Maximal Comparable and Incomparable Sets in Boolean Algebras

by

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Thesis directed by Prof. J. Donald Monk

We consider the minimal possible sizes of both maximal comparable and maximal incomparable subsets of Boolean algebras. Comparability is given upper and lower bounds for familiar quotients of powerset algebras. The main upper bound is proved using a construction reminiscent of the construction of the reals from Dedekind cuts. Incomparability is placed in relation to the types of dense sets occurring, resulting in several upper bounds. Specifically, the existence of a countable dense set implies the existence of a countable maximal incomparable set, the latter being constructed using a game. A weaker result is proved for uncountable density with the aid of the diamond principle leaving open the question of whether the bound holds in ZFC.

Dedication

For my mother and father.

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Chapter 1

Introduction

This thesis is concerned with maximal comparable and incomparable subsets of Boolean algebras, whose precise definitions are given in the corresponding chapters. Here maximal means maximal with respect to set inclusion. The existence of maximal sets follows from an application of Zorn's lemma which is familiar to any student of algebra. However, one could imagine, for example, a maximal comparable set being built up piecemeal: begin with a singleton set, pass to an unordered pair set by adding an element comparable to the member of the singleton, again pass to a larger set by adding an element comparable to both members of the unordered pair, and so forth. This process may continue into the transfinite with myriad choices to be made at each stage. By varying the sequence of choices one sees that the collection of all maximal comparable sets is populated by diverse members. One method for constructing a maximal set may terminate sooner than another and in this way maximal sets can have different cardinalities.

For each fixed algebra, the collection of these cardinalities forms a set known as the spectrum, with separate spectra for comparable sets and incomparable sets. As such, this collection has a supremum and a minimum. The supremum has so far received more study, see [7]. The present topic is the minimum. We restrict attention only to infinite comparable and incomparable sets so that the minimum is always an infinite cardinal. These minima are denoted Length_{mm} and Inc_{mm}. Here the subscript denotes min-max as in minimal size of a maximal set.

Length_{mm} and Inc_{mm} are cardinal valued operations defined on the class of all Boolean algebras and are manifestly constant within isomorphism classes. For this reason, they are known as cardinal invariants. The general study of such operations on the class of Boolean algebras is taken in [7]. Of course, invariants are widely studied in other areas, such as in topology where their role is often to distinguish non-homeomorphic spaces. They can fruitfully play an analogous role in Boolean algebra as well but perhaps it is prudent to mention another reason why they are of interest.

If a Boolean algebra is given abstractly by writing $\langle A, +, \cdot, -, 0, 1\rangle$, one can always realize this algebra concretely as a subalgebra of a powerset algebra [5, Cor 2.9]. This means there is some set X and an injection $f : A \to \mathcal{P}(X)$ such that for all $a, b \in A$, $f(a + b) = f(a) \cup f(b)$, $f(a \cdot b) = f(a) \cap f(b)$, $f(-a) = X \setminus f(a)$, $f(0) = \emptyset$ and $f(1) = X$. Conversely, any subset of $\mathcal{P}(X)$ closed under pairwise unions, pairwise intersections, relative complements and containing $\emptyset$ and X is a Boolean algebra (this list of requirements is not meant to be minimal). Although there are many other possible interpretations and definitions of the class of Boolean algebras, this shows that it is possible to conceive of the study of Boolean algebras as the study of subsets of powersets which are closed under the common set theoretic operations of finite union, finite intersection and relative complement.

Beginning with proof of the independence of the continuum hypothesis, it became clear that much of the incompleteness of ZFC was accounted for by the powerset operation. The powerset axiom, by asserting that the collection of all subsets of a given set is itself a set in the same sense as the original set seems to provide an ambiguity. What is the cardinality of such a powerset? What sizes are possible for maximal almost disjoint sets? What types of gaps occur under the order $\subseteq^*$? Questions about subsets of a powerset with this property or that and their investigation often find the boundary of ZFC.

It is unsurprising then that Boolean algebras provide both an occasion and a framework for studying such questions and this probably accounts for much of the interest in cardinal invariants. Some invariants are truly invariant in that they are calculable in ZFC, some may vary only insomuch as cardinal arithmetic is perturbed by additions to ZFC and others may vary even as cardinal arithmetic remains fixed. In what follows, the invariants under examination are mostly calculated precisely in ZFC or reduced to cardinal arithmetic. However, some calculations require hypothesis about other invariants (e.g. density and saturation), some leave gaps between lower and upper bounds and one is calculated using the diamond principle. All of these point to further lines of investigation as regards maximal comparable and maximal incomparable sets.

The remainder of this section is a short survey of the most important new results to be proved. In some cases, what is actually proved is more general than what is recorded here. The more general formulation can of course be found in the relevant chapter.

Theorem 2.1.1. *If* $\lambda \leq \kappa$*,* $\mathrm{cf}(\lambda) \leq \mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\lambda})$*.*

Theorem 2.1.3. *If* A *is atomless and satisfies the countable separation property then* $\mathfrak{c} \leq \mathrm{Length}_{mm}(A)$*.*

Theorem 2.2.16. *Let* $\mu < \mathrm{cf}(\kappa)$ *satisfy* $\mathrm{cf}(\kappa) \leq 2^{\mu} \leq \kappa$*. Then* $\mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}) \leq 2^{\mu}$*.*

Corollary 2.1.5. $\mathrm{Length}_{mm}(\mathcal{P}(\omega_1)/\mathrm{Fin}(\omega_1)) = \mathfrak{c}$.

Theorem 3.1.8. *For any infinite cardinal* κ*,* $\mathrm{Inc}_{mm}(\mathrm{FinCof}(\kappa)) = \aleph_0$*.*

Theorem 3.1.10. *There is an atomic Boolean algebra with no countably infinite maximal incomparable subset.*

Corollary 3.2.9. *If* A *is an infinite Boolean algebra with countable density, then* $\mathrm{Inc}_{mm}(A) = \aleph_0$*.*

Corollary 3.2.11. *Let* S *be an infinite tree of height at most* ω *and* A *be any Boolean algebra densely embedding* $\mathrm{TreeAlg}(S)$*. Then* $\mathrm{Inc}_{mm}(A) \leq |S|$*.*

Corollary 3.3.9. *Let* κ *be a regular uncountable cardinal. Assume* $\diamondsuit_{\kappa}$*. Let* A *be a* κ*-complete Boolean algebra with density at most* κ*. Then* $\mathrm{Inc}_{mm}(A) \leq \kappa$*.*

Chapter 2

Comparability

Our work begins with a study of comparability.

Definition. *Let A be a Boolean algebra. A subset $L \subseteq A$ is called a* ***chain*** *if it is linearly ordered by the order on A and L is called* ***maximal comparable*** *if L is maximal with respect to inclusion among all infinite chains contained in A. If L is a chain and $X \subseteq L$, define $\sup_L(X)$ and $\inf_L(X)$ to be the least upper bound and greatest lower bound of X, respectively, if such things exist.*

There is no necessity that $\sup_L(X) = \sum X$ except if L is maximal.

Lemma 2.0.1. *Let A be a Boolean algebra, $L \subseteq A$ be a maximal comparable set and $X \subseteq L$. If $\sum X$ exists then $\sum X \in L$ and $\sum X = \sup_L(X)$.*

Proof. Since $\sum X$ is the least upper bound of X in A and L inherits its order from A, if $\sum X \in L$ then $\sup_L(X) = \sum X$. So it is only necessary to prove $\sum X \in L$.

Suppose $\sum X \notin L$. A contradiction is reached by showing that every member of L is comparable to $\sum X$ because then $L \cup \{\sum X\}$ is a chain of which L is a proper subset violating maximality. So take $l \in L$. Since $X \subseteq L$, l is comparable to every member of X. If $l \leq x$ for some $x \in X$ then $l \leq \sum X$. If there is no such x then $l \geq x$ for every $x \in X$ which shows that $\sum X \leq l$. Thus, $\sum X \in L$. □

Definition. *Let A be a Boolean algebra and $L \subseteq A$ be a chain. A pair $\langle X_0, X_1 \rangle$ is called a* ***cut*** *of L if $L = X_0 \cup X_1$ and for all $\langle x_0, x_1 \rangle \in X_0 \times X_1$, $x_0 < x_1$.*

Thus, if $c \in B \setminus L$ is comparable to every member of L then

$$\langle \{l \in L : l < c\}, \{l \in L : c < l\} \rangle$$

is a cut of L.

Lemma 2.0.2. *Let A be a Boolean algebra and $L \subseteq A$ be a chain. Assume that for every cut $\langle X_0, X_1 \rangle$ of L, $\sum X_0$ and $\prod X_1$ both exist, are equal and are in L. Then L is maximal.*

Proof. Assume L has the prescribed property and yet is not maximal. Then $L \subsetneq L'$ for some chain $L' \subseteq A$. Take $c \in L' \setminus L$ and set $X_0 = \{l \in L : l < c\}$ and $X_1 = \{l \in L : c < l\}$. Then c is comparable to every member of L which ensures $\langle X_0, X_1 \rangle$ is a cut of L. By assumption, $\sum X_0$ and $\prod X_1$ exist, are equal and are in L. By definition c is an upper bound for X_0 so $\sum X_0 \leq c$. But $\sum X_0 \in L$ and $c \notin L$ so $\sum X_0 < c$. Similarly, $c < \prod X_1$. Then $\sum X_0 < c < \prod X_1$ which violates the fact that $\sum X_0 = \prod X_1$. □

Lemma 2.0.3. *Let A be a Boolean algebra, $L \subseteq A$ be a chain and $Y \subseteq X \subseteq L$. Assume $\sum Y$, $\sup_L(X)$ and $\sup_L(Y)$ all exist and are equal. Then $\sum X$ exists and equals $\sum Y$. The same statement holds with every occurrence of $\sum$ replaced by $\prod$ and every occurrence of* sup *replaced by* inf*, respectively.*

Proof. Since $\sum Y = \sup_L(X)$, $\sum Y$ is an upper bound for X in A. Let $z \in A$ be an upper bound for X. Then z is an upper bound for Y because $Y \subseteq X$. Thus, $\sum Y \leq z$. This true of any $z \in A$ shows $\sum Y$ is the least upper bound for X in A. □

We will mainly be interested in the algebra $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$, defined as follows.

Definition. *$[\kappa]^{<\lambda}$ is the set $\{X \subseteq \kappa : |X| < \lambda\}$ and $[\kappa]^{\lambda}$ is the set $\{X \subseteq \kappa : |X| = \lambda\}$.*

Clearly, $[\kappa]^{<\lambda}$ forms an ideal in $\mathcal{P}(\kappa)$.

Definition. *For any sets X, Y and cardinal κ, define $X \subseteq_\kappa Y$ to mean that $|X \setminus Y| < \kappa$ and $X =_\kappa Y$ to mean that $X \subseteq_\kappa Y$ and $Y \subseteq_\kappa X$.*

For $X, Y \in \mathcal{P}(\kappa)$, $X \subseteq_\kappa Y$ is equivalent to $X/[\kappa]^{<\kappa} \leq Y/[\kappa]^{<\kappa}$ and $X =_\kappa Y$ is equivalent to $X/[\kappa]^{<\kappa} = Y/[\kappa]^{<\kappa}$.

Definition. *For cardinals κ and λ, recall that $\kappa^{<\lambda} = \sup_{\nu<\lambda} \kappa^\nu$.*

Definition. *For any Boolean algebra, define* $\text{Length}_{spec}(A) = \{|I| : I \subseteq A \textit{ is maximal comparable}\}$, $\text{Length}(A) = \sup(\text{Length}_{spec}(A))$ *and* $\text{Length}_{mm}(A) = \min(\text{Length}_{spec}(A))$.

Theorem 2.0.4 (Scherer)**.** *If $\mu < \text{cf}(\kappa)$ then* $\text{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\mu}) \leq \kappa^{<\kappa}$.

Proof. Let $L_0 = \{\mu \cdot \alpha/[\kappa]^{<\mu} : \alpha \in \kappa\} \cup \{\kappa/[\kappa]^{<\mu}\}$. By Zorn's Lemma, let $L \supseteq L_0$ be a maximal comparable set. Take $m/[\kappa]^{<\mu} \in L \setminus L_0$. Since $\kappa/[\kappa]^{<\mu} \in L_0$, $|\kappa \setminus m| \geq \mu$. The fact that $\mu < \text{cf}(\kappa)$ implies the first μ elements of $\kappa \setminus m$ form a bounded set in κ. Thus, there is some $\alpha \in \kappa$ such that

$$\mu \leq |(\mu \cdot \alpha) \cap (\kappa \setminus m)| = |(\mu \cdot \alpha) \setminus m|.$$

Thus, $(\mu\cdot\alpha)/[\kappa]^{<\mu} \not\leq m/[\kappa]^{<\mu}$ and since L is a chain it must be that $m/[\kappa]^{<\mu} \leq (\mu\cdot\alpha)/[\kappa]^{<\mu}$. So $|m \setminus (\mu \cdot \alpha)| < \mu$. Thus,

$$|m| \leq |(\mu \cdot \alpha) \cap m| + |m \setminus (\mu \cdot \alpha)| \leq \mu + \mu \cdot |\alpha| < \kappa.$$

So L is represented by elements from $[\kappa]^{<\kappa} \cup \{\kappa\}$. □

Corollary 2.0.5 (Scherer)**.** $\text{Length}_{mm}(\mathcal{P}(\omega_1)/\operatorname{Fin}(\omega_1)) \leq \mathfrak{c}$.

Proof. Let $\kappa = \omega_1$ and $\mu = \omega$. Then $\kappa^{<\kappa} = \omega_1^\omega = \mathfrak{c}$. □

Lemma 2.0.6. *Suppose $\lambda \leq \kappa$. If $X \subseteq \mathcal{P}(\kappa)/[\kappa]^{<\lambda}$ and $|X| < \text{cf}(\lambda)$ then $\sum X$ exists and is represented modulo $[\kappa]^{<\lambda}$ by $\bigcup X_0$ where $X_0 \subseteq \mathcal{P}(\kappa)$ is any set of coset representatives of the elements of X. Similarly,* $\inf X$ *exists and is represented by $\bigcap X_0$.*

Proof. Write $\nu = |X|$, $X_0 = \{x_\alpha : \alpha < \nu\}$ and $U = \bigcup_{\alpha<\nu} x_\alpha$. If $|U \setminus y| \geq \lambda$ then since

$$U \setminus y = \bigcup_{\alpha<\nu} x_\alpha \setminus y$$

and $\nu < \mathrm{cf}(\kappa)$, it must be that $|x_\alpha \setminus y| \geq \lambda$ for some $\alpha < \nu$. Thus, if $U/[\kappa]^{<\lambda} \not\leq y/[\kappa]^{<\lambda}$ then $x_\alpha/[\kappa]^{<\lambda} \not\leq y/[\kappa]^{<\lambda}$ for some α. The contrapositive of this statement is precisely that $U/[\kappa]^{<\lambda}$ is the least upper bound for X in $\mathcal{P}(\kappa)/[\kappa]^{<\lambda}$.

The case for $\inf(X)$ is nearly identical using the equation

$$y \setminus \bigcap_{\alpha<\nu} x_\alpha = \bigcup_{\alpha<\nu} y \setminus x_\alpha.$$

□

2.1 A Lower Bound

Theorem 2.1.1 (Scherer). *If $\lambda \leq \kappa$, $\mathrm{cf}(\lambda) \leq \mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\lambda})$.*

Proof. Let $L \subseteq \mathcal{P}(\kappa)/[\kappa]^{<\lambda}$ be a chain and assume $|L| < \mathrm{cf}(\lambda)$. The goal is to show that L is not maximal. Towards a contradiction assume that it is. Let $L_0 \subseteq \mathcal{P}(\kappa)$ be a full system of representatives for L. That is, $|L_0 \cap l| = 1$ for every $l \in L$. Put

$$E = \bigcup \{l \setminus m \,:\, l, m \in L_0 \text{ and } l \subseteq_\lambda m\}$$

Since $|L_0| < \mathrm{cf}(\lambda)$ and $|l \setminus m| < \lambda$ for all $l, m \in L_0$ such that $l \subseteq_\lambda m$, $|E| < \lambda$. For any $\alpha \in \kappa \setminus E$, define $X_\alpha = \{l \in L_0 \,:\, \alpha \notin l\}$, $Y_\alpha = \{l \in L_0 \,:\, \alpha \in l\}$ and $C_\alpha = \langle \{l/[\kappa]^{<\lambda} \,:\, l \in X_\alpha\}, \{l/[\kappa]^{<\lambda} \,:\, l \in Y_\alpha\} \rangle$

For every $\alpha \in \kappa \setminus E$, C_α is a cut. Indeed, it is clear that every $l \in L_0$ must lie in X_α or Y_α. If $l \in X_\alpha$, $m \in Y_\alpha$ and $m \subseteq_\lambda l$, then $m \setminus l \subseteq E$ but $\alpha \in m \setminus l$ while $\alpha \notin E$. Thus, $l \subseteq_\lambda m$ whenever $l \in X_\alpha$ and $m \in Y_\alpha$, which proves C_α is a cut.

Since $|X_\alpha| + |Y_\alpha| < \mathrm{cf}(\lambda)$, $\sum_{l \in X_\alpha} l/[\kappa]^{<\lambda}$ and $\prod_{l \in Y_\alpha} l/[\kappa]^{<\lambda}$ both exist, they each must lie in L and are represented by $\bigcup X_\alpha$ and $\bigcap Y_\alpha$, respectively. By dint of the fact that L is maximal, $\sum_{l \in X_\alpha} l/[\kappa]^{<\lambda} = \prod_{l \in Y_\alpha} l/[\kappa]^{<\lambda}$. In particular,

$$\left| \bigcap Y_\alpha \setminus \bigcup X_\alpha \right| < \lambda.$$

As $|L| < \mathrm{cf}(\lambda) \leq \lambda \leq \kappa$, there must be some $S \in [\kappa \setminus E]^\kappa$ and $s \in L_0$ such that $\sum_{l \in X_\alpha} l/[\kappa]^{<\kappa} = s/[\kappa]^{<\kappa}$ for all $\alpha \in S$. For every $\alpha \in S$ either $s \in X_\alpha$ or $s \in Y_\alpha$ so take $T \in [S]^\kappa$ such that either $s \in X_\alpha$ for all $\alpha \in T$ or $s \in Y_\alpha$ for all $\alpha \in T$.

Case 1: $s \in X_\alpha$ for all $\alpha \in T$. Then for every $\alpha \in T$, $X_\alpha = \{l \in L_0 : l \subseteq_\kappa s\}$ and $Y_\alpha = \{l \in L_0 : s \subsetneq_\kappa l\}$. So $X_\alpha = X_\beta$ and $Y_\alpha = Y_\beta$ for all $\alpha, \beta \in T$. Thus, $T \cap l = \emptyset$ for all $l \in X_\alpha$ and $T \subseteq l$ for all $l \in Y_\alpha$. Verily, if $\alpha \in T$, $l \in X_\alpha$ and $\beta \in T \cap l$ then by $X_\alpha = X_\beta$, $l \in X_\beta$ which contradicts $\beta \in l$ so it must be that $T \cap l = \emptyset$. Similarly, if $\alpha \in T$, $l \in Y_\alpha$ and $\beta \in T \setminus l$ then by $Y_\alpha = Y_\beta$, $l \in Y_\beta$ which contradicts $\beta \notin l$ and it must needs be that $T \setminus l = \emptyset$. But then the following absurdity arises:

$$T \subseteq \bigcap Y_\alpha \setminus \bigcup X_\alpha.$$

Case 2: $s \in Y_\alpha$ for all $\alpha \in T$. A similar argument shows $X_\alpha = \{l \in L_0 : l \subsetneq_\kappa x\}$ and $Y_\alpha = \{l \in L_0 : s \subseteq_\kappa l\}$ resulting in the same absurdity. $\square$

Definition. *A linear order L is* ***densely ordered*** *if for every $a, b \in L$ there is some $c \in L$ such that $a < c < b$.*

Definition. *Let A be a Boolean algebra and $a \in A$. Define the* ***relative subalgebra*** *of A with respect to a to be $A \restriction a = \{b \in A : b \leq a\}$.*

Definition. *Let A be a Boolean algebra and $S \subseteq A$. Say that $D \subseteq S$ is* ***dense*** *in S if for all $a \in S \setminus \{0\}$, D intersects $(A \restriction a) \setminus \{0\}$ non-trivially. If S is not specified then D being dense means D is dense in A.*

Define $d(A) = \min\{|D| : D \subseteq A \text{ is dense}\}$ and for $a \in A$, $d_A(a) = d(A \restriction a)$.

Definition. *Let A be a Boolean algebra. A non-zero $a \in A$ is called an* ***atom*** *if $A \restriction a = \{0, a\}$. $\mathrm{At}(A)$ is the collection of all atoms in A. A is called* ***atomic*** *if $\mathrm{At}(A)$ is dense. A is called* ***atomless*** *if $\mathrm{At}(A) = \emptyset$.*

Lemma 2.1.2. *If A is an atomless Boolean algebra and $L \subseteq A$ is a maximal comparable set then L is densely ordered.*

Proof. If L is not densely ordered then there are $a, b \in L$ with $a < b$ and no $c \in L$ such that $a < c < b$. Since A is atomless, there is some $d \in A$ with $0 < d < b \cdot -a$. Then $a < a + d < b$ so $L \cup \{a + d\}$ is a strictly larger chain than L, contradiction. □

Definition. *A Boolean algebra A has the* ***countable separation property*** *if for any two at most countable subsets X and Y of A satisfying $x \cdot y = 0$ for all $\langle x, y\rangle \in X \times Y$ there is some $a \in A$ such that $x \leq a$ for all $x \in X$ and $y \leq -a$ for all $y \in Y$.*

By [5, Lm 5.27], A has the countable separation property if and only if: for all $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ from A, such that $x_n \leq x_{n+1} \leq y_{n+1} \leq y_n$ for all $n \in \omega$, there is some $a \in A$ such that $x_n \leq a \leq y_n$ for all $n \in \omega$. Note that it need not be the case that $\sum x_n = a$ or $\prod y_n = a$. The countable separation property is weaker than σ-completeness.

A special case of the following, which never-the-less captures the idea of the full proof, was first provided by Joel Hamkins [2].

Theorem 2.1.3 (Scherer)**.** *If A is atomless and satisfies the countable separation property then $\mathfrak{c} \leq \text{Length}_{mm}(A)$.*

Proof. Let $L \subseteq A$ be a maximal comparable set. Since A is atomless, L is densely ordered. Consider the complete binary tree ${}^{<\omega}2$. Define a map $F : {}^{<\omega}2 \to \{[a, b] : a, b \in L\}$ level by level. Let $F(\emptyset) = [0_A, 1_A]$ to complete the definition of F on $\text{Lev}_0({}^{<\omega}2)$. Suppose F has been defined on $\text{Lev}_n({}^{<\omega}2) = {}^n2$ and that for all $f \in {}^n2$, $F(f) = [a, b]$ where $a < b$. Given $f \in {}^n2$, pick $c, d \in L$ such that $a < c < d < b$ where $F(f) = [a, b]$. This is possible because L is densely ordered. Let $F(f \cup \{\langle n, 0\rangle\}) = [a, c]$ and $F(f \cup \{\langle n, 1\rangle\}) = [d, b]$. This completes the definition of F.

Now take $f \in {}^\omega2$. Let $F(f \restriction n) = [a_n^{(f)}, b_n^{(f)}]$ for all $n \in \omega$. Since A has the countable separation property and $a_n^{(f)} \leq a_{n+1}^{(f)} \leq b_{n+1}^{(f)} \leq b_n^{(f)}$ for all $n \in \omega$, there is some $x^{(f)} \in \bigcap_{n\in\omega}[a_n^{(f)}, b_n^{(f)}]$.

If $g \in {}^\omega2$ and $f \neq g$, take n minimal such that $f(n) \neq g(n)$. Without loss of generality suppose $f(n) = 0$. Then by construction,

$$a_n^{(f)} \leq a_{n+1}^{(f)} \leq x^{(f)} \leq b_{n+1}^{(f)} < a_{n+1}^{(g)} \leq x^{(g)} \leq b_{n+1}^{(g)} \leq b_n^{(g)}.$$

Consequently, $x^{(f)} < x^{(g)}$. Conclusion: $\left|\{x^{(f)} : f \in {}^{\omega}2\}\right| = \mathfrak{c}$. □

Corollary 2.1.4 (Scherer). *If* $\lambda \leq \kappa$, $\mathfrak{c} \leq \text{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\lambda})$.

Proof. It is well known that $\mathcal{P}(\kappa/[\kappa]^{<\lambda})$ is atomless and has the countable separation property. A proof of the latter is included for completeness.

Suppose $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$ are sequences from $\mathcal{P}(\kappa)/[\kappa]^{<\lambda}$ such that $a_n \subseteq_\lambda a_{n+1} \subseteq_\lambda b_{n+1} \subseteq_\lambda b_n$ for all $n \in \omega$. Let

$$c = \bigcup_{n\in\omega} \left(a_n \cap \bigcap_{k\in n} b_k \right)$$

Then for all $n \in \omega$,

$$a_n \setminus c \subseteq a_n \setminus \left(a_n \cap \bigcap_{k\in n} b_k \right) = a_n \setminus \bigcap_{k\in n} b_k = \bigcup_{k\in n} a_n \setminus b_k$$

which is smaller than λ hypothesis. Also, for any $n \in \omega$,

$$c \setminus b_n = \left(\bigcup_{k\leq n} a_k \cap \bigcap_{j\in k} b_j \right) \setminus b_n \subseteq \bigcup_{k\leq n} a_k \setminus b_n$$

which is once more smaller than λ by hypothesis. □

Together with Corollary 2.0.5 this gives:

Corollary 2.1.5 (Scherer). $\text{Length}_{mm}(\mathcal{P}(\omega_1)/\operatorname{Fin}(\omega_1)) = \mathfrak{c}$.

2.2 An Upper Bound

Definition. *If α is an ordinal and L is any linearly ordered set, the* ***lexicographical order*** *on ${}^{\alpha}L$ is given by declaring $x < y$ if $x(\beta) < y(\beta)$ where $\beta = \min\{\eta \in \alpha : x(\eta) \neq y(\eta)\}$.*

The lexicographical order is a linear order and much of the coming material about linear orders is probably familiar. See [1, Section 4.2].

Fix a cardinal μ and consider the set ${}^{\mu}2$ under the lexicographical order. Fix $x, y \in {}^{\mu}2$, define $x \sim y$ and $y \sim x$ if either $x = y$ or if there is some $\alpha < \mu$ such that:

(1) $x \restriction \alpha = y \restriction \alpha$,

(2) $x(\alpha) = 0$,

(3) $x(\beta) = 1$ for $\beta > \alpha$,

(4) $y(\alpha) = 1$ and

(5) $y(\beta) = 0$ for $\beta > \alpha$.

Also define $[x]_{\sim} = \{y \in {}^{\mu}2 : x \sim y\}$.

Lemma 2.2.1. *Let $x, y, z \in {}^{\alpha}2$. If $x < y < z$ then $x \restriction \beta = y \restriction \beta = z \restriction \beta$ where $\beta = \min\{\eta \in \alpha : x(\eta) \neq z(\eta)\}$.*

Proof. By definition, $x \restriction \beta = z \restriction \beta$. Let $\gamma = \min\{\eta \in \alpha : x(\eta) \neq y(\eta)\}$. If $\gamma < \beta$ then by $x < y$, $x(\gamma) < y(\gamma)$ but $x \restriction \beta = z \restriction \beta$ which implies that $z < y$. This contradicts $y < z$ □

Lemma 2.2.2. *If $x \sim y$ then there is no $z \in {}^{\mu}2$ such that $x < z < y$.*

Proof. Take $\alpha = \min\{\beta \in \mu : x(\beta) \neq y(\beta)\}$. Then $x \restriction \beta = z \restriction \beta = y \restriction \beta$ and $z(\alpha) \in \{x(\alpha), y(\alpha)\}$. But $x(\beta) = 1$ for all $\beta > \alpha$ which makes $z(\alpha) = x(\alpha)$ inconsistent with $x < z$. Similarly, $z(\alpha) = y(\alpha)$ is inconsistent with $z < y$. □

Lemma 2.2.3. *If $x \sim y$ and $y \sim z$ then $|\{x, y, z\}| < 3$.*

Proof. Suppose $x \sim y$ and $y \sim z$ yet no two of x, y, z are equal. By symmetry, after renaming elements we may either suppose that $x < y < z$ or $x < z < y$. The latter case is already known to be impossible. So suppose $x < y < z$. Take $\alpha_x = \min\{\beta \in \mu : x(\beta) \neq y(\beta)\}$ and

$\alpha_z = \min\{\beta \in \mu : z(\beta) \neq y(\beta)\}$. Then $x \sim y$ together with $x < y$ implies $y(\eta) = 0$ for all $\eta > \alpha_x$. Similarly, $y \sim z$ together with $y < z$ implies $y(\eta) = 1$ for all $\eta > \alpha_z$. This is absurd. □

Corollary 2.2.4. *The relation $\sim$ is an equivalence relation whose equivalence classes have at most two elements and are convex. Consequently, ${}^{\mu}2/\sim$ (defined to be $\{[x]_\sim : x \in {}^{\mu}2\}$) is linearly ordered by declaring $[x]_\sim < [y]_\sim$ if $x \nsim y$ and $x < y$.*

Proof. First note that if $\sim$ is an equivalence relation, the preceding lemma shows that its equivalence classes have at most two elements and are convex. The relation is reflexive and symmetric by definition. Suppose $x \sim y$ and $y \sim z$. The previous lemma says that at least two of x, y, z must be equal. If $x = z$ then $x \sim z$ by definition. Otherwise, y must be one of x or z and $x \sim z$ is a restatement of either $x \sim y$ or $y \sim z$ depending on whether $y = z$ or $y = x$.

To see that ${}^{\mu}2/\sim$ is linearly ordered take $x, y \in {}^{\mu}2$ such that $x \nsim y$ and $x < y$. If $w \in [x]_\sim$, $z \in [y]_\sim$ and $z \leq w$ then by the fact that $[y]_\sim$ is convex, $w \in [y]_\sim$ and therefore $[x]_\sim = [y]_\sim$ defying the fact that $x \nsim y$. This shows that the order on ${}^{\mu}2/\sim$ is well defined and it must be a simple order because the order on ${}^{\mu}2$ is. □

Let $R = {}^{\mu}2/\sim$ and define $0_R = [\mu \times \{0\}]_\sim$ and $1_R = [\mu \times \{1\}]_\sim$. Since $\mu \times \{0\} = \inf({}^{\mu}2)$ and $\mu \times \{1\} = \sup({}^{\mu}2)$, $0_R = \inf(R)$ and $1_R = \sup(R)$.

Lemma 2.2.5. *The map $I : {}^{\mu}2 \to {}^{\mu}2$ given by $x \mapsto \{\langle \alpha, 1 - x(\alpha) \rangle : \alpha \in \mu\}$ is an order reversing bijection and constant on $\sim$-equivalence classes. Therefore R is isomorphic to the reverse order on R.*

Proof. To see that I is a bijection one only needs to note that $I \circ I$ is the identity map. Let $x, y \in {}^{\mu}2$ and $x < y$ and $\beta = \min\{\alpha \in \mu : x(\alpha) \neq y(\beta)\}$ and note that $\beta = \min\{\alpha \in \mu : (I(x))(\alpha) \neq (I(y))(\alpha)\}$. Since $x(\beta) = 0$ and $y(\beta) = 1$, $(I(x))(\beta) = 1$ and $(I(y))(\beta) = 0$ so $I(y) < I(x)$.

Now suppose $x \sim y$. Then it has just been noted condition (1) hold for $I(x)$ and $I(y)$. I transposes conditions (2) and (4) and conditions (3) and (5) hence these hold for $I(y)$ and $I(x)$, respectively. □

By this Lemma, it is only necessary to prove suprema of the prescribed type and an analogous result will follow for infima.

Lemma 2.2.6. *For every $X \subseteq R$, $\sup X$ and $\inf X$ exist.*

Proof. Without loss of generality, assume that X has no maximum. Define a sequence $\{a_\alpha : \alpha < \mu\}$ recursively. Suppose that $\{a_\beta : \beta < \alpha\}$ is given. Set $b_\alpha = \bigcup_{\beta<\alpha} a_\beta$ and

$$i_\alpha = \sup\{x(\alpha) : [x]_\sim \in X \text{ and } x \restriction \alpha = b_\alpha\}.$$

Define $a_\alpha = b_\alpha \cup \{\langle \alpha, i_\alpha \rangle\}$. This completes the definition of $\{a_\alpha : \alpha < \mu\}$. Put $a = \bigcup_{\alpha<\mu} a_\alpha$. Note that since a_α is a function with domain $\alpha + 1$ and range L, a is a function with domain μ and range L. The lemma is proved by showing $\sup X = [a]_\sim$.

To see that $[a]_\sim$ is an upper bound, take $[x]_\sim \in X$ and suppose $x \neq a$. Let $\alpha = \min\{\beta \in \mu : x(\beta) \neq a(\beta)\}$. Then $a \restriction \alpha = b_\alpha = x \restriction \alpha$. Therefore $x(\alpha) \leq i_\alpha = a(\alpha)$. But since $x(\alpha) \neq a(\alpha)$ this shows $x(\alpha) < a(\alpha)$ so $x < a$. Thus, for any $[x]_\sim \in X$, $[x]_\sim \leq [a]_\sim$.

Now suppose that $r \in {}^\mu L$ and $r < a$. Then there is an $\alpha \in \mu$ such that $r \restriction \alpha = a \restriction \alpha$ and $r(\alpha) < a(\alpha)$. But then by the definition of a_α, there is some $[x]_\sim \in X$ such that $x \restriction \alpha = b_\alpha$ and $r(\alpha) < x(\alpha)$. Since $b_\alpha = a \restriction \alpha$, $r \restriction \alpha = x \restriction \alpha$ and therefore $r < x$. Thus $[r]_\sim \leq [x]_\sim$. Since X has no max there is some $[y]_\sim \in X$ such that $[x]_\sim < [y]_\sim$. But then $[r]_\sim < [y]_\sim$ which shows that the former is not an upper bound for X. This establishes that $[a]_\sim$ is a least upper bound. □

Lemma 2.2.7. *For every $x \in {}^\mu 2$ and $i \in 2$ there is some $w_i \in {}^\mu 2$ such that $x \sim w_i$ and $w_i^{-1}[\{i\}]$ has no maximum.*

Proof. If $x^{-1}[\{i\}]$ has no maximum let $w_i = x$. Otherwise, take $\alpha = \sup(x^{-1}[\{i\}])$ and put $w_i = x \restriction \alpha \cup \{\langle \alpha, 1 - i\} \cup (\mu \setminus (\alpha + 1) \times \{i\})$. □

Lemma 2.2.8. *R is densely ordered.*

Proof. Suppose $x, y \in {}^{\mu}2$ and $[x]_\sim < [y]_\sim$. Without loss of generality, neither of $x^{-1}[\{0\}]$ or $y^{-1}[\{1\}]$ has a maximum. Take $\alpha = \min\{\beta \in \mu : x(\beta) \neq y(\beta)\}$. Then $x(\alpha) = 0$ and $y(\alpha) = 1$. Since $\alpha \neq \sup(x^{-1}[\{0\}])$, take $\beta > \alpha$ such that $x(\beta) = 0$. Define $z = x \restriction \beta \cup ((\mu \setminus \beta) \times \{1\})$. Then immediately $x < z < y$. □

Lemma 2.2.9. *R has no strictly increasing sequence of length* μ^+.

Proof. If R had such a sequence then ${}^{\mu}2$ would have one as well and this is known to be impossible. Indeed, take $\{x_\alpha : \alpha < \mu^+\} \subseteq {}^{\mu}2$ strictly increasing, γ minimal such that $|\{x_\alpha \restriction \gamma : \alpha < \mu^+\}| = \mu^+$ and $\Gamma = \{\alpha < \mu^+ : x_\alpha \restriction \gamma \neq x_\beta \restriction \gamma \text{ for all } \beta < \alpha\}$. Then for every $\alpha \in \mu^+$ there is exactly one $\beta \in \Gamma$ such that $x_\alpha \restriction \gamma = x_\beta \restriction \gamma$ so $|\Gamma| = \mu^+$ and $x_\alpha \restriction \gamma \neq x_\beta \restriction \gamma$ for all $\alpha, \beta \in \Gamma$. Define $\Gamma(\alpha) = \min(\Gamma \setminus (\alpha + 1))$ and $\eta_\alpha = \min\{\beta \in \gamma : x_\alpha(\beta) \neq x_{\Gamma(\alpha)}(\beta)\}$. There must be some $\xi \in \gamma$ such that $|\{\alpha \in \Gamma : \eta_\alpha = \xi\}| = \mu^+$.

Consider $\alpha, \beta \in \Gamma$ with $\eta_\alpha = \eta_\beta = \xi$ and $\alpha < \beta$. If $x_\alpha \restriction (\xi + 1) = x_\beta \restriction (\xi + 1)$ then $x_\beta < x_{\Gamma(\alpha)}$. However, it must be that $\Gamma(\alpha) \leq \beta$. Therefore $x_\alpha \restriction (\xi + 1) \neq x_\beta \restriction (\xi + 1)$. Finally, define $\Delta_i = \{\alpha \in \Gamma : \eta_\alpha = \xi \text{ and } x_\alpha(\xi) = i\}$. For some $i \in 2$, $|\Delta_i| = \mu^+$ and consequently $x_\alpha \restriction \xi \neq x_\beta \restriction \xi$ for all $\alpha, \beta \in \Delta_i$ contradicting the minimality of γ. □

Lemma 2.2.10. *For every $r \in R$ there are sets $Y_0 \subseteq [0_R, r)$ and $Y_1 \subseteq (r, 1_R]$ each with cardinality at most μ such that* $\sup(Y_0) = \inf(Y_1) = r$.

Proof. Suppose that for every $Y_0 \in [[0_R, r)]^{\leq \mu}$, $\sup(Y_0) < r$. Construct a strictly increasing sequence $\{x_\alpha : \alpha < \mu^+\} \subseteq [0_R, r)$ as follows: given an initial segment $\{x_\alpha : \alpha < \beta\} \subseteq [0_R, r)$, choose x_β such that $\sup_{\alpha<\beta} x_\alpha < x_\alpha < r$. This is always possible by density. But then $\{x_\alpha : \alpha < \mu^+\}$ contradicts the previous lemma. □

Lemma 2.2.11. *Suppose θ is a cardinal satisfying $\theta < 2^\mu$. Then there is $\{I_\alpha : \alpha < \theta\} \subseteq {}^{\mu}2$ such that for any distinct $\alpha, \beta < \theta$ and $\delta < \mu$ there is ξ such that $I_\alpha(\delta + \xi) \neq I_\beta(\xi)$.*

Proof. Case 1: there is some $\nu < \mu$ such that $\theta < 2^\nu$. Take $J \in [{}^\nu 2]^\theta$ and let $\{J_\alpha : \alpha < \theta\}$ be a one-one enumeration of J. For $\alpha < \theta$, $\eta < \mu$ and $\pi < \nu$ define $I_\alpha(\nu \cdot \eta + \pi) = J_\alpha(\pi)$. Since every ordinal can uniquely be written in the form $\nu \cdot \eta + \pi$ with $\pi < \nu$, this is a valid definition for a sequence $\{I_\alpha : \alpha < \theta\}$. Now suppose that $\alpha, \beta < \theta$ are distinct and $\delta < \mu$ is given. Write $\delta = \nu \cdot \eta + \pi$ and take ζ such that $J_\alpha(\zeta) \neq J_\beta(\zeta)$. Let $\xi = \nu + \zeta$. Then $\delta + \xi = \nu \cdot (\eta + 1) + \zeta$ so that $I_\alpha(\delta + \xi) = J_\alpha(\zeta) \neq J_\beta(\zeta) = I_\beta(\xi)$.

Case 2: there is no $\nu < \mu$ such that $\theta < 2^\nu$. Define $\{I_\alpha : \alpha < \theta\}$ recursively. Assume $\{I_\alpha : \alpha < \beta\}$ is given for some $\beta < \theta$. For $\delta < \mu$ define

$$A_{\alpha,\delta} = \{J \in {}^\mu 2 : J(\delta + \xi) = I_\alpha(\xi) \text{ for all } \xi < \mu\}$$

and

$$B_{\alpha,\delta} = \{J \in {}^\mu 2 : J(\xi) = I_\alpha(\delta + \xi) \text{ for all } \xi < \mu\}.$$

Then $|A_{\alpha,\delta}| \leq \left|{}^\delta 2\right| \leq \theta$ and $|B_{\alpha,\delta}| = 1$, so

$$\left| \bigcup_{\alpha<\beta} \bigcup_{\delta<\mu} A_{\alpha,\delta} \cup B_{\alpha,\delta} \right| \leq \theta + \mu < 2^\mu.$$

So there is some $I_\beta \in {}^\mu 2$ that is not in $A_{\alpha,\delta} \cup B_{\alpha,\delta}$ for any $\alpha < \beta$ or $\delta < \mu$. This completes the definition of $\{I_\alpha : \alpha < \theta\}$ and the construction guarantees that it satisfies the conclusion of the lemma. Indeed, take distinct $\alpha, \beta < \theta$ and $\delta < \mu$. If $\alpha < \beta$, then $I_\beta \notin B_{\alpha,\delta}$ implies $I_\alpha(\delta + \xi) \neq I_\beta(\xi)$ for some ξ. If $\beta < \alpha$ then $I_\alpha \notin A_{\beta,\delta}$ implies $I_\alpha(\delta + \xi) \neq I_\beta(\xi)$ for some ξ. In either case there some $\xi < \mu$ such that $I_\alpha(\delta + \xi) \neq I_\beta(\xi)$. □

Lemma 2.2.12. *Suppose μ and θ are any cardinals satisfying $\theta \leq 2^\mu$. Then there is a map $L : {}^\mu 2 \to \theta$ such that for any $x, y \in {}^\mu 2$ if $[x]_\sim < [y]_\sim$ then $\sup\{L(z) : x < z < y\} = \theta$.*

Proof. Case 1: $\theta < 2^\mu$. Take $\{I_\alpha : \alpha < \theta\}$ as in the previous lemma. Define

$$L(x) = \begin{cases} \alpha & \text{if there are } \alpha < \theta \text{ and } \eta < \mu \text{ such that } I_\alpha(\xi) = x(\eta + \xi) \text{ for all } \xi < \mu \\ 1 & \text{otherwise.} \end{cases}$$

To see that this is well defined, assume that there are distinct $\alpha, \beta < \theta$ and $\eta, \gamma < \mu$ such that $I_\alpha(\xi) = x(\eta + \xi)$ and $I_\beta(\xi) = x(\gamma + \xi)$ for all $\xi < \mu$. Without loss of generality, suppose that $\eta < \gamma$ and with $\gamma = \eta + \delta$. Then $I_\alpha(\delta + \xi) = x(\eta + \delta + \xi) = I_\beta(\xi)$ for all $\xi < \mu$ and this contradicts the construction of $\{I_\alpha : \alpha < \theta\}$. Thus, L is well defined.

Now assume that $x, y \in {}^\mu 2$ and $[x]_\sim < [y]_\sim$. Put $\alpha = \min\{\beta \in \mu : x(\beta) \neq y(\beta)\}$. Then $x(\alpha) = 0$ and $y(\alpha) = 1$. Since $x \nsim y$, there is some $\beta > \alpha$ such that $x(\beta) = 0$ or $y(\beta) = 1$.

Consider the possibility $x(\beta) = 0$. Take $\xi < \theta$. Define

$$z_w(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma < \beta \\ 1 & \text{if } \gamma = \beta \\ I_\xi(\eta) & \text{if } \gamma = \beta + \eta \end{cases}$$

By construction, $x < z < y$ and $L(z) = \xi$. Thus, $\xi \leq \sup\{L(z) : x < z < y\}$ for every $\xi < \theta$ which shows that $\sup\{L(z) : x < z < y\} = \theta$.

The alternative that $y(\beta) = 1$ is very similar.

Case 2: $\theta = 2^\mu$. Let $L : {}^\mu 2 \to \theta$ be any bijection. Take $x, y \in {}^\mu 2$ such that $[x]_\sim < [y]_\sim$. As in case 1, put $\alpha = \min\{\beta \in \mu : x(\beta) \neq y(\beta)\}$ and note there is some $\beta > \alpha$ such that $x(\beta) = 0$ or $y(\beta) = 1$.

First suppose $x(\beta) = 0$. For any $w \in {}^\mu 2$

$$z(\gamma) = \begin{cases} x(\gamma) & \text{if } \gamma < \beta \\ 1 & \text{if } \gamma = \beta \\ w(\eta) & \text{if } \gamma = \beta + \eta. \end{cases}$$

Then $x < z_w < y$ and $z_w \neq z_{w'}$ if $w \neq w'$. Thus, $\{L(z) : x < z < y\}$ has size 2^μ and is therefore unbounded in θ.

$y(\beta) = 1$ is dealt with similarly. $\square$

For the remainder of this section, assume that $\mu < \mathrm{cf}(\kappa) \leq 2^\mu \leq \kappa$. Let $\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ be a non-decreasing sequence of cardinals in κ such that $\sum_{\alpha<\mathrm{cf}(\kappa)} \kappa_\alpha = \kappa$. Then for any $I \in [\mathrm{cf}(\kappa)]^{\mathrm{cf}(\kappa)}$, $\sum_{\alpha\in I} \kappa_\alpha = \kappa$. Let $L : {}^\mu 2 \to \mathrm{cf}(\kappa)$ be a map as in the previous lemma. Choose $\{x_\alpha : \alpha < 2^\mu\}$ to be a one-one enumeration of ${}^\mu 2$ and $\lambda_\alpha = \kappa_{L(x_\alpha)}$. By the choice of L, $\sum_{x<x_\alpha<y} \lambda_\alpha = \kappa$ for any $x, y \in {}^\mu 2$ such that $[x]_\sim < [y]_\sim$. Write

$$K = \bigcup_{\alpha<2^\mu} \lambda_\alpha \times \{x_\alpha\}.$$

to get a set K such that $|K| = \kappa$. Define $f : {}^\mu 2 \to K$ by

$$f(x) = \{(\eta, y) \in K : y < x\} = \bigcup_{x_\alpha<x} \lambda_\alpha \times \{x_\alpha\}.$$

Then clearly $x \leq y$ implies $f(x) \subseteq f(y)$.

Lemma 2.2.13. *$|f(y) \setminus f(x)| = \kappa$ if and only if $[x]_\sim < [y]_\sim$.*

Proof. First assume $|f(y)\setminus f(x)| = \kappa$. If $y \leq x$ then $f(y) \subseteq f(x)$ would imply $|f(y)\setminus f(x)| = 0$ so it must be that $x \leq y$. It only remains to check that $x \nsim y$. Assume the contrary. Then $[x]_\sim = [y]_\sim$ is a convex set of size two. This means there is no $z \in {}^\mu 2$ such that $x < z < y$. Therefore

$$f(y) \setminus f(x) = \left(\bigcup_{x_\alpha<y} \lambda_\alpha \times \{x_\alpha\}\right) \setminus \left(\bigcup_{x_\alpha<x} \lambda_\alpha \times \{x_\alpha\}\right) = \bigcup_{x\,\leq\, x_\alpha\,<\,y} \lambda_\alpha \times \{x_\alpha\} = \lambda_\alpha \times \{x_\alpha\}$$

where $x = x_\alpha$. But $|\lambda_\alpha \times \{x_\alpha\}| = \lambda_\alpha < \kappa$. Contradiction.

Now assume that $[x]_\sim < [y]_\sim$. It must be proved that $|f(y) \setminus f(x)| = \kappa$. Then $x < y$ and $x \nsim y$ which shows that $\sum_{x<x_\alpha<y} \lambda_\alpha = \kappa$. So

$$|f(y) \setminus f(x)| = \left|\bigcup_{x\,\leq\, x_\alpha\,<\,y} \lambda_\alpha \times \{x_\alpha\}\right| = \kappa$$

□

By this lemma, the map $F : [x]_\sim \mapsto f(x)/[\kappa]^{<\kappa}$ is well defined and order preserving from R to $\mathcal{P}(K)/[K]^{<\kappa}$.

Lemma 2.2.14. *Take $Y \subseteq R$. Set $[s]_\sim = \inf(Y)$ and $[r]_\sim = \sup(Y)$. Then*

$$f(s) =_\kappa \bigcup_{[y]_\sim \in Y} f(y) \qquad \text{and} \qquad f(t) =_\kappa \bigcap_{[y]_\sim \in Y} f(y).$$

Proof. $[s]_\sim \leq [y]_\sim$ implies $s \leq y$ which in turn implies $f(s) \subseteq f(y)$. Thus, $f(s) \subseteq \bigcup_{[y]_\sim \in Y} f(y)$ is automatic.

On the other hand,

$$\bigcap_{[y]_\sim \in Y} f(y) = \bigcap_{[y]_\sim \in Y} \bigcup_{x_\alpha < y} \lambda_\alpha \times \{x_\alpha\} = \bigcup_{x_\alpha \leq s} \lambda_\alpha \times \{x_\alpha\}.$$

So

$$\bigcap_{[y]_\sim \in Y} f(y) \setminus f(s) = \lambda_\alpha \times \{x_\alpha\}$$

where $x_\alpha = s$. Since $\lambda_\alpha < \kappa$, this shows $f(s) =_\kappa \bigcap_{[y]_\sim \in Y} f(y)$.

For the other direction simply note that

$$\bigcup_{[y]_\sim \in Y} \bigcup_{x_\alpha < y} \lambda_\alpha \times \{x_\alpha\} = \bigcup_{x_\alpha < r} \lambda_\alpha \times \{x_\alpha\} = f(r)$$

□

Theorem 2.2.15 (Scherer)**.** *$F[R]$ is a maximal comparable subset of $\mathcal{P}(K)/[K]^{<\kappa}$.*

Proof. By Lemma 2.0.2, it is enough to check that for every cut $\langle X_0, X_1 \rangle \subseteq F[R]$, $\sum X_0$ and $\prod X_1$ both exist are equal and are in $F[R]$. Let $r = \sup F^{-1}[X_0]$. Since F is order preserving, $\langle F^{-1}[X_0], F^{-1}[X_1] \rangle$ is a cut in R. Thus, $r \in F^{-1}[X_0] \cup F^{-1}[X_1] = R$.

Case 1: $r \in F^{-1}[X_0]$. Then $r = \max(F^{-1}[X_0])$ and by the definition of a cut, $r < s$ for every $s \in F^{-1}[X_1]$. Thus, $F^{-1}[X_1] \subseteq (r, 1_R]$. By the definition of r, $F^{-1}[X_0] \subseteq [0_R, r]$. Now, $F^{-1}[X_0] \subseteq [0_R, r]$, $F^{-1}[X_1] \subseteq (r, 1_R]$, $F^{-1}[X_0] \cup F^{-1}[X_1] = R$ and $[0_R, r] \cap (r, 1_R] = \emptyset$. This can only be possible if $F^{-1}[X_0] = [0_R, r]$ and $F^{-1}[X_1] = (r, 1_R]$. By Lemma 2.2.10, there is some $Y \in [F^{-1}[X_1]]^{\leq\mu}$ such that $\inf(Y) = r$. Since $r = \max(F^{-1}[X_0])$, $F(r) = \max(X_0)$. So by Lemma 2.0.6,

$$\sum X_0 = F(r) = \left(\bigcap_{[y]_\sim \in Y} f(y) \right) / [K]^{<\kappa} = \prod F[Y]$$

Since r is a lower bound for $F^{-1}[X_1]$, $F(r)$ is a lower bound for X_1. Now because $F[Y] \subseteq X_0$, $\prod F[Y] = \prod X_0$.

Case 2: $r \in F^{-1}[X_1]$. One creates a proof through comsetic changes to Case 1. $\square$

Since $|R| = 2^\mu$ and $|K| = \kappa$, the following theorem results.

Theorem 2.2.16 (Scherer)**.** *Let* $\mu < \mathrm{cf}(\kappa)$ *satisfy* $\mathrm{cf}(\kappa) \leq 2^\mu \leq \kappa$. *Then* $\mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}) \leq 2^\mu$.

Chapter 3

Incomparability

We now proceed to the examination of incomparability, mainly as it relates to density. First we collect some facts that will be used freely.

Definition. *Let A be a Boolean algebra. For any $x, y \in A$, x and y are called* ***incomparable*** *if $x \not\leq y$ and $y \not\leq x$. Equivalently, x and y are incomparable if $x \cdot -y \neq 0$ and $y \cdot -x \neq 0$. A subset $I \subseteq A$ is called* ***pairwise incomparable*** *if any pair of elements drawn from it are incomparable.*

Lemma 3.0.1. $\mathrm{At}(A)$ *is pairwise incomparable.*

Proof. Take $a, b \in \mathrm{At}(A)$ and suppose that $a \leq b$. Then $\{0, a\} = A \restriction a \subseteq A \restriction b = \{0, b\}$. Since $a \neq 0$ and $b \neq 0$, $a = b$. So any distinct pair from $\mathrm{At}(A)$ must be incomparable. □

Definition. *For any Boolean algebra, A, define $\mathcal{I}(A) = \{a \in A : (A \restriction a) \cap \mathrm{At}(A) = \emptyset\}$.*

Lemma 3.0.2. $\mathcal{I}(A)$ *is an ideal of A.*

Proof. If $j \leq i$ then $A \restriction j \subseteq A \restriction i$, so $i \in \mathcal{I}(A)$ implies $j \in \mathcal{I}(A)$. Further, for any $i, j \in \mathcal{I}(A)$, if there were $a \in \mathrm{At}(A)$ such that $a \leq i + j$, then one of $a \cdot i$ or $a \cdot j$ is non-zero. But $\{a \cdot i, a \cdot j\} \subseteq A \restriction a = \{0, a\}$ which means one of $a \cdot i$ or $a \cdot j$ is a. But that implies a is below one of i or j, so there could be no such a. □

Definition. *Let A be a Boolean algebra and $S \subseteq A$. A subset $I \subseteq S$ is called* ***maximal incomparable*** *with respect to S if I is maximal with respect to inclusion among all infinite* *pairwise incomparable subsets of S. If S is not specified, then maximal incomparable means maximal incomparable with respect to A.*

Lemma 3.0.3. *Let A be a Boolean algebra. If $I \subseteq A$ is a pairwise incomparable set, $x \in \{0,1\}$ and $x \in I$ then $I = \{x\}$. Thus, if I is infinite then $I \cap \{0,1\} = \emptyset$.*

Proof. 0 and 1 are each comparable to every element of A, so any pairwise incomparable set containing either must have the same as its only member. □

Lemma 3.0.4. *Let A be a Boolean algebra and $S \subseteq A$. Suppose that $T \subseteq S$ is a (possibly finite) pairwise incomparable set. Then T is maximal with respect to inclusion among all (possibly finite) pairwise incomparable subsets of S if and only if for all $s \in S$ there is some $t \in T$ that is comparable with s.*

Proof. Suppose that T is maximal with respect to inclusion among all (possibly finite) pairwise incomparable subsets of S. Every $s \in T$ is comparable with itself, so take $s \in S \setminus T$. Since T is maximal with respect to inclusion among all (possibly finite) pairwise incomparable subsets of S, $T \cup \{s\}$ is not pairwise incomparable. So there are distinct $u, v \in T \cup \{s\}$ that are comparable. T is pairwise incomparable so one of u or v is not in T, and that one must be s. Therefore, s is comparable to some member of T.

Now suppose that T is not maximal with respect to inclusion among all (possibly finite) pairwise incomparable subsets of S. Then there is some $U \subseteq S$ with $T \subsetneq U$ that is a pairwise incomparable set. Take $s \in U \setminus T$. Then $T \cup \{s\}$ is pairwise incomparable which shows that for all $t \in T$, s and t are incomparable. □

Corollary 3.0.5. *If T is infinite, then T is maximal incomparable with respect to S if and only if for all $s \in S$ there is some $t \in T$ that is comparable with s.*

Definition. *For any Boolean algebra, define* $\mathrm{Inc}_{spec}(A) = \{|I| : I \subseteq A \text{ is maximal incomparable}\}$, $\mathrm{Inc}(A) = \sup(\mathrm{Inc}_{spec}(A))$ *and* $\mathrm{Inc}_{mm}(A) = \min(\mathrm{Inc}_{spec}(A))$.

Lemma 3.0.6. *Let A be a Boolean algebra. If I is maximal incomparable with respect to $\mathcal{I}(A)$, then $\mathrm{At}(A) \cup I$ is maximal incomparable with respect to A.*

Proof. $\mathrm{At}(A)$ and I are each pairwise incomparable so to see that $\mathrm{At}(A) \cup I$ is pairwise incomparable take $a \in \mathrm{At}(A)$ and $b \in I$. By definition, $a \not\leq b$ and in particular $a \neq b$. If $b \leq a$ then since $a \neq b$, $b = 0$. As I is infinite, there is some $c \in I \setminus \{b\}$ but $b = 0$ would mean $b \leq c$. By hypothesis, I must be pairwise incomparable so a and b are incomparable.

To see that $\mathrm{At}(A) \cup I$ is maximal, take $a \in A$. If $(A \upharpoonright a) \cap \mathrm{At}(A) \neq \emptyset$, then $b \in (A \upharpoonright a) \cap \mathrm{At}(A)$ is a member of $\mathrm{At}(A)$ that is comparable with a. If $(A \upharpoonright a) \cap \mathrm{At}(A) = \emptyset$, then $a \in \mathcal{I}(A)$ which implies there is some $b \in \mathcal{I}(A)$ that is comparable to a. □

Definition. *Let A be a Boolean algebra and $G \subseteq A$. Define $\langle G \rangle = \bigcap\{B \subseteq A : G \subseteq B$ and B is a subalgebra$\}$.*

Definition. *Let A be a Boolean algebra, $a \in A$ and $\varepsilon \in \{-1, +1\}$. Then*

$$\varepsilon a = \begin{cases} a & \text{if } \varepsilon = +1 \\ -a & \text{if } \varepsilon = -1 \end{cases}$$

The following argument is due to Michael Hrusak [3].

Lemma 3.0.7. *Let A be a Boolean algebra and I a (possibly finite) pairwise incomparable set disjoint from $\{0, 1\}$. If $|\langle I \rangle| < \min\{d_A(a) : a \in A \setminus \{0\}\}$ then I is not maximal with respect to inclusion among all (possibly finite) pairwise incomparable sets.*

Proof. Fix $i \in I$. Since $i \notin \{0, 1\}$, both i and $-i$ are non-zero. By the size restriction on $\langle I \rangle$, there is $x_\varepsilon \in A \upharpoonright \varepsilon i$ such that $(A \upharpoonright x_\varepsilon) \cap \langle I \rangle = \{0\}$ for $\varepsilon \in \{-1, +1\}$. Define $j = x_{-1} + (i \cdot -x_{+1})$. The proof is finished by showing there is no $k \in I$ comparable to j.

Case 1: $k = i$.

$$0 < x_{-1} = x_{-1} \cdot -i = j \cdot -i.$$

and

$$0 < x_{+1} = i \cdot x_{+1} = i \cdot (-x_{-1} \cdot -i + x_{+1}) = i \cdot (-x_{-1} \cdot (-i + x_{+1})) = i \cdot -j.$$

Case 2: $k \neq i$. Since $i \cdot -k \in \langle I \rangle$, $i \cdot -k \not\leq x_{+1}$ so

$$0 < i \cdot -x_{+1} \cdot -k \leq j \cdot -k.$$

Similarly, since $k \cdot -i \not\leq x_{-1}$,

$$0 < k \cdot -x_{-1} \cdot -i \leq k \cdot -x_{-1} \cdot -i + k \cdot -x_{-1} \cdot x_{+1} = k \cdot (-x_{-1} \cdot (-i + x_{+1})) = k \cdot -j.$$

□

Theorem 3.0.8 (Hrusak). *For any infinite Boolean algebra, A,* $\min\{d_A(a) : a \in A \setminus \{0\}\} \leq \mathrm{Inc}_{mm}(A)$.

Proof. If I is an infinite pairwise incomparable set then $|I| = |\langle I \rangle|$ and $I \cap \{0,1\} = \emptyset$. □

Corollary 3.0.9. *If A is atomless, the only finite pairwise incomparable subsets of A that is maximal with respect to inclusion among all (possibly finite) pairwise incomparable subsets of A are $\{0\}$ and $\{1\}$.*

Proof. Let I be a finite pairwise incomparable subset of A. If $0 \in I$ then since I is pairwise incomparable it must be that $I = \{0\}$. Similarly, if $1 \in I$ then $I = \{1\}$. Thus, we may suppose that $I \cap \{0,1\} = \emptyset$. Since A is atomless, $\aleph_0 \leq \min\{d_A(a) : a \in A \setminus \{0\}\}$. I being finite implies $\langle I \rangle$ is also finite so by Hrusak's lemma I is not maximal in the sense prescribed. □

3.1 Atomic algebras

Theorem 3.1.1. *If A is an infinite atomic Boolean algebra then* $\mathrm{Inc}_{mm}(A) \leq |\mathrm{At}(A)|$.

Proof. Since A is atomic, $|A| \leq 2^{|\mathrm{At}(A)|}$ so A infinite implies $\mathrm{At}(A)$ is infinite. Thus by Lemma 3.0.1, $\mathrm{At}(A)$ is an infinite pairwise incomparable set. As it is also dense, it is maximal. □

Theorem 3.1.2. *Let A be an atomic Boolean algebra. If there is some infinite $B \subseteq \mathrm{At}(A)$ such that $\sum B$ exists then* $\mathrm{Inc}_{mm}(A) \leq |B|$.

Proof. If $\sum B = 1$ then $B = \mathrm{At}(A)$ and the conclusion is merely $\mathrm{Inc}_{mm}(A) \leq |\mathrm{At}(A)|$ which is known. So suppose $\sum B < 1$. Then $I = B \cup \{-\sum B\}$ is maximal incomparable. Indeed, I is pairwise disjoint hence pairwise incomparable. If $a \in A$ is incomparable to every $b \in B$ then $a \cdot b = 0$ for all $b \in B$ so $a \cdot \sum B = 0$ and $a \leq -\sum B$. Thus, I is maximal. □

Corollary 3.1.3. $\mathrm{Inc}_{mm}(\mathcal{P}(\kappa)) = \aleph_0$.

Definition. *Let X be any set.* $\mathrm{Fin}(X) = \{Y \subseteq X : |Y| < \aleph_0\}$, $\mathrm{Cof}(X) = \{Y \subseteq X : |X \setminus Y| < \aleph_0\}$ *and* $\mathrm{FinCof}(X) = \mathrm{Fin}(X) \cup \mathrm{Cof}(X)$.

$\mathrm{FinCof}(X)$ forms a Boolean algebra using the normal set operations. This algebra excludes as many infinite sums as possible for an atomic algebra and in that way is the opposite of the case considered in the previous theorem, at least so far as atomic algebras go. We now turn our attention towards it.

Lemma 3.1.4. *Let κ be a cardinal and $X \subseteq \mathrm{FinCof}(\kappa)$ be infinite. Put $X_0 = X \cap \mathrm{Fin}(\kappa)$ and $X_1 = \{\kappa \setminus x : x \in X \cap \mathrm{Cof}(\kappa)\}$. Then X is maximal incomparable if and only if:*

1. *X_0 and X_1 are each pairwise incomparable.*

2. *For all $\langle x_0, x_1 \rangle \in X_0 \times X_1$, $x_0 \cap x_1 \neq \emptyset$.*

3. *X_0 and X_1 are maximal subsets of $\mathrm{Fin}(\kappa)$ with properties 1. and 2., in other words, for any $x \in \mathrm{Fin}(\kappa)$ the following holds for $i \in 2$: if x is incomparable to every member of X_i then $x \cap y = \emptyset$ for some $y \in X_{1-i}$.*

Proof. First, assume that 1., 2. and 3. hold. Take $x, y \in X$. If $x, y \in X \cap \mathrm{Fin}(\kappa)$ then by 1. x and y are incomparable. If $x, y \in X \cap \mathrm{Cof}(\kappa)$ then by 1. $\kappa \setminus x$ and $\kappa \setminus y$ are incomparable which shows x and y are incomparable. If $x \in X \cap \mathrm{Fin}(\kappa)$ and $y \in x \cap \mathrm{Cof}(\kappa)$, then by 2. $x \cap (\kappa \setminus y) = x \setminus y \neq \emptyset$ so $x \nsubseteq y$ but since x is finite and y infinite, $y \nsubseteq x$ as well. Thus, X is an infinite pairwise incomparable set. Take $z \in \mathrm{FinCof}(\kappa)$ and assume that z is incomparable to every member of X.

Case 1: $z \in \mathrm{Fin}(\kappa)$. Then z is incomparable to every member of X_0 so by 3. there is some $y \in X_1$ such that $z \cap y = \emptyset$. But $y = \kappa \setminus w$ for some $w \in X$ so $z \subseteq w$. Contradiction.

Case 2: $z \in \mathrm{Cof}(\kappa)$. Put $w = \kappa \setminus z$. Then w is incomparable to every member of X_1 because z is incomparable to every member of $X \cap \mathrm{Cof}(\kappa)$. Thus, by 3., there is some $y \in X_0$ such that $w \cap y = \emptyset$. But then $y \subseteq z$. Contradiction. This shows that if 1. 2. and 3. hold then X is maximal.

Conversely, suppose that X is maximal incomparable. Then clearly X_0 and X_1 are each incomparable. If $\langle x_0, x_1 \rangle \in X_0 \times X_1$, then since $x_0 \nsubseteq \kappa \setminus x_1$, $x_0 \cap x_1 \neq \emptyset$. Thus 1. and 2. hold. To see that 3. holds, take $x \in \mathrm{Fin}(\kappa)$, $i \in 2$ and suppose that x is incomparable to every member of X_i.

Case 1: $i = 0$. Since x is finite and incomparable with every member of X_0 it follows $x \notin X$. By maximality, x is comparable to some member of X, say y. By assumption, $y \notin X_0$. Then $y \in \mathrm{Cof}(\kappa)$ and since $x \in \mathrm{Fin}(\kappa)$ is must be that $x \subseteq y$. Further, $\kappa \setminus y \in X_1$ and $x \cap (\kappa \setminus y) = \emptyset$.

Case 2: $i = 1$. If $\kappa \setminus x \in X$ then $\kappa \setminus x \in X \cap \mathrm{Cof}(\kappa)$ and $x \in X_1$, contradiction. So $x \notin X$. Again, $\kappa \setminus x$ is comparable to some $y \in X$. If $y \in X \cap \mathrm{Cof}(\kappa)$ then x and $\kappa \setminus y$ are comparable which is against our assumption. Thus, $y \in X \cap \mathrm{Fin}(\kappa) = X_0$. Since $y \in \mathrm{Fin}(\kappa)$ and $\kappa \setminus x \in \mathrm{Cof}(\kappa)$ it must be that $y \subseteq \kappa \setminus x$. Then $y \cap x = \emptyset$ and $y \in X_0$. □

Lemma 3.1.5. *Let $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ be sequences of elements of* $\mathrm{Fin}(\kappa)$ *such that for all $n \in \omega$:*

(1) $\{x_i : i \leq n\}$ is incomparable and $x_n \cap y_j \neq \emptyset$ for all $j < n$,

(2) $\{y_i : i \leq n\}$ is incomparable and $y_n \cap x_j \neq \emptyset$ for all $j \leq n$.

Set $X_0 = \{x_n : n \in \omega\}$ and $X_1 = \{y_n : n \in \omega\}$. Then X_0 and X_1 satisfy conditions 1. and 2. of Lemma 3.1.4 although 3. may fail.

Proof. The fact that X_0 and X_1 are incomparable is obvious. Take $\langle z_0, z_1 \rangle \in X_0 \times X_1$ where

$z_0 = x_n$ and $z_1 = \kappa \setminus y_m$ for some $n, m \in \omega$. If $m < n$ then (1) ensures that $x_n \cap y_m \neq \emptyset$. If $n \leq m$ then (2) ensures that $y_m \cap x_n \neq \emptyset$. In any case, 2. holds. $\square$

Let $E, O \subseteq \omega$ be the sets of even and odd numbers, respectively. Define $E_n = E \cap n$, $O_n = O \cap n$, $x_n = O_{2n} \cup \{2n\}$ and $y_n = E_{2n+1} \cup \{2n+1\}$ for all $n \in \omega$.

n	x_n	y_n
0	$\{0\}$	$\{0,1\}$
1	$\{1,2\}$	$\{0,2,3\}$
2	$\{1,3,4\}$	$\{0,2,4,5\}$
3	$\{1,3,5,6\}$	$\{0,2,4,6,7\}$
4	$\{1,3,5,7,8\}$	$\{0,2,4,6,8,9\}$

Lemma 3.1.6. *Given the above definitions,*

(1) $\{x_n : n \in \omega\}$ *and* $\{y_n : n \in \omega\}$ *satisfy conditions 1. and 2. of Lemma 3.1.5,*

(2) if $S \subseteq \omega$ *satisfies* $x_n \nsubseteq S$ *and* $y_n \cap S \neq \emptyset$ *for all* $n \in \omega$ *then* $S = O$ *and*

(3) if $S \subseteq \omega$ *satisfies* $y_n \nsubseteq S$ *and* $x_n \cap S \neq \emptyset$ *for all* $n \in \omega$ *then* $S = E$.

Proof. (1) Suppose $j < n$. Then by definition $2n \in x_n \setminus x_j$ and $2j \in x_j \setminus x_n$ which shows $\{x_j : i \leq n\}$ is incomparable. Also $x_n \cap y_j = \{2j+1\}$ which completes the verification of condition 1. from Lemma 3.1.5.

Similarly, $2n+1 \in y_n \setminus y_j$ and $2j+1 \in y_j \setminus y_n$ whenever $j < n$ which confirms that $\{y_i : i \leq n\}$ is incomparable. The equation $y_n \cap x_j = \{2j\}$ holds whether $j < n$ or $j = n$. Thus, condition 2. from Lemma 3.1.5 is seen to hold.

(2) Suppose $S \subseteq \omega$ satisfies $x_n \nsubseteq S$ and $y_n \cap S \neq \emptyset$ for all $n \in \omega$. Proceed by induction to show that $2n \notin S$ and $2n+1 \in S$ for all $n \in \omega$. The case $n = 0$ is clear because by assumption $x_0 = \{0\} \nsubseteq S$ and $y_0 = \{0,1\} \cap S \neq \emptyset$, corresponding to the facts that $0 \notin S$ and $1 \in S$. Now suppose the statement holds for all $k < n$. So $S \cap 2n = O_{2n}$

and since $x_n \nsubseteq S$ it must be that $2n \notin S$. So $E_{2n+1} \cap S = \emptyset$ and the assumption that $y_n \cap S \neq \emptyset$ gives $2n + 1 \in S$.

(3) This is similar to part (2) but we include the proof. Under the assumption that $S \subseteq \omega$ satisfies $y_n \nsubseteq S$ and $x_n \cap S \neq \emptyset$ for all $n \in \omega$, prove $2n \in S$ and $2n+1 \notin S$ by induction. For $n = 0$, note that $x_0 = \{0\} \cap S \neq \emptyset$ shows $0 \in S$ and $y_0 = \{0, 1\} \nsubseteq S$ shows $1 \notin S$. Suppose the claim is true for $k < n$. Then $S \cap 2n = E_{2n}$ and since $x_n \cap S \neq \emptyset$ it must be that $2n \in S$. So $E_{2n+1} \subseteq S$. In order for $y_n \nsubseteq S$ it must be that $2n + 1 \notin S$.

□

Corollary 3.1.7. *$X_0 = \{x_n : n \in \omega\}$ and $X_1 = \{y_n : n \in \omega\}$ satisfy condition 1. 2. and 3. of Lemma 3.1.4.*

Proof. It is previously established that conditions 1. and 2. are satisfied. Take $x \in \mathrm{Fin}(\kappa)$. First suppose that x is incomparable to x_n and yet $x \cap y_n \neq \emptyset$ for every $n \in \omega$. Put $S = x \cap \omega$. Then surely, $x_n \nsubseteq S$ because $S \subseteq x$ and $\emptyset \neq y_n \cap x = (y_n \cap \omega) \cap x = y_n \cap (x \cap \omega) = y_n \cap S$. This true of every $n \in \omega$ implies $S = O$ which implies $x \notin \mathrm{FinCof}(\kappa)$, a contradiction.

Now suppose that x is incomparable to y_n and yet $x \cap x_n \neq \emptyset$ for every $n \in \omega$. With $S = x \cap \omega$, it is now the case that $y_n \nsubseteq S$ and $x_n \cap S \neq \emptyset$ for all $n \in \omega$ and this time $S = E$ which implies $x \notin \mathrm{FinCof}(\kappa)$. □

Theorem 3.1.8 (Scherer)**.** *For any infinite cardinal κ,* $\mathrm{Inc}_{mm}(\mathrm{FinCof}(\kappa)) = \aleph_0$.

So in the case of atomic algebras both completeness and abject incompleteness result in small incomparability being countable. This however, is not a consequence of atomicity. To construct an atomic algebra with small incomparability uncountable, countable infinite sums of atoms must in all cases be avoided but sums of larger arity must be allowed. This is done below.

Lemma 3.1.9. *Let κ be uncountable, $I \in [\mathrm{FinCof}(\kappa)]^{\aleph_0}$ be pairwise incomparable in the algebra $\mathcal{P}(\kappa)$ and*

$$C = \bigcup_{i \in I \cap \mathrm{Fin}(\kappa)} i \cup \bigcup_{i \in I \cap \mathrm{Cof}(\kappa)} \kappa \setminus i.$$

Then there is some $J \subseteq C$ such that for every $X \subseteq \kappa \setminus C$ with $|X| = |\kappa \setminus X| = \kappa$, $\{J \cup X\} \cup I$ is pairwise incomparable in $\mathcal{P}(\kappa)$.

Proof. Let $I_f = I \cap \mathrm{Fin}(\kappa)$ and $I_c = I \cap \mathrm{Cof}(\kappa)$.

Case 1: I_f is infinite. Let $\{i_n : n < \delta\}$ be a one-one enumeration of I_c where $\delta \leq \omega$. If δ is finite set $\delta' = \delta + 1$, otherwise put $\delta' = \omega$. Define an descending chain of infinite subsets of I_f, $\{L_n : n < \delta'\}$, and an ascending chain of subsets of κ, $\{h_n : n < \delta'\}$, as follows. Put $L_0 = I_f$ and $h_0 = \emptyset$. Suppose $n < \delta$, L_n is an infinite subset of I_f and h_n satisfies $h_n \subsetneq i$ for all $i \in L_n$. By the incomparability of I, $i \nsubseteq i_n$ and therefore $i \cap (\kappa \setminus i_n) \neq \emptyset$ for all $i \in L_n$. Since $\kappa \setminus i_n$ is finite, there is some $d_n \in \kappa \setminus i_n$ such that $L_{n+1} = \{i \in L_n : d_n \in i \cap (\kappa \setminus i_n)\}$ is infinite. Put $h_{n+1} = h_n \cup \{d_n\}$. Then the assumption that L_n is infinite and $h_n \subsetneq i$ for all $i \in L_n$ is preserved at every $n < \delta'$.

Set $J = \bigcup_{n<\delta'} h_n$. First, note that $i \nsubseteq h_n$ for all $i \in I_f$. Otherwise, $i \subseteq h_n$ for some $n < \delta'$. But since L_n is infinite, we can take $j \in L_n \setminus \{i\}$ to violate pairwise incomparability by way of the inequality $i \subseteq h_n \subseteq j$. Second, note that $J \nsubseteq i_n$ for all $n < \delta$ because $d_n \in h_{n+1} \cap (\kappa \setminus i_n) \subseteq J \cap (\kappa \setminus i_n)$.

Now take any $X \subseteq (\kappa \setminus C)$ such that $|X| = |\kappa \setminus X| = \kappa$. Since $J \cup X$ is infinite it will never be that $J \cup X \subseteq i$ for any $i \in I_f$. If $i \subseteq J \cup X$ for some $i \in I_f$ then $i = i \cap C \subseteq (J \cup X) \cap C = J$ which has already been precluded. Similarly, $i \subseteq X \cup J$ can not hold for $i \in I_c$ because $\kappa \setminus X$ is uncountable, C is countable and so $(\kappa \setminus X) \setminus C = \kappa \setminus (X \cup C) \subseteq \kappa \setminus (X \cup J)$ is uncountable. Finally, $J \nsubseteq i$ implies $J \cup X \nsubseteq i$ for $i \in I_c$.

Case 2: I_f is finite. Then I_c is infinite. Let $I' = \{\kappa \setminus i : i \in I\}$, $C' = \bigcup_{i \in I' \cap \mathrm{Fin}(\kappa)} i \cup \bigcup_{i \in I' \cap \mathrm{Cof}(\kappa)} \kappa \setminus i$, $I'_f = I' \cap \mathrm{Fin}(\kappa)$ and $I'_c = I' \cap \mathrm{Cof}(\kappa)$. Then I'_f is infinite and the proof of case 1 gives $J' \subseteq C'$ such that for every $X \subseteq \kappa \setminus C'$ with $|X| = |\kappa \setminus X| = \kappa$, $\{J' \cup X\} \cup I'$ is pairwise incomparable.

Let $J = C \setminus J'$ and take $X \subseteq \kappa \setminus C$ such that $|X| = |\kappa \setminus X| = \kappa$. Note that $I'_f = \{\kappa \setminus i : i \in I \cap \mathrm{Cof}(\kappa)\}$ and $I'_c = \{\kappa \setminus i : i \in I \cap \mathrm{Fin}(\kappa)\}$. Thus,

$$C' = \bigcup_{i \in I' \cap \mathrm{Fin}(\kappa)} i \cup \bigcup_{i \in I' \cap \mathrm{Cof}(\kappa)} \kappa \setminus i = \bigcup_{i \in I \cap \mathrm{Cof}(\kappa)} \kappa \setminus i \cup \bigcup_{i \in I \cap \mathrm{Fin}(\kappa)} i = C.$$

So $X \subseteq C'$ and $\{J' \cup X\} \cup I'$ is pairwise incomparable. The cases $J \cup X \subseteq i$ for $i \in I_f$ or $i \subseteq J \cup X$ for $i \in I_c$ are discounted on the grounds that $J \cup X$ is infinite and coinfinite. If $i \subseteq J \cup X$ for $i \in I_f$ then $i = i \cap C \subseteq (J \cup X) \cap C = J$ so $J' = C \setminus J \subseteq C \setminus i$. But then $J' \cup X \subseteq C \setminus i \cup (\kappa \setminus C) = \kappa \setminus i$, which is against the choice of J'. If $J \cup X \subseteq i$ for $i \in I_c$ then $J = (J \cup X) \cap C \subseteq i \cap C$ so $\kappa \setminus i = C \setminus i \subseteq C \setminus J = J' \subseteq J' \cup X$ where $\kappa \setminus i \in I'$ which again violates the choice of J'. □

Theorem 3.1.10 (Scherer)**.** *There is an atomic Boolean algebra with no countably infinite maximal incomparable subset.*

Proof. First note that $(\mathfrak{c}^+)^{\aleph_0} = \mathfrak{c}^+$ by the Hausdorff formula [4, p. 57].

The construction proceeds by building an ascending chain $\{A_\alpha : \alpha < \mathfrak{c}^+\}$ of subalgebras of $\mathcal{P}(\mathfrak{c}^+)$ such that no countably infinite pairwise incomparable subset of $\bigcup_{\alpha<\mathfrak{c}^+} A_\alpha$ is maximal. By way of bookkeeping, choose a surjection $B : \mathfrak{c}^+ \to \mathfrak{c}^+ \times [\mathfrak{c}^+]^{\aleph_0}$ with the property that for all $\langle \alpha, x\rangle \in \mathfrak{c}^+ \times [\mathfrak{c}^+]^{\aleph_0}$, $|B^{-1}[\{\langle \alpha, x\rangle\}]| = \mathfrak{c}^+$. Start the construction with $A_0 = \mathrm{FinCof}(\mathfrak{c}^+)$.

Suppose that for some $\beta < \mathfrak{c}^+$, we are given an initial segment $\{A_\alpha : \alpha < \beta\}$ together with bijections $a_\alpha : A_\alpha \to \mathfrak{c}^+$ for every $\alpha < \beta$ and that the terms of this segment satisfy:

1. $|A_\alpha / \mathrm{Fin}(A_\alpha)| \leq \mathfrak{c}$ and

2. for every $y \in A_\alpha \setminus \mathrm{FinCof}(\mathfrak{c}^+)$, $|y| = |\mathfrak{c}^+ \setminus y| = \mathfrak{c}^+$.

The present task is to construct A_β and a_β preserving these properties. Start by writing $U_\beta = \bigcup_{\alpha<\beta} A_\alpha$ and $B(\beta) = \langle \gamma, x\rangle$.

Case 1: $\gamma < \beta$ and $a_\gamma^{-1}[x]$ is incomparable. Write $I = a_\gamma^{-1}[x]$, $I_0 = I \cap \operatorname{FinCof}(\mathfrak{c}^+)$ and $I_1 = I \setminus I_0$. Put

$$C = \bigcup_{i \in I_0 \cap \operatorname{Fin}(\mathfrak{c}^+)} i \cup \bigcup_{i \in I_0 \cap \operatorname{Cof}(\mathfrak{c}^+)} \mathfrak{c}^+ \setminus i.$$

Take J as in Lemma 3.1.9 and let N_β be a full system representatives for $U_\beta / \operatorname{Fin}(\mathfrak{c}^+) \setminus \{\operatorname{Fin}(\mathfrak{c}^+), \mathfrak{c}^+/\operatorname{Fin}(\mathfrak{c}^+)\}$, so $|N_\beta \cap x| = 1$ for every $x \in U_\beta/\operatorname{Fin}(\mathfrak{c}^+) \setminus \{\operatorname{Fin}(\mathfrak{c}^+), \mathfrak{c}^+/\operatorname{Fin}(\mathfrak{c}^+)\}$. Since $\mathfrak{c}^+$ is regular, $|A_\alpha/\operatorname{Fin}(\mathfrak{c}^+)| \leq \mathfrak{c}$ for every $\alpha < \beta$ and $\beta < \mathfrak{c}^+$, it follows $|N_\beta| \leq \mathfrak{c}$ so write $N_\beta = \{y_\xi : \xi < \mathfrak{c}\}$ with repeats allowed.

We now construct sequences $\{\alpha_\xi : \xi < \mathfrak{c}^+\}$, $\{\beta_{\xi,\eta} : \langle \xi, \eta\rangle \in \mathfrak{c}^+ \times \mathfrak{c}\}$ to supply elements for an X to pair with J. Take $\alpha_0 = \sup\{\eta + 1 : \eta \in C\}$. Suppose α_ξ is given. By the fact that $y_\xi/\operatorname{Fin}(\mathfrak{c}^+) \notin \{\operatorname{Fin}(\mathfrak{c}^+), \mathfrak{c}^+/\operatorname{Fin}(\mathfrak{c}^+)\}$, $y_\xi \notin \operatorname{FinCof}(\mathfrak{c}^+)$. So by assumption, $|y_\xi| = |\mathfrak{c}^+ \setminus y_\xi| = \mathfrak{c}^+$. Therefore, y_ξ and $\mathfrak{c}^+ \setminus y_\xi$ are unbounded in $\mathfrak{c}^+$ and it is possible to iteratively choose $\beta_{\xi,\eta} > \max(\{\alpha_\xi\} \cup \{\beta_{\xi,\zeta} : \zeta < \eta\})$ such that $\beta_{\xi,2\eta} \in y_\eta$ and $\beta_{\xi,2\eta+1} \notin y_\eta$ for all $\eta \in \mathfrak{c}$. Put $\alpha_{\xi+1} = \sup_{\eta\in\mathfrak{c}} \beta_{\xi,\eta}$. If γ is limit, let $\alpha_\gamma = \sup_{\xi<\gamma} \alpha_\xi$. By regularity, α_ξ and $\beta_{\xi,\eta}$ lie in $\mathfrak{c}^+$ for every $\xi < \mathfrak{c}^+$ and $\eta < \mathfrak{c}$.

Let

$$X = \{\beta_{2\xi,2\eta} : \langle \xi, \eta\rangle \in \mathfrak{c}^+ \times \mathfrak{c}\} \cup \{\beta_{2\xi,2\eta+1} : \langle \xi, \eta\rangle \in \mathfrak{c}^+ \times \mathfrak{c}\}.$$

Now for every $\xi \in C$, $\xi < \alpha_0 \leq \min(X)$ so $X \subseteq \mathfrak{c}^+ \setminus C$ and the following inequalities show that $|X \setminus y_\xi| = |y_\xi \setminus X| = |X \cap y_\xi| = |X| = |\mathfrak{c}^+ \setminus X| = \mathfrak{c}^+$,

$$\{\beta_{2\xi,2\eta} : \xi \in \mathfrak{c}^+\} \subseteq X \cap y_\eta \subseteq X$$

$$\{\beta_{2\xi,2\eta+1} : \xi \in \mathfrak{c}^+\} \subseteq X \setminus y_\eta$$

$$\{\beta_{2\xi+1,2\eta} : \xi \in \mathfrak{c}^+\} \subseteq y_\eta \setminus X \subseteq \mathfrak{c}^+ \setminus X$$

Put $b_\beta = J \cup X$. Define $A_\beta = \langle U_\beta \cup \{b_\beta\}\rangle$ and choose a bijection $a_\beta : A_\beta \to \mathfrak{c}^+$. Conditions 1. and 2. must be verified for A_β. The fact that $|A_\beta/\operatorname{Fin}(\mathfrak{c}^+)| \leq \mathfrak{c}$ is clear because $A_\beta/\operatorname{Fin}(\mathfrak{c}^+) = \langle U_\beta/\operatorname{Fin}(\mathfrak{c}^+) \cup \{b_\beta/\operatorname{Fin}(\mathfrak{c}^+)\}\rangle$ and it was previously noted that $|U_\beta/\operatorname{Fin}(\mathfrak{c}^+)| \leq \mathfrak{c}$. Towards 2., take $y \in A_\beta \setminus \operatorname{FinCof}(\mathfrak{c}^+)$ and suppose that $|y| < \mathfrak{c}^+$ or $|\mathfrak{c}^+ \setminus y| < \mathfrak{c}^+$. Since

A_β is a subalgebra of $\mathcal{P}(\mathfrak{c}^+)$, there will be a y satisfying one of these inequalities if and only if there is a y satisfying the other. So we may as well assume that $|y| < \mathfrak{c}^+$. Write $y = (z_0 \cap b_\beta) \cup (z_1 \setminus b_\beta)$ for $z_0, z_1 \in U_\beta$ [5, Cor 4.7]. Since $y \notin \mathrm{Fin}(\mathfrak{c}^+)$, z_i is infinite for some $i \in 2$. Now $J \subseteq C$ and C is countable so the fact that $|X| = |\mathfrak{c}^+ \setminus X| = \mathfrak{c}^+$ implies $|b_\beta| = |\mathfrak{c}^+ \setminus b_\beta| = \mathfrak{c}^+$. Thus, if z_i is coinfinite then $|z_i \cap b_\beta| = |z_i \setminus b_\beta| = \mathfrak{c}^+$ which is impossible because $|y| < \mathfrak{c}^+$. Ergo, $z_i/\mathrm{Fin}(\mathfrak{c}^+) \notin \{\mathrm{Fin}(\mathfrak{c}^+), \mathfrak{c}^+/\mathrm{Fin}(\mathfrak{c}^+)\}$ so $z_i/\mathrm{Fin}(\mathfrak{c}^+) = y_\xi/\mathrm{Fin}(\mathfrak{c}^+)$ for some $\xi < \mathfrak{c}$. But $|y_\xi \cap X| = |y_\xi \setminus X| = \mathfrak{c}^+$ and this contradicts $|y| < \mathfrak{c}^+$.

Case 2: $\neg$(Case 1). Let $A_\beta = U_\beta$ and choose any bijection $a_\beta : A_\beta \to \mathfrak{c}^+$.

This completes the construction of $\{A_\alpha : \alpha < \mathfrak{c}^+\}$ and $\{a_\alpha : \alpha < \mathfrak{c}^+\}$. Let $I \subseteq \bigcup_{\alpha<\mathfrak{c}^+} A_\alpha$ be a countably infinite pairwise incomparable set. Take β large enough that $I \subseteq A_\beta$ and $\gamma > \beta$ such that $B(\gamma) = \langle \beta, a_\beta[I] \rangle$. Then when A_γ is being constructed, case 1 is in effect. So by Lemma 3.1.9, b_γ is incomparable to every member of $I \cap \mathrm{FinCof}(\mathfrak{c}^+)$. Consider $i \in I \setminus \mathrm{FinCof}(\mathfrak{c}^+)$. Then $i/\mathrm{Fin}(\mathfrak{c}^+) = y/\mathrm{Fin}(\mathfrak{c}^+)$ for some $y \in N_\gamma$ and $|i \setminus b_\gamma| = |y \setminus b_\gamma| = |b_\gamma \setminus y| = |b_\gamma \setminus i| = \mathfrak{c}^+$. $\square$

3.2 Tree algebras

Definition. *A **tree** is a partially ordered set $(T, \leq)$ such that for every $t \in T$, $\{s \in T : s < t\}$ is well ordered. If T is a tree and $t \in T$, $\mathrm{ht}_T(t)$ is the ordinal giving the order type of $\{s \in T : s < t\}$. For any ordinal α, $\mathrm{Lev}_\alpha(T) = \{t \in T : \mathrm{ht}_T(t) = \alpha\}$. The elements of $\mathrm{Lev}_0(T)$ are called **roots**. Finally, the **height** of T is defined to be* $\sup\{\mathrm{ht}_T(t) + 1 : t \in T\}$.

Definition. *Let T be a tree. The **tree algebra** on T,* $\mathrm{TreeAlg}(T)$ *is the subalgebra of $\mathcal{P}(T)$ generated by $\{\{s \in T : t \leq s\} : t \in T\}$.*

See [5, Section 16] for the basic facts of tree algebras.

Let T be a tree of height ω with more than one root such that for every $n \in \omega$ and $t \in \mathrm{Lev}_n(T)$ there are distinct $s_0, s_1 \in \mathrm{Lev}_{n+1}(T)$ with $t < s_0$ and $t < s_1$. For every $t \in T$, fix some such choice of s_0 and s_1 and denote them by $\overleftarrow{t}$ and $\overrightarrow{t}$. Furthermore, for any $m, n \in \omega$

with $m \leq n$ and $t \in \text{Lev}_n(T)$ let $t \upharpoonright m$ be the unique $s \in \text{Lev}_m(T)$ such that $s \leq t$.

Given this tree define $I_0 = \{\{t\} : t \in \text{Lev}_0(T)\}$. Now supposing I_n is at hand for some $n \in \omega$, define I_{n+1} to be the set of all $i \in [\text{Lev}_{n+1}(T)]^{n+2}$ such that there is some $j \in I_n$ with the following properties:

1) for every $s \in j$ there is some $t \in i$ such that $s < t$,

2) there is some $t \in i$ such that $s \not< t$ for all $s \in j$.

Whenever these two properties hold between sets $j \in I_n$ and $i \in I_{n+1}$, say that i amends j. Some observations are immediate from this definition. First, that $I_n \subseteq [\text{Lev}_n(T)]^{n+1}$ for all $n \in \omega$. Second, that if i amends j then $|i| = |j| + 1$.

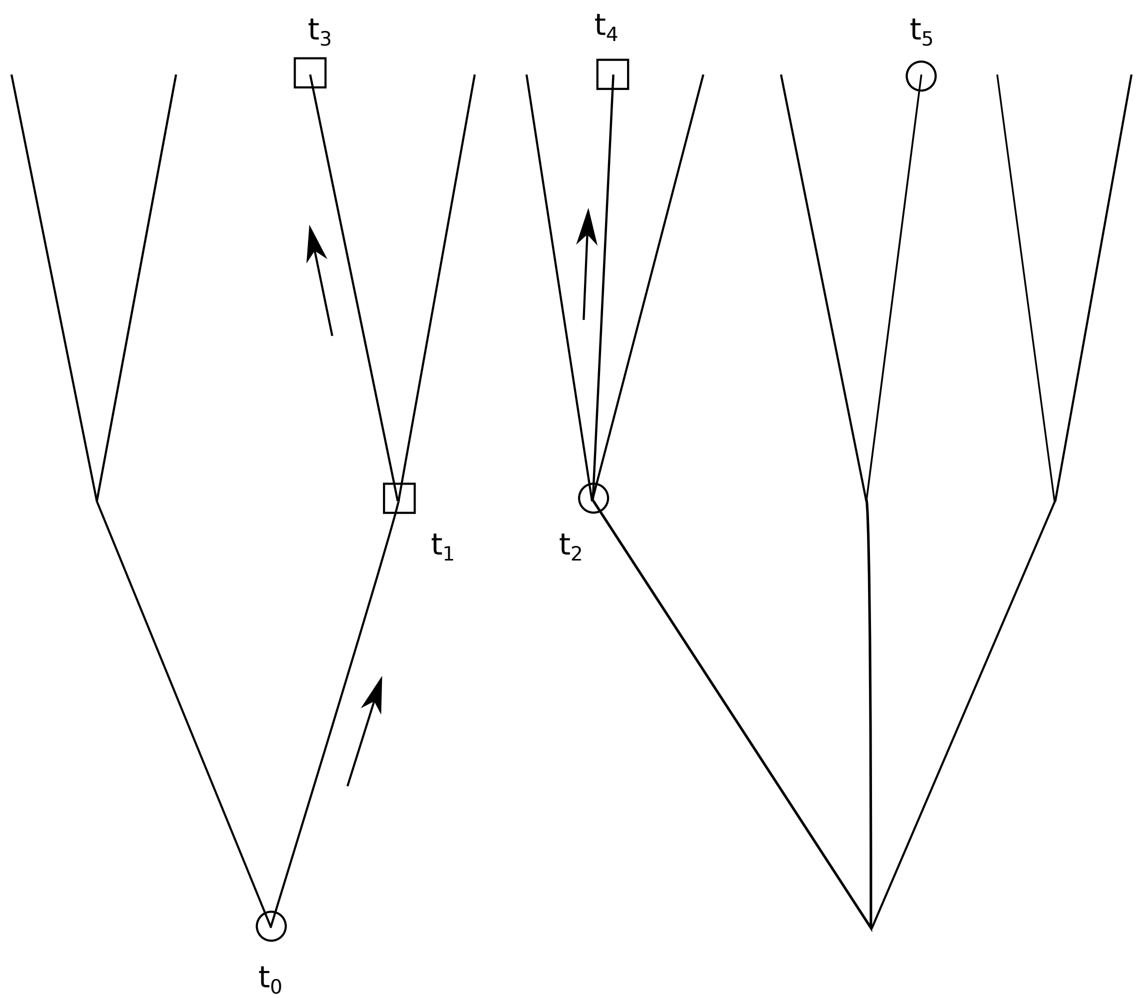

Figure 3.1: $\{t_1, t_2\}$ amends $\{t_0\}$ and is amended by $\{t_3, t_4, t_5\}$.

Squares witness condition 1) and circles condition 2).

Lemma 3.2.1. *Let $j \leq k$, $i_0 \in I_j$, $i_1 \in I_k$ and suppose that $i_0 \neq i_1$. Then*

i) There is some $t_0 \in \mathrm{Lev}_k(T)$ such that

$$t \leq t_0 \text{ for some } t \in i_0, \text{ and } t_0 \notin i_1$$

ii) There is some $t_1 \in i_1$ such that

$$t \nleq t_1 \text{ for all } t \in i_0.$$

Proof. Work by induction on $k - j$.

If $k - j = 0$ then i_0 and i_1 are distinct $k + 1$ element subsets of $\mathrm{Lev}_k(T)$. So we can take $t_0 \in i_0 \setminus i_1$ and $t_1 \in i_1 \setminus i_0$. Since for any $s_0, s_1 \in \mathrm{Lev}_k(T)$, $s_0 \leq s_1$ is equivalent to $s_0 = s_1$, the requirements become first that

$$t = t_0 \text{ for some } t \in i_0, \text{ and } t_0 \notin i_1$$

and second that

$$t \neq t_1 \text{ for all } t \in i_0.$$

These are trivially satisfied by the choice of t_0 and t_1. So suppose for the remainder that $k - j > 0$.

Case 1: i_1 amends i_0. Then t_1 is provided by requirement 2) of the definition of i_1 amending i_0. Choose any $s \in i_0$ and $t \in i_1$ with $s < t$. Since s has at least two immediate successors, it is possible to choose $t_0 \in \mathrm{Lev}_k(T)$ with $s < t_0$ and $t_0 \neq t$. It remains to be seen that $t_0 \notin i_1$. Assume the contrary. For every $u \in i_0$, let $S_u = \{v \in i_1 : u < v\}$. By definition $|S_u| \geq 1$ for every $u \in i_0$ and by assumption $|S_s| \geq 2$. Now if u and u' are distinct elements of i_0 then $S_u \cap S_{u'} = \emptyset$. Otherwise there would be $v \in S_u \cap S_{u'}$, in which case $u < v$ and $u' < v$ would imply that u and u' are comparable which is absurd because $\{u, u'\} \subseteq i_0 \subseteq \mathrm{Lev}_j(T)$. Thus, $\{S_u : u \in i_0\}$ is a collection of pairwise disjoint sets. So

$\left|\bigcup_{u\in i_0} S_u\right| = \sum_{u\in i_0} |S_u|$. This sum has $|i_0| = j+1$ terms each of which are at least 1, and one of which (namely S_s) is at least 2. Therefore, the sum is at least $j+2$. However, since i_1 amends i_0, $|i_1| = |i_0| + 1 = j+2$. Thus $\bigcup_{u\in i_0} S_u = i_1$. Consider then that there must be some $u \in i_0$ with $t_1 \in S_u$, defying the choice of t_1. We have therefore found a contradiction stemming from the assumption that $t_0 \in i_1$.

Case 2: i_1 amends some $i \neq i_0$. Then by the inductive hypothesis, there is some $t_1' \in i$ such that $t \not\leq t_1'$ for all $t \in i_0$. Take $t_1 \in i_1$ with $t_1' < t_1$. Supposing that $t \leq t_1$ for some $t \in i_0$, then t and t_1' would be comparable. But $i_0 \subseteq \mathrm{Lev}_j(T)$, $i \subseteq \mathrm{Lev}_{k-1}(T)$ and $j \leq k-1$ so the possibility that $t_1' < t$ is eliminated. Since $t \leq t_1'$ violates the choice of t_1', it must be that $t \not\leq t_1$.

Again by the inductive hypothesis there is some $t_0' \in \mathrm{Lev}_{k-1}(T)$ such that $t \leq t_0'$ for some $t \in i_0$ and $t_0' \notin i$. One of $\overrightarrow{t_0'}$ or $\overleftarrow{t_0'}$ must lie outside i_1. To see this, once again consider $S_u = \{v \in i_1 : u < v\}$. Since $|S_u| \geq 1$ for all $u \in i$, and $S_u \cap S_{u'} = \emptyset$ for all distinct $u, u' \in i$, $\left|\bigcup_{u\in i} S_u\right| \geq |i| = |i_1| - 1$. If both $\overrightarrow{t_0'}$ and $\overleftarrow{t_0'}$ were in i_1, then since at most one element of i_1 lies outside $\bigcup_{u\in i} S_u$, one of $\overrightarrow{t_0'}$ or $\overleftarrow{t_0'}$ must lie in S_u for some $u \in i$. But then u and t_0' are comparable. Since $\{u, t_0'\} \subseteq \mathrm{Lev}_{k-1}(T)$, this would mean $u = t_0'$ violating the fact that $t_0' \notin i$. Thus, $\overrightarrow{t_0'}$ or $\overleftarrow{t_0'}$ can not both be in i_1 so we can pick t_0 to be whichever is not. □

Consider the following game. In round k, Player 1 will choose $t_k \in T$ and Player 2 will choose $e_k \in I_k$. They are bound by the following rules:

I. If $k > 0$ then Player 1 must choose t_k such that $\mathrm{ht}_T(t_k) > \mathrm{ht}_T(t_{k-1})$,

II. If $k > 0$ then Player 1 must choose t_k such that $t \not\leq t_k$ for all $t \in e_{k-1}$.

III. For every $k > 0$, Player 2 must choose e_k such that it amends e_{k-1}.

Lemma 3.2.2. *Let $t \in \mathrm{Lev}_n(T)$ for some $n \in \omega$,*

i) $|\mathrm{Lev}_k(T)| \geq 2^{k+1}$, *ii)* $|\{s \in \mathrm{Lev}_{n+j}(T) : t \leq s\}| \geq 2^j$,

iii) *If* $j \in I_k$, $i \subseteq \mathrm{Lev}_{k+1}(T)$, $|i| \leq k+2$ *and* i *satisfies conditions 1) and 2) of amending* j *then* $|i| = k+2$ *so* $i \in I_{k+1}$. *Hence also in this case* i *amends* j.

iv) *Each player has a valid move for every round* k.

Proof. i) and ii) are obvious consequences of that fact that $|\mathrm{Lev}_0(T)| > 1$ and every element of T has at least two successors. So suppose i and j are as stipulated in iii). For every $s \in j$ choose $t_s \in i$ such that $s \leq t_s$. Also pick $t_\perp \in i$ such that $s \not\leq t_\perp$ for all $s \in j$. If $t_s = t_{s'}$ then s and s' are comparable elements of $\mathrm{Lev}_k(T)$ which shows $s = s'$. Therefore $s \neq s'$ implies $t_s \neq t_{s'}$. Since $t_\perp \neq t_s$ by its definition,

$$|i| \geq |\{t_s : s \in j\} \cup \{t_\perp\}| = |j| + 1 = k + 2.$$

Therefore $|i| = k+2$, and i fulfills all the requirements for $i \in I_{k+1}$ and for i to amend j.

As for iv), at round $k = 0$, the players are completely free in their choices. So suppose $k > 0$. Since $e_{k-1} \in [\mathrm{Lev}_{k-1}(T)]^k$ and $k < 2^k$ for all $k \in \omega$, there is some $s \in \mathrm{Lev}_{k-1}(T) \setminus e_{k-1}$. If for any $t \in e_{k-1}$, $t \leq \overrightarrow{s}$ then s and t would be comparable elements of $\mathrm{Lev}_{k-1}(T)$, implying $s = t \in e_{k-1}$. Thus $t \not\leq \overrightarrow{s}$ for all $t \in e_{k-1}$. Consequently, $e_k = \left\{\overrightarrow{t} : t \in e_{k-1}\right\} \cup \{\overrightarrow{s}\}$ is a valid move for Player 2, by part iii).

Set $n = \mathrm{ht}_T(t_{k-1}) + 1$ and note that $0 \leq \mathrm{ht}_T(t_0) < \mathrm{ht}_T(t_1) \cdots < \mathrm{ht}_T(t_{k-1}) < n$ so $k \leq n$. We have already seen that there must be some $s \in \mathrm{Lev}_{k-1}(T) \setminus e_{k-1}$. Taking $t_k \in \mathrm{Lev}_n(T)$ such that $s \leq t_k$ fulfills Player 1's obligations. □

We say Player 2 wins the game if at some round k:

$$\forall s \in e_k \exists j \leq k (t_j \leq s)$$

otherwise Player 1 wins.

Lemma 3.2.3. *Player 2 has a strategy to win the game no matter what Player 1 does.*

Proof. Let us play a game and describe what choices Player 2 should make at each round. Suppose Player 1 begins by playing $t_0 \in \mathrm{Lev}_N(T)$ for some $N \in \omega$. Put $L = N + 1$. Player 2 will divide her play into two phases.

Phase 1: On round 0, Player 2 begins by setting $f_0^0 = t_0 \upharpoonright 0$ and playing $e_0 = \{t_0 \upharpoonright 0\}$. This is valid because e_0 is surely a singleton subset of $\mathrm{Lev}_0(T)$.

Suppose that $0 < k \leq L$ and that Player 1 has offered t_k for play. Player 2 looks at her scratch paper and sees $e_{k-1} = \{f_0^{k-1}, f_1^{k-1}, ..., f_{k-1}^{k-1}\}$. She defines

$$f_j^k = \begin{cases} t_j \upharpoonright k & \text{if } k \leq \mathrm{ht}_T(t_j) \\ \\ \overrightarrow{f_j^{k-1}} & \text{otherwise} \end{cases}$$

for $j < k$ and $f_k^k = t_k \upharpoonright k$. The definition of f_k^k is valid because $0 \leq \mathrm{ht}_T(t_0) < \mathrm{ht}_T(t_1) < \cdots < \mathrm{ht}_T(t_k)$ which implies $\mathrm{ht}_T(t_k) \geq k$ so $f_k \upharpoonright k$ is a well defined element of T. She plays $e_k = \{f_0^k, f_1^k, ..., f_k^k\}$.

Does this move follow the rules? We must check that $e_k \in I_k$ and that it amends e_{k-1}. Since $e_k = \{f_0^k, f_1^k, ..., f_k^k\}$, $|e_k| \leq k + 1$. By part iii) of the Lemma 3.2.2, it only remains to check that e_k satisfies requirements 1) and 2) with respect to amending e_{k-1}.

Turning to requirement 1), observe that so long as Player 2 follows the given rule for defining f_j^k one has that $f_j^k = t_j \upharpoonright k$ whenever $j \leq k \leq \mathrm{ht}_T(t_j)$. So if $j \leq k - 1$ and $k \leq \mathrm{ht}_T(t_j)$ then $f_j^{k-1} = t_j \upharpoonright (k-1) < t_j \upharpoonright k = f_j^k$. Since $f_j^{k-1} < \overrightarrow{f_j^{k-1}}$ by definition, we get that $f_j^{k-1} < f_j^k$ for all $j < k$. Thus, requirement 1) of e_k amending e_{k-1} is satisfied.

If $t \in e_{k-1}$ satisfied $t \leq f_k^k$, then since $f_k^k = t_k \upharpoonright k \leq t_k$, we would have $t \leq t_k$. But Rule II states that $t \not\leq t_k$ for all $t \in e_{k-1}$. Therefore $t \not\leq f_k^k$ for all $t \in e_{k-1}$ which gives requirement 2) of e_k amending e_{k-1}.

Player 2 continues in this manner until the end of round L at which point she pauses to reflect on the following claim.

Claim 1. *For every $k \in \omega$ and $j \leq k$, f_j^k and t_j are comparable.*

Proof. When $k = 0$, $f_0^0 = t_0 \restriction 0 \leq t_0$. Suppose $k > 0$ and the claim holds for $k - 1$. If $j = k$ or if $k \leq \text{ht}_T(t_j)$ then $f_j^k = t_j \restriction k \leq t_k$. If $j < k$ and $k > \text{ht}_T(t_j)$ then one of two possibilities arises. The first possibility is that $k - 1 = \text{ht}_T(t_j)$. Then since f_j^{k-1} and t_j are comparable elements of $\text{Lev}_{k-1}(T)$, they are equal. The second possibility is that $k - 1 > \text{ht}_T(t_j)$. In this case, $\text{ht}_T(t_j) < \text{ht}_T(f_j^{k-1})$ and they are comparable, so $t_j < f_j^{k-1}$. Combining these possibilities, $t_j \leq f_j^{k-1}$. So $t_j \leq f_j^{k-1} < \overrightarrow{f_j^{k-1}} = f_j^k$, fulfilling the claim. □

Player 2 has arranged that either $f_0^N \leq t_0$ or else $t_0 \leq f_0^N$. But $\text{ht}_T(f_0^N) = N = \text{ht}_T(t_0)$. So $t_0 = f_0^N$. Since $L = N+1 > \text{ht}_T(t_0)$, $f_0^{N+1} = \overrightarrow{f_0^N} = \overrightarrow{t_0}$. She defines $h_L = f_0^N$ and $g_0^L = f_0^L$ so that $e_L = \{g_0^L, f_1^L, f_2^L, ..., f_L^L\}$ and $f_0^N \leq H_l < g_0^L$. Phase 2 commences at the beginning of round $L + 1$.

Phase 2: Let $M = \max\{\text{ht}_T(t_j) \; : \; j \leq N\}$. Suppose that $L < k \leq M$ and Player 1 offers t_k for play.

As before, player 2 defines

$$f_j^k = \begin{cases} t_j \restriction k & \text{if } k \leq \text{ht}_T(t_j) \\ \\ \overrightarrow{f_j^{k-1}} & \text{otherwise} \end{cases}$$

for $1 \leq j \leq L$. She also defines $g_j^k = \overleftarrow{g_j^{k-1}}$ for $j < k - L$.

Since $k > L$, $k - L$ is a natural number, $k - 1 = N + k - (1 + N) = N + k - L$ and

$$|\{t \in \text{Lev}_{k-1}(T) \; : \; f_0^N \leq t\}| = |\{t \in \text{Lev}_{N+k-L}(T) \; : \; f_0^N \leq t\}|$$

$$\geq 2^{k-L} > k - L \geq |\{g_0^{k-1}, g_1^{k-1}, ..., g_{k-1-L}^{k-1}\}|$$

She is therefore at liberty to take $h_k \in \text{Lev}_{k-1}(T)$ distinct from each of $g_0^{k-1}, g_1^{k-1}, ..., g_{k-1-L}^{k-1}$ such that $f_0^N \leq h_k$, define $g_{k-L}^k = \overrightarrow{h_k}$ and submit $e_k = \{g_0^k, g_1^k, ..., g_{k-L}^k, f_1^k, f_2^k, ..., f_L^k\}$ for play.

We must once more act as referee and check that Player 2 has followed the rules. As before, the fact that $e_k = \{g_0^k, g_1^k, ..., g_{k-L}^k, f_1^k, f_2^k, ..., f_L^k\}$, ensures that $|e_k| \leq k + 1$. So it must be seen that e_k satisfies requirements 1) and 2) vis-à-vis e_{k-1}.

As the definition of f_j^k has not changed, neither has the fact that $f_j^{k-1} < f_j^k$. Since $g_j^k = \overrightarrow{g_j^{k-1}}$ for all $j < k-L$, the fact that $g_j^{k-1} < g_j^k$ completes requirement 1) of e_k amending e_{k-1}. To see that $t \not\leq g_{k-L}^k$ for all $t \in e_{k-1}$, it is enough to show that $h_k \notin e_{k-1}$. This is true because $e_{k-1} \cup \{h_k\} \subseteq \mathrm{Lev}_{k-1}(T)$ and if $t \leq g_{k-L}^k$ for some $t \in e_{k-1}$ then t and h_k would be comparable, hence equal, which would result in $h_k \in e_{k-1}$. Now by choice, $h_k \notin \{g_0^{k-1}, g_1^{k-1}, ..., g_{k-L-1}^{k-1}\}$ so what remains is to see that $h_k \notin \{f_1^{k-1}, f_2^{k-1}, ..., f_N^{k-1}\}$. If the opposite were true, namely that $h_k = f_j^{k-1}$ for some $j \in \{1, 2, ..., N\}$ then since $f_0^N \leq h_k$, $f_j^N \leq f_j^{k-1}$ and $\{f_0^N, f_j^N\} \subseteq \mathrm{Lev}_N(T)$ one finds that $f_0^N = f_j^N$. How could this be? It has been established that $e_N = \{f_0^N, f_1^N, ..., f_N^N\}$ is an $N+1$ element set since it is in I_N. Since $1 \leq j$, f_0^N and f_j^N could nowise be equal. Thus $h_k \notin \{f_1^{k-1}, f_2^{k-1}, ..., f_N^{k-1}\}$ which completes requirement 2) of e_k amending e_{k-1}.

Game Over: Player 2 has won at the end of round M. Indeed, since the definition of f_j^M did not change in Phase 2, Claim 1 remains in effect and f_j^M and t_j are comparable for $j = 1, 2, ..., N$. But since $\mathrm{ht}_T(t_j) \leq M$ by definition and $\mathrm{ht}_T(f_j^M) = M$, it must be that $t_j \leq f_j^M$. For every $k \geq L$, Player 2 has ensured that $f_0^N \leq h_k < g_{k-L}^k$ and that $g_j^{k-1} < g_j^k$ for $j < L - k$ whenever $L < k \leq M$. Together these facts guarantee that $f_0^N < g_j^M$ for $j \leq M - L$ and since $e_M = \{g_0^M, g_1^M, \cdots, g_{M-L}^M, f_1^M, \cdots, f_N^M\}$ she is victorious. $\square$

Now suppose that A is a Boolean algebra and $f : T \to A$ is an order reversing embedding, meaning that for all $s, t \in T$, $s \leq t$ if and only if $f(t) \leq f(s)$.

Lemma 3.2.4. *If $f[T]$ is dense then for all distinct $t_0, t_1 \in T$, either $f(t_0)$ and $f(t_1)$ are comparable or they are disjoint.*

Proof. Assume that $f(t_0)$ and $f(t_1)$ are not disjoint. By density there is some $s \in T$ such that $f(s) \leq f(t_0) \cdot f(t_1)$. Because f is order reversing, this means that $t_0 \leq s$ and $t_1 \leq s$. So t_0 and t_1 are comparable. Again, because f is order reversing, $f(t_0)$ and $f(t_1)$ must also be comparable. $\square$

For any $S \in [T]^{<\omega}$, define $\phi(S) = \sum_{t \in S} f(t)$. Also define $I = \bigcup_{n \in \omega} I_n$.

Lemma 3.2.5. *If $f[T]$ is dense then for all distinct $i_0, i_1 \in I$, $\phi(i_0)$ and $\phi(i_1)$ are incomparable.*

Proof. Take distinct $i_0, i_1 \in I$ where I was defined at the beginning of this subsection. Without loss of generality we may suppose that $i_0 \in I_j$, $i_1 \in I_k$ and $j \leq k$. By part i) of Lemma 3.2.1, there is some $t_0 \in \mathrm{Lev}_k(T)$ such that $t \leq t_0$ for some $t \in i_0$ and $t_0 \notin i_1$. Since $i_1 \subseteq \mathrm{Lev}_k(T)$, t_0 is incomparable to every element of i_1. Hence, $f(t_0)$ is disjoint from $\phi(i_1)$. The requirement that $t \leq t_0$ for some $t \in t_0$ ensures that $f(t_0) \leq f(t) \leq \phi(i_0)$. Thus, $f(t_0) \leq \phi(i_0) \cdot -\phi(i_1)$ showing that $\phi(i_0) \not\leq \phi(i_1)$.

By part ii) of Lemma 3.2.1 there is some $t_1 \in i_1$ such that $t \not\leq t_1$ for all $t \in i_0$. If $t \in i_0$ then $\mathrm{ht}_T(t) = j \leq k = \mathrm{ht}_T(t_1)$ so t and t_0 could only be comparable if $t \leq t_1$ and since this is not the case they must be incomparable. But then $f(t)$ and $f(t_0)$ are disjoint for all $t \in i_0$ ergo $f(t_0)$ is disjoint from $\phi(i_0)$. Since $f(t_1) \leq \phi(i_1)$ by definition, we have $f(t_1) \leq \phi(i_1) \cdot -\phi(i_0)$ demonstrating that $\phi(i_1) \not\leq \phi(i_0)$. □

Lemma 3.2.6. *Let $J \subseteq A$ be an ideal. If $f[T]$ is dense in J then $\phi[I]$ is maximal incomparable with respect to J.*

Proof. First, since J is an ideal and every member of $\phi[I]$ is a finite sum of members of $f[T]$, $\phi[I] \subseteq J$. Now, $I_k \subseteq [\mathrm{Lev}_k(T)]^{k+1}$ which shows $I_k \cap I_j = \emptyset$ for all distinct $j, k \in \omega$. Further, since Player 2 has a way to satisfy Rule III in every round, $I_k \neq \emptyset$ for all $k \in \omega$. Thus, I is infinite. Combining this with Lemma 3.2.5, gives that $\phi[I]$ is infinite and pairwise incomparable.

Take $j \in J$. Assume that $j \not\leq \phi(e)$ for all $e \in I$. The task is to find $e \in I$ with $\phi(e) \leq j$. We will describe a course of action for Player 1 with the property that $t_k \leq j$ for every round k. When Player 2 wins by playing e_k, the victory condition implies

$$\phi(e_k) = \sum_{t \in e_k} f(t) \leq \sum_{l \leq k} f(t_l) \leq j.$$

On round 0, let Player 1 choose t_0 such that $f(t_0) \leq j$. This is possible by density. Now suppose we have arrived at the beginning of round $k > 0$. By assumption, $j \not\leq \phi(e_{k-1})$.

So let $f(t) \leq j \cdot -\phi(e_{k-1})$. By the fact that every element of T has at least 2 successors, there is some $t_k \geq t$ with $\mathrm{ht}_T(t_k) > \mathrm{ht}_T(t_{k-1})$. Let Player 1 offer t_k for play in round k. Rule I is certainly satisfied by this move. As for Rule II, if there were $t \in e_{k-1}$ such that $t \leq t_k$ then the result would be $f(t_k) \leq f(t) \leq \phi(e_{k-1})$. Since also $f(t_k) \leq -\phi(e_{k-1})$ this would imply $f(t_k) = 0$. But since $f\left(\overleftarrow{t_k}\right) < f(t_k)$ this is clearly impossible. Thus, Rule II is satisfied. □

For the main theorem of this section, we will restate all the hypotheses in effect.

Theorem 3.2.7 (Scherer)**.** *Let T be a tree of height ω with more than one root such that every member of T has at least two successors. Let A be a Boolean algebra. Suppose there is an order reversing embedding $f : T \to \mathcal{I}(A)$ whose image is dense in $\mathcal{I}(A)$. Then* $\mathrm{Inc}_{mm}(A) \leq |\operatorname{At}(A)| + |T|$.

Proof. By Lemmas 3.2.6 and 3.3, $\operatorname{At}(A) \cup \phi[I]$ is maximal incomparable. Since $I \subseteq [T]^{<\omega}$, $|\phi[I]| \leq |I| \leq |T|$. So $|\operatorname{At}(A) \cup \phi[I]| \leq |\operatorname{At}(A)| + |T|$. □

For any tree T and $t \in T$, define $b_t = \{s \in T : t \leq s\}$.

Lemma 3.2.8. *Let A be an atomless Boolean algebra with a countable dense subset. Then there is a tree T of height ω with more than one root such that every member of T has at least two successors and an order reversing embedding $f : T \to A$ such that $f[T]$ is dense in A.*

Proof. Let $T = {}^{<\omega}\omega \setminus \{\emptyset\}$. Then T is a tree of height ω with ω-many roots such that every member has ω-many successors. Let $A = \mathrm{TreeAlg}(T)$. Define $g : T \to A$ by $b(t) = b_t$. Since every member of T has infinitely many successors, A is atomless and $g[T]$ is dense in A [5, Rmk 16.10].

Let D be a countable dense subset of A. Then $C = \langle D \rangle$ is an atomless countable Boolean algebra. Therefore, there is an isomorphism $h : A \to C$ [5, Cor 5.16] So $f = g \circ h : T \to A$ is an order reversing embedding who image is dense in A. □

Corollary 3.2.9 (Scherer)**.** *If A is an infinite Boolean algebra with countable density, then* $\mathrm{Inc}_{mm}(A) = \aleph_0$.

Proof. If A is atomic then $\mathrm{At}(A)$ is a maximal incomparable set of size $\aleph_0$ so for the remainder assume that A is not atomic. Countable density immediately implies $|\mathrm{At}(A)| \leq \aleph_0$ so it remains to construct a tree of the appropriate type to satisfy Theorem 3.2.7. Let $D \subseteq A$ be a countable dense set.

Case 1: $\mathcal{I}(A)$ has a maximal element. Let $m = \sum \mathcal{I}(A)$. Then $\mathcal{I}(A) = A \upharpoonright m$ and consequently $A \upharpoonright m$ is atomless and $D \cap (A \upharpoonright m)$ is a countable dense subset of it. Thus, there is a tree, T, and order reversing embeding $f : T \to \mathcal{I}(A)$ as in Lemma 3.2.8. This satisfies the hypotheses of Theorem 3.2.7.

Case 2: $\mathcal{I}(A)$ has no maximal element. Let $B = \langle AtlI(A) \cap D \rangle$.

Claim 2. *B is atomless.*

Proof. First, note that since $\mathcal{I}(A)$ is an ideal, $\mathcal{I}(A) \cup \{-j : j \in \mathcal{I}(A)\}$ is a subalgebra of A containing the generating set of B. Thus, $B \subseteq \mathcal{I}(A) \cup \{-j : j \in \mathcal{I}(A)\}$. Take $a \in B$. Either $a \in \mathcal{I}(A)$ or $-a \in \mathcal{I}(A)$.

If $a \in \mathcal{I}(A)$, then a is not an atom of A so there is some $b \in A$ with $0 < b < a$. Since D is dense there is some $d \in D$ such that $0 < d < b$. Thus, $d \in \mathcal{I}(A) \cap D \subseteq B$ which demonstrates that a is not an atom.

So suppose $-a \in \mathcal{I}(A)$. Since $\mathcal{I}(A)$ has no maximal element there is some $b \in \mathcal{I}(A)$ such that $-a < b$. Therefore, $b \cdot a \neq 0$ and there is some $d \in D$ such that $0 < d < b \cdot a$. Thus, $d < b$ which shows $d \in \mathcal{I}(A) \cap D \subseteq B$ and $0 < d < a$ which once again demonstrates a is not an atom. $\square$

Since B is countable and atomless, there is a tree, T, and order reversing embedding $f : T \to \mathcal{I}(A)$ as in Lemma 3.2.8. This satisfies the hypotheses of Theorem 3.2.7. $\square$

Corollary 3.2.10 (Scherer)**.** $\mathrm{Inc}_{mm}(\mathrm{IntAlg}(\mathbb{R})) = \aleph_0$.

Proof. $\{[q, r) : q, r \in \mathbb{Q}\}$ is dense in $\mathrm{IntAlg}(\mathbb{R})$. □

In the following corollary, the tree S is free of any hypothesis from Theorem 3.2.7.

Corollary 3.2.11 (Scherer)**.** *Let S be an infinite tree of height at most ω and A be any Boolean algebra densely embedding* $\mathrm{TreeAlg}(S)$*. Then* $\mathrm{Inc}_{mm}(A) \leq |S|$*.*

Proof. Without loss of generality, suppose $\mathrm{TreeAlg}(S) \subseteq A$. Also, assume that S has a single root. Note that the construction given in the proof of Proposition 16.7 in [5] does not change the height of trees with infinite height, hence the assumption loses no generality.

We will construct a subtree of S suitable for the application of 3.2.7. By density, $\mathrm{At}(A) = \mathrm{At}(\mathrm{TreeAlg}(S))$. Since $\mathrm{TreeAlg}(S)$ is generated by sets of the form $b_t = \{s \in S : t \leq s\}$ for $t \in S$, every atom in A is a term depending on finitely many $t \in S$. Thus, $|\mathrm{At}(A)| \leq |S|$.

If $\mathcal{I}(A) = \{0\}$, then A is atomic in which case $\mathrm{Inc}_{mm}(A) \leq |S|$ by Corollary 3.2.11. So, for the remainder, suppose $\mathcal{I}(A) \neq \{0\}$. Let $g : S \to A$ be given by $g(t) = b_t$. Define $U = g^{-1}[\mathcal{I}(A)]$.

Claim 3. *$g[U]$ is dense in $\mathcal{I}(A)$.*

Proof. Take $j \in \mathcal{I}(A) \setminus \{0\}$. By density, there is some $k \in \mathrm{TreeAlg}(S)$ such that $0 < k \leq j$. By [5, Lm 16.3] and the fact that S has a single root, k can be written in normal form. Consequently, there is $b \in \mathrm{TreeAlg}(S)$ with $0 < b \leq k$, $t \in S$ and a finite anti-chain $A \subseteq b_t$ such that $b = b_t \cdot -\sum_{s\in A} b_s$.

Let V be the set of successors of t. If V is finite then $b_t \cdot -\sum_{v\in V} b_v$ is an atom [5, Rmk 16.10] below $b_t \cdot -\sum_{s\in A} b_s$ which itself is below j. Thus, V must be infinite and we can choose $u \in V$ that is below no element of A. So $0 < g(u) \leq j$, hence $g(u) \in \mathcal{I}(A)$ and $u \in U$. □

No element of U can have finitely many successors or else its image under g would be above an atom. Since $\mathcal{I}(A) \neq \{0\}$ and $g[U]$ is dense in $\mathcal{I}(A)$, U must be non-empty.

Therefore, U must have height ω. Define $T = U \setminus \text{Lev}_0(U)$. Then T has height ω, it has more than one root, and each of its members has at least two successors. Write $f = g \restriction T$ to fulfill the hypotheses of Theorem 3.2.7. Finally, $\text{Inc}_{mm}(A) \leq |\text{At}(A)| + |T| \leq |S| + |S| = |S|$. $\square$

3.3 Diamond

Definition. *Let κ be a cardinal. A subset $C \subseteq \kappa$ is called a **club** if*

(1) C is unbounded in κ and

(2) for every limit ordinal $\alpha \in \kappa$, if $C \cap \alpha$ is unbounded in α then $\alpha \in C$.

*A subset $S \subseteq \kappa$ is called **stationary** if S non-trivially intersects every club contained in κ.*

Definition. *Let κ be a cardinal. A $\diamondsuit_\kappa$**-sequence** is a sequence $\{S_\alpha : \alpha < \kappa\}$ such that $S_\alpha \subseteq \alpha$ for all α and for every $X \subseteq \kappa$ the set $\{\alpha \in \kappa : S_\alpha = X \cap \alpha\}$ is stationary.*

$\diamondsuit_\kappa$ is the assertion that a $\diamondsuit_\kappa$-sequence exists. For any uncountable regular cardinal κ, the statement $\diamondsuit_\kappa$ is consistent with ZFC [Jech, pg 442]. For the remainder of this section, fix an uncountable regular cardinal κ and assume $\diamondsuit_\kappa$ is true and witnessed by the sequence $\{S_\alpha : \alpha < \kappa\}$.

Definition. *For any Boolean algebra, A, the **saturation** of A is* $\text{sat}(A) = \min\{\mu : |X| < \mu$ *for all pairwise disjoint* $X \subseteq A\}$ *and for any $b \in A$, define* $\text{sat}_A(b) = \text{sat}(A \restriction b)$. *A subset $S \subseteq A$ is called a **partition** of A if S is maximal with respect to inclusion among all pairwise disjoint subsets of A. For $b \in A$, a partition of $A \restriction b$ is called a partition of b.*

Lemma 3.3.1. $\text{sat}(A) = \min\{\mu : |X| < \mu$ *for all partitions, $X \subseteq D$, of* $A\}$.

Proof. Put $\lambda = \min\{\mu : |X| < \mu$ for all partitions, $X \subseteq D$, of $A\}$. First, $\{X : X \subseteq D$ is a partition of $A\} \subseteq \{X : X$ is a partition of $A\} \subseteq \{X : X$ is pairwise disjoint$\}$ so $\text{sat}(A) \leq \lambda$.

Let X be a pairwise disjoint subset of A. For every $x \in X$ there is some pairwise disjoint $D_x \subseteq D$ such that $\sum D_x = x$ [5, Lm 4.9]. Then by Zorn's Lemma, $E = \bigcup_{x \in X} D_x$ can be extended to pairwise disjoint set maximal among all those contained in D, say F. If $\sum F \neq 1$ then there is a non-zero x disjoint from every member of F. Take $d \in D$ with $d \leq x$. Then $F \cup \{d\}$ is contradicts the maximality of F. This contradiction implies F is a partition hence $|F| < \lambda$. But $|X| \leq |E| \leq |F|$ which shows $\lambda \leq \mathrm{sat}(A)$. □

Let A be a κ-complete Boolean algebra with a dense set, D, of size at most κ. Let k_0 and k_1 be subsets of κ such that $k_0 \cup k_1 = \kappa$, $k_0 \cap k_1 = \emptyset$ and $|k_0| = |k_1| = \kappa$. Choose surjections $h_i : k_i \to D$ for $i \in 2$. For any $X \in [\kappa]^{<\kappa}$, define $H_i(X) = \sum h_i[X \cap k_i]$ for $i \in 2$. Note that since $S_\alpha \subseteq \alpha$, $H_i(S_\alpha)$ is defined for all $\alpha \in \kappa$. Finally, let λ be any infinite cardinal smaller than κ.

Lemma 3.3.2. *Let* $j_0, j_1 \in A$. *If* $j_0 < j_1$ *then* $\mathrm{sat}_A(j_0) + 1 \leq \mathrm{sat}_A(j_1)$.

Proof. Suppose $j_0 < j_1$ and let $X \subseteq A \restriction j_0$ be a partition of j_0. So X is a pairwise disjoint family such that $\sum X = j_0$. Then $X \cup \{j_1 \setminus j_0\}$ is a partition of j_1. □

Lemma 3.3.3. *Let* $j_0, j_1 \in A$. *Then* $\mathrm{sat}_A(j_0 + j_1) \leq \mathrm{sat}_A(j_0) + \mathrm{sat}_A(j_1)$.

Proof. Let $X \subseteq A \restriction (j_0 + j_1)$ be a partition of $j_0 + j_1$. Then $X_i := \{j_i \cdot x : x \in X\} \setminus \{0\}$ is a partition of j_i for $i \in 2$ [5, Lm 1.33]. Since every $x \in X$ meets one or both of j_0, j_1 non-trivially, and since $|X_i| \leq \mathrm{sat}_A(j_i)$ for $i \in 2$

$$|X| \leq |X_0 \cup X_1| \leq |X_0| + |X_1| \leq \mathrm{sat}_A(j_0) + \mathrm{sat}_A(j_1).$$

This true for every partition of $j_0 + j_1$ completes the proof. □

Define $J = \{j \in A : \mathrm{sat}_A(j) \leq \lambda\}$.

Corollary 3.3.4. *J is an ideal.*

Proof. λ was chosen to be infinite. □

Define $\{i_\alpha : \alpha < \kappa\}$ as follows. Suppose an initial sequence $\{i_\beta : \beta < \alpha\}$ is given. Define $i_\alpha = H_0(S_\alpha)$ if each of the following criteria holds:

(1) $H_i(S_\alpha) \notin J$ for all $i \in 2$.

(2) $H_0(S_\alpha) \cdot H_1(S_\alpha) = 0$.

(3) For all $\beta < \alpha$, if $i_\beta \neq 0$ then $H_0(S_\alpha) \cdot -i_\beta \notin J$ and $H_1(S_\alpha) \cdot i_\beta \notin J$.

If any of these criteria fail, define $i_\alpha = 0$. Define $G_\lambda = \{\alpha \in \kappa : i_\alpha \neq 0\}$ and $I = \{i_\alpha : \alpha \in G_\lambda\}$.

Lemma 3.3.5. *$\{i/J : i \in I\}$ is pairwise incomparable in A/J.*

Proof. Take $\beta < \alpha < \kappa$ such that $0 \notin \{i_\alpha, i_\beta\}$. Note that i_α/J and i_β/J being incomparable in A/J is equivalent to $i_\alpha \cdot -i_\beta \notin J$ and $i_\beta \cdot -i_\alpha \notin J$. The first condition is met by observing $H_0(S_\alpha) \cdot -i_\beta \notin J$ and $H_0(S_\alpha) = i_\alpha$. The second condition is met by observing $H_0(S_\alpha) \cdot H_1(S_\alpha) = 0$ implies $H_1(S_\alpha) \leq -i_\alpha$ and therefore $H_1(S_\alpha) \cdot i_\beta \notin J$ implies $i_\beta \cdot -i_\alpha \notin J$. □

Lemma 3.3.6. *Let $a \in A$ such that $a, -a \notin J$. Then a/J is comparable to some element of $\{i/J : i \in I\}$.*

Proof. Towards a contradiction, assume a/J is incomparable to i/J for every $i \in I$. By $a, -a \notin J$, $\lambda < \min(\mathrm{sat}_A(a), \mathrm{sat}_A(-a))$ and there are subsets $X_0, X_1 \in [D]^{\geq\lambda}$ that partition a and $-a$, respectively. Define $X = \bigcup_{i \in 2} h_i^{-1}[X_i]$. Then $H_0(X) = a$ and $H_1(X) = -a$. For each $i \in 2$ and $x \in D$, define $g_i(x) = \min h_i^{-1}[\{x\}]$.

Claim 4. *If $i \in I$ then there $\gamma, \eta \in \kappa$ such that $H_0(X \cap \gamma) \cdot -i \notin J$ and $H_1(X \cap \eta) \cdot i \notin J$.*

Proof. Define $i_0 = -i$ and $i_1 = i$. Let $j \in 2$. By the regularity of κ it is sufficient to find $Y_j \in [X]^{\leq\lambda}$ such that $H_j(Y_j) \cdot i_j \notin J$. We know by assumption that $H_j(X) \cdot i_j \notin J$.

Let $Z_j = \{d \in X_j : d \cdot i_j \neq 0\}$. If $|Z_j| < \lambda$, let $Y_j = g_j[Z_j]$, otherwise let $Y_j \in [g_j[Z_j]]^\lambda$. In the first case, $h_j[Y_j] = Z_j$ and therefore

$$H_j(Y_j) \cdot -i_j = \sum Z_j \cdot -i_j = \sum X_j \cdot -i_j = H_j(X) \cdot -i_j \notin J.$$

In the second case, $\{h_j(y) \cdot -i_j : y \in Y_j\}$ is a partition of size λ of $H_j(Y_j) \cdot -i_j$. □

Let C be the set of all $\alpha \in \kappa$ such that for all $\beta \in G_\lambda \cap \alpha$, $H_0(X \cap \beta) \cdot -i \notin J$ and $H_1(X \cap \beta) \cdot i \notin J$.

Claim 5. *C is unbounded.*

Proof. Let $\alpha < \kappa$. Define a sequence $\{\beta_n : n \in \omega\} \subseteq \kappa$ as follows. Put $\beta_0 = \alpha$. Now suppose β_n is given. For every $\mu < \beta_n$, choose $\eta_{\mu,n}, \gamma_{\mu,n} \in X$ such that $i_\mu \cdot H_1(X \cap \eta_{\mu,n}) \notin J$ and $H_0(X \cap \gamma_{\mu,n}) \cdot -i_\mu \notin J$. Now choose

$$\beta_{n+1} > \sup\left(\{\beta_n\} \cup \{\eta_{\mu,n} : \mu < \beta_n\} \cup \{\gamma_{\mu,n} : \mu < \beta_n\}\right).$$

This construction does not leave κ because κ is regular and uncountable. Set $\beta = \sup_{n<\omega} \beta_n$.

Since $\alpha < \beta$, it remains to be seen that $\beta \in C$. So take $\nu \in G_\lambda \cap \beta$. Since $\nu < \beta$, there is $n \in \omega$ such that $\nu < \beta_{n+1}$. Thus, $0 < i_\nu \cdot H_1(X \cap \gamma_{\nu,n}) \leq i_\nu \cdot H_1(X \cap \beta)$ and $0 < H_0(X \cap \gamma_{\nu,n}) \cdot -i_\nu \leq H_0(X \cap \beta) \cdot -i_\nu$. This true for arbitrary $\nu \in G_\lambda \cap \beta$ shows that $\beta \in C$. □

Claim 6. *C is closed.*

Proof. Let $\alpha < \kappa$ be a limit ordinal and assume that $\sup(C \cap \alpha) = \alpha$. Take $\beta \in G_\lambda \cap \alpha$. By assumption, there is some $\gamma \in C \cap \alpha$ such that $\beta < \gamma$. Then $\beta \in G_\lambda \cap \alpha \cap \gamma = G_\lambda \cap \gamma$. So by the definition of $\gamma \in C$, $0 < i_\beta \cdot H_1(X \cap \gamma) \leq i_\beta \cdot H_1(X \cap \alpha)$ and $0 < H_0(X \cap \gamma) \cdot -i_\beta \leq H_0(X \cap \alpha) \cdot -i_\beta$. This true for arbitrary $\beta \in G_\gamma \cap \alpha$ shows that $\alpha \in C$. □

C is therefore a club. Choose $Y_i \in [X_i]^\lambda$ and let $\delta_i = \sup\{g_i(y) + 1 : y \in Y_i\}$. Since κ is regular, $\delta_i \in \kappa$. Hence, $C' = C \setminus \max(\delta_0, \delta_1)$ is a club. By $\diamondsuit_\kappa$, there is some $\alpha \in C'$ such that $S_\alpha = X \cap \alpha$.

Claim 7. $\alpha \in G_\lambda$.

Proof. We must satisfy the three conditions required for $i_\alpha = H_0(S_\alpha)$.

(1) By definition,

$$H_i(S_\alpha) = \sum h_i[S_\alpha \cap k_i] = \sum h_i[X \cap \alpha \cap k_i]$$

and

$$X \cap \alpha \cap k_i = \left(\bigcup_{j \in 2} h_j^{-1}[X_j] \right) \cap \alpha \cap k_i = h_i^{-1}[X_i] \cap \alpha$$

since $\mathrm{dom}(h_i) = k_i$. For any $y \in Y$, $g_i(y) < \delta_i < \alpha$ and hence $g_i(y) \in h_i^{-1}[X_i] \cap \alpha$. Therefore $y \leq H_i(S_\alpha)$. Since $|Y_i| \geq \lambda$ and Y_i is disjoint, we have $H_i(S_\alpha) \notin J$.

(2) $H_0(S_\alpha) \cdot H_1(S_\alpha) \leq H_0(X) \cdot H_1(X) = a \cdot -a = 0$.

(3) Take $\beta < \alpha$ such that $i_\beta \neq 0$. Then $\beta \in G_\lambda \cap \alpha$. By the definition of C, $H_0(X \cap \alpha) \cdot -i_\beta \notin J$ and $i_\beta \cdot H_1(X \cap \alpha) \notin J$.

□

But then $i_\alpha \in I$ and yet $i_\alpha = H_0(S_\alpha) = H_0(X \cap \alpha) \leq H_0(X) = a$. Contradiction. □

Lemma 3.3.7. *If* $2 < |A/J|$ *then* $\{i/J : i \in I\}$ *is maximal incomparable in* A/J.

Proof. The condition $2 < |A/J|$ is equivalent to the existence of some $a \in A$ such that $a, -a \notin J$. By the previous Lemma, there is some $i \in I$ such that a/J and i/J are comparable and hence $\{i/J : i \in I\} \neq \emptyset$. Trivially, $0/J$ and $1/J$ are comparable to i/I. So in fact $\{i/J : i \in I\}$ is maximal with respect to inclusion among all (possibly finite) pairwise incomparable subsets of A/J.

Furthermore, by construction, for any $i_\alpha \in I$, we have $H_0(S_\alpha) = i_\alpha$ and $H_1(S_\alpha) \leq -i_\alpha$ while $H_j(S_\alpha) \notin J$ for each $j \in 2$. Thus, $i, -i \notin J$ for all $i \in I$. So $\{0/J, 1/J\} \cap \{i/J : i \in I\} = \emptyset$.

By Corollary 3.0.9, the proof is completed by observing that A/J is atomless since this implies $\{i/J : i \in I\}$ is infinite. To see this take $a \in A$ such that $a \notin J$. Write $a = \sum X$ where $|X| \geq \lambda$ and choose $Y \in [X]^\lambda$ such that $|X \setminus Y| \geq \lambda$. Define $b = \sum Y$. Then $0/J < b/J < a/J$. □

Theorem 3.3.8 (Scherer)**.** *Let κ be a regular uncountable cardinal and λ be an infinite cardinal less than κ. Assume $\diamondsuit_\kappa$. Let A be a κ-complete Boolean algebra with density at most κ. Then* $\mathrm{Inc}_{mm}(A/\{i \in A : \mathrm{sat}_A(i) \leq \lambda\}) \leq \kappa$.

Corollary 3.3.9 (Scherer)**.** *Let κ be a regular uncountable cardinal. Assume $\diamondsuit_\kappa$. Let A be a κ-complete Boolean algebra with density at most κ. Then* $\mathrm{Inc}_{mm}(A) \leq \kappa$.

Proof. If A has a countably infinite set of atoms, S, then $S \cup \{-\sum S\}$ is a maximal incomparable set of size less than κ. So suppose $\mathrm{At}(A)$ is finite. Let $x = -\sum \mathrm{At}(A)$. Define $B = A \restriction x$. Clearly B is atomless. Therefore, for every $b \in B \setminus \{0\}$, $B \restriction b$ is atomlesss which shows that b has an infinite partition and it follows $\omega_1 \leq \mathrm{sat}_B(b)$. Thus, $\{i \in B : \mathrm{sat}_B(b) \leq \omega\} = \{0\}$. So applying the theorem to B with $\lambda = \omega$ gives a maximal incomparable $I \subseteq B$ such that $|I| \leq \kappa$. Since $B = \mathcal{I}(A)$, $\mathrm{At}(A) \cup I$ is maximal incomparable in A by Lemma . □

Corollary 3.3.10 (Scherer)**.** *Let κ be a regular uncountable cardinal and λ be an infinite cardinal less than κ. Assume $\diamondsuit_\kappa$. Then* $\mathrm{Inc}_{mm}\left(\mathcal{P}(\kappa)/[\kappa]^{<\lambda}\right) \leq \kappa$.

Proof. Take $X \subseteq \kappa$. Then $\{\{x\} : x \in X\}$ is a partition of X of the largest possible size so $\mathrm{sat}_{\mathcal{P}(\kappa)}(X) = |X|^+$. Thus, $\{i \in \mathcal{P}(\kappa) : \mathrm{sat}_{\mathcal{P}(\kappa)}(i) \leq \lambda\} = [\kappa]^{<\lambda}$. Since the density of $\mathcal{P}(\kappa)$ is κ, the result follows. □

An example of this corollary is $\mathcal{P}(\omega_1)/\mathrm{Fin}(\omega_1)$.

Lemma 3.3.11. *Let $A = \mathcal{P}(\omega_1)/\mathrm{Fin}(\omega_1)$. Then $d_A(a) = \mathfrak{c}$ for all $a \in A \setminus \{0\}$.*

Proof. Take $a \in A \setminus \{0\}$ and $D \subseteq A \restriction a$ a dense set. Write $a = X/\operatorname{Fin}(\omega_1)$, take $Y \in [X]^{\aleph_0}$ and write $b = Y/\operatorname{Fin}(\omega_1)$. By [6, Thm 1.3, p. 48], there is some a pairwise disjoint (in the sense of A), $S \in [A \restriction b]^{\mathfrak{c}}$. For every $s \in S$, take $d_s \in D \cap (A \restriction s) \setminus \{0\}$. Then $\mathfrak{c} \leq |\{d_s : s \in S\}| \leq |D| \leq d_A(a)$. □

In this case, Theorem 3.0.8 implies $\mathfrak{c} \leq \operatorname{Inc}_{mm}(\mathcal{P}(\omega_1)/\operatorname{Fin}(\omega_1))$ and $\diamondsuit_{\omega_1}$ implies that $\mathfrak{c} = \omega_1$. It's unclear if $\operatorname{Inc}_{mm}(\mathcal{P}(\omega_1)/\operatorname{Fin}(\omega_1)) \leq \mathfrak{c}$ is necessary even without $\diamondsuit_\kappa$. In general, $\diamondsuit_\kappa$ implies $\kappa^\lambda = \kappa$ for whenever $0 < \lambda < \kappa$ so in the presence of the assumption that A has density at most κ, $\operatorname{sat}(A) \leq \kappa$ implies $|A| \leq \kappa$. So the result is most useful on algebras with saturation larger than κ.

Chapter 4

Conclusion

We conclude by listing some questions that are raised by the bounds obtained. Start by considering the bounds for comparability:

Theorem 2.1.1. *If* $\lambda \leq \kappa$, $\mathrm{cf}(\lambda) \leq \mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\lambda})$.

Theorem 2.2.16. *Let* $\mu < \mathrm{cf}(\kappa)$ *satisfy* $\mathrm{cf}(\kappa) \leq 2^{\mu} \leq \kappa$. *Then* $\mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}) \leq 2^{\mu}$.

Theorem 2.1.3. *If A is atomless and satisfies the countable separation property then* $\mathfrak{c} \leq \mathrm{Length}_{mm}(A)$.

Question 1. *What is* $\mathrm{Length}_{mm}(\mathcal{P}(\aleph_\omega)/[\aleph_\omega]^{<\aleph_\omega})$ *and in general is there some κ with* $\mathrm{cf}(\kappa) = \omega$ *and* $\mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}) > \mathfrak{c}$*?*

Question 2. *Is it consistent to have a strongly inaccessible κ such that* $\mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}) = \kappa$*?*

Question 3. *Is there some κ with* $\mathrm{Length}_{mm}(\mathcal{P}(\kappa)/[\kappa]^{<\kappa}) > 2^{\mathrm{cf}(\kappa)}$*?*

Additionally, we ask to what extent is it possible to formulate these bounds algebraically say in terms of density, completeness and separation properties.

As for incomparability, consider:

Corollary 3.2.9. *If A is an infinite Boolean algebra with countable density, then* $\mathrm{Inc}_{mm}(A) = \aleph_0$.

Corollary 3.3.9. *Let κ be a regular uncountable cardinal. Assume $\diamondsuit_\kappa$. Let A be a κ-complete Boolean algebra with density at most κ. Then* $\mathrm{Inc}_{mm}(A) \leq \kappa$.

Since it also holds that $\mathrm{Inc}_{mm}(A) \leq |\,\mathrm{At}(A)|$ for atomic algebras, one wonders:

Question 4. *Is there an algebra with $d(A) < \mathrm{Inc}_{mm}(A)$?*

Even if not,

Question 5. *Is Corollary 3.3.9 in ZFC?*

Question 6. *Is there an analogue to Corollary 3.3.9 using a weaker club guessing principle in ZFC?*

Finally, one wonders if the following corollary extends to trees of height greater than ω:

Corollary 3.2.11. *Let S be an infinite tree of height at most ω and A be any Boolean algebra densely embedding* $\mathrm{TreeAlg}(S)$. *Then* $\mathrm{Inc}_{mm}(A) \leq |S|$.

The difficulty in this seems to lie in the proof of Lemma 3.2.1, specifically that it uses the fact that two distinct sets of the same finite size are incomparable which fails for infinite sets.

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