Model-Theoretic Limits and the Hrushovski Construction

University of Colorado at Boulder

Mathematics Department

Mick Walker

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Committee Members:
Thesis Advisor: Keith Kearnes - Mathematics Department
Honors Representative: Magdalena Czubak - Mathematics Department
Outside Representative: Joshua Grochow - Computer Science Department

Abstract

Roland Fraïssé provided a model-theoretic framework for constructing a homogeneous model from a countable class of structures through amalgamation. This process was then expanded upon by Hrushovski, who modified the construction to operate under weaker conditions. These approaches allowed the creation of strongly minimal models with pathological geometries in order to disprove conjectures that aimed to classify strongly minimal models, including Zil'ber's Conjecture. It is the aim of this paper to present an introduction into model theory and model-theoretic nature of limits, including the Fraïssé and Hrushovski constructions. This will be in the context of understanding methods for the creation of homogeneous structures, and understanding where these limit structures have appeared in mathematics.

"...and it was too late: their power over it was no longer absolute" - Richard Siken

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Introduction

The aim of this thesis is to provide a cursory introduction to model theory and the constructions of Fraïssé and Hrushovski of homogeneous and rich models. To accomplish this, we will begin with a brief introduction in model theory, building all the concepts we will need to discuss limit structures. While it is certainly not an exhaustive review of model theory, our discussion should be sufficiently thorough so that a reader without a background in logic will be able to follow the later discussion. We will then discuss the Fraïssé construction as well as some example constructions, before moving on to the Hrushovski construction and its motivations.

In general, we will use script letters $\mathcal{M}, \mathcal{N}$ to refer to models, and capital letters M, N to refer to the underlying sets. In addition, script letters like $\mathcal{T}, \mathcal{L}$ will also be used to denote certain collections in model theory, though these will be explicitly described. For classes of certain models, we will use bold letters such as $\mathbf{K}$. We will denote our constants with the letters a_i, b_i, or c_i for $i \in \mathbb{N}$, and our variables will be x_i or v_i. If we have a fixed arbitrary number of constants or variables, we will use an overline, such as $\overline{x}$, to denote the tuple $(x_0, x_1, ..., x_n)$ for some $n \in \mathbb{N}$.

Chapter 1

Model Theory

To begin studying models, we must introduce the method of describing them. For this we rely on a formal notion of a language.

Definition 1.1. A *language* $\mathcal{L}$ is a collection of symbols. We start with three important subcollections, the set of constants $\mathcal{C}$, the set of function symbols $\mathcal{F}$, and the set of relation symbols $\mathcal{R}$. In addition, for each relation $R_i \in \mathcal{R}$ and function $f_j \in \mathcal{F}$ we have a corresponding arity n_i and n_j that gives the number of arguments. Furthermore, we also include variable symbols $x_1, x_2, ...$ and logical syntax symbols such as $\neg, \vee, \wedge$.

These languages are common throughout mathematics, with some examples including:

1. $\{\cdot, 1\}$, where $\cdot$ is a binary function symbol, giving the language of groups;
2. $\{+, \cdot, 0, 1\}$, with binary function symbols $+, \cdot$ and constant symbols $0, 1$ giving the language of fields and rings;
3. $\{<\}$, where $<$ is a binary relation symbol, giving the language of posets.
4. $\{E\}$, where E is a binary relation symbol, giving the language of graphs.

A language provides the necessary tools for the description of mathematical objects. However, languages themselves do not encode any information themselves, rather they must be realized in a structure.

Definition 1.2. An $\mathcal{L}$-*structure* $\mathcal{M}$ is given by a nonempty set M referred to as the domain, constants $c^M \in M$ for every $c \in \mathcal{C}$, functions $f^M : M^n \to M$ for every $f \in \mathcal{F}$, and relations $R^M \subseteq M^n$ for every relational symbol $R \in \mathcal{R}$.

These models are prevalent throughout mathematics, and provide enough structure to encode many algebraic and relational structures. Using our above examples, some models include:

1. The group formed by S_3 in the language $\{\cdot, 1\}$, where the domain is the set of permutations of three elements, the operation $\cdot$ is composition of permutations, and 1 is the identity permutation.

2. The ring of integers in the language $\{+, \cdot, 0, 1\}$, where the domain is $\mathbb{Z}$, the function symbols $+, \cdot$ correspond to addition and multiplication respectively, and the constant symbols 0 and 1 correspond to the integers 0 and 1, respectively.

3. The dense linear order formed by the rationals, where the domain is $\mathbb{Q}$ and the relation symbol $<$ corresponds to the usual ordering of rational numbers.

4. The complete graph K_3 on the set $V = \{0, 1, 2\}$ with an irreflexive symmetric edge relation E where $E(x_1, x_2)$ iff x_1 and x_2 are neighbors in K_3.

Moving forward, we will assume that all languages will include $=$ as a binary relation symbol, and models will realize this as equality as one expects, where it contains only the ordered pairs (x, x) for all $x \in M$. To understand how models interact and relate, we can talk of morphisms between models of a shared language.

Definition 1.3. A *homomorphism* is a function between models with signature $\mathcal{L}$ such that all structure is preserved. For a homomorphism h between models $\mathcal{M}$ and $\mathcal{N}$ we have for all functional symbols $f_{\mathcal{M}}(\overline{a}) = b$ implies $f_{\mathcal{N}}(h(\overline{a})) = h(b)$, for all relational symbols $R_{\mathcal{M}}(\overline{a})$ implies $R_{\mathcal{N}}(h(\overline{a}))$, and for all constants c we have $c_{\mathcal{N}} = h(c_{\mathcal{M}})$.

This definition of homomorphism can be thought of as a generalization of the homomorphisms of algebraic structures. If our models are groups in the language $\{\cdot, 1\}$, then a homomorphism is one that respects the binary operation and maps the identity to the identity. If we consider rings in the language $\{+, \cdot, 0, 1\}$, then homomorphisms are those that preserve addition and ring multiplication, as well as the constants 0 and 1. Now that we have morphisms that preserve structure, we can further define morphisms that give a sense of inclusion.

Definition 1.4. An *embedding* is a homomorphism h between models $\mathcal{M}$ and $\mathcal{N}$ with the additional condition that $R_{\mathcal{M}}(\overline{a})$ if and only if $R_{\mathcal{N}}(h(\overline{a}))$.

We can see that an embedding is necessarily injective by taking R to be equality. If $M \subseteq N$ and the inclusion map is an embedding, then we say $\mathcal{M}$ is a *substructure* of $\mathcal{N}$. If an embedding is surjective, it is an *isomorphism*.

We now have notions of structures and morphisms between them, so we now turn to methods of describing the model, and elements within it. For this we will define formulas, which are formed from elements of $\mathcal{L}$, logical symbols $\neg, \wedge, \vee$, quantifiers $\exists, \forall$, and variables v_i. To define them, we first define terms and atomic formulas.

Definition 1.5. *$\mathcal{L}$-terms* is the minimal collection $\mathcal{T}$ defined recursively as follows:

1. $c \in \mathcal{T}$ for all constant terms c.

2. $x_i \in \mathcal{T}$ for all variables x_i.

3. $f(t_1, t_2, \ldots, t_n) \in \mathcal{T}$ for every function symbol f and terms $t_1, t_2, \ldots, t_n$.

We will at times write terms as as $t(x_1, \ldots, x_n)$ where each x_i is a variable, that may or may not appear in t. Other notation will include bound variables in this list, but for clarity moving forward we will not.

To discuss relations between these terms, we have to further expand our set of symbols and define a new concept.

Definition 1.6. An *atomic formula* is an expression of the form $R(t_1, t_2, ..., t_n)$ for a relation symbol R and terms $t_1, t_2, ..., t_n$. Since equality is in our language, this necessarily includes $t_1 = t_2$.

This gives now a method of connecting our model-theoretic statements to the realm of logical statements. From here we can equip ourselves with logical symbols and quantifiers to further expand the ways we are able to discuss models and substructures within them. We introduce constants $\exists$ and $\forall$ to quantify over elements, as well as logical syntax $\neg, \vee, \wedge$ to combine our atomic formulas into more complex sentences.

Definition 1.7. The *set of $\mathcal{L}$-formulas $\mathcal{F}$* is defined recursively.

1. If ϕ is an atomic formula, then $\phi \in \mathcal{F}$.
2. If $\phi \in \mathcal{F}$, then $\neg\phi \in \mathcal{F}$.
3. If $\phi, \psi \in \mathcal{F}$, then $\phi \wedge \psi \in \mathcal{F}$, and similarly for $\vee$.
4. If $\phi \in \mathcal{F}$, then $\forall x_i \phi \in \mathcal{F}$, and similarly for $\exists$. We say that x_i is bound by the quantifier $\forall$ (or $\exists$) in this case.

We take care to note how quantifiers interact with the variables. Variables that are not bound are considered free variables. We will write formulas as $\phi(x_1, x_2, ..., x_n)$ where the x_i are free variables. Other works may at times include bound variables in these representations, but for clarity we will not. A formula with no free variables is a *sentence*.

With these new constructions, we have a means of describing structure of models. An important question is if these formulas are true in a given model. To formalize this, we recursively define yet another notion, that of satisfaction.

Definition 1.8. Fix a model $\mathcal{M}$, and formula $\phi(\overline{x})$, and tuple $\overline{a}$. We write $\phi(\overline{a})$ to denote the formula obtained when each free occurrence of x_i is replaced by a_i in ϕ. We say $\mathcal{M}$ *satisfies* $\phi(\overline{a})$, written as $\mathcal{M} \models \phi(\overline{a})$, if the following conditions are met.

1. If ϕ is of the form $t_1 = t_2$ for terms t_1, t_2, then $\mathcal{M} \models \phi(\overline{a})$ iff ${t_1}^{\mathcal{M}}(\overline{a}) = {t_2}^{\mathcal{M}}(\overline{a})$.
2. If ϕ is of the form $\neg\psi$ for formula ψ, then $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{M} \not\models \psi(\overline{a})$.
3. If ϕ is of the form $\psi_1 \wedge \psi_2$ then $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{M} \models \psi_1(\overline{a})$ and $\mathcal{M} \models \psi_2(\overline{a})$.
4. If ϕ is of the form $\psi_1 \vee \psi_2$ then $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{M} \models \psi_1(\overline{a})$ or $\mathcal{M} \models \psi_2(\overline{a})$.
5. If ϕ is of the form $R(t_1, t_2, ..., t_n)$ for terms t_i, then $\mathcal{M} \models \phi(\overline{a})$ iff

$$(t_1^{\mathcal{M}}(\overline{a}), t_2^{\mathcal{M}}(\overline{a}), ..., t_n^{\mathcal{M}}(\overline{a})) \in R^{\mathcal{M}}$$

6. If ϕ is of the form $\phi(\overline{x}) = \exists x_{n+1}\psi(\overline{x}, x_{n+1})$, then $\mathcal{M} \models \phi(\overline{a})$ iff there exists some $a_{n+1} \in M$ where $\mathcal{M} \models \psi(\overline{a}, a_{n+1})$.

7. If ϕ is of the form $\phi(\overline{x}) = \forall x_{n+1}\psi(\overline{x}, x_{n+1})$, then $\mathcal{M} \models \phi(\overline{a})$ iff for every $a_{n+1} \in M$ we have $\mathcal{M} \models \psi(\overline{a}, a_{n+1})$.

Definition 1.9. Two $\mathcal{L}$ models $\mathcal{M}$ and $\mathcal{N}$ are called *elementarily equivalent* if for all $\mathcal{L}$-sentences ϕ we have $\mathcal{M} \models \phi$ if and only if $\mathcal{N} \models \phi$.

Lemma 1.10. If two models are isomorphic, then they are elementarily equivalent. However, the converse does not always hold.

Proof. Since formulas are necessarily finite, we proceed by recursion. By definition of isomorphism, we have that f preserves all terms and atomic formulas. Now it suffices to show that it preserves satisfaction under the 7 cases given above. Assume now that $\mathcal{M}$ and $\mathcal{N}$ are equivalent for all formulas of a shorter depth than ϕ.

1. If ϕ is of the form $t_1 = t_2$ then $t_1^{\mathcal{M}}(\overline{a}) = t_2^{\mathcal{M}}(\overline{a})$ iff $t_1^{\mathcal{N}}(f(\overline{a})) = t_2^{\mathcal{N}}(f(\overline{a}))$, so $\mathcal{M} \models \phi(\overline{a})$ if and only if $\mathcal{N} \models \phi(f(\overline{a}))$.

2. If ϕ is of the form $\neg\psi$, then $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{M} \not\models \psi(\overline{a})$, and by assumption this is equivalent to $\mathcal{N} \not\models \psi(f(\overline{a}))$ iff $\mathcal{N} \models \phi(f(\overline{a}))$.

3. If ϕ is of the form $\psi_1 \wedge \psi_2$, then $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{M} \models \psi_1(\overline{a})$ and $\mathcal{M} \models \psi_2(\overline{a})$. By assumption this is equivalent to $\mathcal{N} \models \psi_1(f(\overline{a}))$ and $\mathcal{N} \models \psi_2(f(\overline{a}))$, which occurs if and only if $\mathcal{N} \models \psi(f(\overline{a}))$.

4. If ϕ is of the form $\psi_1 \vee \psi_2$, then $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{M} \models \psi_1(\overline{a})$ or $\mathcal{M} \models \psi_2(\overline{a})$. By assumption this is equivalent to $\mathcal{N} \models \psi_1(f(\overline{a}))$ or $\mathcal{N} \models \psi_2(f(\overline{a}))$, which occurs if and only if $\mathcal{N} \models \psi(f(\overline{a}))$.

5. If ϕ is of the form $R(t_1, t_2, \ldots, t_n)$ for terms t_i, then $\mathcal{M} \models \phi(\overline{a})$ iff
$$(t_1^{\mathcal{M}}(\overline{a}), t_2^{\mathcal{M}}(\overline{a}), ..., t_n^{\mathcal{M}}(\overline{a})) \in R^{\mathcal{M}}$$
As f is an embedding, this occurs iff
$$(t_1^{\mathcal{N}}(f(\overline{a})), t_2^{\mathcal{N}}(f(\overline{a})), ..., t_n^{\mathcal{N}}(f(\overline{a}))) \in R^{\mathcal{N}}$$
Hence the statement is $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{N} \models \phi(\overline{a})$.

6. If ϕ is of the form $\phi(\overline{x}) = \exists x_{n+1}\psi(\overline{x}, x_{n+1})$, then $\mathcal{M} \models \phi(\overline{a})$ iff $a_{n+1} \in M$ and $\mathcal{M} \models \psi(\overline{a}, a_{n+1})$. We see then that $\mathcal{N} \models \psi(f(\overline{a}), f(a_{n+1}))$. However, this is only one direction. Assume $\mathcal{N} \models \psi(f(\overline{a}), b_{n+1}))$. Then $\mathcal{M} \models \psi(\overline{a}, f^{-1}(b_{n+1}))$, giving $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{N} \models \phi(f(\overline{a}))$.

7. If ϕ is of the form $\phi(\overline{x}) = \forall x_{n+1}\psi(\overline{x}, x + 1)$, then we will instead the contrapositive. As $\mathcal{M} \not\models \phi(\overline{a})$ iff there exists a a_{n+1} such that $\mathcal{M} \not\models \psi(\overline{a}, a_{n+1})$, which occurs iff $\mathcal{N} \not\models \psi(f(\overline{a}), f(a_{n+1}))$, and conversely we have $\mathcal{N} \not\models \psi(f(\overline{a}), b_{n+1}))$ iff $\mathcal{M} \not\models \psi(\overline{a}, f^{-1}(b_{n+1}))$, hence $\mathcal{M} \not\models \phi(\overline{a})$ iff $\mathcal{N} \not\models \phi(f(\overline{a})))$.

Thus we see at every recursive step of satisfaction $\mathcal{M}$ and $\mathcal{N}$ satisfy ϕ if and only if the other does, so $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent. To see that there are models that are elementarily equivalent without being isomorphic, consider $\mathcal{M} = (\mathbb{R}, <)$ and $\mathcal{N} = (\mathbb{Q}, <)$ in the theory of dense linear orders without endpoints. We can easily see that because the underlying universes are not of the same cardinality, then the models are not isomorphic. However, Cantor's Isomorphism Theorem gives us that any two models of the dense linear order satisfy the same sentences. $\square$

We now seek to describe similar models, especially those that satisfy the same sentences. For this, we need a few more notions.

Definition 1.11. A set of sentences Σ is *consistent* if it is unable to deduce a logical contradiction.

Gödel's Completeness Theorem gives us that this is equivalent to *satisfiability*, where a set of sentences Σ is *satisfiable* if there is some model $\mathcal{M}$ such that $\mathcal{M} \models \sigma$ for every $\sigma \in \Sigma$. That is, a set of sentences is consistent iff it is satisfiable. Because these notions are equivalent, will use consistency interchangeably with satisfaction.

Definition 1.12. A *theory* is a collection $\mathcal{T}$ of sentences closed under deduction. A theory is *complete* if for all sentences $\phi \in \mathcal{L}$ we have either $\phi \in \mathcal{T}$ or $\neg\phi \in \mathcal{T}$.

While we do not require the collection to be consistent, any inconsistent theory is not satisfiable, so when we refer to theories with models, this requires the theory to be consistent. We defer a discussion of deduction, as it mostly falls outside the scope of this paper, but refer to [End01] for a deeper exploration.

Rather than give an infinite set of sentences, below we will give sentences that generate a given theory under deduction. We refer to this as a theory being *axiomatized* by a given set of sentences. Corresponding to our languages and models previously described, some theories include:

1. The theory of groups, axiomatized the sentences

$$\forall x(1 \cdot x = x \cdot 1 = x)$$
$$\forall x \forall y \forall z(x \cdot (y \cdot z) = (x \cdot y) \cdot z)$$
$$\forall x \exists y(x \cdot y = 1 = y \cdot x)$$

2. The theory of rings with unity, which is axiomatized by the sentences of groups for the operation $+$ and constant 0 along with

$$\forall x(1 \cdot x = x \cdot 1 = x)$$
$$\forall x \forall y \forall z(x \cdot (y + z) = x \cdot y + x \cdot z)$$
$$\forall x \forall y \forall z((x + y) \cdot z = x \cdot z + y \cdot z)$$
$$\forall x \forall y \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$$

3. The theory of dense linear orders, axiomatized by the sentences

$$\begin{aligned}
&\forall x(\neg(x < x)) \\
&\forall x \forall y \forall z((x < y) \wedge (y < z) \rightarrow (x < z)) \\
&\forall x \forall y((x < y) \vee (y < x) \vee (x = y)) \\
&\forall x \forall y((x < y) \rightarrow \exists z((x < z) \wedge (z < y)))
\end{aligned}$$

4. The theory of undirected irreflexive graphs, given by the sentences

$$\begin{aligned}
&\forall x(\neg E(x, x)) \\
&\forall x \forall y(E(x, y) \leftrightarrow E(y, x))
\end{aligned}$$

We now have methods of discussing models, and sentences the models satisfy. We can now employ the same tools we have developed in order to discuss elements and tuples of the models themselves. Fix some model $\mathcal{M}$ and subset A. Let $\mathcal{L} \cup A$ refer to the language $\mathcal{L}$ such that a constant symbol is added for each $a \in A$.

Definition 1.13. An *n-type* defined over A is a collection of formulas free over the same set of n variables in the language $\mathcal{L} \cup \{A\}$ such that the formulas are closed under logical consequence. Given an n-type τ, if for all formulas we have $\phi \in \tau$ or $\neg\phi \in \tau$, then τ is a *complete type*. If there exists some n-tuple $\overline{a} = (a_1, ..., a_n)$ such that $\mathcal{M} \models \phi(\overline{a})$ for all $\phi \in \tau$, then we say that τ is *realized*. We will refer to the set of complete n-types of of $\mathcal{M}$ defined over A as $S_n^{\mathcal{M}}(A)$.

Types will play a large role moving forward, as we will seek to construct models where types are realized. We formalize this in the following way:

Definition 1.14. A model M is *κ-saturated* for some cardinal κ if for every $A \subseteq M$ with $|A| < \kappa$, then the complete 1-types over A are realized in $\mathcal{M}$.

We note here some literature instead requires all n-types, including incomplete types, to be realized, but these statements are equivalent. Any incomplete type is contained in a complete type, and induction on 1-types allows you to construct n-types for arbitrary n. Furthermore, saturation is very dependent on the cardinal κ. For example, the model $(\mathbb{R}, <)$ is κ-saturated as is it models the theory of dense linear orders, however it is not κ^+ saturated as the type given by

$$\tau = \{1 - 1/n < x\}_{n \in \mathbb{N}^+} \cup \{x < 1\}$$

is not realized. One now may ask the question, is it possible to extend $(\mathbb{R}, <)$ to a larger model that includes such types? Yes we can, and this notion can be strengthened as follows.

Proposition 1.14.1. *For some model $\mathcal{M}$ and $\kappa \geq \aleph_0$ there exists some κ^+-saturated $\mathcal{N}$ such that $\mathcal{M}$ elementarily embeds in $\mathcal{N}$.*

Proof. First, we see that for any collection of unrealized 1-types $\{\tau_i\}_{i \in I}$ defined over $A \subseteq M$, then we can always construct the model $\mathcal{M}'$ that realizes each τ_i. Since types are necessarily consistent, any finite set of them is realizable, so by including the formulas true in M we have there is some elementary extension $\mathcal{M}'$ of $\mathcal{M}$ that realizes each type in $\{\tau_i\}_{i\in I}$. In this way, we can add construct extensions recursively to realize types. To construct a saturated model, we are going to construct a chain of extensions and make use of regular cardinals to saturate the model. We define for $\alpha < \kappa^+$ the structures

$$\begin{aligned} M_0 &:= M \\ M_\alpha &:= \bigcup_{\beta<\alpha} M_\lambda \text{ for limit ordinal } \alpha \\ M_{\alpha+1} &= \text{ an extension that realizes all types of } M_\alpha \text{ for successor ordinals } \alpha+1 \end{aligned}$$

Define now

$$\mathcal{N} = \bigcup_{\alpha<\kappa^+} \mathcal{M}_\alpha$$

We claim that this is saturated. Let $A \subseteq M$ with $|A| < \kappa^+$. Since κ^+ is regular then $A \subseteq M_\alpha$ for some α. We have then that all types over A are realized in $M_{\alpha+1}$, hence is realized in $\mathcal{N}$, giving that this model is saturated. $\square$

At times we want a weaker condition: homogeneity. To define this, we first need to define partial elementary embeddings.

Definition 1.15. Consider $\mathcal{L}$-models $\mathcal{M}, \mathcal{N}$ with $B \subseteq M$. A *partial elementary embedding* is a function $f : B \to \mathcal{N}$ such that for all $\mathcal{L}$-formulas φ we have $\mathcal{M} \models \varphi(\bar{b})$ iff $\mathcal{N} \models \varphi(f(\bar{b}))$ for all finite B-sequences $\bar{b}$.

Homogeneity can then be defined as follows:

Definition 1.16. A structure $\mathcal{M}$ is *κ-homogeneous* if for all partial elementary embeddings $f : A \to \mathcal{M}$ with $A \subseteq M$ with $|A| < \kappa$, then for any $b \in \mathcal{M}$ there is a partial embedding $f^* : A \cup \{b\} \to \mathcal{M}$ that extends f. A model $\mathcal{M}$ is *homogeneous* if it is $|M|$-homogeneous.

We verify that this condition is indeed weaker than κ-saturation.

Proposition 1.16.1. *A κ-saturated structure is κ-homogeneous.*

Proof. Take some model $\mathcal{M}$ with $|A| < \kappa$ and a partial embedding $f : A \to \mathcal{M}$. Take now $b \in M \setminus A$. Define the set of formulas

$$\Gamma = \{\phi(v, f(\bar{a})) : \mathcal{M} \models \phi(b, \bar{a})\}$$

Note that $\mathcal{M} \models \exists x\phi(x, \bar{a})$ and since f is partial elementary then $\mathcal{M} \models \exists x\phi(x, f(\bar{a}))$. Since $\mathcal{M}$ is saturated then there is some element b' that realizes this statement, so we define f^* on $A \cup \{b\}$ to be $f^*(a) = f(a)$ for $a \in A$ and $f^*(b) = b'$. By construction this forms a partial embedding, and so $\mathcal{M}$ is homogeneous. $\square$

The importance of homogeneity comes from the following result, which is at times given as the definition of homogeneous:

Proposition 1.16.2. *In a homogeneous model, any partial elementary embedding $f : A \to \mathcal{M}$ with $|A| < |M|$ extends to an automorphism σ on $\mathcal{M}$.*

The proof of this statement requires a sufficiently nontrival amount of background work that we defer the reader to [Bou98].

It is the construction of homogeneous models that motivates the following sections. In this way homogeneous models are those that realize elementary embeddings in an algebraic manner. We can restate this property slightly, noting that any isomorphism between substructures $\phi : A \to A'$ and embedding $f' : A' \to \mathcal{M}$ extends to an embedding $f : A \to \mathcal{M}$. This allows us to note that any isomorphism between finitely generated substuctures of a homogeneous model extends to an automorphism of the structure itself. Moving forward, we aim to construct homogeneous models that contain a class of smaller models as substructures.

Chapter 2

Fraïssé Limits

2.1 Fraïssé Construction

Definition 2.1. For a given model $\mathcal{M}$ we define the *age* of $\mathcal{M}$, written as $\mathrm{Age}(\mathcal{M})$ is the class of all finitely generated structures that can be embedded into $\mathcal{M}$. At times we will only concern ourself with the elements in the age up to isomorphism.

Definition 2.2. A class $\mathbf{K}$ of models satisfies the *hereditary property* if for any finitely generated substructure $\mathcal{N}$ of some $\mathcal{M} \in \mathbf{K}$ we have $\mathcal{N}$ is isomorphic to some element of $\mathbf{K}$.

Definition 2.3. A class $\mathbf{K}$ of models satisfies the *joint embedding property* if for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ there exists some $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{A}$ and $\mathcal{B}$ embed into $\mathcal{C}$.

Definition 2.4. A class $\mathbf{K}$ of models satisfies the *amalgamation property* if for all $\mathcal{A} \in \mathbf{K}$ with embeddings f_i into $\mathcal{B}_i \in \mathbf{K}$ for $i = 1, 2$ there are embeddings g_i from $\mathcal{B}_i$ into some $\mathcal{C} \in \mathbf{K}$ where $g_1 \circ f_1(A) = g_2 \circ f_2(A)$.

In general, amalgamation does not imply joint embedding. To show this, we can consider fields. The class of algebraically closed fields does not satisfy joint embedding property as two fields of distinct characteristics cannot be embedded into one field due to subfield preserving characteristic. However, algebraically closed fields do satisfy the amalgamation property, as any two field extensions $\mathcal{B}_1, \mathcal{B}_2$ of a common field $\mathcal{A}$ can be embedded into a common field. For this, take $\mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2$. The tensor product is a nontrivial ring, being the tensor of two vector spaces over $\mathcal{A}$, and so has a maximal ideal I. It follows that

$$\mathcal{C} = \mathcal{B}_1 \otimes_{\mathcal{A}} \mathcal{B}_2 / I$$

is a field. The canonical homomorphisms $\pi_i : \mathcal{B}_1 \to \mathcal{C}$ where $\pi_i : b \mapsto (b \otimes 1) + I$ are necessarily injective, as the kernel is an ideal and hence is trivial or entire, and we see $\pi_i(1) = 1_{\mathcal{C}} \neq 0$.

Lemma 2.5. Any age $\mathrm{Age}(\mathcal{M})$ satisfies the hereditary property and joint embedding property.

Proof. Let $\mathcal{M}$ be some model with finitely generated substructure $\mathcal{N}$. If there is a finitely generated substructure $\mathcal{N}' \leq \mathcal{N} \leq \mathcal{M}$, then we necessarily have $\mathcal{N}'$ is a finitely generated substructure of $\mathcal{M}$, and hence is contained in Age($\mathcal{M}$).

If $\mathcal{A}$ is generated by $(a_1, a_2, ..., a_n)$ and $\mathcal{B}$ is generated by $(b_1, b_2, ..., b_n)$ take $\mathcal{C}$ to be the structure generated by $(a_1, ..., a_n, b_1, ..., b_n)$ and take the inclusion maps to be the embeddings. $\square$

It is natural to now ask that if every age satisfies these two conditions, then is every class satisfying these three conditions the age of some model? In general, no. If the size of the class is uncountable, then it is possible to construct classes of models that cannot be the age of some model. See [Kud18] for such a construction.

If instead we restrict our attention to countable classes, then our required conditions are sufficient to ensure that our class is the age of some stucture that is unique up to isomorphism.

Theorem 2.6. *For a countable class* $\boldsymbol{K}$ *with HP, JEP, and AP, then there is a unique structure, up to isomorphism, that has age* $\boldsymbol{K}$*. We call this the Fraïssé limit of the class* $\boldsymbol{K}$*.*

Proof. We will first show uniqueness. In order to do this, we are going to introduce a new form of homogeneity which is in general is weaker, but we will see is equivalent to homogeneity for countable structures.

Definition 2.7. A structure C is *weakly homogeneous* if for all finitely generated substuctures A, B with $A \subseteq B$ we have that if for any embedding $f : A \to C$ there exists an embedding $g : B \to C$ such that $g|_A = f$.

Suppose that we have two countable structures, C and D, that have the same age $\mathbf{K}$ and are homogeneous. Let $(C_n)_{n<\omega}$ and $(D_n)_{n<\omega}$ be chains of finitely generated substructures such that $C = \bigcup C_n$ and $D = \bigcup D_n$. We construct now a sequence of isomorphisms $(f_n)_{n<\omega}$ such that the domain will include each C_n and the image will include each D_n. For ease of induction, define $C_0 = \emptyset = D_0 = f_0$.

Now assuming f_k has been defined for all $k < 2n$, we define f_{2n} to include C_n. For this note that there is an isomorphism of finitely generated substructures $g_n : C_n \to D' \subseteq D$. We see then that $g_n(C_{2n-1})$ maps to a finitely generated substructure of D through $f_{2n-1} \circ g_n^{-1}$, as f_{2n-1} is an isomorphism of finitely generated substructures. Since D is weakly homogeneous, we have that there is an embedding $h_n : g_n(C_{2n}) \to D$. Consider now defining $f_{2n} = h_n \circ g_n$, which naturally extends f_{2n-1} and includes C_n.

To construct f_{2n+1}, we define a similar process to include D_n in the image of the isomorphism. We note there is an isomorphism $a_n : D_n \to C'$ of finitely generated substructures. Then we see that $a_n(D_{n-1})$ embeds into C through $f_{2n}^{-1} \circ a_n^{-1}$. Since C is weakly homogeneous, we have that there is an embedding $b_n : a_n(D_n) \to C$ that extends $f_{2n}^{-1} \circ a_n^{-1}$. We then can define $f_{2n+1}^{-1} = b_n \circ a_n$, which by construction extends f_{2n}^{-1} and includes D_n in the image of f_{2n+1}.

We can now construct $(f_n)_{n<\omega}$, and we take $f = \bigcup_{n<\omega} f_n$ to be the isomorphism between C and D. We have now that any two countable weakly homogeneous structures with the same age are isomorphic, and since a homogeneous structure is weakly homogeneous, then this guarantees the uniqueness of the Fraïsse limit of $\mathbf{K}$.

We will also note that here it is possible to see that weak homogeneity is equivalent to homogeneity for a countable structure $\mathcal{M}$. To do this, take $C = D = \mathcal{M}$ for some model $\mathcal{M}$, $C_0 = A \subseteq M$, $D_0 = B \subseteq M$, and $f_0 = g : A \to B$ for some isomorphism g of finitely generated structures A and B. We see then that the constructed f is an automorphism of $\mathcal{M}$ extending g.

We will now show existence of a countable homogeneous structure with age $\mathbf{K}$. Let $\mathbf{K}$ be a countable class of models satisfying HP, JEP, and AP. We are going to construct a chain of models where the union is our desired structure. First, choose an enumeration of pairs $P = (A_i, B_i)_{i<\omega}$ such that for any $A \subseteq B \in K$ we have $(A, B) = (A_i, B_i)$ for some i, and choose a bijection $\pi : \omega \times \omega \to \omega$ such that $\pi(i, j) \geq i$. We construct the chain as follows: Let $C_0 \in K$. Assume now we have C_k for $k \leq n$. Define the list

$$P_k = \{(f_{k,j}, (A, B)_{k,j})\}_{j<\omega}$$

Where $(A, B)_{k,j} \in P$ and there is an embedding $f_{k,j} : A_{k,j} \to C_k$. Assume that $n = \pi(i, j)$. Define C_{n+1} now as the amalgam of of $B_{i,j}$ with C_n over $A_{i,j}$. We claim that $C = \bigcup_{i<\omega} C_\omega$ is the desired homogeneous structure with age K.

In order to see that C is homogeneous, see that by construction it is necessarily weakly homogeneous, as for every element of P we extended an embedding of A_i to an embedding on B_i. From above we know that this is sufficient for homogeneity as C is countable. To see that it has age $\mathcal{K}$, see that for any finitely generated substructure C', its set of generators must be contained in finitely many C_i, and so we can take the largest such $C_{i_{max}}$. Since this was the result of finite amalgamation over element of $\mathbf{K}$, it is an element of $\mathbf{K}$, and since our collection is closed under finite substructures, $C' \in \mathbf{K}$. Similarly, since every $K \in \mathbf{K}$ is in $(K, K) \in P$, then we can see at some finite j the structure K is embedded into C_j. The image of K in C_j then has the isomorphism type of K and is a finitely generated substructure of $\mathcal{C}$, so we have that the age of $\mathcal{C}$ is precisely $\mathbf{K}$. $\square$

2.1.1 Back and Forth

Often times we are unable to prove elementary equivalence or isomorphism between two structures, despite them having similar structures. To resolve this, we look towards a notion of "testing" if two structures are similar.

Definition 2.8. An *Ehrenfeucht-Fraïssé game* between $\mathcal{A}$ and $\mathcal{B}$ of length γ, written as $G_\gamma(\mathcal{A}, \mathcal{B})$ is a method of constructing a partial embedding f between two models, $\mathcal{A}$ and $\mathcal{B}$ where the domain or range of f is expanded at every step. It is played as follows:

- Player 1 selects some element, either $a \in A$ or $b \in B$.
- Player 2 selects some element in the other set, such that if Player 1 selects some $a \in A$ then Player 2 selects some $b \in B$ and vice versa.

- The domain of f is expanded to include a and $f(a) := b$.

This process is repeated for a number of turns given by a cardinal γ. Player 2 wins if at the end of the game f is a *partial embedding*, or that f is a bijection that preserves all functions and relations of the language $\mathcal{L}$. Player 1 wins if f fails to be a partial embedding.

It is clear there will be times when Player 2 can win, such as if the two structures are isomorphic. To formalize this notion of Player 2 being able to win regardless of what Player 1 chooses, we introduce some notions.

Definition 2.9. The *state* of the game is a tuple consisting of all moves made up to a given point, such as $(a_1, b_1, b_2, a_2, b_3, a_3)$. The *state space* of the game is the collection of all possible states.

With this, we can now define strategies in the game.

Definition 2.10. A *strategy* is a function τ from the state space that outputs a valid move. For our game, a strategy for Player 1 may output any move whereas a strategy for Player 2 outputs an element from A if the last move was an element from B, and vice versa. A *winning strategy* for Player i is a strategy τ such that if Player i follows τ, then no matter the choice of the other player, Player i is guaranteed to win.

We can see that in the case of $\mathcal{A}$ and $\mathcal{B}$ being isomorphic, then Player 2 has a winning strategy, namely to follow the isomorphism. If in an Ehrenfeucht-Fraïssé game of length γ there exists some winning strategy for Player 2, then we write that $\mathcal{A} \sim_\gamma \mathcal{B}$. We now show some results on this equivalence.

Proposition 2.10.1. *If $\mathcal{A} \cong \mathcal{B}$, then $\mathcal{A} \sim_\gamma \mathcal{B}$ for all ordinals γ.*

Proof. Assume there is some isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$. Then let τ be the strategy such that if Player 1 picks $a \in A$, then Player 2 picks $\varphi(a)$, and if Player 1 picks $b \in B$, then Player 2 picks $\varphi^{-1}(b)$. The constructed f will then be φ restricted to some domain, which will clearly be a partial embedding. $\square$

Proposition 2.10.2. *$\sim_\gamma$ is an equivalence relation.*

Proof. To see reflexivity, note $\mathcal{A} \cong \mathcal{A}$ so by above $\mathcal{A} \sim_\gamma \mathcal{A}$. Symmetry follows from the fact that $G_\gamma(\mathcal{A}, \mathcal{B}) = G_\gamma(\mathcal{B}, \mathcal{A})$, so a winning strategy in one gives a winning strategy in the other.

Transitivity requires more care. Assume $\mathcal{A} \sim_\gamma \mathcal{B}$ and $\mathcal{B} \sim_\gamma \mathcal{C}$. Let τ_1 be the winning strategy for Player 2 in $G_\gamma(\mathcal{A}, \mathcal{B})$ and τ_2 the winning strategy for Player 2 in $G_\gamma(\mathcal{B}, \mathcal{C})$. We define a strategy τ_3 as follows. If Player 1 selects a, then Player 2 selects $\tau_2(G_{\tau_1(a)})$ where $G_{\tau_1(a)}$ is the game state given by the previous state and move $\tau_1(a)$, and if Player 1 selects c, then Player 2 selects $\tau_1(G_{\tau_2(a)})$. We see then that the generated $f : A \to C$ would be expressible as $f_1 \circ f_2$, where f_i is the function generated by τ_i. Since each f_i preserves the functions and relations, it is clear their composition does as well. $\square$

Proposition 2.10.3. *Two weakly-homogeneous structures with the same age are back-and-forth equivalent.*

Proof. The uniqueness proof from Theorem 2.6 provides the direct tools to construct a winning strategy for Player 2. Let D_{n-1} and C_{n-1} be the elements of D and C already chosen by Player 1 and f_{n-1} the partial isomorphism already constructed, which can all be empty if necessary. If Player 1 chooses some $c_n \in C$, then we can take $C_n = C_{n-1} \cup \{c_n\}$. Player 2 then uses the given amalgamation method to extend the f_{n-1} to f_n to include C_n in the domain of f, and so Player 2 chooses $f_n(c_n)$. Similarly, if Player 1 chooses some d_n, then we define $D_n = D_{n-1} \cup \{d_n\}$ and extend f_{n-1} to f_n to include D_n in the image, allowing Player 2 to choose $f_n^{-1}(d_n)$. □

2.1.2 Strong Amalgamation

It is possible to guarantee further properties of the Fraïssé limit by imposing additional restrictions on the class of structures. We will now explore one such property. By strengthening our amalgamation requirement, we can guarantee that the limit model has "no algebraicity". We formalize this as follows:

Definition 2.11. An element b is *algebraic* over A in a model $\mathcal{M}$ if there exists a formula ψ with parameters in A such that $\mathcal{M} \models \psi(a)$ and $\{x : \mathcal{M} \models \psi(x)\}$ is finite.

With this we can then define a closure operator.

Definition 2.12. In a model $\mathcal{M}$ we define the *algebraic closure* of $A \subseteq M$, denoted $\mathrm{acl}(A)$, as the collection of $x \in M$ such that x is algebraic over A.

Definition 2.13. A structure $\mathcal{M}$ has *no algebraicity* if for every $A \subset M$ we have $\mathrm{acl}(A) = A$.

In order to ensure that our model has no algebraicity, we are required to strengthen our amalgamation requirement to ensure that no additional information appears during the amalgamation of structures. We formalize this as follows.

Definition 2.14. A class $\mathcal{K}$ of models satisfies the *strong amalgamation property* if it satisfies the amalgamation property such that $g_1(B_1) \cap g_2(B_2) = g_i \circ f_i(A)$ for $i = 1, 2$.

Proposition 2.14.1. *If a countable class $\boldsymbol{K}$ satisfying HP and JEP additionally satisfies the strong amalgamation property, then the Fraïssé limit $\mathcal{M}$ has no algebraicity.*

Proof. Since formulas use a finite number of coefficients, it is enough to show for finite $A \subseteq M$ we have $\mathrm{acl}(A) = A$. From here on we will identify such sets with the models they generate. Consider now some $x \in M \setminus A$ and take $B_0 := A \cup \{x\}$. Consider the structure B_1 to be defined as the amalgam of B_0 with B_0 over A, taking f_i to be the inclusion maps. Since $x \notin f_i(A)$, by strong amalgamation there are two images of x in B_1. Consider now the amalgam B_k of B_{k-1} and B_0 over A. Assuming there are k images of x in $k-1$ and 1 image of x in B_0, then by strong amalgamation there are $k+1$ images of x in B_k.

By induction, there are $n+1$ images of x in B_n. Since n is arbitrary, we have that there exist embeddings that contain an arbitrarily large number of distinct elements that x may be sent to under an embedding. By homogeneity, this implies that x has arbitrarily large orbit under an automorphisms σ that fixes A. If x was in a definable finite subset, then the size of the orbit would be bounded by the magnitude of the set. Since x was an arbitrary element in A^c, there is no A-definable finite set outside of A. This gives that $\mathrm{acl}(A) = A$, as needed. □

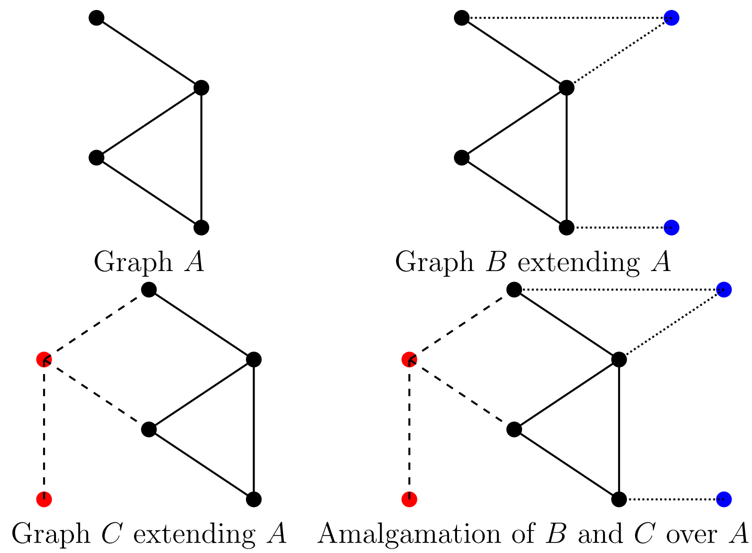

Figure 2.1: The Amalgamation of Two Graphs

2.2 Rado's Graph

We present an example of a Fraïssé limit in graph theory. Recall the theory of graphs, as we defined earlier in the text. Consider the class of all finite graphs $\mathbf{C}$. We can see easily that it satisfies the hereditary property, as any finitely-generated subgraph of a finite graph is necessarily finite.

To show the joint embedding property and amalgamation property, we will give a construction of the amalgamation for which we can take $A = \emptyset$ to obtain the joint embedding construction. Let $\mathcal{A}$ be a finite graph with embeddings f_1, f_2 into finite graphs $\mathcal{B}_1, \mathcal{B}_2$. We now construct $\mathcal{C}$ as follows:

Define the vertices of $\mathcal{C}$ to be given by the set $C = A \cup (B_1 \setminus f_1(A)) \cup (B_2 \setminus f_2(A))$. Then define edges on C to such that for $c_1, c_2 \in C$ we have $E_{\mathcal{C}}(c_1, c_2)$ if and only if one of the following conditions hold:

1. $c_1, c_2 \in A$ and $E_{\mathcal{A}}(c_1, c_2)$.
2. $c_1, c_2 \in B_i \setminus f_i(A)$ and $E_{\mathcal{B}_i}(c_1, c_2)$
3. $c_1 \in B_i \setminus f_i(A)$ and $c_2 \in A$, and $E_{\mathcal{B}_i}(c_1, f_i(c_2))$, or similarly for $c_1 \in A$ and $c_2 \in B_i$.

In this manner we are taking the union of $\mathcal{B}_i$ such the embeddings of $\mathcal{A}$ in each $\mathcal{B}_i$ are identified with one another. We note that by this construction, we can see that the graph itself satisfies the strong amalgamation property, as $g_1(B_1) \cap g_2(B_2) = (g_i \circ f_i)(A)$. We have then that there exists a Fraïssé limit of the class $\mathbf{C}$ that has no algebraicity.

Chapter 3

The Hrushovski Construction

In 1993 Dr. Ehud Hrushovski expanded on the Fraïssé construction, where he weakened the amalgamation to only be necessary for certain inclusions, which were referred to as strong inclusions. Our discussion of this will be brief and primarily expository, though we will provide review of combinatorial geometry and select topics in model theory to understand the significance of the construction.

3.1 Combinatorial Geometry

The Hrushovski construction was created to address Zil'ber's Trichotomy Conjecture, which posited the geometry of certain models could be categorized into three separate cases. To discuss the construction itself, we need to begin by defining a pregeometry.

Definition 3.1. A *pregeometry* (X, cl) is a set X equipped with a function cl, commonly referred to as the *closure*, such that

1. $A \subseteq X \Rightarrow A \subseteq \mathrm{cl}(A)$. (Reflexivity / Extensivity)
2. $A \subseteq B \subseteq X \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$. (Monotonicity)
3. $A \subseteq X \Rightarrow \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. (Transitivity / Idempotent)
4. $a, b \in X$ gives $a \in \mathrm{cl}(A \cup \{b\}) \setminus \mathrm{cl}(A) \Rightarrow b \in \mathrm{cl}(A \cup \{a\})$. (Exchange)
5. $a \in \mathrm{cl}(A) \Rightarrow a \in \mathrm{cl}(A_0)$ for some finite $A_0 \subseteq A$. (Finite Character)

Definition 3.2. We define a *geometry* to be a pregeometry with the additional conditions that

$$\mathrm{cl}(\emptyset) = \emptyset \qquad \text{and} \qquad \mathrm{cl}(\{x\}) = \{x\}$$

We seek to define a geometry on models, so we look towards our previous definitions of closure and see if they form a candidate for a pregeometry. A good initial candidate is that of algebraic closure. We will see that this closure satisfies most of the requirements of a pregeometry.

Proposition 3.2.1. *The algebraic closure acl is reflexive, monotonic, transitive, and of finite character.*

Proof.

- Reflexivity: For any $a \in A$ we see that a is the unique element satisfying the formula $x = a$.
- Monotonicity: Assume $A \subseteq B$ and take some $a \in \operatorname{acl}(A)$ with a satisfying $\psi(x)$. As ψ takes parameters from $A \subseteq B$ then we can take ψ itself as a witness for a being algebraic over B.
- Finite Character: Any formula ψ necessarily uses a finite number of symbols, and so if ψ takes parameters from A it takes parameters from some finite $A_0 \subseteq A$.
- Transitivity: From reflexivity we have one direction of set containment. It will suffice to show that $\operatorname{acl}(\operatorname{acl}(A)) \subseteq \operatorname{acl}(A)$. For this take some $a \in \operatorname{acl}(\operatorname{acl}(A))$ that satisfies ψ such that $\exists_n x : \psi(x)$, where $\exists_n$ is shorthand for "exists exactly n elements".

  By finite character we know a is algebraic over some finite set $\{a_1, ..., a_r\} = A_0 \subseteq \operatorname{acl}(A)$ where we may write $\psi(a_1, ..., a_r, x)$. Let a_i be algebraic over A with $\mathcal{M} \models \varphi_i(a_i)$. Consider the formula given by

  $$\psi'(x) = \exists x_1 \exists x_2 ... \exists x_r \left(\bigwedge_{i=1}^{r} \varphi_i(x_i) \right) \wedge \psi(x_1, x_2, ..., x_r, x) \wedge \exists_n y(\psi(x_1, x_2, ..., x_r, y))$$

  We can see that ψ' is defined over A, and that $\mathcal{M} \models \psi'(a)$. To see that it is finitely satisfied, let $n_i = |\{x : \mathcal{M} \models \psi_i(x)\}|$. We see that there are $\prod_{i=1}^{n} n_i$ possible tuples $(x_1, ..., x_r)$ that satisfy the first term, and each tuple may define at most n elements by the last term, so at most $n \prod_{i=1}^{r} n_i$ elements satisfy $\psi'(x)$.

$\square$

One may note that we did not give a proof of acl satisfying the exchange condition, and that is because it does not. Rather, we must impose an additional condition on the underlying model in order for algebraic closure to satisfy the condition.

Definition 3.3. A structure $\mathcal{M}$ is *strongly minimal* if every definable subset of M is finite or cofinite. A set A is *strongly minimal* if every set definable over A is finite or cofinite.

We can see that this condition is sufficient for algebraic closure to form a pregeometry.

Lemma 3.4. A strongly minimal structure, equipped with algebraic closure, satisfies the exchange property.

Proof. Let $\mathcal{M}$ be a strongly minimal model with $A \subseteq M$ and $a, b \in M$. Assume now that $a \in \operatorname{acl}(A \cup \{b\}) \setminus \operatorname{acl}(A)$. Additionally, assume up to contradiction that $b \notin \operatorname{acl}(A \cup \{a\})$. As $a \in \mathit{acl}(A \cup \{b\})$, it follows that there is some formula $\psi(b, x)$ with parameters from A such that

$$\mathcal{M} \models \psi(b, a) \wedge \exists_n v\, \psi(b, v)$$

Consider now the formula $\psi(x,a) \wedge \exists_n v\, \psi(x,v)$. Since b is not algebraic over $A \cup \{a\}$, we must have that an infinite number of $m \in M$ satisfying this equation. As $\mathcal{M}$ is strongly minimal, then this set B is cofinite, such that for all $b' \in B$ we have

$$\mathcal{M} \models \psi(b', a) \wedge \exists_n v\, \psi(b', v)$$

Since $a \notin \mathrm{acl}(A)$ we have that there is an infinite set A' such that for all $a' \in A'$

$$\mathcal{M} \models \exists_{>n} b' \, (\psi(b', a') \wedge \exists_n v\, \psi(b', v))$$

Choose $a_0, a_1, ..., a_n \in A'$. Then since B is cofinite there exists some b_0 such that

$$\mathcal{M} \models \bigwedge_{i=0}^{n} \psi(b_0, a') \wedge \exists_n v\, \psi(b_0, v)$$

A contradiction. $\square$

3.2 Zil'ber's Conjecture

In 1982 Zil'ber noted that for strongly minimal uncountably categorical structures, it appeared that the geometry formed was always one of three types, up to a geometric equivalence. To define such an equivalence, we need to define the localization of a geometry.

Definition 3.5. Given a pregeometry (X, cl), we define the localization of the pregeometry at D to be $X_D = X$ and $\mathrm{cl}_D(B) = \mathrm{cl}_D(D \cup B)$.

It is easy to see that this is a pregeometry, with all but exchange being trivial. To see the exchange property, see that if $a \in \mathrm{cl}_D(A \cup \{b\}) \setminus \mathrm{cl}_D(A)$ then we have

$$a \in \mathrm{cl}((D \cup A) \cup \{b\}) \setminus \mathrm{cl}(D \cup A)$$

And applying exchange on cl gives the required result. In order to ensure that this forms a geometry as well, we modify the set and closure slightly. We define $X^{\sim} = (X - \mathrm{cl}(\emptyset))/\sim$, where $x, y \in X$ satisfy $x \sim y$ iff $\mathrm{cl}(x) = \mathrm{cl}(y)$. Then $\mathrm{cl}^{\sim}([A]) = [\mathrm{cl}(A)]$. We can see by the exchange property this forms a geometry. This allows us to then define locally isomorphic geometries.

Definition 3.6. Two geometries (X, cl_X) and (Y, cl_Y) are locally isomorphic if there exist finite sets X_0 and Y_0 such that $(X_{X_0}, \mathrm{cl}_{X_0})^{\sim}$ is isomorphic to $(Y_{Y_0}, \mathrm{cl}_{Y_0})^{\sim}$.

In essence the two geometries are isomorphic, up to the inclusions of a finite number of parameters in each set. With this, we can define now geometric equivalence.

Definition 3.7. Two structures $\mathcal{M}$ and $\mathcal{N}$ are geometrically equivalent if their induced geometries are locally isomorphic.

Until this point, there appeared to only be three strongly minimal structures, up to geometric equivalence. Zil'ber formalized this observation in the following conjecture: Any strongly minimal structure, when equipped with acl as a closure operation, was one of the following types, up to geometric equivalence.

1. The structure X has degenerate geometry, where $\mathrm{cl}(A) = A$ for all $A \subseteq X$.
2. The structure is an infinite vector space $(X, +, \cdot, \alpha)_{\alpha \in F}$ over some fixed division ring F
3. The structure is an algebraically closed field $(X, +, \cdot, c)_{c \in F_0}$ with certain elements of a subfield F_0 distinguished.

We will provide now some additional details in seeing that the structures we have described are strongly minimal.

1. Take X to be an infinite set without structure. Then the only A-definable elements are A, so we have that $\mathrm{cl}(A) = A$.
2. We use without proof that the theory of infinite vector spaces over a field allows quantifier elimination, that is every formula φ is equivalent to some φ' without quantifiers. All such formulas in one variable $\phi(x)$ are equivalent to a boolean combination of formulas like $rx = a$ for $r \in K$ and $a \in V$. Since these define either a singleton or all of X, the structure is strongly minimal.
3. Similar to above, the theory of algebraically closed fields allows quantifier elimination. This allows us to see that formulas are equivalent to boolean combinations of polynomials $p(x) = 0$, which only is able to define finite solution sets or the entire space, hence algebraically closed fields are strongly minimal.

Frequently the first two cases are linked together, and the trichotomy conjecture states that any "non-locally modular" strongly minimal model is geometrically equivalent to an algebraically closed field. To understand this, we introduce a notion of dimension.

Definition 3.8. Given a structure X we can define a dimension function dim as follows: Given a subset D we define the dimension $\dim(D)$ to be the cardinality of the largest set $A \subseteq D$ such that for each $a \in A$ we have $a \notin \mathrm{cl}(A \setminus \{a\})$. That is, the dimension of D is the cardinality the largest independent subset of D.

We can immediately see some properties of this dimension function, namely $\dim(\emptyset) = 0$, $\dim(\{a\}) \leq 1$, and that for $A \subset B$ we have $\dim(A) \leq \dim(B)$. When considering only finite sets, we additionally have

$$\dim(A \cup B) + \dim(A \cap B) \leq \dim(A) + \dim(B)$$

This follows from a major result from matroid theory that states rank satisfies the above inequality, and a pregeometry forms a finitary matroid, where dimension coincides with rank. Because dim satisfies this inequality, we call dim submodular, for reasons that will become clear.

It is not immediately clear that this function is well defined, since we have not shown maximally independent sets A_1 and A_2 in D have the same cardinality. We will defer the reader to [Hod93] for more information on this. For the geometries described above, the dimension function gives the cardinality of the set D in the degenerate case, gives the dimension of the smallest subspace containing D for linear case, and gives the transcendence degree of D over F_0 for the case of the algebraically closed field.

With this definition of dimension we can define modularity.

Definition 3.9. A geometry (X, cl) is *modular* if for all closed $X_1, X_2 \subset X$ the geometry satisfies

$$\dim(X_1 \cup X_2) + \dim(X_1 \cap X_2) = \dim(X_1) + \dim(X_2)$$

A geometry is *locally modular* if it satisfies the above condition when $\dim(X_1 \cap X_2) > 0$. Equivalently, a geometry is locally modular iff the localization at a singleton $\mathrm{cl}_{\{x\}}$ is modular.

Returning to the structures we have been discussing, we check to see which are modular.

1. Let (X, cl) be degenerate. From inclusion and exclusion we trivially have

   $$\dim(X_1 \cup X_2) + \dim(X_1 \cap X_2) = |X_1 \cup X_2| + |X_1 \cap X_2| = |X_1| + |X_2| = \dim(X_1) + \dim(X_2)$$

2. Let (X, cl) be that of an infinite vector space. Similar to above, from linear algebra we know that

   $$\dim(X_1 \cup X_2) + \dim(X_1 \cap X_2) = \dim(X_1 + X_2) + \dim(X_1 \cap X_2) = \dim(X_1) + \dim(X_2)$$

   so the pregeometry of infinite vector spaces is modular.

3. Let (X, cl) be that of an algebraically closed field of transcendence degree ≥ 4. Take e, a, b, x independent over F_0. Now take the sets

   $$X_1 = \mathrm{cl}(F_0(e, a, b)) \quad \text{and} \quad X_2 = \mathrm{cl}(F_0(e, x, ax + b))$$

   See that $\dim(X_1 \cup X_2) = \dim(F_0(e, a, x, b)) = 4$, and $\dim(X_1 \cap X_2) = \dim(F_0(e)) = 1$. Since

   $$5 = \dim(X_1 \cup X_2) + \dim(X_1 \cap X_2) < \dim(X_1) + \dim(X_2) = 6$$

   We see that the geometry of large enough fields are not modular.

We can see that geometries of the first two types are modular, whereas the third is not. Because of this, Zil'ber's conjecture was frequently described as that all strongly minimal models with a non-locally modular geometry are geometrically equivalent to a field. To disprove Zil'ber's conjecture, Hrushovski designed a strongly minimal model whose geometry was not modular but did not have group structure, let alone field structure.

3.3 The Hrushovski Construction

The construction used by Hrushovski to construct his strongly minimal set is similar to that of Fraïssé, but weakens amalgamation. Instead, he introduces a notion of a closed substructure.

Notation. We write $\mathcal{A} \leq \mathcal{B}$, or $\mathcal{A}$ is *closed* in B. We require the relation $\leq$ to be contained in the relation $\subseteq$, while also being transitive and preserved under intersections, so that $A_1, A_2 \leq B$ implies $A_1 \cap A_2 \leq B$. We will also say that if $A \leq B$ that A is *strongly included* in B.

With this we only require amalgamation on strong inclusions, that is for $\mathcal{A}$ such that $\mathcal{A} \leq \mathcal{B}_i$ for $i = 1, 2$ with strong embeddings f_1, f_2, there exists $\mathcal{C}$ such that $\mathcal{B}_i \leq \mathcal{C}$ and for strong embeddings $g_i : \mathcal{B}_i \to \mathcal{C}$ we have $g_1 \circ f_1 = g_2 \circ f_2$. With this, our method of construction of a unique limit, as described in the Fraïssé construction section, is still possible, where we now only amalgamate over strong inclusions. We will call it the *Hrushovski limit* for brevity, though this terminology is not standard. This does change properties of the limit, however, and instead of homogeneity we instead have a new property, which we refer to as richness.

Definition 3.10. A structure $\mathcal{M}$ is *rich* over a class $\mathbf{K}$ if for each $A, B \in \mathbf{K}$ we have $A \leq \mathcal{M}$ and that there is a an embedding $B \to \mathcal{M}$ where the image of B is closed and the embedding is the identity on A.

The proof of the existence and uniqueness of the Hrushovski limit is the same method as that of the Fraïssé limit, and similarly the proof that the model is rich is similar to that of the proof homogeneity. Because of the similarity to previous topics, we will omit the details of the construction, and instead focus on the construct itself, and the geometries it presented.

3.3.1 The Strongly Minimal Set

To begin with the construction, we consider a language $\mathcal{L}$ equipped with only a ternary relation R, and we only consider models where R is totally irreflexive and totally symmetric. Because our language contains no function symbols, the finitely generated substructures considered will be precisely the finite substructures.

To construct the strongly minimal set, Hrushovski defined δ as $\delta(A) = |A| - |R(A)|$, where $R(A)$ is the set of unordered triples in A that satisfy R. Strong inclusion, written as $A \leq M$ was then defined for finite A as $A \leq M$ if $\delta(A) \leq \delta(B)$ for all finite B such that $A \subseteq B \subseteq M$.

Proposition 3.10.1. *The function δ defines a strong inclusion where $A \leq B$ iff $\delta(A) \leq \delta(C)$ for all finite C such that $A \subseteq C \subseteq B$.*

Proof. An important property of this is that the function is *submodular*, or satisfies

$$\delta(A_1 \cup A_2) + \delta(A_1 \cap A_2) \leq \delta(A_1) + \delta(A_2)$$

To see this, note that since $|A_1 \cup A_2| + |A_1 \cap A_2| = |A_1| + |A_2|$, then it is sufficient to only consider the number of triples of R each side contains. Assume a triple appears in exactly one of A_1 or A_2. Then it appears once on the righthand side, and once on the left in $A_1 \cup A_2$.

If a triple appears in both A_1 and A_2 then it appears in both $A_1 \cup A_2$ and $A_1 \cap A_2$. The inequality follows immediately after this.

We will now show that this definition does satisfy our conditions of strong inclusion. We will first show that $A \leq B$ implies $A \cap D \leq B \cap D$ for all $D \subset M$. For this, note $A \leq B$ and only if $\delta(A \cup D') - \delta(A) \geq 0$ for all $D' \subseteq B$. By the above property, $A \leq B$ if and only if for all D' we have

$$\delta(D') - \delta(A \cap D') \geq \delta(A \cup D') - \delta(A) \geq 0$$

Now assume that $A \leq B$. We seek to show that $A \cap D \leq B \cap D$, or equivalently

$$\delta(E) - \delta((A \cap C) \cap E) \geq 0$$

For all $E \subseteq B \cap D$. Note simply

$$\delta(E \cap D) - \delta(A \cap D \cap E) = \delta(E) - \delta(A \cap E) \geq 0$$

As $E \subseteq B$, so $A \leq B$ implies $A \cap D \leq B \cap D$, as desired.

We will now verify that this definition of $\leq$ satisfies our conditions, namely transitivity and closed under intersections. To see transitivity, let $A \leq B \leq C$. Consider now an arbitrary $D \subseteq C$, then we have $A \cap D \leq B \cap D \leq D$, so $\delta(A \cap D) \leq \delta(D)$, which by our first step above gives $A \leq C$. To see preservation under intersection, let $A_1, A_2 \leq B$. Since $A_1 \leq B$ then $A_1 \cap A_2 \leq A_2$, and since $A_2 \leq C$ then $A_1 \cap A_2 \leq C$. $\square$

From here, we define a new class $\mathcal{C}$ of structures to be the collection $\{\mathcal{M} : 0 \leq \mathcal{M}\}$. We then take $\mathcal{C}_{fin}$, which contains only the finite models in $\mathcal{C}$. This will not actually be the class used to construct the strongly minimal model, however. We first need to define a new function μ. For this we need some a new notion.

Definition 3.11. A structure A in our language is *simply algebraic* over B if $B \leq A \cup B$, $A \cap B = \emptyset$, and that A is the smallest nonempty subset A' of A such that $\delta(A' \cup B) = \delta(B)$. A is *minimally simply algebraic* over B if B is the smallest set that A is simply algebraic over.

We need one more concept to define our function μ, which will be that of atomic type, which is closely and clearly connected to our old notion of type.

Definition 3.12. An *atomic n-type* is a consistent collection of atomic formulas and negations of atomic formulas over the same set of n variables that is closed under deduction. The atomic type of a structure is the collection of atomic formulas and negations of atomic formulas true of a given structure.

Since $A = \mathrm{cl}(\overline{a})$ for some finite tuple $\overline{a}$, we take the atomic type of A to be the atomic type of $\overline{a}$. From here we let the atomic type of $(A_i, B) = (A_i/B)$ be defined as the atomic sentences of $A = \mathrm{cl}(\overline{a})$ with parameters in B. We can now define our function μ as any integer-valued function defined on the atomic type of pairs of (A, B) where A is nontrivial and minimally simply algebraic over B, such that $\mu(A, B) \geq \delta(B)$. This restriction is stronger than needed, but for now it suffices. We can now define our class $\mathbf{C}$ that we will form the Hrushovski construction from.

We define $\mathbf{C}_\mu$ to be the collection of finite $\mathcal{L}$-structures $\mathcal{M}$ that satisfy the following properties.

1. $\emptyset \leq \mathcal{M}$.
2. Take pairwise disjoint subsets B, A_i for $i \in [n]$ of $\mathcal{M}$ with A_i nontrivial such that the atomic type of (A_i, B) is constant with respect to i. Assume A_i minimally simply algebraic over B, then $n \leq \mu(A_i, B)$.

This second condition, while austere, ensures not only that the given class $\mathbf{C}_\mu$ has HP, JEP, and strong amalgamation, it additionally ensures that the geometries of of the limit structures structures of $\mathbf{C}_\mu$ are not isomorphic. The proof of these two facts is highly nontrivial, and is the content of [Hru92]. Instead of repeating Hrushovski's work here, we will focus on the results, particularly the discussion of the geometry of $\mathcal{M}^\mu$, the Hrushovski limit of the class $\mathbf{C}_\mu$. We will take not only that $\mathcal{M}^\mu$ is the limit, but also immediate results of $\mathcal{M}^\mu$ being both saturated and strongly minimal.

We will now explore why this is a counterexample to Zil'ber's Conjecture. For this we will first introduce a new geometry.

Definition 3.13. Let $(E_i)_{i \in I}$ be a finite number of finite dimensional closed sets in a geometry (X, cl). Let also

$$E_\emptyset = \bigcup E_i \text{ and } E_S = \bigcap_{i \in S} E_i \text{ for } S \subset I$$

We say a geometry is *flat* if for all such $(E_i)_{i \in I}$ we have

$$\sum_{S \subseteq I} (-1)^{|S|} \dim(E_s) \leq 0$$

With this, we can now show that $\mathcal{M}^\mu$ does not have a geometry of the three categories above.

Proposition 3.13.1. *$\mathcal{M}^\mu$ is not locally modular.*

Proof. One fact we will use is that in a modular structure, the union of two closed spaces is closed. This proof requires a deeper analysis of geometries than present here, but [Zie13] provides a good explanation. We now define a structure M_{nm} in the language $\mathcal{L}$ with underlying set $\{a_1, a_2, b_1, b_2, c\}$ and sets $\{\{a_1, a_2, c\}, \{b_1, b_2, c\}\}$ satisfying R. We see that $A = \{a_1, b_1\}$ and $B = \{a_2, b_2\}$ are closed, as $\delta(A)$ is minimal and same for $\delta(B)$, but their union $A \cup B = \{a_1, a_2, b_1, b_2\}$ is not closed, as $\delta(M_{nm}) < \delta(A \cup B)$.

To see that M_{nm} is contained in $\mathcal{M}^\mu$ for any μ, we first note that by construction $0 \leq M_{nm}$, as at least 2 distinct elements are needed per relation, so $\delta(A) \geq 0$ for all $A \subseteq M_{nm}$. To see that M_{nm} satisfies the second condition, note c is the only element appearing in multiple relations, and so δ is increasing with regard to set inclusion on all sets that exclude c. We also note the inclusion of c only decreases δ if all other elements are present, so the only set not strongly included in M_{nm} is $A \cup B$ as described above. We can similarly see that all other sets are closed in all of their supersets.

From here, we will categorize all pairs (A, B) where A is minimally simply algebraic over B. We have that $A = \{x_1\}$ is minimially simply algebraic over $B = \{x_2, x_3\}$ where $\{x_1, x_2, x_3\} \in R$. Since this implies there is at most A_i such that (A_i, B). Since $\mu(A_i, B) \geq \delta(B) = 2 > 1 = (A_i, B)$, then we see that for all μ we have M_{nm} is contained in $\mathbf{C}_\mu$, hence is strongly embedded in $\mathcal{M}_\mu$. To see $\mathcal{M}_\mu$ cannot be locally modular, simply consider the structure $M_{nm} \cup \{d\}$, $A \cup \{d\}$ and $B \cup \{d\}$ for some d independent of M_{nm}. □

According to the conjecture, this would be sufficient to imply that M^μ has a geometry isomorphic to that of an algebraically closed field, but we will show that it does not even contain a group structure, let alone that of a field.

Proposition 3.13.2. *A strongly minimal set with a flat geometry does not have an infinite group structure.*

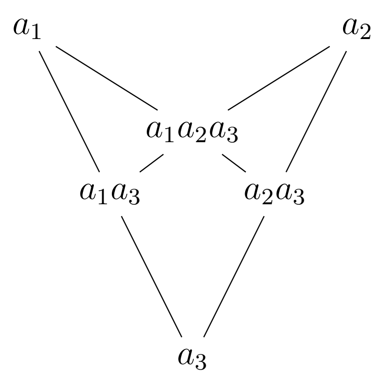

Figure 3.1: Zil'ber Configuration

Proof. Let (X, cl) be a geometry. Assume there exists a group $G \subseteq X$ of dimension g, where the dimension is induced from the closure operator. Choose now the independent elements $a_1, a_2, a_3 \in G$. Consider the configuration of the elements as in Figure 3.1, which is commonly referred to as the "Zil'ber Configuration", and let E_1, E_2, E_3, E_4 be the four lines of the configuration given as $E_1 = \mathrm{cl}\{a_1, a_1a_3, a_3\}$, $E_2 = \mathrm{cl}\{a_2, a_2a_3, a_3\}$, $E_3 = \mathrm{cl}\{a_1, a_1a_2a_3, a_2a_3\}$, and $E_4 = \mathrm{cl}\{a_2, a_1a_2a_3, a_1a_3\}$.

With the definitions above, we see that the dimension of each E_s will be the number of independent elements times times g, as an independent generating set would exist for each space spanned by an independent element. Thsi gives that $\dim(E_\emptyset) = 3g$, $\dim(E_i) = 2g$, $\dim(E_{i,j}) = g$, and $\dim(E_{i,j,k}) = 0$, as We then know by the flatness of (X, cl) that

$$\sum_{S \subseteq I} (-1)^{|S|} \dim(E_s) = (-1)^0(3g) + 4(-1)^1(2g) + 6(-1)^2(g) = 3g - 8g + 6g = g \leq 0$$

And so the dimension of G is zero, and hence must be finite since the structure is strongly minimal. □

It follows that a flat geometry is not locally modular, and is not admitting a group structure. All that is left to see is that the strongly minimal M^μ is flat.

Proposition 3.13.3. *$\mathcal{M}^\mu$ is flat.*

Proof. Let I be a finite indexing set and take closed finite dimensional $(E_i)_{i\in I}$. Choose $G_i \leq \mathcal{M}_\mu$ such that $\mathrm{cl}(G_i) = E_i$, so that $\dim(E_i) = \delta(G_i)$. We compute then

$$\sum_{S\subseteq I}(-1)^{|S|}\dim(E_S) = \sum_{S\subseteq I}(-1)^{|S|}|G_S| - \sum_{S\subseteq I}(-1)^{|S|}|R(G_S)|$$

By inclusion-exclusion we have that the first term is zero, and similarly by the same process on the sets $R(G_i)$ we have that the second term is

$$\sum_{S\subseteq I}(-1)^{|S|}|R(G_s)| = \left|R\left(\bigcup_{i\in I} G_i\right)\right| - \left|\bigcup_{i\in I} R(G_i)\right|$$

Giving that the sum is $|\bigcup R(G_i)| - |R(\bigcup G_i)|$. This is necessarily nonpositive, as the set on the left is contained in the the set on the right. □

With this, we have that $\mathcal{M}^\mu$ is both non-locally modular and has geometry not that of a field, in contradiction to Zil'ber's Conjecture.

3.4 Subsequent Results

This first blow to Zil'ber's Conjecture was certainly not the last, and we will take this opportunity to discuss is a relaxed manner results that followed.

The initial construction of the strongly minimal set did not provide a single counterexample, but rather Hrushovski showed that by varying μ one could construct a continuum of these flat geometries. This in itself seemed to be promising for the classification of geometries, as taken together they could be considered to be of a fourth type, which only slightly expanded the trichotomy. It was even noted in the original paper that the geometries found in the $\mathcal{M}^\mu$ had a defining property: CM triviality. Hrushovski described this property as the geometry forbidding any rich structure, such that the intersection of two distinct lines is finite, the intersection of two distinct planes is contained in the union of finitely many lines, and that infinitely many lines passed through any given pair of point and plane.

Sadly, this fourth category of CM-trivial geometries was not sufficient, and Hrushovski again showed that the geometries were far more diverse than the classification. In [Hru93] Hrushovski presented a method to fuse strongly minimal theories, and gave as an example a structure that interpreted two algebraically closed fields of differing characteristic. Not only did this structure itself prevent classification, the method of fusion was sufficiently strong that it has been able to provide counterexamples to most classification conjectures made since then. Some recent work [Wag10] has explored these fusions and their extent to generate new structures, but no lasting classification has been made.

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