}$$

Note that this action involves both branch charges q_e and node fluxes ϕ_v. Such an action may appear unusual from the point of view of textbook Lagrangian mechanics, where normally one writes a Lagrangian in terms of coordinates and velocities, and which will contain more "velocity" terms than the single linear-in-velocity term found in Eq. (2.6). However, as we will thoroughly explain in this manuscript, the form of the Lagrangian in the integrand of the action above is fully analogous to the Lagrangian $L = p\dot{x} - H(x, p)$, whose Euler-Lagrange equations reproduce Hamilton's equations directly. Indeed, the action in Eq. (2.6) manifestly encodes within it the *Hamiltonian mechanics* of the circuit. One can therefore interpret Eq. (2.6) directly within the framework of Hamiltonian mechanics: rather than defining conjugate momenta by differentiating a Lagrangian, we will explain how to directly read off a Poisson bracket and Hamiltonian function directly from (2.6). In particular, the function E_{tot} (upon fixing all constrained variables, which we provide a prescription to do) is the Hamiltonian itself, and equal to the sum of inductive and capacitive energies in the circuit. Critically, the energy of elements are expressed in their native coordinates: inductive elements' energies are expressed in terms of fluxes ϕ_v, while capactive elements' energies are expressed in terms of charges q_e. Furthermore, Ω_{ev} is the *incidence matrix* for the *capacitive subgraph* of the circuit (see Fig. 2.2). $q_e \Omega_{ev} \dot{\phi}_v$ is naturally interpreted using symplectic geometry, and implies both a classical Poisson bracket, and a quantization prescription.

The rest of the paper will derive Eq. (2.6) in Section 2.2, explain how to subsequently quantize circuits in Section 2.3, and then show how to identify the physical degrees of freedom in numerous example circuits in Section 2.4. We have written this paper in a pedagogical and self-contained way; no prior knowledge is required in either circuit quantization or mathematical physics (beyond textbook Lagrangian and Hamiltonian mechanics).

2.2 Classical formalism

We now begin a gentle introduction to the classical mechanics of quantum circuits. Our focus will be to motivate the derivation of Eq. (2.6); however, we provide background knowledge both into the experimental systems and also the mathematics of Hamiltonian mechanics, as is necessary to appreciate Eq. (2.6).

2.2.1 Circuit elements and degrees of freedom

First, we review the definition of branch charges and node fluxes. When describing superconducting circuits, we assume that the circuit elements are connected by perfect superconducting wires without inductive or resistive contributions. We call a part of the circuit that contains a circuit element a branch, and an intersection of the superconducting wires a node. We will soon mathematically describe the circuits as graphs, where branches will be associated to edges in the graph, and nodes as vertices of the graph. We will use the former terminology throughout the paper to conform with the tradition used in quantum circuits. As an homage to the mathematics, however, we will use the letter e to denote a generic branch (edge), and v to denote a generic node (vertex).

Now, consider a two-terminal superconducting circuit element defined between two nodes of a circuit. The node voltage $V_v(t)$ is the voltage at a given node, and the branch current $I_e(t)$ is the current flowing through the circuit element. We assume that it is sufficient to use only the voltage at nodes and the current across the element to describe the system and ignore the current and voltage distribution inside the elements, i. e., we use a lumped element approximation for the circuit.[2] Next,

[2]We will always separate out the inductive and capacitive elements, rather than working with circuit elements

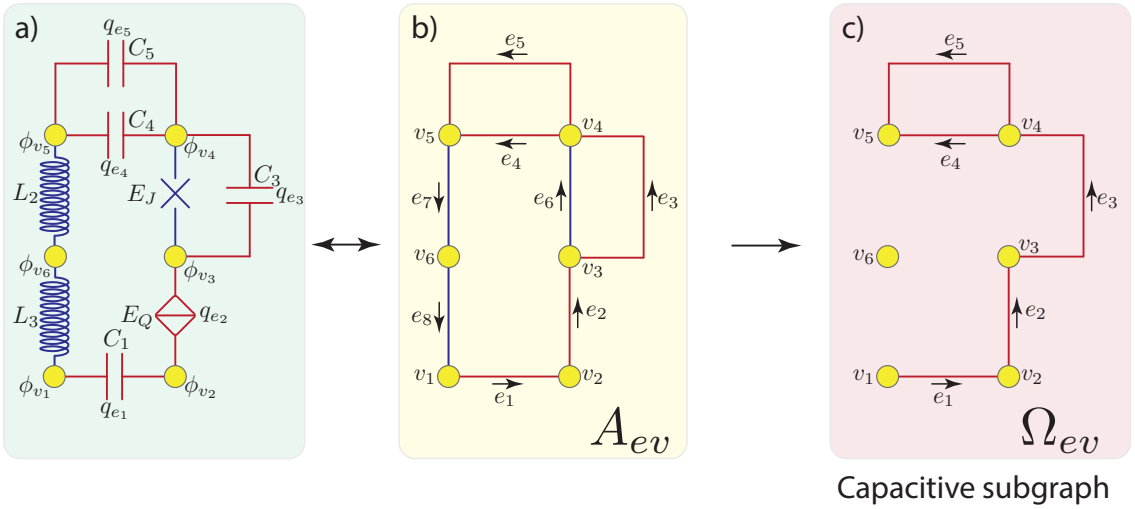


Figure 2.2: **Graph theory and quantum circuits.** (a) An arbitrary circuit containing all four types of superconducting circuit elements: capacitors, inductors, Josephson junctions and quantum phase slips. The node fluxes are ϕ_{v_i}, while the branch charges across the capacitive elements are q_{e_i}. (b) The graph of the full circuit. The vertices of the graph are label as v_i, while the edges are denoted as e_i. Inductive branches are colored with blue lines, while capacitive ones are highlighted with red lines. The incidence matrix of the graph of the full circuit is A_{ev}. (c) The capacitive subgraph of the circuit containing only capacitive edges. The capacitive incidence matrix is Ω_{ev}.

we define the generalized node flux ϕ_v and the branch charges q_e as the time integral of the voltage and the current

$$\phi_v(t) = \int_{-\infty}^{t} \mathrm{d}\tau V_v(\tau), \tag{2.7a}$$

$$q_e(t) = \int_{-\infty}^{t} \mathrm{d}\tau I_e(\tau). \tag{2.7b}$$

It is also common to define the generalized branch fluxes, as the difference between the fluxes of the corresponding nodes; if branch e connects nodes v_i and v_j, the branch flux is

$$\phi_e(t) = \phi_{v_i}(t) - \phi_{v_j}(t). \tag{2.8}$$

The most standard variables in which one performs circuit quantization are ϕ_v or ϕ_e, starting in the Lagrangian formalism. However, such coordinates cannot describe circuits with quantum phase slip elements due to their non-invertible charge-voltage relationship. To address that challenge, loop charges have been also used as an alternative approach to describe charge degrees of freedom in the Lagrangian description, in the special case where the graph of the circuit is planar [117].

The various circuit elements establish different relations between the branch variables and their derivatives [see Fig. 2.3]. For example, linear capacitors and inductors connect linearly two variables

$$\dot{q}_e = \frac{1}{L}\phi_e, \tag{2.9a}$$

$$\dot{\phi}_e = \frac{1}{C}q_e, \tag{2.9b}$$

where C is the capacitance, L is the inductance, and for simplicity, we dropped the explicit time-dependence of the variables in the notations.

whose impedances are complicated functions of frequency. In principle, one sometimes does not wish to do this, but lumping inductors and capacitors together will complicate the quantization prescription in this paper.

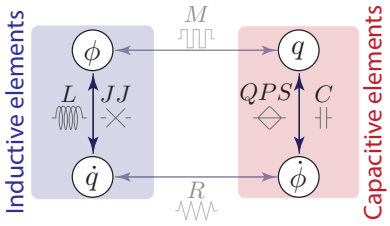


Figure 2.3: **Circuit elements and their constitutive relations.** The circuit elements that we consider in this work connect two variables (flux, charge, and their derivatives). Inductive elements relate flux ϕ and the current $\dot{q}$, such as linear inductors (L) and Josephson junctions (JJ). On the other hand, capacitive elements connect charge q with voltage $\dot{\phi}$, such as linear capacitors (C) and quantum phase slip elements (QPS). The memristor (M) [24] and resistor (R) connect directly charge with flux or their derivatives.

The two most well-known nonlinear and nondissipative superconducting circuit elements are the (multi-channel, low-transmission) Josephson junctions and the quantum phase slip junctions, where the connection between the variables is sinusoidal instead of linear

$$\dot{q}_e = I_C \sin\left(2\pi \frac{\phi_e}{\phi_0}\right), \tag{2.10a}$$

$$\dot{\phi}_e = V_Q \sin\left(2\pi \frac{q_e}{2e}\right). \tag{2.10b}$$

Here, I_C is the critical current of the Josephson junction, and V_Q is the voltage amplitude of the quantum phase slip junction. There are other more complicated relations, e.g. for high-transmission Josephson junctions, containing higher harmonics in the current-phase relationship [8].

In this work, we focus only on non-dissipative circuits; thus, the two types of elements that we are concerned with are the capacitive and inductive elements. The energy stored in the elements can be expressed as

$$E = \int_{-\infty}^{t} \mathrm{d}\tau I_e(\tau) V_e(\tau) = \int_{-\infty}^{t} \mathrm{d}\tau \dot{q}_e(\tau) \dot{\phi}_e(\tau). \tag{2.11}$$

We then see that the energy of the capacitive elements $E_{\mathcal{C}}$ depends only on the branch charge q_e such that $E_{\mathcal{C}} = E_{\mathcal{C}}(q_e)$, while the energy of the inductive elements $E_{\mathcal{I}}$ is a function of only the branch

flux ϕ_e and $E_{\mathcal{I}} = E_{\mathcal{I}}(\phi_e)$. For example,

$$\text{linear capacitor} : E_{\mathcal{C}}(q_e) = \frac{q_e^2}{2C}, \tag{2.12a}$$

$$\text{quantum phase slip} : E_{\mathcal{C}}(q_e) = -E_Q \cos\left(2\pi \frac{q_e}{2e}\right), \tag{2.12b}$$

$$\text{linear inductor} : E_{\mathcal{I}}(\phi_e) = \frac{\phi_e^2}{2L}, \tag{2.12c}$$

$$\text{Josephson junction} : E_{\mathcal{I}}(\phi_e) = -E_J \cos\left(2\pi \frac{\phi_e}{\phi_0}\right). \tag{2.12d}$$

where $E_J = \phi_0 I_C/(2\pi)$ and $E_Q = 2eV_Q/(2\pi)$ are the Josephson and quantum phase slip energies. The derivative of the energy with respect to the coordinates gives the voltage for capacitive elements, and the current for inductive elements

$$V_e(t) = \frac{\partial E_{\mathcal{C}}(q_e)}{\partial q_e}, \tag{2.13a}$$

$$I_e(t) = \frac{\partial E_{\mathcal{I}}(\phi_e)}{\partial \phi_e}. \tag{2.13b}$$

2.2.2 Circuits and graph theory

Electrical network graph theory [97] has played an important role in the existing theory of circuit quantization [11, 17, 30, 118]. Following this approach, a quantum circuit can be considered as a directed graph, where the two-node circuit elements correspond to the edges of the graph, and the vertices of the graph are the points of the circuit where the elements connect. We consider an a particular example of a circuit (see Fig. 2.2) that has k nodes and K branches, and we denote the set of nodes as $\mathcal{V}$ and the set of branches as $\mathcal{E}$. We assume that we do not lump together inductive and capacitive elements, so that we can classify each branch as one or the other. Suppose there are N capacitive branches and $K - N$ inductive branches. The set of capacitive branches is $\mathcal{C} \subset \mathcal{E}$, and the set of inductive branches is $\mathcal{I} \subset \mathcal{E}$; thus, $|\mathcal{E}| = K$, $|\mathcal{C}| = N$, and $|\mathcal{I}| = K - N$.

A key object of the graph is the incidence matrix A_{ev} that provides information on the interconnection of the circuit elements. In particular, the rows/columns of the matrix correspond to

the edges/vertices, and the value of a matrix element is $+1$ (-1) if an edge points toward (from) a vertex, otherwise it is 0:

$$A_{ev} = \begin{cases} 1, & \text{if } e \to v \\ -1, & \text{if } e \leftarrow v \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

By using A_{ev}, we can write down equations of motion in a compact way, which will eventually lead us to our quantization prescription.

To see this, we define the branch charges q_e only on the capacitive edges of the graph, whereas we assign node fluxes ϕ_v to all nodes. We consider circuits without time-dependent external fluxes and gate voltages for now.

Let us first explain why, in fact, the flux variables should naturally live on vertices. Kirchhoff's voltage rule states that in the absence of external magnetic fields around any loop of branches $e_1 \to e_2 \to \cdots \to e_n \to e_1$ of (any) length n:

$$\sum_{j=1}^{n} \phi_{e_j} = 0. \tag{2.15}$$

Observe that if $\phi_e(t) = \phi_v(t) - \phi_u(t)$ whenever $e = u \to v$ [see Eq. (2.8)], then this constraint would automatically be obeyed. Mathematically, one can actually prove that *all* solutions to the constraints in Eq. (2.15) are of this form.[3] It thus makes sense to think of the dynamical variables as the ϕ_v, which are much less constrained, rather than ϕ_es, which obey all of the constraints in Eq. (2.15).

Kirchhoff's current law states that for any node v, the exiting and entering currents are equal

$$\sum_{e:e \text{ entering } v} I_e(t) = \sum_{e:e \text{ exiting } v} I_e(t). \tag{2.16}$$

[3]This uses homology theory, or the abstract form of Stokes' Theorem [85].

Using the incidence matrix A_{ev} defined above, we can write Eq. (2.16) as

$$\sum_e A_{ev} I_e(t) = 0. \tag{2.17}$$

To obtain a more explicit version of Kirchhoff's laws, we need to consider the energy of the various elements. The total inductive energy of the system is

$$E_{\mathcal{I},\text{tot}} = \sum_{e=u\to v\in\mathcal{I}} E_{\mathcal{I},e}(\phi_e) = \sum_{e=u\to v\in\mathcal{I}} E_{\mathcal{I},e}(\phi_v - \phi_u), \tag{2.18}$$

where we have defined $E_{\mathcal{I},e}$ to be the energy function for the inductor on edge e. Similarly, there is energy stored in the capacitive branches:

$$E_{\mathcal{C},\text{tot}} = \sum_{e\in\mathcal{C}} E_{\mathcal{C},e}(q_e), \tag{2.19}$$

and the total energy of the system is

$$E_{\text{tot}} = E_{\mathcal{C},\text{tot}} + E_{\mathcal{I},\text{tot}}. \tag{2.20}$$

Here, we emphasize that the energies can be general functions of the charge branch variables or node flux variables. Note also that E_{tot} is not necessarily a Hamiltonian function, since the variables ϕ_v and q_e are not conjugate in any obvious way. We will explain how to obtain a Hamiltonian from E_{tot} by the end of the section.

Now, we describe the equations of motion, which provide the glue between the charge and flux variables, and link time derivatives of ϕ_v and q_e to the energies above. First, if $e = u \to v \in \mathcal{C}$ is a pair of nodes that are connected through a capacitive branch e, the voltage between the nodes equals to the voltage drop across the connecting capacitors. Using (2.13a), we find

$$\dot{\phi}_v - \dot{\phi}_u - \frac{\partial E_{\mathcal{C},e}(q_e)}{\partial q_e} = 0 \quad \text{for all } e = u \to v \in \mathcal{C}. \tag{2.21}$$

Second, at each node, we apply Kirchhoff's current law. For a capacitive edge we have $I_e = \dot{q}_e$, while for an inductive edge Eq. (2.13b) implies that

$$I_e - \frac{\partial E_{\mathcal{I},e}(\phi_v - \phi_u)}{\partial \phi} = 0 \quad \text{for all } e = u \to v \in \mathcal{I}. \tag{2.22}$$

Hence we arrive at our second equations of motion upon plugging in to Eq. (2.17)

$$\sum_{e=u' \to v' \in \mathcal{I}} A_{ev} \frac{\partial E_{\mathcal{I},e}(\phi_{v'} - \phi_{u'})}{\partial \phi} + \sum_{e \in \mathcal{C}} A_{ev} \dot{q}_e = 0 \quad \text{for all } v \in \mathcal{V}. \tag{2.23}$$

Observe that these equations of motion follow from the principle of least action applied to the Lagrangian

$$L = -E_{\text{tot}}(q_e \text{ for } e \in \mathcal{C}, \phi_v \text{ for } v \in \mathcal{V}) + \sum_{e \in \mathcal{C}, v \in \mathcal{V}} q_e \Omega_{ev} \dot{\phi}_v, \tag{2.24}$$

where Ω_{ev} is the incidence matrix whose rows correspond to *capacitive edges only*, and columns correspond to *all vertices*. We will explain how to interpret such Lagrangians as encoding a Hamiltonian dynamical system in Section 2.2.3. While entry-wise $\Omega_{ev} = A_{ev}$, we use the Ω_{ev} notation to emphasize that now we only care about *capacitive edges*. E_{tot} is simply the energy of all circuit elements. Indeed, Eq. (2.21) comes from the Euler-Lagrange equation of motion of ϕ_v, while Eq. (2.23) comes from the equation of motion of q_e. This may seem like a miracle! But we hope that by the end of this paper, the reader will walk away thinking that Eq. (2.24) is in fact *the most natural Lagrangian for a circuit*. Firstly, it elegantly allows us to encode ϕ_v as degrees of freedom on nodes, while q_e are degrees of freedom on branches. Secondly, and much more importantly, it also encodes within it the universal prescription for circuit quantization.

As we describe in the upcoming sections, it is possible to remove non-dynamical degrees of freedom relying on the geometrical properties of the capacitive incidence matrix Ω_{ev}. After removing those variables, we arrive at a set of q_i and ϕ_j variables that are the linear combination of the original charge branch and flux node variables, where, crucially, the number of charge and flux variables are

equal. With these variables, the Lagrangian reads

$$L = -H(q_i, \phi_j) + \sum_{i,j} q_i \tilde{\Omega}_{ij} \dot{\phi}_j, \tag{2.25}$$

where the total energy term corresponds to the Hamiltonian function $H(q_i, \phi_j) = E_{\text{tot}}(q_i, \phi_j)$ and the second term in the Lagrangian contains a square invertible matrix $\tilde{\Omega}_{ij}$ that determines symplectic matrix of the circuit. In fact, we will show in Section 2.2.5 how to choose variables wherein $\tilde{\Omega}_{ij}$ is the identity matrix.

2.2.3 From Hamiltonian mechanics to symplectic geometry

To understand how Eq. (2.25) leads us to circuit quantization, we need to take a step back and comment on a crucial analogy. Consider for the moment the classical mechanics textbook problem of an object with N degrees of freedom with coordinates $(x^1, \ldots, x^N)$ and canonically conjugate momenta $(p^1, \ldots, p^N)$ that is described by a Hamiltonian $H(x^i, p^i)$. In this section we will use raised or lowered indices to emphasize the connections with mathematics: raised indices correspond to vector fields on phase space, while lowered indices correspond to differential forms (this is historically known as contravariant vs. covariant vectors). Observe that if we define the Lagrangian as

$$L = -H(x^i, p^i) + \sum_{i=1}^{N} p^i \dot{x}^i, \tag{2.26}$$

and the action as $S = \int \mathrm{d}t\, L$, the Euler-Lagrange equations reproduce Hamilton's equations:

$$0 = \frac{\delta S}{\delta p^i} = \frac{\partial L}{\partial p_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{p}_i} = \dot{x}^i - \frac{\partial H}{\partial p^i}, \tag{2.27a}$$

$$0 = \frac{\delta S}{\delta x^i} = \frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} = -\dot{p}^i - \frac{\partial H}{\partial x^i}. \tag{2.27b}$$

Now, suppose that we have an invertible matrix M_{ij}, and we define a "Lagrangian" such that

$$L = -H(x^i, p^i) + \sum_{i,j=1}^{N} p^i M_{ij} \dot{x}^j. \tag{2.28}$$

Again we find a kind of Hamiltonian mechanics, but with a slightly modified form of Hamilton's equations. Denoting the elements of the matrix inverse M^{-1} with raised indices so that

$$\sum_{j=1}^{N} M^{ij} M_{jk} = \sum_{j=1}^{N} M_{kj} M^{ji} = \delta^i_k, \tag{2.29}$$

we find that

$$0 = \dot{x}^i - \sum_{j=1}^{N} M^{ij} \frac{\partial H}{\partial p^j}, \tag{2.30a}$$

$$0 = -\dot{p}^i - \sum_{j=1}^{N} \frac{\partial H}{\partial x^j} M^{ji}. \tag{2.30b}$$

In simple terms, the M matrix tells us that the canonical conjugate variables are not p^i and x^i, but rather p^i and $\sum_j M_{ij} x^j$.

If we define the Poisson bracket for two functions f and g as

$$\{f, g\} = \sum_{i,j=1}^{N} M^{ij} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p^j} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial p^j} \right), \tag{2.31}$$

then Hamilton's equations can be re-written for an arbitrary function f as

$$\dot{f} = \{f, H\}. \tag{2.32}$$

As we highlight in Section 2.3, such theories can be quantized in a general way, subject to certain physical assumptions[4]. Importantly, with a few caveats, our classical Lagrangian for a circuit, which

[4] In the case of circuits, one such assumption is the existence of a unique solution to Kirchhoff's laws. It is further simplifying to consider only phase spaces $\mathbb{R}^{2n}$ for some n. Quantization can be possible in principle even without these properties, but less straightforwardly.

is $L = -H(q_i, \phi_j) + \sum_{i,j} q_i \tilde{\Omega}_{ij} \dot{\phi}_j$ has precisely the form of Eq. (2.28).

It is helpful to re-formulate the previous paragraph in a more abstract language. Let us collect the position and momentum coordinates into a single variable $\xi^I = (x^1, \ldots, x^N, p^1, \ldots, p^N)$, and define the matrix

$$\omega_{IJ} = \begin{pmatrix} 0 & M_{ij} \\ -M_{ji} & 0 \end{pmatrix}. \tag{2.33}$$

Note that $\omega_{IJ} = -\omega_{JI}$ is antisymmetric, invertible, in our special case a constant, and it is called the *symplectic form.*[5] Mathematicians define Hamiltonian mechanics in terms of a Hamiltonian function H, which generates time evolution via the Poisson brackets in Eq. (2.32), and a symplectic form ω. Defining the inverse of ω_{IJ} as ω^{IJ} as before:

$$\sum_{J=1}^{2N} \omega^{IJ} \omega_{JK} = \sum_{J=1}^{2N} \omega_{KJ} \omega^{JI} = \delta^I_K, \tag{2.34}$$

we can re-write the Poisson bracket as

$$\{f, g\} = \sum_{I,J=1}^{2n} \frac{\partial f}{\partial \xi^I} \omega^{IJ} \frac{\partial g}{\partial \xi^J}. \tag{2.35}$$

The pair of a manifold with coordinates ξ^I and symplectic form ω is called a *symplectic manifold*. Such a symplectic manifold is required for a notion of Hamiltonian mechanics to exist [4, 109]. Importantly however, *any* symplectic manifold gives rise to the structures of Hamiltonian mechanics – even ones where there are no global canonical conjugate pairs of coordinates. The mathematical theory of geometric quantization [121] shows that there is a way to quantize all such systems.

[5] More formally, a symplectic form is required to be a closed, non-degenerate 2-form. To the geometer, our ω_{IJ} easily satisfies these criteria.

The key point is that “Lagrangians" of the form of Eq. (2.28),

$$L = -H(x^i, p^i) + \sum_{i,j=1}^{N} p^i M_{ij} \dot{x}^j, \tag{2.36}$$

which are similar to our circuit Lagrangian in Eq. (2.24),

$$L = -H(q_i, \phi_j) + \sum_{i,j} q_i \tilde{\Omega}_{ij} \dot{\phi}_j, \tag{2.37}$$

are immediately understood in the language of Hamiltonian mechanics and symplectic geometry. In particular, we simply *read out the Hamiltonian function* $H(q_i, \phi_j)$ *and a Poisson bracket* $\{\phi_i, q_j\} = \tilde{\Omega}^{-1}_{ji}$, which allows us to elegantly transition from classical to quantum mechanics.

2.2.4 Symplectic geometry of a circuit

After having reviewed the basics of symplectic geometry, we return to the question of how to construct the Hamiltonian $H(q_i, \phi_j)$ function of an arbitrary circuit from its total energy $E_{\text{tot}}(q_e, \phi_v)$ and the connectivity of the elements Ω_{ev}. In this section, we focus on the general approach, while in Sec. 2.4 we provide examples. A mathematically precise discussion, with proofs of all claims and careful definitions is relegated to Appendix A.1.

To begin, it is important to notice that the incidence matrix Ω_{ev} appearing in the Lagrangian of the circuit [see Eq. (2.24)] is not invertible. This is in contrast to the definition of the symplectic matrix, which is constructed from an invertible matrix M_{ij} [see Eq. (2.28)]. Thus, at this point, it is not possible to carry out a Legendre transformation to arrive from the Lagrangian to a Hamiltonian with conjugate flux and charge pairs. The root of the problem is that we overcounted the degrees of freedom the way we constructed the Lagrangian. However, as we prove in Appendix A.1 and summarize in Section 2.2.5, we can consistently and efficiently remove variables associated with constraints to obtain an invertible matrix (and a symplectic form) from the incidence matrix. Crucially, this procedure only depends on the geometrical structure of the capacitive subgraph of

the circuit: the locations of inductively shunted islands and capacitive loops (see Fig. 2.4).

There are three general methods to reduce the number of variables in our approach, which we highlight here (and give examples of in Sec. 2.4). Firstly, we may find a left null vector, l_e, of the incidence matrix: a linear combination of the branches such that

$$\sum_e l_e \Omega_{ev} = 0. \tag{2.38}$$

Geometrically, these null vectors correspond to loops in the circuit, where all branches in a loop have capacitive elements on them (see proof in Appendix A.1). Physically, the Euler-Lagrange equations for these null vectors lead to the physical constraint that the voltages in such a **capacitive loop** vanishes

$$\sum_e l_e \frac{\partial E_C}{\partial q_e} = 0. \tag{2.39}$$

This constraint fixes one of the charge variables in terms of the others in the loop. We denote the set of such capacitive loops as Δ_C. For a loop $Z \in \Delta_C$, the form of the null vectors is

$$l_e = \begin{cases} \pm 1 & e \in Z \\ \\ 0 & e \notin Z \end{cases}. \tag{2.40}$$

The ± 1 sign is based on the orientation of the edges in the loop: all signs are $+1$ when the edges are all oriented so they touch tip-to-tail: see the examples for more details in Sec. 2.4.

Secondly, the right null vectors r_v of Ω_{ev} also imply constraints. These are the combinations of nodes such that

$$\sum_v \Omega_{ev} r_v = 0. \tag{2.41}$$

The geometrical meaning of these vectors is that they represent **inductively shunted islands**. The constraint associated with these vectors is that the total current entering the island must equal

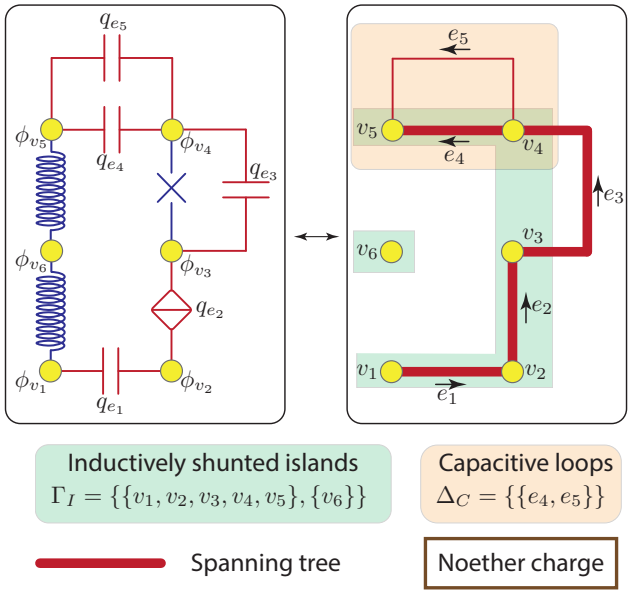


Figure 2.4: **Symplectic geometry of a circuit.** When describing a circuit with its capacitive graph, we can define two types of geometrical objects that correspond to null vectors of the capacitive incidence matrix of the circuit: inductively shunted islands (green-filled rectangular), and capacitive loops (orange-filled rectangular). The variables associated with these null vectors must be removed to be able to quantize the circuit. Additional variables can be removed based on the Noether charges of the circuit. The edges that are part of a (non-unique) spanning tree are highlighted with wide lines.

the current exiting the island. In this case, the Euler-Lagrange equations read

$$\sum_v \frac{\partial E_{\mathcal{I}}}{\partial \phi_v} r_v = 0. \tag{2.42}$$

We denote the set of all subsets of vertices that correspond to such inductively shunted islands as $\mathit{\Gamma}_I$. Note that for any such island $J \in \mathit{\Gamma}_I$, we have $J \subseteq \mathcal{V}$. The explicit form of the null vector r_v becomes

$$r_v = \begin{cases} 1 & v \in J \\ \\ 0 & v \notin J \end{cases}. \tag{2.43}$$

These left and right null vectors correspond to non-dynamical variables that can be removed from the Lagrangian (see Appendix A.2). After removing the variables associated with the left and right null vectors of Ω_{ev}, the Lagrangian will have fewer coordinates. The linear combinations of coordinates which remain are the non-null vectors of Ω_{ev}, which becomes a non-degenerate matrix $\tilde{\Omega}_{ij}$ in the subspace of remaining modes. The total energy E_{tot}, restricted to the corresponding constrained subspace, is the Hamiltonian H for the circuit. Hence, we can find a symplectic form for the remaining coordinates, and a Hamiltonian function to quantize.

Removal of the left and right null vectors of Ω does not in general compromise our ability to quantize a circuit.[6] One can explicitly show that an effective Hamiltonian persists after "integrating out" the non-dynamical variables. In particular, because the non-dynamical variables (by construction) do not enter $q_e \Omega_{ev} \dot{\phi}_v$, they do not affect the symplectic form on phase space, which remains well-defined. As we explain in Appendix A.2, replacing the Hamiltonian with an effective Hamiltonian where constrained variables are fixed by their equations of motion does not compromise the Hamiltonian structure of the theory, so long as the constraints have a unique solution. If the constraints do not have *any* solution, it may be the case that the circuit itself is unphysical: for example, a loop of capacitive elements is placed into the circuit and Kirchoff's voltage law cannot be

[6] This is because for physical circuits (at least that we are aware of) the classical phase space is always $\mathbb{R}^{2n}$ or a quotient thereof (cylinder, torus, etc.).

satisfied by the constitutive relations along that loop..[7] It the constraints have multiple solutions, the circuit is singular; we discuss this case in Section 2.4.7.

Let us now detail a few simple ways to remove degrees of freedom. Suppose that, as in Fig. 2.4, there is a vertex (or more generally, a set of vertices) that are only connected to the rest of the circuit via capacitive edges. For simplicity here, let us focus on the case where, as in Fig. 2.4, it is a single vertex v_2. Then, the Lagrangian L in Eq. (2.24), and therefore the Hamiltonian H, is invariant under constant shifts in ϕ_{v_2}:

$$H(\phi_{v_2}) = H(\phi_{v_2} + c). \tag{2.44}$$

Noether's Theorem states that such continuous symmetry allows one to remove one dynamical variable (i.e. one q and one ϕ) from the problem.[8] Removal of such a degree of freedom is contingent on the assumption that any external probes one couples to the circuit will not depend on this "Noether charge", or couple it to the dynamical degrees of freedom.

We remark that every vertex is in *some* inductively shunted island, so

$$0 = \sum_{J \in \Gamma_I} \sum_{v \in J} r_v \frac{\partial E_{\mathcal{I}}}{\partial \phi_v} = \sum_{v \in \mathcal{V}} \frac{\partial E_{\mathcal{I}}}{\partial \phi_v}. \tag{2.45}$$

This rather trivial expression has an important physical interpretation: for every circuit, there is always at least one node degree of freedom that is nondynamical. In particular, we may fix

$$\sum_v \phi_v = c \tag{2.46}$$

to any desired value c without physical consequence. This is a gauge degree of freedom, which is often leveraged by choosing a "ground node" that takes on a value of zero at all times.

[7] This could arise if each element is a battery, e.g.

[8] More formally, there exists a quotient of the classical phase space (intuitively amounting to ignoring the dynamics of the conserved quantity and its conjugate) which remains a symplectic manifold. In the mathematical literature, this is known as the Marsden-Weinstein-Meyer Theorem, and the method of symplectic reduction.

2.2.5 Choosing canonically conjugate variables

At this point, we can formally attempt to quantize the theory, as we describe in Sec. 2.3, by replacing Poisson brackets with quantum commutators. But, as we will see when quantizing the theory, it is desirable to find n pairs of "canonical coordinates":

$$\{x_i, p_j\} = \delta_{ij}. \tag{2.47}$$

This is because, in quantum mechanics, Poisson brackets become quantum mechanical commutators. However, when calculating the Poisson brackets using Eq. (2.31) in our formalism, we find that the Poisson brackets correspond to the element of the symplectic matrix

$$\{\phi_j, q_i\} = \tilde{\Omega}^{-1}_{ij}. \tag{2.48}$$

This does not jeopardize our ability to quantize the circuit, but it is still desirable to find coordinates where Eq. (2.47) holds. In this section, we show how to find charge and flux variables that achieve this. For simplicity, we will focus on an example presented in Fig. 2.4, and relegate the general argument to Appendix A.1.

Our argument is exclusively about the second term in the Lagrangian of the system in Eq. (2.6), which reads as $\sum_{e,v} q_e \Omega_{ev} \dot{\phi}_v$. We aim to find a set of (Φ_i, Q_i) variables for which this term takes the form of $\sum_i Q_i \dot{\Phi}_i$. As we discussed before in Sec. 2.2.4, in part this will mean removing all left and right null vectors. Remarkably, in the construction that follows, these null vectors will be automatically removed.

First, we choose a spanning tree $\mathcal{T} \subseteq \mathcal{C}$ of the capacitive subgraph. Here a spanning tree corresponds to a set of capacitive branches $\mathcal{T} \subset \mathcal{C}$ so that every node adjacent to some branch in $\mathcal{C}$ is adjacent to some branch in $\mathcal{T}$, but without any cycles. Schematically, one can manufacture a spanning tree by simply choosing an edge to delete from every cycle in Δ_C. For example in Figure

2.4, we can take the spanning tree to be

$$\mathcal{T} = \{e_1, e_2, e_3\}. \tag{2.49}$$

An alternative choice is $\{e_1, e_2, e_4\}$, and the choice made does not affect the spectrum or dynamics of the resulting circuit (the resulting Hamiltonians differ by a canonical transformation). Recalling the definition of branch flux in Eq. (2.8), we define branch fluxes Φ_f for $f \in \mathcal{T}$ as our fundamental degrees of freedom. One can explicitly show that

$$\sum_{e\in\mathcal{C}} \sum_{v\in\mathcal{V}} q_e \Omega_{ev} \dot{\phi}_v = \sum_{f\in\mathcal{T}} Q_f \dot{\Phi}_f, \tag{2.50}$$

where Q_f is a linear combination of the original q_e variables with integer coefficients $0, \pm 1$. In our example, we find

$$\sum_{e\in\mathcal{C}} \sum_{v\in\mathcal{V}} q_e \Omega_{ev} \dot{\phi}_v = q_{e_1}\dot{\Phi}_{e_1} + q_{e_2}\dot{\Phi}_{e_2} + (q_{e_3} + q_{e_4})\dot{\Phi}_{e_3} = Q_{e_1}\dot{\Phi}_{e_1} + Q_{e_2}\dot{\phi}_{e_2} + Q_{e_3}\dot{\Phi}_{e_3}. \tag{2.51}$$

We will show how to use this procedure in the additional examples of Sec. 2.4.

Observe that in this construction, we have immediately removed one linear combination of node fluxes on each disconnected subgraph of $\mathcal{C}$. In Fig. 2.4, we can see the following right null vectors are no longer dynamical degrees of freedom: ϕ_{v_6}, $\phi_{v_1} + \phi_{v_2} + \phi_{v_3}$, $\phi_{v_4} + \phi_{v_5}$. The three branch flux variables on the spanning tree are linearly independent to these non-dynamical modes. Similarly, the linear combinations of charges that are removed are the unphysical ones corresponding to charges flowing around capacitive loops. In our example, this combination of branch charges $q_{e_3} - q_{e_4}$ is orthogonal to the physical degree of freedom $Q_{e_3} = q_{e_3} + q_{e_4}$ that arose in Eq. (2.51). Happily, all the needed left and right null vectors of Ω_{ev} are automatically removed by this "spanning tree construction" of choosing good coordinates.

2.3 Circuit quantization

With the understanding of how to use our formalism to describe arbitrary non-dissipative circuits at the classical level, we now discuss how to carry out circuit quantization. Suppose that the circuit is described by the Lagrangian

$$L = -H(q_i, \phi_j) + \sum_{i,j} q_i \tilde{\Omega}_{ij} \dot{\phi}_j = -H(Q_f, \Phi_f) + \sum_{f \in \mathcal{T}} Q_f \dot{\Phi}_f. \tag{2.52}$$

Note that we have used the spanning tree construction of Sec. 2.2.5 to choose good variables to quantize. The equations of motion for the non-dynamical coordinates are constraints that should also be solved before quantization. Let us first suppose that, after such constraints have been implemented, the dynamical variables Q_f and Φ_f are real-valued, and the classical phase space is $\mathbb{R}^{2n}$, if the number of conjugate pairs Q_f and Φ_f is n.

Referring to the definition of the Poisson brackets in Eq. (2.31), we see that

$$\{\Phi_i, Q_j\} = \delta_{ij}. \tag{2.53}$$

To quantize the circuit, we define commutation relations between the charge operator $\hat{Q}_i$ and the flux operator $\hat{\Phi}_j$ as

$$[\hat{\Phi}_i, \hat{Q}_j] = \mathrm{i}\hbar\delta_{ij}, \tag{2.54}$$

as long as both $\hat{Q}_i$ and $\hat{\Phi}_j$ are non-compact (i.e., not periodically identified) variables.

The quantum mechanical Hamiltonian is simply $\hat{H}(\hat{Q}_i, \hat{\Phi}_i)$, where as in the classical setting, we must first restrict to the constrained subspace by solving for left/right null vectors of Ω_{ev}. Since in our theory, all circuit elements are purely capacitive or purely inductive, there is no ambiguity about the operator ordering of non-commuting $\hat{Q}_i$ and $\hat{\Phi}_i$, so $\hat{H}$ is a uniquely specified operator. This completes our formulation of circuit quantization for non-dissipative circuits.

Such a simple and intuitive solution to circuit quantization is possible because we are able to

find a globally constant Poisson bracket on the classical phase space. There do exist Hamiltonian systems outside of the scope of this paper where this task cannot be achieved.[9] The most notable property of our quantization procedure, and our formalism on the whole, is that Eq. (2.54) is agnostic to the form of the Hamiltonian; it depends only on the capacitive subgraph of the circuit.

We now revisit our assumption that the classical phase space was $\mathbb{R}^{2n}$. When considering circuits, it may be the case that some of the flux coordinates are periodically identified:

$$\Phi_i \sim \Phi_i + \phi_0, \tag{2.55}$$

where ϕ_0 is the flux quantum. For example, it is generally assumed that flux across a Josephson junction shunted by a capacitor is periodic.[10] It is well-known [19] that in such circuits, Φ_i is not a well-defined operator; the well-defined operators become $\exp[2\pi \mathrm{i}\Phi/\phi_0 \cdot n]$ for integer n. Our quantization prescription does not change in this scenario: one simply avoids writing Eq. (2.54) and instead writes

$$[\mathrm{e}^{2\pi \mathrm{i}\hat{\Phi}/\phi_0}, \hat{Q}/2e] = -\mathrm{e}^{2\pi \mathrm{i}\hat{\Phi}/\phi_0}, \tag{2.56}$$

which is now expressed in terms of globally-defined operators.

Let us remark on what has transpired from a mathematical perspective. For simplicity let us assume a single dynamical Q and Φ variables; the argument immediately generalizes to the higher dimensional case. The original classical phase space is $M = \mathbb{R}^2$. The periodic identification of Φ corresponds to identifying points in phase space when the Φ coordinates are related as in Eq. (2.55). At the classical level, this turns the phase space into $\mathbb{R} \times \mathrm{S}^1$, where $Q \in \mathbb{R}$ and $\Phi \in \mathrm{S}^1$. Here $\Phi \in \mathrm{S}^1$ lives on a circle, which is equivalent to the real line with all points shifted by ϕ_0 identified. Because the manifold $\mathbb{R} \times \mathrm{S}^1$ is a non-singular quotient of $\mathbb{R}^2$, there exists [70] a symplectic form ω on $\mathbb{R} \times \mathrm{S}^1$ which is equal to the inclusion of the original symplectic form on $\mathbb{R}^2$. In more physical terms, this means that we can use the same commutation relations to quantize the reduced phase space, provided

[9] A simple example is a system whose classical phase space is the sphere. This system is quantizable and gives a single spin-j Hilbert space of dimension $2j+1$ [121], when the volume of the sphere is $2\pi\hbar(2j+1)$.

[10] However, there are other approaches that assume no periodicity in this case [114].

we only study well-defined functions as in Eq. (2.56). Note that in quantum mechanics, Φ becoming periodic means that Q becomes integer-valued.

In this paper, we study classical phase spaces that are quotients of $\mathbb{R}^{2n}$ by periodically identifying Φ coordinates only. The framework of geometric quantization may prove important if one can build circuits where both a Q and Φ degree of freedom should be periodically identified at the classical level. We leave this intriguing possibility to future work. The mathematics of geometric quantization for torus phase spaces can then become relevant [121].

2.4 Examples and generalizations

In this section, we provide examples of how to use our formalism to efficiently derive a quantizable Hamiltonian for various circuits. We emphasize that many of these examples have already been studied using other methods; the purpose of this section is to demonstrate that the formalism we have developed, from a rather different starting point, can both reproduce and ultimately extend the existing methods in the literature.

2.4.1 Inductively and capacitively shunted islands

As a first example to understand how we can eliminate variables associated with unphysical degrees of freedom, we consider an inductively shunted island. An inductively shunted island contains a set of nodes that lie on a path consisting of only capacitive branches or a single node that is connected only to inductive elements. For formal definitions and other relevant discussions, see Appendix A.1. As we discussed before, an inductively shunted island corresponds to a right null vector of the incidence matrix Ω_{ev}. Figure 2.5(a) shows an example of a circuit that has two inductively shunted islands: a node is connected to two inductors, L_1 and L_2, and a quantum phase slip element with energy E_Q is also shunted by the inductors. The circuit has three node variables but using the constraints for right null vectors [see Eq. (2.42)], we can eliminate two flux variables.

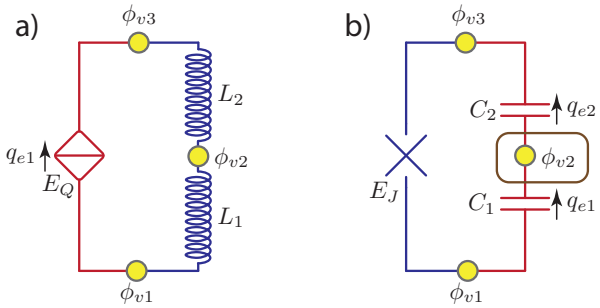


Figure 2.5: **Inductively and capacitively shunted islands.** (a) The inductively shunted island at node flux ϕ_{v_2} corresponds to the right null vector of the incidence matrix Ω_{ev}; thus, we need to remove such variable to be able to arrive to a self-consistent Hamiltonian. (b) The capacitively shunted island at ϕ_{v_2} corresponds to a Noether charge in the circuit. It is possible but not required to remove such a variable to be able to define conjugate pairs.

To start, we write down the incidence matrix of the capacitive subgraph

$$\Omega = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}, \tag{2.57}$$

where the single row corresponds to the q_{e_1} branch charge, and the three columns refer to the three flux node variables. The capacitive and the inductive energies are

$$E_{\mathcal{C}} = -E_Q \cos\left(2\pi \frac{q_{e_1}}{2e}\right), \tag{2.58a}$$

$$E_{\mathcal{I}} = \frac{1}{2L_1}\left(\phi_{v_1} - \phi_{v_2}\right)^2 + \frac{1}{2L_2}\left(\phi_{v_2} - \phi_{v_3}\right)^2. \tag{2.58b}$$

Thus, based on Eq. (2.6) the Lagrangian is

$$\begin{aligned} L &= \sum_{e,v} q_e \Omega_{ev} \dot{\phi}_v - E_{\mathcal{I}} - E_{\mathcal{C}} \\ &= q_{e_1}(\dot{\phi}_{v_3} - \dot{\phi}_{v_1}) + E_Q \cos\left(2\pi \frac{q_{e_1}}{2e}\right) - \frac{1}{2L_1}\left(\phi_{v_1} - \phi_{v_2}\right)^2 - \frac{1}{2L_2}\left(\phi_{v_2} - \phi_{v_3}\right)^2. \end{aligned} \tag{2.59}$$

We notice that there are two inductively shunted islands and hence two right null vectors

$$\Gamma_I = \{\{v_1, v_3\}, \{v_2\}\} \longleftrightarrow r_v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{2.60}$$

indicating that $\phi_{v_1} + \phi_{v_3}$ and ϕ_{v_2} are non-dynamical variables. Furthermore, based on the constraints imposed by the right null vectors[11] [see Eq. (2.42)], we can write that

$$\frac{\phi_{v_1} - \phi_{v_2}}{L_1} + \frac{\phi_{v_3} - \phi_{v_2}}{L_2} = 0. \tag{2.61}$$

[11]The constraints provided by each of the two null vectors are the same in this example, but need not be in general.

After some algebra, we can simplify the Lagrangian such as

$$L = Q\dot{\Phi} + E_Q \cos\left(2\pi\frac{Q}{2e}\right) - \frac{1}{2(L_1+L_2)}\Phi^2, \tag{2.62}$$

where $\Phi = \phi_{v_3} - \phi_{v_1}$, and $Q = q_{e_1}$. We can see that this geometrical method reproduced the well-known result of how to add inductors together. The system is left with one degree of freedom, and the symplectic form (the first term in the Lagrangian) indicates that the conjugate variables are $\{\Phi, Q\} = 1$. Finally, the Hamiltonian is

$$H = -E_Q \cos\left(2\pi\frac{Q}{2e}\right) + \frac{1}{2(L_1+L_2)}\Phi^2. \tag{2.63}$$

We remark that the conjugate pairs in this example were necessarily Q and Φ because there is only one capacitive branch and thus it must have been included in any spanning tree.

As a second example, we consider a capacitively shunted island, for example, a node between two capacitors [see Fig. 2.5(b)]. A capacitively shunted island is a set of vertices that can be traversed by moving only along branches with inductive elements. As before, a node connected only to capacitors constitutes its own island. In our formalism, the presence of such an island does not lead to a null vector of the adjacency matrix. Thus, removing such variables is not necessary to define a symplectic form and to carry out quantization (see Appendix A.3 for an example). However, we can remove such degrees of freedom since capacitively shunted islands correspond to Noether currents, which represent an additional constraint. Physically, this constraint corresponds to Kirchhoff's current law, i. e., the current through a network of capacitors is conserved.

In this example [see Fig. 2.5(b)], the circuit contains a Josephson junction and two capacitors in a loop; the Lagrangian describing this circuit is given by Eq. (2.6)

$$L = q_{e_1}(\dot{\phi}_{v_2} - \dot{\phi}_{v_1}) + q_{e_2}(\dot{\phi}_{v_3} - \dot{\phi}_{v_2}) + E_J \cos\left(2\pi\frac{\phi_{v_1} - \phi_{v_3}}{\phi_0}\right) - \frac{1}{2C_1}q_{e_1}^2 - \frac{1}{2C_2}q_{e_2}^2. \tag{2.64}$$

Our spanning tree[12] construction provides for us the fact that the variables

$$\begin{aligned} Q_{e_1} &= q_{e_1}, \\ Q_{e_2} &= q_{e_2}, \\ \Phi_{e_1} &= \phi_{v_2} - \phi_{v_1}, \\ \Phi_{e_2} &= \phi_{v_3} - \phi_{v_2} \end{aligned} \tag{2.65}$$

form canonical conjugate pairs with

$$\{\Phi_{e'}, Q_e\} = \delta_{ee'}. \tag{2.66}$$

Written in terms of these variables, the Lagrangian is

$$L = Q_{e_1}\dot{\Phi}_{e_1} + Q_{e_2}\dot{\Phi}_{e_2} + E_J \cos\left(2\pi \frac{\Phi_{e_1} + \Phi_{e_2}}{\phi_0}\right) - \frac{1}{2C_1}Q_{e_1}^2 - \frac{1}{2C_2}Q_{e_2}^2. \tag{2.67}$$

At this point, the circuit can be quantized. However, we can remove one more variable by noticing that the constraint due to the Noether current is

$$\frac{\delta S}{\delta \phi_{v_2}} = \dot{q}_{e_1} - \dot{q}_{e_2} = \dot{Q}_{e_1} - \dot{Q}_{e_2} = 0. \tag{2.68}$$

This is easy to understand as simply the conservation of charge on the two inner plates connecting the capacitors. A nonzero constant of integration would only represent a time-independent charge trapped between the plates therein. In this case, we can write

$$Q_{e_1}\dot{\Phi}_{e_1} + Q_{e_2}\dot{\Phi}_{e_2} = Q(\dot{\Phi}_{e_1} + \dot{\Phi}_{e_2}), \tag{2.69}$$

where we redefine $Q = Q_{e_1} = Q_{e_2}$ and $\Phi = \Phi_{e_1} + \Phi_{e_2}$ with $\{\Phi, Q\} = 1$. This choice is just a reflection

[12] We remark that, again, the choice of spanning tree on this circuit is unique.

of the fact that a free particle is “integrated out" by using Eq. (2.68). Thus, the Hamiltonian reads

$$H = \frac{1}{2}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) Q^2 - E_J \cos\left(2\pi \frac{\Phi}{\phi_0}\right). \tag{2.70}$$

In this way, we see that Noether charges provide instructions on how to add capacitive circuit elements in series.

2.4.2 The dualmon qubit

We continue the series of examples with the circuit that motivated our discussion, the dualmon circuit [69]. In this device, a Josephson junction and a quantum phase slip element form a loop [see Fig. 2.6(a)]. In the following, we analyze the circuit in the absence of offset charges and external fluxes. Later, we show how these external parameters can be added to our formalism.

First, we define the flux variables at the two nodes, ϕ_{v_1} and ϕ_{v_2}, and the branch charge across the capacitive element, q_{e_1}. The incidence matrix of the capacitive subgraph is simply

$$\Omega = \begin{pmatrix} -1 & 1 \end{pmatrix}. \tag{2.71}$$

The circuit has one inductively shunted island, and no capacitive loop, thus the nullvectors are

$$\Gamma_I = \{\{v_1, v_2\}\} \longleftrightarrow r_v = \begin{pmatrix} 1 \\ \\ 1 \end{pmatrix} \tag{2.72a}$$

$$\Delta_C = \{\} \longleftrightarrow l_e = \begin{pmatrix} 0 \end{pmatrix}. \tag{2.72b}$$

Based on Eq. (2.42), the constraint arising from the right null vector is trivial, and does not reduce the number of variables in the circuit. However, the identification of the right null vector itself formally removes a degree of freedom, which can also be seen in that the variable $\phi_{v_1} + \phi_{v_2}$ never

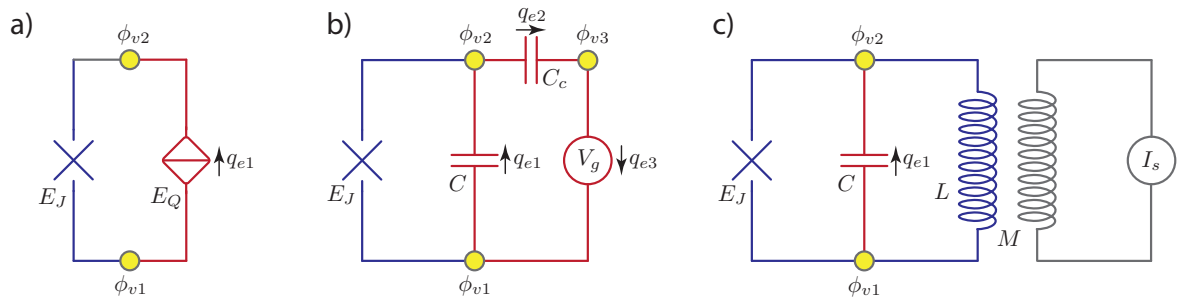


Figure 2.6: **Examples for quantum circuits in the framework of symplectic geometry.** The branches are colored based on the type of element they contain; red: capacitive elements, blue: inductive elements. The circuits are (a) dualmon circuit, (b) offset-charge-sensitive transmon[16], (c) external-flux-sensitive fluxonium.

appears in the Lagrangian. Formally, this null vector is appropriately removed by the spanning tree construction.

From the spanning tree construction, we see that $Q = q_{e_1}$ is conjugate to $\Phi = \phi_{v_2} - \phi_{v_1}$ so that $\{\Phi, Q\} = 1$. Furthermore, if the Josephson energy is E_J and the quantum phase energy is E_Q, the capacitive and inductive energies in the circuit are

$$E_{\mathcal{C}} = -E_Q \cos\left(2\pi \frac{Q}{2e}\right), \tag{2.73a}$$

$$E_{\mathcal{I}} = -E_J \cos\left(2\pi \frac{\Phi}{\phi_0}\right). \tag{2.73b}$$

Using Eq. (2.6), we write the Lagrangian of the circuit as

$$\begin{aligned} L &= \sum_{e,v} q_e \Omega_{ev} \dot{\phi}_v - E_{\mathcal{I}} - E_{\mathcal{C}} \\ &= Q\dot{\Phi} + E_Q \cos\left(2\pi \frac{Q}{2e}\right) + E_J \cos\left(2\pi \frac{\Phi}{\phi_0}\right). \end{aligned} \tag{2.74}$$

Finally, the Hamiltonian function takes the form of

$$H = -E_Q \cos\left(\frac{2\pi}{2e} Q\right) - E_J \cos\left(\frac{2\pi}{\phi_0} \Phi\right). \tag{2.75}$$

2.4.3 Offset-charges and external voltages in the transmon

Now, we show how we can incorporate the offset charges in our description through the example of the offset-charge sensitive transmon or Cooper pair box [see Fig. 2.6(b)]. The circuit contains a single Josephson junction with Josephson energy of E_J shunted by a capacitor C, and coupled with capacitance C_c to a classical gate voltage V_g that models the effects of offset charges. A key observation is that *in our formalism, we include voltage sources by treating them as additional capacitive edges.* While they do not end up leading to new degrees of freedom, this is how they are straightforwardly handled in our framework.

In our example, the circuit has three nodes (v_i, where $i = 1, 2, 3$), and three capacitive branches, including the voltage source (e_i, where $i = 1, 2, 3$). Thus, the capacitive incidence matrix is

$$\Omega = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \tag{2.76}$$

where the columns correspond to the three vertices and rows to the three branches. By inspection, we note that the inductively shunted islands, capacitive loops, and the corresponding null vectors in the circuit are

$$\Gamma_I = \{\{v_1, v_2, v_3\}\} \longleftrightarrow r_v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{2.77a}$$

$$\Delta_C = \{\{e_1, e_2, e_3\}\} \longleftrightarrow l_e = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}. \tag{2.77b}$$

The capacitive and inductive energies are

$$E_{\mathcal{C}} = \frac{q_{e_1}^2}{2C} + \frac{q_{e_2}^2}{2C_c} + q_{e_3} V_g, \tag{2.78a}$$

$$E_{\mathcal{I}} = -E_J \cos\left(2\pi \frac{\phi_{v_2} - \phi_{v_1}}{\phi_0}\right). \tag{2.78b}$$

Thus, based on Eq. (2.6) the Lagrangian of the circuit reads

$$\begin{aligned} L &= \sum_{e,v} q_e \Omega_{ev} \dot{\phi}_v - E_{\mathcal{I}} - E_{\mathcal{C}} \\ &= q_{e_1}(\dot{\phi}_{v_2} - \dot{\phi}_{v_1}) + q_{e_2}(\dot{\phi}_{v_3} - \dot{\phi}_{v_2}) + q_{e_3}(\dot{\phi}_{v_1} - \dot{\phi}_{v_3}) - \frac{q_{e_1}^2}{2C} - \frac{q_{e_2}^2}{2C_c} - q_{e_3} V + E_J \cos\left[\frac{2\pi}{\phi_0}(\phi_{v_2} - \phi_{v_1})\right]. \end{aligned} \tag{2.79}$$

In this example, the choice of spanning tree is not unique. We will choose

$$\mathcal{T} = \{e_1, e_2\} \tag{2.80}$$

as a spanning tree, and because the sum of the branch fluxes in the loop vanishes

$$\phi_{e_3} = -\phi_{e_1} - \phi_{e_2}. \tag{2.81}$$

Then we introduce the new variables

$$\begin{aligned} Q_{e_1} &= q_{e_1} - q_{e_3}, \\ Q_{e_2} &= q_{e_2} - q_{e_3}, \\ \Phi_{e_1} &= \phi_{e_1}, \\ \Phi_{e_2} &= \phi_{e_2}. \end{aligned} \tag{2.82}$$

We are free to rewrite

$$L = Q_{e_1} \dot{\Phi}_{e_1} + Q_{e_2} \dot{\Phi}_{e_2} - \frac{(Q_{e_1} + q_{e_3})^2}{2C} - \frac{(Q_{e_2} + q_{e_3})^2}{2C_c} - q_{e_3} V_g + E_J \cos\left[\frac{2\pi}{\phi_0} \Phi_{e_1}\right]. \tag{2.83}$$

Only the capacitive loop (left null vector) gives a nontrivial constraint based on Eq. (2.39)

since

$$\frac{Q_{e_1}+q_{e_3}}{C}+\frac{Q_{e_2}+q_{e_3}}{C_c}+V_g=0. \tag{2.84}$$

which can be used to fix q_{e_3} in terms of Q_{e_1} and Q_{e_2}. Further, there is a Noether current which produces the constraint

$$0=\frac{\delta S}{\delta\phi_{v_3}}=\dot{q}_{e_2}-\dot{q}_{e_3}=\dot{Q}_{e_2}=0. \tag{2.85}$$

Choosing the constant of integration to be zero, and defining $Q=Q_{e_1}$ and $\Phi=\Phi_{e_1}$ with $\{\Phi,Q\}=1$, we arrive at the Lagrangian in the form of

$$L=Q\dot{\Phi}-\frac{(Q-C_cV_g)^2}{2(C+C_c)}+E_J\cos\left(2\pi\frac{\Phi}{\phi_0}\right), \tag{2.86}$$

after dropping a constant term. Thus, we can write the Hamiltonian function in the well-known form

$$H=\frac{(q-C_cV_g)^2}{2(C+C_c)}-E_J\cos\left(2\pi\frac{\Phi}{\phi_0}\right) \tag{2.87}$$

From this point, it is straightforward to quantize H even with compact variable Φ [see Eq. (2.56)].

2.4.4 External flux in the fluxonium

Now, we turn our attention to the case of external fluxes. It is generally straightforward to include external flux biases; here, we model it by coupling the circuit inductively to a loop with current I_s flowing [see Fig. 2.6(c)]. If the mutual induction is M, the relevant energy term is

$$E=\frac{M}{L}I_s(\phi_{v_1}-\phi_{v_2}). \tag{2.88}$$

For the sake of brevity, we will simply write out the Lagrangian of the fluxonium following

similar procedures as in the first two examples:

$$L = q_{e_1}(\dot{\phi}_{v_2} - \dot{\phi}_{v_1}) + \frac{q_{e_1}^2}{2C} + E_J \cos\left(2\pi \frac{\phi_{v_2} - \phi_{v_1}}{\phi_0}\right) - \frac{1}{2L}(\phi_{v_2} - \phi_{v_1})^2 - \frac{M}{L} I_s(\phi_{v_2} - \phi_{v_1}), \quad (2.89)$$

which, after finding the spanning tree, can be further written as

$$L = Q\dot{\Phi} + \frac{Q^2}{2C} + E_J \cos\left(2\pi \frac{\Phi}{\phi_0}\right) - \frac{1}{2L}(\Phi - \phi_{\text{ext}})^2, \quad (2.90)$$

where $\Phi = \phi_{v_2} - \phi_{v_1}$, $Q = q_{e_1}$, and $\phi_{\text{ext}} = -MI_s$. We have neglected an overall constant contribution to L. Thus, the Hamiltonian of the circuit reads

$$H = \frac{Q^2}{2C} - E_J \cos\left(2\pi \frac{\Phi}{\phi_0}\right) + \frac{1}{2L}(\Phi - \phi_{\text{ext}})^2, \quad (2.91)$$

where the sole conjugate pair consists of Q and Φ. We will examine the case of time-dependent external biases ϕ_{ext} in Sec. 2.4.6.

2.4.5 Josephson junctions with quantum phase slips

Let us now look at a more complicated example, which cannot be quantized using the existing paradigm. Consider the circuit drawn in Figs. 2.2 and 2.4, which has both nonlinear capacitors and nonlinear inductors. Despite this, it is not singular [106]. In order to build a well-defined Lagrangian, one may expect to incorporate both node-flux and loop-charge [117, 123] variables. Using our approach, we will show that such a construction is both algorithmic and transparent. We will directly obtain a Hamiltonian operator for this quantum circuit.

We begin by writing down a Lagrangian using the incidence matrix discussed in Section 2.2.3:

$$\begin{aligned} L = {} & q_{e_1}(\dot{\phi}_{v_2} - \dot{\phi}_{v_1}) + q_{e_2}(\dot{\phi}_{v_3} - \dot{\phi}_{v_2}) + q_{e_3}(\dot{\phi}_{v_4} - \dot{\phi}_{v_3}) + (q_{e_4} + q_{e_5})(\dot{\phi}_{v_5} - \dot{\phi}_{v_4}) \\ & - \frac{1}{2C_1} q_{e_1}^2 + E_C \cos\left(2\pi \frac{q_{e_2}}{2e}\right) - \frac{1}{2C_3} q_{e_3}^2 - \frac{1}{2C_4} q_{e_4}^2 - \frac{1}{2C_5} q_{e_5}^2 \\ & + E_J \cos\left(2\pi \frac{\phi_{v_4} - \phi_{v_3}}{\phi_0}\right) - \frac{1}{2L_2}(\phi_{v_6} - \phi_{v_5})^2 - \frac{1}{2L_3}(\phi_{v_1} - \phi_{v_6})^2. \end{aligned} \quad (2.92)$$

There is a single capacitive cycle and three capacitively shunted islands, which give rise to the independent constraints

$$0 = \frac{\delta S}{\delta \phi_{v_6}} = \frac{\phi_{v_6} - \phi_{v_5}}{L_2} + \frac{\phi_{v_6} - \phi_{v_1}}{L_3} \tag{2.93a}$$

$$0 = \frac{\delta S}{\delta q_{e_4}} - \frac{\delta S}{\delta q_{e_5}} = \frac{q_{e_4}}{C_4} - \frac{q_{e_5}}{C_5}. \tag{2.93b}$$

The only further simplification is the presence of a Noether current that will simplify the capacitors added in series. The relevant constraint is given by

$$0 = \frac{\delta S}{\delta \phi_{v_2}} = \dot{q}_{e_1} - \dot{q}_{e_2}, \tag{2.94}$$

and we elect to set the constant of integration to zero. Taking advantage of all of the constraints in Eqs. (2.93-2.94), we see

$$\begin{aligned} L = {} & q_{e_1}(\dot{\phi}_{v_3} - \dot{\phi}_{v_1}) + q_{e_3}(\dot{\phi}_{v_4} - \dot{\phi}_{v_3}) + (q_{e_4} + q_{e_5})(\dot{\phi}_{v_5} - \dot{\phi}_{v_4}) \\ & - \frac{1}{2C_1} q_{e_1}^2 + E_C \cos\left(2\pi \frac{q_{e_1}}{2e}\right) + E_J \cos\left(2\pi \frac{\phi_{v_4} - \phi_{v_3}}{\phi_0}\right) - \frac{1}{2C_3} q_{e_3}^2 - \frac{1}{2(C_4 + C_5)} (q_{e_4} + q_{e_5})^2 \\ & - \frac{1}{2(L_2 + L_3)} (\phi_{v_1} - \phi_{v_5})^2. \end{aligned} \tag{2.95}$$

Now, we relabel variables according to the identification

$$\Phi_{e_i} = \begin{cases} \phi_{v_3} - \phi_{v_1} & e = e_1 \\ \phi_{v_4} - \phi_{v_3} & e = e_3 \\ \phi_{v_5} - \phi_{v_4} & e = e_4 \end{cases} \tag{2.96a}$$

$$Q_e = \begin{cases} q_{e_1} & e = e_1 \\ q_{e_3} & e = e_3 \\ q_{e_4} + q_{e_5} & e = e_4 \end{cases} \tag{2.96b}$$

and our Lagrangian takes the form

$$\begin{aligned} L = & Q_{e_1}\dot{\Phi}_{e_1} + Q_{e_3}\dot{\Phi}_{e_3} + Q_{e_4}\dot{\Phi}_{e_4} \\ & - \frac{1}{2C_1}Q_{e_1}^2 + E_C \cos\left(\frac{2\pi}{2e}Q_{e_1}\right) - \frac{1}{2C_3}Q_{e_3}^2 - \frac{1}{2(C_4+C_5)}Q_{e_4}^2 \\ & + E_J \cos\left(\frac{2\pi}{\phi_0}\Phi_{e_3}\right) - \frac{1}{2(L_2+L_3)}(\Phi_{e_1}+\Phi_{e_3}+\Phi_{e_4})^2. \end{aligned} \tag{2.97}$$

Finally, it is straightforward to write down the Hamiltonian

$$\begin{aligned} H = & \frac{1}{2C_1}Q_{e_1}^2 - E_C \cos\left(\frac{2\pi}{2e}Q_{e_1}\right) + \frac{1}{2C_3}Q_{e_3}^2 + \frac{1}{2(C_4+C_5)}Q_{e_4}^2 \\ & -E_J \cos\left(\frac{2\pi}{\phi_0}\Phi_{e_3}\right) + \frac{1}{2(L_2+L_3)}(\Phi_{e_1}+\Phi_{e_3}+\Phi_{e_4})^2 \end{aligned} \tag{2.98}$$

with

$$[\Phi_{e_i}, Q_{e_j}] = \mathrm{i}\hbar\delta_{ij}. \tag{2.99}$$

Since it is desirable to have a Hamiltonian which is quadratic in derivatives, we elect to write

$$Q_{e_i} = -\mathrm{i}\hbar\frac{\partial}{\partial\Phi_{e_i}} \qquad i = 3, 4 \tag{2.100a}$$

$$\Phi_{e_i} = \mathrm{i}\hbar\frac{\partial}{\partial Q_{e_i}} \qquad i = 1 \tag{2.100b}$$

which is of course allowed by Eq. (2.99). This choice ensures that no more than two derivatives appear in H, which may make it easier to incorporate our algorithm into existing software packages for circuit quantization.

2.4.6 Time-dependent external charges or fluxes

In the example above, we introduced the external flux by inductively coupling the circuit to an external current source. There is an alternative way in our description to introduce external flux: we can add one or more new branches with a voltage source to a loop. To understand this construction, we recall that Faraday's law states that in the presence of time-dependent magnetic fields, the sum

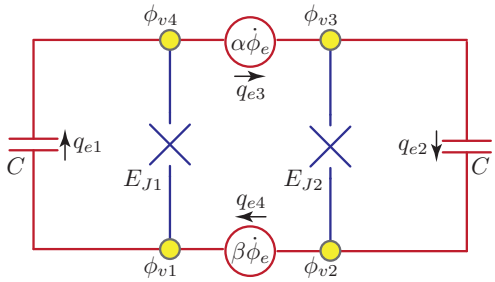


Figure 2.7: **Circuit in time-dependent external flux.** A flux-tunable transmon with two Josephson junctions shunted by two capacitors. The external flux in the loop enclosed by the two junctions can be modeled as one or more additional batteries in a loop, as long as the total flux provided by these batteries equals the external flux. The batteries are modeled as capacitive elements.

of voltages in a loop equals the rate of change of the magnetic field. Thus, if e_i $(i = 1, 2, \ldots, n)$ are the physical capacitive branches in a loop

$$\sum_{i=1}^{n} \phi_{e_i} + \phi_{\text{ext}} = 0, \tag{2.101}$$

where ϕ_{ext} is the external flux piercing the loop. This suggests that we can think of the external flux as just another branch in the loop with an additional fixed flux ϕ_{ext}. However, a natural question arises at this point: where should one put this additional branch in the loop? As discussed in Refs. [15, 122], the various Hamiltonians are linked by a gauge transformation.

Figure 2.7 shows an example of how one can place the "flux batteries" in a circuit to capture the external flux. The circuit is flux-tunable transmon, where two Josephson junctions, E_{J1} and E_{J2}, are shunted by capacitors C. Notice that we have the freedom to place the external flux batteries in various ways, for example, here we choose to put two batteries with fluxes of $\alpha\phi_{\text{ext}}$ and $\beta\phi_{\text{ext}}$ in the loop. The condition of $\alpha + \beta = 1$ ensures that the total external flux in the loop is ϕ_{ext}. This approach makes the circuit artificially a four-node circuit, but using the constraints outlined in this paper, we can end up with a single degree of freedom.

To start, we recall that batteries are capacitive elements in our formalism, and using the

variables in Fig. 2.7, we write down the Lagrangian

$$\begin{aligned} L = {} & q_{e_1}(\dot{\phi}_{v_4} - \dot{\phi}_{v_1}) + q_{e_2}(\dot{\phi}_{v_2} - \dot{\phi}_{v_3}) + q_{e_3}(\dot{\phi}_{v_3} - \dot{\phi}_{v_4}) + q_{e_4}(\dot{\phi}_{v_1} - \dot{\phi}_{v_2}) + \\ & + E_{J1}\cos\left(2\pi\frac{\phi_{v_1} - \phi_{v_4}}{\phi_0}\right) + E_{J2}\cos\left(2\pi\frac{\phi_{v_3} - \phi_{v_2}}{\phi_0}\right) - \frac{1}{2C}\left(q_{e_1}^2 + q_{e_2}^2\right) + \\ & - \beta\dot{\phi}_{\text{ext}} q_{e_4} - \alpha\dot{\phi}_{\text{ext}} q_{e_3}, \end{aligned} \tag{2.102}$$

where the last two lines are the contribution of the voltage sources. Using the constraint arising from the capacitive loop $\Delta_C = \{\{e_1, e_4, e_2, e_3\}\}$, and integrating out q_{e_3} and q_{e_4}, we arrive at the Lagrangian

$$L = Q\dot{\Phi} - \frac{1}{2}(\alpha - \beta)Q\dot{\phi}_{\text{ext}} + E_{J1}\cos\left(2\pi\frac{\Phi - \alpha\phi_{\text{ext}}}{\phi_0}\right) + E_{J2}\cos\left(2\pi\frac{\Phi + \beta\phi_{\text{ext}}}{\phi_0}\right) - \frac{1}{4C}Q^2, \tag{2.103}$$

where $Q = q_{e_1} - q_{e_2}$ and $\Phi = \phi_{v_3} - \phi_{v_1}$. And finally, the Hamiltonian is

$$H = \frac{1}{4C}Q^2 - E_{J1}\cos\left(2\pi\frac{\Phi - \alpha\phi_{\text{ext}}}{\phi_0}\right) - E_{J2}\cos\left(2\pi\frac{\Phi + \beta\phi_{\text{ext}}}{\phi_0}\right) + \frac{1}{2}(\alpha - \beta)Q\dot{\phi}_{\text{ext}}, \tag{2.104}$$

with conjugate pairs $\{\Phi, Q\} = 1$.

Notice that the last term in Eq. (2.104) depends on our choice of how to distribute the batteries in the loop. For example, in the "irrotational gauge" [15, 122], when $\alpha = \beta = \frac{1}{2}$, there is no term linear in Q. We can transform between the different gauges at the classical level by a time-dependent type-2 canonical transformation from $(Q, \Phi) \to (Q', \Phi')$. For example, if we take $\alpha = 1$ and $\beta = 0$, the Hamiltonian is

$$H = \frac{1}{4C}Q^2 - E_{J1}\cos\left(2\pi\frac{\Phi - \phi_{\text{ext}}}{\phi_0}\right) - E_{J2}\cos\left(2\pi\frac{\Phi}{\phi_0}\right) + \frac{1}{2}Q\dot{\phi}_{\text{ext}}, \tag{2.105}$$

while in the irrotational gauge

$$\tilde{H} = \frac{1}{4C}\tilde{Q}^2 - E_{J1}\cos\left(2\pi\frac{\tilde{\Phi} - \frac{1}{2}\phi_{\text{ext}}}{\phi_0}\right) - E_{J2}\cos\left(2\pi\frac{\tilde{\Phi} + \frac{1}{2}\phi_{\text{ext}}}{\phi_0}\right). \tag{2.106}$$

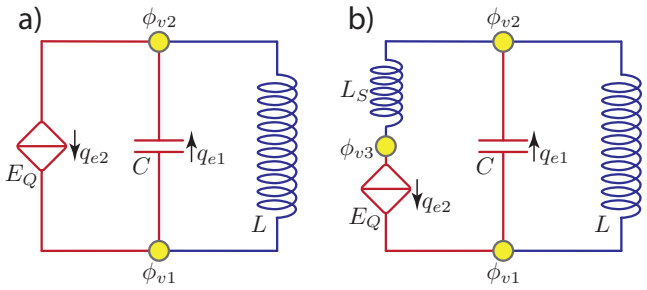


Figure 2.8: **Singular circuit.** (a) Example for a one-mode circuit where a quantum phase element has no series inductance attached to it. The circuit has a capacitive loop, $\mathcal{C} = \{\{q_{e_1}, q_{e_2}\}\}$ leading to a non-analytical constraint. (b) When a series inductance is included in the circuit, the capacitive loop is broken, and the system has two degrees of freedom and an analytical Hamiltonian.

In this particular example, the generating function is

$$G(\Phi, \tilde{Q}, t) = \tilde{Q} \cdot \left(\Phi - \frac{\phi_{\text{ext}}}{2} \right), \tag{2.107}$$

which implies

$$Q = \frac{\partial G}{\partial \Phi} = \tilde{Q}, \tag{2.108a}$$

$$\tilde{\Phi} = \frac{\partial G}{\partial \tilde{Q}} = \Phi - \frac{1}{2}\phi_{\text{ext}}, \tag{2.108b}$$

$$\tilde{H} = H + \frac{\partial G}{\partial t} = H - \frac{1}{2} Q \dot{\phi}_{\text{ext}}. \tag{2.108c}$$

2.4.7 Singular circuits

When a Josephson junction is not accompanied by a parallel capacitor, or a quantum phase slip element has no series inductor attached to it, the resultant circuit can become singular [106]. In this section, we analyze an example for such a singular circuit, which leads to a non-analytical Hamiltonian.

The circuit is presented in Fig. 2.8(a), and it has a quantum phase slip, a capacitor and an

inductor all in parallel. Following our procedures, we arrive at a Lagrangian of

$$L = q_{e_1}(\dot{\phi}_{v_2} - \dot{\phi}_{v_1}) + q_{e_2}(\dot{\phi}_{v_1} - \dot{\phi}_{v_2}) - \frac{1}{2C}q_{e_1}^2 + E_Q \cos\left(2\pi\frac{q_{e_2}}{2e}\right) - \frac{1}{2L}\left(\phi_{v_1} - \phi_{v_2}\right)^2. \tag{2.109}$$

By looking at the geometry of the circuit, we notice that there is one capacitive loop $\mathcal{C} = \{\{q_{e_1}, q_{e_2}\}\}$, which based on Eq. (2.39) gives rise to the constraint

$$V_Q \sin\left(2\pi\frac{q_{e_2}}{2e}\right) + \frac{q_{e_1}}{C} = 0. \tag{2.110}$$

When $\pi V_q/e > C$, this constraint cannot be uniquely solved. More generally, the constraint cannot be solved analytically in terms of simple functions, although it can be solved straightforwardly numerically. A detailed discussion on this topic can be found in Ref. [106]. We denote

$$Q = q_{e_1} - q_{e_2} \tag{2.111}$$

and write

$$V_Q \sin\left(2\pi\frac{q_{e_2}}{2e}\right) + \frac{Q + q_{e_2}}{C} = 0. \tag{2.112}$$

By denoting a solution of Eq. (2.112) as $q_{e_2}(Q)$, and introducing $\Phi = \phi_{v_2} - \phi_{v_1}$, the Hamiltonian becomes

$$H = \frac{1}{2C}\left[Q + q_{e_2}(Q)\right]^2 - E_Q \cos\left[2\pi\frac{q_{e_2}(Q)}{2e}\right] + \frac{1}{2L}\Phi^2 \tag{2.113}$$

where the conjugate pairs are $\{\Phi, Q\} = 1$.

In circuits which can be realized in current experiments, this singular behavior is not present because quantum phase slip elements always have a series inductor component L_S [see Fig. 2.8(b)]. This additional element transfers the singular one-mode circuit into a two-mode circuit. The key observation is that the presence of the series inductance breaks the capacitive loop, and removes the left null vector of the circuit, thus, the constraint of Eq. (2.110) is lifted. After a few steps, the

Hamiltonian reads

$$H = \frac{1}{2C}Q_{e_1}^2 + \frac{1}{2L}\Phi_{e_1}^2 + \frac{1}{2L_S}(\Phi_{e_1} + \Phi_{e_2})^2 - E_Q \cos\left[2\pi\frac{Q_{e_2}}{2e}\right] \tag{2.114}$$

where

$$\begin{aligned} Q_{e_1} &= q_{e_1}, \\ Q_{e_2} &= q_{e_2}, \\ \Phi_{e_1} &= \phi_{v_2} - \phi_{v_1}, \\ \Phi_{e_2} &= \phi_{v_1} - \phi_{v_3}, \end{aligned} \tag{2.115}$$

and the conjugate pairs are $\{\Phi_{e_1}, Q_{e_1}\} = 1$ and $\{\Phi_{e_2}, Q_{e_2}\} = 1$.

As was explained in detail in [106], the quantum mechanical spectra of the singular vs. non-singular Hamiltonians can drastically differ. The mechanism for this is analogous to the Born-Oppenheimer approximation for molecules, in which the light electrons cannot be approximated as sitting near the classical minima of the ion-electron potential: instead the wave function becomes spread out in configuration space. In the example above, when $L_S \to 0$, any low-energy quantum mechanical eigenstate $\psi(Q_{e_1}, Q_{e_2})$ will become highly delocalized in the $Q_{e_1} + Q_{e_2}$ direction, such that the effective potential cannot be approximated by (2.113).

A similar argument can be made for the case of parallel capacitors for Josephson junctions.

2.4.8 Connecting to the existing Lagrangian formalism

Finally, let us briefly discuss how our formalism straightforwardly reproduces the existing Lagrangian formalism in suitable limits. As one example, consider a circuit with only linear capacitors, and inductive elements of any kind. The linear capacitor at the capacitive branch $e \in \mathcal{C}$ has capacitance C_e, while the inductive element at an inductive branch $e \in \mathcal{I}$ is described by an energy function g_e. In our formalism, the Lagrangian is

$$L(q_e, \phi_v, \dot{\phi}_v) = \sum_{e\in\mathcal{C}, v\in\mathcal{V}} q_e \Omega_{ev} \dot{\phi}_v - \sum_{e\in\mathcal{C}} \frac{q_e^2}{2C_e} - \sum_{e\in\mathcal{I}} g_e(\phi_e), \tag{2.116}$$

where Ω_{ev} is the usual capacitive incidence matrix. Since L is a quadratic function of q_e, but L does not depend on the time derivatives of the charges $\dot{q}_e$, we can easily "integrate out" q_e. (This statement remains true in a quantum mechanical path integral). Solving the Euler-Lagrange equation for q_e, we find

$$\Omega_{ev}\dot{\phi}_v = \frac{q_e}{C_e}. \tag{2.117}$$

Then defining a capacitance matrix as

$$C_{uv} = \sum_e C_e \Omega_{eu} \Omega_{ev}, \tag{2.118}$$

we find a Lagrangian expressed only in terms of node flux variables, as is standard in the literature [118]:

$$L(\phi_v, \dot{\phi}_v) = \sum_{u,v \in \mathcal{V}} \frac{1}{2} C_{uv} \dot{\phi}_u \dot{\phi}_v - \sum_{e \in \mathcal{I}} g_e(\phi_e). \tag{2.119}$$

2.5 Outlook

In this paper, we have developed a universal theory of circuit quantization for all LC circuits. Our approach allows for the quantization of singular circuits, with arbitrary graph topology, and with arbitrary time-dependent sources. The approach is inspired by symplectic geometry and graph theory, and the quantization prescription depends only on the topology of the capacitive subgraph, but not on which elements are linear or nonlinear. The "spanning tree construction" leads to a straightforward quantization prescription using a set of canonically conjugate coordinates that could be efficiently implemented in future software packages that perform generic circuit quantization.

Looking forward, this approach will provide an efficient algorithm for computing the numerical spectra of complicated hybrid circuits simultaneously involving Josephson junctions, quantum phase slips, and other arbitrary nonlinear elements that can be classified as inductive or capacitive. Obtaining these spectra will be a critical step in identifying the behavior of quantum phase slip and other nonlinear capacitive elements in a circuit quantum electrodynamics setup, opening new avenues

to design and create novel superconducting devices beyond the current architectures. Further, this formalism should extend naturally to nonlinear mechanical oscillators [59, 76, 98, 107, 113] and ideal non-reciprocal elements [95], and may also be extensible to the quantization of transmission lines [96].

Note added.— After our work became public, other authors [93] proposed another new framework for circuit quantization.

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Chapter 3

Flux-Charge symmetric theory of superconducting circuits

3.1 Introduction

An ordinary non–dissipative circuit, when sufficiently cooled such that all wires become superconducting, will exhibit macroscopic manifestations of quantum mechanics with a finite number of degrees of freedom. Such circuits are referred to as superconducting circuits; they are of interest for a variety of reasons, not least of which is the design and implementation of qubits for use in a future quantum computer [13, 31, 60, 61, 65]. To that end, the theory of superconducting circuits has been very successful in producing a number of qubits with desirable properties, including the transmon, 0–π and fluxonium qubits [43, 48, 62, 78]. On more fundamental grounds, superconducting circuits are also of use for quantum simulation of exotic physics, such as particle motion on spaces with negative curvature [64].

In the theory of superconducting circuits, charges and fluxes take the role of momenta and positions in classical Hamiltonian mechanics [17]. More formally, they are canonically conjugate degrees of freedom. For simple circuits, it is often straightforward to identify the conjugate pairs of charges and fluxes, and then promote these variables to operators, as in textbook quantum mechanics. This procedure is referred to as circuit quantization. However, for complicated circuits, namely those containing both phase slips [84] and Josephson junctions [6, 9], the standard approach [26] to circuit quantization relies on using a *Lagrangian formulation* of the problem that breaks the explicit structure, such as the presence of canonical transformations that do not change the underlying structure behind Hamiltonian mechanics. In fact, only very recently have some authors [90, 93,

96] begun to develop an intrensically Hamiltonian [42] approach to describing the classical (and quantum) mechanics of superconducting circuits. Still, these approaches have continued to treat charge and flux on an arguably asymmetric footing.

This paper presents a theory of circuit quantization that is manifestly symmetric under the exchange of charge and flux degrees of freedom, and treats capacitors and inductors on an equal footing. An advantage of this approach is that it makes certain “mysterious" properties of the algorithmic method of [90], such as the appearance of conserved quantities, more manifest. Moreover, our approach provides an extremely transparent description of *circuit dualities* [58, 117] for planar circuits, and – in some special cases – for non-planar circuits as well.

3.2 Classical theory of circuits

We consider circuits made out of elements that are either purely inductive or purely capacitive. An inductive element has a constitutive relation of the form

$$I = \frac{\partial E}{\partial \phi}, \tag{3.1}$$

where E is the energy stored in the element, and ϕ is the magnetic flux through it. Likewise, a circuit element with a constitutive relation

$$V = \frac{\partial E}{\partial q} \tag{3.2}$$

is a capacitor: here q is the charge accumulated across the capacitor, while V is the voltage. Note that these definitions include arbitrary nonlinear elements such as the Josephson junction and the quantum phase slip junction. From the perspective of circuit quantization, these nonlinear elements are no obstruction to any of the results that we present.

Loosely speaking, $I = \dot{q}$ and $V = \dot{\phi}$ (here dots denote time derivatives). This suggests that inductors and capacitors host conjugate degrees of freedom in a superconducting circuit.

3.2.1 Symplectic geometry approach to circuit quantization

It is our goal to produce a quantization perscription for superconducting circuits that treats capacitors and inductors (equivalently charges and fluxes) on equal footing. To ensure full generality in our construction, we will begin with a short review of a recently developed [90] symplectic approach to quantization of circuits, which is applicable to arbitrarily nonlinear circuits built out of inductors and capacitors. Besides having the advantage of being more general (the new approach can quantize circuits for which a standard Lagrangian does not exist), we will show in this paper that the approach of [90] also beautifully encodes the mathematics of circuit duality, when re-written in a flux-charge-symmetric manner.

In [90], it was found that a Hamiltonian dynamical system – namely, a Hamiltonian function H and a symplectic form and/or Poisson bracket on a suitable phase space – is neatly encoded in a simple Lagrangian that depends on the incidence matrix of a circuit. A circuit can be thought of mathematically as a directed graph with vertices v and directed edges e, which have a start and end vertex. On each edge e, we place either an inductive or capacitive element whose energy E depends either on the flux difference ϕ_e across the circuit element, or the charge q_e which has accumulated due to current flow across the edge. The directedness of each edge is important to capture the correct sign of ϕ_e and q_e. For the remainder of this manuscript, we will use the symbol $\mathcal{E}$ to refer to the set of edges in a circuit. Generally, the circuit to which $\mathcal{E}$ pertains will be clear from context. We define the incidence matrix

$$A_{ev} = \begin{cases} 1 & e \text{ arrives at } v \\ -1 & e \text{ leaves from } v \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

which encodes a convention for positive current and voltage across an edge. There is another closely related matrix, Ω, called the reduced incidence matrix, which is a restriction of A to edges containing

only capacitors. In terms of Ω, the action encoding the dynamics of G may be written

$$S = \int \mathrm{d}t \left[\sum_{e \in \mathcal{C}, v} q_e \Omega_{ev} \dot{\phi}_v - \sum_{e \in \mathcal{C}} E_e^C(q_e) - \sum_{e \in \mathcal{I}} E_e^L(A_{ev}\phi^v) \right] \tag{3.4}$$

with capacitive branches in the set $\mathcal{C}$ and inductive branches in the set $\mathcal{I}$. Note that $\mathcal{E} = \mathcal{C} \cup \mathcal{I}$. Here, E_e^C (E_e^L) is the energy of the capacitor (inductor) on branch e for a given configuration.

Quantization of a circuit is achieved by enumerating and removing the null vectors of Ω and the corresponding variables. The resulting invertible matrix Ω can be related to a symplectic form and Poisson bracket, which then formally lead to a canonical quantization prescription of the circuit [90].

3.2.2 Topology of a graph

In this paper, we will need to review a little more graph theory than was presented above. Indeed, in the language of mathematics, circuits are directed graphs where the edge set $\mathcal{E} = \mathcal{C} \cup \mathcal{I}$ has a decomposition into capacitors and inductors. We now briefly review some of the relevant graph theory, and refer the reader to Appendix B.1 for a more formal discussion.

The most important aspect of a graph for us here will be the structure of cycles, or loops. For an undirected graph, it is possible to define cycle addition [45, 50] as an additive operation on a vector space over the field $\mathbb{Z}_2$. This operation imbues us with a natural notion of linear independence between cycles. This notion of independence will remain relevant for circuits, which are directed graphs, since the “arrow" on each edge is used to keep track of the direction of current flow, but not whether an object is a loop or not.

For a circuit with $|\mathcal{E}|$ edges, and $|\mathcal{V}|$ vertices, there are $|\mathcal{E}| - |\mathcal{V}| + 1$ linearly independent loops, assuming that the circuit is connected. For reasons that will become clear, we will find it expedient to find a basis for this space with *one redundant loop*: i.e., we choose $|\mathcal{E}| - |\mathcal{V}| + 2$ loops and collect them into set

$$\mathcal{L} = \{l_1, l_2, \ldots, l_{|\mathcal{E}|-|\mathcal{V}|+2}\}. \tag{3.5}$$

To each loop assign an orientation, such that an edge in the loop l may either be oriented with l or against l. Such assignments are encoded in an **orientation matrix**

$$B_{le} = \begin{cases} 1 & l \text{ and } e \text{ are oriented alike} \\ -1 & l \text{ and } e \text{ are oriented unalike} \\ 0 & e \text{ does not lie on the boundary of } l \end{cases} . \tag{3.6}$$

For an example, consider the graph drawn in Fig. 3.1(a). The orientation matrix for this graph is given by

$$B = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{matrix} \end{array} \tag{3.7}$$

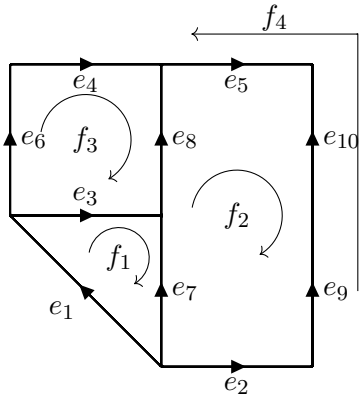
Since we have defined $\mathcal{L}$ in such a way that it always contains $|\mathcal{E}| - |\mathcal{V}| + 2$ elements, it is always the case that

$$|\mathcal{L}| - |\mathcal{E}| + |\mathcal{V}| = 2. \tag{3.8}$$

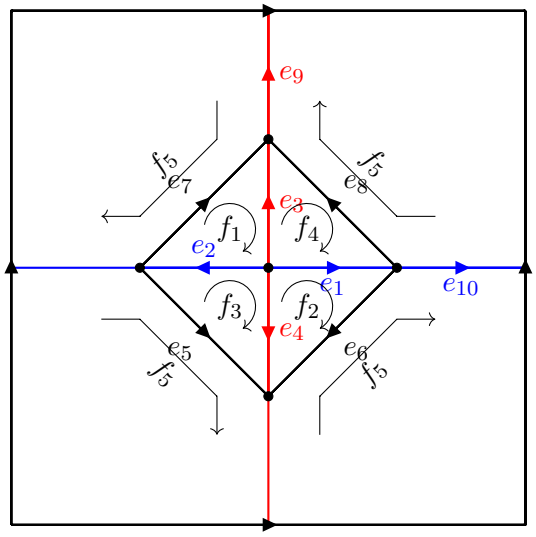
Euler's formula should be called to mind by (3.8). As we will see, this is no accident.

Every graph (and thus every circuit) can be drawn (more formally, embedded) on some closed, orientable surface[1] with no crossing edges. One can always find such a surface, wherein this drawing of a graph partitions the surface of embedding into some number of regions isomorphic to the unit disk. We will call every such two dimensional disk a *face* of the circuit. For a particular drawing of

[1]The surfaces one should keep in mind are the Riemann surface of genus g, including the sphere ($g = 0$) and the torus ($g = 1$). Such surfaces will always be sufficient since the topology of two dimensional orientable manifolds is fully characterized by their genus.



(a) An embedding of a graph with orientations chosen for faces and branches. The outer face is labeled f_4. All faces are oriented alike if we consider this plane to be a patch of a sphere (namely we identify points at infinity as the same point).



(b) An embedding of K_5 on the torus. The vertical border at the top of the drawing is to be identified with the vertical border at the bottom of the drawing. Likewise, the horizontal boundaries are to be identified. K_n is the fully connected graph on n vertices, and for $n \geq 5$, K_n is nonplanar. Nonplanar graphs cannot be embedded on the sphere, but only on a torus with a postive number of handles. A valid choice of "topological loops"(See footnote 2) is given by the blue and red edges.

Figure 3.1: Examples of faces and edges in a (a) planar and (b) non-planar graph.

a circuit on a surface of genus g, we will write that the number of faces induced by the drawing is $|\mathcal{F}|$. Euler's formula reads

$$|\mathcal{F}| - |\mathcal{E}| + |\mathcal{V}| = 2 - 2g. \tag{3.9}$$

Evidently, (3.9) together with (3.8) implies that

$$|\mathcal{L}| = |\mathcal{F}| + 2g. \tag{3.10}$$

Of course, one may have suspected from the outset that $|\mathcal{L}|$ was closely related to $|\mathcal{F}|$ since the boundary of a unit disk is a circle, a one–dimensional object.

We emphasize that the distinction between loops and faces (at least for planar graphs) is very minor. For planar graphs, there is a precise correspondence between faces and loops, since there is a (contractible) loop at the boundary of each face. For nonplanar graphs such as K_5 (See Fig. 3.1(b)), there are necessarily some number of loops which are topological.[2] As we will discuss in later sections, the presence of such "topological" loops is of some import to the existence of a dual circuit. More precisely, the set of loops at the boundary of some face in a drawing of a planar graph on the sphere (or the plane) number $|\mathcal{E}| - |\mathcal{V}| + 2$, and every edge exists in the common boundary of exactly two faces. Moreover, it is possible to demand that all the loops chosen in this way are "oriented alike" such that planar graphs satisfy[3]

$$\sum_{l \in \mathcal{L}} B_{le} = 0. \tag{3.11}$$

As we will see, choosing loops in correspondence with faces will be of great utility for us, even for nonplanar graphs. It will always be our convention to choose $|\mathcal{F}|$ loops corresponding to the faces of some embedding of a circuit, and we will also always choose them to be oriented alike.

[2] By refering to a loop as "topological" we mean to denote that the loop is not contractible on the surface of embedding. For planar graphs, all loops are can be expressed as some combination of contractible loops.

[3] We emphasize that this particularly simple redundancy in $|\mathcal{L}|$ is a consequence of choosing loops in correspondence to faces. In principle, it is possible to choose a spanning set of loops with a less trivial redundancy, but this complicates calculations and provides no simplification. Any choice of loops for a planar graph can be shown to correspond to the faces of *some* drawing, so this choice is also perfectly general.

Another important matter of convention arises when considering the sign of B_{le} when it is nonzero. Consider the figure in Fig. 3.1(a), and the loop consisting of edges e_1, e_3, and e_7. Intuitively, it should be the case that a suitable definition of B for this graph has the property that $B_{l_1 e}$ is nonzero only for e in $\{e_1, e_3, e_7\}$. With this preference in mind, it remains to fix the sign of $B_{l_1 e}$ for such edges e. Our convention will be that an orientation should be chosen for l_1 so that $B_{l_1 e}$ is equal to one if e is oriented like l_1 and negative one otherwise. In this particular example (as determined by the semicircular arrow surrounding the label f_1), the matrix elements of B should be $B_{l_1 e_1} = B_{l_1 e_3} = -B_{l_1 e_7} = 1$.

For nonplanar graphs, it is always possible to choose an embedding such that every edge appears on the boundary of precisely two faces (and thus included in precisely two loops chosen in correspondence with faces). However, it is not always possible to choose an embedding such that every edge appears in precisely two loops in $\mathcal{L}$. As an example, consider edge e_9 in the circuit drawn in Fig. 3.1b. e_9 appears on the boundary of f_5, and on the boundary of no other face. However, e_9 appears on the boundary of f_5 twice. When this occurs, we take the convention that $B_{l_5 e_9} = 0$,[4] since “each side" of e_9 is seen by the same face, and $+1 - 1 = 0$.

In order to define an orientation matrix for a nonplanar graph, one needs to choose apropriate topological loops and repeat the procedure. For the graph drawn in Fig. 3.1(b), one topological loop

[4]Here, l_5 is used to denote the loop on the boundary of the face f_5.

is drawn in blue, while the other is drawn in red. The orientation matrix for this graph is given by

$$B = \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{array}{c} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \\ l_7 \end{array} \end{array} \tag{3.12}$$

It turns out that B (for any drawing of G) has the property that

$$\sum_e B_{le} A_{ev} = 0. \tag{3.13}$$

In the case of (circuits that can be embedded as) planar graphs, this is an elementary result from algebraic topology [50]: B and A are the discrete differential acting on 0-forms (defined on vertices) and 1-forms (defined on edges) respectively (see Appendix B.1). For non-planar graphs, the matrix B includes the discrete differential acting on 1-forms, *and* the non-trivial elements of the first cohomology group. While we aren't aware of any elegant geometric interpretation of this combined object, (3.13) continues to hold. Note, however that for nonplanar graphs, the condition (3.11) cannot hold. However, there is some analogous expression corresponding to the set of "face–like"

loops in $\mathcal{L}$. In the case of the circuit drawn in Fig. 3.1(b), (3.12) admits

$$\sum_{i=1}^{5} B_{l_i e} = 0. \tag{3.14}$$

It is always possible to incorporate some such condition into a choice of loops in $\mathcal{L}$.

We now introduce our third and final matrix. First, note that it follows from (3.13) that, for a set $P \subset \mathcal{E}$,

$$\sum_{e \in P} B_{le} A_{ev} = -\sum_{e \notin P} B_{le} A_{ev}. \tag{3.15}$$

This fact will be of great use to us. In particular, there will be a natural partition P for a circuit: simply count only the capacitive, or only the inductive, edges! Hence for any circuit, we define the **connection matrix**

$$M_{lv} = \frac{1}{2} \sum_{e \in \mathcal{C}} B_{le} A_{ev} - \frac{1}{2} \sum_{e \in \mathcal{I}} B_{le} A_{ev}. \tag{3.16}$$

Consider the circuit drawn in Fig. 3.2, which we will call G. In the following discussion, we will repeatedly return to G as an example for the sake of concreteness. As we will see, the connection

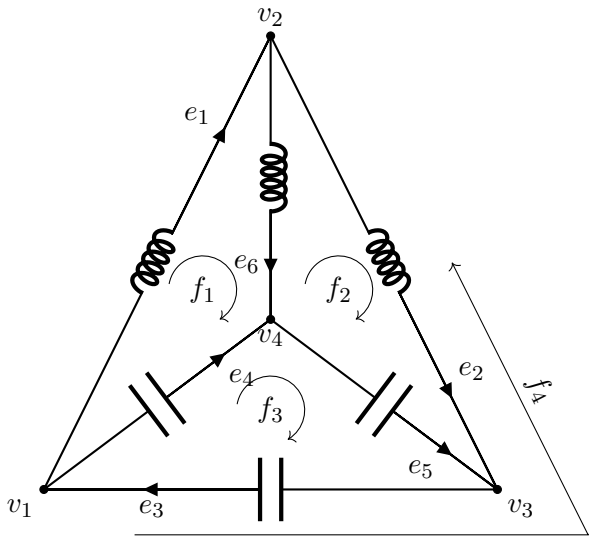


Figure 3.2: A circuit with four faces (counting the external face), six edges, and four vertices. For this circuit, $\mathcal{I} = \{e_1, e_6, e_2\}$ and $\mathcal{C} = \{e_3, e_4, e_5\}$. To see that all faces are "oriented alike", imagine the circuit embedded on the sphere: all faces are oriented into the sphere.

matrix of a circuit has a number of remarkable properties. For the circuit G in Fig. 3.2,

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \end{array} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{matrix} \tag{3.17}$$

$$B = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{matrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{matrix} \tag{3.18}$$

and

$$M = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{matrix} \tag{3.19}$$

It is possible to build a robust theory of circuit quantization using B in place of the incidence matrix of a circuit A or, indeed, in conjunction with B. While this point is conceptually appealing on its own, it also turns out to be necessary to understand circuit duality in full generality. The main result of this paper is that the matrix M, containing information inherited from both A and B, encodes the symplectic form of the classical phase space, which may be used to quantize a Hamiltonian without requiring the user to proclaim that charges are more fundamental than fluxes or vice–versa.

3.2.3 A symmetric circuit Lagrangian

The charge accumulated on a particular capacitive branch is given by $q_e = \sum_l q_l B_{le}$, which may be used to connect to the formalism of [90], as we will see later. Moreover, these loop charge variables are precisely the ones described in [117], with the exception of the fact that we "count" the external loop. Of course, this is only a matter of convenience since one is always free to fix one loop charge to vanish as a gauge degree of freedom, in direct analogy with one's freedom to choose a ground. Now, for any circuit,

$$\frac{1}{2}\sum_{e\in\mathcal{C}} q_l B_{le} A_{ev} \dot{\phi}_v - \frac{1}{2}\sum_{e\in\mathcal{I}} q_l B_{le} A_{ev} \dot{\phi}_v = \sum_{e\in\mathcal{C}} q_l B_{le} A_{ev} \dot{\phi}_v = -\sum_{e\in\mathcal{I}} q_l B_{le} A_{ev} \dot{\phi}_v, \tag{3.20}$$

according to (3.15). While (3.20) contains only trivial mathematical information, we notice that the first expression in (3.20) is not preferential toward inductive or capacitive branches, while the second two expressions *are*.

For the remainder of this manuscript, we will write E_e^L (E_e^C) to denote the energy of an inductive (capacitive) branch e in some circuit. We assume that such energies may only depend on the flux across (the charge on) some branch.[5] Explicitly, we require that, for $e = (v_1, v_2)$, E_e^I depends only on $\phi_{v_1} - \phi_{v_2}$. Analogously, for a branch adjacent to the loops l_1 and l_2, we require that E_e^C may depend only on $q_{l_1} - q_{l_2}$. As we will show shortly, this demand is equivalent to the imposition of Kirchoff's rules. We claim that the Lagrangian

$$L = \sum_{l,v} q_l M_{lv} \dot{\phi}_v - \sum_{e \in \mathcal{C}} E_e^C \left(\sum_l q_l B_{le} \right) - \sum_{e \in \mathcal{I}} E_e^L \left(\sum_v A_{ev} \phi_v \right) \tag{3.21}$$

is a general Lagrangian which can be used to describe arbitrary LC circuits, and which contains a Hamiltonian which is easily quantizable.

Let us examine the equations of motion implied by the principle of least action applied to $S = \int \mathrm{d}t\, L$:

$$\begin{aligned} 0 = \frac{\delta S}{\delta q_l} &= \sum_v M_{lv} \dot{\phi}_v - \sum_{e \in \mathcal{C}} \frac{\partial E_e^C}{\partial q_l} \\ 0 = \frac{\delta S}{\delta \phi_v} &= -\sum_l \dot{q}_l M_{lv} - \sum_{e \in \mathcal{I}} \frac{\partial E_e^L}{\partial \phi_v}. \end{aligned} \tag{3.22}$$

We rewrite the first line using the definition of M, to see that

$$0 = -\sum_{e \in \mathcal{I}, v} B_{le} A_{ev} \dot{\phi}_v - \sum_{e \in \mathcal{C}} \frac{\partial E_e^C}{\partial q_l}. \tag{3.23}$$

Another way of writing (3.23) is

$$0 = \sum_{e \in \mathcal{C}, v} B_{le} A_{ev} \dot{\phi}_v - \sum_{e \in \mathcal{C}} \frac{\partial E_e^C}{\partial q_l}. \tag{3.24}$$

[5] There do exist ideal non-reciprocal circuit elements such as gyrators that are not of this form []. We leave any possible generalization of our results to such circuits to future work.

Further, since we have demanded that E_e^C may only depend on $\sum_l B_{le} q_l$, we can see that (3.24) implies

$$0 = \sum_{e \in \mathcal{C}} B_{le} \left[\sum_v A_{ev} \dot{\phi}_v - \frac{\partial E_e^C(q)}{\partial q} \right]. \tag{3.25}$$

Given that $\sum_v A_{ev} \dot{\phi}_v$ can be interpreted as a voltage difference across branch e, we see that (3.25) is compatible with the constitutive relation for capacitors:

$$\sum_v A_{ev} \dot{\phi}_v = \frac{\partial E_e^C(q)}{\partial q}. \tag{3.26}$$

In fact, we will show in Corollary 80 that (up to a certain exception to be discussed later), the *only* solution to (3.25) is (3.26). Substituting

$$\sum_{e \in \mathcal{C}} \frac{\partial E_e^C}{\partial q_l} = \sum_{e \in \mathcal{C}, v} B_{le} A_{ev} \dot{\phi}_v, \tag{3.27}$$

we can see that (3.23) becomes

$$\sum_{e,v} B_{le} A_{ev} \dot{\phi}_v = 0. \tag{3.28}$$

In other words, (3.23) is also consistent with Kirchoff's voltage rule around a particular loop l.

Likewise, we rewrite the second line of (4.37) as

$$0 = -\sum_{l, e \in \mathcal{C}} \dot{q}_l B_{le} A_{ev} - \sum_{e \in \mathcal{I}} \frac{\partial E_e^L}{\partial \phi_v} = \sum_{l, e \in \mathcal{I}} \dot{q}_l B_{le} A_{ev} - \sum_{e \in \mathcal{I}} \frac{\partial E_e^L}{\partial \phi_v}. \tag{3.29}$$

A similar manipulation to the one in (3.23) and (3.24) can be used to show that (3.29) correctly incorporates the constitutive relation for inductors, as well as Kirchoff's current law on a particular node v.

3.2.4 A Hamiltonian theory from a Lagrangian theory

Classically, the dynamics of a circuit are entirely determined by the Lagrangian in (3.21). In order to quantize, however, we must produce a symplectic form, which will in turn imply quantum

commutation relations between charge and flux operators. Practically, this is a matter of enumerating the null vectors of M and integrating out related non–dynamical variables as was done in [90]. One of the notable properties of the formalism at hand is that the null vectors of M do not depend asymmetrically on inductors and capacitors.

In lieu of a formal discussion, we will focus on a particular example to demonstrate the simplicity of our results. For a formal discussion, see Appendix B.2. For the time being, we will consider the circuit drawn in Fig. 3.2, which we will call G. We will suppose that all circuit elements are linear and equal, though we emphasize that this is purely a matter of convenience. The Lagrangian for G is

$$\begin{aligned} L = &(q_{l_3} - q_{l_4})(\dot{\phi}_{v_1} - \dot{\phi}_{v_3}) + (q_{l_3} - q_{l_1})(\dot{\phi}_{v_4} - \dot{\phi}_{v_1}) + (q_{l_3} - q_{l_2})(\dot{\phi}_{v_3} - \dot{\phi}_{v_4}) \\ &- \frac{1}{2C}(q_{l_3} - q_{l_4})^2 - \frac{1}{2C}(q_{l_3} - q_{l_1})^2 - \frac{1}{2C}(q_{l_3} - q_{l_2})^2 \\ &- \frac{1}{2L}(\phi_{v_2} - \phi_{v_1})^2 - \frac{1}{2L}(\phi_{v_3} - \phi_{v_2}) - \frac{1}{2L}(\phi_{v_4} - \phi_{v_2})^2. \end{aligned} \tag{3.30}$$

Left null vectors of M, namely those vectors γ_l such that $\sum_l \gamma_l M_{lv} = 0$ for all v, correspond to cycles either made entirely of capacitors or entirely of inductors. In G, there is only a single such cycle, namely the cycle bounding the loop l_3. Constraints of this kind may involve the resolution of some constraint as is the case now:

$$0 = \frac{\delta S}{\delta q_{l_3}} = \frac{1}{C}(q_{l_3} - q_{l_4}) + \frac{1}{C}(q_{l_3} - q_{l_1}) + \frac{1}{C}(q_{l_3} - q_{l_2}). \tag{3.31}$$

The constraint in (3.31) corresponds to the demand that voltage must vanish about a loop, and further that q_{l_3} must be determined by the other loop charges. In other words, q_{l_3} is not dynamical. If, instead of capacitors, there were a cycle of inductors, so that there was no voltage constraint, the relevant q variable would simply be cancelled in L. This corresponds to a row of M which is identically zero in every entry.

On the other hand, right null vectors of M correspond to *cuts* of G which are entirely inductive or entirely capacitive. A cut of G is a set of edges that connect a subset of $\mathcal{V}$ to its complement. In

G, there is a single such example, namely the cut consisting of the branches incident upon v_2. Notice that $\dot{\phi}_{v_2}$ does not appear in (3.30) at all, so ϕ_{v_2} plays the role of a Lagrange multiplier. Recognizing that

$$0 = \frac{\delta S}{\delta \phi_{v_2}} \tag{3.32}$$

gives rise to another constraint. This constraint may be interpreted as a demand that current must be conserved at the node v_2, and that ϕ_{v_2} must therefore be fixed in terms of other fluxes.

There is another pair of null vectors which are omnipresent in the theory of circuits. Namely, energies may only depend upon differences of loop current (node flux), so the sums $\sum_v \phi_v$ and $\sum_{l\in\mathcal{F}} q_l$ are fixed by a so–called "gauge freedom".[6] Explicitly,

$$0 = \sum_v \frac{\delta S}{\delta \phi_v} = \sum_l \frac{\delta S}{\delta q_l}. \tag{3.33}$$

Since both of these functional derivatives vanish identically, a constraint can never be imposed by these particular null vectors.

After resolving the constraints in (3.31) and (3.32), we are left with

$$L = Q_1\dot{\Phi}_1 + Q_2\dot{\Phi}_2 - \frac{1}{3C}\left(Q_1^2 - Q_1Q_2 + Q_2^2\right) - \frac{1}{3L}\left(\Phi_1^2 - \Phi_1\Phi_2 + \Phi_2^2\right) \tag{3.34}$$

with

$$\begin{aligned} Q_1 &= q_{l_2} - q_{l_4} \\ Q_2 &= q_{l_2} - q_{l_1} \\ \Phi_1 &= \phi_{v_1} - \phi_{v_3} \\ \Phi_2 &= \phi_{v_4} - \phi_{v_1} \end{aligned}, \tag{3.35}$$

and

$$\{\Phi_j, Q_i\} = \delta_{ij}. \tag{3.36}$$

[6] We remark that $\sum_{l\in\mathcal{L}} q_l$ is not necessarily nondynamical. This leads to subtleties when discussing circuit dualities, which we will discuss in Section 3.4.

Therefore, we are free to write

$$H = \frac{1}{3C}\left(Q_1^2 - Q_1Q_2 + Q_2^2\right) + \frac{1}{3L}\left(\Phi_1^2 - \Phi_1\Phi_2 + \Phi_2^2\right), \tag{3.37}$$

with quantum commutation relations

$$[\Phi_j, Q_i] = \mathrm{i}\hbar\delta_{ij}. \tag{3.38}$$

The process of "integrating out" degrees of freedom associated to null vectors that arise during this process is conceptually straightforward. Still, as was the case in [90], nonlinear constraint equations may arise in the presence of nonlinear circuit elements without accompanying parasitic elements; these nonlinear equations may not have unique solutions, implying ambiguities in the correct phase space to quantize. Since it was argued in [106] that the addition of parasitic elements (which are present in any real experiment) qualitatively changes the spectrum, we do not focus on the technical question of picking the correct phase space in the presence of nonlinear constraints further in this manuscript.

There is another subtlety that arises in an attempt to formalize a generic quantization algorithm. In some circuits, there are conjugate pairs corresponding to a two–dimensional subspace of phase space that is not isomorphic to $\mathbb{R}^2$. The most readily available example is the transmon qubit (See Sec. 3.5.2), where the flux across the Josephson junction is thought to be compact, which implies that the charge across the capacitor is integer–valued. Physically, this implication can be understood by the fact that the energy on the capacitor is a function of the number of Cooper pairs in the condensate on the plates of the capacitor. Mathematically, the consequence of this fact that the operator Φ is not Hermitan. The problem of quantizing a Hamiltonian on a phase space of arbitrary topology is beyond the scope of this paper. However, we remark that the particular (and most experimentally relevant) example of a phase space isomorphic to $\mathbb{R} \times S^1$ is understood [29], and the

resolution is that the operator algebra (3.38) is replaced by

$$[e^{\mathrm{i}\Phi}, Q] = \mathrm{i}\hbar e^{\mathrm{i}\Phi}. \tag{3.39}$$

We will remark on the implications of this fact on circuit duality in Sec 3.4.

3.3 Equivalence to other formulations

Formally, it is true that the circuit Lagrangian (3.21) encodes all of the necessary physics: Kirchoff's rules are either manifestly obeyed, or correspond to the Euler-Lagrange equations of (3.21). It must then be true that the predictions of (3.21) are equivalent to previous theories in the literature. This equivalence is formally demonstrable, as we now summarize; see Appendix B.2 for details and/or justifications of the claims made in this section.

3.3.1 Branch–node formalism

The most common approach to circuit quantization involves writing a Lagrangian in terms of fluxes that live on vertices alone: ϕ_v. If we have linear capacitors, then one can directly integrate out q_l in (3.21) to obtain a Lagrangian for ϕ_v alone with quadratic terms in $\dot{\phi}$.

However, it was recently pointed out in [90] that more general circuits could be quantized by instead starting with a Lagrangian that depends on both $q_{e\in\mathcal{C}}$ (charges on capacitive edges) and ϕ_v. Comparing the flux-charge symmetric formulation to this one is slightly more subtle. Recall that we may write

$$M_{lv} = \sum_{e\in\mathcal{C}} B_{le} A_{ev}. \tag{3.40}$$

Introduce a set of Lagrange multipliers λ_e on capacitive edges, and write:

$$L' = \sum_{e\in\mathcal{C}} q_l B_{le} A_{ev} \dot{\phi}_v - \sum_{e\in\mathcal{C}} E_e^C(Q_e) - \sum_{e\in\mathcal{I}} E_e^L \left(\sum_v A_{ev}\phi_v \right) + \sum_{e\in C} \lambda_e \left(Q_e - \sum_l q_l B_{le} \right). \tag{3.41}$$

The functional derivative

$$0 = \frac{\delta S}{\delta q_l} = \sum_{e \in \mathcal{C}} B_{le} \left[\sum_v A_{ev} \dot{\phi}_v - \lambda_e \right] \tag{3.42}$$

implies that the quantity

$$\gamma_e = \sum_v A_{ev} \dot{\phi}_v - \lambda_e \tag{3.43}$$

forms a right null vector of the matrix B restricted to the set of capacitive edges. Corollary 80 shows that there is one such null vector for each capacitive cut of G. As a consequence, there exists m vectors $|n_i\rangle = \sum_{e \in \mathcal{C}} \sigma_{e,i} |e\rangle$ with $\sigma_{e,i} \in \{-1, 0, 1\}$ with $i = 1, 2, 3, \ldots, m$ such that

$$|\gamma\rangle = \sum_{i=1}^{m} \mu_i |n_i\rangle \tag{3.44}$$

and thus

$$\lambda_e = \sum_v A_{ev} \dot{\phi}_v - \sum_{i=1}^{m} \mu_i \sigma_{e,i} \tag{3.45}$$

for some parameter μ. Thus, it follows that L' may be rewritten

$$L' = \sum_{e \in \mathcal{C}} Q_e A_{ev} \dot{\phi}_v - \sum_{e \in \mathcal{C}} E_e^C(Q_e) - \sum_{e \in \mathcal{I}} \left(\sum_v A_{ev} \phi_v \right) - \sum_{i=1}^{m} \mu_i \sum_e \sigma_{e,i} Q_e. \tag{3.46}$$

A functional derivative

$$0 = \frac{\delta S}{\delta \mu_i} = \sum_e \sigma_{e,i} Q_e \tag{3.47}$$

is easily understood as a consequence of

$$\sum_{l,e} q_l B_{le} \sigma_{e,i} = 0 \tag{3.48}$$

and can be recognized as a constraint that would follow from a Noether current in [90]. Borrowing the terminology used in [90], the right null vectors of B restricted to edges in $\mathcal{C}$ corresponded to a "capacitively shunted island". In the end, after using the Lagrange multiplier μ_i to fix the charges across capacitive cuts, we see that (3.46) reproduces [90].

3.3.2 Face–Edge formalism

A possibly undesirable feature of the Lagrangian in (3.46) is that capacitors are treated differently than inductors. With the flux-charge symmetric framework, however, we can produce a similarly universal theory of circuits that treats inductors as preferential. We write

$$M_{lv} = -\sum_{e\in\mathcal{I}} B_{le}A_{ev} \tag{3.49}$$

In direct analogy with the discussion in Sec. 3.3.1, we again introduce Lagrange multipliers λ_e, but this time they exist only on inductive edges:

$$L' = -\sum_{e\in\mathcal{I}} q_l B_{le}A_{ev}\dot{\phi}_v - \sum_{e\in\mathcal{C}} E_e^C\left(\sum_l q_l B_{le}\right) - \sum_{e\in\mathcal{I}} E_e^L\left(\sum_v \psi_e\right) + \sum_{e\in\mathcal{I}} \lambda_e\left(\psi_e - \sum_v A_{ev}\phi_v\right). \tag{3.50}$$

By taking functional derivatives and exploiting the properties of A, it is possible to perform a calulation very closely mirroring the one in Sec 3.3.1 to produce

$$L' = -\sum_{e\in\mathcal{I},l} q_l B_{le}\dot{\psi}_e - \sum_{e\in\mathcal{C}} E_e^C(Q_e) - \sum_{e\in\mathcal{I}} E_e^L(\psi_e). \tag{3.51}$$

Of note, the role of matrix A in (3.46) is filled by $-B$ in (3.51). From a formal perspective, the matrix B is less well–behaved than A since, for nonplanar graphs, B may have less trivial null vectors than A, and there is no corresponding subtlety for the structure of A, even for nonplanar graphs. (3.51) is a so–called loop–branch[7] Lagrangian.

In the special case when the circuit corresponds to a planar graph, it was noted in [117] that Lagrangians of the form (3.51) are dual to Lagrangians of the form (3.46), as we discuss in Section 3.4. This fact is intimately related to the appearance of B in (3.51) instead of A. Using the flux-charge symmetric Lagrangian (3.21), we will see a more straightforward derivation of such duality.

[7] In the literature, it is sometimes said that any Lagrangian involving charges defined on a loop is called a "loop–charge Lagrangian".

3.4 Circuit duality

The fact that we are able to describe a circuit in either a framework that "prefers" capacitors *or* inductors is reminiscent of classical discussions of circuit duality [25, 58, 68, 77, 119]; see [117] for a discussion of such circuit dualities in the context of Lagrangian mechanics of superconducting circuits. We now show that these dualities are especially transparent in our framework. A formal discussion of the results that follow is found in Appendix B.3.

First we make a few general comments. Any sensible notion of duality should be an involution (operation which is the identity when applied twice) on classical phase space (or quantum Hilbert space) of a circuit. A circuit is said to be self-dual if its Hamiltonian is invariant under this transformation.

One kind of duality arises by a mere "relabeling transformation" on (3.21). With the Lagrangian (3.21) in mind, consider the transformation given by

$$\begin{aligned}
\phi_v &\to \phi_v^* = q_v' \\
q_l &\to q_l^* = \phi_l' \\
A &\to A^* = B^{\mathrm{T}} \\
B &\to B^* = A^{\mathrm{T}} \\
\mathcal{C} &\to \mathcal{C}^* \simeq \mathcal{I} \\
\mathcal{I} &\to \mathcal{I}^* \simeq \mathcal{C} \\
\mathcal{V} &\to \mathcal{V}^* \simeq \mathcal{L} \\
\mathcal{F} &\to \mathcal{L}^* \simeq \mathcal{V}.
\end{aligned} \tag{3.52}$$

The table in Fig.3.3 illustrates the transformation (3.52). Informally, (3.52) swaps fluxes with charges, capacitors with inductors, and nodes with faces. Under (3.52), the connection matrix of G

transforms as[8]

$$M \to M^* = -M^{\mathrm{T}}. \tag{3.53}$$

Therefore, the Lagrangian in (3.21) transforms as

$$L \to L^* = \sum_{l,v} q_l^* M_{fv}^* \dot{\phi}_v^* - \sum_{e \in \mathcal{C}^*} E_e^C \left(\sum_l q_l^* B_{fe}^* \right) - \sum_{e \in \mathcal{I}^*} E_e^L \left(\sum_v A_{ev}^* \phi_v^* \right) \tag{3.54}$$

or

$$L^* = \sum_{l,v} q_v' M_{vl} \dot{\phi}_l' - \sum_{e \in \mathcal{I}} E_e^C \left(\sum_{l \in \mathcal{V}^*} A_{el} \phi_l' \right) - \sum_{e \in \mathcal{C}} E_e^L \left(\sum_{v \in \mathcal{F}^*} q_v' B_{ve} \right) \tag{3.55}$$

where we have integrated the first term by parts after carrying out the substitutions in (3.52).

It appears from (3.52) that duality is an involution (operation that squares to the identity) on phase space. However, we emphasize that this interpretation can be subtle when the phase space is not equivalent to $\mathbb{R}^{2n}$. The most common scenario relevant for superconducting circuits is, as discussed before, the case where some of the coordinates are periodic. In particular, if the classical phase space where a conjugate pair (Q, Φ) lives is $\mathbb{R} \times S^1$, then the phase space associated with the dual conjugate pair (Φ^*, Q^*) is $S^1 \times \mathbb{R}$. Physically, this means that a quantum Hamiltonian with a compact flux variable Φ and integer valued conjugate Q has a dual with a compact charge variable Φ^* and an integer valued flux Q^*. We conclude that in this example, duality maps one phase space to another.

The non-trivial question is whether L^* can be interpreted as the Lagrangian for an *actual circuit*, where q_l^* represent physical loop charges and ϕ_v^* represent physical node fluxes. We address this for the remainder of the section.

3.4.1 Planar circuits

One way to understand the subtlety of this duality transformation is to ask whether it behaves "nicely" on the *drawings* of a circuit. In other words, does L^* describe a physical circuit that we can

[8] We remark that X^* is the dual of some object X under the transformation (3.52) and not, for example, complex conjugation.

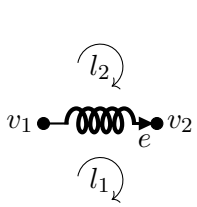
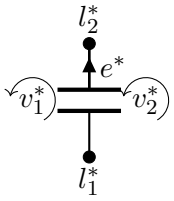
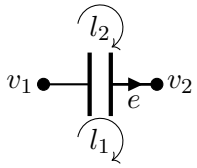
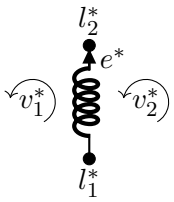
object	dual object
	
	
$A_{ev_1} = -1$	$B^*_{v_1^* e^*} = -1$
$B_{l_1 e} = 1$	$A^*_{e^* l_1^*} = 1$
$M_{l_1 v_1}$	$-M_{v_1^*, l_1^*}$
ϕ_{v_1}	$q_{v_1^*}$
q_{l_1}	$\phi_{l_1^*}$

Figure 3.3: A number of examples showing how the transformation (3.52) effects various aspects of a circuit. We emphasize that, from the perspective of this formalism, nonlinear inductors may be treated simply as inductors on equal footing with linear inductors. In particular, a Josephson junction would be dual to a quantum phase slip.

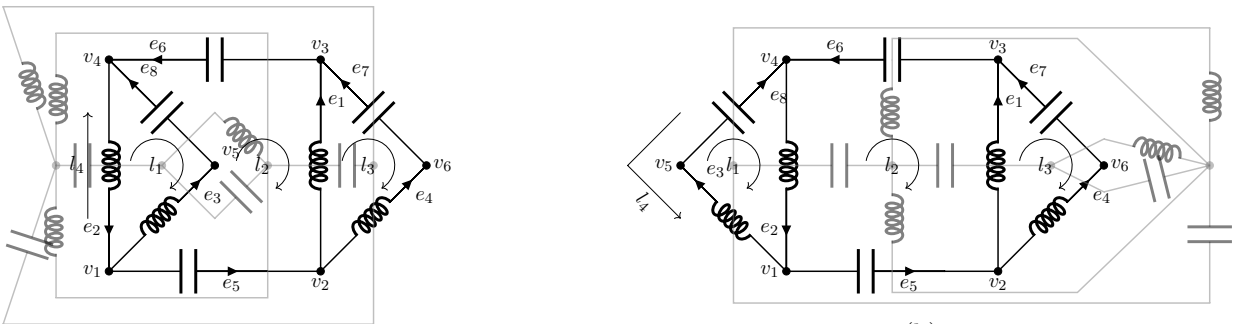


Figure 3.4: The black circuits in (a) and (b) are simply redrawings of the same circuit. The duals of the circuits (a) and (b) are drawn in grey. In both subfigures, labels correspond to the circuit in black.

easily draw? In what follows, we will assume that capacitances and inductances have the same units, or equivalently that charges and fluxes have the same units. This is a matter of convenience that can be adapted to a physical system by making suitable variable redefinitions. The discussion in this section only applies to planar circuits, and in the next section we will discuss the problems that may arise for nonplanar circuits.

The procedure for constructing the dual of a given circuit G is as follows [45, 117]:

(1) Draw G on a plane (or equivalently a sphere, by identifying points at infinite distance on the plane with a single point).

(2) For every face[9] l in G, draw a node inside the l labeled l^*

(3) Every branch e in G lies on the boundary of exactly two faces, say l_1 and l_2. If e is capacitive (inductive), draw an inductive (capacitive) branch labeled e^* connecting l_1^* and l_2^*.

An example of this drawing procedure is given in Fig.3.4.

We remark that B, the orientation matrix of G, depends upon how G is drawn. Since the incidence matrix of the dual of G is the transpose of B, it follows that a single circuit can have multiple dual circuits. Since G's incidence matrix is uniquely defined, all duals of G have the same orientation matrix, and the number of duals of G with distinct incidence matrices is exactly the number of drawings of G with different orientation matrices. For example, in Fig.3.4(b), the dual of G has a node of degree six, which the dual of G in (a) has no such vertex. In this way, we can see that circuits need not have a unique dual. A detailed discussion of this circuit is in Section 3.5.1.

The manner in which incidence and orientation matrices are “exchanged" under duality can be interpreted in the following way: the combinatorial information of a graph is encoded in the topological information of its dual, and the topological information of a graph determines the combinatorial information in its dual. In this way, it is sensible to say that, for graphs (and thus circuits), topology is dual to combinatorics.

[9]Recall that there are no loops that cannot be drawn as faces for a planar graph.

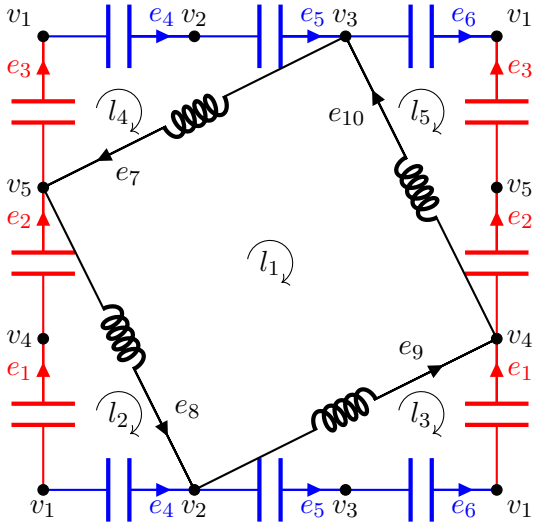


Figure 3.5: A circuit on K_5. The edges with alike labels are to be identified. The loops l_6 and l_7 are drawn in red and blue respectively.

3.4.2 Nonplanar circuits

The transformation (3.52) is the most natural candidate for a duality transformation at the Lagrangian level. Unfortunately, as we now discuss, for nonplanar circuits it is not general clear how to construct a physical circuit for which L^* is its Lagrangian. The difficulty in constructing the dual to a nonplanar circuit arises when attempting to draw the dual circuit, rather than when trying to produce the dual Lagrangian.

For planar graphs, it is possible to choose a suitable set of loops $\mathcal{L}$ by drawing the circuit in question on a surface (namely the plane), and such circuits can be made to have the property that

$$\sum_l B_{le} = 0. \tag{3.56}$$

The importance of this property is that every edge in the circuit participates in precisely two loops within $\mathcal{L}$. On the other hand, for all circuits, including nonplanar ones, we demand

$$\sum_v A_{ev} = 0. \tag{3.57}$$

The duality transformation (3.52) sends

$$\begin{aligned} B \to B^* &= A^{\mathrm{T}} \\ A \to A^* &= B^{\mathrm{T}}. \end{aligned} \tag{3.58}$$

If

$$\sum_{l \in \mathcal{L}} B_{le} \neq 0, \tag{3.59}$$

then $A^* = B^{\mathrm{T}}$ is *not a valid incidence matrix*. The reason for this is the $2g$ "topological loops" that must be included in $\mathcal{L}$ in order to fully specify all of the currents in the nonplanar circuit of interest. See Fig 3.5 for a drawing of a circuit with topological loops chosen and emphasized with color coding. Consider some edge e that participates in a topological loop l_t. e also lies on the boundary of a pair of faces, say l_1 and l_2. Then,

$$\sum_{l} B_{le} = B_{l_t e} + B_{l_1 e} + B_{l_2 e} = B_{l_t e} \neq 0. \tag{3.60}$$

Thus, if one attempted to interpret B^{T} as an incidence matrix, the edge e would "connect" three vertices, rather than two, but then the resulting construction is not an edge. From a mathematical perspective, it is no issue to acknowledge that an edge in a graph participates in many cycles, but there is no sensible notion of an edge that connects many vertices.

Moreover, any notion of graph duality must conserve g, the genus of the surface of embedding. We see, however that

$$B \to B^* = A^{\mathrm{T}} \tag{3.61}$$

gives rise to B^* that *does* have the property that

$$\sum_{l} B^*_{le} = 0, \tag{3.62}$$

which is a hallmark property of planar graphs in the sense that (3.62) holds for and only for planar

graphs.

Hence, there exist nonplanar circuits with no physical dual. We remark that (3.52) is still perfectly well–defined for nonplanar circuits, but that it may not be possible to draw a circuit that gives rise to the resulting Hamiltonian. It may be possible that one can always introduce auxiliary degrees of freedom and integrate them out, such that L^* is the reduced description of a circuit with (nonlinear) capacitive loops or inductive cuts, but we have not found an elegant and fully general construction which allows us to construct an honest circuit dual for a general nonplanar circuit. Resolving this question seems to us to be an important open problem.

While the problem of constructing nonplanar circuits is not a central subject in this paper, we would be remiss not to remark briefly on challenges therein. Nonplanar graphs all contain some number of copies of K_5 or $K_{3,3}$ as subgraphs or minors.[10] A notable property of these graphs, which are necessary ingredients in nonplanarity, is that removing any edge from either K_5 or $K_{3,3}$ gives rise to a planar graph. Thus, for example, a circuit on K_5 could be constructed by placing all but one circuit element on a single layer, and the remaining circuit element could be placed on a second layer connected to the first. That is to say that the construction of at least some nonplanar circuits seems reasonable in principle, provided that multilayer fabrication is possible.

3.5 Examples

In what follows, we will solve a number of examples that are either particularly pedagogical or of particular interest.

3.5.1 A planar circuit

Consider the black circuit drawn in Fig.3.4(a), which we will call G. The face in the dual of G (drawn in grey) corresponding to the vertex v_i in G will be labeled v_i^*. We will refer to the grey circuit as G^*. The capacitance (inductance) of the circuit on element on e_i will be denoted as C_i (L_i). We suppose for the sake of simplicity in the presentation that all circuit elements are linear,

[10] K_5 is the fully connected graph on 5 vertices, and $K_{3,3}$ is the complete bipartite graph on 6 vertices.

but we emphasize that this is unnecessary. We will begin by analyzing G.

To start,

$$A = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{array} \end{array} \tag{3.63}$$

and

$$B = \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{array}{c} l_1 \\ l_2 \\ l_3 \\ l_4 \end{array} \end{array} \tag{3.64}$$

and thus

$$M = \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & -1 \end{pmatrix} \begin{array}{c} l_1 \\ l_2 \\ l_3 \\ l_4 \end{array} \end{array}, \tag{3.65}$$

Now, the Lagrangian for G is given by

$$\begin{aligned} L = {} & (q_4 - q_2)(\dot{\phi}_{v_2} - \dot{\phi}_{v_1}) + (q_2 - q_1)(\dot{\phi}_{v_4} - \dot{\phi}_{v_5}) + (q_4 - q_3)(\dot{\phi}_{v_3} - \dot{\phi}_{v_6}) + (q_4 - q_2)(\dot{\phi}_{v_4} - \dot{\phi}_{v_3}) \\ & - \frac{1}{2L_1}(\phi_{v_3} - \phi_{v_2})^2 - \frac{1}{2L_2}(\phi_{v_1} - \phi_{v_4})^2 - \frac{1}{2L_3}(\phi_{v_5} - \phi_{v_1})^2 - \frac{1}{2L_4}(\phi_{v_6} - \phi_{v_2})^2 \\ & - \frac{1}{2C_5}(q_4 - q_2)^2 - \frac{1}{2C_6}(q_4 - q_2)^2 - \frac{1}{2C_7}(q_4 - q_3)^2 - \frac{1}{2C_8}(q_2 - q_1)^2. \end{aligned} \tag{3.66}$$

There are two right null vectors of M. The first corresponds to

$$0 = \frac{\delta S}{\delta \phi_{v_2}} + \frac{\delta S}{\delta \phi_{v_3}} + \frac{\delta S}{\delta \phi_{v_6}} \tag{3.67}$$

which contains no nontrivial constraint. The other nontrivial null vector corresponds to

$$0 = \frac{\delta S}{\delta \phi_{v_1}} + \frac{\delta S}{\delta \phi_{v_2}} = \frac{1}{L_1}(\phi_{v_2} - \phi_{v_3}) + \frac{1}{L_2}(\phi_{v_1} - \phi_{v_4}) + \frac{1}{L_4}(\phi_{v_1} - \phi_{v_5}) + \frac{1}{L_4}(\phi_{v_2} - \phi_{v_6}). \tag{3.68}$$

Resolving this constraint and defining

$$\begin{aligned} Q_1 &= q_4 - q_2 \\ Q_2 &= q_2 - q_1 \\ Q_3 &= q_4 - q_3 \\ \Phi_1 &= \phi_{v_2} + \phi_{v_4} - \phi_{v_1} - \phi_{v_3} \\ \Phi_2 &= \phi_{v_4} - \phi_{v_5} \\ \Phi_3 &= \phi_{v_3} - \phi_{v_6}, \end{aligned} \tag{3.69}$$

we find immediately

$$\begin{aligned} L = Q_1\dot{\Phi}_1 + Q_2\dot{\Phi}_2 + Q_3\dot{\Phi}_3 - \frac{1}{2}\left(\frac{1}{C_5} + \frac{1}{C_6}\right)Q_1^2 - \frac{1}{2C_7}Q_3^2 - \frac{1}{2C_8}Q_2^2 \\ - \frac{L_\Sigma}{2}\left(\frac{1}{L_1L_2}\Phi_1^2 + \frac{1}{L_1L_3}\left(\Phi_1 - \Phi_2\right)^2 + \frac{1}{L_1L_4}\Phi_3^2 + \frac{1}{L_2L_3}\Phi_2^2 + \frac{1}{L_2L_4}\left(\Phi_1 + \Phi_3\right)^2 + \frac{1}{L_3L_4}\left(\Phi_1 - \Phi_2 + \Phi_3\right)^2\right) \end{aligned} \tag{3.70}$$

with

$$\frac{1}{L_\Sigma} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \frac{1}{L_4}. \tag{3.71}$$

By taking the transformation

$$\begin{aligned} Q_i &\to Q_i^* = -\Phi_i \\ \Phi_i &\to \Phi_i^* = Q_i, \end{aligned} \tag{3.72}$$

we can acquire the Lagrangian (and Hamiltonian) of the grey circuit drawn in Fig 3.4 (a).

We remark that the black circuit drawn in Fig 3.4 (b) is simply a different drawing of the black circuit drawn in Fig 3.4(a), but the grey circuit in Fig 3.4 (b) is not even graph isomorphic to the grey circuit drawn in Fig 3.4 (a). A similarly simple analysis of the black circuit drawn in Fig 3.4 (b) would provide an canonically related Hamiltonian to the one governing G (of course, since the two circuits are identical). It then follows that the Hamiltonian governing the grey circuits must also be canonically related.

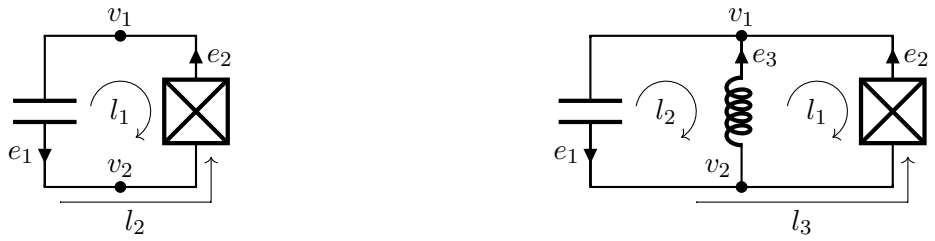


Figure 3.6: Pictured in (a) is the Transmon. In (b) is drawn the fluxonium qubit.

3.5.2 Transmon and Fluxonium qubits

We now turn our attention to a pair of (formally) very similar circuits of practical interest. This pair of examples is useful in that it illustrates the “automatic" solution of a voltage constraint. We will start by considering the fluxonium qubit pictured in Fig. 3.6(b). The A, B, and M matrices

may be read off of the drawing:

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \end{array} \begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix},$$

$$B = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} l_1 \\ l_2 \\ l_3 \end{matrix} \tag{3.73}$$

$$M = \begin{array}{c} \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array} \begin{matrix} l_1 \\ l_2 \\ l_3 \end{matrix}.$$

Thus, the Lagrangian describing the fluxonium qubit is

$$L_{\text{fluxonium}} = (q_{l_2} - q_{l_3})(\dot{\phi}_{v_1} - \dot{\phi}_{v_2}) - \frac{1}{2C}(q_{l_2} - q_{l_3})^2 - \frac{1}{2L}(\phi_{v_1} - \phi_{v_2})^2 + E_J \cos(\phi_{v_1} - \phi_{v_2}). \tag{3.74}$$

After making the definitions

$$\begin{aligned} \Phi &= \phi_{v_1} - \phi_{v_2} \\ Q &= q_{l_2} - q_{l_3}, \end{aligned} \tag{3.75}$$

we see that the (classical) Hamiltonian for the fluxonium qubit is

$$H_{\text{fluxonium}} = \frac{1}{2C}Q^2 + \frac{1}{2L}\Phi^2 - E_J \cos(\Phi). \tag{3.76}$$

Now, for the transmon qubit,

$$M_{\text{transmon}} = \begin{array}{c} \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{matrix} l_1 \\ l_2 \end{matrix} \end{array} . \tag{3.77}$$

Notice that M_{transmon} appears as a submatrix of M in (3.73). This can be understood as a consequence of the fact that Kirchhoff's voltage rule demands that the flux across the linear inductor and Josephson junction in fluxonium must be equal (up to a constant of integration that we set to zero). It is straightforward to produce

$$H_{\text{transmon}} = \frac{1}{2C}Q^2 - E_J \cos(\Phi). \tag{3.78}$$

There is a subtlety regarding the process of quantizing (3.76) and (3.78) after arriving at a classical Hamiltonian. Namely, $H_{\text{fluxonium}}$ is not invariant under translation of Φ by any nonzero constant. In other words, $H_{\text{fluxonium}}$ is inconsistent with a periodically identified "compact" Φ variable. To contrast this observation, it is thought that Φ as it appears in H_{transmon} is a compact variable.

While it is true that the question of how the topology of phase space affects quantization

is not the subject of this paper, we remark that phase space in the fluxonium case is isomorphic (equivalent) to $\mathbb{R}^2$, while it may be the case that the classical phase space in the transmon case is $\mathbb{R} \times S^1$. The dual of the fluxonium qubit likewise must have classical phase space $\mathbb{R}^2$, and the dual of the transmon has an equivalent phase space to that of the original transmon (albeit the periodically-identified variable changes physical interpretation from flux to charge).

3.5.3 A self–dual circuit

Consider the circuit drawn in Fig.3.2. The Hamiltonian produced is given by

$$H(Q, \Phi) = \frac{1}{3C}(Q_1^2 - Q_1 Q_2 + Q_2^2) + \frac{1}{3L}(\Phi_1^2 - \Phi_1 \Phi_2 + \Phi_2^2). \tag{3.79}$$

It is obvious that $H(-\Phi, Q) = H(Q, \Phi)$ provided that we choose units where $C = L$, which is to say that the circuit drawn in Fig.3.2 is, or at least *has* a self–dual drawing.

3.5.4 A circuit on K_5

For the purposes of this example, we will focus on the circuit drawn in Fig. 3.5, which will again be called G. The drawing at hand is slightly unconventional since it is drawn on a polygon with boundaries identified (which is typical in mathematical descriptions of a torus, for example). For the sake of simplicity, we will suppose that all inductances take on a value L and all capacitances take on a value C.

First, we must compute the relevant A, B, and M matrices:

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \\ \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \end{matrix} \end{array}, \tag{3.80}$$

$$B = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \end{matrix} \end{array}$$

Thus,

$$\begin{aligned} L = &\left(q_{l_3} - q_{l_2}\right)\left(\dot{\phi}_{v_2} - \dot{\phi}_{v_5}\right) + \left(q_{l_4} - q_{l_5}\right)\left(\dot{\phi}_{v_3} - \dot{\phi}_{v_5}\right) + \left(q_{l_5} - q_{l_3}\right)\left(\dot{\phi}_{v_4} - \dot{\phi}_{v_5}\right) \\ &- \frac{1}{2C}\left(\left(q_{l_2} - q_{l_3} + q_{l_6}\right)^2 + \left(q_{l_2} - q_{l_5} + q_{l_6}\right)^2 + \left(q_{l_4} - q_{l_5} + q_{l_6}\right)^2\right) \\ &- \frac{1}{2C}\left(\left(q_{l_4} - q_{l_2} + q_{l_7}\right)^2 + \left(q_{l_4} - q_{l_3} + q_{l_7}\right)^2 + \left(q_{l_5} - q_{l_3} + q_{l_7}\right)^2\right) \\ &- \frac{1}{2L}\left(\left(\phi_{v_5} - \phi_{v_3}\right)^2 + \left(\phi_{v_2} - \phi_{v_5}\right)^2 + \left(\phi_{v_4} - \phi_{v_2}\right)^2 + \left(\phi_{v_4} - \phi_{v_3}\right)^2\right). \end{aligned} \tag{3.81}$$

Choosing the variables

$$\begin{aligned} Q_1 &= q_{l_3} - q_{l_2} \\ Q_2 &= q_{l_4} - q_{l_5} \\ Q_3 &= q_{l_5} - q_{l_3} \\ \Phi_1 &= \phi_{v_2} - \phi_{v_5} \\ \Phi_2 &= \phi_{v_3} - \phi_{v_5} \\ \Phi_3 &= \phi_{v_4} - \phi_{v_5} \end{aligned} \tag{3.82}$$

we can rewrite

$$\begin{aligned} L = \sum_{i=1}^{3} Q_i\dot{\Phi}_i &- \frac{1}{2C}\left(\left(q_{l_6} - Q_1\right)^2 + \left(q_{l_6} - Q_1 - Q_3\right)^2 + \left(Q_2 + q_{l_6}\right)^2\right) \\ &- \frac{1}{2C}\left(\left(q_{l_7} + Q_1 + Q_2 + Q_3\right)^2 + \left(q_{l_7} + Q_2 + Q_3\right)^2 + \left(q_{l_7} + Q_3\right)^2\right) \\ &- \frac{1}{2L}\left(\Phi_2^2 + \Phi_1^2 + \left(\Phi_3 - \Phi_1\right)^2 + \left(\Phi_2 - \Phi_3\right)^2\right). \end{aligned} \tag{3.83}$$

From (3.83) it is clear to see that q_{l_6} and q_{l_7} are constrained in terms of Q variables as a result of

$$0 = \frac{\delta S}{\delta q_{l_6}} = \frac{\delta S}{\delta q_{l_7}}. \tag{3.84}$$

After using (3.84), we find

$$\begin{aligned} q_{l_6} &= \frac{1}{3}(2Q_1 - Q_2 + Q_3) \\ q_{l_7} &= \frac{1}{3}(-Q_1 - 2Q_2 - 3Q_3). \end{aligned} \tag{3.85}$$

Then, the Hamiltonian describing G is given by $H(Q_1, Q_2, Q_3, \Phi_1, \Phi_2, \Phi_3) = \sum_{i=1}^{3} Q_i\dot{\Phi}_i - L$. The

Poisson brackets of the system at hand are

$$\{\Phi_i, Q_j\} = \delta_{ij}. \tag{3.86}$$

One may find it peculiar that the "topological loops" q_{l_6} and q_{l_7} are nondynamical in the sense that they are not independent of loops on the surface of the torus. This is no accident. Indeed, for circuits on the fully connected graph on V vertices, K_V, there are $\binom{V}{2}$ edges, and there must be at least $2g$ loops made of only capacitors or only inductors. To see how this is true, observe that the number of independent loops in a circuit G with E edges and V vertices is at least

$$\text{number of loops} \geq n(E, V) = \max\left(E - V + 1, 0\right). \tag{3.87}$$

We emphasize that this bound on $n(E, V)$ holds even for graphs that are not connected. Now, suppose the number of capacitors in a circuit is N_C, while the number of inductors is N_I. There exists a pair of graphs G_C and G_I that consist of all vertices from G and only the capacitive edges or inductive edges respectively. The number of inductive (capacitive) loops in G_I (G_C) is then at least $n(N_I, V)$ ($n(N_C, V)$). Since G_C and G_I are subgraphs of G, it follows that the number of homogenous loops in G is at least $n(N_I, V) + n(N_C, V)$. Moreover, it must be the case that $N_I + N_C = E$. Thus, we have that the number of homogenous loops in G is bounded below by

$$N_l = n(N_I, V) + n(N_C, V) = n(E - N_C, V) + n(N_C, V). \tag{3.88}$$

Now, for fully connected graphs K_V, $E = \binom{V}{2}$, so a circuit on K_V with N_C capacitors has

$$N_l = n\left(\binom{V}{2} - N_C, V\right) + n(N_C, V) \geq 2g \tag{3.89}$$

since

$$\min_x \left[\max\left(\binom{V}{2} - x - V + 1, 0\right) + \max\left(x - V + 1, 0\right)\right] \geq 2g. \tag{3.90}$$

In a few words, a fully connected graph on V vertices can always be drawn on a g–holed torus in such a way that all $2g$ topological loops are null vectors of M. It is, in principle, always possible to synthesize some planar circuit with the same dynamics as a given circuit on K_5.

While it is generally impossible to construct a dual circuit for a nonplanar circuit, it is nonetheless sometimes possible to do something similar. In particular, it is sometimes possible to reduce a nonplanar circuit to an effectively planar circuit by using constraints that follow from Kirchoff's rules (e.g. by adding inductors or capacitors in series or parallel or by using the delta–wye transform [73]) . One way to accomplish this goal is to arrange circuit elements such that the edges lying on a given topological loop contain either all capacitors or all inductors (so that the relevant loop degree of freedom is nondynamical). Then, one can produce a simplified equivalent circuit with fewer edges (by using Kirchoff's rules) than the original. If the simplified circuit is then planar, and has a well–defined dual. In some sense, the resulting dual serves as what would be the dual of the nonplanar circuit. Whether or not this planarization may be carried out apparently depends on whether or not there exist at least $2g$ homogenous loops in a circuit. We mention in passing that no similar result holds for circuits on, say $K_{3,3}$. From this we suspect that the most interesting nonplanar circuits are likely to be those that are most sparsely connected.

3.6 Conclusions

In this paper, we have derived a new Lagrangian description of circuits that treats fluxes and charges on an equal footing, extending our previous work [90] towards a more general theory of circuit quantization. Existing methods can be shown to be equivalent to our formalism. A key feature of this "flux-charge symmetric" formulation of circuit mechanics is that circuit duality becomes a simple relabeling transformation, for planar circuits.

For non-planar circuits, it is not always obvious if a dual circuit is physical. An interesting future research direction is to understand whether it is possible to add *additional* circuit elements such that the dual of a general non-planar circuit is well-defined.

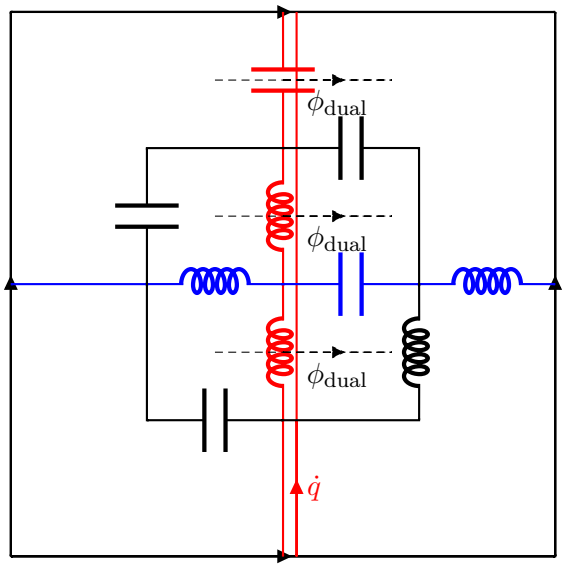


Figure 3.7: A heuristic drawing on an identified polygon representation of a torus. If a charge or flux about some "topological" loop is to be dynamical, then it's dual must also be topological. A "topological current" is drawn and labeled in red. The naive dual to the current $\dot{q}$ would be a voltage drop across the corresponding cut. In the figure above, the nonlocal "topological" voltage resulting from the naive duality map would participate in all three dual edges with the label ϕ_{dual}.

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Chapter 4

Stochastic theory of nonlinear electrical circuits in thermal equilibrium

4.1 Introduction

Qubits made out of superconducting circuits [10, 48, 61, 62, 65, 78] represent a promising candidate qubit for building a scalable fault-tolerant quantum computer. Some schemes even exist to use dissipation to assist in error correction [34, 86]. As with all quantum technologies, one of the important challenges in building robust qubits is protecting them against decoherence with a thermal environment.

A common way that dissipation with environmental degrees of freedom is modeled in the literature is via the Caldeira-Leggett model [18], in which a dissipative bath is modeled by a large number of dissipationless degrees of freedom. This approach is frequently used in the literature on circuit quantization [1, 10, 20, 82, 92, 118]. Yet in a quantum setting, it is rather undesirable to include a very large number of degrees of freedom (which causes enormous computational overhead) simply to account for the existence of dissipation. In principle, it would be more desirable to simply model the dissipative dynamics with a Lindblad master equation, describing the open quantum dynamics of only the relevant circuit degrees of freedom.

Before such a task can be achieved, it is desirable to first develop a thorough understanding of the *classical* dynamics of electrical circuits interacting with a thermal bath, within a framework that is amenable to quantization. For dissipationless LC circuits, this has recently been achieved in an intriniscally Hamiltonian formulation of circuit mechanics [90, 93], building on earlier work [14, 68, 119]. This Hamiltonian formulation contrasts with the more conventional Lagrangian approach used

in much of the superconducting circuit literature [10, 13, 26, 31, 61, 62, 65, 88, 104, 114, 117, 118, 122]. For dissipative circuits with linear resistors, [80, 93, 94] have also incorporated the resistors within a classical Hamiltonian formalism via Rayleigh dissipative functions.

This paper provides an alternative route towards *systematically* incorporating dissipation into the mechanical theory of electrical circuits. Following the recent literature on effective theories for thermal systems [28, 40, 41, 49, 54], we write down a dissipative Lagrangian for the dynamics of an RLC circuit with Gaussian white noise. The approach is closely related to the much older Martin-Siggia-Rose (MSR) Lagrangian [81] for stochastic dynamics. Indeed, a *crucial* aspect of our construction is that it necessarily incorporates both dissipation along with stochastic fluctuations, which are mandated by the fluctuation-dissipation theorem. Indeed, it is not consistent with statistical mechanics to neglect the presence of noise when modeling a dissipative system. Hence, we must develop a description of circuit dynamics for circuits in thermal equilibrium at temperature T, ensuring our description is compatible with the laws of statistical physics. We can accomplish this without needing to worry about the microscopic details of the environment and its coupling to the relevant circuit degrees of freedom. In our view, this makes our starting point more physically natural than a Rayleigh dissipative function, although we will ultimately see how to reproduce (and generalize) the physics of Rayleigh dissipative functions within our framework.

As our formalism necessarily incorporates both dissipative and stochastic physics, it allows us to reproduce the physics of Johnson noise [55, 89, 116], which was an important historical benchmark for testing the fluctuation-dissipation theorem itself (or, more precisely, testing that physical systems are in thermal equilibrium). When resistors are linear, we will explain how the MSR theory reproduces the textbook predictions for Johnson noise in complete generality. However, at least in principle, it is also transparent in our framework how to model *nonlinear resistors*, which imply *multiplicative Johnson noise*. There have been inconsistent predictions in the literature [57, 111] on the correct form of multiplicative Johnson noise; our formalism gives unambiguous predictions which can be derived transparently. In particular, for circuits where resistors have suitable “parasitic" capacitors or inductors alongside them, we are able to give simple analytic expressions for multiplicative Johnson

noise, regardless of the global circuit topology. In the appropriate limiting case, our results agree with [111].

This paper focuses on classical circuits, as the problem is already somewhat involved. Nevertheless, we do expect that our methods could be a starting point for building intrinsically dissipative descriptions of quantum RLC circuits, in future work.

4.2 Review of stochastic and dissipative mechanics

In a physical setting, the fluctuation-dissipation theorem requires both dissipation and stochastic noise, so it is sensible to demand that any systematic formalism for modeling dissipative circuit dynamics can account for both. In this paper, we will focus on modeling circuits that are in thermal equilibrium at finite temperature T, such that the probability of finding the circuit in configuration ξ in equilibrium is

$$P(\xi) = Z^{-1}\mathrm{e}^{-\beta E(\xi)}. \tag{4.1}$$

Here E is the combined energy of the capacitive and inductive elements, while β is inverse temperature. Z is a normalization constant which, as we will see, is not important.

Our interest is in first–order stochastic differential equations (a.k.a. Langevin equations) with Gaussian white noise:

$$\dot{x}_\alpha = f_\alpha(x) + M_{\alpha\mu}\xi_\mu(t) \tag{4.2}$$

where $\xi_\mu(t)$ are independent and identically distributed Gaussian random white noise:

$$\langle \xi_\mu(t)\xi_\nu(s)\rangle = 2\delta_{\mu\nu}\delta(t-s). \tag{4.3}$$

We use $\mu\nu$ indices to denote the noise variables vs. $\alpha\beta$ indices to represent the physical degrees of freedom, because we may have a smaller number of noise variables than coordinates x_α. Here and below we invoke the Einstein summation convention on repeated indices, unless otherwise stated. If $M_{\alpha\mu}$ is a constant matrix independent of x, then this Langevin equation is well-defined;

otherwise, more care is required [51] to regulate the stochastic equation. We will explain later how to correctly regularize the problem for systems in thermal equilibrium. Because we will find it useful to distinguish between constant and non-constant noise, we introduce the following terminology:

Definition 1 (Multiplicative noise)**.** Stochastic equation (4.2) has **multiplicative noise** if $M_{\alpha\mu}(x)$ is not constant.

An important question is what constraints should be placed on (4.2) to ensure that thermal equilibrium (4.1) describes a statistical steady state to which the stochastic dynamics relaxes. We now describe a few related ways to achieve this goal.

4.2.1 The Martin-Siggia-Rose formalism

The first approach we will use to study (4.2) is based on generalizing Lagrangian mechanics to dissipative and stochastic systems. This approach is inspired by the desire to calculate the transition probability $P(x_\alpha(t) = a_\alpha | x_\alpha(0) = b_\alpha)$ – namely, the probability to go from microstate b to microstate a in time t. One popular way of trying to evaluate this transition probability (density function) is by using the Martin-Siggia-Rose (MSR) path integral: [81]

$$P(x_\alpha(t) = a_\alpha | x_\alpha(0) = b_\alpha) = \int\limits_{x(t)=a, x(0)=b} \mathcal{D}x\mathcal{D}\xi \; \delta(\dot{x}_\alpha - f_\alpha(x) - M_{\alpha\mu}\xi_\mu) \exp\left[-\frac{1}{4}\int \mathrm{d}t \; \xi_\mu \xi_\mu\right]. \quad (4.4)$$

The path integral over ξ corresponds to averaging over the random noise, and is subject to appropriate boundary conditions. If we interpret (4.2) in the Ito formulation (see e.g. [51] for details), then we can perform standard path integral manipulations to write

$$P(x_\alpha(t) = a_\alpha | x_\alpha(0) = b_\alpha) = \int\limits_{x(t)=a, x(0)=b} \mathcal{D}x\mathcal{D}\xi\mathcal{D}\pi \; \exp\left[\mathrm{i}\int \mathrm{d}t \left(\pi_\alpha(\dot{x}_\alpha - f_\alpha(x) - M_{\alpha\mu}\xi_\mu) + \frac{\mathrm{i}}{4}\xi_\mu\xi_\mu\right)\right], \quad (4.5)$$

where π is a Lagrange multiplier enforcing the equation of motion. In the presence of Gaussian noise, we can then integrate out Gaussian variable ξ_μ, and we obtain

$$P(x_\alpha(t) = a_\alpha | x_\alpha(0) = b_\alpha) = \int_{x(t)=a, x(0)=b} \mathcal{D}x\mathcal{D}\pi \ \exp\left[\mathrm{i}\int \mathrm{d}t \left(\pi_\alpha(\dot{x}_\alpha - f_\alpha(x)) + \mathrm{i}M_{\alpha\mu}M_{\beta\mu}\pi_\alpha\pi_\beta\right)\right]. \tag{4.6}$$

This transition probability looks exactly like Feynman's path integral, weighted by $\exp\left[\mathrm{i}\int \mathrm{d}t\ L\right]$. The resulting Lagrangian is given a name:

Definition 2 (MSR Lagrangian)**.** An **MSR Lagrangian** $L(\pi_\alpha, x_\alpha)$ is a complex valued function of the form

$$L = \pi_\alpha \dot{x}_\alpha - \mathcal{H}(\pi_\alpha, x_\alpha) \tag{4.7}$$

such that $\mathcal{H}(0, x_\alpha) = 0$ and $\mathrm{Im}(\mathcal{H}) \leq 0$.

In this paper, as in (4.6), $\mathcal{H}$ will be at most quadratic in π_α, because the noise in the stochastic process is Gaussian. While the MSR Lagrangian is what appears in the admittedly nonrigorous path integral, in many useful cases, we will see that it can be a "mnemonic" for a Fokker-Planck equation that is unambiguously defined. We also note that the requirement $\mathcal{H}(0, x_\alpha) = 0$ is motivated by our path integral derivation of the MSR Lagrangian, while the condition $\mathrm{Im}(\mathcal{H}) \leq 0$ will ensure the physical requirement that the noise variance is positive.

The utility of the MSR Lagrangian comes from its ability to very beautifully incorporate the assumption that the system approaches a known steady-state [51], which manifests as a simple symmetry of the MSR Lagrangian. In this paper, we are interested in systems where this steady state is thermal equilibrium (4.1) [28, 40, 41, 49, 54] where, in many relevant settings, the MSR Lagrangian can be shown to be time-reversal symmetric:[1]

Definition 3 (Time-reversal symmetry)**.** Given a desired thermal steady state (4.1), MSR Lagrangian

[1] While one could envision an even more general definition of time-reversal symmetry, the one below will suffice for the present paper.

L is time-reversal symmetric if $\mathcal{H}$, defined in (4.7), obeys

$$\mathcal{H}(\pi_\alpha(t), x_\alpha(t)) = \mathcal{H}(-\eta_\alpha(\pi_\alpha(-t) - \mathrm{i}\beta\partial_\alpha E), \eta_\alpha x_\alpha(-t)). \tag{4.8}$$

for some $\eta_\alpha \in \{\pm 1\}$ (there is no sum on repeated α index here).

At the (not rigorous) level of path integrals, it is simple [51] to show that time-reversal symmetry implies the microscopic detailed balance condition

$$P(x_\alpha(t) = a_\alpha | x_\alpha(0) = b_\alpha)\mathrm{e}^{-\beta E(b_\alpha)} = P(x_\alpha(t) = \eta_\alpha b_\alpha | x_\alpha(0) = \eta_\alpha a_\alpha)\mathrm{e}^{-\beta E(\eta_\alpha a_\alpha)}, \tag{4.9}$$

as long as $E(\eta a) = E(a)$.

In general, (4.6) will certainly not be invariant under (4.8). Indeed, so long as the x_α are all defined on the real line (so that the phase space of the dynamics is topologically trivial), the most general theory we can consider, whose MSR Lagrangian is quadratic in π and is invariant under (4.8), is [51]

$$L = \pi_\alpha \dot{x}_\alpha + \mathrm{i}K_{\beta\alpha}(x)\pi_\beta(\pi_\alpha - \mathrm{i}\beta\partial_\alpha E). \tag{4.10}$$

This is time-reversal symmetric so long as (no sum on repeated indices):

$$K_{\beta\alpha} = K_{\alpha\beta}\eta_\alpha\eta_\beta. \tag{4.11}$$

The MSR Lagrangians above are very useful for both direct comparison to a Langevin equation, as well as to the Fokker-Planck equation, as we will discuss shortly. However, we will also find it useful when building MSR Lagrangians for circuits, to put them in a slightly different form, which we will call “dissipative Lagrangians" to remind the reader of a few technical differences spelled out below. Suppose for simplicity that the matrix K is invertible:

$$K_{\alpha\beta}Q_{\beta\gamma} = T\delta_{\alpha\gamma}, \tag{4.12}$$

where

$$T = \frac{1}{\beta} \tag{4.13}$$

denotes temperature. By making the change of variable

$$\pi_\alpha = Q_{\alpha\beta}\Pi_\beta, \tag{4.14}$$

we can write (4.10) as

$$L = \mathrm{i}TQ_{\beta\alpha}\Pi_\alpha(\Pi_\beta - \mathrm{i}\beta\dot{x}_\beta) + \Pi_\alpha\partial_\alpha E. \tag{4.15}$$

With these new variables, we can modify our previous definitions as follows:

Definition 4 (Dissipative Lagrangian/time-reversal)**.** A dissipative Lagrangian $L(\Pi_\alpha, x_\alpha)$ obeys $L(0, x_\alpha) = 0$, $\mathrm{Im}(L) \geq 0$, and (no sum on α):

$$L(-\eta_\alpha(\Pi_\alpha(-t) - \mathrm{i}\beta\dot{x}_\alpha(-t)), \eta_\alpha x_\alpha(-t)) = \tilde{L}(\Pi_\alpha, x_\alpha) + \mathrm{i}\beta\frac{\mathrm{d}E}{\mathrm{d}t}. \tag{4.16}$$

$\tilde{L}$ must also obey $\tilde{L}(0, x_\alpha) = 0$ and $\mathrm{Im}(\tilde{L}) \geq 0$. If $\tilde{L} = L$, the Lagrangian is time-reversal symmetric.

If (4.10) is time-reversal symmetric by Definition 3, then (4.15) is time-reversal symmetric by Definition 4. Also, Definition 4 can be interpreted as Kubo-Martin-Schwinger symmetry in the Schwinger-Keldysh path integral for dissipative systems [41].

4.2.2 Fokker-Planck equation

We will also find it useful to interpret the stochastic equation (4.2) in an alternative way, wherein we solve a partial differential equation called the Fokker-Planck equation (FPE) for the probability density $P(x, t)$ of finding the system at microstate x at time t. In the setting where $M_{\alpha\mu}$ is constant in (4.2), the derivation of such a Fokker-Planck equation is unambiguous and can be found in textbooks [39]:

Proposition 5 (Fokker-Planck equation)**.** *Given stochastic equation* (4.2) *with constant* $M_{\alpha\mu}$*, the probability density* $P(x,t)$ *obeys the Fokker-Planck equation:*

$$\partial_t P = -\partial_\alpha(f_\alpha P) + \frac{1}{2} M_{\alpha\mu} M_{\beta\mu} \partial_\alpha \partial_\beta P. \tag{4.17}$$

$P(x(t) = a|x(0) = b)$ *is the Green's function to the Fokker-Planck equation* (4.17)*.*

Now, following [51], we will define the following procedure for converting an MSR Lagrangian into a FPE, when noise is multiplicative:

Definition 6 (Fokker-Planck equation for MSR Lagrangian with multiplicative noise)**.** Consider a system with thermal steady state (4.1) and MSR Lagrangian (4.10). If $K_{\beta\alpha}$ is positive definite and invertible, then the Fokker-Planck equation defining the stochastic process is

$$\partial_t P = \partial_\beta \left(K_{\beta\alpha}(x) \left(\partial_\alpha P + \beta P \partial_\alpha E \right) \right). \tag{4.18}$$

A solution to this differential equation is the thermal steady state: $P(x,t) = \exp[-\beta E(x)]$.

The reason we define the FPE first via (4.18) is that we are guaranteed to have a thermal steady state. That is not trivial to achieve for general multiplicative noise, starting from (4.2). In particular, using the Ito regularization of stochastic equations, which is most commonly employed, one wishes to write

$$\partial_t P = \partial_\alpha \partial_\beta \left(\frac{1}{2} D_{\alpha\beta} P \right) - \partial_\alpha \left(f_\alpha P \right). \tag{4.19}$$

Comparing (4.18) and (4.19), we notice that

$$D_{\alpha\beta} = K_{\alpha\beta} + K_{\beta\alpha}, \tag{4.20a}$$

$$f_\alpha = -\beta K_{\alpha\beta} \partial_\beta E + \partial_\beta \frac{K_{\alpha\beta} + K_{\beta\alpha}}{2}. \tag{4.20b}$$

The MSR path integral is "formally" defined assuming regularization (4.19), whereas we want (4.18). In practice, we will always use Definition 6 in the presence of multiplicative noise before carrying out

any physical calculation, although we will see that the language of dissipative and MSR Lagrangians makes obtaining this eventual FPE much more transparent for electrical circuit mechanics.

4.3 Theory of circuit dynamics

Having reviewed a general formalism for dissipative and stochastic dynamics, we now apply these methods to the analysis of classical RLC circuits. In this section, we focus on the simplifying problem of linear resistors, which we will see corresponds to constant (not multiplicative) noise. In this setting we can perform exact manipulations with dissipative Lagrangians, which cleanly connect to standard expectations about Johnson noise in the literature. The more subtle problem of nonlinear resistors, which implies multiplicative noise, is the subject of Section 4.4.

4.3.1 Overview of circuits

We begin by reviewing a mathematical formalism for describing circuits [91]. Firstly, given any set X, let

$$\mathcal{D}(X) = \text{span}\left(|x\rangle \,:\, x \in X\right). \tag{4.21}$$

With this notation in mind, it is natural to view a circuit as a directed graph.

Definition 7 (Graphs)**.** Let $\mathcal{V}$ denote a set of vertices associated to some graph G. The edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is an ordered pair of vertices corresponding to the start and end point of the directed edge.

Definition 8 (Loop)**.** A **loop** l of length n is a special subset of edges of the form

$$l = \{(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1)\} \subseteq \mathcal{E}. \tag{4.22}$$

It is also acceptable for some edges in the loop to be backwards, e.g. $l = \{(v_1, v_2), (v_3, v_2), \ldots\}$. We associate a loop set $\mathcal{L}$ to graph G, such that for arbitrary loop l, there exist $p \geq 1$ loops $m_1, \ldots, m_p \in \mathcal{L}$ such that $l \subseteq m_1 \cup \cdots \cup m_p$.

Definition 9 (Cut)**.** $c \subseteq \mathcal{E}$ is a **cut** of graph G if and only if there exists a partition of the vertex set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ (with $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$) such that

$$c = \{(u, v) \in \mathcal{E} : \{u, v\} \not\subset \mathcal{V}_1 \text{ and } \{u, v\} \not\subset \mathcal{V}_2\}. \tag{4.23}$$

In other words, a cut corresponds to the set of edges that cross between the partition $\mathcal{V}_1$ and $\mathcal{V}_2$.

Definition 10 (Boundary maps)**.** The boundary maps of a graph are the following matrices $A : \mathcal{D}(\mathcal{V}) \to \mathcal{D}(\mathcal{E})$ and $B : \mathcal{D}(\mathcal{E}) \to \mathcal{D}(\mathcal{L})$:

$$A_{ev} = \langle e|A|v\rangle = \begin{cases} 1 & \exists\, u \text{ such that } e = (u, v) \\ -1 & \exists\, u \text{ such that } e = (v, u) \\ 0 & \text{otherwise} \end{cases} \tag{4.24a}$$

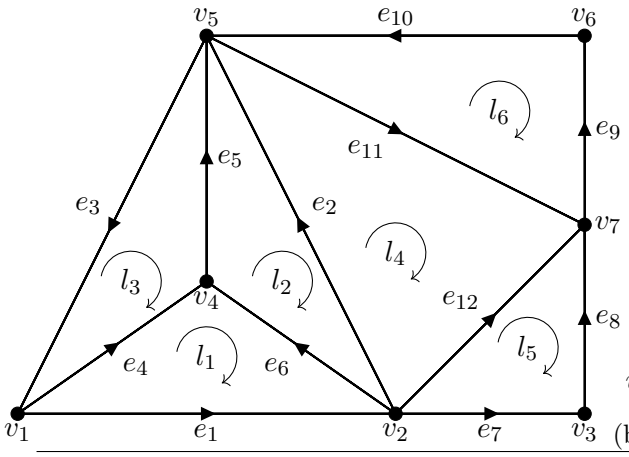
$$B_{le} = \langle l|B|e\rangle = \begin{cases} 1 & e \text{ and } l \text{ are oriented alike} \\ -1 & e \text{ and } l \text{ are oriented unalike} \\ 0 & e \text{ and } l \text{ do not intersect.} \end{cases} \tag{4.24b}$$

Notice that the definition of the loop set $\mathcal{L}$ in Definition 8 is chosen to ensure that the matrix B_{le} defined above provides a map from arbitrary loops to edges. Definition 10, among other upcoming definitions, is illustrated in a simple example in Figure 4.1. We state without proof the following well known fact from graph theory, which amounts to the statement that a loop has no boundary:

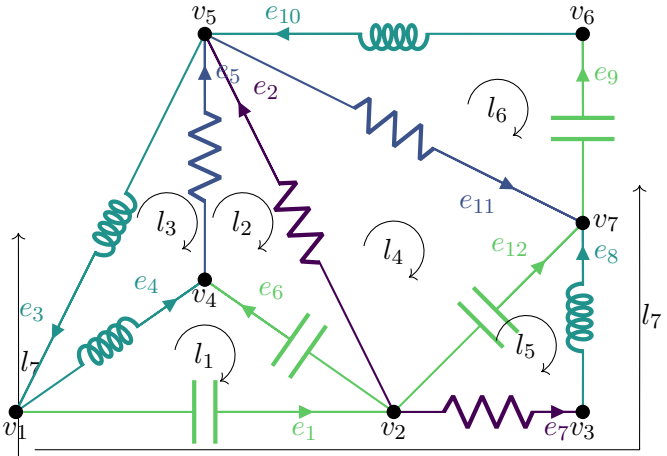
Proposition 11. $\mathcal{D}(\mathcal{V}) \xrightarrow{A} \mathcal{D}(\mathcal{E}) \xrightarrow{B} \mathcal{D}(\mathcal{L})$ *is a short exact sequence over vector spaces, and*

$$\mathrm{Ker}(B) = \mathrm{Im}(A). \tag{4.25}$$

We are now ready to define a circuit as a graph with edges of different “flavors". Obviously, the definition below is not meant to capture everything that could reasonably called an electrical circuit; instead, it is serves to clearly delineate the scope of what we will study in this paper.



(a) A directed graph with $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\}$, and $\mathcal{L} = \{l_1, l_2, l_3, l_4, l_5, l_6, l_7\}$.

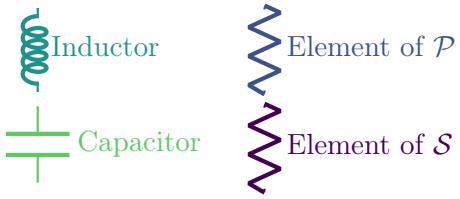


(b) For the circuit at hand, the edges are colored according to set inclusion. The sets of interest for this diagram are $\mathcal{C} = \{e_1, e_6, e_{12}, e_9\}$, $\mathcal{I} = \{e_3, e_4, e_8, e_{10}\}$, $\mathcal{R} = \mathcal{E} \setminus (\mathcal{C} \cup \mathcal{I})$, $\mathcal{S} = \{e_2, e_7\}$, and $\mathcal{P} = \{e_5, e_{11}\}$. The sets $\mathcal{S}$ and $\mathcal{P}$ are chosen to be consistent with the algorithm given in Section 4.3.3.1, but any choice of $\mathcal{S}$ and $\mathcal{P}$ such that $\mathcal{S} \cup \mathcal{P} = \mathcal{R}$ is physically acceptable.

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{matrix} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \\ e_9 \\ e_{10} \\ e_{11} \\ e_{12} \end{matrix}$$

$$B = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} \end{matrix} \\ \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array} \begin{matrix} l_1 \\ l_2 \\ l_3 \\ l_4 \\ l_5 \\ l_6 \\ l_7 \end{matrix}$$

(c) The matrices A and B for the graph and circuit drawn in (a) and (b).



(d) Depicted above is the color scheme that will be employed in all diagrams in this paper.

Figure 4.1: Pictured above are a pair of drawings. The drawing (a) is simply a graph, while (b) is a full–fledged circuit. Since the boundary maps A and B depend only upon the underlying graph structure of a circuit, A and B may be read just as easily off of the drawing in (a) as off of the drawing (b). On the other hand, various projectors onto edge sets may be read only off of the drawing in (b).

Definition 12 (Circuit)**.** In this paper, we define a circuit to be a graph G with a partition of the edge set $\mathcal{E} = \mathcal{C} \cup \mathcal{I} \cup \mathcal{R}$, where these three sets are all disjoint. Physically, this partition of edges into sets $\mathcal{R}$, $\mathcal{C}$, and $\mathcal{I}$ will correspond to resistive, capacitive, and inductive elements of the circuit, respectively. We define projectors on $\mathcal{D}(\mathcal{E})$ which restrict to one of the three types of edges:

$$P_I = \sum_{e \in \mathcal{I}} |e\rangle\langle e|, \tag{4.26a}$$

$$P_C = \sum_{e \in \mathcal{C}} |e\rangle\langle e|, \tag{4.26b}$$

$$P_R = \sum_{e \in \mathcal{R}} |e\rangle\langle e|. \tag{4.26c}$$

For reasons that will be clear as we develop the formalism, we will further partition the set $\mathcal{R}$ into the disjoint sets $\mathcal{S}$ and $\mathcal{P}$: $\mathcal{R} = \mathcal{S} \cup \mathcal{P}$ with $\mathcal{S} \cap \mathcal{P} = \emptyset$. The sets $\mathcal{S}$ and $\mathcal{P}$ will, roughly, correspond to resistors that are in series with a desired element and resistors that are in parallel with a desired element. These "desired" elements will be specific, useful parasitic elements when we discuss nonlinear circuits in Section 4.4. The choice of $\mathcal{S}$ and $\mathcal{P}$ is not unique, and we defer a detailed discussion of good choices to Algorithm 18. Nevertheless, we define $P_R = P_S + P_P$ where

$$P_S = \sum_{e \in \mathcal{S}} |e\rangle\langle e|, \tag{4.27a}$$

$$P_P = \sum_{e \in \mathcal{P}} |e\rangle\langle e|. \tag{4.27b}$$

The dynamical degrees of freedom of a circuit are a subset of flux variables ϕ_v defined at each vertex of a graph, and charge variables q_l defined on each loop in a graph. More abstractly we could think of $|\phi\rangle \in \mathcal{D}(\mathcal{V})$ and $|q\rangle \in \mathcal{D}(\mathcal{L})$. For each edge $e \in \mathcal{I}$ or $e \in \mathcal{C}$, we define an energy

$$E_e = \begin{cases} E_e^I \left(\sum_v A_{ev} \phi_v \right) & e \in \mathcal{I} \\ E_e^C \left(\sum_l q_l B_{le} \right) & e \in \mathcal{C} \end{cases}. \tag{4.28}$$

$$R = \sum_{e \in \mathcal{R}} R_e |e\rangle\langle e|. \tag{4.29}$$

As a slight abuse of notation, we will write

$$R^{-1} = \sum_{e \in \mathcal{R}} \frac{1}{R_e} |e\rangle\langle e|. \tag{4.30}$$

While R^{-1} is not precisely the inverse of R, it satisfies $RR^{-1} = R^{-1}R = P_R$, and in our formalism R will never appear without P_R. To build our formalism for dissipative circuits, we need to make one further assumption:

Assumption 13 (Thermal equilibrium)**.** (4.1) describes the statistical steady state of the circuit, where

$$E = \sum_{e \in \mathcal{I} \cup \mathcal{C}} E_e. \tag{4.31}$$

The circuit is in thermal equilibrium, with energy stored in inductors and capacitors.

4.3.2 The dissipative Lagrangian

We are now ready to write down a locally valid dissipative Lagrangian (Definition 4) for general RLC circuits, as defined in the previous subsection. As has been discussed previously [90, 93, 106], it is possible that there are global singularities which require care to resolve, and we do not focus on resolving all such possible singularities in the present paper. We now introduce a summary of the notation we will use in what follows:

Definition 14 (Dissipative Lagrangian for RLC circuit)**.** The physical degrees of freedom in a circuit are

$$|X\rangle = \begin{pmatrix} \sum_v \phi_v |v\rangle \\ \\ \sum_l q_l |l\rangle\rangle \end{pmatrix} = \begin{pmatrix} |\phi\rangle \\ \\ |q\rangle \end{pmatrix}, \tag{4.32}$$

and we define “noise variables"

$$|\Pi\rangle = \begin{pmatrix} \sum_v \pi_v |v\rangle \\ \\ \sum_l \sigma_l |l\rangle\rangle \end{pmatrix} = \begin{pmatrix} |\pi\rangle \\ \\ |q\rangle \end{pmatrix} \tag{4.33}$$

for each. The following matrices will also appear frequently:

$$\Lambda = \begin{pmatrix} A & 0 \\ \\ 0 & B^\intercal \end{pmatrix} \tag{4.34a}$$

$$\Gamma = \begin{pmatrix} P_P R^{-1} & P_S + P_C \\ \\ P_P + P_I & P_S R \end{pmatrix} \tag{4.34b}$$

$$K = \Lambda^\intercal \Gamma \Lambda \tag{4.34c}$$

$$K^\pm = \frac{1}{2}(K \pm K^\intercal). \tag{4.34d}$$

Intuitively: Λ plays the role of a boundary map on the phase space (whose coordinates are captured by $|X\rangle$). With this notation, the dissipative Lagrangian for an RLC circuit is

$$L = \langle \Pi | K | \dot{X} \rangle + \mathrm{i}T \langle \Pi | K^+ | \Pi \rangle + \langle \Pi | h \rangle, \tag{4.35}$$

where we have defined $|h\rangle$ to be the gradients of the nondissipative energy:

$$|h\rangle = \begin{pmatrix} \sum_v \dfrac{\partial E}{\partial \phi_v} |v\rangle \\ \\ \sum_l \dfrac{\partial E}{\partial q_l} |l\rangle. \end{pmatrix} \tag{4.36}$$

The definiton above serves as one of the main results of the paper. It is, in a very precise

sense, the correct Lagrangian for a dissipative circuit. As we will see, it encodes the correct fluctuation-dissipation theorem (i.e. Johnson noise) for resistors, as well as incorporating the physical assumptions of Kirchoff's current and voltage laws. In Section 4.3.6, we will see that our results reduce to those based on Rayleigh dissipative functions when stochastic fluctuations are neglected. We now explain that:

Observation 15. *The Euler-Lagrange equations for $\dot{\phi}_v$ and $\dot{q}_l$ associated with (4.35) are only satisfied if Kirchoff's current and voltage laws for a circuit are obeyed, with each element of the circuit obeying the correct constitutive relations.*

To justify Observation 15, first notice that the equations of motion for $\dot{\phi}_v$ and $\dot{q}_l$ are:

$$\begin{aligned}
0 = \frac{\delta S}{\delta \pi_v} &= \begin{pmatrix} \langle v| & 0 \end{pmatrix} \left[K|\dot{X}\rangle + 2\mathrm{i}TK^{+}|\Pi\rangle + |h\rangle \right] \\
&= \langle v|A^{\intercal}P_P R^{-1}A|\dot{\phi}\rangle + \langle v|A^{\intercal}P_S B^{\intercal}|\dot{q}\rangle + \langle v|A^{\intercal}P_C B^{\intercal}|\dot{q}\rangle + \frac{\partial E}{\partial \phi_v} + 2\mathrm{i}T\langle v|A^{\intercal}P_P R^{-1}A|\pi\rangle \\
&= \sum_e A_{ev}\left[\mathbb{I}[e\in P]\frac{1}{R_e}\left(\dot{\phi}_e + 2\mathrm{i}T\pi_e\right) + \mathbb{I}[e\in S]\dot{q}_e + \mathbb{I}[e\in\mathcal{C}]\dot{q}_e + \mathbb{I}[e\in\mathcal{I}]\frac{\partial E_e}{\partial \phi_e}\right] \qquad (4.37a) \\
0 = \frac{\delta S}{\delta \sigma_l} &= \begin{pmatrix} 0 & \langle l| \end{pmatrix} \left[K|\dot{X}\rangle + 2\mathrm{i}TK^{+}|\Pi\rangle + |h\rangle \right] \\
&= \langle l|B(P_P + P_I)A|\dot{\phi}\rangle + \langle l|BP_S RB^{\intercal}|\dot{q}\rangle + \frac{\partial E}{\partial q_l} + 2\mathrm{i}T\langle l|BP_S B^{\intercal}|\sigma\rangle \\
&= \sum_e B_{le}\left[\mathbb{I}[e\in P]\dot{\phi}_e + \mathbb{I}[e\in I]\dot{\phi}_e + \mathbb{I}[e\in S]R_e\left(\dot{q}_e + 2\mathrm{i}T\sigma_e\right) + \mathbb{I}[e\in\mathcal{C}]\frac{\partial E_e}{\partial q_e}\right] \qquad (4.37b)
\end{aligned}$$

While these equations are not in the "standard form" of a Langevin equation presented in (4.2), they are actually in a highly amenable form to explain how they encode stochastic Kirchoff's laws. Indeed, recall that

$$P_P + P_S + P_C + P_I = \mathbb{I}, \qquad (4.38a)$$

$$BA = 0. \qquad (4.38b)$$

As these two facts imply, e.g.,

$$B(P_P + P_I)A = -B(P_S + P_C)A, \tag{4.39}$$

we then write

$$0 = \frac{\delta S}{\delta \pi_v} = \sum_{e \in \mathcal{P}} A_{ev} \left[\frac{1}{R_e}(\dot{\phi}_e + 2\mathrm{i}T\pi_e) - \dot{q}_e \right] + \sum_{e \in \mathcal{I}} A_{ev} \left[\frac{\partial E}{\partial \phi_e} - \dot{q}_e \right] \tag{4.40a}$$

$$0 = \frac{\delta S}{\delta \sigma_l} = \sum_{e \in \mathcal{S}} B_{le} \left[R_e(\dot{q}_e + 2\mathrm{i}T\sigma_e) - \dot{\phi}_e \right] + \sum_{e \in \mathcal{C}} B_{le} \left[\frac{\partial E}{\partial q_e} - \dot{\phi}_e \right]. \tag{4.40b}$$

Since we have demanded that the deterministic, energetic part of the Lagrangian depends on q_l only as $\sum_l q_l B_{le}$ and ϕ_v only as $\sum_v A_{ev}\phi_v$, we have written

$$\frac{\partial E}{\partial \phi_v} = \sum_{e \in \mathcal{I}} A_{ev} \frac{\partial E}{\partial \phi_e}. \tag{4.41}$$

We have made a similar observation for the derivative of E with respect to q_l. Notice that if each term in brackets in (4.40) vanished separately, then Kirchoff's current and voltage laws would all be obeyed. To see that this is indeed the case, we prove:

Proposition 16 (Gauge symmetry)**.** *The equations of motion* (4.40)*, and the dissipative Lagrangian* (4.35)*, are both invariant under the following* ***gauge symmetries*** *(redundancies in coordinate descriptions):*

- *Given any* $|l\rangle \in \mathcal{D}(\mathcal{L})$ *such that* $(P_P + P_I)B^{\mathsf{T}}|l\rangle = B^{\mathsf{T}}|l\rangle$*, for arbitrary function* $f(t)$*,* L *is invariant under*

$$|q\rangle \to |q\rangle + f(t)|l\rangle. \tag{4.42}$$

  In other words, each loop in $\mathcal{P} \cup \mathcal{I}$ *removes one* q_l *degree of freedom.*

- *Given any* $|u\rangle \in \mathcal{D}(\mathcal{V})$ *such that* $(P_S + P_C)A|u\rangle = A|u\rangle$, L *is invariant under*

$$|\phi\rangle \to |\phi\rangle + f(t)|u\rangle. \tag{4.43}$$

In other words, each cut in $\mathcal{S} \cup \mathcal{C}$ *removes one* ϕ_v *degree of freedom.*

Therefore, the constitutive relations in (4.40) *hold on each edge of the circuit.*

Proof. The first line of (4.40) may be read as

$$0 = \sum_e A^\intercal (P_I + P_P)|\gamma\rangle \tag{4.44}$$

with $|\gamma\rangle = \sum_e \gamma_e |e\rangle$ taking on the values

$$\gamma_e = \begin{cases} \dfrac{1}{R_e}(\dot{\phi}_e - 2\mathrm{i}T\pi_e) - \dot{q}_e & e \in P \\ \dfrac{\partial E}{\partial \phi_e} - \dot{q}_e & e \in \mathcal{I} \end{cases}. \tag{4.45}$$

We now invoke the following useful fact, stated without proof:

Proposition 17 ([91], Corollary B.5)**.** *If* $|\gamma\rangle \neq 0$ *satisfies* (4.44)*, there exists some vector* $|\psi\rangle$ *such that*

$$(P_I + P_P)|\gamma\rangle = B^\intercal |\psi\rangle. \tag{4.46}$$

Likewise, if $|\gamma'\rangle \neq 0$ *satisfies*

$$0 = B(P_I + P_P)|\gamma'\rangle \tag{4.47}$$

then there exists some vector $|\varphi\rangle$ *such that*

$$(P_I + P_P)|\gamma'\rangle = A|\varphi\rangle. \tag{4.48}$$

For the case (4.44), we can now use (4.46) to conclude that $|\gamma\rangle \neq 0$ only if (4.42) is obeyed

for some loop contained in $\mathcal{P} \cup \mathcal{I}$, as stated in the proposition. Such a loop modifies (4.40a) to

$$\frac{\delta S}{\delta \pi_v} = \sum_{e \in P} A_{ev} \left[\frac{1}{R_e} (\dot{\phi}_e + 2\mathrm{i} T \pi_e) - \dot{q}_e \right] + \sum_{e \in \mathcal{I}} A_{ev} \left[\frac{\partial E}{\partial \phi_e} - \dot{q}_e \right] - \sum_{e \in \mathcal{P} \cup \mathcal{I}} \dot{f}(t) \psi_l B_{le} A_{ev} \tag{4.49}$$

for some coefficients $\psi_l \in \{0, \pm 1\}$ that encode the particular loop in $\mathcal{P} \cup \mathcal{I}$. Since the loop by construction has edges entirely within $\mathcal{P} \cup \mathcal{I}$, we can use $BA = 0$ to show that the last term above vanishes. Therefore, we are free to take the solution where $\gamma_e = 0$, and the constitutive relations hold.

The result that the constitutive relation for resistors in $\mathcal{S}$ and capacitors holds is acquired very similarly, and we do not present it explicitly.

Lastly we confirm that (4.35) is invariant. By construction of K in (4.34), notice that the two gauge transformations (when packaged into $|X\rangle$) are right null vectors of K. The gauge transformation (4.42) cannot change the q variables on any element of $\mathcal{C}$, since the loop is contained entirely in $\mathcal{P} \cup \mathcal{I}$; hence, $|h\rangle$ is independent of (4.42).[2] A similar argument holds for flux gauge transformations. □

Since the constitutive relations for circuit elements are always obeyed, it follows that circuits are physically equivalent to their simplified (via ordinary addition of circuit elements in series and parallel) versions. While the simplification of networks via constitutive relations is certainly well understood, it it not necessarily transparent from the dissipative Lagrangian alone how this arises. We will see how to achieve this in Section 4.3.3.3.

4.3.3 Removing constraints

In many approaches to dissipative circuit mechanics [14, 68, 93, 119], Kirchhoff's laws are incorporated into a theory in the form of constraints, generalized forces, etc. Indeed, as will describe in detail, the dissipative Lagrangian (4.35) itself is generically subject to constraints – not all degrees

[2] This is easy to see by picking $|l\rangle$ in (4.42) as a basis vector in $\mathcal{L}$, which implies that E_C^e must all be independent of $q_l = \langle q|l\rangle$.

of freedom are independent. A systematic removal of constrained degrees of freedom played a crucial role in finding an efficient algorithm for finding the Hamiltonian formulation of circuit mechanics, and ultimately circuit quantization of an equal number of charges and flux degrees of freedom. Clearly in general, (4.35) need not have an equal number of ϕ_v and q_l degrees of freedom. We therefore endeavor to find a complete classification, and systematic prescription to remove, all constrained variables in a transparent way.

In our framework, which will closely follow [90, 91], one motivation for the importance of removing constraints can be more concretely seen as follows. Lagrangians of the form (4.35) are not in the MSR form (4.10). In order to study fully nonlinear stochastic circuit dynamics, it is essential to obtain a theory of the form (4.10) so that we may invoke Definition 6. Naively, all that is required to massage a quadratic (in noise variables) dissipative Lagrangian into a quadratic MSR Lagrangian is the ability to invert K. However, K is singular. Nonetheless, if K were of the form

$$K = \begin{pmatrix} \tilde{K} & 0 \\ \\ 0 & 0 \end{pmatrix} \tag{4.50}$$

in some basis with $\tilde{K}$ invertible, we have hope of obtaining (4.10) by treating the lower block of $|\Pi\rangle$ and $|X\rangle$ as Lagrange multipliers that enforce constraints on this set of degrees of freedom, rendering them not independent. By enforcing such constraints, we could then reduce the degrees of freedom to those in the block where $\tilde{K}$ is invertible. Happily, this task may be accomplished in some level of generality.

4.3.3.1 A good choice of $\mathcal{S}$ and $\mathcal{P}$

In our endeavor to remove constraints, it turns out that our freedom to choose $\mathcal{S}$ and $\mathcal{P}$ according to convenience is surprisingly powerful. While it is generally true that constraints arising from (4.35) may be solved equivalently with any choice of $\mathcal{S}$ and $\mathcal{P}$, the problem of formally resolving all possible constraints with a generic choice is nontrivial. As such, we will leverage our freedom of

choice and specify an algorithm for choosing $\mathcal{S}$ and $\mathcal{P}$ that is amenable to both formal discussion and practical calculation. While eventually we will give an explicit enumeration of all constraints present in a generic disspative Lagrangian, and prove that they are all solvable, we first present our particular choice of $\mathcal{S}$ and $\mathcal{P}$ in the form of an algorithm:

Algorithm 18. Order the edges in $\mathcal{R}$ such that $\mathcal{R} = \{e_1, e_2, \ldots, e_{|\mathcal{R}|}\}$. Define the set $\mathcal{S}^{(0)}$ to include every edge in $\mathcal{R}$, and define $\mathcal{P}^{(0)}$ to be empty. For $i = 1$ to $i = |\mathcal{R}|$, perform the following recursive operation:

- If e_i is in a loop involving only edges in $\mathcal{S}^{(i-1)}$ and $\mathcal{C}$, then define

$$\mathcal{S}^{(i)} = \mathcal{S}^{(i-1)} \setminus \{e_i\} \tag{4.51a}$$

$$\mathcal{P}^{(i)} = \mathcal{P}^{(i-1)} \cup \{e_i\} \tag{4.51b}$$

- Otherwise, define

$$\mathcal{S}^{(i)} = \mathcal{S}^{(i-1)} \tag{4.52a}$$

$$\mathcal{P}^{(i)} = \mathcal{P}^{(i-1)} \tag{4.52b}$$

Let $\mathcal{S} = \mathcal{S}^{(|\mathcal{R}|)}$ and $\mathcal{P} = \mathcal{P}^{(|\mathcal{R}|)}$.

The following results will make clear the utility of the choice of $\mathcal{S}$ and $\mathcal{P}$ espoused above.

Proposition 19. *Let $C \subset \mathcal{E}$ be a cut of some graph G, and let $L \subset \mathcal{E}$ be a loop of the same graph. There exists some natural number $n \in \mathbb{N}$ such that*

$$|C \cap L| = 2n. \tag{4.53}$$

In other words, a loop and a cut cannot have an odd number of edges in common.

Proof. A cut C induced a partition of $\mathcal{V}$ into the sets $\mathcal{V}_1$ and $\mathcal{V}_2$. Define a function $f : \mathcal{V} \to \mathbb{Z}_2$ such

that

$$f(v) = \begin{cases} 0 & v \in \mathcal{V}_1 \\ & \\ 1 & v \in \mathcal{V}_2 \end{cases}. \tag{4.54}$$

Let the vertices $v_1, v_2, \ldots, v_k$ be the vertices traversed by the loop L. If an edge $e = (u, v)$ is in C, then

$$f(u) - f(v) = 1 \mod 2 \tag{4.55}$$

since e is an edge that connects $\mathcal{V}_1$ to $\mathcal{V}_2$. The sum, with $v_{-1} = v_k$,

$$\sum_{i=1}^{k} (f(v_i) - f(v_{i-1})) = 0 \mod 2 \tag{4.56}$$

since it is a telescoping sum. However, (4.56) also counts the number of edges in common between L and C mod 2. Hence, the number of shared edges must be even if (4.56) holds; this implies (4.53). □

Corollary 20. *If $\mathcal{S}$ and $\mathcal{P}$ are chosen according to Algorithm 18, then:*

(1) If there is a loop of edges $l \subset \mathcal{E}$ such that $l \cap (\mathcal{I} \cup \mathcal{P}) = \emptyset$, then $l \subseteq \mathcal{C}$.

(2) If there is a cut of edges $c \subset \mathcal{E}$ such that $c \cap (\mathcal{C} \cup \mathcal{S}) = \emptyset$, then $c \subseteq \mathcal{I}$.

Proof. Point 1 is satisfied by construction; it is only point 2 which must be demonstrated. Choose a cut c satisfying the hypothesis of point 2 and suppose $|c \cap \mathcal{P}| = n > 0$. Every edge in $\mathcal{P}$ is in *some* loop l otherwise consisting only of edges in $\mathcal{C} \cup \mathcal{S}$. By Prop. 19, $|c \cup l| = 0 \mod 2$. However, there is only one edge in l outside of $\mathcal{C} \cup \mathcal{S}$, so c must also contain some edge in $\mathcal{S} \cup \mathcal{C}$. The assumption that $|\mathcal{C} \cap \mathcal{P}| > 0$ was a contradiction, hence $\mathcal{C} \subseteq \mathcal{I}$. □

For the remainder of the paper, we take for granted that we can find such a good partition of $\mathcal{R}$ into $\mathcal{S}$ and $\mathcal{P}$, which satisfies the criteria of Corollary 20.

4.3.3.2 Spanning tree coordinates

We now analyze K. Firstly, it is very straightforward to show that K is positive semidefinite. The symmetric part of K is given by

$$K^{+} = \Lambda^{\intercal} \begin{pmatrix} P_P R^{-1} & \frac{1}{2}\mathbb{I} \\ \frac{1}{2}\mathbb{I} & P_S R \end{pmatrix} \Lambda. \tag{4.57}$$

Since

$$\Lambda^{\intercal} \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \Lambda = 0, \tag{4.58}$$

we may write

$$K^{+} = \begin{pmatrix} A^{\intercal} R^{-1} P_P A & 0 \\ 0 & B P_S R B^{\intercal} \end{pmatrix}, \tag{4.59}$$

which is clearly positive semidefinite provided that R is positive definite (which follows from $R_e \geq 0$).

Definition 21 (Connection matrix)**.** The antisymmetric part of K is related to the **connection matrix**

$$M = \frac{1}{2} B(P_C + P_S - P_P - P_I)A. \tag{4.60}$$

In particular,

$$K^{-} = \begin{pmatrix} 0 & M \\ -M^{\intercal} & 0 \end{pmatrix}. \tag{4.61}$$

Using this new definition, we see that Proposition 17 shows the left (right) null vectors of M are cycles or cuts of G that lie entirely within the sets $\mathcal{C} \cup \mathcal{S}$ or $\mathcal{P} \cup \mathcal{I}$. This leads us to quote the following important result from [90]:

Theorem 22 (Spanning tree coordinates)**.** *Consider circuit G with decomposition $\mathcal{S}$ and $\mathcal{P}$ obeying Corollary 20. Let G' be the (possibly disconnected) subcircuit of G consisting only of the edges in $\mathcal{C} \cup \mathcal{S}$. Let $\mathcal{T} = \{e_1, e_2, \ldots, e_k\}$ be a spanning tree of G'. Define for each $e \in \mathcal{T}$*

$$\Phi_e = \sum_v A_{ev}\phi_v. \tag{4.62}$$

There exists a matrix ρ such that

$$F = \frac{1}{2}\sum_{l,v}\left(\sum_{e\in\mathcal{C}\cup\mathcal{S}} - \sum_{e\in\mathcal{I}\cup\mathcal{S}}\right) q_l B_{le} A_{ev}\phi_v = \sum_{e\in\mathcal{T}} Q_e \Phi_e \tag{4.63}$$

with

$$Q_e = \sum_l \rho_{el} q_l. \tag{4.64}$$

Proof. Using the fact that

$$\frac{1}{2}B(P_C + P_S - P_I - P_P)A = B(P_C + P_S)A, \tag{4.65}$$

we rewrite

$$F = \sum_{l,v}\sum_{e\in\mathcal{C}\cup\mathcal{S}} q_l B_{le} A_{ev}\phi_v. \tag{4.66}$$

With our choice of $\mathcal{T}$ in hand, we further rewrite

$$F = \sum_{l,v}\sum_{e\in\mathcal{T}} q_l B_{le}\Phi_e + \sum_{l,v}\sum_{e\notin\mathcal{T}} q_l B_{le} A_{ev}\phi_v. \tag{4.67}$$

Since the path between any pair of nodes in a tree is unique, it follows that for each $e \in (\mathcal{S}\cup\mathcal{C}) \setminus \mathcal{T}$, there exist some coefficients $\lambda_{ee'}$ such that

$$\sum_v A_{ev}\phi_v = \sum_{e'\in\mathcal{T}} \lambda_{ee'}\Phi_{e'} \tag{4.68}$$

and thus

$$F = \sum_{l,v} \sum_{e \in \mathcal{T}} \left(q_l B_{le} + \sum_{e' \in (\mathcal{C} \cup \mathcal{S}) \setminus \mathcal{T}} q_l B_{le'} \lambda_{e'e} \right) \Phi_e. \tag{4.69}$$

The object in parentheses defines Q_e along with ρ. □

Definition 23 (Spanning tree variable)**.** Let G be an RLC circuit with dissipative Lagrangian L. Further suppose that $\mathcal{S}$ and $\mathcal{P}$ adhere to Corollary 20. The variable definitions Q and Φ furnished by Theorem 22 will be referred to as **spanning tree variables**.

The choice of spanning tree variables for a given circuit need not be unique. Different choices of spanning tree variables are physically equivalent, and in the discussion that follows we simply choose one set to work with. Notice that we could have alternatively chosen a spanning tree in $\mathcal{P} \cup \mathcal{I}$, but we have chosen the convention that is most similar to [90]. For details of these points, we refer the reader to the relevant discussion in [90].

4.3.3.3 Reducing degrees of freedom in the dissipative Lagrangian

Qualitatively, spanning tree variables are a formally and practically convenient choice of variables because of the algorithmic nature of their definition. One can always define spanning tree variables for some circuit in a way that does not demand any creativity, and the Lagrangian for some circuit is always significantly simplified by such a choice. As a passing remark, the manipulations above are very amenable to automation and could easily be used as a subroutine in some software, such as [23]. Variables which are not expressible as a linear combination of spanning tree variables may be integrated out of any circuit Lagrangian provided that null vector constraints are soluble. Unlike the statements above, the proof that constraints are always soluble provided in earlier works are not applicable to (4.35), and must be revisited. It is this task to which we now turn our attention.

Proposition 24. *Given dissipative Lagrangian* (4.35) *of some nonsingular circuit that abides by Corollary 20, if* $|\gamma\rangle \neq 0$ *is a vector such that*

$$K^- |\gamma\rangle = 0, \tag{4.70}$$

then the equation

$$0 = \sum_l \langle l|\gamma\rangle \frac{\delta S}{\delta \sigma_l} + \sum_v \langle v|\gamma\rangle \frac{\delta S}{\delta \pi_v} \tag{4.71}$$

may be used to eliminate one linear combination of physical variables from L by algebraic solution.

Proof. Proposition 16 shows that null vectors of K^- which are cuts of edges in $\mathcal{C} \cup \mathcal{S}$, and loops of edges in $\mathcal{I} \cup \mathcal{P}$, are trivial and can be removed. Since Corollary 20 forbids from appearing any loop involving both edges in $\mathcal{C}$ and $\mathcal{S}$, or a cut involving edges from $\mathcal{I}$ and $\mathcal{P}$, the remaining null vectors of K^- are a loop of edges contained entirely in $\mathcal{C}$, or a cut of edges contained entirely in $\mathcal{I}$. Notice that these latter possibilities are also null vectors of K^+.

For a loop l of capacitors, the charge on one capacitor must be determined in terms of the others as is required to satisfy Kirchhoff's voltage law. The assumption of nonsingularity is equivalent to the demand that such loop constraints always have a unique solution. More precisely, notice that in (4.40), the π_v equation would not depend on $q_l = \langle q|l\rangle$, meaning that

$$0 = \sum_{e \in l} B_{le} \left[\frac{\partial E}{\partial q_e} - \dot{\phi}_e \right] = \frac{\partial E}{\partial q_l}. \tag{4.72}$$

The sum over $\dot{\phi}_e$ vanishes due to $BA = 0$ (voltages around a loop vanish), while the second equality above follows from the relation between q_l and q_e in (4.40).

The analysis of a cut involving only inductors follows analogously; if the cut partitions $\mathcal{V}$ into $\mathcal{V}_1 \cup \mathcal{V}_2$, we find that

$$0 = \sum_{v \in \mathcal{V}_1} \frac{\partial E}{\partial \phi_v} = \sum_{v \in \mathcal{V}_2} \frac{\partial E}{\partial \phi_v}. \tag{4.73}$$

In either case, (4.72) or (4.73) lead to the fixing of one constrained degree of freedom for each independent such loop or cut. Although naively there are two constraints in (4.73), upon choosing the first such cut to be $\mathcal{V}_1 = \mathcal{V}$ (which implies the freedom to pick a grounded node), each additional identified cut only removes one further degree of freedom [90]. □

Corollary 25. *Suppose L is a dissipative Lagrangian with $\mathcal{S}$ and $\mathcal{P}$ chosen according to Algorithm*

18. Choose a spanning tree $\mathcal{T} \subset \mathcal{C} \cup \mathcal{S}$. L can be written in terms of only spanning tree variables,

$$\Phi_e = \sum_v A_{ev}\phi_v \tag{4.74a}$$

$$\Xi_e = \sum_v A_{ev}\pi_v \tag{4.74b}$$

$$Q_e = \sum_l \rho_{el}q_l \tag{4.74c}$$

$$\Sigma_e = \sum_l \rho_{el}\sigma_l \tag{4.74d}$$

where ρ is the matrix implicitly defined in Theorem 22 and the variables Ξ_e and Σ_e are the spanning tree noise variables. Further, there exist functions $\bar{h}$, $\tilde{h}$, and $E(Q, \Phi)$ such that

$$\langle \Pi | h \rangle = \sum_{e\in\mathcal{T}} \Sigma_e \bar{h}_e(Q_{e_1}, Q_{e_2}, \dots) + \sum_{e\in\mathcal{T}} \Xi_e \tilde{h}_e(\Phi_{e_1}, \Phi_{e_2}, \dots) \tag{4.75a}$$

$$\bar{h}_e = \frac{\partial E}{\partial Q_e} \tag{4.75b}$$

$$\tilde{h}_e = \frac{\partial E}{\partial \Phi_e}. \tag{4.75c}$$

In terms of these variables, there exists a matrix $Y : \mathcal{D}(\mathcal{E}) \to \mathcal{D}(\mathcal{P})$ such that

$$\begin{aligned} L = &\sum_{e\in\mathcal{T}} \left[\Xi_e \left(\dot{Q}_e + \frac{\partial E}{\partial \Phi_e} \right) - \Sigma_e \left(\dot{\Phi}_e - \frac{\partial E}{\partial Q_e} \right) \right] \\ &+ \sum_{e,e'\in\mathcal{T},\epsilon\in\mathcal{P}} \left[\Xi_e Y_{\epsilon e} R_\epsilon^{-1} Y_{\epsilon e'} (\dot{\Phi}_{e'} + \mathrm{i}T\Xi_{e'}) \right] + \sum_{e\in\mathcal{S}} \left[\Sigma_e R_e (\dot{Q}_e + \mathrm{i}T\Sigma_e) \right]. \end{aligned} \tag{4.76}$$

Proof. Algorithm 18 implies that any loops in $\mathcal{C} \cup \mathcal{S}$ are subsets of $\mathcal{C}$ alone. Hence, we can always choose a spanning tree $\mathcal{T}$ so that $\mathcal{S} \subset \mathcal{T}$. By Proposition 24, after eliminating the nonsingular

constrained variables,

$$L = \langle \Pi | K | \dot{X} \rangle + \mathrm{i}T \langle \Pi | K^{+} | \Pi \rangle + \langle \Pi | h \rangle$$
$$= \sum_{e,e' \in \mathcal{T}} \begin{pmatrix} \Xi_e & \Sigma_e \end{pmatrix} \begin{pmatrix} \sum_{\epsilon \in \mathcal{P}} Y_{\epsilon e} R_\epsilon^{-1} Y_{\epsilon e'} & \delta_{ee'} \\ -\delta_{ee'} & R_e \mathbb{I}[e \in \mathcal{S}] \delta_{ee'} \end{pmatrix} \begin{pmatrix} \dot{\Phi}_e + \mathrm{i}T \Xi_e \\ \dot{Q}_e + \mathrm{i}T \Sigma_e \end{pmatrix} + \sum_{e \in \mathcal{T}} \Xi_e \frac{\partial E}{\partial \Phi_e} + \Sigma_e \frac{\partial E}{\partial Q_e}, \tag{4.77}$$

where the matrix

$$Y_{\epsilon e} = B_{l(\epsilon) e} \mathbb{I}[e \in \mathcal{S} \cup \mathcal{C}] \tag{4.78}$$

is defined by the fact that for each edge ϵ in $\mathcal{P}$, there is a unique loop $l(\epsilon)$ such that $l(\epsilon) \setminus (\mathcal{C} \cup \mathcal{S}) = \{\epsilon\}$. □

As we will see, (4.76) will be a powerful tool in what follows. As was the case in [90], such spanning tree variables may be treated as independent, yet it could be possible to *further reduce* the number of degrees of freedom. As one example, in the absence of resistors, if E is independent of Φ_e, then the Euler-Lagrange equations above imply that $\dot{Q}_e = 0$, so Q_e is constant. In this Hamiltonian setting [90], it is always possible to further remove Φ_e as a degree of freedom via the method of symplectic reduction. In the dissipative setting, it is slightly more involved: the second line of (4.76) becomes relevant as well. Regardless, we will not stress this possibility of further reducing the number of degrees of freedom in the discussion that follows, until Section 4.5.1 and 4.5.2. Doing so is a matter of convenience.

Recall from the discussion of the relationship between the MSR path integral and the Fokker-Planck equation that to write the latter, we need to calculate the inverse of the matrix

$$\tilde{K} = \begin{pmatrix} Y^\intercal P_P R^{-1} Y & \mathbb{I} \\ -\mathbb{I} & P_S R \end{pmatrix} \tag{4.79}$$

which appears in (4.76). This is always possible:

Proposition 26. *The matrix $\tilde{K}$ defined in (4.79) is invertible.*

Proof. Since K is positive semidefinite, so too must be $\tilde{K}$. More pointedly, the diagonal blocks of $\tilde{K}$ must be positive semidefinite, and because the lower two blocks of (4.79) commute, we find that

$$\det \tilde{K} = \det\left(Y^{\intercal} P_P R^{-1} Y P_R R + \mathbb{I}\right) \geq 1. \tag{4.80}$$

So, indeed $\tilde{K}$ is invertible. □

Proposition 26 shows that the problem posed in the discussion around (4.50) always has a solution. With a firm grasp on the nonsingular submatrix of K we can, in principle, derive the correct nonlinear fluctuation-dissipation theorem for any dissipative circuit (within the definitions of this paper) of interest. Still, it would be very desirable to discern a compact and convenient formula for $\tilde{K}^{-1}$ in general. Unfortunately, the problem of inverting $\tilde{K}$ in general is nontrivial because the block $Y^{\intercal} P_P R^{-1} Y$ can be complicated. We do not expect that a general expression for the inverse of $\tilde{K}$ is a simple problem because of the following example. Consider a single loop RC circuit consisting of one resistor and one capacitor in parallel between the vertices u and v. Then attach an arbitrary resistor network across the vertices u and v. The problem of inverting $\tilde{K}$ amounts to the simplification of the arbitrary resistor network which is a problem that is known to be difficult [71]. In the case of linear resistors, this problem will not prevent us from deriving the consequences of the fluctuation-dissipation theorem in Section 4.3.4, but this problem is an obstruction in deriving a result of such generality in the case of nonlinear resistors in Section 4.4, without further restrictions on the circuit.

4.3.3.4 Time-reversal symmetry

Now, we show by direct calculation that the dissipative time–reversal transformation (Definition 4) leaves the dissipative circuit Lagrangian invariant provided that all inductive and capacitive

energies are time–reversal invariant. The relevant time–reversal transformation $\mathfrak{T}$ is given by

$$t \to -t, \tag{4.81a}$$

$$Q_e \to Q_e, \tag{4.81b}$$

$$\Phi_e \to -\Phi_e, \tag{4.81c}$$

$$\Xi_e \to -\Xi_e + \frac{\mathrm{i}}{T}\dot{\Phi}_e, \tag{4.81d}$$

$$\Sigma_e \to \Sigma_e - \frac{\mathrm{i}}{T}\dot{Q}_e. \tag{4.81e}$$

This transformation acts on the Lagrangian (4.76) as

$$\begin{aligned}
L \to L' = \mathfrak{T}\, L = &\sum_{e\in\mathcal{T}} \left[\left(-\Xi_e + \frac{\mathrm{i}}{T}\dot{\Phi}_e\right)\left(-\dot{Q}_e + \mathfrak{T}\frac{\partial E}{\partial \Phi_e}\right) - \left(\Sigma_e - \frac{\mathrm{i}}{T}\dot{Q}_e\right)\left(\dot{\Phi}_e - \mathfrak{T}\frac{\partial E}{\partial Q_e}\right)\right] \\
&+ \sum_{e,e'\in\mathcal{T},\epsilon\in\mathcal{P}} \left[\left(-\Xi_e + \frac{\mathrm{i}}{T}\dot{\Phi}_e\right) Y_{\epsilon e} R_\epsilon^{-1} Y_{\epsilon e'}\left(\dot{\Phi}_{e'} + \mathrm{i}T\left(-\Xi_{e'} + \frac{\mathrm{i}}{T}\dot{\Phi}_e\right)\right)\right] \\
&+ \sum_{e\in\mathcal{S}} \left[\left(\Sigma_e - \frac{\mathrm{i}}{T}\dot{Q}_e\right) R_e\left(-\dot{Q}_e + \mathrm{i}T\left(\Sigma_e - \frac{\mathrm{i}}{T}\dot{Q}_e\right)\right)\right] \\
&= \sum_{e\in\mathcal{T}} \left[\Xi_e\left(\dot{Q}_e - \mathfrak{T}\frac{\partial E}{\partial \Phi_e}\right) - \Sigma_e\left(\dot{\Phi}_e - \mathfrak{T}\frac{\partial E}{\partial Q_e}\right)\right] \\
&+ \sum_{e,e'\in\mathcal{T},\epsilon\in\mathcal{P}} \left[\left(\mathrm{i}T\Xi_e + \dot{\Phi}_e\right) Y_{\epsilon e} R_\epsilon^{-1} Y_{\epsilon e'} \Xi_{e'}\right] + \sum_{e\in\mathcal{S}} \left[\left(\mathrm{i}T\Sigma_e + \dot{Q}_e\right) R_e \Sigma_e\right] \\
&+ \sum_{e\in\mathcal{T}} \left[\frac{\mathrm{i}}{T}\dot{\Phi}_e\left(-\dot{Q}_e + \mathfrak{T}\frac{\partial E}{\partial \Phi_e}\right) + \frac{\mathrm{i}}{T}\dot{Q}_e\left(\dot{\Phi}_e - \mathfrak{T}\frac{\partial E}{\partial Q_e}\right)\right].
\end{aligned} \tag{4.82}$$

The last term in (4.82) is a total time derivative of exactly the form required by Definiton 4. If

$$\mathfrak{T}E = E, \tag{4.83}$$

then we simply recover (4.16). Note that (4.83) holds if and only if E is invariant under the transformation that takes each Φ_e to $-\Phi_e$ at once. We remark that many circuits of practical interest are time–reversal symmetric. Any circuit made of linear inductors, linear capacitors, Josephson junctions, quantum phase slips, and resistors is time–reversal symmetric.

4.3.4 Johnson noise for linear resistors

Let us now show that the MSR path integral reproduces the standard formulas for Johnson noise through linear resistors [116]; nonlinear resistors are discussed in Section 4.4. For each edge $e \in \mathcal{R}$, define a variable ζ_e, and

$$V = \sum_{e\in\mathcal{S},l} \sqrt{R_e}\sigma_l B_{le}\zeta_e + \sum_{e\in\mathcal{P},v} \frac{1}{\sqrt{R_e}}\zeta_e A_{ev}\pi_v + \sum_{e\in\mathcal{R}} \frac{\mathrm{i}}{4T}\zeta_e^2. \tag{4.84}$$

The Lagrangian

$$L = \langle \Pi | K | \dot{X} \rangle + \langle \Pi | h \rangle + V \tag{4.85}$$

has a number of useful properties that follow from the definitions above. The first property to note is that, since ζ_e appears at only quadratic order, it can be integrated out. The ζ equation of motion is

$$0 = \frac{\delta S}{\delta \zeta_e} = \begin{cases} \frac{\mathrm{i}}{2T}\zeta_e + \sqrt{R_e}\sum_l \sigma_l B_{le} & e \in \mathcal{S} \\ \frac{\mathrm{i}}{2T}\zeta_e + \frac{1}{\sqrt{R_e}}\sum_v A_{ev}\phi_v & e \in \mathcal{P} \end{cases} \tag{4.86}$$

After integrating out all ζ variables, we find (4.35).

However, if we refrain from integrating out the ζ variables, it is then possible to derive Langevin equations in a more transparent form. Define

$$\mathcal{D}\Pi = \prod_{v\in\mathcal{V}} \mathcal{D}\pi_{v_i} \prod_{l\in\mathcal{L}} \mathcal{D}\sigma_l \tag{4.87a}$$

$$\mathcal{D}X = \prod_{v\in\mathcal{V}} \mathcal{D}\phi_v \prod_{l\in\mathcal{L}} \mathcal{D}q_l \tag{4.87b}$$

$$\mathcal{D}\zeta = \prod_{e\in\mathcal{R}} \mathcal{D}\zeta_e \tag{4.87c}$$

$$\delta\left(\frac{\partial L}{\partial \Pi}\right) = \prod_{v\in\mathcal{V}} \prod_{l\in\mathcal{L}} \delta\left(\frac{\partial L}{\partial \pi_v}\right) \delta\left(\frac{\partial L}{\partial \sigma_l}\right). \tag{4.87d}$$

With these definitions in hand, we can use the standard δ function identity to evaluate the integral

with respect to Π variables, yielding a transition probability (as in (4.4)):

$$P(X(t)|X(0)) = \int \mathcal{D}\Pi\mathcal{D}X\mathcal{D}\zeta \exp\left[\mathrm{i}\int \mathrm{d}t L\right] = \int \mathcal{D}X\mathcal{D}\zeta\delta\left(\frac{\partial L}{\partial \Pi}\right) e^{-\frac{1}{4T}\sum_{e\in\mathcal{R}}\int \zeta_e^2}. \tag{4.88}$$

The manipulation above implies that we interpret the resulting stochastic equations in the Ito formalism [51]. The system of stochastic differential equations is determined by the argument of the δ function:

$$0 = K|\dot{X}\rangle + |h\rangle + \sum_l \frac{\partial V}{\partial \sigma_l}|l\rangle + \sum_v \frac{\partial V}{\partial \pi_v}|v\rangle. \tag{4.89}$$

The Langevin equation (4.89) is sufficient to recover a fluctuation–dissipation theorem for RLC circuits with linear resistors in general. To show this concretely, notice that (4.89) contains only additive noise, since we have assumed for this section that each resistance is a constant. In this setting, the Langevin equation (4.89) is unambiguously defined and we do not need to worry about operator ordering in the FPE. Rescale

$$\bar{\zeta}_e = \sqrt{\rho_e}\zeta_e \tag{4.90}$$

with

$$\rho_e = \begin{cases} R_e & e \in \mathcal{S} \\ \frac{1}{R_e} & e \in \mathcal{P}, \end{cases} \tag{4.91}$$

such that $\bar{\zeta}_e$ has units of current if e is in $\mathcal{P}$ and units of voltage if e is in $\mathcal{S}$. Using this rescaling, we may rewrite (4.89) as

$$0 = \frac{\delta S}{\delta \pi_v} = A_{ev}\left[\sum_{e\in\mathcal{P}}\left(\frac{1}{R_e}\dot{\phi}_e + \bar{\zeta}_e\right) + \sum_{e\in\mathcal{S}\cup\mathcal{C}}\dot{q}_e + \sum_{e\in\mathcal{I}}\frac{\partial E}{\partial \phi_e}\right], \tag{4.92a}$$

$$0 = \frac{\delta S}{\delta \sigma_l} = B_{le}\left[\sum_{e\in\mathcal{S}}(R_e\dot{q}_e + \bar{\zeta}_e) + \sum_{e\in\mathcal{I}\cup\mathcal{P}}\dot{\phi}_e + \sum_{e\in\mathcal{C}}\frac{\partial E}{\partial q_e}\right]. \tag{4.92b}$$

where we have again used the notation in (4.41). Upon using (4.88) to calculate expectation values,

we find

$$\langle \bar{\zeta}_e(t)\bar{\zeta}_{e'}(0)\rangle = 2T\rho_e\delta_{ee'}\delta(t). \tag{4.93}$$

Thus, we can interpret (4.93) as an expression that encodes the variance of both voltage and current fluctuations across all resistors in a generic LRC circuit (with linear resistors only). It is possible to recover textbook formulas describing the fluctuations in voltage measured over a small range by performing a Fourier transform and integrating over some range of frequencies: the power spectrum of voltage fluctuations in a small range of frequencies $[-f_0, f_0]$ of a resistor on edge e in $\mathcal{S}$ is given by

$$S_e = \int_{-f_0}^{f_0} \mathrm{d}f \left| \int_{-\infty}^{\infty} \mathrm{d}t\ \mathrm{e}^{\mathrm{i}2\pi f_0 t}\langle \bar{\zeta}_e(t)\bar{\zeta}_{e'}(0)\rangle \right|^2 = 4R_eT\Delta f_0. \tag{4.94}$$

4.3.5 Circuit dualities

Circuit duality is a map defined on drawings of circuits such that the physical observables of one circuit are a relabeling of the observables of its dual. In classical Hamiltonian systems, duality is a canonical transformation. In our dissipative formalism, circuit duality will be similarly transparent in the setting of planar graphs (nonplanar circuit dualities appear to be not fully understood [91]). Note the following definition of a planar circuit:

Definition 27 (Planar circuits and faces)**.** A circuit (graph) is **planar** if it can be embedded (drawn) on the surface of a two-dimensional sphere without any two edges intersecting (except at a vertex). Given such an embedding, a **face** corresponds to a loop which encloses no vertices. We define the loop set $\mathcal{L}$ for a planar graph to consist of the set of all faces, when drawn on a sphere.

We now quote the following simple result from graph theory (see e.g. [91]):

Proposition 28. *The matrix B has one left null vector, and the matrix A has one right null vector.*

The simplest representation of circuit duality is graphical. To take the dual of a planar dissipative circuit, execute the following algorithm:

Algorithm 29. Suppose G is a planar dissipative circuit. To construct G^*, the dual of G,

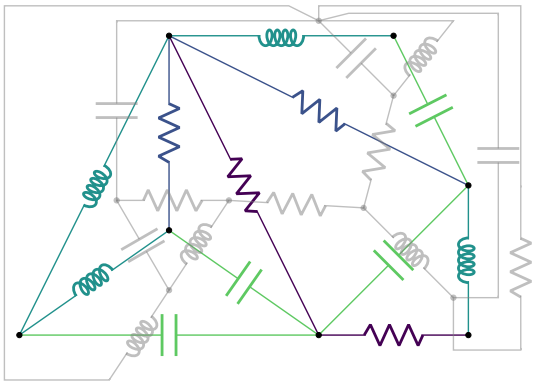


Figure 4.2: A circuit drawn with elements colored according to set inclusion. Its dual, constructed by Algorithm 29, is drawn in grey, with labels omitted on both circuits.

(1) Draw G on the plane with all points at infinity identified such that each l in $\mathcal{L}$ is on the boundary of some face of the drawing of G. For planar circuits, this is always possible [91].

(2) In every interior face and on the lone exterior face, draw a vertex. If the loop at the boundary of a face is labeled l, label the vertex l^*.

(3) Every edge e is on the boundary of exactly two faces. If e is in the loops l and l', draw an edge between l^* and $(l')^*$ and label it e^*.

(4) If e is an inductor (capacitor), make e^* a capacitor (inductor). If e is a resistor with resistance R_e in $\mathcal{S}$ ($\mathcal{P}$), make e^* a resistor with resistance $R_{e^*} = R_e^{-1}$ in $\mathcal{P}$ ($\mathcal{S}$).

(5) Each face in the drawing of G^* encloses exactly one vertex v in G. Label the loop in G^* as v^*.

An example of a dual circuit constructed by using Algorithm 29 is provided in Figure 4.2.

Definition 30 (Dual circuit)**.** Let G be a dissipative circuit. The **dual circuit** G^* has properties

defined via

$$\phi_v \to (\phi_v)^* = q'_{v^*}, \tag{4.95a}$$

$$q_l \to (q_l)^* = \phi'_{l^*}, \tag{4.95b}$$

$$\pi_v \to (\pi_v)^* = \sigma'_{v^*}, \tag{4.95c}$$

$$\sigma_l \to (\sigma_l)^* = \pi'_{l^*}, \tag{4.95d}$$

$$A \to A^* = (B')^\intercal, \tag{4.95e}$$

$$B \to B^* = (A')^\intercal, \tag{4.95f}$$

$$\mathcal{V} \to \mathcal{V}^* = \mathcal{L}', \tag{4.95g}$$

$$\mathcal{E} \to \mathcal{E}^* = \mathcal{E}', \tag{4.95h}$$

$$\mathcal{L} \to \mathcal{L}^* = \mathcal{V}', \tag{4.95i}$$

$$\mathcal{C} \to \mathcal{C}^* = \mathcal{I}', \tag{4.95j}$$

$$\mathcal{I} \to \mathcal{I}^* = \mathcal{C}', \tag{4.95k}$$

$$\mathcal{S} \to \mathcal{S}^* = \mathcal{P}', \tag{4.95l}$$

$$\mathcal{P} \to \mathcal{P}^* = \mathcal{S}', \tag{4.95m}$$

$$R_e \to (R_e)^* = \frac{1}{R'_e}. \tag{4.95n}$$

The Lagrangian of the dual circuit is (4.35) expressed in the dual coordinates.

The transformation given in Definition 30 is a relabeling transformation. In this sense, it is clear that L^* is equivalent to L. Moreover, L^* is a dissipative Lagrangian that describes the circuit G^* constructed by means of Algorithm 29. Hence, there is a very transparent generalization of circuit duality in our formalism. In particular, we notice that the dual of the dual circuit is the original one: $G = (G^*)^*$, and that the dual circuit obeys all of the nice properties of the original circuit (such as the existence of spanning tree coordinates), because:

Proposition 31. *Corollary 20, and therefore Theorem 22, hold for the dual circuit G^*.*

Proof. This result follows immediately from the identification of sets in (4.95), together with (*1*) Proposition 28 which guarantees that A^* and B^* are valid boundary maps, and (*2*) the fact that a loop and cut are mapped on to each other under the duality transformation [91]. □

4.3.6 Comparison to formalism based on Rayleigh dissipative function

The Rayleigh dissipative function is a construction that may be used to incorporate dissipative terms into Lagrangian and Hamiltonian mechanics [79, 80, 93, 112]. We now explain that one can "interpret" our approach, at least for linear resistors, in terms of such a Rayleigh dissipative function. As the discussion below does not rely on any specific details of circuit mechanics, we will discuss a general Hamiltonian dynamical system with canonical Poisson brackets; the relation between a Rayleigh system of the form below, and circuit mechanics, can be found in [79, 80, 93, 112].

Definition 32 (Rayleigh system)**.** Consider a Lagrangian

$$L(p_\alpha, x_\alpha) = \sum_\alpha \dot{x}_\alpha p_\alpha - E \tag{4.96}$$

with E a generic function of x_α and p_α. Given positive semidefinite matrices M and N, define

$$\mathfrak{R} = \sum_{\alpha,\beta} \left[M_{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta + N_{\alpha\beta} \dot{p}_\alpha \dot{p}_\beta \right]. \tag{4.97}$$

The pair $(L, \mathfrak{R})$ together with the equations of motion

$$0 = \frac{\partial \mathfrak{R}}{\partial \dot{x}_\alpha} + \dot{p}_\alpha + \frac{\partial E}{\partial x_\alpha} \tag{4.98a}$$

$$0 = -\frac{\partial \mathfrak{R}}{\partial \dot{p}_\alpha} + \dot{x} - \frac{\partial E}{\partial p_\alpha} \tag{4.98b}$$

is called a **Rayleigh system**.

Observation 33. *Let* $(L, \mathfrak{R})$ *be a Rayleigh system with* $\mathfrak{R}$ *defined in* (4.97)*. The dissipative*

Lagrangian

$$L' = \sum_{\alpha,\beta} \begin{pmatrix} \pi_\alpha & \sigma_\alpha \end{pmatrix} \begin{pmatrix} M_{\alpha\beta} & \delta_{\alpha\beta} \\ -\delta_{\alpha\beta} & N_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \dot{x}_\beta + \mathrm{i}T\pi_\beta \\ \dot{p}_\beta + \mathrm{i}T\sigma_\beta \end{pmatrix} + \sum_\alpha \left[\pi_\alpha \frac{\partial E}{\partial x_\alpha} + \sigma_\alpha \frac{\partial E}{\partial p_\alpha} \right] \tag{4.99}$$

has the property that

$$0 = \left. \frac{\partial L'}{\partial \pi_\alpha} \right|_{\pi=\sigma=0} = \frac{\partial \mathfrak{R}}{\partial \dot{x}_\alpha} + \dot{p}_\alpha + \frac{\partial E}{\partial x_\alpha} \tag{4.100a}$$

$$0 = \left. \frac{\partial L'}{\partial \sigma_\alpha} \right|_{\pi=\sigma=0} = -\frac{\partial \mathfrak{R}}{\partial \dot{p}_\alpha} + \dot{x} - \frac{\partial E}{\partial p_\alpha}. \tag{4.100b}$$

The constraints on M and N in (4.97), together with the relative signs in front of $\mathfrak{R}$ in (4.98), are imposed in dissipative Lagrangian L' transparently, since $\mathrm{Im}(L') \geq 0$.

In short, the framework of dissipative Lagrangians is a more physically transparent way of implementing the same methodology of a Rayleigh dissipation function in Hamiltonian mechanics. We strongly prefer the presentation we have done above, which emphasizes the deep *physical* connection betewen the dissipative Lagrangian and statistical mechanics via the fluctuation-dissipation theorem. We remind the reader that due to the existence of the spanning tree coordinates, we can systematically map dissipationless circuit mechanics onto the form (4.96).

4.4 Nonlinear resistors

We now return to a discussion of nonlinear resistors. A fully general discussion of nonlinear resistors is challenging without introducing additional degrees of freedom and constraints. As we now explain, for circuits with suitable parasitic inductors or capacitors, it is possible to exhaustively analyze the dynamics.

Our first task is to make precise statements about what kinds of dissipative circuit elements may be described by our formalism. In general, resistors are described by a so–called current–voltage

curve. For a generic resistor, there exists some function $f(\dot{\phi}, \dot{q})$ such that the relationship between current and voltage is specified by

$$f(\dot{\phi}, \dot{q}) = 0. \tag{4.101}$$

However, some resistors have current–voltage relationships that are poorly behaved. For this reason, it will be useful for us to restrict our attention to circuits made of elements with special current–voltage relationships.

Definition 34 (Current fixed)**.** A dissipative circuit element is called **current fixed** if there exists some function $I(\dot{\phi})$ such that the current across the element satisfies

$$\dot{q} = I(\dot{\phi}). \tag{4.102}$$

Definition 35 (Voltage fixed)**.** A dissipative circuit element is called **voltage fixed** if there exists some function $V(\dot{q})$ such that the voltage across the element satisfies

$$\dot{\phi} = V(\dot{q}). \tag{4.103}$$

Of course, a resistor may be both voltage fixed and current fixed. An example of some such circuit element is the linear resistor, which admits

$$I(\dot{\phi}) = \frac{1}{R}\dot{\phi} \tag{4.104a}$$

$$V(\dot{q}) = R\dot{q}. \tag{4.104b}$$

Definition 36 (Doubly fixed)**.** A dissipative circuit element that is both voltage fixed and current fixed is called **doubly fixed**.

For a dissipative circuit element that is either voltage fixed or current fixed, it is possible to

define a function R such that a version of Ohm's law is made to hold:

$$R := \begin{cases} \dfrac{\dot{\phi}}{I(\dot{\phi})} & e \text{ is current fixed} \\ \dfrac{V(\dot{q})}{\dot{q}} & e \text{ is voltage fixed.} \end{cases} \tag{4.105}$$

When we refer to nonlinear resistances, this is tentatively the quantity to which we refer, although we will ultimately see some subtleties related to the fluctuation-dissipation theorem. For circuit elements which are doubly fixed, one is free to choose to represent R either as a pure function of $\dot{\phi}$ or as a pure function of $\dot{q}$.

We aim to identify a restricted class of circuit where the matrix $\tilde{K}$ from (4.79) is easy to invert. In order to proceed, we need to make some additional restrictions on the circuits we study. First, we introduce a few more relevant definitions.

Definition 37 (Parallel/series circuit elements)**.** We say that circuit elements in a circuit G on edges e and e' are in parallel (series) if every cut (loop) containing e also contains e' and vice versa.

Note that this definition precludes us from talking about series/parallel lumped elements, but for the discussion that follows this restriction is unimportant. With this definition in mind, we observe that:

Proposition 38. *In a circuit with more than two edges, if e and e' are in parallel, then neither edge is in series with any edge. Likewise, if e and e' are in series, then neither edge is in parallel with any edge.*

Proof. Suppose $e = (u, v)$ and e' are in parallel. Then $\langle e'|A|v\rangle$ and $\langle e'|A|u\rangle$ must be nonzero lest there be some cut involving e but not e'. Thus, either $e' = (u, v)$ or $e' = (v, u)$. It follows that the only edge that can be in series with e is e', and only if u and v are the only vertices in G. The remainder of the theorem follows analogously. □

Corollary 39. *If e and e' are edges in series (parallel) then there is a cut (loop) involving only e and e'.*

Definition 40. A dissipative circuit is called a **clean circuit** if:

(1) Every resistor is current fixed, voltage fixed, or doubly fixed.

(2) Every resistor that is current fixed but not voltage fixed is in parallel with a linear capacitor.

(3) Every resistor that is voltage fixed but not current fixed is in series with a linear inductor.

(4) Every resistor that is doubly fixed is in parallel with a linear capacitor or in series with a linear inductor. By Corollary 39 only one of these possibilities can be realized.

From a physical perspective, we regard our restriction to clean circuits as mild in the sense that it is common practice to incorporate parasitic inductances and capacitances for resistors, such that they have an effective frequency-dependent impedance: see e.g. [1, 20, 82, 86, 93, 118]. We also remark that our requirement that parasitic elements be linear is not wholly a matter of convenience. If parasitic capacitances are allowed to be nonlinear, singularity may arise if $\partial E/\partial q$ or $\partial E/\partial \phi$ is not invertible. But for the sake of simplicity in the notation (namely to make this inverse simple), we just stick to parasitic linear elements.

Let G be a clean circuit with $\mathcal{S}$ and $\mathcal{P}$ chosen according to Algorithm 18. The resistive edges in $\mathcal{S}$ are all in series with an inductor, and the resistive edges in $\mathcal{P}$ are all in parallel with a capacitor. Furthermore, all edges in $\mathcal{S}$ are voltage fixed, while all edges in $\mathcal{P}$ are current fixed. We may build the dissipative Lagrangian for G according to

$$L = \langle \Pi | K | \dot{X} \rangle + \mathrm{i} T \langle \Pi | K | \Pi \rangle + T \langle \Pi | \mu \rangle. \tag{4.106}$$

For every edge e in $\mathcal{S}$, there is some vertex v such that

$$0 = \frac{\delta S}{\delta \pi_v} = \dot{q}_e + T h_v. \tag{4.107}$$

Moreover, since $e \in \mathcal{S}$, it is voltage fixed, and thus has a resistance $R_e(\dot{q})$. Thus, it is possible to use (4.107) to write $R_e(-T h_v)$. In other words, the dependence upon the current of the resistance on

edge e may be expressed in terms on only ϕ variables, rather than the naively expected dependence on $\dot{q}$. Furthermore, in any valid choice of spanning tree variables, h_v is expressible as a function of only spanning tree variables. A similar argument may be used to show that the resistance of an edge in $\mathcal{P}$ may be expressed in terms of only charge spanning tree variables.

Choose a spanning tree $\mathcal{T}$ in $\mathcal{S} \cup \mathcal{C}$. Every edge e in $\mathcal{P}$ is in parallel with some edge in $\mathcal{C}$. We will use a $\hat{}$ symbol to denote the fact that $\hat{e} \in \mathcal{C}$ is in parallel with $e \in \mathcal{P}$. This notation is helpful because spanning tree variables are indexed by edges in $\mathcal{C} \cup \mathcal{S}$, but we need a compact way to express a sum that runs over $\mathcal{P}$. With this notation and our chosen spanning tree in mind, the dissipative Lagrangian for a clean circuit may be expressed as

$$L = \sum_{e \in \mathcal{T}} \left[\Xi_e \left(\dot{Q}_e + \frac{\partial E}{\partial \Phi_e} \right) - \Sigma_e \left(\dot{\Phi}_e - \frac{\partial E}{\partial Q_e} \right) \right] + \sum_{e \in \mathcal{P}} \frac{1}{R_e(Q)} \Xi_{\hat{e}} (\dot{\Phi}_{\hat{e}} + 2\mathrm{i}T\Xi_{\hat{e}}) + \sum_{e \in \mathcal{S}} \Sigma_e R_e(\Phi)(\dot{Q}_e + 2\mathrm{i}T\Sigma_e). \tag{4.108}$$

With the definitions

$$|Q\rangle = \sum_{e \in \mathcal{T}} |e\rangle Q_e \tag{4.109a}$$

$$|\Phi\rangle = \sum_{e \in \mathcal{T}} |e\rangle \Phi_e \tag{4.109b}$$

$$|\Xi\rangle = \sum_{e \in \mathcal{T}} |e\rangle \Xi_e \tag{4.109c}$$

$$|\Sigma\rangle = \sum_{e \in \mathcal{T}} |e\rangle \Sigma_e \tag{4.109d}$$

$$M_1 = \sum_{e \in \mathcal{P}} \frac{1}{R_e} |\hat{e}\rangle\langle\hat{e}| \tag{4.109e}$$

$$M_2 = \sum_{e \in \mathcal{S}} R_e |e\rangle\langle e|, \tag{4.109f}$$

we may rewrite (4.108) as a matrix equation in the form

$$L = \begin{pmatrix} \langle \Xi | & \langle \Sigma | \end{pmatrix} \begin{pmatrix} M_1 & -\mathbb{I} \\ \\ \mathbb{I} & M_2 \end{pmatrix} \begin{pmatrix} |\dot{\Phi}\rangle + \mathrm{i}T|\Xi\rangle \\ \\ |\dot{Q}\rangle + \mathrm{i}T|\Sigma\rangle \end{pmatrix} + \sum_i \Xi_i \frac{\partial E}{\partial \Phi_i} + \Sigma_i \frac{\partial E}{\partial Q_i} \tag{4.110}$$

Since we have demonstrated that the matrix

$$\tilde{K} = \begin{pmatrix} M_1 & -\mathbb{I} \\ \\ \mathbb{I} & M_2 \end{pmatrix} \tag{4.111}$$

is generally invertible, we may take advantage of the fact that $M_1 M_2 = 0$, to write

$$\tilde{K}^{-1} = \begin{pmatrix} M_2 & \mathbb{I} \\ \\ -\mathbb{I} & M_1 \end{pmatrix}. \tag{4.112}$$

Now, by a linear transformation of noise variables by $\tilde{K}^{-1}$,

$$L = \begin{pmatrix} \langle \Xi' | & \langle \Sigma' | \end{pmatrix} \begin{pmatrix} |\dot{\Phi}\rangle \\ \\ |\dot{Q}\rangle \end{pmatrix} + \mathrm{i}T \begin{pmatrix} \langle \Xi' | & \langle \Sigma' | \end{pmatrix} \begin{pmatrix} M_2 & \mathbb{I} \\ \\ -\mathbb{I} & M_1 \end{pmatrix} \begin{pmatrix} |\Xi'\rangle - \mathrm{i}|\tilde{h}\rangle \\ \\ |\Sigma'\rangle - \mathrm{i}|\bar{h}\rangle \end{pmatrix} \tag{4.113}$$

By inspection, (4.113) is of the standard MSR form (4.10). From this point, a FPE may be derived from (4.113) according to the prescription given in Sec 4.2. For $e \in \mathcal{S}$, the argument of R_e is some linear combination of fluxes divided by an inductance.

It is not generally the case that R_e is a function of only Φ_e. However, R_e must depend upon Φ_e since Φ_e is a spanning tree variable, and the inductor in series with e must only depend upon differences of flux across nodes in spanning tree. Likewise, for $e \in \mathcal{P}$, the argument of R_e is a

linear combination of spanning tree charges divided by a capacitance. An example where this linear combination can be nontrivial (i.e. more than one spanning tree variable in the argument) is given in Section 4.5.1. Still, if R_{e_3} is a voltage fixed nonlinear resistor, then R_{e_3} may be expressed as a function only of h_{e_2}.

$$\partial_t P = \sum_{e\in\mathcal{T}} \left(\frac{\partial P}{\partial \Phi_e}\frac{\partial E}{\partial Q_e} - \frac{\partial P}{\partial Q_e}\frac{\partial E}{\partial \Phi_e} \right) + \sum_{e\in\mathcal{S}} \frac{\partial}{\partial \Phi_e}\left[R_e(\tilde{h}_e)\left(T\frac{\partial P}{\partial \Phi_e} + \frac{\partial E}{\partial \Phi_e}P \right) \right] + \sum_{e\in\mathcal{P}} \frac{\partial}{\partial Q_{\hat{e}}}\left[\frac{1}{R_e(\tilde{h}_{\hat{e}})}\left(T\frac{\partial P}{\partial Q_{\hat{e}}} + \frac{\partial E}{\partial Q_{\hat{e}}}P \right) \right] \tag{4.114}$$

In order to read off noise correlations from this FPE, it is necessary to rearrange (4.114) in the form

$$\partial_t P = \sum_{e\in\mathcal{T}} \left(\frac{\partial P}{\partial \Phi_e}\frac{\partial E}{\partial Q_e} - \frac{\partial P}{\partial Q_e}\frac{\partial E}{\partial \Phi_e} \right) + \sum_{e\in\mathcal{S}} \left(\frac{\partial}{\partial \Phi_e}\left[\left(R_e\frac{\partial E}{\partial \Phi_e} - T\frac{\partial R_e}{\partial \Phi_e} \right) P \right] + T\frac{\partial^2}{\partial \Phi_e^2}(R_e P) \right) + \sum_{e\in\mathcal{P}} \left(\frac{\partial}{\partial Q_{\hat{e}}}\left[\left(R_e^{-1}\frac{\partial E}{\partial Q_{\hat{e}}} - T\frac{\partial R_e^{-1}}{\partial Q_{\hat{e}}} \right) P \right] + T\frac{\partial^2}{\partial Q_{\hat{e}}^2}(R_e^{-1} P) \right). \tag{4.115}$$

With the FPE in the form (4.115), we can read off the experimentally observable form of Ohm's Law. For example, let us consider a resistor in $\mathcal{S}$, such that the current through the resistor is fixed by its series linear inductor. From (4.115) we can calculate

$$\partial_t \langle \Phi_e \rangle = \int \left(\prod_e \mathrm{d}\Phi_e \mathrm{d}Q_e \right) \Phi_e \left[\frac{\partial}{\partial \Phi_e}\left(\left(\frac{\partial E}{\partial Q_e} + R_e\frac{\partial E}{\partial \Phi_e} - T\frac{\partial R_e}{\partial \Phi_e} \right) P \right) + \cdots \right] = -\left\langle R_e\frac{\partial E}{\partial \Phi_e} - T\frac{\partial R_e}{\partial \Phi_e} \right\rangle \tag{4.116}$$

where $\cdots$ denote terms in the FPE which will not affect the answer because their derivative motif annihilates Φ_e upon integration by parts over all coordinates, and where $\langle \cdots \rangle$ denotes the average over the noise (e.g. averaging over P). To see that (4.116) is an effective Ohm's law, note that the current across e is $-\frac{\partial E}{\partial \Phi_e}$. Rearranging terms for clarity,

$$\partial_t \langle \Phi_e \rangle = \left\langle \left(R_e - T\left(\frac{\partial E}{\partial \Phi_e} \right)^{-1} \frac{\partial R_e}{\partial \Phi_e} \right) \left(-\frac{\partial E}{\partial \Phi_e} \right) \right\rangle. \tag{4.117}$$

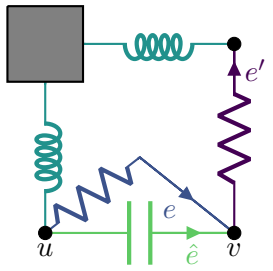


Figure 4.3: A part of a circuit which illustrates the subtleties with extracting the proper form of Ohm's law for nonlinear resistors. The box shaded in gray is meant to denote some arbitrary configuration of circuit elements. Elements explicitly drawn are colored according to set inclusion.

In the case that R_e is constant, this reduces to

$$\partial_t \langle \Phi_e \rangle = R_e \left\langle -\frac{\partial E}{\partial \Phi_e} \right\rangle \tag{4.118}$$

which is the textbook form of Ohm's law.

On the other hand, for edges e in $\mathcal{P}$, the Ohm's law takes on a superficially different form due to the asymmetrical way that our spanning tree construction treats edges in $\mathcal{S}$ and $\mathcal{P}$. Explicitly,

$$I_e = \left\langle \left(R_e^{-1} - T \left(\frac{\partial E}{\partial Q_{\hat{e}}} \right)^{-1} \frac{\partial R_e}{\partial Q_{\hat{e}}} \right) \frac{\partial E}{\partial Q_{\hat{e}}} \right\rangle \tag{4.119}$$

where (4.119) has been derived in exactly the same manner as (4.116). The left hand side of (4.119) is given by

$$I_e = \left\langle \frac{\partial E}{\partial \Phi_{\hat{e}}} \right\rangle - \partial_t \langle Q_{\hat{e}} \rangle \tag{4.120}$$

and may be interpreted as the current through edge e. To understand the difference between (4.119) and (4.116), consider the colored subcircuit drawn in Figure 4.3. Since e is an edge in $\mathcal{P}$, and we have restricted our attention to clean circuits, $\hat{e}$ is in our chosen spanning tree, while e is not. Further, since there is a linear capacitor on $\hat{e}$,

$$\frac{\partial E}{\partial Q_{\hat{e}}} = \frac{Q_{\hat{e}}}{C}, \tag{4.121}$$

which is the voltage across $\hat{e}$. Since e and $\hat{e}$ are in parallel, $\frac{\partial E}{\partial Q_{\hat{e}}}$ is also the voltage across e. Hence,

the quantity $\frac{\partial E}{\partial \Phi_{\hat{e}}}$ may be interpreted as the total current flowing from u to v. As a point of contrast, since e' is in $\mathcal{S}$, $Q_{e'}$ is a spanning tree variable, but no terms in E may depend upon $Q_{e'}$ since e' is a resistor. Thus,

$$\frac{\partial E}{\partial Q_{e'}} = 0. \tag{4.122}$$

Lastly, let us discuss the implications of the positivity condition on R_e, again focusing for simplicity on resistors $e \in \mathcal{S}$. It is clear that the positivity condition which ensures that systems are dissipative is that

$$R_e \geq 0, \tag{4.123}$$

so that the FPE (4.115) is well-defined and has positive noise variance. In contrast, we can use the chain rule to write the effective $I - V$ curve measured for the resistor is

$$\dot{\Phi}_e = -\frac{\partial E}{\partial \Phi_e} R_e + T \frac{\partial R_e}{\partial \Phi_e} = \dot{Q}_e \left(R_e - T \frac{\partial R_e}{\partial E} \right). \tag{4.124}$$

To see that the second equality holds, note that

$$\frac{\partial E}{\partial \Phi_e} = h_e = \frac{1}{L_e} \sum_{e'} \Phi_{e'} \tag{4.125}$$

and R_e is a function of only h_e. Further,

$$\frac{1}{h_e} \frac{\partial}{\partial \Phi_e} R_e(h_e) = \left. \frac{\partial}{\partial E} R_e(\sqrt{2E_e/L_e}) \right|_{E_e = L_e h_e^2/2}. \tag{4.126}$$

The “ohmic resistance" (i.e. the one that appears in an experimentally measured Ohm’s law) is given by

$$\tilde{R}_e = \left. \left(R_e - T \frac{\partial R_e}{\partial E_e} \right) \right|_{E_e = L_e h_e^2/2}. \tag{4.127}$$

Notice that while $R_e \geq 0$ is required by positivity of fluctuations, in contrast we do *not* require that $\tilde{R}_e \geq 0$.

4.5 Examples

We now give a few simple examples of how to apply our formalism to small circuits, to provide concrete illustrations of the abstract ideas in the sections above.

4.5.1 An RLC circuit

Consider the circuit drawn in Figure 4.4. We will begin by constructing a Lagrangian and producing its equations of motion by direct simplification and identification of unnecessary degrees of freedom. Our convention for choosing $\mathcal{S}$ and $\mathcal{P}$ dictates that $\mathcal{S} = \{e_3\}$ and $\mathcal{P} = \emptyset$. From this choice it is straightforward to produce

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & R & -R \\ 0 & 1 & -1 & -R & R \end{pmatrix} \tag{4.128}$$

and thus

$$\begin{aligned} L &= (\pi_{v_2} - \pi_{v_3})(\dot{q}_{l_1} - \dot{q}_{l_2}) - (\dot{\phi}_{v_2} - \dot{\phi}_{v_3})(\sigma_{l_1} - \sigma_{l_2}) + R(\sigma_{l_1} - \sigma_{l_2})(\dot{q}_{l_1} - \dot{q}_{l_2} + \mathrm{i}T(\sigma_{l_1} - \sigma_{l_2})) \\ &\quad + \frac{1}{L}(\pi_{v_3} - \pi_{v_2})(\phi_{v_3} - \phi_{v_2}) + \frac{1}{C}(\sigma_{l_1} - \sigma_{l_2})(q_{l_1} - q_{l_2}) \\ &= (\pi_{v_2} - \pi_{v_3})\left[\dot{q}_{l_1} - \dot{q}_{l_2} + \frac{1}{L}(\phi_{v_2} - \phi_{v_3})\right] + (\sigma_{l_1} - \sigma_{l_2})\left[-(\dot{\phi}_{v_2} - \dot{\phi}_{v_3}) + R(\dot{q}_{l_1} - \dot{q}_{l_2}) + \mathrm{i}TR(\sigma_{l_1} - \sigma_{l_2}) + \frac{1}{C}(q_{l_1} - q_{l} \\ &= \Xi\left(\dot{Q} + \frac{1}{L}\Phi\right) - \Sigma\left(\dot{\Phi} - \frac{1}{C}Q\right) + \Sigma R(\dot{Q} + \mathrm{i}T\Sigma) \end{aligned} \tag{4.129}$$

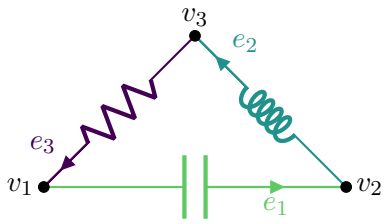


Figure 4.4: A minimal one–loop RLC circuit. Edges are colored according to set inclusion as in Fig. 4.1.

with the definitions

$$Q = q_{l_1} - q_{l_2} \tag{4.130a}$$

$$\Phi = \phi_{v_2} - \phi_{v_3} \tag{4.130b}$$

$$\Xi = \pi_{v_2} - \pi_{v_3} \tag{4.130c}$$

$$\Sigma = \sigma_{l_1} - \sigma_{l_2}. \tag{4.130d}$$

On the other hand, if we use spanning tree variables from the outset, we find

$$L' = \Xi_{e_1}\left(\dot{Q}_{e_1} + \frac{1}{L}(\Phi_{e_1} + \Phi_{e_3})\right) + \Xi_{e_3}\left(\dot{Q}_{e_3} + \frac{1}{L}(\Phi_{e_1} + \Phi_{e_3})\right) - \Sigma_{e_1}\left(\dot{\Phi}_{e_1} - \frac{1}{C}Q_{e_1}\right) - \Sigma_{e_3}\dot{\Phi}_{e_3}$$
$$+ \Sigma_{e_3}R(\dot{Q}_{e_3} + \mathrm{i}T\Sigma_{e_3}). \tag{4.131}$$

While (4.129) and (4.131) encode the same physics, there is a notable difference between them. In (4.131), Q_{e_1} and Q_{e_3} are apparently independent, but physically they must be the same since the circuit in Figure 4.4 has only one loop. In (4.131), this fact is encoded by the fact that

$$0 = \frac{\delta S}{\delta \Xi_{e_1}} - \frac{\delta S}{\delta \Xi_{e_3}} = \dot{Q}_{e_1} - \dot{Q}_{e_3}. \tag{4.132}$$

A similar manipulation shows that $\Sigma_{e_1} = \Sigma_{e_3}$, which allows us to rewrite L' as

$$L' = (\Xi_{e_1} + \Xi_{e_3}) \left(\dot{Q}_{e_1} + \frac{1}{L}(\Phi_{e_1} + \Phi_{e_3}) \right) + \Sigma_{e_1} \left(\dot{\Phi}_{e_1} + \dot{\Phi}_{e_3} - \frac{1}{C} Q_{e_1} \right) + \Sigma_{e_1} R(\dot{Q}_{e_1} + \mathrm{i}T\Sigma_{e_1}) \quad (4.133)$$

which recovers (4.129) with the identification

$$Q = Q_{e_1} \quad (4.134a)$$

$$\Sigma = \Sigma_{e_1} \quad (4.134b)$$

$$\Phi = \Phi_{e_1} + \Phi_{e_3} \quad (4.134c)$$

$$\Xi = \Xi_{e_1} + \Xi_{e_3}. \quad (4.134d)$$

Hence, for a given problem, there are two routes to a maximally simplified Lagrangian. The first is to write down L according to the definition (4.35) and make variable definitions based on its simplified form after cancelling like terms. The second is to start with (4.76) and identify any Noether current constraints that are present and redefine variables after integrating out the constrained degrees of freedom. We emphasize that if one starts from (4.76) as was done in producing (4.131), one need not integrate out Noether current constraints in order to produce an FPE and any applicable FDTs. Noether current constraints are exactly analogous to those that arise in [90], and there is one for each cut consisting only of edges in $\mathcal{C} \cup \mathcal{S}$.

The manipulations of Section 4.4 applied to (4.131) lead to an FPE

$$\partial_t P = \frac{Q_{e_1}}{C} \frac{\partial P}{\partial \Phi_{e_1}} - \frac{\Phi_{e_1} + \Phi_{e_3}}{L} \frac{\partial P}{\partial Q_{e_1}} - \frac{\Phi_{e_1} + \Phi_{e_3}}{L} \frac{\partial P}{\partial Q_{e_3}} + R \left[\frac{\Phi_{e_1} + \Phi_{e_3}}{L} \frac{\partial P}{\partial \Phi_{e_3}} + TR \frac{\partial^2 P}{\partial \Phi_{e_3}^2} \right]. \quad (4.135)$$

This is superficially different from the FPE determined by (4.129)

$$\partial_t P = \frac{Q}{C} \frac{\partial P}{\partial \Phi} - \frac{\Phi}{L} \frac{\partial P}{\partial Q} + R \left[\frac{\Phi}{L} \frac{\partial P}{\partial \Phi} + TR \frac{\partial^2 P}{\partial \Phi^2} \right]. \quad (4.136)$$

Nevertheless, (4.135) encodes the same physics as (4.136), while it also keeps track of a pair of

"redundant" degrees of freedom. To see that this is the case, change variables so that (4.135) is written in terms of

$$\Phi_{\pm} = \Phi_{e_1} \pm \Phi_{e_3}. \tag{4.137}$$

(4.135) may be rewritten as

$$\begin{aligned}\partial_t P = \frac{Q_{e_1}}{C}\left(\frac{\partial P}{\partial \Phi_+} + \frac{\partial P}{\partial \Phi_-}\right) - \frac{\Phi_+}{L}\frac{\partial P}{\partial Q_{e_1}} - \frac{\Phi_+}{L}\frac{\partial P}{\partial Q_{e_3}} \\ + R\left[\frac{\Phi_+}{L}\left(\frac{\partial P}{\partial \Phi_+} - \frac{\partial P}{\partial \Phi_-}\right) + TR\left(\frac{\partial^2 P}{\partial \Phi_+^2} + \frac{\partial^2 P}{\partial \Phi_-^2} - 2\frac{\partial^2 P}{\partial \Phi_+ \partial \Phi_-}\right)\right].\end{aligned} \tag{4.138}$$

Given any solution P of (4.138), define

$$P' = \int \mathrm{d}\Phi_- \mathrm{d}Q_{e_3} P. \tag{4.139}$$

Direct calculation shows that

$$\partial_t P' = \frac{Q_{e_1}}{C}\frac{\partial P'}{\partial \Phi_+} - \frac{\Phi_+}{L}\frac{\partial P'}{\partial Q_{e_1}} + R\left[\frac{\Phi_+}{L}\frac{\partial P'}{\partial \Phi_+} + TR\frac{\partial^2 P'}{\partial \Phi_+^2}\right], \tag{4.140}$$

namely P' is a solution of (4.136). In subsequent examples, we will not belabor the various routes toward simplification that exist with the understanding that such techniques all lead to the same place.

4.5.2 An unclean circuit

Consider the circuit drawn in Figure 4.5. We will show that, although the cleanliness of a circuit guarantees that the inversion of $\tilde{K}$ in (4.79) is simple, it is nonetheless possible to perform

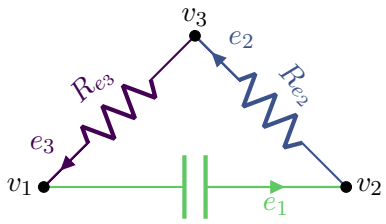


Figure 4.5: A minimal example of a circuit that is not a clean circuit. Note that every circuit consisting of one dissipative element and one nondissipative element is clean. Edges are colored according to set inclusion.

matrix inversion by hand for a circuit that is not clean. For this circuit

$$\mathcal{C} = \{e_1\} \tag{4.141a}$$

$$\mathcal{P} = \{e_2\} \tag{4.141b}$$

$$\mathcal{S} = \{e_3\} \tag{4.141c}$$

We could have equivalently chosen

$$\mathcal{S} = \{e_2\} \tag{4.142a}$$

$$\mathcal{P} = \{e_3\}. \tag{4.142b}$$

Our convention does not specify which of these choices is preferable, even in the case that the resistors at hand are nonlinear. Our spanning tree construction yields

$$\tilde{K} = \begin{array}{c} \begin{array}{cccc} e_1 & e_3 & e_1 & e_3 \end{array} \\ \begin{pmatrix} \frac{1}{R_{e_2}} & \frac{1}{R_{e_2}} & 1 & 0 \\ \frac{1}{R_{e_2}} & \frac{1}{R_{e_2}} & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & R_{e_3} \end{pmatrix} \begin{array}{c} e_1 \\ e_3 \\ e_1 \\ e_3 \end{array} \end{array} \tag{4.143}$$

which is invertible. Explicitly,

$$L = \begin{pmatrix} \Xi_{e_1} & \Xi_{e_3} & \Sigma_{e_1} & \Sigma_{e_3} \end{pmatrix} \begin{pmatrix} \frac{1}{R_{e_2}} & \frac{1}{R_{e_2}} & 1 & 0 \\ \frac{1}{R_{e_2}} & \frac{1}{R_{e_2}} & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & R_{e_3} \end{pmatrix} \begin{pmatrix} \dot{\Phi}_{e_1} + \mathrm{i}T\Xi_{e_1} \\ \dot{\Phi}_{e_3} + \mathrm{i}T\Xi_{e_3} \\ \dot{Q}_{e_1} + \mathrm{i}T\Sigma_{e_1} \\ \dot{Q}_{e_3} + \mathrm{i}T\Sigma_{e_3} \end{pmatrix} + \Sigma_{e_1}\frac{Q_{e_1}}{C}. \tag{4.144}$$

Once again

$$0 = \frac{\delta S}{\delta \Xi_{e_1}} - \frac{\delta S}{\delta \Xi_{e_3}} = \dot{Q}_{e_1} - \dot{Q}_{e_3}, \tag{4.145}$$

and $\Sigma_{e_1} = \Sigma_{e_2}$ follows analogously. Further defining

$$\Phi = \Phi_{e_1} + \Phi_{e_3} \tag{4.146a}$$

$$\Xi = \Xi_{e_1} + \Xi_{e_3} \tag{4.146b}$$

$$Q = Q_{e_1} = Q_{e_3} \tag{4.146c}$$

$$\Sigma = \Sigma_{e_1} = \Sigma_{e_3} \tag{4.146d}$$

we write

$$L = \begin{pmatrix} \Xi & \Sigma \end{pmatrix} \begin{pmatrix} \frac{1}{R_{e_2}} & 1 \\ -1 & R_{e_3} \end{pmatrix} \begin{pmatrix} \dot{\Phi} + \mathrm{i}T\Xi \\ \dot{Q} + \mathrm{i}T\Sigma \end{pmatrix} + \frac{1}{C}\Sigma Q. \tag{4.147}$$

Now, the path integral manipulation

$$\int \mathcal{D}\Phi\mathcal{D}\Xi\mathcal{D}\Sigma \exp\left[\mathrm{i}\int \mathrm{d}t \left(\frac{1}{R_{e_2}}\Xi - \Sigma\right)\dot{\Phi} + \frac{1}{R_{e_2}}\mathrm{i}T\Xi^2 + \Xi\dot{Q}\right] = \int \mathcal{D}\Sigma \exp\left[\mathrm{i}\int \mathrm{d}t\, \Sigma R_{e_2}(\dot{Q} + \mathrm{i}T\Sigma)\right], \tag{4.148}$$

corresponding to integrating out Ξ, allows us to simplify the dissipative Lagrangian to

$$L = \Sigma(R_{e_2} + R_{e_3})(\dot{Q} + \mathrm{i}T\Sigma) + \frac{1}{C}\Sigma Q. \tag{4.149}$$

Finally, the resulting FPE is given by

$$\partial_t P = \frac{\partial}{\partial Q}\frac{1}{R_{e_2} + R_{e_3}}\left[\frac{Q}{C}P + T\frac{\partial P}{\partial Q}\right]. \tag{4.150}$$

Alternatively, we could forgone the simplification afforded by (4.148) and proceeded to write an FPE directly from (4.147). Simplifications at the Lagrangian level are generally less difficult than simplifications in the FPE directly.

If one elects to add the resistors on edge e_2 and e_3 together using constitutive relations, then

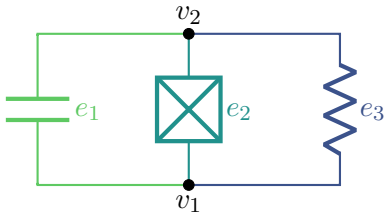


Figure 4.6: A circuit consisting of three edges, colored according to set inclusion. Edge e_2 contains a Josephson junction with $E = -E_J \cos \phi_{e_2}$.

the resulting simplified circuit is clean. Hence (4.114) may be invoked to yield

$$\partial_t P' = \frac{Q}{C} \frac{\partial P'}{\partial \Phi} + \frac{\partial}{\partial Q} \frac{1}{R_{e_2} + R_{e_3}} \left[\frac{Q}{C} P' + T \frac{\partial P'}{\partial Q} \right]. \tag{4.151}$$

To see that (4.150) and (4.151) are equivalent, we can integrate out Φ, analogously to (4.140).

4.5.3 A circuit with a nonlinear resistor

Consider the circuit depicted in Figure 4.6, which we will refer to as G. We suppose that R is current fixed so that

$$R = R(\dot{\phi}_{e_3}). \tag{4.152}$$

Since G is a clean circuit, it follows that $\dot{\phi}_{e_3}$ is expressible in terms of some derivative of E. In terms of spanning tree variables, we have that

$$\dot{\phi}_{e_3} = \dot{\Phi} = \frac{Q}{C}. \tag{4.153}$$

Thus, we may write

$$R = R\left(\frac{Q}{C}\right). \tag{4.154}$$

We will suppress the dependence of R upon Q henceforth.

One may express the Lagrangian of G in the form (4.108) to produce

$$L = \begin{pmatrix} \Xi & \Sigma \end{pmatrix} \begin{pmatrix} \frac{1}{R_e} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{\Phi} + \mathrm{i}T\Xi \\ \dot{Q} + \mathrm{i}T\Sigma \end{pmatrix} - \Xi E_J \sin(\Phi) + \frac{1}{C}\Sigma Q. \tag{4.155}$$

Invoking (4.114) immediately yields

$$\partial_t P = \frac{Q}{C}\frac{\partial P}{\partial \Phi} + E_J \sin(\Phi)\frac{\partial P}{\partial Q} + \frac{\partial}{\partial Q}\left[\frac{1}{R}\frac{Q}{C}P - T\frac{\partial R^{-1}}{\partial Q}P + T\frac{\partial}{\partial Q}\left(\frac{1}{R}P\right)\right] \tag{4.156}$$

which agrees with [111], after a suitable change of variables from R to $\tilde{R}$ as in Section 4.4. Moreover, it is possible to read the variance of fluctuations in the Ito formalism

$$\langle \xi(t)\xi(t')\rangle = \frac{2T}{R}\delta(t-t') \tag{4.157}$$

where we emphasize once again that R depends upon Q.

4.6 Conclusion

We have revisited the Lagrangian mechanics of dissipative RLC circuits. This subject has an old history [14, 68, 119] which has recently been revisited in the context of superconducting circuit quantization [93]. In this prior literature, the deep relationship between dissipative Lagrangians and Johnson noise is not transparent, and first and foremost the purpose of this paper is to remedy this issue. Along the way, we hope that our algorithmic methods for building dissipative Lagrangians and removing constrained degrees of freedom could be practical.

Our work is rather suggestive of a fruitful path forward for systematically modeling superconducting circuits with decoherence as open quantum systems, using the KMS-invariant Schwinger-Keldysh path integrals which represent the natural quantum mechanical analogues [28, 40, 41, 49, 54] of the dissipative Lagrangians discussed in this paper. However, as pointed out in [51], in the

presence of multiplicative noise there appear to be subtleties in path integral regularization. These subtleties are directly related to the two notions of resistance, R_e and $\tilde{R}_e$, highlighted in Section 4.4, and we hope that a careful resolution of such subtleties can enable precision experimental tests of the quantum fluctuation-dissipation theorem in a superconducting circuit.

Given the issues raised above with the path integral formulation of the quantum problem, a desirable alternative could be to construct the Lindbladians which manifestly drive the quantum system to thermal equilibrium. For nonlinear circuits, unfortunately, the challenge of exactly diagonalizing the Hamiltonian renders it difficult to systematically identify local Lindbladian dynamics that exactly protect a thermal steady state [47]. If it is possible to nevertheless identify the specific jump operators that generalize (4.115) to an open quantum system, then one may find a powerful new technique for accurately accounting for the effects of decoherence in superconducting circuit, without the need for a microscopic model of the environment. Exciting recent progress [22] along these lines suggests that at least *numerically* it may be possible to efficiently model dissipative circuits with a thermal steady state, although a clearer analytical understanding, and physical interpretation, of the appropriate jump operators would be highly desirable.

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Appendix A

Appendices of [90]

A.1 Mathematical formalism

In this appendix, we provide a mathematically precise discussion of the results highlighted in the main text. First, we prove the necessary graph theoretic properties of Ω_{ev}. Second, we prove facts about circuit quantization, including the existence of the "spanning tree construction" of canonical coordinates for arbitrary circuits, and identifying the number of Noether conserved charges.

A.1.1 Graph theory for circuits

In previous discussions, we have elected to use terminology familiar in the superconducting circuit community. We will continue to do so here. For the mathematically inclined, we refer to graph–theoretical edges as branches and we refer to vertices as nodes.

Definition 41. Let $\mathcal{V}$ and $\mathcal{E}$ be discrete node and branch sets. **Chains** on $\mathcal{V}$ are elements of

$$\mathcal{D}(\mathcal{V}) = \left\{ \sum_{v\in\mathcal{V}} m_v |v\rangle \,\middle|\, m_v \in \mathbb{Z} \right\} \tag{A.1}$$

where $|v\rangle$ is a vector in a vector space of dimension $|\mathcal{V}|$. Chains on $\mathcal{E}$ are elements of a set $\mathcal{D}(\mathcal{E})$ defined analogously.

Definition 42. Define the **incidence map** $\mathcal{A} : \mathcal{E} \to \mathcal{V} \times \mathcal{V}$, such that if $\mathcal{A}(e) = (u, v)$, branch e is oriented from u to v. We'll really be using the **incidence matrix** $A : \mathcal{D}(\mathcal{V}) \to \mathcal{D}(\mathcal{E})$, defined (using

bra-ket notation) as

$$\langle e|A|v\rangle = \begin{cases} 1 & \mathcal{A}(e) = (u, v) \text{ for some } u \\ -1 & \mathcal{A}(e) = (v, u) \text{ for some } u \\ 0 & \text{otherwise} \end{cases} . \tag{A.2}$$

In the framework of homology, the incidence matrix A can be thought of as the boundary map ∂. Let $A_{ev} = \langle e|A|v\rangle$. Less formally, $A_{ev} = 1$ if edge e ends on v, while $A_{ev} = -1$ if e starts on v. We say that $(\mathcal{V}, \mathcal{E}, A)$ is a **directed graph**.

Definition 43. Let $G = (\mathcal{V}, \mathcal{E}, A)$ be a directed and connected graph. Partition $\mathcal{E}$ so that $\mathcal{E} = \mathcal{C} \cup \mathcal{I}$ and $\mathcal{C} \cap \mathcal{I} = \emptyset$. We define $(\mathcal{V}, \mathcal{E}, A, \mathcal{C})$ as a **circuit**. Intuitively, we will put a capacitive element on all branches in $\mathcal{C}$, and an inductive element on all branches in $\mathcal{I}$.

We note that the choice to require G to be connected is without loss of generality since a circuit with multiple connected components would lead to a separable problem in later discussion. The object which we define to be a circuit consists of a directed graph and a determination of which branches contain capacitors.

Definition 44. The **reduced incidence matrix** $\Omega : \mathcal{D}(\mathcal{V}) \to \mathcal{D}(\mathcal{C})$ obeys $\langle e|\Omega|v\rangle = \langle e|A|v\rangle$. The dimensions of Ω are distinct, and in what follows $\Omega_{ev} = \langle e|\Omega|v\rangle$ will appear frequently.

It is possible that multiple branches begin and end at the same place: i.e. we could have $\mathcal{A}(e_1) = \mathcal{A}(e_2)$ for some $e_1 \neq e_2$. We could also have $e_1 \in \mathcal{I}$ and $e_2 \in \mathcal{C}$.

The following definitions will prove useful in what follows:

Definition 45. An **(unoriented) cycle** of length l, $\delta = (e_1, \ldots, e_l)$, is an ordered list of l branches in $\mathcal{C}$, with the property that there exist nodes $v_1, \ldots, v_l$ such that $\mathcal{A}(e_1) = \{v_1, v_2\}$, $\mathcal{A}(e_2) = \{v_2, v_3\}$, …, $\mathcal{A}(e_l) = \{v_l, v_1\}$. Here the notation means we do not care about the ordering: $\{u, v\} = \{v, u\}$. Indeed, we do not care about the orientation of edges on a cycle for this discussion. Intuitively, δ is a loop made out of only capacitors in the circuit. Let Δ_C denote the set of cycles in $\mathcal{C}$. For the

length l cycle defined above, we write

$$|\delta\rangle = |(e_1, \ldots, e_l)\rangle = \sum_{j=1}^{l} \sigma_j |e_j\rangle \in \mathcal{D}(\mathcal{C}), \tag{A.3}$$

where

$$\sigma_j = \begin{cases} 1 & \mathcal{A}(e_j) = (v_j, v_{j+1}). \\ -1 & \mathcal{A}(e_j) = (v_{j+1}, v_j). \end{cases} . \tag{A.4}$$

We remark that the choice of σ_i implies that, for δ in Δ_C, $\langle\delta|A|v\rangle = 0$ for all $|v\rangle$ in $\mathcal{D}(\mathcal{V})$. Qualitatively, one could say that σ_i is chosen so that the chain corresponding to a given cycle is in some manner "oriented" even if the cycle itself is not.

Definition 46. An **inductively shunted island** in circuit $(\mathcal{V}, \mathcal{E}, A, \mathcal{C})$ is a subset $U \subseteq \mathcal{V}$ with the following property: $u_1 \in U$ and $u_2 \in U$ both hold if and only if there exists a path from u_1 to u_2 along branches in $\mathcal{C}$. Note that a node only connected to inductors forms its own island. Let $\Gamma_I(\mathcal{V}, \mathcal{E}, A, \mathcal{C})$ (denoted as Γ_I for shorthand hereafter) be the set of all such islands. For island $i \in \Gamma_I$, write

$$|i\rangle = \sum_{v \in i} |v\rangle \in \mathcal{D}(V). \tag{A.5}$$

We make an analogous definition for inductors, which will be useful in future discussion.

Definition 47. Let $(\mathcal{E}, \mathcal{V}, \mathcal{C}, A)$ be a circuit. Consider the undirected graph $G_L = (\mathcal{V}, \mathcal{I})$ formed out of only the inductive branches. We say that an **capacitively shunted island** j is a connected subgraph of G_L, which is not a proper subgraph of any other connected subgraph of G_L. If the set of all such capacitively shunted islands is Γ_C, then

$$G_L = \bigcup_{j \in \Gamma_C} (j \cap \mathcal{V}, j \cap \mathcal{I}). \tag{A.6}$$

We remark that G_L contains all nodes that are in G and in particular even those connected to no inductors. Such nodes constitute their own capacitively shunted island in the same manner

that nodes connected only to capacitors make up their own capacative island. The set of inductive islands derived from some circuit is unique.

Combining all of the definitions above, the following theorem summarizes the critical properties of Ω.

Theorem 48. *The generically non-square matrix Ω has the following properties:*

(1) $\langle\alpha|\Omega = 0$ (i.e. $\langle\alpha|$ is a left null vector of Ω) if and only if

$$|\alpha\rangle = \sum_{\delta\in\Delta_C} \alpha_\delta|\delta\rangle, \quad \alpha_\delta \in \mathbb{Z}. \tag{A.7}$$

Note that the right hand side in general contains linearly dependent vectors.

(2) $\Omega|\beta\rangle = 0$ (i.e. $|\beta\rangle$ is a right null vector of Ω) if and only if

$$|\beta\rangle = \sum_{i\in\Gamma_I} \beta_i|i\rangle, \quad \beta_i \in \mathbb{Z}. \tag{A.8}$$

(3) Ω contains a non-degenerate submatrix $\bar{\Omega}$, which is a $n \times n$ square matrix where

$$n = |\mathcal{V}| - |\Gamma_I|. \tag{A.9}$$

Proof. We prove the three parts in turn:

(1) It is straightforward to see that (A.7) is a null vector of Ω, using linearity and the fact that for any cycle δ of length l,

$$\langle\delta|\Omega = (\langle v_1| - \langle v_2|) + (\langle v_2| - \langle v_3|) + \cdots + (\langle v_l| - \langle v_1|) = 0. \tag{A.10}$$

The converse is implied by the homology of the undirected graph $(\mathcal{V}, \mathcal{C})$, but we prove it explicitly. It will prove convenient to (without loss of generality) choose the orientations

of edges $|e\rangle$ such that for all e, $\langle\alpha|e\rangle \geq 0$.[1] Pick an edge e such that $\langle\alpha|e\rangle > 0$. If $\langle e|\Omega = \langle v| - \langle u|$, then there must exist an edge e', with $\langle\alpha|e'\rangle \neq 0$ and $\langle e'|\Omega|v\rangle = -1$, since $\langle e|\Omega|v\rangle = 1$ but $\langle\alpha|\Omega|v\rangle = 0$. Build the bra $\langle\psi| = \langle e| + \langle e'|$, and observe that

$$\langle\psi|\Omega = \langle v'| - \langle u|. \tag{A.11}$$

If $\langle v'| \neq \langle u|$, then we keep going: look for another edge in $\langle\alpha|$ of the form $\langle e''|\Omega = \langle v''| - \langle v'|$ – such an edge must exist since $\langle\alpha|$ is a null vector, etc. Then update $|\psi\rangle \to |\psi\rangle + |e''\rangle$. Since in each step of this process,

$$\sum_e \langle e|\psi\rangle \to 1 + \sum_e \langle e|\psi\rangle, \tag{A.12}$$

eventually, this process must terminate because we will (if it does not) run out of edges in $\langle\alpha|$ to include. When the process does terminate and we find a left null vector $\langle\psi|$ with $\langle\psi|\Omega = 0$. If $\langle\alpha| - \langle\psi| = 0$, then $\langle\alpha|$ simply corresponds to a chain associated with a cycle $\delta \in \Delta_C$. Otherwise, $\langle\alpha| - \langle\psi|$ is a non-trivial vector where we can simply repeat the argument. Since the sum of coefficients of $\langle\alpha| - \langle\psi|$ is smaller than the sum in $\langle\alpha|$, the process will terminate. By construction, each $\langle\psi|$ that we find in this process formed a cycle $\delta \in \Delta_C$, meaning our null vector can be expressed as a chain of the form (A.7).

(2) Pick some $e \in \mathcal{C}$. We know that $\langle e|\Omega|\beta\rangle = 0$, which means that if $\mathcal{A}(e) = (u, v)$, then $\langle u|\beta\rangle\beta_u = \beta_v = \langle v|\beta$. For any two vertices in island $i \in \Gamma_I$, we can find a path between them (call its length l): $(e_1, \ldots, e_l)$. Applying this argument to all edges along the path, we conclude that if $v_{1,2} \in i$, $\beta_{v_1} = \beta_{v_2}$. Thus, the most generic null vector is of the form (A.8). It is also straightforward to check that (A.8) is always a right null vector.

(3) This follows from the rank-nullity theorem, the fact that $\mathcal{D}(\mathcal{V})$ is a $|\mathcal{V}|$-dimensional vector space, and the fact that the number of linearly independent null vectors in (A.8) is $|\Gamma_I|$.

Thus we prove all three claims. □

[1] This can be done by simply defining $|-e\rangle = -|e\rangle$ as an edge oriented in the opposite direction.

The substance of Theorem 48 is that it is possible to define a symplectic form (and thus to quantize) immediately after enumerating the null vectors of Ω, which are counted exhaustively by the theorem above. Later results will provide simplifications to circuits in general, but none of the following results are strictly necessary to generically quantize non–dissipative circuits. Point 3 of the theorem above immediately implies the following corollaries:

Corollary 49. *Consider the circuit $(\mathcal{V}, \mathcal{E}, A, \mathcal{C})$. Define*

$$g = |\mathcal{C}| - |\mathcal{V}| + 1. \tag{A.13}$$

g is the graph–theoretic equivalent of a topological genus, and it counts the number of loops in a graph.

$$g - 1 = |\Delta_C| - |\Gamma_I|. \tag{A.14}$$

Proof. The rank nullity theorem guarantees that the row–rank and the column–rank of Ω are equal, and $|\mathcal{V}| - |\Gamma_I|$ counts the row–rank of Ω while $|\mathcal{E}| - |\Delta_C|$ counts the column–rank therein. □

A.1.2 Circuit quantization on general graphs

The discussion in Appendix A.1.1 was entirely self–contained. At this point, we shift our focus to the relation of graph theory to the dynamical systems of interest. What follows will depend on discussion that can be found broadly in Section 2.2.

Theorem 50. *Let $G = (\mathcal{E}, \mathcal{V}, A, \mathcal{C})$ be a circuit. It is always possible to define $|\mathcal{C}| - |\Delta_C|$ variables $Q_i = A_{ie} q^e$ and $\Phi_i = B_{iv} \phi^v$ so that*

$$\sum_{e,v} q_e \Omega_{ev} \dot{\phi}_v = \sum_{i=1}^{|\mathcal{C}|-|\Delta_C|} Q_i \dot{\Phi}_i = \sum_{i=1}^{|\mathcal{C}|-|\Delta_C|} \sum_{e,v} A_{ie} B_{iv} q_e \dot{\phi}_v. \tag{A.15}$$

A_{ie} and B_{iv} can be obtained by picking a spanning tree $\mathcal{T} \subseteq \mathcal{C}$. The transformations between the variables corresponding to different spanning trees are canonical.

Proof. Choose a spanning tree $\mathcal{T}$ of $\mathcal{C}$ and write

$$\mathcal{C} \setminus \mathcal{T} = \{a_1, a_2, \ldots, a_{|\Delta_C|}\}. \tag{A.16}$$

The spanning tree has the property that every node adjacent to some branch in $\mathcal{C}$ is adjacent to some branch in $\mathcal{T}$, and $\mathcal{T}$ contains no cycles. Succinctly, if v_1 and v_2 are connected by capacitors, there is a unique shortest path from v_1 to v_2 traversing only on branches in $\mathcal{T}$. The choice of $\mathcal{T}$ is necessarily non–unique and we will later show that the choice of $\mathcal{T}$ cannot have physical implications. By the uniqueness in $\mathcal{T}$ of a path between two vertices, every branch flux from $\mathcal{C} \setminus \mathcal{T}$ can be expressed as an linear combination of branch fluxes from $\mathcal{T}$ with integer coefficients. (Recall the definition of branch flux ϕ_e in (2.8).) Namely, there exists some (generally rectangular) matrix $K \in \mathbb{R}^{|\mathcal{T}| \times |\mathcal{C} \setminus \mathcal{T}|}$ satisfying

$$\phi_e = \sum_{f \in \mathcal{T}} K_{ef} \phi_f \tag{A.17}$$

for $e \in \mathcal{C} \setminus \mathcal{T}$. By construction, $K_{ef} \in \{-1, 0, 1\}$.

We recognize that

$$\sum_{e,v} q_e \Omega_{ev} \dot{\phi}_v = \sum_{f \in \mathcal{T}} q_f \dot{\phi}_f + \sum_{e \notin \mathcal{T}} \sum_{f \in \mathcal{T}} q_e K_{ef} \dot{\phi}_f. \tag{A.18}$$

Define

$$M = \begin{pmatrix} \mathbb{I}_{|\mathcal{T}|} & K \end{pmatrix}^{\mathrm{T}} \tag{A.19}$$

and rewrite

$$Q_f = \sum_e q_e M_{ef}. \tag{A.20}$$

Since M has rank $|\mathcal{T}|$, the Q_f variables are independent, and they span a vector space of dimension $|\mathcal{T}| = |\mathcal{C}| - |\Delta_C|$, so we see that all left null vectors of Ω_{ev} have been removed by this definiton. A similar argument shows that the ϕ_f variables are also independent, and thus all right null vectors have been removed.

Now suppose another spanning tree, $\mathcal{T}'$, is chosen .[2] Suppose further that $\mathcal{T}$ and $\mathcal{T}'$ differ by a single edge, so that

$$\mathcal{T} = \{f_0\} \cup \mathcal{A} \tag{A.21}$$

and

$$\mathcal{T}' = \{f_1\} \cup \mathcal{A} \tag{A.22}$$

with both f_0 and f_1 absent from $\mathcal{A} \subset \mathcal{C}$. Further demand that f_0 and f_1 are in the same cycle. Since $\mathcal{T}$ and $\mathcal{T}'$ are both spanning trees, there exists some matrix Σ whose entries are elements of $\{1, -1, 0\}$ so that

$$\phi_f = \sum_{f'} \Sigma_{ff'} \phi_{f'} \tag{A.23}$$

for f' in $\mathcal{T}'$ and f in $\mathcal{T}$. Moreover, necessarily $|\mathcal{T}| = |\mathcal{T}'|$ so Σ is square. Moreover, Σ is the identity on edges common to $\mathcal{T}$ and $\mathcal{T}'$. Since f_1 is an element of $\mathcal{T}' \setminus \mathcal{T}$,

$$\phi_{f_1} = \sum_{f \in \mathcal{T}} K_{f_1 f}(\mathcal{T}) \phi_f, \tag{A.24}$$

where $K(T)$ denotes the K-matrix constructed in (A.17) for $\mathcal{T}$. Since f_0 is the unique element of $\mathcal{T} \setminus \mathcal{T}'$ and f_0 is in the same cycle as f', necessarily the element $K_{f_1 f_0}(\mathcal{T}) \neq 0$, as otherwise there would be a cycle in $\mathcal{T}$. So

$$\phi_{f_1} = \sum_{f \neq f_0 \in \mathcal{T}} K_{f' f}(\mathcal{T}) \phi_f + K_{f_1 f_0}(\mathcal{T}) \phi_{f_0} \tag{A.25}$$

and thus

$$K_{f_1 f_0}(\mathcal{T}) \phi_{f_0} = \phi_{f_1} - \sum_{f \neq f_0 \in \mathcal{T}} K_{f_1 f}(\mathcal{T}) \phi_f. \tag{A.26}$$

[2] There are $M \leq \prod_{\delta \in \Delta_C} |\delta|$ possible choices of spanning tree in a circuit.

Evidently,

$$\Sigma_{ff'} = \begin{pmatrix} \mathbb{I}_{|\mathcal{T}\cap\mathcal{T}'|} & 0 \\ 0 & K_{f_1 f_0}(\mathcal{T}) \end{pmatrix} \begin{pmatrix} \mathbb{I}_{|\mathcal{T}\cap\mathcal{T}'|} & 0 \\ K_{f_1 f}(\mathcal{T}) & 1 \end{pmatrix}. \tag{A.27}$$

Schematically, $\Sigma = \mathbf{D}(\mathbb{I} + \mathbf{N})$ with $\mathbf{N}^2 = 0$ and $\mathbf{D}^2 = \mathbb{I}$. Immediately, $\Sigma^{-1} = (\mathbb{I} - \mathbf{N})\mathbf{D}$.

Now, clearly,

$$\sum_{f\in\mathcal{T}} Q_f \dot{\phi}_f = \sum_{f\in\mathcal{T}} \sum_{f'\in\mathcal{T}'} Q_f \Sigma_{ff'} \phi_{f'} = Q'_{f'} \dot{\phi}_{f'} \tag{A.28}$$

with

$$Q'_{f'} = \sum_{f\in\mathcal{T}} Q_f \Sigma^f_{f'}. \tag{A.29}$$

Now the transformation

$$\begin{aligned} \mathbf{Q} &\to \mathbf{Q}' = \mathbf{Q}\Sigma \\ \boldsymbol{\phi} &\to \boldsymbol{\phi}' = \Sigma^{-1}\boldsymbol{\phi} \end{aligned} \tag{A.30}$$

is clearly canonical.

Lastly, for general $\mathcal{T}$ and $\mathcal{T}'$, we can find a sequence $\mathcal{T} \to \mathcal{T}_1 \to \cdots \to \mathcal{T}'$ where at each step, $\mathcal{T}_j$ and $\mathcal{T}_{j+1}$ differ by a single edge. The composition of canonical transformations corresponding to each step is still canonical. □

In the proof of Theorem 50, the Q_f variables are independent and always number $|\mathcal{C}| - |\Delta_C|$. Moreover, the transformation between different choices of spanning tree is canonical. Together, these two facts imply that the same non-dynamical degrees of freedom are removed by *every* choice of spanning tree. By theorem 48 these are necessarily those corresponding to the current about each loop of capacitors. Similarly, the null vectors associated to inductively shunted islands are always the same.

Theorem 50 depends only upon Theorem 48 and previous graph–theoretic constructions. In practice, it is often the case that one naturally identifies conjugate variables by applying Theorem 53 instead, without the need to apply the above result. Further, it may pose a technical challenge to

write the Hamiltonian associated with some circuit in terms of only Q_e variables as defined above (see Section 2.4.7), but as a matter of principle the Hamiltonian will always exist.

One utility of Theorem 50 arises in the circumstance where one wishes to periodically identify some subset of variables on a finite interval. Upon doing so, it is required that the conjugate of the periodic variable be integer–valued and theorem 50 guarantees that this is the case. We state the following two observations to emphasize this point:

Observation 51. *Consider the phase space*

$$M = \mathbb{R}^{|\mathcal{V}|} \times \mathbb{R}^{|\mathcal{C}|}. \tag{A.31}$$

Given the reduced incidence matrix Ω, define the 2-form on M

$$\omega = \sum_{e\in\mathcal{C}} \sum_{v\in\mathcal{V}} \Omega_{ev} \mathrm{d}q_e \wedge \mathrm{d}\phi_v. \tag{A.32}$$

Then there exists a submanifold $\bar{M}$ of M, which is symplectic, diffeomorphic to (equivalent to) the cotangent bundle $\mathbb{T}^*\mathbb{R}^n = \mathbb{R}^{2n}$, with ω a globally constant symplectic form. On $\bar{M}$,

$$\omega = \sum_{f\in\mathcal{T}} \mathrm{d}Q_f \wedge \mathrm{d}\phi_f \tag{A.33}$$

as guaranteed by Theorem 50.

Observation 52. *Let $G = (\mathcal{E}, \mathcal{V}, A, \mathcal{C})$ be a circuit, and let ω be the symplectic form ω on $T^*\mathbb{R}^n$ provided by Cor. 51 through Thm. 48. If some number of variables are identified as periodic, the symplectic form on the resulting quotient manifold, ω' is given by the quotient map.*

This result follows immediately from the quotient manifold theorem [70]. We remark that the symplectic form produced by the quotient map is not exact, in the physically relevant case where we quotient by a free group action of $\mathbb{Z}^k$, so that $\mathbb{R}^k/\mathbb{Z}^k = \mathrm{T}^k$ becomes a torus.

Now, we count the number of degrees that can be removed by Noether's Theorem for a general

circuit.

Theorem 53. *Let $(\mathcal{E}, \mathcal{V}, A, \mathcal{C})$ be a circuit. Consider the symplectic form and Hamiltonian provided by (2.24). If Γ_C is the set of capacitively shunted islands, there exist $|\Gamma_C| - 1$ Noether charges.*

Proof. Enumerate the (unique) elements of $\Gamma_C = \{J_1, J_2, \ldots, J_{|\Gamma_C|}\}$. Without loss of generality, take $J_1 = \{u_1, \ldots, u_l\}$. If any inductive edge couples to a $u_i \in J_1$, the edge also connects to another vertex in J_1 (by definition of a capacitively shunted island). Therefore, since H only depends on ϕ_vs via inductive edges,

$$H(q_e, \phi_v) = H(q_e, \phi_v + c_v) \tag{A.34}$$

where

$$c_v = \begin{cases} 1 & v \in J_1 \\ \\ 0 & v \notin J_1 \end{cases}. \tag{A.35}$$

From Lagrangian (2.24), evaluate the Euler-Lagrange equation

$$0 = \sum_{v \in \mathcal{V}} c_v \frac{\delta S}{\delta \phi_v} = -\sum_{v \in \mathcal{V}} c_v \left[\sum_{e \in \mathcal{C}} \dot{q}_e \Omega_{ev} - \frac{\partial H}{\partial \phi_v} \right] = -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{v \in J_1} \sum_{e \in \mathcal{C}} q_e \Omega_{ev}. \tag{A.36}$$

We conclude that

$$Q_{J_i} = \sum_{v \in J_i} \sum_{e \in \mathcal{C}} q_e \Omega_{ev} \tag{A.37}$$

is a constant of motion.

Since we can do this for all $|\Gamma_C|$ islands, we may naively conclude that there are $|\Gamma_C|$ independent Noether charges. However, notice that

$$\sum_{J_i \in \Gamma_C} Q_{J_i} = \sum_{v \in \mathcal{V}} \sum_{e \in \mathcal{C}} q_e \Omega_{ev} = 0, \tag{A.38}$$

since every edge enters one vertex and exits one vertex. Therefore, there is a constraint that not all Q_{J_i}s can be independent. Since we assume that the circuit is connected, there will be no other

constraints on any Q_{J_i}, since any subset of the capacitively shunted islands (that is not the entire circuit) must be connected to another part of the circuit by at least one capacitive edge, meaning that the generalization of (A.38) does not vanish for any other subset of Γ_C. □

A.2 Solubility of energetic constraints

Theorem 50 implies that for a generic circuit, the term $\sum_{e\in\mathcal{C},v} q_e\Omega_{ev}\dot{\phi}_v$ can be rewritten as

$$\sum_{e\in\mathcal{C},v} q_e\Omega_{ev}\dot{\phi}_v = \sum_{\alpha,\beta}\xi_\alpha\omega^{\alpha\beta}\dot{\xi}_\beta. \tag{A.39}$$

Here α,β run only over the degrees of freedom formed in, e.g., the spanning tree construction of Sec. 2.2.4, and $\omega^{\alpha\beta}$ is the canonical symplectic form, which is antisymmetric and invertible. Notice that ξ_α can contain both charge and flux degrees of freedom within it. In other words, the action for a circuit can generically be written as

$$S = \int \mathrm{d}t\left[\sum_{\alpha,\beta}\xi_\alpha\omega^{\alpha\beta}\dot{\xi}_\beta - H(\xi_1,\xi_2,\ldots,\xi_N,\eta_1,\eta_2,\ldots,\eta_M)\right] \tag{A.40}$$

with η_M corresponding to the left and right null vectors of Ω_{ev}, which correspond to non-dynamical variables, i.e. constraints.Functional derivatives of S yield

$$\begin{aligned}\frac{\delta S}{\delta\xi_\alpha} &= \sum_\beta \omega^{\alpha\beta}\dot{\xi}_\beta - \frac{\partial H}{\partial\xi_\alpha},\\ \frac{\delta S}{\delta\eta_\alpha} &= -\frac{\partial H}{\partial\eta_\alpha}.\end{aligned} \tag{A.41}$$

If the principle of least action is to be obeyed, it follows that there must exist some $\bar{\eta} = (\bar{\eta}_1,\bar{\eta}_2,\ldots,\bar{\eta}_M)$ such that

$$\left.\frac{\partial H}{\partial\eta_\alpha}\right|_{\eta=\bar{\eta}} = 0 \tag{A.42}$$

Suppose for the moment that a unique solution $\bar{\eta}$ exists that satisfies (A.42) for each $\alpha =$

$1, 2, \ldots, M$, such that $\partial\bar{\eta}_a/\partial\xi_\alpha$ is finite. Let us define the effective Hamiltonian

$$H_{\text{eff}}(\xi) = H(\xi, \bar{\eta}(\xi)). \tag{A.43}$$

We claim that the action

$$S = \int \mathrm{d}t \left[\sum_{\alpha,\beta} \xi_\alpha \omega^{\alpha\beta} \dot{\xi}_\beta - H_{\text{eff}}(\xi_1, \xi_2, \ldots, \xi_N) \right], \tag{A.44}$$

which explicitly now encodes a Hamiltonian dynamical system due to the invertibility of $\omega_{\alpha\beta}$, is equivalent to the one which we wrote down above. To see this, notice that the equations of motion for this action simply become

$$\frac{\delta S}{\delta \xi_\alpha} = \sum_\beta \omega^{\alpha\beta}\dot{\xi}_\beta - \frac{\partial H_{\text{eff}}}{\partial \xi_\alpha} = \sum_\beta \omega^{\alpha\beta}\dot{\xi}_\beta - \left.\frac{\partial H}{\partial \xi_\alpha}\right|_{\eta=\bar{\eta}} - \sum_{a=1}^{M} \left.\frac{\partial H}{\partial \eta_a}\frac{\partial \eta_a}{\partial \xi_\alpha}\right|_{\eta=\bar{\eta}} = \sum_\beta \omega^{\alpha\beta}\dot{\xi}_\beta - \left.\frac{\partial H}{\partial \xi_\alpha}\right|_{\eta=\bar{\eta}}, \tag{A.45}$$

where we have used (A.42) in the last step. We conclude that the theory in (A.40) describes Hamiltonian dynamics on a symplectic manifold $\mathbb{R}^N$ (or quotient thereof, if some of the coordinates are periodically identified).

It remains to discuss what happens if our assumption of a unique solution $\bar{\eta}$ is not satisfied. The first possibility is that there infinitely many solutions due to a "gauge freedom" – for example, this arises when considering the "gauge" freedom to shift $\phi_v \to \phi_v + c$ for some constant c. In this case, we simply define our symplectic manifold by fixing a gauge (e.g. $\phi_u = 0$ at one vertex u) and continue. The second possibility is that there are *no* solutions to a constraint (A.42). Because the constraint variables η are always sums of either flux or charge like variables only, the only way this can arise is due to an inductive island at which the inductive constitutive relations are not compatible with Kirchoff's current Law, or a capacitive loop at which the capacitive constitutive relations are not compatible with Kirchoff's voltage law. Therefore, we regard this second possibility as unphysical, and do not proceed to analyze the dynamics or quantize the circuit.

The final, and most subtle, possibility, is that there are multiple solutions to the equation.

This is what happens for the singular circuit of Sec. 2.4.7. The analysis of how to quantize such singular circuits necessarily goes beyond the framework of the present paper. Perhaps the simplest strategy would be to simply follow by fiat one branch of solutions $\bar{\eta}$, and quantize the model for the corresponding choice of H_{eff}; we do not guarantee however that this is the physically correct prescription, and in general this is a challenging problem. However, as we noted in the main text, in physical superconducting circuits, parasitic elements always remove such ambiguities, and after this point our framework can be applied straightforwardly.

A.3 Quantization without Noether charges

We have remarked that Theorem 48 is necessary to carry out the quantization procedure but that Theorem 53 is not. We will provide an example that makes this distinction clear. The Lagrangian corresponding to the circuit drawn in Figure 2.5(b) is given by

$$L = q_{e_1}(\dot{\phi}_{v_2} - \dot{\phi}_{v_1}) + q_{e_2}(\dot{\phi}_{v_3} - \dot{\phi}_{v_2}) - \frac{1}{2C_1}q_{e_1}^2 - \frac{1}{2C_2}q_{e_2}^2 - \frac{1}{2L}(\phi_{v_3} - \phi_{v_1})^2. \tag{A.46}$$

To be explicit, the matrix Ω is given by

$$\Omega = \begin{pmatrix} -1 & 1 & 0 \\ \\ 0 & 1 & -1 \end{pmatrix} \tag{A.47}$$

and we see readily that Ω has the right null vector corresponding to a uniform sum over nodes, which informs us that the time evolution of $\phi_{v_1} + \phi_{v_2} + \phi_{v_3}$ is not fixed by L. In the spanning tree construction, we choose dynamical variables Φ_{e_1} and Φ_{e_2} to be branch fluxes on the unique spanning tree; the standard branch charges $Q_{e_1} = q_{e_1}$ and $Q_{e_2} = q_{e_2}$ are the canonical conjugate variables. The Lagrangian becomes

$$L = Q_{e_1}\dot{\Phi}_{e_1} + Q_{e_2}\dot{\Phi}_{e_2} - \frac{1}{2C_1}Q_{e_1}^2 - \frac{1}{2C_2}Q_{e_2}^2 - \frac{1}{2L}(\Phi_{e_1} + \Phi_{e_2})^2. \tag{A.48}$$

The quantum Hamiltonian is

$$H = -\frac{\hbar^2}{2C_1}\frac{\partial^2}{\partial \Phi_{e_1}^2} - \frac{\hbar^2}{2C_2}\frac{\partial^2}{\partial \Phi_{e_2}^2} + \frac{1}{2L}(\Phi_{e_1} + \Phi_{e_2})^2. \tag{A.49}$$

We recognize (A.49) as a harmonic oscillator coupled to a free particle, after a suitable coordinate charge. The free particle degree of freedom could have been removed from the start by using Theorem 53: the Noether charge is associated with charge conservation on the subcircuit trapped between the two capacitors. Neglecting the Noether charge when performing quantization implicitly assumes that the decoupled degree of freedom does not couple to an external probe of interest.

Appendix B

Appendices of [91]

B.1 Graph theory

Where possible, the demonstrations in this appendix will follow [90] as closely as possible.

Definition 54 (Graph)**.** Take a set $\mathcal{V}$ and let $\mathcal{E}$ be the set of ordered pairs of elements of $\mathcal{V}$. We say that $G = (\mathcal{V}, \mathcal{E})$ is a (directed) **graph**.

In all of what follows, we will assume that graphs and structures built upon graphs are connected. This assumption will not henceforth be stated, but we remark that it is not required by every result we prove.

Definition 55 (Embedding)**.** Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. Choose a two dimensional orientable surface S, and associate one (distinct) point in S for every element of $\mathcal{V}$: crudely we can think of $\mathcal{V} \subset S$. Every edge $e = (u, v) \in \mathcal{E}$ corresponds to a smooth map $e : [0, 1] \to S$ such that $e(0) = u$ and $e(1) = v$. Assuming that for $x, x' \in (0, 1)$, $e(x) \neq e(x')$ (i.e. no two lines ever cross), S is partitioned into a set of surfaces. If such a drawing exists, we call it an **embedding** of G on S, or simply an embedding of G.

A single graph can have many embeddings. The combinatorial information in G is independent of the manner in which it is embedded. If a graph G can be embedded upon a surface S at all, it is always possible to draw some embedding of G on S such that the surface S is partitioned into regions isomorphic to the unit disk, though we remark that Definition 55 does not require this property. Nonetheless, these “nice" embeddings will be the subject of our discussion.

Definition 56 (Face)**.** In an embedding of G on S, each "patch" topologically equivalent to the unit disk is a **face**. For two faces f_1 and f_2 and edge e of G embedded on S, we write $f_1 \overset{e}{\leftrightarrow} f_2$ to mean that e is an edge on the common boundary of the disk corresponding to f_1 and the disk corresponding to f_2.

A graph G is a purely combinatorial object. We emphasize that while there is a universal notion of topology in the one-dimensional setting, the topology of the embedding is non-universal. Perhaps surprisingly at first glance, even the subsets of edges in $\mathcal{E}$ which correspond to the boundaries of faces in an embedding is not unique, and can depend on the choice of embedding. As we will see in later discussion, these ambiguities certainly will have no physical consequences in circuit quantization, although they can lead to different looking Lagrangians or Hamiltonians at first glance!

Definition 57 (Loop)**.** Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. For the purposes of this definition, we regard $e = (u, v)$ and $e' = (v, u)$ as equivalent. A **loop** γ is a subset $\gamma \subset \mathcal{E}$, so that (properly orienting each edge)

$$\gamma = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_0)\}. \tag{B.1}$$

We say that γ is of length n if $|\gamma| = n$.

Every face's boundary is a loop, so there is a natural injection from the set of faces to the set of loops.

Definition 58 (Linearly independent loops)**.** Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. Let $\mathit{\Gamma} = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a set of n loops in G. We say that the set $\mathit{\Gamma}$ is linearly dependent if there exists $m \in \{1, \ldots, n\}$ and set $\iota = \{i_1, i_2, \ldots, i_p\} \subset \{1, 2, 3, \ldots, n\} \setminus \{m\}$, such that

$$\gamma_m = \gamma_{i_1} \mathit{\Delta} \gamma_{i_2} \mathit{\Delta} \ldots \mathit{\Delta} \gamma_{i_p} \tag{B.2}$$

where

$$A \mathit{\Delta} B = (A \cup B) \setminus (A \cap B). \tag{B.3}$$

Otherwise, we say that the loops in Γ are **linearly independent**. We say that the rank of a set of loops Γ is the number of linearly independent loops in Γ.

Definition 59 (Cut)**.** Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. Partition $\mathcal{V}$ into m sets $\mathcal{V}_i$ with $i = 1, 2, 3 \ldots, m$ so that

$$\mathcal{V} = \bigcup_{i=1}^{m} \mathcal{V}_i \tag{B.4}$$

and

$$\emptyset = \mathcal{V}_i \cap \mathcal{V}_j. \tag{B.5}$$

for $i \neq j$. Such a partition $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \ldots, \mathcal{V}_m)$ of $\mathcal{V}$ is called a **cut** of G.

Definition 60 (Edge–cut)**.** Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. Let $(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m)$ be a cut of G. Define the set $\mathcal{K} \subset \mathcal{E}$ such that

$$\mathcal{K} = \{(u, v) \in \mathcal{E} \text{ s.t. } \exists i \text{ s.t. } \{u, v\} \subset \mathcal{V}_i\}. \tag{B.6}$$

The set $\mathcal{E} \setminus \mathcal{K}$ is called the **edge–cut** of G induced by $(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m)$.

The notions of a cut and an edge–cut are formally equivalent and every result of this manuscript may be stated and proven favoring one notion over the other. Nonetheless, it is useful to posses an understanding of both conventions since some ideas are easier to understand in terms of cuts and others are easier to understand in terms of edge–cuts.

Definition 61 (Genus)**.** Let $G = (\mathcal{V}, \mathcal{E})$ be a graph. We say that G has genus g if there exists no embedding of G on any surface S of topological genus $g' < g$. This is not the same definition as the usual graph genus from graph theory.

Definition 62 (Embedded graph)**.** Let G be a graph, and S an orientable surface. Suppose that there exists some embedding of G on S, i.e. on a sphere with g holes. Under this embedding, denote the set of faces of G embedded in S as $\mathcal{F}$. We write $G_S = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ to be the **embedded graph** G on S.

In much of what follows, we will suppress some of the above defined terminology when either the embedding of G on S is of no consequence, or when the embedding of G on S is clear from context. In such cases, we will write $G = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ and we will refer to G simply as a graph.

Theorem 63. *Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be an embedded graph. Denote the loop on the boundary of the face f_i as γ_i. If G is connected, the set*

$$\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{|\mathcal{F}|}\} \tag{B.7}$$

is linearly dependent and the set $\Gamma \setminus \{\gamma_1\}$ is linearly independent.

Proof. Every edge in G appears on the boundary of precisely two faces, so $\gamma_1 = \gamma_2 \Delta \gamma_3 \Delta \ldots \Delta \gamma_{|\mathcal{F}|}$. For any γ_j obeying $\gamma_1 \cap \gamma_j \neq \emptyset$, γ_j is linearly independent of the rest, since every edge appears in no other face. Removing such γ_j, we can inductively deduce the linear independence of the remaining γs, since the graph is connected. □

The following two important, and classic, results, are stated without proof:

Theorem 64 (Euler's formula)**.** *Let $G = (\mathcal{V}, \mathcal{E})$, and suppose that G has genus g. Let S be a compact, orientable discretization of a Riemann surface of genus g. For any embedding of G on S, we have*

$$|\mathcal{V}| - |\mathcal{E}| + |\mathcal{F}| = 2 - 2g. \tag{B.8}$$

Corollary 65. *Let $G = (\mathcal{V}, \mathcal{E})$ be a graph and let Γ be a linearly independent set of loops in G. Then*

$$|\Gamma| \leq g_G = |\mathcal{E}| - |\mathcal{V}| + 1. \tag{B.9}$$

In effect, Cor. 65 serves to indicate that, for nonplanar graphs, it is possible to construct a linearly independent set of loops that is greater in size than the set of faces of a graph. As we will see, this fact is crucial to our discussion of circuit dualities for nonplanar graphs.

Theorem 66. *Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be an embedded, nonplanar graph with genus $g > 0$. There exist $2g$ loops that are independent of all of the loops bounding faces f in $\mathcal{F}$.*

Proof. By Theorem 64,

$$|\mathcal{F}| + 2g = g_G + 1. \tag{B.10}$$

As we have seen in Theorem 63, the set of loops bounding faces in $\mathcal{F}$ is $|\mathcal{F}| - 1$, and Corollary 65 tells us that it is possible to construct a set of loops of rank g_G. □

Definition 67 (Loop set)**.** Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be an embedded graph of genus g. Choose $2g$ loops independent of all loops at the boundary of some face in $\mathcal{F}$, say $l_1, \ldots, l_{2g}$. Denote the loop at the boundary of f_i in $\mathcal{F}$ as l_{2g+i}. The set

$$\mathcal{L} = \{l_1, l_2, \ldots, l_{2g+|\mathcal{F}|}\} \tag{B.11}$$

is called the **loop set** of G.

Definition 68 (Extended embedded graph)**.** Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{F})$ be an embedded graph, and let $\mathcal{L}$ be the loop set of G. The object $G' = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ is called an **extended embedded graph**.

Definition 69 (Planar graph)**.** A graph G is **planar** if it has genus $g = 0$.

We remark that, for planar graphs, $\mathcal{F} = \mathcal{L}$.

Planar graph theory is well–studied. In the circuit literature, many formal results are limited to planar graphs in consideration because such circuits are both simpler to draw and analyze, let alone build in experiment. Note that according to Definition 55, a plane is not a suitable surface on which to embed a graph because the “external face" cannot be isomorphic to a unit disk. This problem is resolved by considering the embedding of a planar graph on a sphere or equivalently by identifying the points at infinity to be equivalent. Of course in practice, we will draw planar graphs in the plane, since the plane is equivalent to the sphere if we identify all points at spatial infinity with the same point. So henceforth, we will consider planar graphs to be embedded on the unit sphere.

Nonplanar graphs have genus $g > 0$, by definition. They do have much in common with planar graphs, once we find the right perspective. To talk about graph duality, it is necessary to specify surfaces on which we consider embedding the graph. Generally, for a graph of genus g, we elect to embed upon a "sphere with g handles", or equivalently a "torus with g holes".

Definition 70 (Chains)**.** Let X be a finite set. For every $x \in X$, define a real vector $|x\rangle$ and define

$$\mathcal{D}(X) = \text{span}\left(\{|x\rangle : x \in X\}\right). \tag{B.12}$$

We say that $\mathcal{D}(X)$ is the **set of chains** over X, and we say that $|x\rangle$ is a chain.

Definition 71 (Incidence matrix)**.** Let $G = (\mathcal{V}, \mathcal{E})$ be a directed graph. Define the linear map $A : \mathcal{D}(\mathcal{V}) \to \mathcal{D}(\mathcal{E})$ so that

$$\langle e|A|v\rangle = \begin{cases} 1 & e \text{ is incident upon } v \\ -1 & e \text{ leaves } v \\ 0 & \text{otherwise} \end{cases}. \tag{B.13}$$

We say that A is the **incidence matrix** of G and we often write $\langle e|A|v\rangle = A_{ev}$.

Definition 72 (Orientation matrix)**.** Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ be an extended embedded graph. Orient all faces of G alike.[1] Choose some orientation for the $2g$ loops in $\mathcal{L}$ that do not bound a face of G. Define the linear map $B : \mathcal{D}(\mathcal{E}) \to \mathcal{D}(\mathcal{L})$ so that

$$\langle l|B|e\rangle = \begin{cases} 1 & e \text{ borders } l \text{ and } e \text{ is oriented with } l \\ -1 & e \text{ borders } l \text{ and } e \text{ is oriented against } l \\ 0 & \text{otherwise} \end{cases}. \tag{B.14}$$

We say that B is the **orientation matrix** of G.

[1] Our convention is that loops bounding faces ought to be oriented such that the right hand rule points out of the surface.

An abstract graph G has a unique incidence matrix A, but not necessarily a unique orientation matrix B. On the other hand, an extended embedded graph on a surface S, G_S has both a unique A and a unique B.

Theorem 73. *Let G be an extended embedded graph with incidence matrix A and orientation matrix B. The rank of A is $|\mathcal{V}| - 1$ and the rank of B is $|\mathcal{L}| - 1$.*

Proof. Suppose

$$\sum_v A_{ev} c_v = 0. \tag{B.15}$$

with c_v in $\mathbb{R}^{|\mathcal{V}|} \setminus \{0\}$. Then, there exists at least v such that c_v is nonzero, and if A_{ev} is nonzero then there exists u so that $e = (u, v)$ or $e = (v, u)$. In either case, $A_{ev} = -A_{eu}$ and thus $c_u = c_v$. Continue in this way to discover that $c_v = 1$ for all $v \in \mathcal{V}$. The proof that the only left null vector of B is given by $c_f = 1$ for all $f \in \mathcal{F}$ is exactly analogous. We remark that $\mathcal{F} \neq \mathcal{L}$ in general; hence the left null vector of B corresponds to a sum over loops bounding faces only (see Theorem 63). □

Theorem 74. *If G be an extended embedded graph with incidence matrix A and orientation matrix B,*

$$\sum_{e \in \mathcal{E}} B_{le} A_{ev} = 0. \tag{B.16}$$

Proof. This fact is an elementary result in the theory of CW–complices.[2] Nonetheless, we provide an explicit demonstration. Choose a loop l and a vertex v. If v is not connected to any edges in l, the result is trivial so suppose that there exist edges e and e' in[3] l and vertices u_1 and u_2 so that one of the following possibilities holds:

(1) $e = (u_1, v)$ and $e' = (v, u_2)$

(2) $e = (v, u_1)$ and $e' = (v, u_2)$

(3) $e = (u_1, v)$ and $e' = (u_2, v)$

[2] Explicitly, this is a discrete version of the statement that exterior derivatives are nilpotent.

[3] Each vertex along a loop must be hit an even number of times, or the loop would not be closed. If a vertex is hit $2m$ times, then there will always exist m pairs of e and e' for which this argument holds.

(4) $e = (v, u_1)$ and $e' = (u_2, v)$.

Now, in cases 1. and 4., $B_{le} = B_{le'}$ and $A_{ev} = -A_{e'v}$. In cases 2. and 3., the opposite is true. In any case,

$$B_{le}A_{ev} + B_{le'}A_{e'v} = 0 \tag{B.17}$$

This concludes the proof. □

Theorem 75. *Let G be an extended embedded graph with incidence matrix A and orientation matrix B. Then,*

$$\mathrm{Ker}(B) = \mathrm{Im}(A). \tag{B.18}$$

Proof. From Theorem 74, it is clear that

$$\mathrm{Im}(A) \subseteq \mathrm{Ker}(B). \tag{B.19}$$

For the other direction, note that

$$\mathrm{Rank}(A) = |\mathcal{V}| - 1, \tag{B.20a}$$

$$\mathrm{Rank}(B) = |\mathcal{L}| - 1. \tag{B.20b}$$

Since extended embedded graphs satisfy

$$|\mathcal{V}| - |\mathcal{E}| + |\mathcal{L}| = 2, \tag{B.21}$$

we see that

$$\dim(\mathrm{Ker}(B)) = |\mathcal{E}| - \mathrm{Rank}(B) = |\mathcal{E}| - |\mathcal{L}| + 1 = |\mathcal{V}| - 1 = \mathrm{Rank}(A) = \dim(\mathrm{Im}(A)). \tag{B.22}$$

It follows from (B.18) and (B.22) that $\mathrm{Ker}(B)$ is spanned by elements of $\mathrm{Im}(A)$. □

Another way to state Theorem 75 is that for any vector γ in $\mathbb{R}^{|\mathcal{E}|}$ satisfying

$$\sum_e B_{le}\gamma_e = 0, \tag{B.23}$$

there exists some other vector δ in $\mathbb{R}^{|\mathcal{V}|}$ such that

$$\gamma_e = \sum_v A_{ev}\delta_v. \tag{B.24}$$

Likewise any left null vector γ of A is of the form $\gamma = \delta^T B$.

Informally, A maps vertices (or integer linear combinations of vertices) to edge–cuts of a graph G, and such edge–cuts are precisely the null vectors of B. More precisely, if $(\mathcal{V}_1, \mathcal{V}_2)$ is a cut of G, then $\langle e|A\sum_{v\in\mathcal{V}_1}|v\rangle$ is nonzero if and only if e is in the edge–cut induced by $(\mathcal{V}_1, \mathcal{V}_2)$. The sign may either be positive or negative and is determined by the relative orientation of edges leaving or entering $\mathcal{V}_1$. Likewise, the matrix B maps cycles of G to linear combinations of faces of G. As a remark, we note that, by cutting every edge in a graph, $\mathcal{V}$ is partitioned into singlet sets, and since the graphs of interest are also circuits, there is some sense in which the cycle consisting of all edges *is* indeed a loop. However, the vector $\sum_{e\in\mathcal{E}}|e\rangle$ is not in the range of A.

B.2 Formal approach to symmetric quantization

In this appendix, we discuss more formally the flux-charge symmetric theory of circuit quantization.

B.2.1 Properties of circuits and the connection matrix

In this section, we describe the full formalism for symmetric quantization on arbitrary graphs. In what follows, we consider a circuit on a planar graph with a fixed but arbitrary embedding. In analogy to graphs as combinatorial objects, we regard circuits as graphs with “colored" edges. That is to say that we demand that edges contain precisely a single circuit element and that circuit elements are either inductive or capacitive.

Definition 76 (Circuit)**.** Let $G = (\mathcal{V}, \mathcal{E}, \mathcal{L})$ be an extended embedded graph with incidence matrix A and orientation matrix B. Partition the set $\mathcal{E}$ into the sets $\mathcal{C}$ and $\mathcal{I}$ such that edges housing inductors (capacitors) go into the set $\mathcal{I}$ ($\mathcal{C}$). The tuple $(\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ is called a **circuit**.

Definition 77 (Connection matrix)**.** Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Define the matrix $M : \mathcal{D}(\mathcal{V}) \to \mathcal{D}(\mathcal{L})$ so that

$$M_{lv} := \langle l|M|v\rangle = \frac{1}{2}\sum_{e\in\mathcal{C}} B_{le}A_{ev} - \frac{1}{2}\sum_{e\in\mathcal{I}} B_{le}A_{ev}. \tag{B.25}$$

We say that M is the **connection matrix** of G.

Unlike A and B, M has no intuitive interpretation which can be easily read off of a circuit (at least that we have found). Still, in practice, it is straightforward to simply calculate it. Since $\mathcal{C} \cup \mathcal{I} = \mathcal{E}$, it follows that

$$M = \sum_{e\in\mathcal{C}} B_{le}A_{ev} = -\sum_{e\in\mathcal{I}} B_{le}A_{ev}. \tag{B.26}$$

It will often useful to rewrite M using (B.26). Furthermore, for planar circuits,

$$\sum_{l\in\mathcal{L}} M_{lv} = \sum_{v\in\mathcal{V}} M_{lv} = 0 \tag{B.27}$$

since $\sum_v A_{ev} = 0$ and $\sum_l B_{le} = 0$. For nonplanar circuits, the set of faces $\mathcal{F} \subset \mathcal{L}$ has the property that

$$\sum_{l\in\mathcal{F}} B_{le} = 0. \tag{B.28}$$

Our first goal is to prove the following result:

Corollary 78. *Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. We say that a loop is homogenous if the edges are all capacitors or all inductors. Similarly a cut is called homogenous if its induced edge–cut consists only of inductors or only of capacitors. The following conditions hold.*

- *$M|\varphi\rangle = 0$ if and only if $(\mathcal{V}_1, \mathcal{V}_2)$ is a homogeneous cut of G and $|\varphi\rangle = \sum_{v\in\mathcal{V}_1} |v\rangle$*

- $\langle\psi|M = 0$ *if and only if* γ *is a homogeneous loop and* $\langle\psi| = \sum_{e\in\gamma}\langle e|$.

This result is the corollary of a more abstract mathematical result:

Theorem 79. *Let* V_1, V_2, *and* V_3 *be vector spaces and let* $A : V_1 \to V_2$ *and* $B : V_2 \to V_3$ *be linear maps satisfying*[4]

$$\ker(B) = \mathrm{im}(A) \tag{B.29}$$

The matrix $M = BPA : V_1 \to V_3$ *has the following properties:*

(1) $M|\varphi\rangle = 0$ *if and only if there exist vectors* $|+\rangle$ *and* $|-\rangle$ *such that* $|\varphi\rangle = |+\rangle + |-\rangle$ *and* $PA|\pm\rangle = \pm A|\pm\rangle$*; moreover,* $|\pm\rangle$ *are separately also right null vectors of* M.

(2) $\langle\psi|M = 0$ *if and only if there exist vectors* $|+\rangle$ *and* $|-\rangle$ *such that* $|\psi\rangle = |+\rangle + |-\rangle$ *and* $\langle\pm|BP = \pm\langle\pm|B$*; moreover,* $\langle\pm|$ *are separately also left null vectors of* M.

Proof. We prove point 1 of the list, as point 2 is proven analogously. Suppose

$$M|\varphi\rangle = BPA|\varphi\rangle = 0. \tag{B.30}$$

Define the projectors

$$K_\pm = \frac{1}{2}(\mathbb{I} \pm P) \tag{B.31}$$

which have the property that $PK_\pm = \pm K_\pm$, $K_\pm^2 = K_\pm$, and $K_+ + K_- = \mathbb{I}$. It follows that

$$0 = BPA|\varphi\rangle = BP(K_+ + K_-)A|\varphi\rangle = B(K_+ - K_-)A|\varphi\rangle. \tag{B.32}$$

Of course, we also know that

$$0 = BA|\varphi\rangle = B(K_+ + K_-)A|\varphi\rangle, \tag{B.33}$$

[4] In other words, $B|\varphi\rangle = 0$ for $|\varphi\rangle \in V_2$ if and only if for some $|\varphi'\rangle \in V_1$, $|\varphi\rangle = A|\varphi'\rangle$.

meaning that

$$BK_+A|\varphi\rangle = BK_-A|\varphi\rangle = 0. \tag{B.34}$$

By (B.29) we conclude that there exist $|\pm\rangle$ for which

$$A|\pm\rangle = K_\pm A|\varphi\rangle. \tag{B.35}$$

Evidently,

$$A(|+\rangle + |-\rangle) = (K_+ + K_-)A|\varphi\rangle = A|\varphi\rangle. \tag{B.36}$$

Since any right null vector of A, $|n\rangle$, satisfies

$$PA|n\rangle = \pm A|n\rangle = 0, \tag{B.37}$$

we see that $|+\rangle$ and $|-\rangle$ are only determined up to the addition of right null vectors of A, should any exist. Therefore, $A(|+\rangle + |-\rangle) = (K_+ + K_-)A|\varphi\rangle = A|\varphi\rangle$, which implies that $|\varphi\rangle = |+\rangle + |-\rangle$ (for appropriate definitions of $|+\rangle$ and $|-\rangle$ corresponding to the freedom to add right null vectors of A to either). Left multiply (B.35) by P to conclude that $PA|\pm\rangle = \pm A|\pm\rangle$. Since $PA|\pm\rangle$ is proportional to $A|\pm\rangle$ and $BA = 0$, it follows that $M|\pm\rangle = 0$, which proves the desired statements.

□

A straightforward, but useful, consequence of this result is:

Corollary 80. *Adopt the definitions made in the proof of Theorem 79. Consider the matrix*

$$W = BK_+. \tag{B.38}$$

If $|\varphi\rangle$ *satisfies*

$$W|\varphi\rangle = 0, \tag{B.39}$$

then there exist vectors $|\theta\rangle$ *and* $|\psi\rangle$ *such that*

$$|\varphi\rangle = A|\theta\rangle + K_-|\psi\rangle \tag{B.40}$$

with $K_+A|\theta\rangle = A|\theta\rangle$.

As a passing remark, the hypothesis of Theorem 79 is satisfied by the vector spaces and boundary maps in any short exact sequence of vector spaces together with some partition of the intermediate vector space. For our purposes, Theorem 79 serves to enumerate all of the null vectors of M. To see how Theorem 79 applies to M, define $P : \mathcal{D}(\mathcal{E}) \to \mathcal{D}(\mathcal{E})$ such that

$$P_{ee'} = (-2\mathbb{I}[e \in \mathcal{I}] + 1)\delta_{ee'} \tag{B.41}$$

where $\mathbb{I}$ is an indicator function that vanishes if its argument is untrue and is otherwise equal to one. Then,

$$M = \frac{1}{2}BPA. \tag{B.42}$$

To make more clear our enumeration of null vectors of M, we make the following definitions:

Definition 81 (Homogeneous cut)**.** Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Let $C = (\mathcal{V}_1, \mathcal{V}_2)$ be a cut of G. C is a homogeneous cut if the set

$$\mathfrak{C} = \{e \subset \mathcal{E} \ : \ \exists v_1 \in \mathcal{V}_1 \text{ and } v_2 \in \mathcal{V}_2 \text{ s.t. } e \in \{(v_1, v_2), (v_2, v_1)\}\} \tag{B.43}$$

has the property that

$$\mathfrak{C} \subset \mathcal{C} \tag{B.44}$$

or

$$\mathfrak{C} \subset \mathcal{I}. \tag{B.45}$$

In the former case we say that $\mathfrak{C}$ is capacitive and in the latter case we say that $\mathfrak{C}$ is inductive.

Definition 82 (Homogeneous loop)**.** Let γ be a loop in the sense of Definition 57. If $\gamma \subset \mathcal{C}$ or $\gamma \subset \mathcal{I}$, we say that γ is a **homogeneous loop**.

Corollary 83. *Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Let Δ_I (Δ_C) be the set of homogeneous inductive (capacitive) loops of G, and let Γ_I (Γ_C) be the set of homogenous inductive (capacitive) loops in G. Then,*

$$|\mathcal{F}| - |\Delta_I| - |\Delta_C| = |\mathcal{V}| - |\Gamma_I| - |\Gamma_C| \tag{B.46}$$

Proof. This is an immediate consequence of Thm. 79 together with the Rank–Nullity theorem. □

Corollary 83 was a result nearly achieved in [90], but the relevant discussion relied upon the enumeration of null vectors *and* a number of Noether currents. One of the merits of this approach is that Noether currents in the formalism of [90] are promoted to null vectors.

Thus, the number of degrees of freedom in a circuit is equal to the number of loops in a circuit which are neither purely inductive nor purely capacitive.

B.2.2 Formal circuit Lagrangian

Here, we restrict our attention to planar graphs. These results can be made to hold for nonplanar graphs with some minor modifications, which will be made explicit in appendix B.3.

Definition 84 (Symmetric circuit Lagrangian)**.** Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Let M be the connection matrix of M and define $|\mathcal{C} \cup \mathcal{I}|$ functions labeled E_e which describe the energy of the circuit element on branch e. The function

$$L = \sum_{l \in \mathcal{L}, v \in \mathcal{V}} q_l M_{fv} \dot{\phi}_v - \sum_{e \in \mathcal{C}} E_e \left(\sum_l q_l B_{le} \right) - \sum_{e \in \mathcal{I}} E_e \left(\sum_v A_{ev} \phi_v \right) \tag{B.47}$$

is called the **symmetric circuit Lagrangian** for G, or "the Lagrangian for G" for short.

We see that, from (B.47), $S = \int \mathrm{d}t L$ is symmetric in $\mathcal{C}$ and $\mathcal{V}$. As we will discuss later, this choice of variables leads to a very straightforward circuit duality transformation.

For the following result, we will need to rely upon the results of Section 3.3.1 as well as a number of definitions originally made in [90].

Theorem 85. *Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Let L be the symmetric circuit Lagrangian of G. Define Γ_C (Γ_I) to be the set of capacitive (inductive) cuts of G, and let Δ_C (Δ_I) to be the set of capacitive (inductive) cycles of G. It is always possible to define $|\mathcal{V}| - |\Gamma_I| - |\Gamma_C| - 1$ variables $Q_i = \sum_l D_{il} q_l$ and $\Phi_i = \sum_v S_{iv} \phi_v$ so that*

$$\sum_{l,v} q_l M_{lv} \dot{\phi}_v = \sum_{i=1}^{|\mathcal{V}|-|\Gamma_I|-|\Gamma_C|-1} Q_i \dot{\Phi}_i. \tag{B.48}$$

All possible choices of S and D are related by canonical transformations.

Proof. Define $q_e = \sum_l q_l B_{le}$ for $e \in \mathcal{C}$, and define $\Omega_{ev} = A_{ev}$ for $e \in \mathcal{C}$ and then apply Theorems 10 and 13 from [] directly. □

B.3 Circuit duality

The contents of this appendix depend broadly on the results of Appendix B.2 and serve to formalize the claims in Section 3.4. Duality as a map can be sensibly defined on Lagrangians, graphs, and structures from the theory of topological algebra. While we take a minimal perspective here, the results of this section are very simply expressed as a property of chain–complex isomorphisms.

Definition 86 (Hamiltonian duality transformation)**.** Suppose H is a Hamiltonian function of variables Φ_i and Q_i for $i = 1, 2, \ldots, N$, equipped with Poisson brackets

$$\{\Phi_i, Q_j\} = \delta_{ij}. \tag{B.49}$$

The transformation

$$\begin{aligned} Q_i &\to Q_i' = -\Phi_i \\ \Phi_i &\to \Phi_i' = Q_i \end{aligned} \tag{B.50}$$

is called a **Hamiltonian duality transformation**. We will write $H(Q', \Phi') = H^*$.

Clearly, Hamiltonian duality transformations are canonical since $\{Q_i', \Phi_j'\} = \{Q_i, \Phi_j\}$. Certainly, at the level of Hamiltonian mechanics, it is straightforward to take the dual of any Hamiltonian arising from a circuit Lagrangian in the spirit of the formalism of this work. However, the challenge of constructing the circuit (or circuits) that produce H^* is the subject of this appendix. Moreover, it is not always possible to produce a (physically sensible) circuit that accomplishes this task – at least using any known algorithm for constructing a dual circuit.

Definition 87 (Dual Circuit)**.** Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Define

$$\begin{aligned} \mathcal{V}^* &= \mathcal{L} \\ \mathcal{I}^* &= \mathcal{C} \\ \mathcal{C}^* &= \mathcal{I} \\ \mathcal{L}^* &= \mathcal{V} \\ A^* &= B^{\mathrm{T}} \\ B^* &= A^{\mathrm{T}} \end{aligned} \tag{B.51}$$

and finally

$$G^* = (\mathcal{V}^*, \mathcal{C}^*, \mathcal{I}^*, \mathcal{L}^*, A^*, B^*). \tag{B.52}$$

We say that G^* is the **dual circuit** of G. For an element v of $\mathcal{V}$, we write the corresponding element of $\mathcal{L}^*$ as v^*. For an element l of $\mathcal{L}$, we write the corresponding element of $\mathcal{V}^*$ as l^*.

Our reason for using this terminology will become clear shortly. Both a combinatorial object and a topological object are encoded in G^*. That is to say that the combinatorial properties of G^* are encoded in the structure of A^*. While it is straightforward to recover the combinatorial structure of G^* by looking at the matrix A^*, it is less obvious how one might recover a particular embedding of G^* by using B^*. Though we will not belabour this point presently, we remark that a particular embedding of G^* is recoverable by a gluing procedure where every element of $\mathcal{F}$ is represented by a patch isomorphic to the unit disk, and then patches are glued together by identifying segments on

the boundary of different patches, in a way that is consistent with the content of A^*. This is always possible. For planar graphs, this procedure always accomplished by the following procedure:

Definition 88 (Embedding of dual circuit)**.** Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Further suppose G is embedded upon a sphere. We construct the **embedded dual circuit** of G, G^* as follows:

(1) For every loop l in $\mathcal{L}$, draw a vertex labeled l^* (inside of the face whose boundary is l).

(2) For every pair of loops l_0 and l_1 in $\mathcal{L}$, draw an inductive (capacitive) edge between l_0^* and l_1^* for every capacitive (inductive) edge in both l_0 and l_1. For such an edge e of G, label the corresponding edge in G^* as e^*.

(3) For every edge e^* in G^*, give e^* an orientation so that when the surface is drawn (locally) on the plane, the cross product between e and e^* is always positive.

Definitions 88 and 87 are equivalent for planar graphs. We emphasize that the construction of a dual graph is intrinsically dependent upon the embedding of G chosen. We will make this point explicit with the next observation.

Observation 89. *Let G be a planar graph. Choose two embeddings of G on S, G_S and G'_S. G^*_S and $(G_{S'})^*$ need not be graph isomorphic.*

Proof. We provide a proof by example in Figure 3.4(a)–(b). In order to view a circuit as a graph, one only needs to ignore all of the circuit elements in the circuit so that every branch becomes simply a graph theoretic edge. □

Observation 90. *Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Let $G^* = (\mathcal{V}^*, \mathcal{C}^*, \mathcal{I}^*, \mathcal{L}^*, A^*, B^*)$ be the dual circuit of G. Define the matrix $M^* : \mathcal{V}^* \to \mathcal{L}^*$ with matrix elements*

$$M^*_{v^*,l^*} = \frac{1}{2} \sum_{e^* \in \mathcal{C}^*} B^*_{v^* e^*} A^*_{e^* l^*} - \frac{1}{2} \sum_{e^* \in \mathcal{I}^*} B^*_{v^* e^*} A^*_{e^* l^*}. \tag{B.53}$$

Then, $M^* = -M^{\mathrm{T}}$.

Proof. By relabeling,

$$\langle v^*|M^*|l^*\rangle = \frac{1}{2}\sum_{e^*\in\mathcal{C}^*} B^*_{v^*e^*}A^*_{e^*l^*} - \frac{1}{2}B^*_{v^*e^*}A^*_{e^*l^*} = \frac{1}{2}\sum_{e\in\mathcal{I}} A_{ve}B_{el} - \frac{1}{2}\sum_{e\in\mathcal{C}} A_{ve}B_{el} = -\langle v|M^{\mathrm{T}}|l\rangle. \quad \text{(B.54)}$$

Since $\langle v^*|M^*|l^*\rangle = \langle v|M^{\mathrm{T}}|l\rangle$, we say simply $-M^{\mathrm{T}} = M^*$. □

Theorem 91. *Let $G = (\mathcal{V}, \mathcal{C}, \mathcal{I}, \mathcal{L}, A, B)$ be a circuit. Let G^* be the dual circuit of G. For each edge e in $\mathcal{I}\cup\mathcal{C}$, suppose that the energy associated with edge e is a function $E_e : \mathbb{R} \to \mathbb{R}$. For each edge e^* in $\mathcal{I}^*\cup\mathcal{C}^*$, fix $E_{e^*} : \mathbb{R} \to \mathbb{R}$ so that*

$$E_{e^*}(x) = E_e(x). \quad \text{(B.55)}$$

The Lagrangian for G is related to the Lagrangian for G^* are related by a relabeling transformation.

Proof. The Lagrangian for G is given by

$$L = \sum_{l\in\mathcal{L}}\sum_{v\in\mathcal{V}} Q_l M_{lv}\dot{\phi}_v - \sum_{e\in\mathcal{I}} E_e\left(\sum_v A_{ev}\phi_v\right) - \sum_{e\in\mathcal{C}} E_e\left(\sum_l q_l B_{le}\right). \quad \text{(B.56)}$$

On the other hand, the Lagrangian for G^* is given by

$$L^* = \sum_{v^*\in\mathcal{L}^*}\sum_{l^*\in\mathcal{V}^*} q_{v^*}M^*_{v^*l^*}\dot{\phi}_{l^*} - \sum_{e^*\in\mathcal{I}^*} E_e\left(\sum_{l^*\in\mathcal{V}^*} A^*_{e^*l^*}\phi_{l^*}\right) - \sum_{e^*\in\mathcal{V}^*} E_e\left(\sum_{v^*\in\mathcal{L}^*} K_{v^*}B^*_{v^*e^*}\right). \quad \text{(B.57)}$$

The transformation

$$\begin{gathered} q_{v^*} \to \phi_v \\ \phi_{l^*} \to q_l \\ M^*_{v^*l^*} \to -M_{lv} \\ \mathcal{C}^* \to \mathcal{I} \\ \mathcal{I}^* \to \mathcal{C} \end{gathered} \quad \text{(B.58)}$$

can easily be seen to relate L^* to L after integrating the first term by parts. □

Corollary 92. *Let G be an embedded circuit with Lagrangian L and G^* be the dual circuit of G.*

By theorem 91 and 85, it is always possible to find $N = |\mathcal{V}| - |\Gamma_I| - |\Gamma_C| - 1$ *variables so that the Lagrangian for* G *may be written*

$$L = \sum_{i=1}^{N} Q_i \dot{\Phi}_i - H(Q, \Phi). \tag{B.59}$$

It is always possible to write the Lagrangian for G^* *as*

$$L^* = -\sum_{i=1}^{N} \Phi_i \dot{Q}_i - H(-\Phi, Q). \tag{B.60}$$

Proof. Since

$$\sum_{i=1}^{N} Q_i \dot{\Phi}_i = \sum_{v,l} q_l M_{lv} \dot{\phi}_i, \tag{B.61}$$

it follows that, under the transformation (B.58),

$$Q_i \dot{\Phi}_i \to -\Phi_i \dot{Q}_i. \tag{B.62}$$

□

We remark that, for nonplanar graphs, the Lagrangian transformations given in Cor. 92 is perfectly well–defined. It is also true that the transformation (B.58) is *also* perfectly well–behaved. The issue is simply that, for nonplanar graphs, there is no systematic analogue of Defninition 88 that applies to nonplanar graphs in any meaningful sense. The reason for this is that edges in so–called topological loops would, in the dual of a nonplanar graph, can have an odd number of endpoints, which contradicts our definition of what an edge is. Nonetheless, it is sometimes possible to "planarize" a nonplanar circuit and then take the dual after planarization, as we saw for K_5 in the main text.