University of Colorado

Honors Thesis

The Combinatorial Geometry of Rook Polytopes

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Contents

1 Introduction

The question of how many ways can rooks be placed onto a chessboard such that none may take another was first posed by Dudeney in [5], and has been studied since. The problem is closely related to the transportation problem, which is also well studied, see [3, 4]. The transportation problem asks a logistics problem: Suppose we have factories A, B, C and ports F, G, and H. Factory A produces 120 cartons of goods per week, Factory B produces 100 cartons per week, and C produces 50 per week, while port F can ship 30 cartons per week, port G ships 80 cartons per week and port H ships 160 cartons per week. We could construct these matrices to illustrate possible ways of transporting our goods to our ports

$$\begin{bmatrix} 30 & 80 & 10 \\ 0 & 0 & 100 \\ 0 & 0 & 50 \end{bmatrix}, \quad \begin{bmatrix} 10 & 40 & 50 \\ 10 & 30 & 80 \\ 10 & 10 & 30 \end{bmatrix}.$$

We, however, are primarily interested in the geometry of the polytopes created from these matrices. A polytope is the logical extension of the idea of a polygon or polyhedron. Where a polygon is 2-dimensional, and a polyhedron 3-dimensional, a polytope is n-dimensional. So we take the collection of all of the possible $m \times n$ matrices generated in this way and assign each entry a coordinate in $m \times n$-dimensional space. The possible fillings of these transportation matrices over $\mathbb{R}_{\geq 0}$ form a polytope.

In a particular special case of the transportation polytope, we have what is known as the Birkhoff polytope, which is denoted as $\mathcal{B}_n$. This is the case where we have a $n \times n$ transportation polytope and each column or row bound sums to 1. Note that in general, the entry in the ith row and jth column of a matrix will be referred to as x_{ij}. So we express the bounds as

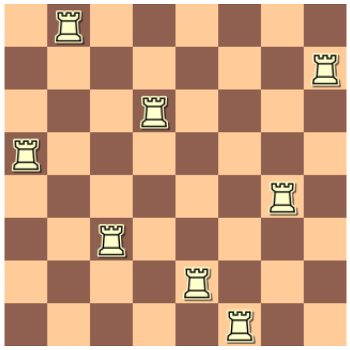
$$1 = \sum_{k=1}^{n} x_{ik} \quad \text{and} \quad 1 = \sum_{k=1}^{n} x_{kj}.$$

The Birkhoff polytope is sometimes referred to as the collection of all $n \times n$ doubly stochastic matrices, or the matching polytope. For more on Birkhoff polytopes see [1, 2, 7, 9].

The rook polyopes, which we call $\mathcal{R}_{m\times n}$, that we are primarily concerned with in this paper are closely related to the Birkhoff polytopes. Instead of forcing an equality on the row bounds and column bounds, we use an inequality, so that

$$1 \geq \sum_{k=1}^{n} x_{ik} \quad \text{and} \quad 1 \geq \sum_{k=1}^{n} x_{kj}.$$

The Rook polytope itself has some interesting combinatorial properties. It gets its name from the Rook monoid or the Rook polynomial, and each vertex of the Rook polytope corresponds to a way to place rooks onto an $m \times n$ chessboard such that no rook can attack another rook.



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

A placement of rooks and its corresponding matrix.

Clearly the $n \times n$ Birkhoff polytope $\mathcal{B}_n$ is contained in the Rook polytope of the same size, $\mathcal{R}_{n\times n}$. The Birkhoff polytope is generally well studied, but while the vertices, edges and facets are well categorized, it is difficult to say anything in general about any other type of face. As such, the intention of this paper is to build a framework for enumerating the faces of the Rook polytope. We introduce some background information on polytopes in general, then we introduce two algorithms for generating facets from a subset of vertices. We also discuss ways to examine the geometric relationship between a pair of vertices in $\mathcal{R}$, and prove a method of finding and enumerating smallest face containing a specific pair of vertices.

I would like to thank the 2016 CU Boulder Summer REU in Mathematics for the opportunity and funding to begin research for this thesis.

2 Background

To study the rook polytope, we need to first begin with some background knowledge about polytopes in general. First, we will start with some definitions on the representation of polytopes, alongside some specific cases of the rook polytope. Next, we will discuss a more combinatorial way of representing polytopes, the face lattice.

2.1 Polytopes

Earlier we defined a polytope somewhat naively, here we seek to give a more precise notion. Since we are interested only in convex polytopes, we should first consider convex sets. The rook polytope $\mathcal{R}$ is convex, so we will frequently refer to a convex polytope as a polytope. For more discussion on polytopes and their representation in general, see [10].

Definition 2.1.1. A **convex set** is a set, C, in a vector space such that for all $a, b \in C$ and all $x \in [0, 1]$, the point $(1 - x)a + xb$ is also in C.

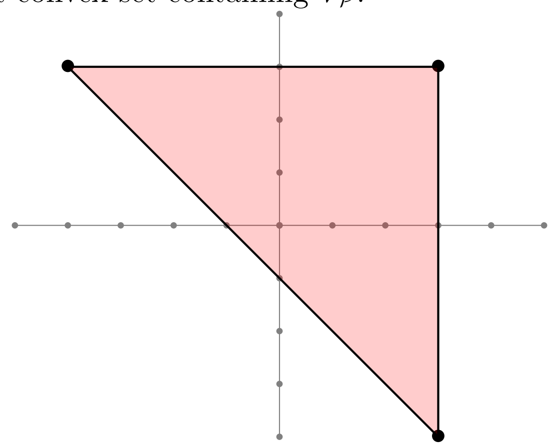
There are two primary ways to represent a polytope, either by vertices or half-spaces. First, let us consider the vertex representation of a polytope:

Definition 2.1.2. The **vertex representation** of a polytope, $\mathcal{P}$, is the set $V_\mathcal{P}$ of the vertices $\mathcal{P}$ where the minimal convex set containing $V_\mathcal{P}$ is $\mathcal{P}$.

Example 2.1.3. Suppose we had an 2 dimensional polytope with

$$V_{\mathcal{P}} = \{(-4,3),(3,3),(3,-4)\}.$$

We then find the smallest convex set containing $V_{\mathcal{P}}$:



$\mathcal{P}$ given $V_{\mathcal{P}} = \{(-4,3),(3,3),(3,-4)\}$.

This means that one way of uniquely identifying polytopes is by identifying every one of the vertices. In the case of the rook polytope, enumerating the vertices, or counting the ways to place non-attacking rooks, is a solved problem. The vertices of $\mathcal{R}_{m\times n}$ are given by the elements of the corresponding rook matrix, and counted using the rook polynomial from Riordan, [8], as

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{k}k!.$$

Alternately, we can also consider a polytope to be identified by its faces with maximal dimensions. In order to do that, we define a half-space.

Definition 2.1.4. A **half-space** is a part of an affine space (that does not necessarily pass through the origin) that lies entirely on one side of an affine hyperplane. Note that an affine hyperplane h can be written as $h = \{x \in \mathbb{R}^n : \vec{n}\cdot\vec{x} = a\}$, while a half-space s can be written as $s = \{x \in \mathbb{R}^n : \vec{n}\cdot\vec{x} \le a\}$

Using half-spaces, we can consider this representation of a polytope:

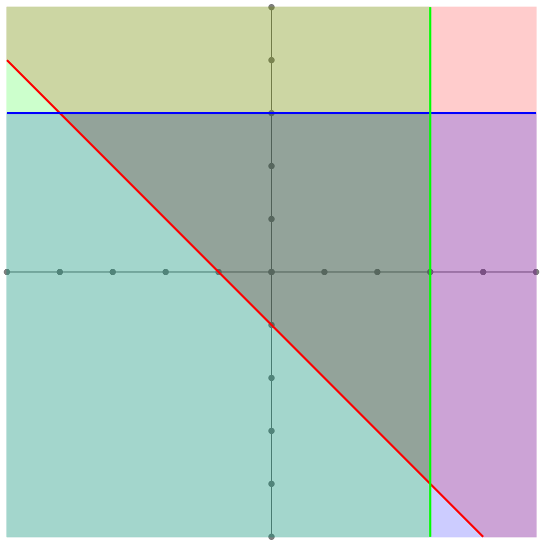
Definition 2.1.5. The **half-space representation** of a d-dimensional polytope, $\mathcal{P}$, is the set of half-spaces $S_{\mathcal{P}}$ such the intersection of all half spaces $s \in S$ is $\mathcal{P}$. Alternately

$$\left\{x \in \mathbb{R}^d : x \in \bigcap_{i=1}^{k} s_i\right\}$$

Example 2.1.6. Suppose we had a 2 dimensional polytope with

$$S_{\mathcal{P}} = \{\{x \le 3\},\{y \ge 3\},\{-x-y \le 1\ \}\}.$$

We then find the intersection of the half-spaces:



$\mathcal{P}$ given $S_{\mathcal{P}} = \{\{x \leq 3\}, \{y \geq 3\}, \{-x - y \leq 1\}\}$.

The half-spaces that represent $\mathcal{P}$ also correspond with a set of **bounding hyperplanes** H_b, where $h \in H_b$ is the boundary of some half-space in the half-space representations of $\mathcal{P}$.

We refer to a specific coordinate in the ith row and jth column of a point in $\mathcal{R}_{m\times n}$ as x_{ij}. So the half-space representation of the $m \times n$ rook polytope are the inequalities $x_{ij} \leq 1$, $x_{ij} \geq 0$, $x_{1j} + x_{2j} + \ldots + x_{mj} \leq 1$, and $x_{i1} + x_{i2} + \ldots + x_{in} \leq 1$ for all $1 \leq i \leq m$ and $1 \leq j \leq m$. Note that we can express $x_{ij} \leq 1$ as $-x_{ij} \geq -1$ and $x_{i1} + x_{i2} + \ldots + x_{in} \leq 1$ as $-x_{i1} - x_{i2} - \ldots - x_{in} \geq -1$, so in general we refer to our inequalities as $\vec{n}x \geq a_0$ where $\vec{n}$ is an $m \times n$ vector normal to the bounding hyperplane of the inequality, pointing into the polytope.

We note four primary sorts of bounding hyperplanes in H_b of $\mathcal{R}_{m\times n}$:

- A bounding hyperplane is **minimal** if, for some fixed i, j, it is written:

$$h = \{v \in \mathbb{R}^{m\times n} : x_{ij} = 0\}.$$

- A bounding hyperplane is **maximal** if, for some fixed i, j, it is written:

$$h = \{v \in \mathbb{R}^{m\times n} : x_{ij} = 1\}.$$

- A bounding hyperplane is **row cutting** if, for some fixed i, it is written:

$$h = \left\{x \in \mathbb{R}^{m\times n} : \sum_{\ell=1}^{n} x_{i\ell} = 1\right\}.$$

- A bounding hyperplane is **column cutting** if, for some fixed j, it is written:

$$h = \left\{v \in \mathbb{R}^{m\times n} : \sum_{\ell=1}^{m} x_{\ell j} = 1\right\}.$$

In the same vein as the half-spaces, we can generally refer to our bounding hyperplanes as $h = \{x \in \mathbb{R}^{m \times n} : \vec{n} \cdot \vec{x} = a\}$. These four types of hyperplane are disjoint if $m, n \geq 2$. All bounding hyperplanes of the rook polytope are one of these.

Definition 2.1.7. A **face** of $\mathcal{P}$ is the intersection of some hyperplane H with $\mathcal{P}$ such that for all $\vec{v} \in V_{\mathcal{P}}$ and all $\vec{h} \in H$, $\vec{v} \geq \vec{h}$, where if $\vec{v}, \vec{u} \in \mathbb{R}^k$, $\vec{v} = (v_1, \ldots, v_k)$, $\vec{u} = (u_1, \ldots, u_k)$ $\vec{v} \geq \vec{u}$ if for all $1 \leq i \leq k$ $v_i \leq u_i$.

A **vertex** face is a 0-dimensional, an **edge** an 1-dimensional face, and a **facet** is a $n-1$-dimensional face. If two k-dimensional faces lie on a common $k+1$-dimensional face, we say they are **adjacent**.

2.2 Face Lattices

The face lattice is a tool that is frequently used in the study of polytopes, and properties of the face lattice provide structure for further observations. In general, for further discussion of the face lattice, you should see [10]. On our end, we should first consider the idea of a poset.

Definition 2.2.1. A **poset**, P, is a relation on a set S such that P is reflexive $(((a, a) \in P\ \forall\ a \in S)$, transitive $((a, b), (b, c) \in P \Rightarrow (a, c) \in P)$, and antisymmetric $((a, b), (b, a) \in P \Rightarrow a = b)$.

A poset is an ideal place to start when categorizing a polytope because there is a natural order of containment between faces– for example an edge contains two adjacent vertices. We can also trace containment between more than one dimension using a chain.

Definition 2.2.2. A **chain**, C, in a poset P is a a sequence $(a_1, a_2, \ldots, a_\ell)$ such that for all $1 \leq i \leq \ell$, $a_i \in S$ and for all $1 \leq k < \ell$, $(a_k, a_{k+1}) \in P$ with length $|C| - 1$.

Consider, for example, a vertex contained in an edge which is contained in a 2-face. This is a chain contained in the poset of a polytope with length $3 - 1 = 2$, having two steps between the 2-face and the 0-face. Next, we want to enforce more structure on our poset.

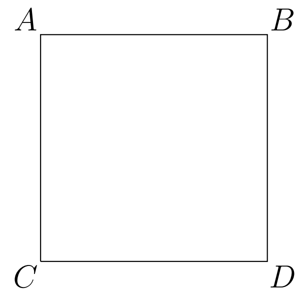
Definition 2.2.3. A **graded lattice** is a poset with unique minimal and maximal elements, where every two elements $a, b \in P$ have a unique upper and lower bound in P, and every maximal chain has the same length. The unique maximal element is $\mathcal{P}$, and the unique minimal element $\emptyset$. The unique upper bound of $a, b \in P$ is called the **join** of a, b or $a \vee b$, and the unique lower bound of $a, b \in P$ is the **meet** of a, b or $a \wedge b$.

So now we can define the face lattice:

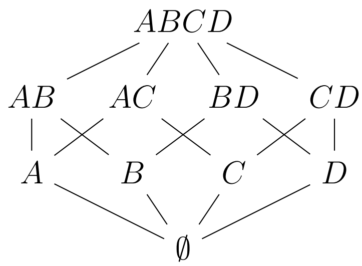
Definition 2.2.4. The **face lattice**, $L(\mathcal{P})$, of a convex polytope $\mathcal{P}$ is the poset on the set of faces of P, partially ordered by inclusion.

We can clearly see that the poset on a polytope is bounded. We also claim that the poset on the polytope is graded and a lattice.

Example 2.2.5. For example, consider the square $ABCD$



We can consider a poset ordered by containment of faces and we get the following face lattice.



Using this definition, we can start to analyze links between the face lattice and our polytopes.

Definition 2.2.6. A graded lattice has a **rank** function where for $a \in S$, $r(a)$ is the length of the maximal chain from $\bar{0}$ to a. We refer to $r(P)$ as the length of the poset.

The rank of a node in the face lattice can be used to find the dimension of the corresponding face of our polytope. In fact, for a node $f \in L(\mathcal{P})$ corresponding to a face $F \in \mathcal{P}$, $r(f) = \dim(F) + 1$.

Since we are concerned with the vertices and facets of polytopes as the most obvious ways to categorize them, there should be some clearly equivalent object in the face lattice.

Definition 2.2.7. If P is a graded lattice, we call the minimal elements of $P \backslash \bar{0}$ as **atoms**, and the maximal elements of $P \backslash \bar{1}$ as the **coatoms**.

We can see a direct bijection between the vertices of $\mathcal{P}$ and the atoms of $L(\mathcal{P})$ and a similar bijections between the facets of $\mathcal{P}$ and the coatoms of $L(\mathcal{P})$.

Definition 2.2.8. A lattice is **atomic** if every element $a \in P$ is a join $a = x_1 \vee x_2 \vee \ldots \vee x_k$ of $k \geq 0$ atoms. Similarly, a lattice is **coatomic** if every element $a \in P$ is a meet $a = y_1 \wedge y_2 \wedge \ldots \wedge y_k$ of $k \geq 0$ coatoms.

The face lattice is atomic/coatomic if and only if the vertex/half-space representation of a polytope is unique. The atomic/coatomic property also proves that every face $F \in \mathcal{P}$ is itself a polytope.

The following theorem from Ziegler resolves many of our assumptions from earlier:

Theorem 2.2.9 (Ziegler, [10])**.** *Let $\mathcal{P}$ be a convex polytope.*

1. *For every polytope $\mathcal{P}$, the face poset $L(\mathcal{P})$ is a graded lattice of length* $\dim(\mathcal{P})+1$ *with rank function* $r(f)=\dim(f)+1$.

2. *Every interval $[g,g]$ of $L(\mathcal{P})$ is the face lattice of a convex polytope of dimension $r(f)-r(g)-1$.*

3. *("Diamond property") Every interval of length 2 has exactly four elements. That is, if $g \subseteq f$ with $r(f)-r(g)=2$, then there are exactly two faces h with $g \subset h \subset f$.*

4. *The opposite poset $L(\mathcal{P})^{op}$ is also the face poset of a convex polytope.*

5. *The face lattice $L(\mathcal{P})$ is both atomic and coatomic.*

3 Faces of the Rook Polytope

So now that we have a few different ways of examining polytopes, both geometrically and combinatorially, we want to start categorizing faces of the rook polytope. First, we will introduce an algorithm for finding the set of bounding hyperplanes $H_{\{u,v\}}$ from a pair of vertices such that the intersection of the hyperplanes contains the vertices, then we will show that the intersection $H_{\{u,v\}} \cap \mathcal{R}$ is itself a face of $\mathcal{R}$.

3.1 Tic-Tac-Toe to identify bounding hyperplanes

Since the vertex representation of $\mathcal{R}$ is solved, its natural to try to find the relation of any two vertices within $\mathcal{R}$. We begin by building an algorithm that takes some set of vertices V_f and find the facets that each contain all $v \in V_f$.

Algorithm 3.1.1. Given some set of vertices $V_f \subseteq V_{\mathcal{R}}$ we define a set of hyperplanes $H_{\{u,v\}}$ as follows:

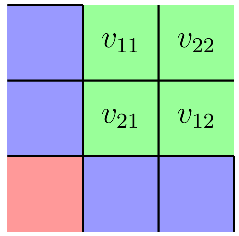
- If no $v \in V_f$ has $v_{ij}=1$, we take the minimal hyperplane $h=\{v \in \mathbb{R}^{m\times n} : x_{ij}=0\}$.
- If $v_{ij}=1$ for all $v \in V_f$, we take the maximal hyperplane $h=\{v \in \mathbb{R}^{m\times n} : x_{ij}=1\}$.
- If all $v \in V_f$ have a $v_{kj}=1$ for some $1 \le k \le i$ then we take the cutting hyperplane

$$h=\Big\{x \in \mathbb{R}^{m\times n} : \sum_{j=1}^{n} x_{ij}=1\Big\}.$$

- If all $v \in V_f$ have a $v_{i\ell}=1$ for some $1 \le \ell \le j$ then we take the cutting hyperplane

$$h=\Big\{v \in \mathbb{R}^{n\times m} : \sum_{i=1}^{m} x_{ij}=1\Big\}.$$

We want to construct a special type of array to keep track of what hyperplanes we want to have.



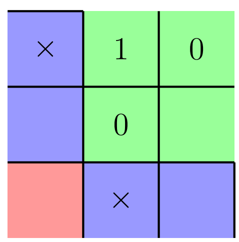
Tic-Tac-Toe boxes for $\mathcal{R}_{2\times 2}$.

We consider an augmented version of the rook matrices. The green boxes correspond with the rook matrices in $m \times n$, and allow us to keep track of the minimal and maximal hyperplanes by placing either a 0 or 1 respectively. For each cutting hyperplane in $\mathcal{R}$ we mark the blue box corresponding to the row or column. This system gives us one way of finding the the set of hyperplanes that are satisfied by all v in V_f.

Example 3.1.2. Suppose we have $V_f = \{v, u\}$ with

$$u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we have



Which yields the hyperplanes:

- $x_{11} = 1$
- $x_{12} = 0$
- $x_{21} = 0$
- $x_{11} + x_{12} = 1$
- $x_{11} + x_{21} = 1$

3.2 Finding edges from hyperplanes and vertices

Now that we have a method of generating a set of hyperplanes from a pair of vertices, we want to show that $H_{\{u,v\}}$ forms a face of $\mathcal{R}$. It's worth noting that this result is true for the general case of a convex polytope as well. The following results can be found in [10] in a slightly different form.

Proposition 3.2.1. *If $h_1, h_2, \ldots, h_k$ are bounding hyperplanes of $\mathcal{R}$ then $\mathcal{R} \cap h_1 \cap h_2 \cap \ldots \cap h_k$ is a face of $\mathcal{R}$.*

Proof. Let d be the dimension of $\mathcal{R}$. Let $n_1, n_2, \ldots, n_k$ be normal vectors of $h_1, h_2, \ldots, h_k$ respectively that are pointing into $\mathcal{R}$ and $v \in \mathcal{R} \cap h_1 \cap h_2 \cap \ldots \cap h_k$. Let H be the affine hyperplane with normal vector $n_1 + n_2 + \ldots + n_k$ passing through v. We want to show that

$$\mathcal{R} \cap H = h_1 \cap h_2 \cap \ldots \cap h_k,$$

and all of $\mathcal{R}$ lies on one side of H.

First note that for any $h_i \in H_b$, $\mathcal{R} \cap h_i$ is a face. So choose $\{h_i, h_j \in h_1, h_2, \ldots, h_k\}$ where and $1 \leq i < j \leq k$. Take the affine hyperplane H with normal vector $n_i + n_j$ that passes through $v \in (\mathcal{R} \cap h_i) \cap (\mathcal{R} \cap h_j)$. If $u \in h_i \cap h_j$ then $u \in H$, so

$$\mathcal{R} \cap h_i \cap h_j \subseteq \mathcal{R} \cap H.$$

Note also that if

$$h_i = \{x \in \mathbb{R}^d : \vec{n_i}x = a_i\},$$
$$h_j = \{x \in \mathbb{R}^d : \vec{n_j}x = a_j\},$$

then

$$H = \{x \in \mathbb{R}^d : (\vec{n_i} + \vec{n_j})x = a_i + a_j\}.$$

And if $u \in \mathcal{R}$ then $u \in \{x \in \mathbb{R}^d : \vec{n_i}x \geq a_i\}$ and $u \in \{x \in \mathbb{R}^d : \vec{n_j}x \geq a_j\}$, so if $u \in \mathcal{R} \cap H$ then $u \in \mathcal{R} \cap h_i \cap h_j$, so

$$\mathcal{R} \cap H \subseteq \mathcal{R} \cap h_i \cap h_j.$$

Therefore,

$$\mathcal{R} \cap H = \mathcal{R} \cap h_i \cap h_j.$$

This shows that the intersection of any two bounding hyperplanes forms a face of $\mathcal{R}$. Next, we assume that the intersection of $k-1$ hyperplanes forms a face of $\mathcal{R}$, and we want to use that to show that the intersection of k hyperplanes forms a face of $\mathcal{R}$.

So suppose we have H' such that

$$R \cap H' = h_1 \cap h_2 \cap \ldots \cap h_{k-1},$$

then

$$R \cap H' \cap h_k = h_1 \cap h_2 \cap \ldots \cap h_{k-1} \cap h_k.$$

But $H' \cap h_k$ is just an intersection of two hyperplanes that form faces of $\mathcal{R}$, so there is some H such that

$$\mathcal{R} \cap H = \mathcal{R} \cap H' \cap h_k.$$

□

Proposition 3.2.2. *Every face of $\mathcal{R}$ can be found as a unique intersection of bounding hyperplanes with $\mathcal{R}$.*

Proof. Recall from Theorem 2.2.9 that the face lattice of $\mathcal{R}$ is coatomic. This means that for any $f \in L(\mathcal{R})$, $f = c_0 \wedge c_1 \wedge \ldots \wedge c_k$ of k coatoms. The coatoms of $L(\mathcal{R})$ are the facets of $\mathcal{R}$, or some $h_i \cap \mathcal{R}$ where $h_i \in H_b$. Therefore, for every face $F \in \mathcal{R}$, $F = \mathcal{R} \cap h_1 \cap h_2 \cap \ldots \cap h_k$ of k hyperplanes in H_b, which are unique from the definition of meet. □

Remark. In particular, two vertices of a $m \times n$ rook polytope $u, v \in V$ are adjacent if there is some intersection of bounding hyperplanes $h \in H$ such that $V \cap h_1 \cap h_2 \cap \ldots \cap h_k = \{u, v\}$

Example 3.2.3. Suppose we have

$$v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The hyperplanes we should choose are $x_{12} = 0$, $x_{21} = 0$, $-x_{11} - x_{12} = -1$, $-x_{11} - x_{21} = -1$. The normal vectors of the previous hyperplanes are

$$(0, 1, 0, 0), (0, 0, 1, 0), (-1, -1, 0, 0), (-1, 0, -1, 0).$$

Summing, we get $\vec{n} = (-2, 0, 0, 0)$ which gives the hyperplane

$$-2x_{11} = 0.$$

We want this hyperplane to intersect v, so we move it out to

$$-2x_{11} = -2.$$

The intersection of this hyperplane with the vertices of the 2×2 rook polytope are exactly $\{u, v\}$, and all other vertices lie on one side of the hyperplane, so u, v are adjacent.

Corollary 3.2.4. *The Birkhoff polytope $\mathcal{B}_n$ is a face of the rook polytope $\mathcal{R}_{n\times n}$.*

Proof. We want to show that there is some hyperplane $H_{\mathcal{B}_n}$ such that

$$H_{\mathcal{B}_n} \cap \mathcal{R}_{n\times n} = \mathcal{B}_n.$$

We note that the vertices of the Birkhoff polytope correspond to the case where we choose all $2n$ cutting hyperplanes to be satisfied. So find $H_{\mathcal{B}_n}$ as in Theorem 3.2.1. This gives us $H_{\mathcal{B}_n} = \{x \in \mathbb{R}^d : \vec{x} = n\}$. So,

$$H_{\mathcal{B}_n} \cap \mathcal{R}_{n\times n} = \mathcal{B}_n.$$

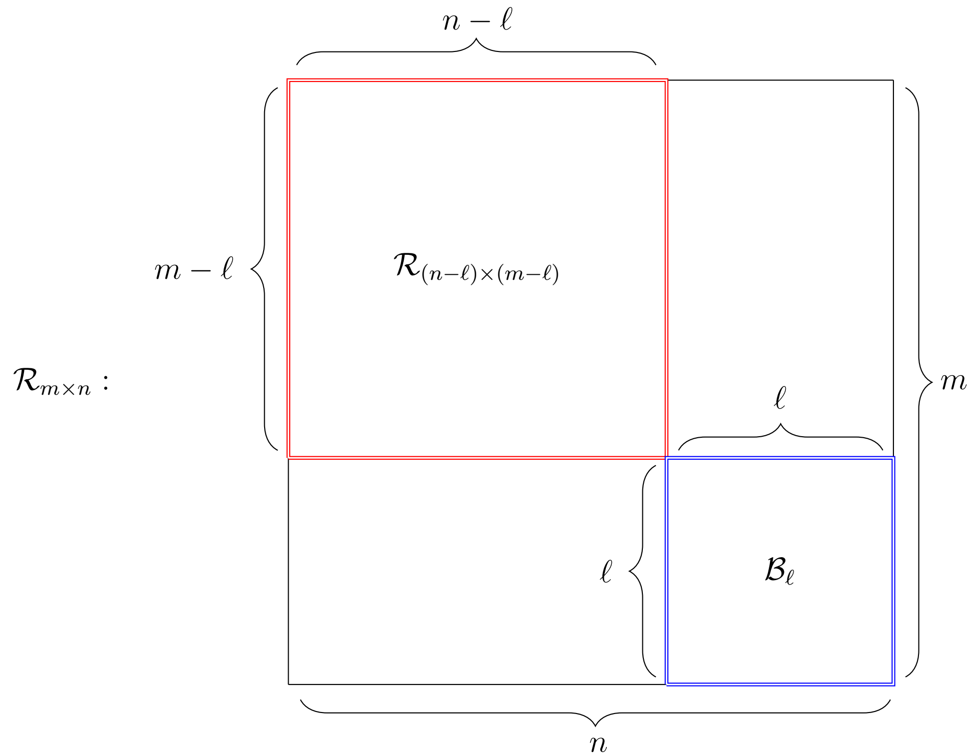
□

In fact, $\mathcal{R}_{m\times n}$ contains

$$\left(\sum_{k=0}^{m-\ell} \binom{m-\ell}{k} \binom{n-\ell}{k} k! \right) \binom{n}{\ell} \binom{m}{\ell}$$

faces that are congruent to $\mathcal{B}_\ell$.

Note that the ℓ rows and ℓ columns of the Birkhoff polytope can be chosen from any of the m rows and n columns of $\mathcal{R}_{m\times n}$, and while the rest of those rows and columns are forced to be zero, the $\mathcal{R}_{(n-\ell)\times(m-\ell)}$ can be any possible vertex of the rook polytope, hence our formula.



$\mathcal{R}_{m\times n}$ portioned into $\mathcal{R}_{(m-\ell)\times(n-\ell)}$ and $\mathcal{B}_\ell$.

Conjecture 3.2.5. The sub-poset of Birkhoff faces in the Rook polytope is a meet-semilattice, that is that it has an unique lower bound, but no unique upper bound. This semilattice is additionally atomic, with atoms equal to $V_{\mathcal{R}} - \{\vec{0}\}$.

4 Support Graphs

There are other algorithms out there that give us insight on the geometric structure of the rook polytope. One such structure is called the support graph, and it gives a clear way of enumerating edges. In this section we will spend some time defining the support graph, then define an algorithm for finding $H_{\{u,v\}}$ given u, v via the support graphs. We will then expand on some results from using the support graphs on the rook polytopes, how to identify edges and hypercube style faces.

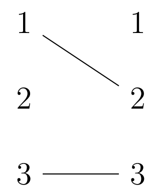
4.1 Definitions

For further machinery to examine our polytopes, we borrow the idea of a support graph from study of transportation polytopes [3].

Definition 4.1.1. The **support graph**, $B(v)$, of a vertex $v \in V_{\mathcal{R}}$ of the rook polytope $\mathcal{R}_{m\times n}$ is a bipartite graph with partitions R, C of $V_{B(v)}$ where $R = \{1, 2, \ldots, m\}$ and $C = \{1, 2, \ldots, n\}$, and number of edges between $r \in R$ and $c \in C$ is equal to the entry in the rth row and cth column of the matrix of v.

Example 4.1.2. Suppose we have $v \in \mathcal{R}_{3\times 3}$ where

$$v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ then } B(v) =$$



It is very useful to consider overlaid support graphs of pairs of vertices.

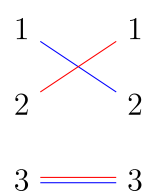
Definition 4.1.3. The support graph $B(u + v)$ is $B(w)$ where $w = u + v$. The **overlaid support graph** $B(u, v)$ is a bicoloring of $B(u + v)$, where the edges from $B(u)$ and the edges from $B(v)$ are distinguished by color. We usually use blue for u and red for v.

We call the **v-degree** of a vertex in $B(u, v)$ the degree of that vertex in $B(v)$ (or the red degree) and the **u-degree** of a vertex is the degree of the corresponding vertex in $B(u)$ (or the blue degree). We can also consider the graph $B(u+v)$, which is similar to $B(u, v)$ except it is monochromatic.

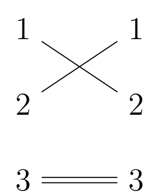
Example 4.1.4.

$$v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$B(u, v) =$



, $B(u + v) =$



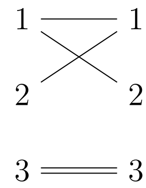
We additionally define a condition for a valid coloring of $B(u + v)$.

Definition 4.1.5. A **valid coloring** of $B(u + v)$ is a bicoloring into $B(u, v)$ where the u-degree or the v-degree of any given vertex is less than or equal to 1.

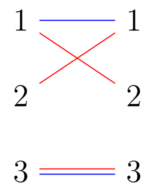
Note that in a support graph $B(u)$ of a vertex of $\mathcal{R}$, the degree of any vertex is either 1 or 0, since otherwise the matrix does not correspond to a vertex in $\mathcal{R}$. So in order to color $B(u + v)$ into a valid $B(u, v)$, both $B(u)$ and $B(v)$ must correspond to vertices of $\mathcal{R}$. This means that the edges of $B(u + v)$ can be partitioned into two spanning subgraphs $B(u)$, $B(v)$ where $u, v \in V_{\mathcal{R}}$.

Example 4.1.6.

$$u+v = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B(u+v) =$$



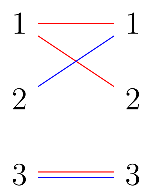
A valid coloring of $B(u+v)$:



$$v = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $u, v \in V_{\mathcal{R}}$, and similarly no r or c has red or blue degree larger than 1.

An invalid coloring of $B(u+v)$:



$$v = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that in this case, $r = 1$ has a red-degree of 2. This means that while $u \in V_{\mathcal{R}}$, $v \notin V_{\mathcal{R}}$, since the first row of v sums to 2.

The support graph can be used to enumerate the faces of the Birkhoff polytope, see [1, 6, 7]. Firstly, the support graphs of the vertices of the Birkhoff polytope are exactly the set of perfect matchings on $K_{n,n}$. Additionally, De Loera notes:

Proposition 4.1.7 (De Loera, E. Kim [3])**.** *Let u and v be distinct vertices of a Birkhoff polytope $\mathcal{B}$. Then the vertices u and v are adjacent if and only if the graph $B(u, v)$ contains a unique cycle.*

We will end up seeing a somewhat similar, but not identical result in the case of the rook polytope.

4.2 Using the support graph to find $H_{\{u,v\}}$

Suppose we have two vertices, u, v. Examine the support graph $B(u+v)$. $B(u+v)$ indicates which hyperplanes $h \in H_f$ should be chosen to find $\{u, v\} \subseteq V \cap h_1 \cap h_2 \cap \ldots \cap h_i$.

Recall that the support graph $B(u)$ is a bipartite graph with partitions R and C, where R corresponds to the rows of the rook monoid corresponding to the vertex, C corresponds to the columns and there is an edge between some $r \in R$ and some $c \in C$ if $x_{rc} = 1$.

Algorithm 4.2.1. Given $\{u, v\}$ with overlaid support graph $B(u, v)$, define a set of hyperplanes H_f as the hyperplanes such that:

- If a node $r \in R$ has degree 2, then we include

$$h = \left\{ x \in \mathbb{R}^{n \times m} : \sum_{j=1}^{n} x_{rj} = 1 \right\}.$$

  Similarly if a node $c \in C$ has degree 2 then we include

$$h = \left\{ v \in \mathbb{R}^{n \times m} : \sum_{i=1}^{m} x_{ic} = 1 \right\}.$$

- If there is no edge between $r \in R$ and $c \in C$ then we include

$$h = \{ v \in \mathbb{R}^{n \times m} : x_{rc} = 0 \}.$$

- If there is a double edge between $r \in R$ and $c \in C$ then we include

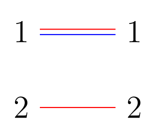
$$h = \{ v \in \mathbb{R}^{n \times m} : x_{rc} = 1 \}.$$

Example 4.2.2. Suppose we have

$$u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then

$B(u, v) =$



.

So we get the following hyperplanes:

- $x_{11} = 1$
- $x_{12} = 0$
- $x_{21} = 0$
- $x_{11} + x_{12} = 1$
- $x_{11} + x_{21} = 1$

Proposition 4.2.3. *Let H_f be the set of hyperplanes generated by Algorithm 4.2.1. The intersection*

$$f = \mathcal{R} \cap \bigcap_{h \in H} h$$

is exactly the smallest face containing $\{u, v\}$.

Proof. First, we want to show that we have not picked any hyperplanes that do not contain $\{u, v\}$. It suffices to show that $f \subseteq h$ for all $h \in H_f$. From Algorithm 3.12 we see we add the corresponding cutting hyperplane h if the degree of $r \in R$ or $c \in C$ is equal to 2. If the degree of some $r \in R$ is 2, that means that both u, v contain a 1 in that row, same for columns, which means that both u, v must lie on h. Similarly, if there is no edge between r and c, then both u and v have a 0 in the corresponding x_{rc} cell, and so both lie on the corresponding minimal hyperplane. Conversely, a double edge clearly indicates that both u, v have $x_{rc} = 1$ and so should lie on the corresponding maximal hyperplane.

Second, we want to show that H_f contains all of the bounding hyperplanes that contain $\{u, v\}$. It suffices to show that if $h \in H_f$, then $f \subseteq h$. Suppose that we have some hyperplane $h' \notin H_f$ where $\{u, v\} \in h'$. Since h' is a bounding hyperplane, it must be minimal, maximal or cutting. If it is a minimal hyperplane, that implies that $x_{rc} = 0$ in both u and v. But if $x_{rc} = 0$ in both u and v then there is not an edge between r and c in $B(u)$ or $B(v)$, and so there is no edge between r and c in $B(u + v)$ which contradicts the initial supposition. Without loss of generality the same is true for h' a maximal hyperplane. So suppose h' is a row cutting hyperplane. This suggests that the sum of some row r is equal to 1 in both u and v. So in $B(u)$, the degree of r is 1, and in $B(v)$ the degree of r is 1. This means that the degree of r in $B(u + v)$ is 2, and we would have chosen the corresponding row-cutting hyperplane. Without loss of generality, the same is true for h' a column cutting hyperplane. So h' cannot be a minimal, maximal, or cutting hyperplane, and therefore h' is not a bounding hyperplane. □

Corollary 4.2.4. *H_f from algorithm 4.2.1 is exactly $H_{\{u,v\}}$ from Algorithm 3.1.1.*

Since both sets define the smallest face containing $\{u, v\}$, they must be the same sets.

4.3 Using the support graph to identify edges

In fact, we can quickly use the support graph $B(u, v)$ to check if u, v are adjacent in $\mathcal{R}$.

Theorem 4.3.1. *Suppose we have vertices $u, v \in V$ of $\mathcal{R}_{m\times n}$. There exists an edge between u and v in $\mathcal{R}_{m\times n}$ if and only if $B(u + v) = B(u' + v')$ implies that $\{u, v\} = \{u', v'\}$.*

Lemma 4.3.2. *$B(u + v) = B(u' + v')$ if and only if the smallest face containing u, v also contains u', v'*

Proof. Suppose that $B(u + v) = B(u' + v')$. Then the hyperplanes that u, v are solutions for are exactly the same hyperplanes that u', v' are solutions for. So the smallest face containing

u, v must also contain u', v'.

Next, suppose that the smallest face containing u, v also contains u', v'. Then the hyperplanes picked from Algorithm 4.2.1 using u, v are the same hyperplanes picked when using u', v'. Therefore, $B(u + v) = B(u' + v')$. □

Proof of Theorem 4.3.1. Suppose that u, v lie on an edge of $\mathcal{R}$. Then there is some set of bounding hyperplanes

$$H_{\{u,v\}} = \{h_i \in H_b : \bigcap_{i \geq 1} h_i \cap V = \{u, v\}\}.$$

By Lemma 4.3.2 we note that since $V_f = \{u, v\}$, $B(u+v) = B(u'+v')$ only if $\{u, v\} = \{u', v'\}$. Since $B(u, v)$ orders by color, $B(u, v) = B(u', v)$ only if $(u, v) = (u', v')$.

Suppose instead that $B(u, v)$ exists and $B(u, v) = B(u', v')$ implies that $u = u'$, $v = v'$. From Lemma 4.3.2, we note that the smallest face containing u, v must also contain u', v'. Since $u = u'$ and $v = v'$, that smallest face has a vertex representation $V_{u,v}$ with a size of 2. Therefore, u, v must lie on a 1-face of $\mathcal{R}$. □

Additionally, we note the following:

Theorem 4.3.3. *Suppose u, v are vertices of $\mathcal{R}$. Then the following are equivalent:*

1. *The vertices u, v lie on an edge of $\mathcal{R}$.*
2. *For all $u', v' \in V_{\mathcal{R}}$, $u + v = u' + v'$ implies $\{u, v\} = \{u', v'\}$.*
3. *$B(u + v)$ can only be colored as $B(u, v)$ or $B(v, u)$.*

Lemma 4.3.4. *The vertices u, v are adjacent in $\mathcal{R}$ if and only if*

$$\{u, v\} = H_{\{u,v\}} \cap V_{\mathcal{R}}.$$

Proof of Lemma. Suppose u, v are adjacent in $\mathcal{R}$. Then the hyperplanes chosen in Algorithm 3.1.1 are exactly $H_{\{u,v\}}$, and the edge between u, v in $\mathcal{R}$ is the smallest face containing u, v. Conversely, assume that $H_{\{u,v\}} \cap V_{\mathcal{R}} = \{u, v\}$. Since $H_{\{u,v\}}$ is the smallest face containing u, v, then u and v are adjacent in $\mathcal{R}$. □

Proof of 4.3.3. $1 \Rightarrow 2$ since if we have $H_{\{u,v\}} \cap V_{\mathcal{R}} = \{u, v\}$, then by Lemma 4.3.2, $B(u+v) = B(u' + v')$ only if $\{u, v\} = \{u', v'\}$.

$2 \Rightarrow 3$, since $B(u, v)$ orders by color, the only two ways to arrange u, v are $(u, v), (v, u)$.

$3 \Rightarrow 1$ since if $B(u + v)$ can only be colored as $B(u, v)$ or $B(v, u)$ that implies that $V_{\{u,v\}} = H_{\{u,v\}} \cap V_{\mathcal{R}}$, and $|V_{\{u,v\}}| = 2$, and so is exactly $\{u, v\}$. □

Theorem 4.3.5. *$B(u, v)$ contains at most one walk that is not a 2-step cycle if and only if $B(u + v)$ can only be colored as $B(u, v)$ or $B(v, u)$.*

Proof. First, note that in a 2-step cycle, switching colors is indistinguishable. Choose $e \in B(u)$ where $e \in B(v)$. Switching its color requires you to switch the color of the same edge in $B(v)$. So $u = u$ still, and $v = v$.

So, suppose we have $B(u + v)$ containing at most one walk that is not a 2-step cycle. So pick some edge $e \in B(u)$ such that $e \notin B(v)$ and change it's color. Now at least one of the vertices $\varepsilon(e) = \{r, c\}$ has v-degree of 2. So change the color of the edges adjacent to e that are now the same color as e. If we have a valid graph, then we have swapped every edge in the walk containing e and we are done. Otherwise there is a new vertex with u or v-degree of 2, and we should continue until we have a valid graph. Since we have swapped every edge in the walk containing e and since there was only one walk, we now have $B(v, u)$.

Conversely, suppose $B(u+v)$ contains more than one unique walk that is not a 2-step cycle. Then choose some $e \in B(u + v)$ such that $e \notin B(v)$. Changing the color of e results in changing every edge in the entire walk. But there is more than one unique walk. So we have $B(u + v)$ colored as $B(u', v')$ such that $\{u', v'\} \neq \{u, v\}$. □

This leaves us with our main result.

Corollary 4.3.6. *Two vertices, u, v in $V_{\mathcal{R}}$ are adjacent if and only if $B(u, v)$ contains at most one walk that is not a 2-step cycle.*

This allows us to quickly check whether or not two vertices lie on a 1-face of $\mathcal{R}$.

4.4 Euclidean Geometry

We can say some things in general about the Euclidean geometry of the rook polytope. Firstly, if we want to construct a tinker-toy like 'skeleton' of the polytope out of just the edges we need just a small set of types of edges of different lengths and a small set of possible angles between them.

We can just use the distance formula to identify the length of an edge e between two vertices $u, v \in \mathcal{R}_{m\times n}$

$$\|e\| = \sqrt{\sum_{i=1}^{mn} (u_i - v_i)^2}.$$

It's pretty quick to note that this length is bounded above at $\sqrt{2m}$, since we can have at most m ones in any given vector of a rook polytope, so if two vertices with m ones have an edge and share no common maximal bounding hyperplane we get $\|e\| = \sqrt{2m}$.

We also can find the small set of possible angles between two edges. Take two edges, e_1

between v and u, and e_2 between v and w. We say $e_1 = v - u$ and $e_2 = v - w$. Since

$$e_1 \bullet e_2 = \|e_1\| \cdot \|e_2\| \cos\theta,$$

we can find

$$\theta = \cos^{-1}\left(\frac{e_1 \bullet e_2}{\|e_1\| \cdot \|e_2\|}\right).$$

Proposition 4.4.1. *For any e_1, e_2 edges that share a vertex in $\mathcal{R}$, the angle between them is between 0 and $\pi/2$.*

Proof. Suppose we have $e_1 = v - u$, $e_2 = v - w$ for some $u, v, w \in V_{\mathcal{R}}$. Note that $u, v, w \in V_m cr$ implies that u, v, w are also found in the set of vertices of the hypercube of the same dimension as $\mathcal{R}$. The angle between $v' - u'$ and $v' - w'$ for any u', v', w' in a hypercube is between 0 and $\pi/2$, so the angle between e_1 and e_2 must also be between 0 and $\pi/2$. □

Proposition 4.4.2. *If the support graph $B(u+v)$, where $u, v \in V$ of the $m \times n$ rook polytope, can be partitioned into subgraphs $B_1, B_2, \ldots, B_k$ where $k \leq n$ and each B_ℓ for $1 \leq \ell \leq k$ contains exactly one walk that is not a 2-step cycle then $H_{\{u,v\}}$ generated by Algorithm 4.2.1 has size 2^k.*

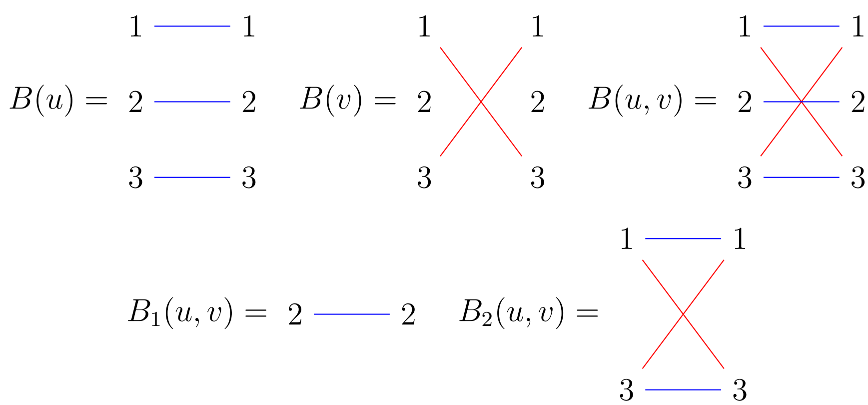
Proof. Note that if $B(u, v)$ can be partitioned into k subgraphs, each containing exactly one unique walk, each of those subgraphs can be colored in exactly two ways– their colors can be swapped. So there are 2^k possible valid colorings of $B(u, v)$. This means that the size of the smallest V_f such that $u, v \in V_f$ must be 2^k. □

Proposition 4.4.3. *If the support graph $B(u+v)$, where $u, v \in V$ of the $m \times n$ rook polytope, can be partitioned into subgraphs B_1, B_2 which each contain exactly one walk that is not a 2-step cycle then u, v are on a 2-face of the rook polytope.*

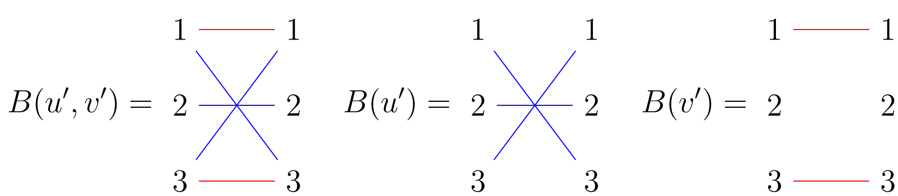
Proof. Suppose we have $B(u, v)$ partitioned into 2 subgraphs containing exactly one unique walk. Then we know that $|V_f| = 2^2 = 4$. We also know that the angle between any two edges should be less than or equal to $\pi/2$. Note that if 4 vertices form a 3-face they are a simplex (specifically a tetrahedron), in which case each vertex lies on a 1 face with each other vertex in the face, and so V_f cannot be the smallest face containing u, v. So if $|V_f| = 4$, then $\dim(V_f) \leq 2$. But if $\dim(V_f) = 1$ then V_f must either contain interior points, which it cannot since it is a subset of $V_{\mathcal{R}}$, or $|V_f| = 2$, which is also false. So $\dim V_f = 2$. □

Example 4.4.4. Suppose we have

$$u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad v = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



So if we swap B_2



$$u' = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad v' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So $|V_f| = 4$ and u, v lie on a 2-face of the 3×3 rook polytope.

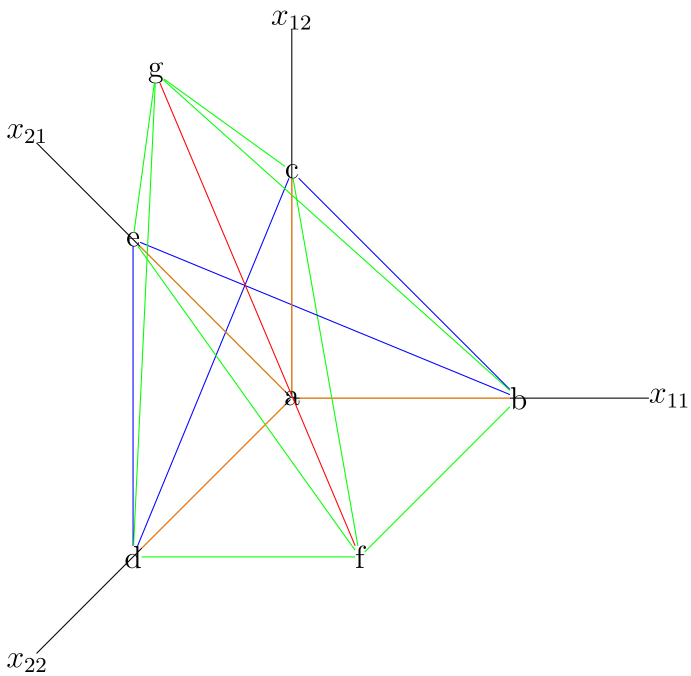
This leads us to believe that, in general, when there are k partitions of $B(u, v)$ containing unique walks that are not two-cycles, u, v lie on a k-face.

Conjecture 4.4.5. If the support graph $B(u+v)$, where $u, v \in V$ of the $m \times n$ rook polytope, can be partitioned into subgraphs $B_1, B_2, \ldots, B_k$ where $k \leq n$ and each B_ℓ for $1 \leq \ell \leq k$ contains exactly one walk that is not a 2-step cycle then u, v are on a k-face of the rook polytope.

If we examine the structure of the example above we can see that we have a face containing u, v, u', v'. The distance between u, u' or v, v' is 2, while the distance between u, v' or v, u' is 1, while the distance between u, v or u', v' is $\sqrt{5}$, which is expected simply from the Pythagorean theorem, and the face is rectangular rather than square. The matrices themselves also bear some interesting properties, the central coordinate is free to be either 0 and 1, while the corner coordinates form a 'semi'-free block, where they can vary between exactly two positions.

There are immediately a handful of different directions I could continue in to answer questions about these rook polytopes. The obvious next step is to prove the conjectures in this paper. This could be accomplished by building more framework to explain the combinatorial implications of various types of edges, and examining the affine independence of various vertices in a given V_f of size 2^k. I am also curious to examine the sublattice of the Birkhoff faces of the polytope; my initial look at the case left me wanting to spend more time on it. I could also attempt to generalize these results to a larger case– since the Birkhoff polytope is a special case of the transportation polytope, what happens when we allow the column bounds and row bounds to exceed 1? The geometry of the shapes is somewhat different, but they still follow very similar rules. Can this research yield a characterization of all types of faces? Paffenholz in [7] finds very similar results in the case of the Birkhoff polytope, where all faces of the Birkhoff polytope to be either simplices or hypercubes or some hybrid of both. The topic, in general, has many questions still unanswered.

A The 2×2 Case



Plot of $\mathcal{R}_{2\times 2}$.

Facets:

$$
\begin{array}{ll}
x_1 = 0 & x_1 + x_2 = 1 \\
x_2 = 0 & x_3 + x_4 = 1 \\
x_3 = 0 & x_1 + x_3 = 1 \\
x_4 = 0 & x_2 + x_4 = 1
\end{array}
$$

Vertices:

$$
a = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
$$

$$
c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad d = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
e = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

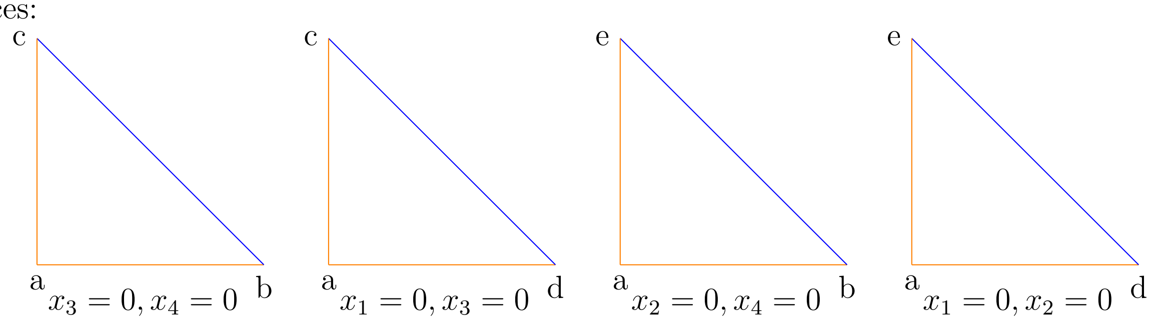
$$
g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
$$

Edges:

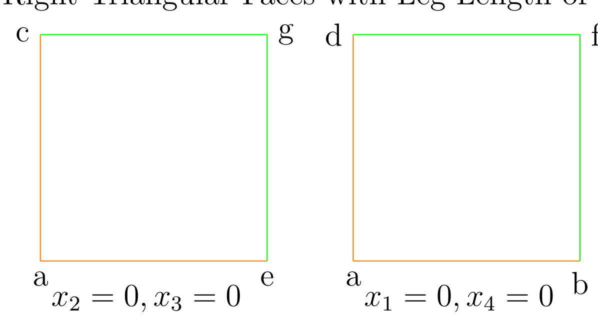
Edge Table

Line	EQ1	EQ2	EQ3	Length	Dim
$\overline{ab}$	$x_2 = 0$	$x_3 = 0$	$x_4 = 0$	1	0,1
$\overline{ac}$	$x_1 = 0$	$x_3 = 0$	$x_4 = 0$	1	0,1
$\overline{ad}$	$x_1 = 0$	$x_2 = 0$	$x_3 = 0$	1	0,1
$\overline{ae}$	$x_1 = 0$	$x_2 = 0$	$x_4 = 0$	1	0,1
$\overline{cg}*$	$x_2 = 0$	$x_3 = 0$	$x_1 + x_2 = 1$	1	1,2
$\overline{eg}*$	$x_2 = 0$	$x_3 = 0$	$x_3 + x_4 = 1$	1	1,2
$\overline{df}*$	$x_1 = 0$	$x_4 = 0$	$x_3 + x_4 = 1$	1	1,2
$\overline{bf}*$	$x_1 = 0$	$x_4 = 0$	$x_1 + x_2 = 1$	1	1,2
$\overline{cb}$	$x_3 = 0$	$x_4 = 0$	$x_1 + x_2 = 1$	$\sqrt{2}$	1
$\overline{cd}$	$x_1 = 0$	$x_3 = 0$	$x_2 + x_4 = 1$	$\sqrt{2}$	1
$\overline{be}$	$x_2 = 0$	$x_4 = 0$	$x_1 + x_3 = 1$	$\sqrt{2}$	1
$\overline{de}$	$x_1 = 0$	$x_2 = 0$	$x_3 + x_4 = 0$	$\sqrt{2}$	1
$\overline{cf}$	$x_1 + x_2 = 1$	$x_2 + x_4 = 1$	$x_3 = 0$	$\sqrt{3}$	1,2
$\overline{ef}$	$x_1 + x_3 = 1$	$x_3 + x_4 = 1$	$x_2 = 0$	$\sqrt{3}$	1,2
$\overline{bg}$	$x_1 + x_2 = 1$	$x_1 + x_3 = 1$	$x_4 = 0$	$\sqrt{3}$	1,2
$\overline{dg}$	$x_2 + x_4 = 1$	$x_3 + x_4 = 1$	$x_1 = 0$	$\sqrt{3}$	1,2
$\overline{fg}*$	$x_1 + x_2 = 1$	$x_3 + x_4 = 1$	More than 3	2	2

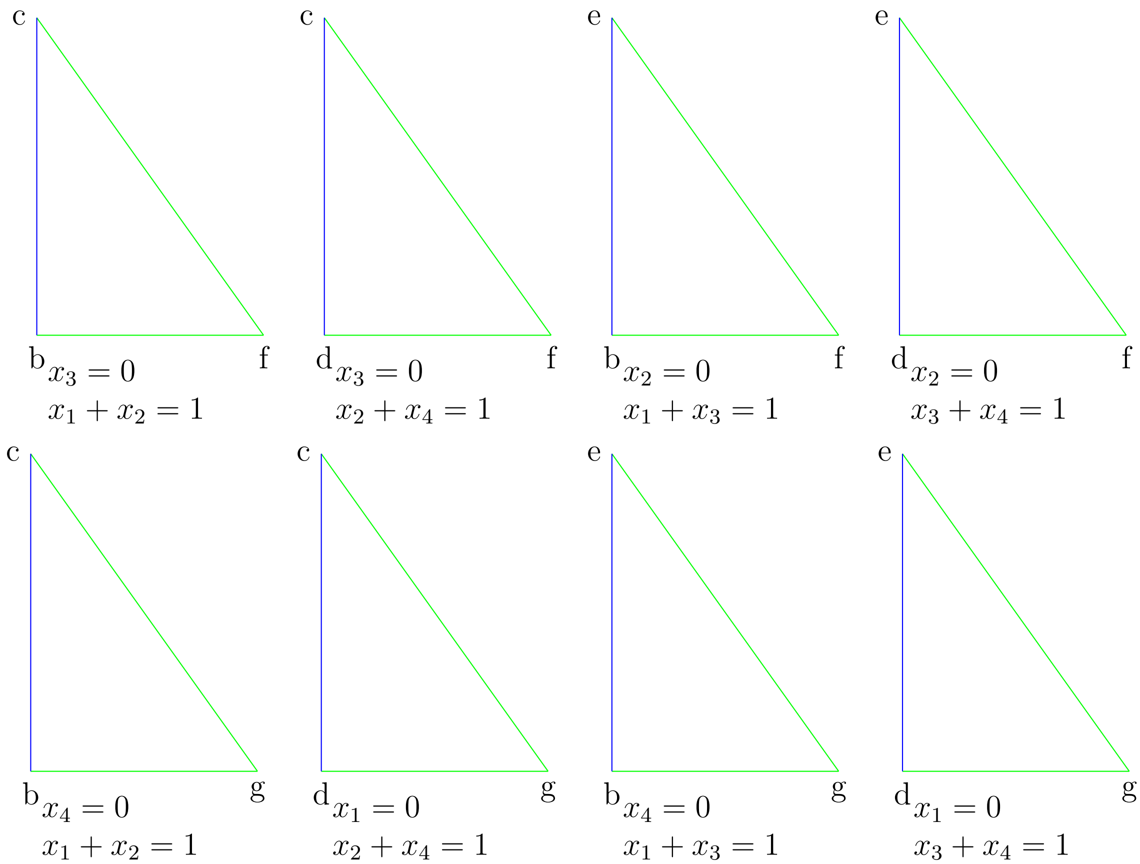
Faces:



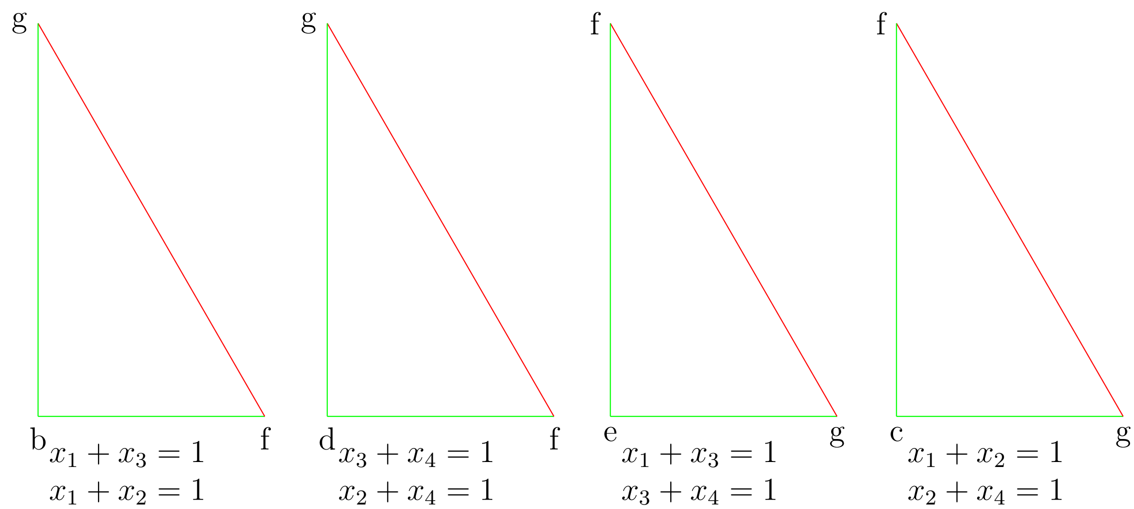
Right Triangular Faces with Leg Length of 1



Unit Square Faces



Right Triangular Faces with Leg Length of 1 and $\sqrt{2}$



Right Triangular Faces with Leg Length of 1 and $\sqrt{3}$

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