# Computing invariant forms for Lie algebras using heaps 

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In this thesis, I present a combinatorial formula for a symmetric invariant quartic form on a spin module for the simple Lie algebra $\mathfrak{d}_{6}$. This formula relies on a description of this spin module as a vector space with weights, and weight vectors, indexed by ideals of a particular heap. I describe a new statistic, the profile, on pairs of heap ideals. The profile efficiently encodes the shape of the symmetric difference between the two ideals and demonstrates the available actions of the Weyl group and Lie algebra on any given pair. From the profile, I identify a property called a crossing. The actions of the Weyl group and Lie algebra on pairs of weights may be interpreted as adding or removing crossings between the corresponding ideals. Using the crossings, I present a formula for the symmetric invariant quartic form on a spin module for $\mathfrak{d}_{6}$, and discuss potential applications to other closely related minuscule representations.

## Dedication

This thesis is dedicated, with innumerable thanks, to all of my teachers, especially the fierce math women who lit the way for me and endlessly challenged me to grow into bigger shoes.

Mary D. Geyer, Becky Lang, June Trachsel, Irena Swanson, and Faan Tone Liu

I strive to be as tenacious and loving as you are.

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## Chapter 1

## Introduction and Definitions

The study of Lie algebras has its roots in the work of S. Lie (1842-1899), who proposed to study groups of continuous transformations (now known as Lie groups) by focusing on the local structure of vector fields. We now know that each Lie group has a corresponding Lie algebra, which arises as the tangent space to the Lie group at the identity. These Lie algebras encode all of the structure of the group in different way. As for groups, some natural questions arise for these objects: What are the basic building blocks, and can they be classified? How can we describe the underlying structure of each Lie algebra? What are the representations of a given Lie algebra, and what is their structure?

The first question was answered (for Lie algebras over $\mathbb{C}$ ) after decades of work by W. Killing (1847-1923) and E. Cartan (1869-1951). This classification of simple Lie algebras requires combinatorial tools, for it relies on the theory of roots and root systems. E. Dynkin (1924-2014) provided a more illuminating version of the classification, using simple roots and Dynkin diagrams, which provided many of the tools we use today to study the structure of Lie algebras and their representations (thus largely answering the second question).

The finite-dimensional irreducible modules for a semisimple Lie algebra over $\mathbb{C}$ have since been classified; they are in one-to-one correspondence with the dominant integral weights of $L$ [1, Theorem 10.21]. These modules can be described in various ways, but many questions remain about their underlying structure. One such question is: if $V$ is a finite-dimensional module for some simple Lie algebra over $\mathbb{C}$, then how do tensor products of $V$ (for example, $V \otimes V$ or $V \otimes V \otimes V$ ) decompose
as products of irreducible modules for $L$ ? This question can be answered using Littelmann's path model [8]. However, a similar question about the dual modules of these tensor powers $\left((V \otimes V)^{*}\right.$, etc.) remains unsolved in general. In particular, do these tensor products contain one-dimensional submodules, and with what multiplicity?

The last question may be rephrased in terms of invariant forms: what $L$-invariant forms exist for a given $L$-module $V$ ? Much work has been done on studying bilinear forms on the irreducible modules; less is known about the cubic and quartic forms. In this thesis, we make progress toward using combinatorial tools to produce explicit formulas for symmetric invariant quartic forms on some irreducible modules for the simple Lie algebras of types $A$ and $D$, relying on a description of these particular modules using heaps over Dynkin diagrams. We construct a new object, the profile of two heap ideals (Definition 2.26), and demonstrate that it has several very useful properties with respect to the usual generators of $L$. Then, in Chapter 3, we use the profile to give an explicit description of a symmetric invariant quartic form for a spin module in type $D_{6}$. We also discuss connections to other related representations.

### 1.1 Lie Algebras

We begin with an introduction to Lie algebras, their structure, and their representation theory. The following definitions are from [1], [4], and [7]; the results throughout this chapter are well-known.

A Lie algebra $L$ is a $\mathbb{k}$-vector space equipped with a bilinear operation

$$
[-,-]: L \times L \rightarrow L
$$

that satisfies
(L1) $[x, x]=0$ for all $x \in L$ (antisymmetry), and
(L2) $[x,[y, z]]+[y,[z, x]]+[z,[x, y]]=0$ for all $x, y, z \in L$ (Jacobi identity).
This bilinear operation is called the Lie bracket. Antisymmetry implies that $[x, y]=-[y, x]$ for all $x, y \in L$.

One familiar example of a Lie algebra is $\mathfrak{g l}(n, \mathbb{C})=\{n \times n$ matrices over $\mathbb{C}\}$, with bracket given by the commutator: $[A, B]=A B-B A$. In this way, by choosing a basis, $\operatorname{End}_{\mathbb{k}}(V)$ is a Lie algebra for any $\mathbb{k}$-vector space $V$. In fact, one can use this bracket to make any associative algebra into a Lie algebra. (The Jacobi identity in this case relies on the associativity of the original algebra.)

A subalgebra of a Lie algebra is a linear subspace closed under bracket. An ideal of a Lie algebra is a subalgebra $I$ with the property that $[x, y] \in I$ whenever $x \in L$ and $y \in I$. (Because of antisymmetry, ideals are automatically two-sided, unlike for rings.) A Lie algebra $L$ is abelian if $[x, y]=0$ for all $x, y \in L$. A simple Lie algebra is a nonabelian Lie algebra with no ideals other than $\{0\}$ and $L$.

The simple Lie algebras over $\mathbb{C}$ were classified, up to isomorphism, by Killing, Engel, and Cartan. They include the four infinite families $\mathfrak{a}_{n}, \mathfrak{b}_{n}, \mathfrak{c}_{n}$ and $\mathfrak{d}_{n}$, as well as five exceptional Lie algebras, $\mathfrak{e}_{6}, \mathfrak{e}_{7}, \mathfrak{e}_{8}, \mathfrak{f}_{4}$, and $\mathfrak{g}_{2}$.

If $L_{1}$ and $L_{2}$ are Lie subalgebras of $L$, we write $\left[L_{1}, L_{2}\right.$ ] for the subalgebra that is generated (as a Lie algebra) by all elements of the form $\left[x_{1}, x_{2}\right]$, where $x_{1} \in L_{1}$ and $x_{2} \in L_{2}$.

If $L$ is a Lie algebra, we write $L^{1}=L$ and $L^{n+1}=\left[L^{n}, L\right]$ for $n>1$. Then $L$ is said to be nilpotent if $L^{n}=0$ for some $n \geq 1$. Abelian Lie algebras are automatically nilpotent, as they satisfy $L^{2}=0$.

For a Lie algebra $L$, we write $L^{(0)}=L$ and $L^{(n+1)}=\left[L^{(n)}, L^{(n)}\right]$ for $n \geq 1$. Then $L$ is said to be solvable if $L^{(n)}=0$ for some $n \geq 1$. An ideal $I$ is said to be solvable if it is solvable when viewed as a Lie algebra in its own right. A Lie algebra is said to be semisimple if it contains no non-zero solvable ideals.

If $H$ is a subalgebra of $L$, the normalizer of $H$ (in $L$ ) is

$$
N(H)=\{x \in L:[x, h] \in H \text { for all } h \in H\} .
$$

A Cartan subalgebra of $L$ is a nilpotent subalgebra $H$ satisfying $H=N(H)$.
For example, if $L=\mathfrak{g l}(n, \mathbb{C})$, we can take $H$ to be the subalgebra consisting of all diagonal
matrices. We might wonder if every Lie algebra has such a subalgebra, and whether these Cartan subalgebras are unique.

Theorem 1.1. Every nonzero Lie algebra contains a Cartan subalgebra. Moreover, provided that $L$ is a finite-dimensional Lie algebra over $\mathbb{C}$, any two Cartan subalgebras of $L$ are conjugate under automorphisms of $L$.

Proof. This is [1, Theorems 3.2 and 3.13].

It is frequently useful to define a Lie algebra using generators and relations. In order to do this, we need the notion of a free Lie algebra on a set of generators. To build such a Lie algebra, we begin with a set $S$ of generators, indexed by a set $I$. We first define a free associative algebra on $S$ by taking all finite words in the alphabet $S$ (including words of length zero), and then taking finite $\mathbb{C}$-linear combinations of these words. The elements of the free associative algebra $F(S)$ thus have the form

$$
\sum_{k \geq 0} \sum_{i_{1}, \ldots i_{k} \in I} c_{i_{1}, \ldots, i_{k}} s_{i_{1}} s_{i_{2}} \cdots s_{i_{k}} .
$$

This is an associative algebra with addition given linearly by combining like terms, multiplication by linearity and concatenation of words from $S$, and scalar multiplication in the usual way. The empty product is called 1 and provides the multiplicative identity for $F(S)$.

From $F(S)$, we construct a Lie algebra, $[F(S)]$, with the same underlying vector space, by using the commutator bracket $[A, B]=A B-B A$ for $A, B \in F(S)$. Now, in order to construct the free Lie algebra on $S$, we note that $S$ is identified with a subset of $F(S)$, and hence also $[F(S)$ ], by simply looking at words of length one. Let $F L(S)$ be the intersection of all Lie subalgebras of [ $F(S)$ ] that contain this copy of $S$. Then $F L(S)$ is known as the free Lie algebra on $S$.

If we have in mind both a set of generators, $S$, and a set of relations, $R$, then we need only to construct the ideal $I_{R}$ of $F L(S)$ generated by all of the relations $R$. The Lie algebra generated by $S$ subject to relations $R$ is now given by $F L(S) / I_{R}$.

### 1.2 Representations

If $L$ and $M$ are Lie algebras over the same field, a Lie algebra homomorphism is a linear map $\varphi: L \rightarrow M$ satisfying

$$
\varphi([x, y])=[\varphi(x), \varphi(y)]
$$

for all $x, y \in L$. A representation of a Lie algebra $L$ is a Lie algebra homomorphism

$$
\rho: L \rightarrow \operatorname{End}_{\mathbb{k}}(V)
$$

where $V$ is a $\mathbb{k}$-vector space. This makes $V$ into an $L$-module.
Conversely, given an $L$-module $V$, we have a corresponding representation, $\rho: L \rightarrow \operatorname{End}_{\mathbb{k}}(V)$, given by $\rho(x)(v)=x \cdot v$. In this way we may view $L$-modules and representations of $L$ as being interchangeable.

We can view a Lie algebra $L$ as being an $L$-module over itself via the adjoint representation ad $: L \rightarrow \operatorname{End}_{\mathfrak{k}}(L)$, given by $(\operatorname{ad} x)(y)=[x, y]$ for $x, y \in L$. Is $S$ is a subalgebra of $L$, then $L$ is an $S$-module by restriction of the domain of ad; if $I$ is an ideal of $L$, then $I$ is an $L$-module since $[x, y] \in I$ whenever $x \in L$ and $y \in I$.

If $V$ is an $L$-module, then an $L$-submodule of $V$ is a linear subspace of $V$ closed under the action of $L$. A nontrivial $L$-module with no nontrivial proper submodules is irreducible. Irreducible modules form the building blocks for representation theory, just as simple Lie algebras serve as building blocks for more complicated Lie algebras.

If $V$ is an $L$-module, then $V \otimes V$ and $V^{*}$, as well as higher tensor powers of $V$, are also equipped with an $L$-module structure. The action of $x \in L$ on a simple tensor $v \otimes w$ of $V \otimes V$ is given by

$$
x \cdot(v \otimes w)=x v \otimes w+v \otimes x w,
$$

while the action of $x \in L$ on an element $q \in V^{*}$ is defined by

$$
(x \cdot q)(v)=q(-x v)
$$

for $v \in V$. These actions follow from the Hopf algebra structure on the universal enveloping algebra of $L$. The universal enveloping algebra will be discussed in Section 1.8.

One way to describe an $L$-module $V$ is using the action of a Cartan subalgebra $H$ on $V$. Based on our example with $L=\mathfrak{g l}(n, \mathbb{C})$, we might expect $H$ to act similarly to diagonal matrices; in particular, $H$ should have eigenvalues for its action on $V$. The analogue of eigenvalues in this situation will be weights.

Definition 1.2. Let $L$ be a Lie algebra with Cartan subalgebra $H$, let $\rho: L \rightarrow \operatorname{End}_{\mathbb{k}}(V)$ be a representation of $L$, and let $\lambda$ be an element of the dual space $H^{*}$; that is, $\lambda$ is a linear map from $H$ to $\mathbb{C}$. We define the $\lambda$-weight space to be

$$
V_{\lambda}=\{v \in V: h \cdot v=\lambda(h) v\}
$$

If $V_{\lambda}$ is a nonzero subspace of $V$, then we say that $\lambda$ is a weight of the representation $\rho$.

We will revisit the use of weights to describe representations once we have established a more thorough description of the internal structure of Lie algebras.

### 1.3 Roots, Root Spaces, and the Killing Form

Given a Lie algebra $L$, we may identify roots by first choosing a Cartan subalgebra $H$ and viewing $L$ as an $H$-module. The roots $\Phi$ are now precisely the weights $\alpha: H \rightarrow \mathbb{C}$ that have nonzero weight space

$$
L_{\alpha}=\{x \in L:[h, x]=\alpha(h) x \text { for all } h \in H\}
$$

When $\alpha$ is a root, we refer to $L_{\alpha}$ as a root space.

Lemma 1.3 ([4, Lemma 10.11(ii)]). The set of roots $\Phi$ spans $H^{*}$.

In order to further describe the relationship between the Lie algebra and its roots, we will now define the Killing form.

If $L$ is a Lie algebra, recall that $L$ is an $L$-module over itself via the adjoint representation. We define a map $\kappa: L \otimes L \rightarrow \mathbb{C}$ by

$$
\kappa(x, y)=\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) .
$$

This map is called the Killing form on L. The Killing form is bilinear and symmetric. It is also associative with respect to the Lie bracket, i.e.

$$
\kappa([x, y], z)=\kappa(x,[y, z])
$$

for any $x, y, z \in L[4$, Definition 9.5].
Lemma 1.4 ([1, Theorem 4.10]). The Killing form of $L$ is non-degenerate if and only if $L$ is semisimple.

Provided that $L$ is complex semisimple, the Killing form is also non-degenerate when restricted to $H$ [4, Lemma 10.1 (iii)].

The Killing form provides an isomorphism between $H$ and $H^{*}$ : given $h \in H$, we have a map $\theta_{h} \in H^{*}$ given by

$$
\theta_{h}(k)=\kappa(h, k)
$$

for $k \in H$. As this map is an isomorphism, given a root $\alpha \in H^{*}$, there must exist $t_{\alpha} \in H$ with $\alpha=\theta_{t_{\alpha}} ;$ that is,

$$
\alpha(k)=\theta_{t_{\alpha}}(k)=\kappa\left(t_{\alpha}, k\right)
$$

for all $k \in H$. As $\kappa$ is non-degenerate, this $t_{\alpha}$ must be unique for each $\alpha$.
We may now define an inner product on roots, given by $(\alpha, \beta)=\kappa\left(t_{\alpha}, t_{\beta}\right)$ for $\alpha, \beta \in \Phi$. This form extends linearly to a real-valued inner product on $H^{*}$ [4, Lemma 10.15].

### 1.4 Root Systems

The data needed to construct a simple Lie algebra over $\mathbb{C}$ can be communicated in various ways. Three useful tools are those of a root system, a Dynkin diagram, and a Cartan matrix. Any one of these objects specifies a semisimple Lie algebra over $\mathbb{C}$ (up to isomorphism).

Each root system is a special subset of an inner product space.

Definition 1.5. A real inner product space is a vector space $E$ over $\mathbb{R}$, equipped with a bilinear $\operatorname{map}(-,-): E \times E \rightarrow \mathbb{R}$ that is
(i) symmetric: if $x, y \in E$, then $(x, y)=(y, x)$, and
(ii) positive definite: if $x \in E$, then $(x, x) \geq 0$, and $(x, x)=0$ only when $x=0$.

Recall that if $E$ is a real inner product space and $\alpha \in E$, then we have a map $s_{\alpha}$, which is the unique linear map from $E$ to $E$ that sends $\alpha$ to $-\alpha$ and fixes (pointwise) the hyperplane orthogonal to $\alpha$. In other words,

$$
s_{\alpha}(v)=v-\frac{2(v, \alpha)}{(\alpha, \alpha)} \alpha
$$

Maps of this form are called reflections.

Definition 1.6. Let $E$ be a real inner product space. A root system is a subset $\Phi \subset E$, satisfying
(i) $\Phi$ is finite, spans $E$, and does not contain the zero vector;
(ii) if $\alpha \in \Phi$ then $\mathbb{R} \alpha \cap \Phi=\{\alpha,-\alpha\}$;
(iii) if $\alpha, \beta \in \Phi$ then $s_{\alpha}(\beta) \in \Phi$;
(iv) if $\alpha, \beta \in \Phi$ then $2(\beta, \alpha) /(\alpha, \alpha) \in \mathbb{Z}$.

If $\alpha \in \Phi$ then $\alpha$ is called a root.

The roots described in Section 1.3 satisfy this definition. The data of $\Phi$ can be represented more concisely by making a choice of a set of simple roots.

Definition 1.7. If $\Phi$ is a root system in $E$, a subset $\Delta \subset \Phi$ is called a simple system provided that $\Delta$ is a basis for $E$, and each $\alpha \in \Phi$ is a linear combination of the elements of $\Delta$, with coefficients all of the same sign. The elements of $\Delta$ are called simple roots, and the corresponding reflections are called simple reflections.

Figure 1.1: A root system of type $B_{2}$. One possible choice of simple roots is shown in bold.


Given a root system $\Phi$ with some specified simple roots $\Delta$, we can construct a corresponding semisimple Lie algebra $L$ using generators and relations. In order to write down these relations, it will be useful to know the entries of the corresponding Cartan matrix.

### 1.5 Cartan Matrix

Definition 1.8. A generalized Cartan matrix is an $n \times n$ square matrix $A$, satisfying
(i) $A_{i j} \in \mathbb{Z}$ for all $i, j$;
(ii) $A_{i i}=2$ for all $i$;
(iii) $A_{i j} \leq 0$ whenever $i \neq j$; and
(iv) $A_{i j}=0$ if and only if $A_{j i}=0$.

From this matrix, we may define a Lie algebra using generators

$$
\left\{e_{i}, f_{i}, h_{i}: 1 \leq i \leq n\right\}
$$

along with the relations:

$$
\begin{gather*}
{\left[h_{i}, h_{j}\right]=0, \quad\left[h_{i}, e_{j}\right]=A_{i j} e_{j}, \quad\left[h_{i}, f_{j}\right]=-A_{i j} f_{j}, \quad\left[e_{i}, f_{j}\right]=\delta_{i j} h_{i},} \\
\underbrace{\left[e_{i},\left[e_{i}, \cdots\left[e_{i}\right.\right.\right.}_{1-A_{i j}}, e_{j}] \cdots]]=0 \quad \text { and } \underbrace{\left[f_{i},\left[f_{i}, \cdots\left[f_{i}\right.\right.\right.}_{1-A_{i j}}, f_{j}] \cdots]]=0 . \tag{1.1}
\end{gather*}
$$

Conversely, given a root system $\Phi$ with simple roots $\Delta=\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$, we have a corresponding $n \times n$ generalized Cartan matrix $A$, where $A_{i j}=\left(\alpha_{i}, \alpha_{j}\right)$.

If $A$ is a generalized Cartan matrix, we say that $A$ has finite type provided that
(i) $\operatorname{det} A \neq 0$;
(ii) there exists a vector $u=\left(u_{1}, \ldots, u_{j}\right)$ such that $u_{i}>0$ and $(A u)_{i}>0$ for all $i$; and
(iii) if $(A u)_{i} \geq 0$ for all $i$ then either $u=0$ or $u_{i}>0$ for all $i$.

If $A$ is a generalized Cartan matrix of finite type, then we will refer to $A$ as a Cartan matrix.

Theorem 1.9 (Serre's Theorem). If $A=\left[A_{i j}\right]$ is a Cartan matrix corresponding to a root system $\Phi$ with $n$ simple roots $\Delta$, and $L$ is the Lie algebra generated (as a Lie algebra) over $\mathbb{C}$ by generators $\left\{e_{i}, f_{i}, h_{i}: 1 \leq i \leq n\right\}$, subject only to the relations shown above (1.1), then $L$ is a finite-dimensional, semisimple Lie algebra over $\mathbb{C}$ with root system $\Phi$ and Cartan matrix $A$. Moreover, all finitedimensional simple Lie algebras over $\mathbb{C}$ arise in this way.

Proof. This is in [7, Section 18.3].

If the Cartan matrix $A$ has finite type, we will also also say that the corresponding Lie algebra has finite type.

### 1.6 Dynkin Diagram

Given a root system with simple roots $\Delta$, we construct a graph, $\Gamma$, that encodes the relationships between the simple roots. The following description is sufficient for Lie algebras of finite type, although Dynkin diagrams can be more complicated in other situations.

The vertices of $\Gamma$ are the simple roots $\alpha \in \Delta$; we draw $(\alpha, \beta)(\beta, \alpha)$ edges between the vertices corresponding to $\alpha$ and $\beta$. If $\alpha$ is a root, then we call $(\alpha, \alpha)$ the length of $\alpha$; if the roots $\alpha$ and $\beta$ are not the same length, we decorate any edges between them with an arrow pointing toward the shorter root.

Figure 1.2: The Dynkin diagram of type $B_{2}$. Refer to Figure 1.1 for the corresponding root system.


Figure 1.3: The Dynkin diagram of type $D_{6}$.


Figure 1.4: The Dynkin diagram of type $D_{6}^{(1)}$.


Connected components of $\Gamma$ correspond to irreducible direct summands of the semisimple Lie algebra $L$. If two root systems give isomorphic Dynkin diagrams, then the root systems are themselves isomorphic [4, Proposition 11.21].

If the Cartan matrix only contains entries from $\{-2,-1,0,1,2\}$, then we say that the root system is doubly laced. If the Cartan matrix only contains entries from $\{-1,0,1\}$, then we say that the root system is simply laced; this means that the Dynkin diagram has only single edges with no decorations.

Figure 1.5: A list of Dynkin diagrams of all simple Lie algebras over $\mathbb{C}$, showing the indexing conventions that we will use later.
$\begin{array}{llllll}\mathfrak{a}_{n} & \bullet & \bullet & \bullet & \cdots & \bullet \\ & 1 & 2 & 3\end{array} \quad \begin{array}{ll}n-1 & n\end{array}$
$\mathfrak{b}_{n} \quad \stackrel{\bullet}{l} \quad \stackrel{\bullet}{2} \quad \stackrel{\bullet}{2} \quad \cdots \xrightarrow[n-1]{\bullet} \quad \stackrel{\rightharpoonup}{\bullet}$
$\mathfrak{c}_{n}$

$\mathfrak{d}_{n}$

$\mathfrak{e}_{6}$

$\mathfrak{e}_{8}$

$\mathfrak{g}_{2}$


### 1.7 Weyl Group

Given a root system, we have a corresponding group that is generated by the reflections

$$
\left\{s_{\alpha}: \alpha \in \Delta\right\} .
$$

Given a Dynkin diagram $\Gamma$ that corresponds to a generalized Cartan matrix $A=\left(a_{i j}\right)$, we can define the Weyl group $W(\Gamma)$ using generators and relations. The set of generators is $S=\left\{s_{i}: i \in \Gamma\right\}$, and the relations are

$$
\begin{aligned}
s_{i}^{2} & =1 \text { for all } i \in \Gamma, \\
s_{i} s_{j} & =s_{j} s_{i} \text { if } a_{i j}=0, \\
s_{i} s_{j} s_{i} & =s_{j} s_{i} s_{j} \text { if } a_{i j}<0 \text { and } a_{i j} a_{j i}=1, \\
s_{i} s_{j} s_{i} s_{j} & =s_{j} s_{i} s_{j} s_{i} \text { if } a_{i j}<0 \text { and } a_{i j} a_{j i}=2 .
\end{aligned}
$$

These relations encode precisely the defining relations between the simple reflections described above.

Example 1.10. Let $\Phi$ be a root system of type $A_{3}$; in particular, write $\Phi=\left\{\varepsilon_{i}-\varepsilon_{j}: 1 \leq i, j \leq 4\right\}$. For simple roots, we choose $\Delta=\left\{\varepsilon_{1}-\varepsilon_{2}, \varepsilon_{2}-\varepsilon_{3}, \varepsilon_{3}-\varepsilon_{4}\right\}$. Now, each simple reflection $s_{\alpha}$ has the effect of exchanging two basis vectors while fixing the other two: for example, $s_{\alpha_{2}}$ exchanges $\varepsilon_{2}$ and $\varepsilon_{3}$ while fixing $\varepsilon_{1}$ and $\varepsilon_{4}$. Thus the Weyl group $W\left(A_{3}\right)$ generated by the three reflections $\left\{s_{\alpha_{1}}, s_{\alpha_{2}}, s_{\alpha_{3}}\right\}$ is isomorphic to the symmetric group $S_{4}$, acting as permutations on the four basis vectors $\left\{\varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}, \varepsilon_{4}\right\}$.

If $L$ is a Lie algebra and $V$ is an $L$-module with weights $\Lambda$, then the Weyl group of $L$ has a natural action on the set $\Lambda$ [5, Proposition 3.6]. We will describe this action in greater detail later, using the notation of heaps. If the action of the Weyl group $W$ on the set of weights $\Lambda$ is transitive, then we say that $V$ is a minuscule $L$-module. In Table 1.1 we give a list of the fundamental weights corresponding to minuscule representations of the simple Lie algebras. The definition and significance of the fundamental weights will be addressed in Section 1.9. For numbering conventions, refer to Figure 1.5.

Table 1.1: A complete list of fundamental weights of minuscule representations for simple Lie algebras.

| Type | Indices of minuscule weights |
| :---: | :---: |
| $\mathfrak{a}_{n}$ | $1,2, \ldots, n$ |
| $\mathfrak{b}_{n}$ | $n$ |
| $\mathfrak{c}_{n}$ | 1 |
| $\mathfrak{d}_{n}$ | $1, n-1, n$ |
| $\mathfrak{e}_{6}$ | 1,5 |
| $\mathfrak{e}_{7}$ | 6 |
| $\mathfrak{e}_{8}$ | none |
| $\mathfrak{f}_{2}$ | none |
| $\mathfrak{g}_{2}$ | none |

Definition 1.11. Let $W$ is a Weyl group and $S$ be the usual set of generating simple reflections that correspond to the vertices of a Dynkin diagram $\Gamma$. If $S^{\prime}$ is some proper subset of $S$, then we write $W_{S^{\prime}}$ for the subgroup of $W$ that is generated by $S^{\prime}$. This is called a parabolic subgroup, and it corresponds to the subgraph of $\Gamma$ that consists of all vertices from $S^{\prime}$, along with all edges between these vertices.

### 1.8 Universal Enveloping Algebra

In order to describe the irreducible representations for finite-dimensional semisimple Lie algebras over $\mathbb{C}$, it will be useful to be able to provide a general construction of such modules. The usual construction relies on the structure of the universal enveloping algebra.

Recall that each finite group $G$ can be thought of as being contained in an associative algebra $\mathbb{C} G$, the group algebra, that has the same representation theory as the original group. Given an arbitrary Lie algebra $L$, we may associate with it an associative algebra $U(L)$ which plays a similar role.

First, if $V$ is a vector space over $\mathbb{C}$, we will denote by $T^{k}(V)$ the $k$-th tensor power (over $\mathbb{C}$ ) of $V$, so that

$$
T^{0}(V)=\mathbb{C}, \quad T^{1}(V)=V, \quad T^{2}(V)=V \otimes V,
$$

and so on. We have an associative multiplication $T^{j}(V) \times T^{k}(V) \rightarrow T^{j+k}(V)$ given by tensor concatenation:

$$
\left(v_{1} \otimes \cdots \otimes v_{j}\right) \cdot\left(w_{1} \otimes \cdots \otimes w_{k}\right)=v_{1} \otimes \cdots \otimes v_{j} \otimes w_{1} \otimes \cdots \otimes w_{k}
$$

extended linearly to sums of simple tensors.
Definition 1.12. We define the tensor algebra of $V$ to be

$$
T(V)=T^{0}(V) \oplus T^{1}(V) \oplus T^{2}(V) \oplus \cdots
$$

with associative multiplication given by linearly extending the operation described above.
In particular, if $L$ is a Lie algebra, we will make use of the tensor algebra $T(L)$, where we are identifying $L$ with its underlying vector space. Note that $L=T^{1}(L)$, so the elements of the Lie algebra $L$ can be identified with elements of the associative algebra $T(L)$.

Definition 1.13. Let $L$ be a Lie algebra over $\mathbb{C}$ and let $T(L)$ be the tensor algebra of the underlying vector space of $L$. Let $J(L)$ be the two-sided ideal of $T(L)$ generated by all elements of the form

$$
x \otimes y-y \otimes x-[x, y]
$$

where $x, y \in L$ and $[x, y]$ is the usual Lie bracket from $L$. We define the universal enveloping algebra of $L$ by $U(L):=T(L) / J(L)$.

Lemma 1.14 ([1, Proposition 9.3]). For any $\mathbb{k}$-vector space $V$, there is a natural bijection between representations $\rho: L \rightarrow \operatorname{End}_{\mathbb{k}^{k}}(V)$, of $L$ as a Lie algebra, and representations $\phi: U(L) \rightarrow \operatorname{End}_{\mathbb{k}}(V)$, of $U(L)$ as an associative algebra.

Theorem 1.15 (Poincaré-Birkhoff-Witt Basis Theorem, [1, Theorem 9.4]). Let L be a Lie algebra with basis $\left\{x_{i}\right\}_{i \in I}$, and let $\sigma: L \rightarrow U(L)$ be the natural map sending the Lie algebra to its universal enveloping algebra, where $\sigma\left(x_{i}\right)=y_{i}$ for some $y_{i} \in U(L)$. Then the elements

$$
y_{i_{i}}^{r_{1}} y_{i_{2}}^{r_{2}} \cdots y_{i_{n}}^{r_{n}}
$$

as $n \geq 0, r_{i} \geq 0$, and $i_{1}, \ldots, i_{n} \in I$ with $i_{1}<i_{2}<\cdots<i_{n}$, form a basis for $U(L)$.
In particular, the set $\left\{y_{i}\right\}_{i \in I}$ is linearly independent, so $\sigma$ is an injective map.

### 1.9 Highest Weight Modules

Irreducible representations of simple Lie algebras over $\mathbb{C}$ can be classified by their highest weights.

As the roots $\Phi$ span $H^{*}$ [4, Lemma 10.11], and the simple roots form a basis for the roots, we have that the simple roots also span $H^{*}$. It follows that the difference between any two weights in $H^{*}$ can be written as a linear combination of simple roots. We use this fact to define a partial order on weights for $L$, writing $\lambda_{1} \geq \lambda_{2}$ whenever $\lambda_{1}-\lambda_{2}$ is a nonnegative linear combination of simple roots. This makes the set of weights for $L$ into a partially ordered set. Given a finite-dimensional $L$-module, we can search for a maximal element of the poset of weights for this representation. If there exists a unique maximal element, then this is called the highest weight of the module.

Lemma 1.16 ([4, Lemma 15.3]). If $V$ is an irreducible L-module, then the set of weights of $V$ contains a unique highest weight.

Equivalently, we say that $v \in V$ is a highest weight vector of highest weight $\lambda$ provided
(i) $v$ is a vector of weight $\lambda$, and
(ii) the Lie algebra generators $e_{\alpha}$ annihilate $v$, for all simple roots $\alpha$.

Conversely, we might like to know which weights occur as highest weights of irreducible modules. It turns out that we can construct modules with a given highest weight, using the universal enveloping algebra.

Definition 1.17. Let $L$ be a finite-dimensional simple complex Lie algebra with Cartan subalgebra $H$ and simple roots $\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}$. For a weight $\lambda: H \rightarrow \mathbb{C}$, define $K_{\lambda}$ to be the left ideal of $U(L)$ generated by

$$
\left\{e_{\alpha}: \alpha \text { is a positive root }\right\} \cup\left\{h_{i}-\lambda\left(h_{i}\right): 1 \leq i \leq n\right\} .
$$

The Verma module of highest weight $\lambda$ is defined by $M(\lambda):=U(L) / K_{\lambda}$. This is a module for $U(L)$ and hence also a module for $L$.

Theorem 1.18 ([1, Theorem 10.9]). The L-module $M(\lambda)$ contains a unique maximal proper submodule.

We will denote this submodule by $J(\lambda)$. It follows that $L(\lambda):=M(\lambda) / J(\lambda)$ is an irreducible $L$-module. In order to describe the $\lambda$ for which $L(\lambda)$ is finite-dimensional, we require a discussion of the fundamental weights and the lattice of integral weights.

Given a set of simple roots $\alpha_{1}, \ldots, \alpha_{n}$ for $L$, and a corresponding basis $h_{1}, \ldots, h_{n}$ for a Cartan subalgebra $H$ of $L$, we write $\omega_{i}$ for the weight given by $\omega_{i}\left(h_{j}\right)=\delta_{i j}$. The weights $\left\{\omega_{1}, \ldots, \omega_{n}\right\}$ are known as fundamental weights, and form a basis for $H^{*}$.

From the fundamental weights, we can define the lattice of integral weights,

$$
X=\left\{\sum_{i=1}^{n} c_{i} \omega_{i}: c_{i} \in \mathbb{Z}\right\}
$$

A particular subset of these,

$$
X^{+}=\left\{\sum_{i=1}^{n} c_{i} \omega_{i}: c_{i} \in \mathbb{Z}_{\geq 0}\right\}
$$

is known as the set of dominant integral weights.

Theorem 1.19. Let $L$ be a Lie algebra over $\mathbb{C}$.
(i) If $\lambda \in X^{+}$, then $L(\lambda)$ is a finite-dimensional L-module.
(ii) If $L$ is finite-dimensional and semisimple, then the finite-dimensional irreducible L-modules over $\mathbb{C}$ are precisely $L(\lambda)$ for $\lambda \in X^{+}$, and these modules are pairwise nonisomorphic.

Proof. This is [1, Theorems 10.20 and 10.21].

Thus it makes sense to discuss a unique finite-dimensional irreducible module, indexed by its highest weight, for a given finite-dimensional semisimple Lie algebra. If the Lie algebra is simple, we will sometimes write these modules using the notation $L$ (type, highest weight); for example, the representation of the Lie algebra $\mathfrak{d}_{6}$ of highest weight $\omega_{6}$ will be denoted $L\left(D_{6}, \omega_{6}\right)$.

We will now explore how to use heaps to encode and visualize the combinatorial structure of the weights for these modules.

## Chapter 2

## Heaps

We will use heaps over graphs as a tool for describing and visualizing the structure of various minuscule representations for simple Lie algebras. Heaps were first described by G. X. Viennot [11] in 1986. The definitions below, which are from Green [6], appear different but are equivalent to Viennot's original formulation.

Recall that a partially ordered set (or poset) is a pair ( $S, \leq$ ), where $S$ is a set and $\leq$ is a binary relation on $S$ that is reflexive, transitive, and antisymmetric. A chain is a subset $C$ of $S$ such that if $x, y \in C$, then $x \leq y$ or $y \leq x$. An ideal of $S$ is a subset $I$ of $S$ such that if $x \in I$ and $y \leq x$, then $y \in I$ also. If $E$ is a poset, we write $J(E)$ for the set of all ideals of $E$, and $\mathcal{B}(E)=J(E) \backslash\{\emptyset, E\}$ for the set of proper ideals of $E$. If $x, y \in S$, we say that $y$ covers $x$ if $x<y$ and there are no elements $z \in S$ with $x<z<y$.

For example, in Figure 2.1, $\{x, z\}$ is an ideal, but $\{x\}$ is not an ideal since $z<x$.

Definition 2.1. A closed interval in $(S, \leq)$ is a subset of the form $[x, y]=\{z \in S: x \leq z \leq y\}$ for some $x, y \in S$. An open interval is a subset of the form $(x, y)=\{z \in S: x<z<y\}$ for some $x, y \in S$. We say that $(S, \leq)$ is locally finite if every possible interval is a finite subset of $S$. A subset $Q$ of $S$ is said to be convex if whenever $x, y \in Q$ and $x \leq z \leq y$, we have $z \in Q$ also.

If our partially ordered set is finite (or locally finite) we often visualize it (or parts of it) using a Hasse diagram. This is possible because the relation on each locally finite poset is equal to the reflexive, transitive closure of its covering relations [6, Appendix A]. The Hasse diagram is a graph
in which elements that are higher in the partial order are drawn above elements that are lower, and a vertical or diagonal line is drawn between $x$ (below) and $y$ (above) whenever $y$ covers $x$.

Figure 2.1: Hasse diagram of the poset $(S, \leq)$ where $S=\{w, x, y, z\}$, with $z \leq x, z \leq y, z \leq w$, $x \leq w$, and $y \leq w$. Per reflexivity, every element is also comparable to itself. The elements $x$ and $y$ cannot be compared in this poset.


A heap is a function from a special partially ordered set to the set of vertices of an underlying graph $\Gamma$. For our purposes, $\Gamma$ will be the Dynkin diagram of the relevant Lie algebra. In general, we require that $\Gamma$ be a graph without any loops.

Definition 2.2. If $E$ is a partially ordered set and $\Gamma$ is a graph without loops, then a heap over $\Gamma$ is a labelling function $\varepsilon: E \rightarrow \Gamma$, satisfying the following.
(H1) For each vertex $x$ of $\Gamma$, and for each edge $\{x, y\}$ of $\Gamma$, the sub-posets $\varepsilon^{-1}(x)$ and $\varepsilon^{-1}(\{x, y\})$ are chains in $E$. (These are called vertex chains and edge chains, respectively.)
(H2) The partial order $\leq$ on $E$ is the minimal partial order that extends the order on these vertex chains and edge chains.

We say that a heap is locally finite (respectively, infinite, finite) if its underlying poset is locally finite (respectively, infinite, finite).

From now on, we will draw the Hasse diagram of a heap with elements labelled directly.

Definition 2.3. If $\varepsilon_{1}: E_{1} \rightarrow \Gamma_{1}$ and $\varepsilon_{2}: E_{2} \rightarrow \Gamma_{2}$ are heaps, then a morphism of heaps is a pair of maps $f=\left(f_{E}, f_{\Gamma}\right)$, where $f_{E}: E_{1} \rightarrow E_{2}$ and $f_{\Gamma}: \Gamma_{1} \rightarrow \Gamma_{2}$, satisfying
(i) $x \leq y \Rightarrow f_{E}(x) \leq f_{E}(y)$;
(ii) whenever $\{a, b\}$ is an edge in $\Gamma_{1}$, either $f_{\Gamma}(a)=f_{\Gamma}(b)$ or $\left\{f_{\Gamma}(a), f_{\Gamma}(b)\right\}$ is an edge in $\Gamma_{2}$;

Figure 2.2: A heap over the graph $\Gamma$. The labelling function $\varepsilon$ is indicated by the downward arrows; for example, $\varepsilon\left(x_{3}\right)=\varepsilon\left(x_{2}\right)=2$.


Figure 2.3: A more efficient way to draw the heap shown in Figure 2.2.

(iii) $f_{\Gamma} \circ \varepsilon_{1}=\varepsilon_{2} \circ f_{E}$.

We say that a morphism $f=\left(f_{E}, f_{\Gamma}\right)$ is an isomorphism of heaps if there is a morphism $g=$ $\left(g_{E}, g_{\Gamma}\right)$, where $g_{E}: E_{2} \rightarrow E_{1}$ and $g_{\Gamma}: \Gamma_{2} \rightarrow \Gamma_{1}$, such that $g_{E} \circ f_{E}=\operatorname{id}_{E_{1}}, f_{E} \circ g_{E}=\mathrm{id}_{E_{2}}$, $g_{\Gamma} \circ f_{\Gamma}=\operatorname{id}_{\Gamma_{1}}$, and $f_{\Gamma} \circ g_{\Gamma}=\operatorname{id}_{\Gamma_{2}}$. In this case, we say that the heaps $\varepsilon_{1}: E_{1} \rightarrow \Gamma_{1}$ and $\varepsilon_{2}: E_{2} \rightarrow \Gamma_{2}$ are isomorphic. If $\Gamma_{1}=\Gamma_{2}$ and $f_{\Gamma}$ is the identity map, then we say that the heaps are isomorphic as heaps over $\Gamma_{1}$.

Definition 2.4. If $\varepsilon_{1}: F \rightarrow \Gamma$ and $\varepsilon_{2}: E \rightarrow \Gamma$ are both heaps over the same graph, we say that $F$ is a subheap of $E$ if there exists a morphism $f=\left(f_{E}, f_{\Gamma}\right)$ from $F$ to $E$ such that $f_{\Gamma}$ is the identity map and $f_{E}$ is injective. The subheap $F$ is said to be convex if $f_{E}(F)$ is a convex subset of $E$.

### 2.1 Full Heaps

Full heaps over Dynkin diagrams serve as a framework for describing certain representations of Lie algebras. Within these full heaps, particular subheaps describe minuscule modules for simple Lie algebras over $\mathbb{C}$. We begin with some necessary definitions and then specialize to the subheaps of interest.

Definition 2.5. Let $I_{1}$ and $I_{2}$ be ideals of the heap $\varepsilon: E \rightarrow \Gamma$, with $I_{1} \subset I_{2}$, and suppose that $I_{2} \backslash I_{1}=\{x\}$. If $\varepsilon(x)=p$, then we write $I_{1} \prec_{p} I_{2}$.

Definition 2.6. Let $\varepsilon: E \rightarrow \Gamma$ be a nonempty locally finite heap. Suppose that $x, y \in E$ satisfy $\varepsilon(x)=\varepsilon(y)=p$, and further suppose that $(x, y) \cap \varepsilon^{-1}(p)=\emptyset$; in other words, no element of $E$ in between $x$ and $y$ is labelled by $p$. In this case, we call $(x, y)$ an open $p$-interval and $[x, y]$ a closed $p$-interval.

For example, in Figure 2.2, we have that $\left(x_{2}, x_{3}\right)=\left\{x_{1}, x_{4}\right\}$ is an open 2-interval and $\left[x_{2}, x_{3}\right]=$ $\left\{x_{1}, x_{2}, x_{3}, x_{4}\right\}$ is a closed 2-interval.

Definition 2.7. Let be a nonempty locally finite heap, and let $A=\left(A_{i j}\right)$ be the Cartan matrix corresponding to $\Gamma$. We say that $E$ is a full heap over $\Gamma$ if the following are satisfied.
(F1) Every vertex chain of $E$ is isomorphic to $\mathbb{Z}$ (as a poset).
(F2) If $\{a, b\}$ is an edge of $\Gamma$ and $x \in E$ with $\varepsilon(x)=a$, then there exists $y \in E$ with $\varepsilon(y)=b$ such that either $x$ covers $y$ or $y$ covers $x$.
(F3) If $[x, y]$ is a closed $p$-interval in $E$, then $\sum_{x \leq z \leq y} A_{p, \varepsilon(z)}=2$.
In the simply laced case, (F3) becomes simpler: each open $p$-interval contains precisely two elements that are labelled by vertices of $\Gamma$ adjacent to $p$.

Figure 2.4: Part of a full heap over the Dynkin diagram of type $D_{6}^{(1)}$. This heap is infinite and has a repeating structure. One repeat is shown in the dashed box.


Lemma 2.8. If $\varepsilon: E \rightarrow \Gamma$ is a full heap over a Dynkin diagram, then for any ideal $I \in J(E)$, exactly one of the following holds:
(i) there exists a unique $I^{\prime} \in J(E)$ with $I^{\prime} \succ_{p} I$;
(ii) there exists a unique $I^{\prime} \in J(E)$ with $I^{\prime} \prec_{p} I$;
(iii) for every ideal $I^{\prime} \in J(E)$, we do not have $I^{\prime} \succ_{p} I$ or $I^{\prime} \prec_{p} I$.

Proof. This is a restatement of [6, Lemma 3.1.2].

### 2.2 Representations from Heaps

With the possibilities of Lemma 2.8 in mind, we construct the vector space

$$
V_{J(E)}:=\mathbb{C}-\operatorname{span}\left\{v_{I}: I \in J(E)\right\} .
$$

Eventually we will place on $V_{J(E)}$ the structure of a module for a Lie algebra.

Definition 2.9. We define the following linear operators on $V_{J(E)}$, described by their action on an arbitrary basis vector $V_{I}$ for some $I \in J(E)$. We have

$$
\begin{aligned}
& X_{p}\left(v_{I}\right)= \begin{cases}v_{I^{\prime}} & \text { if there exists } I^{\prime} \text { with } I^{\prime} \succ_{p} I, \\
0 & \text { if no such } I^{\prime} \text { exists; }\end{cases} \\
& Y_{p}\left(v_{I}\right)= \begin{cases}v_{I^{\prime}} & \text { if there exists } I^{\prime} \text { with } I^{\prime} \prec_{p} I, \\
0 & \text { if no such } I^{\prime} \text { exists; }\end{cases} \\
& H_{p}\left(v_{I}\right)= \begin{cases}v_{I} & \text { if there exists } I^{\prime} \text { with } I^{\prime} \prec_{p} I, \\
0 & \text { if neither such } I^{\prime} \text { exists; }\end{cases} \\
& S_{p}\left(v_{I}\right)= \begin{cases}v_{I^{\prime}} & \text { if there exists } I^{\prime} \text { with } I^{\prime} \succ_{p} I, \\
v_{I^{\prime}} & \text { if there exists } I^{\prime} \text { with } I^{\prime} \prec_{p} I, \\
0 & \text { if neither such } I^{\prime} \text { exists. }\end{cases}
\end{aligned}
$$

These operators are well-defined due to Lemma 2.8.
We may now describe the action of the Weyl group and Lie algebra corresponding to $\Gamma$ on the module $V_{J(E)}$. Let $W(\Gamma)$ be the Weyl group corresponding to the Dynkin diagram $\Gamma$, and let $s_{p}$ be a simple reflection of the Weyl group that corresponds to the vertex $p$ of $\Gamma$. Then $s_{p}$ acts on $V_{J(E)}$ via the map $S_{p}$, defined above.

Lemma 2.10 ([6, Theorem 3.1.13 (ii)]). The action of $W(\Gamma)$ on $V_{J(E)}$ in which each generator $s_{p}$ acts via the operator $S_{p}$ is well-defined and makes $V_{J(E)}$ into a (left) $W(\Gamma)$-module.

If $L$ is a semisimple Lie algebra with root system corresponding to the Dynkin diagram $\Gamma$, then $V_{J(E)}$ is also an $L$-module, as follows.

Lemma 2.11. The action of $L$ on $V_{J(E)}$ in which the generators $e_{p}, f_{p}, h_{p}$ act via the operators $X_{p}, Y_{p}$, and $H_{p}$, respectively, is well-defined and makes $V_{J(E)}$ into an L-module.

Proof. This is a special case of [6, Theorem 4.1.6].

Lemma $2.12([5$, Lemma $3.1(7),(10)])$. In the situation of Lemma 2.11, for each $p$, the operators $e_{p} \circ e_{p}$ and $f_{p} \circ f_{p}$ both act as 0 on $V_{J(E)}$.

Lemma 2.13 ([6, Lemma 3.3.3]). Let $\varepsilon: E \rightarrow \Gamma$ be a full heap over a finite graph $\Gamma$, and let $I \in \mathcal{B}(E)$. Let $S$ be the set of simple reflections that generate $W(\Gamma)$, and take $S^{\prime}$ to be some nonempty proper subset of $S$. Write $[I]_{S^{\prime}}$ for the orbit of I under the parabolic subgroup $W_{S^{\prime}}$. Then the following hold.
(i) The orbit $[I]_{S^{\prime}}$ contains a maximal element $I_{S^{\prime}}^{+}$and a minimal element $I_{S^{\prime}}^{-}$.
(ii) The set $F_{S^{\prime}}=I_{S^{\prime}}^{+} \backslash I_{S^{\prime}}^{-}$is a finite convex subheap of $E$.
(iii) There is an inclusion-preserving bijection $b$ from the ideals of $F_{S^{\prime}}$ to the orbit $[I]_{S^{\prime}}$ given by

$$
b(J)=J \cup I_{S^{\prime}}^{-}
$$

We will refer to $F_{S^{\prime}}$ as a parabolic subheap. For an example, see Figure 2.5.

Lemma 2.14 ([6, Lemma 5.5.2]). Let $\varepsilon: E \rightarrow \Gamma$ be a full heap over an affine Dynkin diagram, let $A$ be the corresponding Cartan matrix, and let $A^{0}$ be the corresponding finite type matrix (i.e., the matrix $A$ with the row and column corresponding to zero removed). Let $\Gamma^{0}$ be the Dynkin diagram corresponding to $A^{0}$. Then all parabolic subheaps of the form $F_{S \backslash\left\{s_{0}\right\}}$ within $E$ are isomorphic as heaps over $\Gamma$ (regardless of which ideal I we started with).

Figure 2.5: Part of a full heap over the Dynkin diagram of type $D_{6}^{(1)}$, which was shown in Figure 1.4. One possible parabolic subheap corresponding to the subset of generators $S^{\prime}=\left\{s_{1}, s_{2}, s_{3}, s_{4}, s_{5}\right\}$ is shown in grey.


We call $\varepsilon: F_{S \backslash\left\{s_{0}\right\}} \rightarrow \Gamma^{0}$ the principal subheap associated to $E$. For an example, see Figure 2.6.

The following lemma shows that these principal subheaps provide precisely the objects we need in order to realize the minuscule representations of simple Lie algebras. In fact, any minuscule representation can be constructed in this way; all are described in [6].

Lemma 2.15 ([6, Lemma 5.5.5]). Let $\varepsilon: E \rightarrow \Gamma$ be a full heap over an affine Dynkin diagram $\Gamma$. Let $\mathfrak{g}$ be the corresponding simple Lie algebra, and let $F$ be the principal subheap of $E$ corresponding to $\mathfrak{g}$ via the process described in Lemma 2.14. Let $V_{F}$ be the $\mathfrak{g}$-module corresponding to $F$. Then $V_{F}$ is a minuscule module.

We will frequently refer to the shape and characteristics of the principal subheap for the spin representation of $\mathfrak{D}_{6}$ of highest weight $\omega_{6}$, shown in Figure 2.7, as well as to the shapes of the principal subheaps for minuscule representations in type $A$.

Recall from Table 1.1 that for each Lie algebra $\mathfrak{a}_{n}$, we have $n$ minuscule representations; these may be constructed as exterior powers of the natural module [1, Theorem 13.7]. In order to

Figure 2.6: Part of a full heap over the Dynkin diagram of type $D_{6}^{(1)}$. One copy of the principal subheap is shown in grey.


Figure 2.7: Principal subheap for a spin representation of the simple Lie algebra $\mathfrak{d}_{6}$. To see how this principal subheap embeds in its full heap, see Figure 2.6.

describe the principal subheaps for type $A$, it will be useful to first define a particular full heap $\varepsilon: E(\mathbb{Z}) \rightarrow \Gamma(\mathbb{Z})$.

Definition 2.16. Let $\Gamma(\mathbb{Z})$ be the graph with vertices given by $\mathbb{Z}$, with $i$ and $j$ adjacent if and only if $|i-j|=1$. Let $E(\mathbb{Z})$ be the set

$$
\{(x, y): x, y \in \mathbb{Z} \text { and } x-y \text { is even }\} .
$$

The partial order

$$
(a, b) \leq(c, d) \Leftrightarrow \text { both } d>b \text { and }|c-a| \leq|d-b|
$$

makes $E(\mathbb{Z})$ into a poset. The labelling function $\varepsilon: E(\mathbb{Z}) \rightarrow \Gamma(\mathbb{Z})$ given by $\varepsilon((x, y))=x$ is a locally finite full heap [6, Lemma 6.1.4].

Figure 2.8: Part of the full heap $\varepsilon: E(\mathbb{Z}) \rightarrow \Gamma(\mathbb{Z})$.


Lemma 2.17 ([6, Proposition 6.2.3]). Let $n \geq 1$ and let $1 \leq k \leq n$. Let $F$ be the subheap of $E(\mathbb{Z})$ consisting of the elements within the rectangular region bounded by the lines

$$
\begin{aligned}
y & =x+1, \\
y & =x+1-2 n+2 k, \\
x+y & =1, \text { and } \\
x+y & =1+2 k .
\end{aligned}
$$

Then $F$ is isomorphic to the principal subheap corresponding to a minuscule representation of $\mathfrak{a}_{n}$ of highest weight $\omega_{k}$.

Figure 2.9: Principal subheap for the minuscule representation with highest weight $\omega_{3}$ of the simple Lie algebra $\mathfrak{a}_{4}$. By Lemma 2.17, all minuscule representations in type $A$ have rectangle-shaped heaps.


When one of these rectangular heaps appears as a subheap of some heap of interest, we will refer to this as a type $A$ subheap. These type $A$ subheaps are frequently subheaps not of $E(\mathbb{Z})$, but of some other principal subheap. For example, the principal subheap for the spin representation of $\mathfrak{d}_{6}$ of highest weight $\omega_{6}$ has several type $A$ subheaps. Here, we use Lemma 2.13 and the idea of a parabolic subheap to identify copies of type $A$ heaps within our spin module principal heap. Although it is possible to do this by eliminating either generator 5 or generator 6 , it is nice to have the Weyl group of type $A_{5}$ with generators indexed 1 through 5 , so we will adopt the convention of removing generator 6 from type $D$. Thus our type $A$ subheaps will be maximal convex subheaps that avoid heap elements labelled 6 .

Figure 2.10: The orbit of the ideal $I$ under the subgroup $W^{\prime}$ with generating set $S^{\prime}=$ $\left\{s_{1}, s_{2}, s_{3}, s_{4}, s_{5}\right\}$ has maximal element $I_{S^{\prime}}^{+}$and minimal element $I_{S^{\prime}}^{-}$. The heap $I_{S^{\prime}}^{+} \backslash I_{S^{\prime}}^{-}$is a type $A$ subheap of this heap.


Green [6, Proposition 8.2.9] explains how to use particular subheaps to produce branching rules for the decomposition of a module on restriction to a Lie subalgebra. For example, we are interested in how the spin modules for $\mathfrak{d}_{6}$ decompose when restricted to $\mathfrak{a}_{5}$. When $n=6$, this decomposition (as $\mathfrak{a}_{5}$-modules) is given by

$$
L\left(D_{6}, \omega_{6}\right)=L\left(A_{5}, \omega_{0}\right) \oplus L\left(A_{5}, \omega_{2}\right) \oplus L\left(A_{5}, \omega_{4}\right) \oplus L\left(A_{5}, \omega_{6}\right)
$$

where, as in [6, Exercise 8.2.15], we adopt the convention that $L\left(A_{5}, \omega_{0}\right)$ and $L\left(A_{5}, \omega_{6}\right)$ are both isomorphic to the one-dimensional trivial module.

### 2.3 Relative Content

The relative content is a useful statistic on pairs of heap ideals. We will first, more generally, define the content of a heap ideal (or any subset of the associated poset of a heap).

Definition 2.18. Let $\varepsilon: E \rightarrow \Gamma$ be a heap over a Dynkin diagram $\Gamma$, and let $F$ be a finite subheap of $E$. The content of $F$ is a $\mathbb{Z}$-linear combination of simple roots,

$$
\chi(F)=\sum_{i=1}^{n} c_{i} \alpha_{i},
$$

where each coefficient $c_{i}$ counts the number of elements of $F$ labelled $i$.

Definition 2.19. For any two heap ideals $L_{1}$ and $L_{2}$ of a heap $E$, we define their symmetric difference, written $L_{1} \Delta L_{2}$, by

$$
L_{1} \Delta L_{2}=\left(L_{1} \cup L_{2}\right) \backslash\left(L_{1} \cap L_{2}\right) .
$$

We can now define the relative content of a pair of ideals. This definition is [6, Definition 3.2.8]. The relative content is well-defined whenever the symmetric difference is finite.

Definition 2.20. For any two subsets $L_{1}$ and $L_{2}$ of the same heap, we define the relative content of $L_{1}$ and $L_{2}$ by

$$
\chi\left(L_{1}, L_{2}\right)=\chi\left(L_{1} \backslash L_{2}\right)-\chi\left(L_{2} \backslash L_{1}\right) .
$$

Figure 2.11: The ideal $L$ is given by all heap elements below the line. The content of the ideal $L$ is $\alpha_{1}+\alpha_{2}+2 \alpha_{3}+3 \alpha_{4}+2 \alpha_{5}+2 \alpha_{6}$.


By restriction, we can think of the type $D$ spin modules as also modules for smaller, type $A$ Lie algebras. The following definition, which helps to separate the type $A$ action on spin modules from the extra generator in type $D$, is new.

Definition 2.21. Let $L$ be a heap ideal corresponding to the spin representation for $\mathfrak{D}_{6}$ of highest weight $\omega_{6}$, and write

$$
\chi(L)=\sum_{i=1}^{n} c_{i} \alpha_{i} .
$$

We define the level of the ideal $L$ to be $c_{6}$, and we define the $\operatorname{sign}$ of the ideal $L$ to be $c_{6} \bmod 2$.

We now discuss a strategy for viewing ideals of the same level as being ideals of the same type $A$ subheap. Figures 2.13 and 2.14 provide examples of this phenomenon.

Figure 2.12: For the two ideals below, $\chi\left(L_{1}, L_{2}\right)=\alpha_{2}-\alpha_{4}-\alpha_{5}$. These two ideals have the same sign and the same level, since there are no heap elements labelled 6 in the symmetric difference.


Lemma 2.22. If $L_{1}$ and $L_{2}$ are heap ideals of the principal subheap corresponding to the spin module for $\mathfrak{d}_{6}$ of highest weight $\omega_{6}$, and $L_{1}$ and $L_{2}$ have the same level, then $L_{1}$ and $L_{2}$ may be identified with ideals of the same type A subheap.

Proof. As all of these type $A$ subheaps are parabolic subheaps, the identification is given by the bijection of Lemma 2.13.

If $L_{1}$ and $L_{2}$ both have level 0 or level 3, then they are equal and may be identified with ideals of the trivial type $A$ subheap, which contains no elements.

If $L_{1}$ and $L_{2}$ both have level 1 or level 2 , then they may be identified with ideals of the type $A$ subheap shown in Figure 2.13 or Figure 2.14, respectively.

Lemma 2.23 ([6, Proposition 5.2.6]). If $L_{1}$ and $L_{2}$ are heap ideals corresponding to weights $\lambda_{1}$ and $\lambda_{2}$ of any minuscule representation, then $\chi\left(L_{1}, L_{2}\right)=\lambda_{1}-\lambda_{2}$.

Corollary 2.24. Adding a single element to a heap ideal is equivalent to adding that root to the corresponding weight. Removing a single element from a heap ideal is equivalent to subtracting that root from the corresponding weight.

Proof. Combining Definition 2.20 with Lemma 2.23, we see that if one ideal is fixed and the

Figure 2.13: The ideals $L_{1}$ and $L_{2}$ both have level 1, so they may both be identified with ideals of the type $A$ subheap indicated by the grey rectangle.


Figure 2.14: The ideals $L_{1}$ and $L_{2}$ both have level 2, so they may both be identified with ideals of the type $A$ subheap indicated by the grey rectangle.

other ideal is manipulated by adding or removing a single element labelled $p$, that weight changes according to the root $\alpha_{p}$.

It will be useful to have notation for the count of heap elements within the symmetric difference of two ideals.

Definition 2.25. The size of the relative content is defined to be

$$
\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=\left|L_{1} \Delta L_{2}\right|
$$

Note that if $\chi\left(L_{1}, L_{2}\right)=\sum_{i} c_{i} \alpha_{i}$, then $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=\sum_{i}\left|c_{i}\right|$.

### 2.4 Profile and Crossings

As shown in Figure 2.11, the contents of an ideal can be delineated using a zigzag line, where the elements in the ideal are those below the line. We will refer to this line as the contour of the ideal. For ideals of the triangular heaps that describe spin representations, we will always begin the contour at the left apex of the triangle, as shown in Figure 2.11.

The profile of two ideals gives a short description of the shape of the difference between the two ideals, and of the shape that is traced out by the two contours. The profile can give insight into the available actions of the Weyl group and Lie algebra on this pair of ideals; the profile can also help to track such actions as they occur. The profile can be computed within any heap over a Dynkin diagram, but it appears to have particularly useful properties in types $A$ and $D$. To the best of our knowledge, this is a new statistic on heap ideals.

Definition 2.26. If $L_{1}$ and $L_{2}$ are heap ideals, we define the profile of ( $L_{1}, L_{2}$ ) according to the following process.
(i) Label the vertices of the Dynkin diagram according to the corresponding coefficients in the relative content $\chi\left(L_{1}, L_{2}\right)$.
(ii) Identify maximal connected components of the Dynkin diagram that share the same coefficient.
(iii) List the components, $C_{1}, \ldots, C_{k}$. We order these according to the smallest numbered vertex of the Dynkin diagram in each component.
(iv) List the corresponding coefficients from the relative content as $p_{1}, \ldots, p_{k}$. This sequence is called the profile, and will be denoted $P\left(L_{1}, L_{2}\right)$.

We will on occasion omit leading or ending zeroes from the profile; thus, profile is well-defined up to zeroes at the ends.

It is straightforward to read the profile directly from the drawing of two ideals on a heap diagram. The profile also encodes much of the information about the types of nontrivial Weyl group actions that are possible on a given pair of weights.

Example 2.27. We now give an example of how to compute the profile for the ideals $L_{1}$ and $L_{2}$ of Figure 2.12. First, given two heap ideals $L_{1}$ and $L_{2}$, corresponding to weights $\lambda_{1}$ and $\lambda_{2}$, we may write the relative content as a $\mathbb{Z}$-linear combination of simple roots. The relative content can now be visualized as a labeling of the vertices of the Dynkin diagram $\Gamma$ by their integer coefficients.

Figure 2.15: The relative content of the heap ideals shown in Figure 2.12 is $\chi\left(L_{1}, L_{2}\right)=\alpha_{2}-\alpha_{4}-\alpha_{5}$. Below, this relative content is displayed as a labeling of the Dynkin diagram of type $D_{6}$.


Now we list maximal connected components $C_{1}, \ldots, C_{k}$ of the Dynkin diagram that share the same coefficient label. We list these using a set ordering of vertices of the Dynkin diagram; our conventions are shown in Figure 1.5.

Lastly, we read the sequence of coefficients of these components, again using the convention of reading in order of the original labelling of vertices of the Dynkin diagram. In this example,

$$
P\left(L_{1}, L_{2}\right)=(0,1,0,-1,0) \quad \text { or } \quad(1,0,-1)
$$

Figure 2.16: Connected components that share the same coefficient label from Figure 2.15.


Lemma 2.28. Let $L_{1}, L_{2}$ be heap ideals for a minuscule representation of type $A$. Then $P\left(L_{1}, L_{2}\right)$ is a sequence that, in each step, either increases by one or decreases by one.

Proof. We will show that in the relative content, successive coefficients must either be equal or differ by 1 . Let $c_{i}$ denote the coefficient of $\alpha_{i}$ in $\chi\left(L_{1}, L_{2}\right)$. We proceed by induction on $i$.

Let $1 \leq i \leq n$. Observing the directions of the lines for the ideals $L_{1}$ and $L_{2}$ as we move from left to right, we can see that if the lines indicating $L_{1}$ and $L_{2}$ move in the same direction (both upward or both downward), then $c_{i+1}=c_{i}$. Otherwise, if $L_{1}$ moves up and $L_{2}$ moves down, then $c_{i+1}=c_{i}+1$; if $L_{1}$ moves down and $L_{2}$ moves up, then $c_{i+1}=c_{i}-1$. Either way, we have that $c_{i+1}$ is either equal to $c_{i}$, or the two coefficients differ by one.

As the Dynkin diagrams in type $A$ are chains, we can see that when $c_{i+1}=c_{i}$, the roots $\alpha_{i+1}$ and $\alpha_{i}$ will fall into the same component of the profile; otherwise, they will fall in adjacent components. Thus the profile is a sequence that, in each step, increases or decreases by one.

Lemma 2.29. Let $L_{1}, L_{2}$ be heap ideals for a minuscule representation of type $A$. Then the first and last entries of $P\left(L_{1}, L_{2}\right)$ are either 0,1 , or -1 .

Proof. This follows from the shape of the principal subheaps in type $A$; the first and last simple roots occur only once as labels, so the coefficients of these must be either 0,1 , or -1 . It follows from Lemma 2.28 that the first and last nonzero coefficients of the profile will be 1 or -1 .

Corollary 2.30. Let $L_{1}, L_{2}$ be heap ideals for one of the spin representations in type $D$. Then $P\left(L_{1}, L_{2}\right)$ is a sequence that, in each step, either increases or decreases by one, with the possible exception of the terms corresponding to any component that contains $\alpha_{n-1}$ or $\alpha_{n}$.

Proof. As the heaps for the spin representations in type $D$ are triangular, if we identify the labels $n-1$ and $n$ as being the same, we may envision this heap as a subheap of a larger rectangular type $A$ heap, as in Figure 2.17. The result holds there, thus it holds within the heap for the spin representation, with the possible exception of the coefficients of $\alpha_{n-1}$ and $\alpha_{n}$ : this is precisely the information we lost in order to pass to type $A$.

Figure 2.17: Envisioning the principal subheap for the spin module for $\mathfrak{D}_{6}$ of highest weight $\omega_{6}$, with labels 5 and 6 both labelled " 5 ", as a subheap of a principal subheap for a larger minuscule representation in type $A$.


Definition 2.31. If $L_{1}$ and $L_{2}$ are heap ideals with $P\left(L_{1}, L_{2}\right)=\left(p_{1}, \ldots, p_{k}\right)$, we define the profile $\operatorname{sum} \sum P\left(L_{1}, L_{2}\right)$ by

$$
\sum P\left(L_{1}, L_{2}\right)=\sum_{i=1}^{k} p_{i} .
$$

Lemma 2.32. Let $L_{1}$ and $L_{2}$ be heap ideals, corresponding to weights $\lambda_{1}$ and $\lambda_{2}$ respectively, for a minuscule representation of type $A$. Write

$$
\chi\left(L_{1}, L_{2}\right)=\sum_{i=1}^{n} c_{i} \alpha_{i}
$$

for the relative content of $\left(L_{1}, L_{2}\right)$. Let $\alpha_{j}$ be a simple root. The following are equivalent.
(i) We have a local minimum at $c_{j}$, that is, $c_{j-1}=c_{j}+1=c_{j+1}$. (We use the convention that $c_{i}=0$ whenever $i<1$ or $i>n$. )
(ii) The vertex $\alpha_{j}$ of $\Gamma$ is a singleton component of the profile $P\left(L_{1}, L_{2}\right)$, and the two neighboring components correspond to smaller coefficients.
(iii) Both $\lambda_{1}+\alpha_{j}$ and $\lambda_{2}-\alpha_{j}$ are also weights of the representation.

Proof. The equivalence of (i) and (ii) is due to the definition of profile, recalling that the Dynkin diagram for type $A$ is a chain.

To see that (i) $\Rightarrow$ (iii), we recognize that the possible arrangement of the two ideals differs based on the sign of $c_{j}$, but in all cases, the contour of $L_{1}$ dips down and the contour of $L_{2}$ caps up as they pass the heap elements labelled $j$. These possibilities are shown in Figure 2.18 below.

Figure 2.18: Possible contour shapes of $L_{1}$ and $L_{2}$ in the situation of Lemma 2.32.


In any of these cases, it is possible to add a heap element labelled $j$ to the ideal $L_{1}$ ( $j$ labels a minimal element of the complement of $L_{1}$ ), and it is possible to remove a heap element labelled $j$ from $L_{2}\left(j\right.$ labels a maximal element of $\left.L_{2}\right)$. Recall from Corollary 2.24 that adding an element to a heap ideal has the same effect as adding the relevant simple root to the corresponding weight, and removing an element from a heap ideal has the same effect as subtracting the relevant simple
root from the corresponding weight of the representation. It follows that $\lambda_{1}+\alpha_{j}$ and $\lambda_{2}-\alpha_{j}$ are also weights of the representation.

Conversely, suppose (iii) holds and $\lambda_{1}+\alpha_{j}$ and $\lambda_{2}-\alpha_{j}$ are also weights of the representation. Then it must be possible to add an element labelled $j$ to the ideal $L_{1}$, and it must be possible to remove an element labelled $j$ from the ideal $L_{2}$. So the complement of $L_{1}$ must contain a minimal element labelled $j$, and $L_{2}$ must contain a maximal element labelled $j$. It follows that we are in one of the four situations in Figure 2.18, and (i) must hold.

We now focus on the actions of the Lie algebra generators $e_{\alpha_{j}}$ and $f_{\alpha_{j}}$ on various pairs of weights. The following results describe in detail the possible effects of these operations on the profile, profile sum, and the size of the relative content.

Corollary 2.33. Let $L_{1}$ and $L_{2}$ be heap ideals with the same level, corresponding to weights $\lambda_{1}$ and $\lambda_{2}$ respectively, for a spin representation in type $D$. Write

$$
\chi\left(L_{1}, L_{2}\right)=\sum_{i=1}^{n} c_{i} \alpha_{i}
$$

for the relative content of $\left(L_{1}, L_{2}\right)$. Let $\alpha_{j}$ be a simple root with $j \neq n$.
(1) The following are equivalent.
(i) We have a local minimum at $c_{j}$, that is, $c_{j-1}=c_{j}+1=c_{j+1}$. (We use the convention that $c_{i}=0$ whenever $i<1$ or $i>n$.)
(ii) The vertex $\alpha_{j}$ of $\Gamma$ is a singleton component of the profile $P\left(L_{1}, L_{2}\right)$, and the two neighboring components correspond to smaller coefficients.
(iii) Both $\lambda_{1}+\alpha_{j}$ and $\lambda_{2}-\alpha_{j}$ are also weights of the representation.
(2) The following are equivalent.
(i) We have a local maximum at $c_{j}$, that is, $c_{j-1}=c_{j}-1=c_{j+1}$.
(ii) The vertex $\alpha_{j}$ of $\Gamma$ is a singleton component of the profile $P\left(L_{1}, L_{2}\right)$, and the two neighboring components correspond to larger coefficients.
(iii) Both $\lambda_{1}-\alpha_{j}$ and $\lambda_{2}+\alpha_{j}$ are also weights of the representation.

Proof. We will begin by proving (1). Since $L_{1}$ and $L_{2}$ are at the same level, by Lemma 2.22, they may both be identified with ideals of the same type $A$ subheap.

As $j \neq n$, actions that affect only heap elements labelled $j$ will keep the two ideals within this subheap. The result now follows from Lemma 2.32.

Part (2) follows by reversing the roles of $L_{1}$ and $L_{2}$.

Lemma 2.34. In the situation of any of the three conditions in Lemma 2.32 above, or in the situation of any of the three conditions in Corollary 2.33 (1) above, it also follows that
(i) $\chi\left(e_{\alpha_{j}} L_{1}, f_{\alpha_{j}} L_{2}\right)=\chi\left(L_{1}, L_{2}\right)+2 \alpha_{j}$,
(ii) $\left|\chi\left(e_{\alpha_{j}} L_{1}, f_{\alpha_{j}} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 2$, and
(iii) $\sum P\left(e_{\alpha_{j}} L_{1}, f_{\alpha_{j}} L_{2}\right)=\left(\sum P\left(L_{1}, L_{2}\right)\right) \pm 2$.

Proof. (i) This follows from the definition of $\chi(-,-)$ and the actions of $e_{\alpha_{j}}$ and $f_{\alpha_{j}}$.
(ii) This follows from (i) and the definition of $|\chi(-,-)|_{\#}$.
(iii) This follows from (i) and the observation that $\alpha_{j}$ is an isolated point, i.e. a component of size one, when we label $\Gamma$ as in the definition of profile. This isolated point changes coefficient by 2, meaning it cannot be absorbed into a neighboring coefficient, because of Lemma 2.28.

Corollary 2.35. In the situation of any of the three conditions in Corollary 2.33 (2) above, it also follows that
(i) $\chi\left(f_{\alpha_{j}} L_{1}, e_{\alpha_{j}} L_{2}\right)=\chi\left(L_{1}, L_{2}\right)+2 \alpha_{j}$,
(ii) $\left|\chi\left(f_{\alpha_{j}} L_{1}, e_{\alpha_{j}} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 2$, and
(iii) $\sum P\left(f_{\alpha_{j}} L_{1}, e_{\alpha_{j}} L_{2}\right)=\left(\sum P\left(L_{1}, L_{2}\right)\right) \pm 2$.

Proof. This result follows from Lemma 2.34 by reversing the roles of $L_{1}$ and $L_{2}$.

Lemma 2.36. Let $L_{1}$ and $L_{2}$ be heap ideals, corresponding to weights $\lambda_{1}$ and $\lambda_{2}$ respectively, for a minuscule representation of type $A$. Write

$$
\chi\left(L_{1}, L_{2}\right)=\sum_{i=1}^{n} c_{i} \alpha_{i}
$$

for the relative content of $\left(L_{1}, L_{2}\right)$. Let $\alpha_{j}$ be a simple root.
(1) If $\lambda_{1}+\alpha_{j}$ is also a weight, but $\lambda_{2}+\alpha_{j}$ and $\lambda_{2}-\alpha_{j}$ are not weights of this representation, then
(i) $\left|\chi\left(e_{\alpha_{j}} L_{1}, L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 1$ and
(ii) $P\left(e_{\alpha_{j}} L_{1}, L_{2}\right)=P\left(L_{1}, L_{2}\right)$.
(2) If $\lambda_{1}-\alpha_{j}$ is also a weight, but $\lambda_{2}+\alpha_{j}$ and $\lambda_{2}-\alpha_{j}$ are not weights of this representation, then
(i) $\left|\chi\left(f_{\alpha_{j}} L_{1}, L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 1$ and
(ii) $P\left(f_{\alpha_{j}} L_{1}, L_{2}\right)=P\left(L_{1}, L_{2}\right)$.

Proof. We will prove (1) below; (2) follows by a similar argument.
Conclusion (i) follows from the definition of $\chi\left(L_{1}, L_{2}\right)$ and the fact that, here, $\chi\left(e_{\alpha_{j}} L_{1}, L_{2}\right)=$ $\chi\left(L_{1}, L_{2}\right)+\alpha_{j}$. In this situation, is possible for $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ to decrease when $L_{1}$ is acted upon by $e_{\alpha_{j}}$ if $L_{1}$ lies below $L_{2}$ in the heap at this point, i.e. if the coefficient $c_{j}$ is negative.

The key to proving (ii) is that we are not in the situation of Lemma 2.32, so by Lemma 2.28, either all three of $c_{j-1}, c_{j}$, and $c_{j+1}$ must be distinct, or $c_{j}$ must be equal to one (or both) of $c_{j-1}$ or $c_{j+1}$.

First, if $c_{j-1}, c_{j}$, and $c_{j+1}$ are all distinct, then they must be an increasing or decreasing sequence of integers by Lemma 2.28. In this case, there is no way to change $c_{j}$ while still meeting the conditions of Lemma 2.28, and we have reached a contradiction. We may assume until further notice that $c_{j}$ is at least equal to one of $c_{j-1}$ or $c_{j+1}$.

If $c_{j-1}=c_{j}=c_{j+1}$, then the two ideals $L_{1}$ and $L_{2}$ must have parallel contours as we trace from left to right through the part of the heap labelled by $j-1, j$, and $j+1$. In order for $e_{\alpha_{j}} \lambda_{1}$ to
be a weight, the complement of $L_{1}$ must have a minimal element labelled $j$. However, as $L_{1}$ and $L_{2}$ have parallel contours, the complement of $L_{2}$ must also have a minimal element labelled $j$, giving a contradiction. So it is not possible for all three coefficients to be equal.

If $c_{j}=c_{j+1}$, then $c_{j-1}=c_{j} \pm 1$. Since the action of $e_{\alpha_{j}}$ will increase $c_{j}$ by one, we must have that $c_{j}+1=c_{j-1}$. Thus this operation removes $\alpha_{j}$ from the component of $\Gamma$ that contains $\alpha_{j+1}$, and reassigns $\alpha_{j}$ instead to the component of $\Gamma$ that contains $\alpha_{j-1}$. This does not change the profile or profile sum.

A similar argument shows that if $c_{j}=c_{j-1}$, then $c_{j+1}=c_{j}+1$ and the profile is again preserved.

Corollary 2.37. Let $L_{1}$ and $L_{2}$ be heap ideals of the same level, corresponding to weights $\lambda_{1}$ and $\lambda_{2}$ respectively, for a spin representation in type $D$. Write

$$
\chi\left(L_{1}, L_{2}\right)=\sum_{i=1}^{n} c_{i} \alpha_{i}
$$

for the relative content of $\left(L_{1}, L_{2}\right)$. Let $\alpha_{j}$ be a simple root with $j \neq n$.
(1) If $\lambda_{1}+\alpha_{j}$ is also a weight, but $\lambda_{2}+\alpha_{j}$ and $\lambda_{2}-\alpha_{j}$ are not weights of this representation, then
(i) $\left|\chi\left(e_{\alpha_{j}} L_{1}, L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 1$ and
(ii) $P\left(e_{\alpha_{j}} L_{1}, L_{2}\right)=P\left(L_{1}, L_{2}\right)$.
(2) If $\lambda_{1}-\alpha_{j}$ is also a weight, but $\lambda_{2}+\alpha_{j}$ and $\lambda_{2}-\alpha_{j}$ are not weights of this representation, then
(i) $\left|\chi\left(f_{\alpha_{j}} L_{1}, L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 1$ and
(ii) $P\left(f_{\alpha_{j}} L_{1}, L_{2}\right)=P\left(L_{1}, L_{2}\right)$.

Proof. Since $L_{1}$ and $L_{2}$ are at the same level, by Lemma 2.22, they may be again identified with ideals of the same type $A$ subheap. As $j \neq n$, actions that affect only heap elements labelled $j$ will keep the two ideals within this subheap. Thus Lemma 2.36 applies here.

Definition 2.38. Let $L_{1}$ and $L_{2}$ be heap ideals of the same level and write $P\left(L_{1}, L_{2}\right)=\left(p_{1}, \ldots, p_{k}\right)$. We say that $L_{1}$ and $L_{2}$ cross if there exists some $\ell$ such that $p_{\ell}=0$ and the coefficients $p_{\ell-1}$ and $p_{\ell+1}$ have like sign.

Figure 2.19: For the heap ideals shown below, we have $\chi\left(L_{1}, L_{2}\right)=\alpha_{2}+\alpha_{4}+\alpha_{5}$ and $P\left(L_{1}, L_{2}\right)=$ $(1,0,1)$. This is an example of a crossing.


This definition for weights of the spin representations for $\mathfrak{d}_{6}$ comes from the bitangent notation used for pairs of opposite weights of the 56 -dimensional minuscule module for $\mathfrak{e}_{7}$. Each pair of weights is labelled by a particular $(i, j)$, where $1 \leq i<j \leq 6$. In bitangent notation, we say that weights labelled $(1,4)$ and $(2,5)$ cross, but weights $(1,2)$ and $(4,5)$ do not cross; neither do weights $(1,5)$ and $(2,4)$. These crossings or non-crossings can be visualized using a diagram like the ones below. A similar idea, in the context of $\mathfrak{e}_{7}$, occurs implicitly in the work of Cooperstein [3]; the ideas here are a new adaptation.

When a spin representation is viewed as a submodule of the 56 -dimensional module for $\mathfrak{e}_{7}$ (by restriction to $\mathfrak{d}_{6}$ ), our definition of crossing coincides with the natural interpretation, shown in Figure 2.20, in terms of bitangents. The corresponding classification of a pair of weights as either a crossing, separate pair, or covering will appear later as Proposition 3.16.

Crossings have the nice property that if an adjacent transposition acts on a pair of bitangents,

Figure 2.20: Possible shapes for a pair of bitangents with distinct indices.

and thus on the corresponding arcs of the diagram above, by moving exactly one endpoint of each arc, this has the effect of either adding or removing a single crossing. The corresponding result for spin modules will be proved later as Lemma 3.20. Lemma 2.34 and Corollary 2.35 show that the profile of two ideals tracks similar information: when $e_{\alpha}$ and $f_{\alpha}$ move two weights corresponding to heap ideals of the same level, a singleton component of the profile changes by $\pm 2$. Thus crossings can be counted via the profile, which also includes other information about the relationship between the two ideals.

## Chapter 3

## Invariant Forms

Given a Lie algebra $L$ and and an $L$-module $V$, one might wonder how tensor products of $V$, as well as their dual modules, decompose as products of irreducible modules for $L$. In particular, do these tensor products contain one-dimensional submodules, and with what multiplicity? One part of this answer involves a hunt for $L$-invariant forms on $V$.

### 3.1 Necessary Conditions for a Symmetric Invariant Quartic Form

Definition 3.1. If $L$ is a Lie algebra and $V$ is an $L$-module, then a map $q: V^{\otimes k} \rightarrow \mathbb{C}$ is an invariant form provided that $q$ is linear in each input and

$$
(x \cdot q)\left(v_{1} \otimes \cdots \otimes v_{k}\right)=0
$$

for all $x \in L$ and $v_{i} \in V$. In this case we also say that $L$ leaves invariant the form $q$.

Here, recall from Section 1.2 that the action of $x \in L$ on the map $q$ is given by acting on each tensor coordinate individually, then summing negatives:
$(x \cdot q)\left(v_{1} \otimes \cdots \otimes v_{k}\right)=-q\left(x v_{1} \otimes v_{2} \otimes \cdots \otimes v_{k}\right)-q\left(v_{1} \otimes x v_{2} \otimes \cdots \otimes x v_{k}\right)-\cdots-q\left(v_{1} \otimes v_{2} \otimes \cdots \otimes x v_{k}\right)$.

An invariant form is symmetric if it is symmetric in the inputs. The invariant forms considered here will be symmetric quartic forms, meaning that $k=4$.

These invariant quartic forms sometimes have curious connections to the overall structure of the Lie algebra. For example, Lurie [9] showed that the Lie algebra $\mathfrak{c}_{7}$ can be defined as the

Lie algebra of $\mathbb{C}$-linear endomorphisms of $L\left(E_{7}, \omega_{6}\right)$ that leave invariant a particular symplectic form and a symmetric quartic form. Cooperstein [3] gave an explicit description of this symmetric invariant quartic form.

Manivel [10] computed the dimension of the space of symmetric invariant quartic forms on the spin representations for type $D$ Lie algebras for small $n$, viewing these dimensions as the multiplicity of the one-dimensional module in a decomposition of $(V \otimes V \otimes V \otimes V)^{*}$. In particular, Manivel showed that when $n=4,6,8,9$ and 10 , the space of symmetric invariant quartic forms is one-dimensional. Manivel provided a conjecture for a recursion and a generating function for these dimensions [10, p.15].

Before proceeding further, we should note that if $V$ is the module $L\left(D_{6}, \omega_{6}\right)$, then there is a symplectic invariant bilinear form on $V$, which is described in [6, Theorem 5.6.3]. If we denote this symplectic form by $p$, then we might try to create a quartic form on $V$ by

$$
q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right)=p\left(v_{1}, v_{2}\right) p\left(v_{3}, v_{4}\right)
$$

While this quartic form is $\mathfrak{d}_{6}$-invariant, it is not symmetric; moreover, symmetrizing in the usual way by averaging over all permutations the indices causes this form to become identically zero. Thus some a new approach is required in order to describe a symmetric invariant quartic form on $L\left(D_{6}, \omega_{6}\right)$.

In this chapter, we compute some necessary relations that follow from the definition of a symmetric invariant quartic form on a minuscule module. Then we focus on the smallest interesting example of Manivel's results: the spin representation $L\left(D_{6}, \omega_{6}\right)$. Using a realization of this representation in terms of heaps, we employ the relative content and profile to provide a formula for the quartic form, which Manivel showed is unique up to scalar multiplication. We also discuss some extensions of this result to the Lie algebras $\mathfrak{a}_{5}$ and $\mathfrak{e}_{6}$.

Throughout this section, $q$ will refer to a symmetric invariant quartic form on a minuscule module $V$.

Lemma 3.2. Let $b_{\lambda}$ denote a weight vector of weight $\lambda$, and let $\alpha$ be any simple root. Then at
most one of $e_{\alpha} b_{\lambda}$ and $f_{\alpha} b_{\lambda}$ is nonzero.

Proof. Recall from Lemma 2.11 and Corollary 2.24 that

$$
e_{\alpha} b_{\lambda}= \begin{cases}b_{\lambda+\alpha} & \text { if } \lambda+\alpha \text { is a weight } \\ 0 & \text { otherwise }\end{cases}
$$

and

$$
f_{\alpha} b_{\lambda}= \begin{cases}b_{\lambda-\alpha} & \text { if } \lambda-\alpha \text { is a weight } \\ 0 & \text { otherwise }\end{cases}
$$

From this description, it is evident that if $e_{\alpha} b \neq 0$, then $f_{\alpha} e_{\alpha} b=b$; similarly, if $f_{\alpha} b \neq 0$, then $e_{\alpha} f_{\alpha} b=b$. By Lemma 2.12, we have $e_{\alpha} \circ e_{\alpha}=0$ and $f_{\alpha} \circ f_{\alpha}=0$. Suppose that $e_{\alpha} b_{\lambda}$ and $f_{\alpha} b_{\lambda}$ are both nonzero. Then $0=e_{\alpha} e_{\alpha} f_{\alpha} b_{\lambda}=e_{\alpha} b_{\lambda} \neq 0$, giving a contradiction.

Lemma 3.3. Let $b_{1}, b_{2}, b_{3}, b_{4}$ be weight vectors. If $\alpha$ is a simple root such that

$$
e_{\alpha} b_{1} \neq 0, \quad e_{\alpha} b_{2} \neq 0, \quad f_{\alpha} b_{3} \neq 0, \quad \text { and } \quad f_{\alpha} b_{4} \neq 0
$$

then

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=q\left(e_{\alpha} b_{1} \otimes e_{\alpha} b_{2} \otimes f_{\alpha} b_{3} \otimes f_{\alpha} b_{4}\right)
$$

Proof. We will write $e=e_{\alpha}$ and $f=f_{\alpha}$. Note that $e b \neq 0$ implies that $f b=0$ and $f e b=b$, and similarly, $f b \neq 0$ implies that $e b=0$ and $e f b=b$.

Using the invariance of $q$, we act on $q$ by $e$ first to see

$$
\begin{aligned}
0 & =(e \cdot q)\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right) \\
& =q\left(-e b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}-b_{1} \otimes e b_{2} \otimes b_{3} \otimes b_{4}\right)
\end{aligned}
$$

Similarly, acting now by $f$, we have

$$
\begin{align*}
& 0=(f \cdot q)\left(-e b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}-b_{1} \otimes e b_{2} \otimes b_{3} \otimes b_{4}\right) \\
&=q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}+e b_{1} \otimes b_{2} \otimes f b_{3} \otimes b_{4}+e b_{1} \otimes b_{2} \otimes b_{3} \otimes f b_{4}\right. \\
&\left.+b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}+b_{1} \otimes e b_{2} \otimes f b_{3} \otimes b_{4}+e b_{1} \otimes e b_{2} \otimes b_{3} \otimes f b_{4}\right) \\
&=2 q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right) \\
&+q\left(e b_{1} \otimes b_{2} \otimes f b_{3} \otimes b_{4}\right)+q\left(e b_{1} \otimes b_{2} \otimes b_{3} \otimes f b_{4}\right)  \tag{3.1}\\
&+q\left(b_{1} \otimes e b_{2} \otimes f b_{3} \otimes b_{4}\right)+q\left(b_{1} \otimes e b_{2} \otimes b_{3} \otimes f b_{4}\right)
\end{align*}
$$

Next, we act on $q$ by $e$, to obtain

$$
\begin{aligned}
0 & =(e \cdot q)\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right) \\
& =q\left(-e b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}-b_{1} \otimes e b_{2} \otimes b_{3} \otimes b_{4}\right)
\end{aligned}
$$

Acting again by $e$ on this gives

$$
\begin{aligned}
0 & =(e \cdot q)\left(-e b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}-b_{1} \otimes e b_{2} \otimes b_{3} \otimes b_{4}\right) \\
& =2 q\left(e b_{1} \otimes e b_{2} \otimes b_{3} \otimes b_{4}\right)
\end{aligned}
$$

Now acting on this by $f$, we have

$$
\begin{aligned}
0= & (f \cdot q)\left(2 e b_{1} \otimes e b_{2} \otimes b_{3} \otimes b_{4}\right) \\
=q & \left(-2 b_{1} \otimes e b_{2} \otimes b_{3} \otimes b_{4}-2 e b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right. \\
& \left.-2 e b_{1} \otimes e b_{2} \otimes f b_{3} \otimes b_{4}-2 e b_{1} \otimes e b_{2} \otimes b_{3} \otimes f b_{4}\right)
\end{aligned}
$$

Acting once more by $f$, we have

$$
\begin{aligned}
0=( & f \cdot q)\left(-2 b_{1} \otimes e b_{2} \otimes b_{3} \otimes b_{4}-2 e b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right. \\
& \left.-2 e b_{1} \otimes e b_{2} \otimes f b_{3} \otimes b_{4}-2 e b_{1} \otimes e b_{2} \otimes b_{3} \otimes f b_{4}\right), \\
=q & \left(2 b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}+2 b_{1} \otimes e b_{2} \otimes f b_{3} \otimes b_{4}+2 b_{1} \otimes e b_{2} \otimes b_{3} \otimes f b_{4}\right. \\
& +2 b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}+2 e b_{1} \otimes b_{2} \otimes f b_{3} \otimes b_{4}+2 e b_{1} \otimes b_{2} \otimes b_{3} \otimes f b_{4} \\
& +2 b_{1} \otimes e b_{2} \otimes f b_{3} \otimes b_{4}+2 e b_{1} \otimes b_{2} \otimes f b_{3} \otimes b_{4}+2 e b_{1} \otimes e b_{2} \otimes f b_{3} \otimes f b_{4} \\
& \left.+2 b_{1} \otimes e b_{2} \otimes b_{3} \otimes f b_{4}+2 e b_{1} \otimes b_{2} \otimes b_{3} \otimes f b_{4}+2 e b_{1} \otimes e b_{2} \otimes f b_{3} \otimes f b_{4}\right), \\
=4 q & \left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)+4 q\left(e b_{1} \otimes e b_{2} \otimes f b_{3} \otimes f b_{4}\right) \\
& +4 q\left(e b_{1} \otimes b_{2} \otimes f b_{3} \otimes b_{4}\right)+4 q\left(e b_{1} \otimes b_{2} \otimes b_{3} \otimes f b_{4}\right) \\
& +4 q\left(b_{1} \otimes e b_{2} \otimes f b_{3} \otimes b_{4}\right)+4 q\left(b_{1} \otimes e b_{2} \otimes b_{3} \otimes f b_{4}\right) .
\end{aligned}
$$

Dividing through by 4 , we have

$$
\begin{align*}
0= & q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)+q\left(e_{\alpha} b_{1} \otimes e_{\alpha} b_{2} \otimes f_{\alpha} b_{3} \otimes f_{\alpha} b_{4}\right) \\
& +q\left(e_{\alpha} b_{1} \otimes b_{2} \otimes f_{\alpha} b_{3} \otimes b_{4}\right)+q\left(e_{\alpha} b_{1} \otimes b_{2} \otimes b_{3} \otimes f_{\alpha} b_{4}\right)  \tag{3.2}\\
& +q\left(b_{1} \otimes e_{\alpha} b_{2} \otimes f_{\alpha} b_{3} \otimes b_{4}\right)+q\left(b_{1} \otimes e_{\alpha} b_{2} \otimes b_{3} \otimes f_{\alpha} b_{4}\right) .
\end{align*}
$$

Now we subtract (3.1) from (3.2) to see that

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=q\left(e_{\alpha} b_{1} \otimes e_{\alpha} b_{2} \otimes f_{\alpha} b_{3} \otimes f_{\alpha} b_{4}\right)
$$

Corollary 3.4. Let $b_{1}, b_{2}, b_{3}, b_{4}$ be weight vectors. If $\alpha$ is a simple root such that $e_{\alpha} b_{1} \neq 0$, $e_{\alpha} b_{2} \neq 0, f_{\alpha} b_{3} \neq 0$, and $f_{\alpha} b_{4} \neq 0$, then

$$
q\left(e_{\alpha} b_{1} \otimes b_{2} \otimes f_{\alpha} b_{3} \otimes b_{4}\right)=q\left(b_{1} \otimes e_{\alpha} b_{2} \otimes b_{3} \otimes f_{\alpha} b_{4}\right)
$$

and

$$
q\left(b_{1} \otimes e_{\alpha} b_{2} \otimes f_{\alpha} b_{3} \otimes b_{4}\right)=q\left(e_{\alpha} b_{1} \otimes b_{2} \otimes b_{3} \otimes f_{\alpha} b_{4}\right)
$$

Proof. This follows from Lemma 3.3 by substituting $\left\{b_{2}, f_{\alpha} b_{3}, e_{\alpha} b_{1}, b_{4}\right\}$ or $\left\{b_{1}, f_{\alpha} b_{3}, e_{\alpha} b_{2}, b_{4}\right\}$ in place of $\left\{b_{1}, b_{2}, b_{3}, b_{4}\right\}$.

Corollary 3.5. Let $v_{1}, v_{2}, v_{3}, v_{4}$ be weight vectors. If $\alpha$ is a simple root such that $e_{\alpha} v_{1} \neq 0$, $e_{\alpha} v_{2} \neq 0, f_{\alpha} v_{3} \neq 0$, and $f_{\alpha} v_{4} \neq 0$, then

$$
0=q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right)+q\left(e_{\alpha} v_{1} \otimes v_{2} \otimes f_{\alpha} v_{3} \otimes v_{4}\right)+q\left(e_{\alpha} v_{1} \otimes v_{2} \otimes v_{3} \otimes f_{\alpha} v_{4}\right) .
$$

Proof. This result follows directly from Equation (3.1) and Corollary 3.4.

Corollary 3.6. Let $v_{1}, v_{1}^{\prime}$ be weight vectors corresponding to weights $\mu_{1}$ and $-\mu_{1}$ respectively. Suppose $\alpha$ is a simple root such that $e_{\alpha} v_{1} \neq 0$, and write $e_{\alpha} v_{1}=v_{2}$ and $f_{\alpha} v_{1}^{\prime}=v_{2}^{\prime}$. Then

$$
0=q\left(v_{1} \otimes v_{1} \otimes v_{1}^{\prime} \otimes v_{1}^{\prime}\right)+2 q\left(v_{1} \otimes v_{1}^{\prime} \otimes v_{2} \otimes v_{2}^{\prime}\right) .
$$

Proof. This result follows from Corollary 3.5 by substituting $\left\{v_{1}, v_{1}, v_{1}^{\prime}, v_{1}^{\prime}\right\}$ for $\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\}$, then simplifying using symmetry of $q$.

Lemma 3.7. Let $v_{1}, v_{2}, v_{3}, v_{4}$ be weight vectors. If $\alpha$ is a simple root such that $e_{\alpha} v_{1} \neq 0$ and $f_{\alpha} v_{2} \neq 0$, but $e_{\alpha}$ and $f_{\alpha}$ act as zero on both $v_{3}$ and $v_{4}$, then

$$
q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right)=-q\left(e_{\alpha} v_{1} \otimes f_{\alpha} v_{2} \otimes v_{3} \otimes v_{4}\right)
$$

Proof. We will again write $e=e_{\alpha}$ and $f=f_{\alpha}$. We act on $q$ by $e$ and then $f$ :

$$
\begin{aligned}
0 & =(e \cdot q)\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right) \\
& =q\left(-e v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right) \\
& =(f \cdot q)\left(-e v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right) \\
& =q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}+e v_{1} \otimes f v_{2} \otimes v_{3} \otimes v_{4}\right) \\
& =q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right)+q\left(e v_{1} \otimes f v_{2} \otimes v_{3} \otimes v_{4}\right) .
\end{aligned}
$$

Lemma 3.8. Let $b_{1}, b_{2}, b_{3}, b_{4}$ be weight vectors of weights $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ respectively.
If $\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4} \neq 0$, then $q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=0$.

Proof. Suppose that $\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4} \neq 0$. Then there exists some root $\alpha_{i}$ such that the coefficient of $\omega_{\alpha_{i}}$ in $\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4}$ is nonzero. Let $c_{1}, c_{2}, c_{3}, c_{4}$ denote the coefficients of $\omega_{\alpha_{i}}$ in $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$
respectively, so that by assumption, $c_{1}+c_{2}+c_{3}+c_{4} \neq 0$. We now act on $q$ by $h:=h_{\alpha_{i}}$ :

$$
\begin{aligned}
0 & =(h \cdot q)\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right), \\
& =q\left(-h b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}-b_{1} \otimes h b_{2} \otimes b_{3} \otimes b_{4}-b_{1} \otimes b_{2} \otimes h b_{3} \otimes b_{4}-b_{1} \otimes b_{2} \otimes b_{3} \otimes h b_{4}\right), \\
& =q\left(-c_{1} b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}-b_{1} \otimes c_{2} b_{2} \otimes b_{3} \otimes b_{4}-b_{1} \otimes b_{2} \otimes c_{3} b_{3} \otimes b_{4}-b_{1} \otimes b_{2} \otimes b_{3} \otimes c_{4} b_{4}\right), \\
& =-\left(c_{1}+c_{2}+c_{3}+c_{4}\right) q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right) .
\end{aligned}
$$

By assumption, $c_{1}+c_{2}+c_{3}+c_{4} \neq 0$, so $q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=0$.

We will use the above results to calculate values of $q$ on various types of 4 -tuples of weights. In the next section, we classify these possible 4 -tuples by the distances between the four weights.

### 3.2 Configurations of Quadruples of Weights

By Lemma 3.8, the only nonzero values of $q$ will occur on 4 -tuples of weights that sum to 0 . We can sort these potential 4-tuples of weights into three classes:
(a) $\{\lambda, \lambda,-\lambda,-\lambda\}$;
(b) $\{\lambda,-\lambda, \mu,-\mu\}$, where $\lambda \neq \pm \mu$;
(c) $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$, where $\lambda_{i} \neq \pm \lambda_{j}$ for $i \neq j$.

Using the convention of Green in [5], we write the weights for the spin representation of $\mathfrak{d}_{6}$ of highest weight $\omega_{6}$ as

$$
\Psi_{D}^{+}=\{( \pm 2, \pm 2, \pm 2, \pm 2, \pm 2, \pm 2):-2 \text { occurs an even number of times }\}
$$

where each $\pm$ is chosen independently, and the simple roots as

$$
\Delta=\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}, \alpha_{4}, \alpha_{5}, \alpha_{6}\right\}
$$

where $\alpha_{i}=4 \varepsilon_{i}-4 \varepsilon_{i+1}$ for $1 \leq i \leq 5$, and $\alpha_{6}=4 \varepsilon_{5}+4 \varepsilon_{6}$. Recall that we denote this representation by $L\left(D_{6}, \omega_{6}\right)$.

Definition 3.9. Given any pair of weights for the spin representation of $\mathfrak{d}_{6}$ of highest weight $\omega_{6}$,
(i) if the Euclidean distance between two weights is $\sqrt{32}$, we say that the pair is skew, and
(ii) if the distance is $\sqrt{64}$, we say the pair is incident.

The only other options for a pair of these weights are that they are negatives of each other or that they are equal. If two weights are negatives of each other, we will refer to them as an opposite pair.

We may also represent these weights as ideals of a heap, as shown in Figure 3.1 below. One way to visualize the correspondence between weights and ideals is to draw the ideals, as in Figure 3.1 below, extending all the way to the far left side of the heap, and then interpret the pattern of + and - signs in the weight as a series of ups and downs, respectively, as we pass from left to right. The last coordinate is chosen so that all of our weights have an even number of minus signs.

An equivalent way to track the correspondence between weights and ideals is to recall from Corollary 2.24 that if $L$ is a heap ideal corresponding to some weight $\lambda$, adding heap elements to $L$ corresponds to adding simple roots to $\lambda$. Thus the ideal containing the entire heap must correspond to the weight $(2,2,2,2,2,2)$, in which all possible simple roots have been added, and the empty ideal corresponds to the weight $(-2,-2,-2,-2,-2,-2)$.

Lemma 3.10. Let $L_{1}$ and $L_{2}$ be distinct heap ideals of the same level, for $L\left(D_{6}, \omega_{6}\right)$, with neither being the empty heap or the whole heap. If the corresponding weights $\lambda_{1}$ and $\lambda_{2}$ are a skew pair, then $\chi\left(L_{1}, L_{2}\right)$ has coefficients $\{0, \pm 1\}$, all of the same sign, and the support of $\chi\left(L_{1}, L_{2}\right)$ is a connected subgraph of the underlying Dynkin diagram $\Gamma$.

Proof. It result follows from arguing as [6, Example 8.4.9] that $\lambda_{1}-\lambda_{2}$ is equal to a root. For $1 \leq i<j \leq 6$, we have

$$
4 \varepsilon_{i}-4 \varepsilon_{j}=\alpha_{i}+\alpha_{i+1}+\ldots+\alpha_{j-1}
$$

If $1 \leq j<i \leq 6$, we have

$$
4 \varepsilon_{i}-4 \varepsilon_{j}=-\alpha_{i}-\alpha_{i+1}-\ldots-\alpha_{j-1}
$$

Figure 3.1: Two ideals of the principal subheap corresponding to the spin representation of $\mathfrak{d}_{6}$ of highest weight $\omega_{6}$. The dotted line shows the ideal corresponding to the weight ( $2,-2,2,2,-2,2$ ), while the solid line shows the ideal that corresponds to $(-2,-2,-2,2,-2,2)$. These two weights differ in two places, so the pair is skew.


In either case, the root has connected support, all with the same coefficient of $\pm 1$.

Corollary 3.11. In the situation of Lemma 3.10, $P\left(L_{1}, L_{2}\right)=( \pm 1)$.

Proof. By Lemma 3.10, $\chi\left(L_{1}, L_{2}\right)$ has coefficients $\{0, \pm 1\}$, all of the same sign, and the support of $\chi\left(L_{1}, L_{2}\right)$ is a connected subgraph of $\Gamma$. It follows that the profile has only one nonzero component, which has coefficient $\pm 1$.

We now examine all possible 4 -tuples of weights for $L\left(D_{6}, \omega_{6}\right)$ of the form $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$, where $\lambda_{i} \neq \pm \lambda_{j}$ for $i \neq j$.

Lemma 3.12. Let $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$ be a 4-tuple of weights, summing to zero, for $L\left(D_{6}, \omega_{6}\right)$, with $\lambda_{i} \neq \pm \lambda_{j}$ for $i \neq j$. Then all four weights are mutually incident.

Proof. Suppose for sake of contradiction that two of the weights, without loss of generality $\lambda_{1}$ and $\lambda_{2}$, are skew. As the Weyl group acts transitively on the weights, we may move our 4 -tuple, preserving distances, so that $\lambda_{1}=\omega_{6}$.

Recall that for $1 \leq i \leq 5$, the simple roots are given by $\alpha_{i}=4 \varepsilon_{i}-4 \varepsilon_{i+1}$, and Weyl group generators act by adding or subtracting simple roots as is possible. This amounts to place permutation on the weights as written. So we may also further permute the coordinates of $\lambda_{2}, \lambda_{3}$, and $\lambda_{4}$, so that $\lambda_{2}=(2,2,2,2,-2,-2)$. Now $\lambda_{3}$ and $\lambda_{4}$ must both begin with four coordinates of -2 in order for the weights to sum to zero. Lastly, in order for the weights to sum to zero, $\lambda_{3}$ and $\lambda_{4}$ must disagree in at least one of the last two coordinates. However, both have an even number of negative signs. This forces one of $\lambda_{3}$ or $\lambda_{4}$ to be $-\omega_{6}$, giving an opposite pair and a contradiction. So it is not possible for any pair of weights to be skew, and the weights must all be pairwise incident.

With this result in mind, we will now refer to such 4-tuples as tetrahedra. These can be divided into three subclasses, described below. As Lemma 3.12 might suggest, the conditions for such a tetrahedron are restrictive.

Lemma 3.13. Let $\lambda$ be a weight for $L\left(D_{6}, \omega_{6}\right)$ and suppose that $\lambda$ contains $m$ negative signs, where $0 \leq m \leq 6$ and $m$ is even. Let $L$ be the heap ideal corresponding to $\lambda$, and let $\ell$ denote the level of $L$, where $0 \leq \ell \leq 3$. Then $m=6-2 \ell$.

Proof. We proceed by induction on the number of heap elements contained in $L$.
When $L$ is the empty ideal, which is of level 0 , we have $\lambda=-\omega_{6}$. This contains six negative signs, so the equation is satisfied.

We now suppose that $L$ contains $j$ elements and satisfies $m=6-2 \ell$, and we consider what happens when we add a new maximal heap element to $L$, in a way that still results in an ideal. (The reason this is possible is that in reverse, starting with the entire heap, we may remove maximal elements one at a time until we reach $L$.)

If the heap element is not labelled by a 6 , then the level of $L$ does not change, as the content of $L$ will not contain any new elements labelled by 6 . If the new heap element is not labelled by 6 , then the number of negative signs in $\lambda$ does not change either, because we are adding to $\lambda$ a root of the form $\alpha_{i}=4 \varepsilon_{i}-4 \varepsilon_{i+1}$. It follows that the equation is still satisfied for the new heap ideal and its corresponding weight.

If the newly added heap element is labelled by a 6 , then the level of $L$ increases to $\ell+1$. Adding an element labelled 6 to $L$ corresponds to adding the root $\alpha_{6}=4 \varepsilon_{5}+4 \varepsilon_{6}$ to $\lambda$. As $\lambda+\alpha_{6}$ must also be a weight, this means that $\lambda$ had negative signs in its final two coordinates, while $\lambda+\alpha_{6}$ has positive signs in the last two coordinates. So the number of negative signs in $\lambda+\alpha_{6}$ is $m-2$. By assumption, $m=6-2 \ell$, so $m-2=6-2(\ell+1)$, as required.

Lemma 3.14. Suppose we have a tetrahedron of weights for $L\left(D_{6}, \omega_{6}\right)$ of the form $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$, where $\lambda_{i} \neq \pm \lambda_{j}$ for $i \neq j$. Then, up to permutation of the four weights, exactly one of the following holds.
(c.1+) Without loss of generality, and up to permutation of the indices $i, j, k$, and $\ell$ over all possibilities between 1 and 6, we have

$$
\begin{aligned}
& \lambda_{1}=\omega_{6}, \\
& \lambda_{2}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{j}\right), \\
& \lambda_{3}=-\omega_{6}+4\left(\varepsilon_{k}+\varepsilon_{\ell}\right), \text { and } \\
& \lambda_{4}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}+\varepsilon_{k}+\varepsilon_{\ell}\right)
\end{aligned}
$$

for some distinct indices $i, j, k$, and $\ell$. Moreover, $\lambda_{2}, \lambda_{3}$, and $\lambda_{4}$ all have level 1 , while $\lambda_{1}$ has level 3, and all four weights have the same sign.
(c.1-) Without loss of generality, and up to permutation of the indices $i, j, k$, and $\ell$, we have

$$
\begin{aligned}
& \lambda_{1}=-\omega_{6} \\
& \lambda_{2}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}\right), \\
& \lambda_{3}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right), \text { and } \\
& \lambda_{4}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{j}+\varepsilon_{k}+\varepsilon_{\ell}\right)
\end{aligned}
$$

for some distinct indices $i, j, k$, and $\ell$. Moreover, $\lambda_{2}, \lambda_{3}$, and $\lambda_{4}$ all have level 2 , while $\lambda_{1}$ has level 0, and all four weights have the same sign.
(c.2) Without loss of generality, and up to permutation of the indices $i, j, k$, and $\ell$, we have

$$
\begin{aligned}
& \lambda_{1}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}\right), \\
& \lambda_{2}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right), \\
& \lambda_{3}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{k}\right), \text { and } \\
& \lambda_{4}=-\omega_{6}+4\left(\varepsilon_{j}+\varepsilon_{\ell}\right)
\end{aligned}
$$

for some distinct indices $i, j, k$, and $\ell$. Moreover, $\lambda_{1}$ and $\lambda_{2}$ have level 2, while $\lambda_{3}$ and $\lambda_{4}$ have level 1.

Note that the tetrahedra of class (c. $1^{-}$) are the negatives of the tetrahedra in (c. $1^{+}$).

Proof. First, it is useful to note that a tetrahedron of weights, each with 6 coordinates of $\pm 2$, that sums to 0 must contain exactly 12 negative signs. As each weight must contain an even number of minus signs, it follows that from any tetrahedron we may read a partition of 12 into four even parts of size at most 6 . The list of possible partitions is given below.

$$
\begin{aligned}
& 0+0+6+6 \\
& 0+2+4+6 \\
& 0+4+4+4 \\
& 2+2+2+6 \\
& 2+2+4+4
\end{aligned}
$$

We note that any partition that has both 0 and 6 as parts implies our tetrahedron must contain both $\omega_{6}$ and $-\omega_{6}$ as weights. It follows that such a partition must correspond to a 4 -tuple of opposite pairs of weights, and does not fall within the scope of this lemma.
$\left(c .1^{+}\right)$We first consider the partition $0+4+4+4$. By Lemma 3.13, this corresponds to a tetrahedron containing $\omega_{6}$ along with three weights of level 1 that have four negative signs each. Suppose that $\lambda_{1}=\omega_{6}$ and $\lambda_{2}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{j}\right)$ for some distinct $i$ and $j$. Then $\lambda_{3}$ and $\lambda_{4}$ must
both have negative $i$ and $j$ entries in order for the four weights to sum to 0 . It follows that $\lambda_{3}=-\omega_{6}+4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)$ for some $k$ and $\ell$ distinct from $i$ and $j$. Now $\lambda_{4}$ is completely determined and we have $\lambda_{4}=-\omega_{6}+4\left(\varepsilon_{j}+\varepsilon_{\ell}\right)$.
(c.1-) Applying the previous argument to the negatives of all the weights shows that if we begin with the partition $2+2+2+6$, we have a tetrahedron where $\lambda_{1}=-\omega_{6}, \lambda_{2}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}\right)$, $\lambda_{3}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)$, and $\lambda_{4}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{j}+\varepsilon_{k}+\varepsilon_{\ell}\right)$ for some distinct indices $i, j, k$, and $\ell$.
(c.2) It remains to decipher the partition $2+2+4+4$. Without loss of generality we may write $\lambda_{1}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}\right)$ for some distinct indices $i$ and $j$. Let $\lambda_{2}$ denote the other weight with two negative signs.

Suppose for the sake of contradiction that $\lambda_{2}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{k}\right)$ for some $k$ distinct from $i$ and $j$, so that $\lambda_{1}$ and $\lambda_{2}$ share one of their negative coordinates. It follows that $\lambda_{3}$ and $\lambda_{4}$ must both have positive signs in coordinate $i$, as well as negative signs in the three remaining coordinates. So, without loss of generality, we have $\lambda_{3}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{j}\right)$ and $\lambda_{4}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{k}\right)$. This describes a set of opposite pairs, so we have a contradiction.

Thus $\lambda_{2}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)$ for some $k$ and $\ell$ distinct from $i$ and $j$. It follows that both $\lambda_{3}$ and $\lambda_{4}$ must both have negative signs in the two non-indexed coordinates. Now, $\lambda_{3}$ cannot equal $-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{j}\right)$ or $-\omega_{6}+4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)$, because in either of these situations we would have an opposite pair. So without loss of generality, we have $\lambda_{3}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{k}\right)$ and $\lambda_{4}=-\omega_{6}+4\left(\varepsilon_{j}+\varepsilon_{\ell}\right)$.

The levels of the weights follow from the above argument, using Lemma 3.13, and the assertions about sign follow in turn from the assertions about weights.

Some examples of these tetrahedra are shown below.

Figure 3.2: Three ideals that, along with the empty ideal, make up a tetrahedron of class (c.1). In the notation used earlier, this tetrahedron is $\{(2,2,2,2,-2,-2),(-2,2,2,-2,2,2),(2,-2,-2,2,2,2),(-2,-2,-2,-2,-2,-2)\}$.


Figure 3.3: Four ideals that make up a tetrahedron of class (c.2). In the notation used earlier, this tetrahedron is $\{(2,2,2,2,-2,-2),(2,-2,2,-2,2,2),(-2,2,-2,-2,2,-2),(-2,-2,-2,2,-2,2)\}$.


Lemma 3.15. Let $W^{\prime}$ denote the subgroup of $W\left(D_{6}\right)$ generated by $s_{1}, \ldots, s_{5}$. Then $W^{\prime}$ acts on tetrahedra of weights, with orbits equal to the three classes described in Lemma 3.14.

Proof. The subgroup of $W\left(D_{6}\right)$ generated by $s_{1}, \ldots, s_{5}$ is isomorphic to $S_{6}$, the symmetric group on 6 objects, and acts by the usual permutations on the coordinates of the weights. The result now follows by the symmetry of the definitions.

Proposition 3.16. If $L_{1}$ and $L_{2}$ are two heap ideals of the same level that correspond to a pair of incident weights for $L\left(D_{6}, \omega_{6}\right)$, neither being the empty ideal or the whole heap (i.e. fitting in a tetrahedron of class (c.2)), then the pair $\left\{L_{1}, L_{2}\right\}$ is one of three types:
(i) crossing: $P\left(L_{1}, L_{2}\right)$ is either $(1,0,1)$ or $(-1,0,-1)$, and $\sum P\left(L_{1}, L_{2}\right)= \pm 2$;
(ii) covering: $P\left(L_{1}, L_{2}\right)$ is either $(1,0,-1)$ or $(-1,0,1)$, and $\sum P\left(L_{1}, L_{2}\right)=0$;
(iii) separate: $P\left(L_{1}, L_{2}\right)$ is either $(1,2,1)$ or $(-1,-2,-1)$, and $\sum P\left(L_{1}, L_{2}\right)= \pm 4$.

Figure 3.4: Pairs of ideals of the same level that correspond to pairs of incident weights. We have $\chi\left(L_{1}, L_{2}\right)=\alpha_{2}+\alpha_{4}+\alpha_{5}$ (crossing), $\chi\left(L_{3}, L_{4}\right)=\alpha_{2}-\alpha_{4}-\alpha_{5}$ (covering), and $\chi\left(L_{5}, L_{6}\right)=$ $\alpha_{2}+2 \alpha_{3}+\alpha_{4}+\alpha_{5}$ (separate).


Proof. We will rely heavily on Lemma 2.23 , which tells us that if $\lambda_{1}$ and $\lambda_{2}$ are weights corresponding to heap ideals $L_{1}$ and $L_{2}$, then $\chi\left(L_{1}, L_{2}\right)=\lambda_{2}-\lambda_{1}$. As $\lambda_{1}$ and $\lambda_{2}$ are a pair of incident weights of the same sign, not the empty ideal or whole heap, we may assume that $\lambda_{1}$ and $\lambda_{2}$ each have four occurrences of 2 and two occurrences of -2 . (Otherwise, we replace both weights by their negatives, and undo at the end, at the expense of a total sign change in each overall expression.)

Since $\lambda_{1}$ and $\lambda_{2}$ are a pair of incident weights, we have

$$
\lambda_{1}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}\right) \quad \text { and } \quad \lambda_{2}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)
$$

for distinct $i, j, k, \ell$, where, without loss of generality, $i<j, k<\ell$, and $i<k$. Thus

$$
\chi\left(L_{1}, L_{2}\right)=\lambda_{2}-\lambda_{1}=-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)+4\left(\varepsilon_{i}+\varepsilon_{j}\right)
$$

(i) If $i<k<j<\ell$, then

$$
\begin{aligned}
\chi\left(L_{1}, L_{2}\right) & =-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)+4\left(\varepsilon_{i}+\varepsilon_{j}\right) \\
& =4\left(\varepsilon_{i}-\varepsilon_{k}\right)+4\left(\varepsilon_{j}-\varepsilon_{\ell}\right) \\
& =\left(\alpha_{i}+\alpha_{i+1}+\cdots+\alpha_{k-1}\right)+\left(\alpha_{j}+\alpha_{j+1}+\cdots+\alpha_{\ell-1}\right) .
\end{aligned}
$$

This is a sum of distinct roots, all with coefficient 1 . As $\alpha_{k}$ is missing, the support of this expression is a disconnected subgraph of $\Gamma$. So $P\left(L_{1}, L_{2}\right)$ is either $(1,0,1)$ or $(-1,0,-1)$.
(ii) If $i<k<\ell<j$, then

$$
\begin{aligned}
\chi\left(L_{1}, L_{2}\right) & =-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)+4\left(\varepsilon_{i}+\varepsilon_{j}\right) \\
& =4\left(\varepsilon_{i}-\varepsilon_{k}\right)-4\left(\varepsilon_{\ell}-\varepsilon_{j}\right) \\
& =\left(\alpha_{i}+\alpha_{i+1}+\cdots+\alpha_{k-1}\right)-\left(\alpha_{\ell}+\alpha_{\ell+1}+\cdots+\alpha_{j}\right) .
\end{aligned}
$$

This is a sum of distinct roots, the first part with coefficient 1 , and the second part with coefficient -1 . Again, $\alpha_{k}$ is missing and the support of this expression is a disconnected subgraph of $\Gamma$. So $P\left(L_{1}, L_{2}\right)$ is either $(1,0,-1)$ or $(-1,0,1)$.
(iii) If $i<j<k<\ell$, then

$$
\begin{aligned}
\chi\left(L_{1}, L_{2}\right) & =-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)+4\left(\varepsilon_{i}+\varepsilon_{j}\right) \\
& =4\left(\varepsilon_{i}-\varepsilon_{k}\right)+4\left(\varepsilon_{j}-\varepsilon_{\ell}\right) \\
& =\left(\alpha_{i}+\alpha_{i+1}+\cdots+\alpha_{k-1}\right)+\left(\alpha_{j}+\alpha_{j+1}+\cdots+\alpha_{\ell-1}\right) \\
& =\left(\alpha_{i}+\cdots+\alpha_{j-1}+\alpha_{j}+\cdots+\alpha_{k-1}\right)+\left(\alpha_{j}+\cdots+\alpha_{k-1}+\alpha_{k}+\cdots+\alpha_{\ell-1}\right) \\
& =\left(\alpha_{i}+\cdots+\alpha_{j-1}\right)+2\left(\alpha_{j}+\cdots+\alpha_{k-1}\right)+\left(\alpha_{k}+\cdots+\alpha_{\ell-1}\right) .
\end{aligned}
$$

This is a linear combination of distinct roots, the first part with coefficient 1 , the second part with coefficient 2, and the third part with coefficient 1. The root indices are a list of
consecutive integers $i, \ldots, \ell-1$, so the support of this expression is a connected subgraph of $\Gamma$. Hence $P\left(L_{1}, L_{2}\right)$ is either $(1,2,1)$ or $(-1,-2,-1)$.

Corollary 3.17. Let $\left\{L_{1}, L_{2}\right\}$ be a pair of heap ideals of the same level that correspond to a pair of incident weights, and let $\left\{M_{1}, M_{2}\right\}$ be another such pair.
(i) If $\sum P\left(L_{1}, L_{2}\right)=\sum P\left(M_{1}, M_{2}\right) \pm 2$, then exactly one of the pairs $\left\{L_{1}, L_{2}\right\}$ and $\left\{M_{1}, M_{2}\right\}$ is a crossing.
(ii) If $\sum P\left(L_{1}, L_{2}\right)=\sum P\left(M_{1}, M_{2}\right)$, then either the pairs $\left\{L_{1}, L_{2}\right\}$ and $\left\{M_{1}, M_{2}\right\}$ are either both crossings, or neither is a crossing.

Proof. This result follows directly from Proposition 3.16, as crossings have profile sum equal to $\pm 2$, and non-crossings have profile sum equal to 0 or $\pm 4$.

Lemma 3.18. Let $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$ be a 4-tuple of weights with $\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4}=0$, and suppose that $s$ is a Weyl group generator. Let $n_{1}$ (respectively, $n_{2}$ ) denote the number of weights in the 4-tuple that s acts on by adding $\alpha$ (respectively, subtracting $\alpha$ ). Then $n_{1}=n_{2}$ and $n_{1}+n_{2}$ is even. Proof. Let $s$ be a Weyl group generator corresponding to a simple root $\alpha$. Recall from Definition 2.9 and Corollary 2.24 that the action of $s$ on a single weight either fixes that weight, adds $\alpha$ to the weight, or subtracts $\alpha$ from the weight. Note that $n_{1}+n_{2}$ is the total number of weights moved by $s$; since we have four weights, $0 \leq n_{1}+n_{2} \leq 4$.

The generator $s$ acts linearly, so we have

$$
\begin{aligned}
0 & =s(0) \\
& =s\left(\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4}\right) \\
& =s\left(\lambda_{1}\right)+s\left(\lambda_{2}\right)+s\left(\lambda_{3}\right)+s\left(\lambda_{4}\right) \\
& =\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4}+n_{1} \alpha-n_{2} \alpha \\
& =\left(n_{1}-n_{2}\right) \alpha .
\end{aligned}
$$

It follows that $n_{1}=n_{2}$, so $n_{1}+n_{2}$ is even.

Lemma 3.19. If $\left\{L_{1}, L_{2}, L_{3}, L_{4}\right\}$ form a tetrahedron of class (c.2), where $L_{1}$ and $L_{2}$ have the same sign, then $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ and $\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$ have the same parity.

Proof. By Lemma 3.14, all tetrahedra of class (c.2) have the form

$$
\begin{aligned}
& \lambda_{1}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}\right) \\
& \lambda_{2}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right) \\
& \lambda_{3}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{k}\right), \text { and } \\
& \lambda_{4}=-\omega_{6}+4\left(\varepsilon_{j}+\varepsilon_{\ell}\right)
\end{aligned}
$$

for some distinct indices $i, j, k$, and $\ell$.
We will use as a base case the tetrahedron in Figure 3.3 , with ideals numbered $L_{1}, L_{2}, L_{3}$, and $L_{4}$ as in the figure. The corresponding weights, where $\lambda_{i}$ corresponds to $L_{i}$, are

$$
\begin{aligned}
& \lambda_{1}=\omega_{6}-4\left(\varepsilon_{5}+\varepsilon_{6}\right) \\
& \lambda_{2}=\omega_{6}-4\left(\varepsilon_{2}+\varepsilon_{4}\right) \\
& \lambda_{3}=-\omega_{6}+4\left(\varepsilon_{2}+\varepsilon_{5}\right), \text { and } \\
& \lambda_{4}=-\omega_{6}+4\left(\varepsilon_{4}+\varepsilon_{6}\right)
\end{aligned}
$$

We observe that $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ and $\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$ are both odd for this tetrahedron.

By Lemma 3.15, we know that the subgroup of $W\left(D_{6}\right)$ generated by $s_{1}, \ldots, s_{5}$ acts transitively on tetrahedra of class (c.2). As mentioned previously, the generators $s_{1}, \ldots, s_{5}$ cannot change the level of any heap ideals, so $s_{i} L_{1}$ and $s_{i} L_{2}$ have the same level, as do $s_{i} L_{3}$ and $s_{i} L_{4}$. Thus it suffices to show that the action of any one of these generators will either preserve the parity of both $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ and $\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$, or change both parities.

By examining the structure of tetrahedra of class (c.2), as described in Lemma 3.15, we will argue that any of the generators $s_{1}, \ldots, s_{5}$ will act on the tetrahedron in one of four ways. By Lemma 3.18, any generator will move an even number of weights.
(i) A generator may act by fixing the tetrahedron setwise, possibly permuting the weights. Even though the weights have been permuted, as they cannot change level, the same-level pairings remain the same. In this case, because $\left\{s_{i} L_{1}, s_{i} L_{2}\right\}=\left\{L_{1}, L_{2}\right\}$ and $\left\{s_{i} L_{3}, s_{i} L_{4}\right\}=\left\{L_{3}, L_{4}\right\}$, we have that

$$
\left|\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \quad \text { and } \quad\left|\chi\left(s_{i} L_{3}, s_{i} L_{4}\right)\right|_{\#}=\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#} .
$$

(ii) A generator may move two weights of the same sign. Without loss of generality, we may assume that $s_{i}$ moves the weights corresponding to $L_{1}$ and $L_{2}$. Since the two weights must move in opposite directions, we use Lemma 2.23 to see that

$$
\begin{aligned}
\chi\left(s_{i} L_{1}, s_{i} L_{2}\right) & =\left(\lambda_{1} \pm \alpha_{i}\right)-\left(\lambda_{2} \mp \alpha_{i}\right) \\
& =\lambda_{1}-\lambda_{2} \pm 2 \alpha_{i} \\
& =\chi\left(L_{1}, L_{2}\right) \pm 2 \alpha_{i} .
\end{aligned}
$$

It follows that $\left|\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 2$. Since $s_{i}$ fixes $L_{3}$ and $L_{4}$, we also have $\left|\chi\left(s_{i} L_{3}, s_{i} L_{4}\right)\right|_{\#}=\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$.
(iii) A generator may move two weights of different signs. Without loss of generality, we may assume that $s_{i}$ moves the weights corresponding to $L_{1}$ and $L_{3}$. Since these two weights must move in opposite directions, we use Lemma 2.23 to see that

$$
\begin{aligned}
\chi\left(s_{i} L_{1}, s_{i} L_{2}\right) & =\left(\lambda_{1} \pm \alpha_{i}\right)-\lambda_{2} \\
& =\lambda_{1}-\lambda_{2} \pm \alpha_{i} \\
& =\chi\left(L_{1}, L_{2}\right) \pm \alpha_{i}
\end{aligned}
$$

and, similarly, $\chi\left(s_{i} L_{3}, s_{i} L_{4}\right)=\chi\left(L_{3}, L_{4}\right) \mp \alpha_{i}$. It follows that $\left|\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm$ 1 and $\left|\chi\left(s_{i} L_{3}, s_{i} L_{4}\right)\right|_{\#}=\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#} \mp 1$; both have changed parity in this case.
(iv) A generator may move all four weights to four new weights, at least one of which is not from the original 4 -tuple. An argument similar to that in part (ii) shows that for each same-sign
pair, the relative content changes by $\pm 2 \alpha_{i}$, and hence the parity of the size of each relative content does not change.

Lemma 3.20. Let $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$ be a 4-tuple of weights with $\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4}=0$, where $\lambda_{1}$ and $\lambda_{2}$ are incident weights corresponding to heap ideals $L_{1}$ and $L_{2}$ of the same sign. Let $s$ be a Weyl group generator, other than $s_{6}$, and suppose that $s$ acts on $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$ by moving $\lambda_{1}$ and $\lambda_{2}$ while fixing $\lambda_{3}$ and $\lambda_{4}$. Then this operation either adds or removes a crossing between $L_{1}$ and $L_{2}$, preserves the parity of $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$. Moreover, this operation preserves the number of crossings between $L_{3}$ and $L_{4}$ and preserves $\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$.

Proof. We first prove that $L_{1}$ and $L_{2}$ have the same level. Suppose for the sake of contradiction that $L_{1}$ and $L_{2}$ have the same sign but not the same level. Then, as there are only four levels, one of the two heap ideals must be either the entire heap or the empty ideal. These ideals have weight $\pm \omega_{6}$ and are not moved by any Weyl group generator other than $s_{6}$. Thus we have a contradiction, and $L_{1}$ and $L_{2}$ must have the same level.

Write $\lambda_{1}$ and $\lambda_{2}$ for the weights corresponding to heap ideals $L_{1}$ and $L_{2}$, respectively. We know that $s_{i}$ adds an element labelled $i$ to one of $\left\{L_{1}, L_{2}\right\}$ and removes an element labelled $i$ from the other. We proceed by cases.
(1) If $s_{i}\left(\lambda_{1}\right)=\lambda_{1}+\alpha_{i}$ and $s_{i}\left(\lambda_{2}\right)=\lambda_{2}-\alpha_{i}$, then we are in the situation of Corollary 2.33, part (1)(iii). Note that $s_{i}\left(\lambda_{1}\right)=e_{\alpha_{i}}\left(\lambda_{1}\right)$ and $s_{i}\left(\lambda_{2}\right)=f_{\alpha_{i}}\left(\lambda_{2}\right)$ in this case. It follows from Lemma 2.34 that $\left|\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 2$ and $\sum P\left(s_{i} L_{1}, s_{i} L_{2}\right)=\left(\sum P\left(L_{1}, L_{2}\right)\right) \pm 2$, so the parity of $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ is preserved. By Corollary 3.17, we have either added or removed a crossing.
(2) If $s_{i}\left(\lambda_{1}\right)=\lambda_{1}-\alpha_{i}$ and $s_{i}\left(\lambda_{2}\right)=\lambda_{2}+\alpha_{i}$, then we are in the situation of Corollary 2.33, part (2)(iii). Note that $s_{i}\left(\lambda_{1}\right)=f_{\alpha_{i}}\left(\lambda_{1}\right)$ and $s_{i}\left(\lambda_{2}\right)=e_{\alpha_{i}}\left(\lambda_{2}\right)$ in this case. It follows from Corollary 2.35 that $\left|\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 2$ and $\sum P\left(s_{i} L_{1}, s_{i} L_{2}\right)=\left(\sum P\left(L_{1}, L_{2}\right)\right) \pm$ 2, so the parity of $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ is preserved. By Corollary 3.17, we have either added or removed a crossing.

Lastly, because $s_{i}$ does not move $L_{3}$ or $L_{4}$, it follows that the number of crossings between $L_{3}$ and $L_{4}$ does not change and $\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$ is preserved.

Lemma 3.21. Suppose that the weights $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$ form a tetrahedron of class (c.2). Write $\left\{L_{1}, L_{2}, L_{3}, L_{4}\right\}$ for the corresponding heap ideals, where $L_{1}$ and $L_{2}$ have the same level, and $L_{3}$ and $L_{4}$ have the same level. Suppose that $s_{i}$ is a Weyl group generator, other than $s_{6}$, that acts on this 4-tuple of weights by moving only $\lambda_{1}$ and $\lambda_{3}$, while fixing $\lambda_{2}$ and $\lambda_{4}$. Then this operation preserves the number of crossings in the 4 -tuple, but changes the parity of $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ and of $\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$. Proof. The key to this result is that each same-level pair of ideals is in the situation of Corollary 2.37.

As $s_{i}$ acts on the tetrahedron by moving $L_{1}$ and $L_{3}$ while fixing the other weights, resulting in a new 4 -tuple with weights that still sum to zero, then $s_{i}$ must have added an element labelled $i$ to one of $\left\{L_{1}, L_{3}\right\}$ and removed an element labelled $i$ from the other. We proceed by cases. (Recall that crossings can only occur between ideals of the same level.)
(a) Suppose that $s_{i}\left(\lambda_{1}\right)=\lambda_{1}+\alpha_{i}$ and $s_{i}\left(\lambda_{3}\right)=\lambda_{3}-\alpha_{i}$. Note that $s_{i}\left(\lambda_{1}\right)=e_{\alpha_{i}}\left(\lambda_{1}\right)$ and $s_{i}\left(\lambda_{3}\right)=$ $f_{\alpha_{i}}\left(\lambda_{3}\right)$ in this case.

The pair $\left\{L_{1}, L_{2}\right\}$ is in the situation of Corollary 2.37 (1), so $P\left(s_{i} L_{1}, s_{i} L_{2}\right)=P\left(L_{1}, L_{2}\right)$. It follows from Corollary 3.17 that no crossings were added or removed for this pair. It also follows that

$$
\left|\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 1
$$

and the parity of size of the relative content has changed.

The pair $\left\{L_{3}, L_{4}\right\}$ is in the situation of Corollary 2.37 (2), so $P\left(s_{i} L_{3}, s_{i} L_{4}\right)=P\left(L_{3}, L_{4}\right)$. It follows from Corollary 3.17 that no crossings were added or removed for this pair. It also follows that

$$
\left|\chi\left(s_{i} L_{3}, s_{i} L_{4}\right)\right|_{\#}=\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#} \pm 1
$$

and the parity of the size of the relative content has changed.
(b) A similar argument shows that the result also holds when $s_{i}\left(\lambda_{1}\right)=\lambda_{1}-\alpha_{i}$ and $s_{i}\left(\lambda_{3}\right)=$ $\lambda_{3}+\alpha_{i}$.

Lemma 3.22. Suppose that the weights $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$ form a tetrahedron of class (c.2). Write $\left\{L_{1}, L_{2}, L_{3}, L_{4}\right\}$ for the corresponding heap ideals, where $L_{1}$ and $L_{2}$ have the same level, and $L_{3}$ and $L_{4}$ have the same level.

Suppose that $s_{i}$ is a Weyl group generator, other than $s_{6}$, that acts on a tetrahedron of class (c.2) by moving all four weights, producing a different tetrahedron of class (c.2). Then the following hold.
(i) The number of crossings in the 4 -tuple $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}$ and the number of crossings in the 4-tuple $\left\{s_{i}\left(\lambda_{1}\right), s_{i}\left(\lambda_{2}\right), s_{i}\left(\lambda_{3}\right), s_{i}\left(\lambda_{4}\right)\right\}$ have the same parity.
(ii) The parity of $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ is the same as the parity of $\left|\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)\right|_{\#}$.
(iii) The parity of $\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$ is the same as the parity of $\left|\chi\left(s_{i} L_{3}, s_{i} L_{4}\right)\right|_{\#}$.

Proof. Recall first that generators other than $s_{6}$ cannot change the level of an ideal, so $\left\{s_{i} L_{1}, s_{i} L_{2}\right\}$ and $\left\{s_{i} L_{3}, s_{i} L_{4}\right\}$ are same-level pairs.

If $s_{i}$ acts on the tetrahedron by moving all four weights, then it must add $\alpha_{i}$ to two of the weights, while subtracting $\alpha_{i}$ from the other two weights. We proceed by cases.
(a) If $s_{i}$ acts on the tetrahedron by adding $\alpha_{i}$ to weights $\lambda_{1}$ and $\lambda_{2}$, while subtracting $\alpha_{i}$ from weights $\lambda_{3}$ and $\lambda_{4}$, then by Lemma 2.23 we have

$$
\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)=\chi\left(L_{1}, L_{2}\right) \quad \text { and } \quad \chi\left(s_{i} L_{3}, s_{i} L_{4}\right)=\chi\left(L_{3}, L_{4}\right) .
$$

As the relative contents for the same-level pairs have not changed, it follows by Corollary 3.17 that the number of crossings and the parity of the size of the relative content for these pairs have not changed either. This proves (i), (ii), and (iii).

A similar argument shows that the result holds if $s_{i}$ acts on the tetrahedron by subtracting $\alpha_{i}$ from weights $\lambda_{1}$ and $\lambda_{2}$ while adding $\alpha_{i}$ to weights $\lambda_{3}$ and $\lambda_{4}$.
(b) We now suppose that $s_{i}$ acts on the tetrahedron by adding $\alpha_{i}$ to two weights that correspond to ideals of different levels. Without loss of generality, we may assume that $s_{i}\left(\lambda_{1}\right)=\lambda_{1}+\alpha_{i}$ and $s_{i}\left(\lambda_{3}\right)=\lambda_{3}+\alpha_{i}$, while $s_{i}\left(\lambda_{2}\right)=\lambda_{2}-\alpha_{i}$ and $s_{i}\left(\lambda_{4}\right)=\lambda_{4}-\alpha_{i}$. Now the pair $\left\{L_{1}, L_{2}\right\}$ is in the situation of Corollary 2.33 with respect to the root $\alpha_{i}$, as is the pair $\left\{L_{3}, L_{4}\right\}$. By Lemma 2.34, we then have

$$
\sum P\left(s_{i} L_{1}, s_{i} L_{2}\right)=\left(\sum P\left(L_{1}, L_{2}\right)\right) \pm 2 \quad \text { and } \quad \sum P\left(s_{i} L_{3}, s_{i} L_{4}\right)=\left(\sum P\left(L_{3}, L_{4}\right)\right) \pm 2 .
$$

By Corollary 3.17, this means that the pair $\left\{L_{1}, L_{2}\right\}$ has either lost or gained a crossing, and the pair $\left\{L_{3}, L_{4}\right\}$ has either lost or gained a crossing. Thus the parity of the overall crossing count remains the same, proving (i).

By Lemma 2.34, we also have that

$$
\left|\chi\left(s_{i} L_{1}, s_{i} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 2 \quad \text { and } \quad\left|\chi\left(s_{i} L_{3}, s_{i} L_{4}\right)\right|_{\#}=\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#} \pm 2
$$

proving (ii) and (iii).

### 3.3 Main Result

Lemma 3.23. Let $b_{1}$ and $b_{1}^{\prime}$ be weight vectors corresponding to weights $\lambda_{1}$ and $-\lambda_{1}$ of $L\left(D_{6}, \omega_{6}\right)$, respectively, and let $b_{2}$ and $b_{2}^{\prime}$ be weight vectors corresponding to $\lambda_{2}$ and $-\lambda_{2}$ respectively. Then we have

$$
q\left(b_{1} \otimes b_{1} \otimes b_{1}^{\prime} \otimes b_{1}^{\prime}\right)=q\left(b_{2} \otimes b_{2} \otimes b_{2}^{\prime} \otimes b_{2}^{\prime}\right) .
$$

Proof. The Weyl group $W\left(D_{6}\right)$ acts transitively on the set of weights, so there exists an element $w$ of the Weyl group that moves $\lambda_{1}$ to $\lambda_{2}$. If we write $w$ as a minimal word in the generators $s_{1}, \ldots, s_{6}$, and act on $\left\{b_{1}, b_{1}, b_{1}^{\prime}, b_{1}^{\prime}\right\}$ by each generator successively, then each of these generators will move all four weight vectors. By Lemma 3.3, each of these actions preserves the value of $q$.

We now normalize $q$ so that

$$
q\left(b \otimes b \otimes b^{\prime} \otimes b^{\prime}\right)=1
$$

whenever $b$ and $b^{\prime}$ are weight vectors labelled by $\lambda$ and $-\lambda$ respectively (i.e. a 4 -tuple of class (a)).

Lemma 3.24. Let $b_{1}, b_{1}^{\prime}, b_{2}$, $b_{2}^{\prime}$ be weight vectors corresponding to weights $\lambda_{1},-\lambda_{1}, \lambda_{2},-\lambda_{2}$ respectively, where $\lambda_{1}=\omega_{6}$ and $\lambda_{2}=-\omega_{6}+2\left(\varepsilon_{i}+\varepsilon_{j}\right)$ for some $1 \leq i<j \leq 6$ (so that $\lambda_{1}$ and $\lambda_{2}$ have the same sign). Let $L_{1}$ and $L_{2}$ be the heap ideals corresponding to $\lambda_{1}$ and $\lambda_{2}$, and let $L_{1}^{\prime}$ and $L_{2}^{\prime}$ be the heap ideals corresponding to $-\lambda_{1}$ and $-\lambda_{2}$. Then

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=\frac{(-1)^{S+1}}{2}
$$

where $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}^{\prime}, L_{2}^{\prime}\right)\right|_{\#}$.
Proof. We first show that $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}^{\prime}, L_{2}^{\prime}\right)\right|_{\#}$. Negation of weights corresponds to vertically reflecting the contours of the corresponding ideals. It follows by symmetry that the size of the relative content will be the same for each same-sign pair. Thus we may compute $S$ using either pair.

We now begin with the case $i=5, j=6$. Let $v_{1}, v_{1}^{\prime}$ be weight vectors corresponding to weights $\mu_{1}=-\omega_{6}$ and $-\mu_{1}=\omega_{6}$, respectively, and let $v_{2}, v_{2}^{\prime}$ correspond to $\mu_{2}=-\omega_{6}+4\left(\varepsilon_{5}+\varepsilon_{6}\right)$ and $-\mu_{2}=\omega_{6}-4\left(\varepsilon_{5}+\varepsilon_{6}\right)$, respectively. We apply Corollary 3.6 with $\alpha=\alpha_{6}$ to see that

$$
\begin{equation*}
-\frac{1}{2}=q\left(v_{1} \otimes v_{1}^{\prime} \otimes v_{2} \otimes v_{2}^{\prime}\right) . \tag{3.3}
\end{equation*}
$$

Let $b_{1}$ and $b_{2}$ be weight vectors corresponding to the weights $\lambda_{1}=\omega_{6}$ and $\lambda_{2}=-\omega_{6}+4\left(\varepsilon_{5}+\right.$ $\varepsilon_{6}$ ), respectively. The corresponding heap ideals $L_{1}$ and $L_{2}$, shown in Figure 3.5, have levels 3 and 1 , respectively, so $b_{1}$ and $b_{2}$ correspond to ideals of the same sign. By Equation 3.3 above and the symmetry of $q$, we have

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=-\frac{1}{2} .
$$

A look at the corresponding heap ideals, shown in Figure 3.5, demonstrates that $S=$ $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=14$, so the equality $q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=\frac{(-1)^{S+1}}{2}$ is satisfied here.

Now we let $i$ and $j$ be distinct indices between 1 and 6 , and let $w$ be a word of minimal length in the generators $s_{1}, \ldots, s_{5}$ that moves coordinates $\{5,6\}$ to coordinates $\{i, j\}$ via the usual permutations. We proceed by induction on the length of $w$. Our base case, when $\ell(w)=0$, is above.

Suppose that for all words of length $n$, the equality $q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=\frac{(-1)^{S+1}}{2}$ holds. Let $w=s_{i_{1}} s_{i_{2}} \cdots s_{i_{n+1}}$ be a word of minimal length $n+1$. Let $\lambda_{1}=s_{i_{2}} \cdots s_{i_{n+1}}\left(\omega_{6}\right)=\omega_{6}$ and $\lambda_{2}=s_{i_{2}} \cdots s_{i_{n+1}}\left(-\omega_{6}+4\left(\varepsilon_{5}+\varepsilon_{6}\right)\right)$. Let $L_{1}$ and $L_{2}$ be heap ideals corresponding to $\lambda_{1}$ and $\lambda_{2}$, respectively, and let $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$. Then by assumption, as $\ell\left(s_{i_{2}} \cdots s_{i_{n+1}}\right)=n$, we have

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=\frac{(-1)^{S+1}}{2}
$$

where as usual, $b_{1}, b_{1}^{\prime}$ are weight vectors of weight $\pm \lambda_{1}$, and $b_{2}, b_{2}^{\prime}$ are weight vectors of weight $\pm \lambda_{2}$.
Now, since $s_{i_{1}} \neq s_{6}$, we have $s_{i_{1}}\left( \pm \omega_{6}\right)= \pm \omega_{6}$, so the weights $\pm \lambda_{1}$ and their corresponding ideals remain fixed, and $s_{i_{1}}$ only moves the other two weights, $\lambda_{2}$ and $-\lambda_{2}$. By Lemma 3.7, this operation will change the sign of $q$. So, if we write $b_{3}, b_{3}^{\prime}$ for the weight vectors of weight $\pm s_{i_{1}}\left(\lambda_{2}\right)$, we have

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{3} \otimes b_{3}^{\prime}\right)=-\frac{(-1)^{S+1}}{2}
$$

As $s_{i_{1}} \neq s_{6}$, it follows that $s_{i_{1}} L_{1}$ and $s_{i_{1}} L_{2}$ still have the same sign. Let $S^{\prime}=\left|\chi\left(s_{i_{1}} L_{1}, s_{i_{1}} L_{2}\right)\right|_{\#}$. As $s_{i_{1}}$ moves $\pm \lambda_{2}$ but not $\pm \lambda_{1}$, it follows that $\left|\chi\left(s_{i_{1}} L_{1}, s_{i_{1}} L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#} \pm 1$, so we have

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{3} \otimes b_{3}^{\prime}\right)=\frac{(-1)^{S^{\prime}+1}}{2}
$$

proving the result.
Corollary 3.25. Let $b_{1}, b_{1}^{\prime}, b_{2}, b_{2}^{\prime}$ be weight vectors corresponding to weights $\lambda_{1},-\lambda_{1}, \lambda_{2},-\lambda_{2}$ respectively, where $\lambda_{1}=-\omega_{6}$ and $\lambda_{2}=\omega_{6}-2\left(\varepsilon_{i}+\varepsilon_{j}\right)$ for some $1 \leq i<j \leq 6$ (so that $\lambda_{1}$ and $\lambda_{2}$ have the same sign). Let $L_{1}$ and $L_{2}$ be the heap ideals corresponding to $\lambda_{1}$ and $\lambda_{2}$, and let $L_{1}^{\prime}$ and $L_{2}^{\prime}$ be the heap ideals corresponding to $-\lambda_{1}$ and $-\lambda_{2}$. Then we have

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=\frac{(-1)^{S+1}}{2}
$$

where $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=\left|\chi\left(L_{1}^{\prime}, L_{2}^{\prime}\right)\right|_{\#}$.
Proof. We apply Lemma 3.24 with the roles of $\lambda_{1}$ and $\lambda_{1}^{\prime}$ exchanged and the roles of $\lambda_{2}$ and $\lambda_{2}^{\prime}$ exchanged.

Figure 3.5: The heap ideals $L_{1}$ and $L_{2}$ below correspond to weights $\lambda_{1}=\omega_{6}$ and $\lambda_{2}=-\omega_{6}+4\left(\varepsilon_{5}+\right.$ $\varepsilon_{6}$ ), respectively.


Lemma 3.26. Let $b_{1}, b_{1}^{\prime}, b_{2}$, $b_{2}^{\prime}$ be weight vectors corresponding to weights $\lambda_{1},-\lambda_{1}, \lambda_{2},-\lambda_{2}$ respectively, where $\lambda_{1}$ and $\lambda_{2}$ are weights of the same sign, with neither being $\pm \omega_{6}$. Let $L_{1}$ and $L_{2}$ be the heap ideals corresponding to $\lambda_{1}$ and $\lambda_{2}$. Suppose that $w \in\left\langle s_{1}, \ldots, s_{5}\right\rangle$. Then the quantity

$$
(-1)^{\left|\chi\left(w L_{1}, w L_{2}\right)\right| \# q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes w b_{2} \otimes w b_{2}^{\prime}\right)}
$$

is independent of the choice of $w$.
Proof. Fix $b_{1}, b_{1}^{\prime}, b_{2}, b_{2}^{\prime}$, and then fix $Q=\left.(-1)^{\left|\chi\left(L_{1}, L_{2}\right)\right|}\right|_{\#}\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)$. We will prove that

$$
(-1)^{\left|\chi\left(w L_{1}, w L_{2}\right)\right|} \| q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes w b_{2} \otimes w b_{2}^{\prime}\right)=Q
$$

for any $w \in\left\langle s_{1}, \ldots, s_{5}\right\rangle$. We proceed by induction on the length of $w$. If $\ell(w)=0$, we have

$$
(-1)^{\left|\chi\left(w L_{1}, w L_{2}\right)\right| \#} q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes w b_{2} \otimes w b_{2}^{\prime}\right)=(-1)^{\left|\chi\left(L_{1}, L_{2}\right)\right|} \| q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=Q .
$$

Suppose that for all words $w$ of length $n$ or less, the desired equality holds. Let $w=$ $s_{i_{1}} s_{i_{2}} \cdots s_{i_{n+1}}$ be a word of length $n+1$. Let $u=s_{i_{2}} \cdots s_{i_{n+1}}$ so that $w=s_{i_{1}} u$. Now let $\mu_{1}=u\left(\lambda_{1}\right)$ and $\mu_{2}=u\left(\lambda_{2}\right)$, and let $M_{1}=u L_{1}$ and $M_{2}=u L_{2}$ be the heap ideals corresponding to $\mu_{1}$ and $\mu_{2}$. Because $\lambda_{1}$ and $\lambda_{2}$ are weights of the same sign, with neither being $\pm \omega_{6}$, we have that $L_{1}$ and $L_{2}$
have the same level. Because our generators do not include $s_{6}$, the heap ideals $M_{1}$ and $M_{2}$ here will also have the same level. Now, by assumption, as $\ell(u)=n$, we have

$$
Q=\left.(-1)^{\left|\chi\left(M_{1}, M_{2}\right)\right|}\right|_{\#} q\left(u b_{1} \otimes u b_{1}^{\prime} \otimes u b_{2} \otimes u b_{2}^{\prime}\right) .
$$

Now we consider the action of the generator $s_{i_{1}}$ on the weights $\mu_{1}$ and $\mu_{2}$. Let $\alpha_{i_{1}}$ denote the simple root corresponding to $s_{i_{1}}$.
(i) If $s_{i_{1}}$ moves both $\mu_{1}$ and $\mu_{2}$ by adding $\alpha_{i_{1}}$ to each of these roots, then $s_{i_{1}}$ moves all four weights in the 4-tuple $\left\{ \pm \mu_{1}, \pm \mu_{2}\right\}$. By Lemma 3.3, this operation will not change the sign of q. So we have

$$
q\left(s_{i_{1}} u b_{1} \otimes s_{i_{1}} u b_{1}^{\prime} \otimes s_{i_{1}} u b_{2} \otimes s_{i_{1}} u b_{2}^{\prime}\right)=q\left(u b_{1} \otimes u b_{1}^{\prime} \otimes u b_{2} \otimes u b_{2}^{\prime}\right) .
$$

Moreover, as $s_{i_{1}}$ moves $\mu_{1}$ to $\mu_{1}+\alpha_{i_{1}}$ and moves $\mu_{2}$ to $\mu_{2}+\alpha_{i_{1}}$, it follows that $\left|\chi\left(s_{i_{1}} M_{1}, s_{i_{1}} M_{2}\right)\right|_{\#}=$ $\left|\chi\left(M_{1}, M_{2}\right)\right|_{\#}$. Thus we have

$$
\begin{aligned}
&(-1)^{\left|\chi\left(w L_{1}, w L_{2}\right)\right|_{\#} q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes\right.}\left.w b_{2} \otimes w b_{2}^{\prime}\right) \\
&=\left.(-1)^{\left|\chi\left(s_{i_{1}} M_{1}, s_{i_{1}} M_{2}\right)\right|}\right|_{\# q\left(s_{i_{1}} u b_{1} \otimes s_{i_{1}} u b_{1}^{\prime} \otimes s_{i_{1}} u b_{2} \otimes s_{i_{1}} u b_{2}^{\prime}\right)} \\
&=\left.(-1)^{\left|\chi\left(M_{1}, M_{2}\right)\right|}\right|_{\#} q\left(u b_{1} \otimes u b_{1}^{\prime} \otimes u b_{2} \otimes u b_{2}^{\prime}\right) \\
&=Q .
\end{aligned}
$$

(ii) A similar argument shows that the result holds if $s_{i_{1}}$ moves both $\mu_{1}$ and $\mu_{2}$ by subtracting $\alpha_{i_{1}}$ from each of these roots.
(iii) If $s_{i_{1}}$ moves both $\mu_{1}$ and $\mu_{2}$ by adding $\alpha_{i_{1}}$ to one weight and subtracting $\alpha_{i_{1}}$ from the other, then $s_{i_{1}}$ moves all four weights in the 4 -tuple $\left\{ \pm \mu_{1}, \pm \mu_{2}\right\}$. By Lemma 3.3, this operation will not change the sign of $q$. So we have

$$
q\left(s_{i_{1}} u b_{1} \otimes s_{i_{1}} u b_{1}^{\prime} \otimes s_{i_{1}} u b_{2} \otimes s_{i_{1}} u b_{2}^{\prime}\right)=q\left(u b_{1} \otimes u b_{1}^{\prime} \otimes u b_{2} \otimes u b_{2}^{\prime}\right) .
$$

As $s_{i_{1}}$ moves $\mu_{1}$ to $\mu_{1} \pm \alpha_{i_{1}}$ and moves $\mu_{2}$ to $\mu_{2} \mp \alpha_{i_{2}}$, it follows from Lemma 2.34 that $\left|\chi\left(s_{i_{1}} M_{1}, s_{i_{1}} M_{2}\right)\right|_{\#}=\left|\chi\left(M_{1}, M_{2}\right)\right|_{\#} \pm 2$. Thus we have

$$
\begin{aligned}
&(-1)^{\left|\chi\left(w L_{1}, w L_{2}\right)\right| \#} q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes\right.\left.w b_{2} \otimes w b_{2}^{\prime}\right) \\
&=\left.(-1)^{\left|\chi\left(s_{i_{1}} M_{1}, s_{i_{1}} M_{2}\right)\right|}\right|_{\# q\left(s_{i_{1}} u b_{1} \otimes s_{i_{1}} u b_{1}^{\prime} \otimes s_{i_{1}} u b_{2} \otimes s_{i_{1}} u b_{2}^{\prime}\right)} \\
&=(-1)^{\left|\chi\left(M_{1}, M_{2}\right)\right|_{\#} \pm 2} q\left(u b_{1} \otimes u b_{1}^{\prime} \otimes u b_{2} \otimes u b_{2}^{\prime}\right) \\
&=Q .
\end{aligned}
$$

(iv) If $s_{i_{1}}$ fixes $\mu_{1}$ and moves $\mu_{2}$ to $\mu_{2}+\alpha_{i_{1}}$, then $s_{i_{1}}$ moves precisely two weights in the 4 -tuple $\left\{ \pm \mu_{1}, \pm \mu_{2}\right\}$. By Lemma 3.7, this operation will change the sign of $q$. So we have

$$
q\left(s_{i_{1}} u b_{1} \otimes s_{i_{1}} u b_{1}^{\prime} \otimes s_{i_{1}} u b_{2} \otimes s_{i_{1}} u b_{2}^{\prime}\right)=-q\left(u b_{1} \otimes u b_{1}^{\prime} \otimes u b_{2} \otimes u b_{2}^{\prime}\right) .
$$

As $s_{i_{1}}$ fixes $\mu_{1}$ and moves $\mu_{2}$ to $\mu_{2}+\alpha_{i_{1}}$, it follows that $\left|\chi\left(s_{i_{1}} M_{1}, s_{i_{1}} M_{2}\right)\right|_{\#}=\left|\chi\left(M_{1}, M_{2}\right)\right|_{\#} \pm 1$. Thus we have

$$
\begin{aligned}
(-1)^{\left|\chi\left(w L_{1}, w L_{2}\right)\right| \#} q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes\right. & \left.w b_{2} \otimes w b_{2}^{\prime}\right) \\
& =(-1)^{\left|\chi\left(s_{i_{1}} M_{1}, s_{i_{1}} M_{2}\right)\right|_{\# q}} q\left(s_{i_{1}} u b_{1} \otimes s_{i_{1}} u b_{1}^{\prime} \otimes s_{i_{1}} u b_{2} \otimes s_{i_{1}} u b_{2}^{\prime}\right) \\
& =(-1)^{\left|\chi\left(M_{1}, M_{2}\right)\right|_{\# \#^{ \pm 1}}}\left(-q\left(u b_{1} \otimes u b_{1}^{\prime} \otimes u b_{2} \otimes u b_{2}^{\prime}\right)\right) \\
& =Q .
\end{aligned}
$$

(v) A similar argument shows that the result holds if $s_{i_{1}}$ fixes $\mu_{1}$ and moves $\mu_{2}$ to $\mu_{2}-\alpha_{i_{1}}$, or if $s_{i_{1}}$ fixes $\mu_{2}$ and moves $\mu_{1}$ by either adding or subtracting $\alpha_{i_{1}}$.

Lemma 3.27. Let $b_{1}, b_{1}^{\prime}, b_{2}, b_{2}^{\prime}$ be weight vectors corresponding to weights $\lambda_{1},-\lambda_{1}, \lambda_{2},-\lambda_{2}$ respectively, where $\lambda_{1}$ and $\lambda_{2}$ are skew weights of the same sign, with neither being $\pm \omega_{6}$. Let $L_{1}$ and $L_{2}$ be the heap ideals corresponding to $\lambda_{1}$ and $\lambda_{2}$. Then

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=\frac{(-1)^{S}}{2}
$$

where $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$.

Proof. First, we observe that if $\lambda_{1}$ and $\lambda_{2}$ are weights of the same sign, with neither being $\pm \omega_{6}$, then they must also have the same level. As $\lambda_{1}$ and $\lambda_{2}$ are a skew pair, they must differ in exactly two places. It follows that either

$$
\lambda_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{j}\right) \quad \text { and } \quad \lambda_{2}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{k}\right)
$$

or

$$
\lambda_{1}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{j}\right) \quad \text { and } \quad \lambda_{2}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{k}\right)
$$

for some distinct indices $i, j$, and $k$. As our lemma statement is invariant under negating all four weights, without loss of generality, we may assume that $\lambda_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{j}\right)$ and $\lambda_{2}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{k}\right)$.

We begin with the case $i=1, j=2, k=3$. Let $\mu_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right)$ and $\mu_{2}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{3}\right)$. Let $v_{1}, v_{1}^{\prime}$ be weight vectors corresponding to weights $\mu_{1}$ and $-\mu_{1}$, respectively, and let $v_{2}, v_{2}^{\prime}$ be weight vectors corresponding to weights $\mu_{2}$ and $-\mu_{2}$, respectively. We apply Corollary 3.6 with $\alpha=\alpha_{2}$ to see that

$$
-\frac{1}{2}=q\left(v_{1} \otimes v_{1}^{\prime} \otimes v_{2} \otimes v_{2}^{\prime}\right) .
$$

A look at the corresponding heap ideals, shown in Figure 3.6, demonstrates that $S=$ $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=1$, so the equality $q\left(v_{1} \otimes v_{1}^{\prime} \otimes v_{2} \otimes v_{2}^{\prime}\right)=\frac{(-1)^{S}}{2}$ is satisfied here.

Now we let $i, j$, and $k$ be distinct indices between 1 and 6 , and let $w$ be a word in the generators $s_{1}, \ldots, s_{5}$ that moves coordinate 1 to coordinate $i$ and moves coordinates $\{2,3\}$ to coordinates $\{j, k\}$ via the usual permutations. The result now follows by Lemma 3.26.

Figure 3.6: The heap ideals $L_{1}$ and $L_{2}$ below correspond to weights $\lambda_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right)$ and $\lambda_{2}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{3}\right)$, respectively.


Lemma 3.28. Let $b_{1}, b_{1}^{\prime}, b_{2}, b_{2}^{\prime}$ be weight vectors corresponding to weights $\lambda_{1},-\lambda_{1}, \lambda_{2},-\lambda_{2}$ respectively, where $\lambda_{1}$ and $\lambda_{2}$ are incident weights of the same sign, with neither being $\pm \omega_{6}$. Let $L_{1}$ and $L_{2}$ be the heap ideals corresponding to $\lambda_{1}$ and $\lambda_{2}$. Then

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=\frac{(-1)^{S+1}}{2}
$$

where $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$.

Proof. First, we observe that if $\lambda_{1}$ and $\lambda_{2}$ are weights of the same sign, with neither being $\pm \omega_{6}$, then they must also have the same level. As $\lambda_{1}$ and $\lambda_{2}$ are an incident pair, they must differ in exactly four places. It follows that either

$$
\lambda_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{j}\right) \quad \text { and } \quad \lambda_{2}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)
$$

or

$$
\lambda_{1}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{j}\right) \quad \text { and } \quad \lambda_{2}=-\omega_{6}+4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)
$$

for some distinct indices $i, j, k$, and $\ell$. Without loss of generality, we may assume that $\lambda_{1}=$ $\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{j}\right)$ and $\lambda_{2}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right)$.

We begin with the case $i=1, j=2, k=3, \ell=4$. Let $\mu_{1}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{2}\right)$ and $\mu_{2}=$ $\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right)$. Let $v_{1}, v_{1}^{\prime}$ be weight vectors corresponding to weights $\mu_{1}$ and $-\mu_{1}$, respectively, and let $v_{2}, v_{2}^{\prime}$ be weight vectors corresponding to weights $\mu_{2}$ and $-\mu_{2}$, respectively. We apply Corollary 3.6 with $\alpha=\alpha_{6}$ to see that

$$
-\frac{1}{2}=q\left(v_{1} \otimes v_{1}^{\prime} \otimes v_{2} \otimes v_{2}^{\prime}\right) .
$$

Let $b_{1}$ and $b_{2}$ be weight vectors corresponding to the weights $\lambda_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right)$ and $\lambda_{2}=\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right)$, respectively. The corresponding heap ideals $L_{1}$ and $L_{2}$, shown in Figure 3.7, both have level 2 , so $b_{1}$ and $b_{2}$ correspond to ideals of the same sign. By our computation above and the symmetry of $q$, we have

$$
q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=-\frac{1}{2} .
$$

A look at the corresponding heap ideals, shown in Figure 3.7, demonstrates that $S=$ $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=4$, so the equality $q\left(b_{1} \otimes b_{1}^{\prime} \otimes b_{2} \otimes b_{2}^{\prime}\right)=\frac{(-1)^{S+1}}{2}$ is satisfied here.

Now we let $i, j, k$, and $\ell$ be distinct indices between 1 and 6 , and let $w=s_{i_{1}} s_{i_{2}} \cdots s_{i_{n}}$ be a word in the generators $s_{1}, \ldots, s_{5}$ that moves the paired coordinates $\{\{1,2\},\{3,4\}\}$ to paired coordinates $\{\{i, j\},\{k, \ell\}\}$ via the usual permutations. The result now follows by Lemma 3.26.

Figure 3.7: The heap ideals $L_{1}$ and $L_{2}$ below correspond to weights $\lambda_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right)$ and $\lambda_{2}=\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right)$, respectively.


Remark 3.29. Let $b_{1}, b_{2}, b_{3}, b_{4}$ be weight vectors corresponding to weights $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ that form a tetrahedron of class (c.2). Write $L_{1}, L_{2}, L_{3}, L_{4}$ for the corresponding heap ideals, where $L_{1}$ and $L_{2}$ have the same sign, and let $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$. By Lemma 3.19, $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$ and $\left|\chi\left(L_{3}, L_{4}\right)\right|_{\#}$ have the same parity. It follows that the parity of $S$ is the same regardless of which same-sign pair of weights we use to calculate $S$.

We will use this observation without comment in the statements of the following lemmas.

Lemma 3.30. Let $b_{1}, b_{2}, b_{3}, b_{4}$ be weight vectors corresponding to weights $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ that form a tetrahedron of class (c.2). Write $L_{1}, L_{2}, L_{3}, L_{4}$ for the corresponding heap ideals, where $L_{1}$ and $L_{2}$ have the same sign. Suppose that $w \in\left\langle s_{1}, \ldots, s_{5}\right\rangle$. Let $S=\left|\chi\left(w L_{1}, w L_{2}\right)\right|_{\#}$ and let $C$ count the number of crossings in $\left\{w L_{1}, w L_{2}, w L_{3}, w L_{4}\right\}$. Then the quantity

$$
(-1)^{C+S} q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes w b_{2} \otimes w b_{2}^{\prime}\right)
$$

is independent of the choice of $w$.

Proof. Fix $b_{1}, b_{2}, b_{3}, b_{4}$, and then fix $Q=(-1)^{C+S} q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)$, where $C$ counts crossings in $\left\{L_{1}, L_{2}, L_{3}, L_{4}\right\}$ and $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$. We will prove that

$$
(-1)^{C+S} q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes w b_{2} \otimes w b_{2}^{\prime}\right)=Q
$$

for any $w \in\left\langle s_{1}, \ldots, s_{5}\right\rangle$, where $C$ and $S$ depend on $w$, as in the statement. We proceed by induction on the length of $w$. If $\ell(w)=0$, the result holds by the definition of $Q$ above.

Suppose that for all words $w$ of length $n$ or less, the desired equality holds. Let $w=$ $s_{i_{1}} s_{i_{2}} \cdots s_{i_{n+1}}$ be a word of length $n+1$. Let $u=s_{i_{2}} \cdots s_{i_{n+1}}$ so that $w=s_{i_{1}} u$. Because $\lambda_{1}$ and $\lambda_{2}$ are weights of the same sign, with neither being $\pm \omega_{6}$, it follows that $L_{1}$ and $L_{2}$ have the same level. Because our generators do not include $s_{6}$, the heap ideals $u L_{1}$ and $u L_{2}$ here will also have the same level. Now, by assumption, as $\ell(u)=n$, we have

$$
Q=(-1)^{C+S} q\left(u b_{1} \otimes u b_{2} \otimes u b_{3} \otimes u b_{4}\right)
$$

where $C$ counts crossings in $\left\{u L_{1}, u L_{2}, u L_{3}, u L_{4}\right\}$ and $S=\left|\chi\left(u L_{1}, u L_{2}\right)\right|_{\#}$.

Now we consider the action of the generator $s_{i_{1}}$ on the weights $\left\{u \lambda_{1}, u \lambda_{2}, u \lambda_{3}, u \lambda_{4}\right\}$.
(i) If $s_{i_{1}}$ moves all four weights, then by Lemma 3.3 , the sign of $q$ will not change. It follows from Lemma 3.22 (i) that when all four weights are moved, the parity of $C$ does not change. It follows from Lemma 3.22 (ii) that when all four weights are moved, the parity of $S$ does not change. So our result holds for the new 4-tuple as well.
(ii) If $s_{i_{1}}$ moves exactly two weights, then by Lemma 3.7 , the sign of $q$ will change. In terms of heaps, within the tetrahedra of class (c.2), there are two ways in which this can occur; these two possibilities are described in Lemma 3.20 and Lemma 3.21. In either case, the parity of $C+S$ is changed. So our result holds for the new 4-tuple.

Lemma 3.31. Suppose $b_{1}, b_{2}, b_{3}, b_{4}$ are weight vectors corresponding to weights $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ that form a tetrahedron of class (c.2). Write $L_{1}, L_{2}, L_{3}, L_{4}$ for the corresponding heap ideals, where $L_{1}$ and $L_{2}$ have the same sign. Then we have

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=(-1)^{C+S+1}
$$

where $C$ is the number of crossings in $\left\{L_{1}, L_{2}, L_{3}, L_{4}\right\}$, and $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$.

Proof. By Lemma 3.14, up to permutation of the four weights, any tetrahedron from class (c.2) has the form

$$
\begin{aligned}
& \lambda_{1}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}\right) \\
& \lambda_{2}=\omega_{6}-4\left(\varepsilon_{k}+\varepsilon_{\ell}\right) \\
& \lambda_{3}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{k}\right), \text { and } \\
& \lambda_{4}=-\omega_{6}+4\left(\varepsilon_{j}+\varepsilon_{\ell},\right)
\end{aligned}
$$

for some distinct indices $i, j, k$, and $\ell$.
We begin with the case $i=1, j=2, k=3, \ell=4$. First, we will apply Corollary 3.5 with
$\alpha=\alpha_{6}$ to the tetrahedron of weight vectors $v_{1}, v_{2}, v_{3}, v_{4}$ with weights

$$
\begin{align*}
& \mu_{1}=-\omega_{6}+4\left(\varepsilon_{2}+\varepsilon_{4}\right), \\
& \mu_{2}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{3}\right),  \tag{3.4}\\
& \mu_{3}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right), \text { and } \\
& \mu_{4}=\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right) .
\end{align*}
$$

The resulting 4 -tuples following Corollary 3.5 are

$$
\begin{align*}
e_{\alpha} \mu_{1} & =\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{3}\right), \\
\mu_{2} & =-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{3}\right),  \tag{3.5}\\
f_{\alpha} \mu_{3} & =-\omega_{6}+4\left(\varepsilon_{3}+\varepsilon_{4}\right), \text { and } \\
\mu_{4} & =\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right),
\end{align*}
$$

and

$$
\begin{align*}
e_{\alpha} \mu_{1} & =\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{3}\right), \\
\mu_{2} & =-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{3}\right),  \tag{3.6}\\
\mu_{3} & =\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right), \text { and } \\
f_{\alpha} \mu_{4} & =-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{2}\right) .
\end{align*}
$$

The heap ideals corresponding to the 4 -tuple of weights shown in equations (3.5) above are shown in Figure 3.8. The heap ideals corresponding to the 4 -tuple of weights shown in equations (3.6) above are shown in Figure 3.9. For each 4 -tuple, $S$ is odd, so by Lemma 3.27, the value of $q$ is $-1 / 2$ on both 4 -tuples. Hence by Corollary 3.5 , we have

$$
\begin{aligned}
0 & =q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right)+q\left(e_{\alpha} v_{1} \otimes v_{2} \otimes f_{\alpha} v_{3} \otimes v_{4}\right)+q\left(e_{\alpha} v_{1} \otimes v_{2} \otimes v_{3} \otimes f_{\alpha} v_{4}\right) \\
& =q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right)-\frac{1}{2}-\frac{1}{2}
\end{aligned}
$$

hence $q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right)=1$.

We now consider the specific tetrahedron of class (c.2) with weights given by

$$
\begin{aligned}
& \nu_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right), \\
& \nu_{2}=\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right), \\
& \nu_{3}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{3}\right), \text { and } \\
& \nu_{4}=-\omega_{6}+4\left(\varepsilon_{2}+\varepsilon_{4}\right) .
\end{aligned}
$$

These are a permutation of the weights in equations (3.4). Let $L_{1}, L_{2}, L_{3}, L_{4}$ be the corresponding heap ideals, and let $b_{1}, b_{2}, b_{3}, b_{4}$ be the corresponding weight vectors. It follows from the symmetry of $q$, along with our computation above, that

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=1
$$

A look at the ideals $L_{1}, L_{2}, L_{3}$, and $L_{4}$, shown in Figure 3.10, demonstrates that $S=$ $\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=4$. We also calculate from Figure 3.10 that $\sum P\left(L_{1}, L_{2}\right)=4$ and $\sum P\left(L_{3}, L_{4}\right)=2$. It follows from Proposition 3.16 that $L_{3}$ and $L_{4}$ cross, but $L_{1}$ and $L_{2}$ do not cross, so we have $C=1$. Hence the equality

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=(-1)^{C+S+1}
$$

is satisfied here.
Now we let $i, j, k$, and $\ell$ be distinct indices between 1 and 6 , and let $w=s_{i_{1}} s_{i_{2}} \cdots s_{i_{n}}$ be a word in the generators $s_{1}, \ldots, s_{5}$ that moves 1 to $i, 2$ to $j, 3$ to $k$, and 4 to $\ell$ via the usual permutations. The result now follows by Lemma 3.30.

Figure 3.8: The heap ideals $L_{1}, L_{2}, L_{3}, L_{4}$ below correspond to weights $e_{\alpha} \mu_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{3}\right)$, $\mu_{2}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{3}\right), f_{\alpha} \mu_{3}=-\omega_{6}+4\left(\varepsilon_{3}+\varepsilon_{4}\right)$, and $\mu_{4}=\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right)$, respectively. These are the weights from equations (3.5).


Figure 3.9: The heap ideals $L_{1}, L_{2}, L_{3}, L_{4}$ below correspond to weights $e_{\alpha} \mu_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{3}\right)$, $\mu_{2}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{3}\right), \mu_{3}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right)$, and $f_{\alpha} \mu_{4}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{2}\right)$, respectively. These are the weights from equations (3.6).


Figure 3.10: The heap ideals $L_{1}, L_{2}, L_{3}, L_{4}$ below correspond to weights $\lambda_{1}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}\right)$, $\lambda_{2}=\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right), \lambda_{3}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{3}\right)$, and $\lambda_{4}=-\omega_{6}+4\left(\varepsilon_{2}+\varepsilon_{4}\right)$, respectively.


Lemma 3.32. Let $b_{1}, b_{2}, b_{3}, b_{4}$ be weight vectors corresponding to weights $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ that form a tetrahedron of class (c.1), where $\lambda_{1}= \pm \omega_{6}$. Write $L_{1}, L_{2}, L_{3}, L_{4}$ for the corresponding heap ideals. Suppose that $w \in\left\langle s_{1}, \ldots, s_{5}\right\rangle$. Let $C$ count the number of crossings in $\left\{w L_{1}, w L_{2}, w L_{3}, w L_{4}\right\}$. Then the quantity

$$
(-1)^{C} q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes w b_{2} \otimes w b_{2}^{\prime}\right)
$$

is independent of the choice of $w$.

Proof. Fix $b_{1}, b_{2}, b_{3}, b_{4}$, and then fix $Q=(-1)^{C} q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)$, where $C$ counts crossings in $\left\{L_{1}, L_{2}, L_{3}, L_{4}\right\}$. We will prove that

$$
(-1)^{C} q\left(w b_{1} \otimes w b_{1}^{\prime} \otimes w b_{2} \otimes w b_{2}^{\prime}\right)=Q
$$

for any $w \in\left\langle s_{1}, \ldots, s_{5}\right\rangle$, where $C$ depends on $w$, as in the statement. We proceed by induction on the length of $w$. If $\ell(w)=0$, the result holds by the definition of $Q$ above.

Suppose that for all words $w$ of length $n$ or less, the desired equality holds. Let $w=$ $s_{i_{1}} s_{i_{2}} \cdots s_{i_{n+1}}$ be a word of length $n+1$. Let $u=s_{i_{2}} \cdots s_{i_{n+1}}$ so that $w=s_{i_{1}} u$. Because our generators do not include $s_{6}$, it follows that $u\left(\lambda_{1}\right)=u\left( \pm \omega_{6}\right)= \pm \omega_{6}$. By Lemma 3.14, $L_{2}, L_{3}$, and $L_{4}$ all have the same level. As none of our generators is $s_{6}$, it follows that $u L_{2}, u L_{3}$, and $u L_{4}$ also all have the same level. By assumption, as $\ell(u)=n$, we have

$$
Q=(-1)^{C} q\left(u b_{1} \otimes u b_{2} \otimes u b_{3} \otimes u b_{4}\right)
$$

where $C$ counts crossings in $\left\{u L_{1}, u L_{2}, u L_{3}, u L_{4}\right\}$.
Now we consider the action of the generator $s_{i_{1}}$ on the weights $\left\{u \lambda_{1}, u \lambda_{2}, u \lambda_{3}, u \lambda_{4}\right\}$. Note that $w L_{1}=u L_{1}=L_{1}$ is either the empty ideal or the ideal containing the entire heap, and thus cannot cross any other ideals. So it only makes sense to count crossings among $w L_{2}, w L_{3}$, and $w L_{4}$.

As $s_{i_{1}} \neq s_{6}$, we have that $s_{i_{1}}$ will never move $u \lambda_{1}=\lambda_{1}= \pm \omega_{6}$. It follows that $s_{i_{1}}$ must move exactly two of the four weights, and these two will not include $\lambda_{1}$. By Lemma 3.7, this will change the sign of $q$. By Lemma 3.20, this operation of moving an incident pair of weights of the same
level has the effect of either adding or removing a single crossing. Thus the result holds for the new tetrahedron.

Lemma 3.33. Let $b_{1}, b_{2}, b_{3}, b_{4}$ be weight vectors corresponding to weights $\lambda_{1}, \lambda_{2}, \lambda_{3}$, and $\lambda_{4}$ that form a tetrahedron of class (c.1), where $\lambda_{1}= \pm \omega_{6}$. Let $L_{1}, L_{2}, L_{3}, L_{4}$ be the corresponding heap ideals, and let $b_{1}, b_{2}, b_{3}, b_{4}$ be the corresponding weight vectors. Then

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=(-1)^{C}
$$

where $C$ is equal to the number of crossings between $L_{2}, L_{3}$, and $L_{4}$.

Proof. We will first restrict ourselves to class (c. $1^{+}$), where $\lambda_{1}=\omega_{6}$. By Lemma 3.14, every tetrahedron of class (c. $1^{+}$) has the form

$$
\begin{aligned}
& \lambda_{1}=\omega_{6}, \\
& \lambda_{2}=-\omega_{6}+4\left(\varepsilon_{i}+\varepsilon_{j}\right), \\
& \lambda_{3}=-\omega_{6}+4\left(\varepsilon_{k}+\varepsilon_{\ell}\right), \text { and } \\
& \lambda_{4}=\omega_{6}-4\left(\varepsilon_{i}+\varepsilon_{j}+\varepsilon_{k}+\varepsilon_{\ell}\right),
\end{aligned}
$$

for some distinct indices $i, j, k$, and $\ell$.
We begin with the case $i=1, j=2, k=4, \ell=6$. First, we will apply Lemma 3.7 with $\alpha=\alpha_{6}$ to the tetrahedron of weight vectors $v_{1}, v_{2}, v_{3}, v_{4}$ with weights

$$
\begin{aligned}
& \mu_{1}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{2}\right), \\
& \mu_{2}=\omega_{6}, \\
& \mu_{3}=-\omega_{6}+4\left(\varepsilon_{4}+\varepsilon_{6}\right), \text { and } \\
& \mu_{4}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{4}+\varepsilon_{6}\right) .
\end{aligned}
$$

The resulting 4 -tuple in Lemma 3.7 is given by

$$
\begin{align*}
e_{\alpha}\left(\mu_{1}\right) & =\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right), \\
f_{\alpha}\left(\mu_{2}\right) & =\omega_{6}-4\left(\varepsilon_{5}+\varepsilon_{6}\right),  \tag{3.7}\\
\mu_{3} & =-\omega_{6}+4\left(\varepsilon_{4}+\varepsilon_{6}\right), \text { and } \\
\mu_{4} & =-\omega_{6}+4\left(\varepsilon_{3}+\varepsilon_{5}\right) .
\end{align*}
$$

This is a tetrahedron of class (c.2). The corresponding heap ideals are shown in Figure 3.11. We calculate that $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}=4, \sum P\left(L_{1}, L_{2}\right)=4$, and $\sum P\left(L_{3}, L_{4}\right)=2$. It follows from Proposition 3.16 that $L_{1}$ and $L_{2}$ do not cross while $L_{3}$ and $L_{4}$ do cross. Thus we have $C=1$, and $S$ is even, so we can apply Lemma 3.31 to see that $q\left(e_{\alpha} v_{1} \otimes f_{\alpha} v_{2} \otimes v_{3} \otimes v_{4}\right)=(-1)^{C+S+1}=1$. It follows from Lemma 3.7 that

$$
q\left(v_{1} \otimes v_{2} \otimes v_{3} \otimes v_{4}\right)=-1
$$

We now consider the specific tetrahedron of class (c. $1^{+}$) with weights given by

$$
\begin{aligned}
& \nu_{1}=\omega_{6}, \\
& \nu_{2}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{2}\right), \\
& \nu_{3}=-\omega_{6}+4\left(\varepsilon_{4}+\varepsilon_{6}\right), \text { and } \\
& \nu_{4}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{4}+\varepsilon_{6}\right) .
\end{aligned}
$$

This is a permutation of the weights $\left\{\mu_{1}, \mu_{2}, \mu_{3}, \mu_{4}\right\}$ above. The corresponding ideals $L_{1}, L_{2}, L_{3}, L_{4}$ are shown in Figure 3.12. Note that $L_{1}$, the ideal containing the entire heap, cannot cross any other ideals, so it only makes sense to count crossings among $L_{2}, L_{3}$, and $L_{4}$. From Figure 3.12 we calculate that $\sum P\left(L_{2}, L_{3}\right)=4, \sum P\left(L_{2}, L_{4}\right)=4$, and $\sum P\left(L_{3}, L_{4}\right)=-2$. It follows from Proposition 3.16 that $L_{3}$ and $L_{4}$ cross, while $L_{2}$ and $L_{3}$ do not cross, and $L_{2}$ and $L_{4}$ do not cross. Thus we have $C=1$ for this particular tetrahedron, and the equality $q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=(-1)^{C}$ is satisfied here.

Now we let $i, j, k$, and $\ell$ be distinct indices between 1 and 6 , and let $w=s_{i_{1}} s_{i_{2}} \cdots s_{i_{n}}$ be a word in the generators $s_{1}, \ldots, s_{5}$ that moves 1 to $i, 2$ to $j, 3$ to $k$, and 4 to $\ell$ via the usual
permutations. The result now follows by Lemma 3.32.
A similar argument proves the result for tetrahedra of class (c. $1^{-}$), where $\lambda_{1}=-\omega_{6}$.

Figure 3.11: The heap ideals $L_{1}, L_{2}, L_{3}, L_{4}$ below correspond to weights $e_{\alpha}\left(\mu_{1}\right)=\omega_{6}-4\left(\varepsilon_{3}+\varepsilon_{4}\right)$, $f_{\alpha}\left(\mu_{2}\right)=\omega_{6}-4\left(\varepsilon_{5}+\varepsilon_{6}\right), \mu_{3}=-\omega_{6}+4\left(\varepsilon_{4}+\varepsilon_{6}\right)$, and $\mu_{4}=-\omega_{6}+4\left(\varepsilon_{3}+\varepsilon_{5}\right)$, respectively. These are the weights from equations (3.7).


Figure 3.12: The heap ideals $L_{1}, L_{2}, L_{3}, L_{4}$ below correspond to the 4-tuple of weights $\lambda_{1}=\omega_{6}$, $\lambda_{2}=-\omega_{6}+4\left(\varepsilon_{1}+\varepsilon_{2}\right), \lambda_{3}=-\omega_{6}+4\left(\varepsilon_{4}+\varepsilon_{6}\right)$, and $\lambda_{4}=\omega_{6}-4\left(\varepsilon_{1}+\varepsilon_{2}+\varepsilon_{4}+\varepsilon_{6}\right)$, respectively.


Theorem 3.34. Let $b_{1}, b_{2}, b_{3}, b_{4}$ be weight vectors corresponding to weights $\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}$ for the spin representation for $\mathfrak{D}_{6}$ of highest weight $\omega_{6}$. Let $q$ be a symmetric invariant quartic form on this representation, normalized so that

$$
q\left(b \otimes b \otimes b^{\prime} \otimes b^{\prime}\right)=1
$$

whenever $b$ and $b^{\prime}$ are weight vectors corresponding to weights $\lambda$ and $-\lambda$ respectively. If $\lambda_{1}+\lambda_{2}+$ $\lambda_{3}+\lambda_{4} \neq 0$, then $q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=0$. If $\lambda_{1}+\lambda_{2}+\lambda_{3}+\lambda_{4}=0$, then precisely one of the following holds.
(a) If $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}=\{\lambda, \lambda,-\lambda,-\lambda\}$ for some weight $\lambda$, then $q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=1$.
(b) If $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}=\{\lambda,-\lambda, \mu,-\mu\}$ for some weights $\lambda$ and $\mu$ with $\lambda \neq \pm \mu$, where $\lambda$ and $\mu$ are weights of the same sign, let $L_{1}$ and $L_{2}$ be the heap ideals corresponding to $\lambda$ and $\mu$, respectively. In addition, let $D=1$ if $\lambda$ and $\mu$ are incident, and let $D=0$ if $\lambda$ and $\mu$ are skew. Then

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=\frac{(-1)^{S+D}}{2}
$$

where $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$.
(c)(c.1) If $\lambda_{i} \neq \pm \lambda_{j}$ for $i \neq j$, and $\lambda_{1}= \pm \omega_{6}$, let $L_{2}, L_{3}, L_{4}$ be the heap ideals corresponding to $\lambda_{2}, \lambda_{3}$, and $\lambda_{4}$. Then

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=(-1)^{C}
$$

where $C$ is equal to the number of crossings between $L_{2}, L_{3}$, and $L_{4}$.
(c.2) If $\lambda_{i} \neq \pm \lambda_{j}$ for $i \neq j$, and none of the weights is equal to $\pm \omega_{6}$, write $L_{1}, L_{2}, L_{3}, L_{4}$ for the corresponding heap ideals, where, by rearranging if necessary, $L_{1}$ and $L_{2}$ have the same sign. Then

$$
q\left(b_{1} \otimes b_{2} \otimes b_{3} \otimes b_{4}\right)=(-1)^{C+S+1}
$$

where $C$ is the number of crossings in $\left\{L_{1}, L_{2}, L_{3}, L_{4}\right\}$, and $S=\left|\chi\left(L_{1}, L_{2}\right)\right|_{\#}$.

Proof. We first prove that these conditions are exclusive and exhaustive. We classify 4 -tuples of weights that sum to 0 based on the number of distinct opposite pairs that appear: if one opposite pair is repeated, we are in class (a); if two distinct opposite pairs appear, we are in class (b); and if there are no opposite pairs of weights, then we are in class (c).

In class (b), where $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right\}=\{\lambda,-\lambda, \mu,-\mu\}$ for some weights $\lambda$ and $\mu$, where $\lambda \neq \pm \mu$, we refer to [6, Lemma 9.3.3], which shows that in this case, $\mu$ and $\lambda$ are skew and $\mu$ and $-\lambda$ are incident, or vice versa. As a weight cannot have the same sign as its negative, if $\lambda$ and $\mu$ have the same sign and are incident (respectively, skew), it follows that $-\lambda$ and $-\mu$ have the same sign and are also incident (respectively, skew). This proves that $D$ is well-defined on 4 -tuples.

In class (c), where $\lambda_{i} \neq \pm \lambda_{j}$ for $i \neq j$, it follows from Lemma 3.12 that all four weights are mutually incident. The classification of these situations is given in Lemma 3.14.

Now, the first assertion is immediate from Lemma 3.8. Part (a) follows from Lemma 3.23. For part (b), the case $D=1, \lambda= \pm \omega_{6}$ is from Lemma 3.24 and Corollary 3.25 ; the case $D=$ $1, \lambda, \mu \neq \pm \omega_{6}$ is from Lemma 3.28, and the case $D=0$ is from Lemma 3.27. For part (c), (c.1) is from Lemma 3.33 and (c.2) is from Lemma 3.31.

### 3.4 Further Directions

The natural extension of the work in this thesis would be to try to use the definitions of relative content, profile, and crossings to describe symmetric invariant quartic forms on spin modules for the Lie algebras $\mathfrak{d}_{n}$ for larger values of $n$. There are several obstacles to this process which may be overcome in the future. First, the potential shapes of the various 4 -tuples of weights become much more complicated. There are many possible distances between weights, and each distance causes a pair of weights to behave in a slightly different way under the action of the Weyl group. Second, the process of finding relations between values of $q$ on the various types of 4-tuples becomes much more difficult. It appears that the results of Section 3.1 on necessary conditions for a symmetric invariant quartic form no longer suffice to relate all possible values of $q$ in the way we have done here. From our current vantage point, it appears that proving general results about the relative content and profile may yield a more efficient strategy for describing invariant forms on larger representations.

It is possible to modify our definition of crossings to work for minuscule representations in types $E_{7}$ and $E_{6}$. Moreover, there should be an easy rule on indices that describes the restriction of our symmetric invariant quartic form to the 20-dimensional minuscule module $L\left(A_{5}, \omega_{3}\right)$. As the reader may have deduced from the proofs about the profile, it seems that the profile is most at home in the minuscule modules for type $A$ Lie algebras. The nice properties of profile and crossings hold within any of these rectangular heaps, in particular, in the square heaps for the minuscule modules $L\left(A_{2 n-1}, \omega_{n}\right)$, where nontrivial 4 -tuples of weights summing to zero do occur. It seems that our statistics have the potential to be used fruitfully in this context.

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