A Nonstandard Approach to Keisler's Order

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We use the methods of model-theoretic nonstandard analysis, in particular enlargements, to study the properties of regular and good ultrafilters and their role within Keisler's order on countable complete first-order theories. This understanding is used to produce alternative proofs of several key theorems in the study of Keisler's order, such as the well-definedness of Keisler's order and the maximality of theories with SOP_2 (Strict Order Property 2). Furthermore, we provide an analysis of the logical structure of the types used in the proof that theories with SOP_2 are Keisler maximal and use this analysis to give an easy-to-state set-theoretic characterization of ultrafilters that are both regular and good.

Contents

Chapter 1

Nonstandard Analysis and Keisler's Order

The goal of this thesis is largely to bring the methods of nonstandard analysis to the study of Keisler's order. In particular, in Chapter 2, we give the background necessary to understand the model theoretic nonstandard analysis that we will be using, as well as give new proofs of foundational results in the study of Keisler's order using these methods. In the process of doing so, we develop methods of applying nonstandard analysis to model theory itself in Section 2.6. In Chapter 3, we refine our attention to SOP_2 (Strict Order Property 2), a classification theoretic notion that is believed to be important in Keisler's order. In particular, we study the possible form of distributions for an important class of types occurring in theories with SOP_2 and use this study to characterize good ultrafilters.

In the rest of this chapter, we will give some background information about Keisler's order and briefly describe why we should expect the perspective of nonstandard analysis to be useful to the study of Keisler's order.

1.1 Ultraproducts and Model Theory

Ultrafilters and ultraproducts have over time become incredibly important and useful objects in the study of model theory. Although the ultraproduct construction had been around for many years beforehand, the prominence of ultraproducts within model theory arguably started with Łoś's theorem (sometimes called the fundamental theorem of ultraproducts) and the understanding that ultraproducts could be used to give a succinct proof of the compactness theorem for first-order logic, one of the most fundamental and powerful results known about first-order model theory. The Keisler–Shelah isomorphism theorem (proved in the context of GCH in [Keisler(1961)] and in ZFC in [Shelah(1971)]), that two first-order structures in a common language are elementary equivalent if and only if they have isomorphic ultrapowers, further solidified the position of ultraproducts and implied that elementary classes can be characterized entirely in terms of ultraproducts and isomorphisms.

As we will be spending a considerable amount of time talking about structures, ultrafilters, ultraproducts, and ultrapowers, we record some conventions that we will be following throughout. We note that it will be assumed that the reader is familiar with the basic concepts of model theory as one might expect to find in an introductory graduate course on the topic (e.g. the material covered in [Chang and Keisler(2012)] or [Marker(2002)]), and we will also expect that the reader is familiar with the definitions of ultrafilters and the ultraproduct construction.

Convention 1.1.1. Let α be a cardinal and β a variable representing an ordinal.

(1) We will use I for a set over which $\mathcal{U}$ is an ultrafilter. These two symbols will be reserved exclusively for this purpose.

(2) $\mathcal{L}$ will be used to denote a language consisting of constants and finitary relations and functions.

(3) We will reserve the use of calligraphic capital roman letters (other than $\mathcal{U}$ and $\mathcal{L}$) for structures in a language $\mathcal{L}$. We will write $\mathcal{M} = (M, S_0, \ldots, S_\alpha)$ where M (in general, the non-calligraphic capital letter representing the structure) is the underlying set of $\mathcal{M}$ and the $(S_\beta)_{\beta \leq \alpha}$ enumerate the interpretations of the symbols of $\mathcal{L}$ in $\mathcal{M}$. Alternatively, if S is a symbol of $\mathcal{L}$, then $S^{\mathcal{M}}$ is the interpretation of S in $\mathcal{M}$.

(4) When $(\mathcal{M}_j)_{j \in I}$ is a sequence of structures in the common language $\mathcal{L}$, we will write $\prod_{j \in \mathcal{U}} \mathcal{M}_j$ for the ultraproduct of the $\mathcal{M}_j$ by the ultrafilter $\mathcal{U}$. Note that the notation we use here is somewhat metaphorical in the sense that $j \in \mathcal{U}$ does not literally make

sense, but is meant to represent the fact that we only care about sets of indices occuring in $\mathcal{U}$ in an ultraproduct.

(5) For a structure $\mathcal{M}$ and ultrafilter $\mathcal{U}$, we will write $\mathcal{M}^{\mathcal{U}}$ for the ultraproduct $\prod_{j\in\mathcal{U}} \mathcal{M}$ and call $\mathcal{M}^{\mathcal{U}}$ the ultrapower of $\mathcal{M}$ by $\mathcal{U}$.

1.1.1 Keisler's Order

During the initial flurry of activity around ultrafilters, Keisler [Keisler(1967)] realized that the level of saturation that occurs in an ultrapower $\mathcal{M}^{\mathcal{U}}$ is related to structural properties of the ultrafilter $\mathcal{U}$ as well as some measure of the combinatorial complexity of types occurring in $\mathcal{M}$. To see where this observation is coming from, it is useful to think about the proof of the compactness theorem using ultraproducts.

Convention 1.1.2. Given a set A, we will use $\mathcal{P}_\omega(A)$ to denote the finite powerset of A. That is,

$$\mathcal{P}_\omega(A) := \{B \in \mathcal{P}(A) : |B| < \omega\}.$$

Given a cardinal κ, we will use the notation $[A]^\kappa$ common in Ramsey theory for the κ-element subsets of A. That is,

$$[A]^\kappa := \{B \in \mathcal{P}(A) : |B| = \kappa\}.$$

Theorem 1.1.3 (Compactness for First-Order Logic)**.** If Φ is collection of sentences in the language $\mathcal{L}$, then Φ has a model if and only if every $\Delta \in \mathcal{P}_\omega(\Phi)$ has a model.

The proof proceeds by creating an ultrafilter $\mathcal{U}$ on the indexing set $I = \mathcal{P}_\omega(\Phi)$. For each $\Delta \in \mathcal{P}_\omega(\Phi)$ define $\Delta_\uparrow = \{\Gamma \in \mathcal{P}_\omega(\Phi) : \Delta \subseteq \Gamma\}$ and choose $\mathcal{U}$ so that

$$\{\Delta_\uparrow : \Delta \in \mathcal{P}_\omega(\Phi)\} \subseteq \mathcal{U}.$$

Now, if $(\mathcal{M}_\Delta)_{\Delta\in\mathcal{P}_\omega(\Phi)}$ is chosen so that $\mathcal{M}_\Delta \vDash \bigwedge \Delta$, Łoś's theorem can be used to show that $\prod_{\Delta\in\mathcal{U}} \mathcal{M}_\Delta$ satisfies Φ.

Suppose that we change the context of the compactness theorem slightly. Instead of taking Φ to be sentences from $\mathcal{L}$, we take $\Phi(x)$ to be a 1-type (not necessarily complete) of some fixed structure $\mathcal{M}$. If $\mathcal{U}$ is constructed analogously to the proof of compactness, must $\widehat{\mathcal{M}} := \mathcal{M}^{\mathcal{U}}$ realize $\Phi(x)$? The answer is yes; if for each $\Delta \in \mathcal{P}_\omega(\Phi(x))$ we take m_Δ to be an element of M so that $\mathcal{M} \models \bigwedge_{\varphi(x)\in\Delta} \varphi(m_\Delta)$, then the same argument as before using Łoś's theorem shows that the equivalence class of the function $f\colon I \to M$ defined by $\Delta \mapsto m_\Delta$ in $\widehat{M}$ realizes $\Phi(x)$.

In these examples, we might want to know how the chosen ultrafilters $\mathcal{U}$ would change if we change the set of formulas Φ. Although the elements of the indexing set might change, the set theoretic structure of the ultrafilter constructed depends only on the cardinality of Φ or $\Phi(x)$. This understanding of $\mathcal{U}$ allows us to see that not only does $\widehat{\mathcal{M}}$ realize $\Phi(x)$, but also any type $\Psi(x)$ where the parameters of $\Psi(x)$ come from $\mathcal{M}$ and where $|\Psi(x)| = |\Phi(x)|$.

We can further extend the question to whether $\widehat{\mathcal{M}}$ realizes all types $\Psi(x)$ where $|\Psi(x)| \leq |\Phi(x)|$. This turns out to be a much more complicated question, depending both on further structure of the ultrafilter $\mathcal{U}$ chosen as well as on the combinatorial complexity of the types that occur in $\widehat{\mathcal{M}}$, and leads to the study of Keisler's order and regular ultrafilters.

Definition 1.1.4 (Regular Ultrafilter)**.** An ultrafilter $\mathcal{U}$ on an infinite set I is called **regular** if there exists a collection $\mathbb{X} \subseteq \mathcal{U}$ such that

(1) $|\mathbb{X}| = |I|$ and

(2) for every $\mathbb{Y} \in (\mathcal{P}(\mathbb{X}) \setminus \mathcal{P}_\omega(\mathbb{X}))$ the set $\bigcap \mathbb{Y}$ is empty.

We call such an $\mathbb{X}$ a **regularizing set for** $\mathcal{U}$.

When the collection of sentences Φ is infinite, the ultrafilter $\mathcal{U}$ constructed in the proof of the compactness theorem is a regular ultrafilter as the collection $\mathbb{X} = \{\Delta_\uparrow : \Delta \in \mathcal{P}_\omega(\Phi)\}$ is a regularizing set for $\mathcal{U}$. Furthermore, given any regular ultrafilter $\mathcal{U}$ and an enumeration of a regularizing set $\mathbb{X} = (X_j)_{j\in I}$, we can construct a function $I \to \mathcal{P}_\omega(I)$ by $k \mapsto \{j : k \in X_j\}$. In the context of the ultrafilters used in the proof of the compactness theorem, this function is $\Delta \mapsto \mathcal{P}(\Delta)$, where the image of Δ is the collection of formulas the model with index Δ is guaranteed to satisfy.

The construction of $\mathcal{U}$ used in the proof of the compactness theorem also shows that regular ultrafilters exist for any infinite indexing set I. Because $|\mathcal{P}_\omega(I)| = |I|$, we can change a regular ultrafilter on $\mathcal{P}_\omega(I)$ to a regular ultrafilter on I by using any bijection between $\mathcal{P}_\omega(I)$ and I.

Regular ultrafilters have an interesting relationship with finiteness. It is implicit in much of the work on Keisler's order, in particular in the existence of distributions (see Section 3.2.2 for an extended discussion of distributions), that if $\mathcal{U}$ is a regular ultrafilter on I and $(M_j)_{j\in I}$, then any $A \subseteq \prod_{j\in\mathcal{U}} M_j$ is "hyperfinite" in the sense that there are finite $F_j \subseteq M_j$ such that the induced inclusion $\prod_{j\in\mathcal{U}} F_j \to \prod_{j\in\mathcal{U}} M_j$ has A in its image. Furthermore, this property characterizes regular ultrafilters, as can be seen for example in the language of nonstandard analysis and hyperprincipal generators in Corollary 2.5.9.

The result in Theorem 1.1.6, due to [Keisler(1967)], shows that whether all "small enough" types are realized in an ultrapower of a structure $\mathcal{M}$ by a regular ultrafilter is a function of the first-order theory of $\mathcal{M}$ and does not depend on the model chosen.

Definition 1.1.5. Suppose that $\mathcal{M}$ is a structure in the language $\mathcal{L}$ and $\mathcal{U}$ is an ultrafilter on the set I. If $\mathcal{M}^{\mathcal{U}}$ is $|I|^+$-saturated, then we say that $\mathcal{U}$ saturates $\mathcal{M}$.

Theorem 1.1.6 ([Keisler(1967)])**.** If $\mathcal{U}$ is a regular ultrafilter, $\mathcal{M}$ and $\mathcal{N}$ are structures in a common countable language, and $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{U}$ saturates $\mathcal{M}$ if and only if $\mathcal{U}$ saturates $\mathcal{N}$.

Based on Theorem 1.1.6, there is a (pre)order on the collection of countable complete first-order theories.

Definition 1.1.7 (Keisler's Order [Keisler(1967)])**. Keisler's Order** is a pre-ordering on the collection of first-order countable complete theories defined by $T_1 \leq T_2$ if and only if every regular ultrafilter $\mathcal{U}$ that saturates models of T_2 also saturates models of T_1. If T is a countable complete first-order theory, we will use $[T]$ to mean the equivalence class of T determined by the pre-order $\leq$ and will call $[T]$ the **Keisler class** of T.

The key open problems in Keisler's order are currently to determine the structure of Keisler's order and to determine model theoretic characterizations of the different classes of theories within

Keisler's order. We give a brief description of some of the currently known details of Keisler's order. Some of these descriptions will use terminology from classification theory, much of which we will not define here.

(1) The stable theories split into exactly two Keisler classes that are comparable. These two classes are strictly below the Keisler class of any unstable theory. That is, the two classes of stable theories are the two smallest Keisler classes [Shelah(1990)].

(2) The existence of a minimum class within Keisler's order is originally due to [Keisler(1967)]. The minimum Keisler class is the class of those theories with the NFCP (Not Finite Cover Property) [Shelah(1990)].

(3) The second smallest Keisler class is the collection of stable theories having the FCP (Finite Cover Property) [Shelah(1996)].

(4) The next largest class is the class of the theory of the random graph [Malliaris(2012)].

(5) Keisler's order is not well-founded [Malliaris and Shelah(2018)] and has a continuum sized antichain [Malliaris and Shelah(2021)].

(6) There is a maximum class in Keisler's order [Keisler(1967)], including all theories with SOP_2 (Strict Order Property 2) [Malliaris and Shelah(2016)].

Of the points listed here, we will be most interested in (6). Although it is known that the maximum Keisler class exists, a model theoretic characterization of the class has not yet been identified, although it is conjectured in [Malliaris and Shelah(2016)] that SOP_2 characterizes the maximal class. For more information about SOP_2, including a definition and a description of types related to SOP_2, see Chapter 3.

However, there is a characterization due to Keisler (under the assumption of the generalized continuum hypothesis [Keisler(1967)]) and Kunen (in ZFC [Kunen(1972)]) of the types of ultrafilters that saturate models of theories within the maximal Keisler class. Such ultrafilters are called "good." In Chapter 2, we will further discuss the definition of good ultrafilters and develop characterizations of regular and good ultrafilters in the language of nonstandard analysis along the lines of [Di Nasso(2015), Section 11.4]. We will then show how these characterizations can be used to think about several aspects of Keisler's order, including giving an alternative proof of the fact that theories with SOP_2 are Keisler maximal. In Chapter 3, we give new set theoretic characterization of when an ultrafilter is good that is seemingly weaker than the original definition of good ultrafilters. In particular, the definition of good ultrafilters requires that all functions $\mathcal{P}_\omega(I) \to \mathcal{U}$ satisfying certain properties have multiplicative refinements, and our characterization is able to leverage the theorem of Malliaris and Shelah that theories with SOP_2 are Keisler maximal to considerably shrink the class of functions needed to be considered in order to guarantee that an ultrafilter is good.

1.2 Nonstandard Analysis and Model Theory

The field of nonstandard analysis was started by Robinson also during the initial period of intense research on the property of ultrafilters. Initially, nonstandard analysis was focused on Robinson's hyperreal numbers; usually defined as an ultrapower of the usual real numbers designed to have infinitesimals, that is, a realization of the type $\{x \neq 0\} \cup \{-1/n < x < 1/n : n \in \omega \setminus \{0\}\}$. Of course, if we try to realize this type by using an ultrapower as we did in the discussion of regular ultrapowers above we will succeed, but we will also realize any other countable type occurring in the real numbers $\mathbb{R}$. This gives a convenient way of working in a more saturated structure that still retains many of the properties of the original structure by Łoś's theorem (nonstandard analysts like to call this "transfer"). Eventually, this leads to the idea of using similar regular ultrapowers to study other structures where it is useful to approximate properties that are never realized in the base structure, similar to how infinitesimals are treated in the hyperreal numbers.

We do not assume that the reader is familiar with the methods of nonstandard analysis. A brief treatment of the basics of enlargements, along with many of the requisite proofs, is given in Sections 2.1 through 2.3. The interested reader may find similar treatments that are more fleshed out in [Chang and Keisler(2012), Chapter 4.4] and in [Goldblatt(1998), Section IV].

One place where the approach of nonstandard analysis can be used is with ultrafilters themselves. Given an ultrafilter $\mathcal{U}$ on the set I, there is a (partial) type $\Phi(x) = \{\text{“}x \in X\text{”} : X \in \mathcal{U}\}$ in the language of set theory that is not realized unless $\mathcal{U}$ is principal (Recall that an ultrafilter $\mathcal{U}$ is principal if and only if there is an $i \in I$ so that $\mathcal{U} = \{X \subseteq I : i \in X\}$). Therefore, if we take a structure in the language of set theory containing at least the elements of $\mathcal{U}$ and I and take a regular ultrapower, we will get a set theoretic structure logically similar to the one in which we started, but which also has a realization of the type $\Phi(x)$ regardless of whether $\mathcal{U}$ is principal! This approach to studying ultrafilters was first used in [Luxemburg(1969)] and described in modern terms in [Di Nasso(2015), Section 11.2].

As one might expect from the analogy with principal ultrafilters, realizations of $\Phi(x)$ within a regular ultrapower act in many ways like a principal generator for $\mathcal{U}$ (see e.g. [Di Nasso(2015), Section 11.2] or Lemma 2.4.2 in the current work). In particular, given a realization of $\Phi(x)$ and the regular ultrafilter used to create the ultrapower, one can recover the original ultrafilter $\mathcal{U}$! This means that all of the properties of $\mathcal{U}$ can be determined from the properties of the realizations of $\Phi(x)$, which we call hyperprincipal generators for $\mathcal{U}$. The main goal of Chapter 2 is to develop characterizations for the hyperprincipal generators of regular and good ultrafilters which are slight strengthenings of the characterizations given in Theorems 11.4.3 and 11.4.6 of [Di Nasso(2015)], and to use these characterizations to help understand the Keisler order. In so doing, we also develop some theory of nonstandard model theory in Section 2.6. Although we would be surprised to learn that nonstandard model theory has not been worked on before, we have been unable to find any references where such work has been done, and so develop the theory from the beginning.

The framing of Keisler's order in terms of nonstandard model theory is particularly enticing. For example, the characterization of regular ultrafilters in Corollary 2.5.9 allows us to think of types $\Psi(x)$ having cardinalities less than I occurring in a regular ultrapower as being a subset of a collection $\widehat{\Psi}(x)$ of hyperformulas that the ultrapower internally believes to be finite. If $\Psi(x)$ is realized, then we can force $\widehat{\Psi}(x)$ to be consistent. Using the apparent finiteness of $\widehat{\Psi}(x)$, we may transfer $\widehat{\Psi}(x)$ to a logically equivalent consistent internally finite collection of formulas in the ultrapower of any elementarily equivalent structure. In essence, Theorem 1.1.6 showing that Keisler's order is a pre-order can be thought of as a consequence of the fact that regular ultrapowers can make infinite sets appear finite and that elementary equivalent structures have the same sets of possible logical configurations of finite sets of formulas. For a more careful version of this argument, see Theorem 2.6.14.

Chapter 2

Enlargements and Hyperprincipal Generators for Ultrafilters

While the ultrapower construction is undeniably useful, it is easy to become caught up in the details and exact definition of ultrapowers and lose sight of the power and simplicity that ultrapowers can bring to a problem. The definition of an **enlargement** helps to address this problem by abstracting the useful properties of regular ultrapowers, focusing on the fact that regular ultrapowers realize relatively small types with parameters from the base model. The trade–off is that enlargements are less concrete than even ultrapowers, but fortunately much of the intuition that one can gain from working with regular ultrapowers carries over to the study of enlargements. This chapter will describe what enlargements are, list some useful properties of enlargements with some selected proofs, and describe how enlargements can be used to gain some insight into ultrafilters themselves.

We begin with some definitions needed to understand what an enlargement is and give a way to construct enlargements. For more in-depth coverage of these topics, [Chang and Keisler(2012), Section 4.4] or [Goldblatt(1998), Section IV] are recommended.

2.1 The Universe Over a Set

Let A be any set, which we will be thinking of as containing the underlying set for some mathematical structure that we wish to study as well as the indexing set that we wish to find an ultrafilter over.

Definition 2.1.1 (Universe Over A)**.** Let $\mathbb{U}_0(A) = A$ and recursively define

$$\mathbb{U}_{n+1}(A) := \mathbb{U}_n(A) \cup \mathcal{P}(\mathbb{U}_n(A)).$$

Then **the universe over** A, denoted $\mathbb{U}(A)$, is the set

$$\mathbb{U}(A) := \bigcup_{n\in\omega} \mathbb{U}_n(A).$$

This definition is very similar to the definition of V_ω in the von Neumann hierarchy of well-founded sets, and similarly can be extended to ordinals larger than ω. Taking this analogy a little further, $\mathbb{U}(A)$ can be thought of as being V_ω in a set theory where every $a \in A$ is **atomic** (contains no elements but is not $\emptyset$). This analogy is especially apt if for all $a, b \in A$ we have that $a \cap V_\omega = \emptyset$ where V_ω is taken in the ambient set theory satisfying ZFC, $\mathrm{rk}(a) = \mathrm{rk}(b)$ (where rank is taken in the ambient set theory), and also $a \cap b = \emptyset$, in which case we will say that A **is atomic in** $\mathbb{U}(A)$**. For the rest of this chapter, we will assume that the set** A **is atomic in** $\mathbb{U}(A)$**.**

Continuing the analogy to the von Neumann hierarchy, we can define a rank function for elements of $\mathbb{U}(A)$.

Definition 2.1.2 (Rank in $\mathbb{U}(A)$)**.** For every $u \in \mathbb{U}(A)$ we define the rank of u over A to be

$$\mathrm{rk}_A(u) := \min\{n \in \omega : u \in \mathbb{U}_n(A)\}.$$

Remarks 2.1.3. Some useful facts about $\mathbb{U}(A)$ are listed here:

(1) The set $\mathbb{U}(A)$ is strongly transitive, i.e. for any $B \in \mathbb{U}(A)$, there is a transitive set $T \in \mathbb{U}(A)$ so that $B \subseteq T \subseteq \mathbb{U}(A)$. In general T may be taken to be the transitive closure of B if the elements of A are thought of as atomic.

(2) $\mathbb{U}(A)$ has most (internal) set theoretic constructions. Beware iterated power sets and any construction using elements of unbounded rank!

(3) $\mathbb{U}(A)$ contains every finitary relation and operation on A (using the usual set theoretic definitions of relation and function), as well as all constants from A. This makes the

universe over A a viable place to study the model theory of structures with underlying set of cardinality $\leq |A|$.

In order to be able to talk about the types of structures with underlying set contained in A, we want to put a structure on $\mathbb{U}(A)$ that is able to talk about any structure that we can put on subsets of A.

Definition 2.1.4 (Universal Language $\mathcal{L}_A$)**.** The **universal language of** A, denoted $\mathcal{L}_A$, is the language consisting of the binary relation symbol $\in$ and a distinct constant symbol c_u for each $u \in \mathbb{U}(A)$.

The binary relation $\in$ and the constants c_u are interpreted in $\mathbb{U}(A)$ as the restriction of the usual $\in$ relation and as u respectively.

2.2 Nonstandard Frameworks and Enlargements

We will need some way of talking about the analog of the natural embedding of $\mathcal{M} \to \mathcal{M}^{\mathcal{U}}$ where $\mathcal{U}$ is an ultrafilter. To do this, we introduce the notion of a nonstandard framework which is sometimes also called an elementary universe embedding. First, we need some technical details that allow us to reasonably talk about maps between universes over different sets A and B, as currently the languages of the universal structures $\mathbb{U}(A)$ and $\mathbb{U}(B)$ will only match if $A = B$.

Convention 2.2.1. If $*\colon A \to B$ is a function between any two sets A and B, we will use ${}^*a$ for any $a \in A$ to mean $*(a)$ and use $*[D]$ to be $\{{}^*d : d \in D\}$ if $D \subseteq \mathbb{U}(A)$.

This convention will help us to avoid having an abundance of parentheses later on. We will only use this convention when the function is named $*$.

Definition 2.2.2 (Pushforward Structure ${}^*\mathbb{U}(A)$)**.** Let $\mathbb{U}(A)$ and $\mathbb{U}(B)$ be the universes over the sets A and B respectively. For any $\in$-homomorphism $*\colon \mathbb{U}(A) \to \mathbb{U}(B)$ define the $\mathcal{L}_A$-structure ${}^*\mathbb{U}(A)$ to have underlying set

$$\{u' \in \mathbb{U}(B) : \exists u \in \mathbb{U}(A),\, u' \in {}^*u\},$$

interpreting $\in$ as the restriction of the ambient relation $\in$, and interpreting c_u as ${}^*u$ for each $u \in \mathbb{U}(A)$.

Remark 2.2.3.

(1) If $u \in \mathbb{U}(A)$, then ${}^*u$ is an element of ${}^*\mathbb{U}(A)$. One way to see this is that ${}^*u \in {}^*\{u\}$ by $*$ being an $\in$-homomorphism.

(2) The notation ${}^*\mathbb{U}(A)$ is different from ${}^*D$ where $D \in \mathbb{U}(A)$, although the definition of ${}^*\mathbb{U}(A)$ mirrors the definition of $*$ that will be given in Theorem 2.3.6.

The structure ${}^*\mathbb{U}(A)$ can be thought of as a reduct of the substructure of $\mathbb{U}(B)$ with underlying set

$$C = \{u' : \exists u \in \mathbb{U}(A),\, u' \in {}^*u\}$$

to the language consisting of the symbols

$$\{\in\} \cup \{c_{u'} : u' \in C\}.$$

Further note that this definition guarantees that

$${}^*c_u^{\mathbb{U}(A)} = c_u^{{}^*\mathbb{U}(A)}$$

no matter what the map $*$ is. However, $*$ may not be an $\in$-homomorphism.

It is worth pointing out that the set C defined above is often referred to as the set of **internal elements** of $\mathbb{U}(B)$ and $\mathbb{U}(B)\setminus C$ is referred to as the set of **external elements** of $\mathbb{U}(B)$. Of course, which elements are internal or external depends on the given map $*$, but $*$ will usually be clear from context.

Although nonstandard frameworks want to capture the idea of an elementary extension, nonstandard frameworks will only preserve a certain fragment of the first-order theory. In practice, the fragment preserved by a nonstandard framework is enough to capture the relationship between a structure and its ultrapower. This fragment is defined by the following property:

Definition 2.2.4 (Bounded Quantifier Formulas)**.** A formula $\varphi(x, \bar{y})$ in a language $\mathcal{L}$ is said to have bounded quantifiers if whenever $\forall x$ or $\exists x$ appears in the formula, the scope of the quantifier has the form

$$x \in t(\bar{y}) \to \psi(x, \bar{y}) \quad \text{or} \quad x \in t(\bar{y}) \wedge \psi(x, \bar{y})$$

respectively where t is some term of the language and ψ is a bounded quantifier formula in the same language as φ.

Definition 2.2.5 (Nonstandard Framework)**.** Let A, B be sets and $*\colon \mathbb{U}(A) \to \mathbb{U}(B)$ so that the restriction $*\colon \mathbb{U}(A) \to {}^*\mathbb{U}(A)$ has the following properties:

(1) $(* \upharpoonright A)$ is an injective map $A \hookrightarrow B$.

(2) ${}^*\emptyset = \emptyset$.

(3) (Transfer) For every bounded quantifier sentence φ in the language $\mathcal{L}_A$ we have that

$$(\mathbb{U}(A) \vDash \varphi) \iff ({}^*\mathbb{U}(A) \vDash \varphi).$$

Then we say that $*$ is a nonstandard framework for $\mathbb{U}(A)$.

The transfer property will be our analog of Łoś's Theorem, and the first two conditions require that $*$ is well-behaved on the "atomic" elements of $\mathbb{U}(A)$. This well-behavedness is guaranteed to extend to the rest of $\mathbb{U}(A)$ via induction and the transfer principle.

Nonstandard frameworks can easily be shown to exist: if $A = B$, then the identity map $\mathrm{id}\colon \mathbb{U}(A) \to \mathbb{U}(A)$ is a nonstandard framework. However, such trivial examples of a nonstandard framework are not particularly useful, in a similar way to the fact that taking an ultrapower using a principal ultrafilter does not yield interesting results. In order to exclude the possibility that the nonstandard framework is simply mirroring our original universe, we need to add in a property that will guarantee that ${}^*\mathbb{U}(A)$ will realize more types than the original universe $\mathbb{U}(A)$.

Definition 2.2.6 (Finite Intersection Property [FIP])**.** A set D is said to have the **finite intersection property or FIP** if for every finite subset $\Gamma \subseteq D$ the intersection $\bigcap \Gamma$ is non-empty.

Definition 2.2.7 (Enlargements)**.** Let $*\colon \mathbb{U}(A) \to \mathbb{U}(B)$ be a nonstandard framework. We say that $*$ (or ${}^*\mathbb{U}(A)$) is an **enlargement of** $\mathbb{U}(A)$ if for every $D \in \mathbb{U}(A)$ that is a collection of non-atomic elements with the finite intersection property there is some

$$b \in \bigcap_{d \in D} {}^*d,$$

Remarks 2.2.8. We note that:

(1) $b \in {}^*\mathbb{U}(A)$, as $b \in {}^*d$ for every $d \in D \subseteq \mathbb{U}(A)$ and hence $b \in {}^*\mathbb{U}(A)$.

(2) If $\mathcal{A}$ is a first-order structure with underlying set A and $\Phi(x)$ is a type over $\mathcal{A}$, then the collection $\{\varphi(\mathcal{A}) : \varphi(x) \in \Phi(x)\}$ where $\varphi(\mathcal{A}) = \{a \in A : \mathcal{A} \vDash \varphi(a)\}$ has the FIP, and so the intersection

$$\bigcap_{\varphi(x) \in \Phi(x)} {}^*\varphi(A)$$

is inhabited in ${}^*A$, that is an element of ${}^*A$ is in the intersection. Transfer can be used to show that any element of this intersection satisfies all of the formulas in $\Phi(x)$.

2.3 Constructing an Enlargement

The construction of an enlargement passes through the construction of a bounded ultrapower, so we start with the bounded ultrapower construction.

Definition 2.3.1 (Bounded Ultrapowers)**.** Let A be a set, $\mathbb{U}(A)$ the universe over A, and $\mathcal{U}$ a non-principal ultrafilter over the cardinal α. Then the **bounded ultrapower of** $\mathbb{U}(A)$ **with respect to** $\mathcal{U}$ is the collection of elements $[f]_{\mathcal{U}}$ from the ultrapower $(\mathbb{U}(A))^{\alpha}/\mathcal{U}$ so that there is some $g : \alpha \to \mathbb{U}(A)$ with $g \in [f]_{\mathcal{U}}$ so that $\{\mathrm{rk}_A(g(\beta)) : \beta \in \alpha\}$ is bounded in ω. Equivalently, $[f]_{\mathcal{U}}$ is in the bounded ultrapower if there exists some $n \in \omega$ so that $\{\beta \in \alpha : \mathrm{rk}_A(f(\beta)) = n\} \in \mathcal{U}$. When this is the case, we will say that $\mathrm{rk}_A([f]_{\mathcal{U}}) = n$. We denote the underlying set of the bounded ultrapower as $\mathcal{B}(A)$.

There is nothing in the definition of bounded ultrapower that requires that the base structures be identical in every index, so this definition can be generalized to a definition of bounded ultraproduct, but we will not need this level of generality.

Definition 2.3.2 (Diagonal Embedding)**.** Let A be a set, $\mathbb{U}(A)$ the universe over A, and $\mathcal{B}(A)$ the bounded ultrapower of $\mathbb{U}(A)$ with respect to the non-principal ultrafilter $\mathcal{U}$ over the cardinal α. Then the **diagonal embedding** of $\mathbb{U}(A)$ in $\mathcal{B}(A)$, denoted Δ, is the map $\Delta\colon \mathbb{U}(A) \to \mathcal{B}(A)$ defined by $u \mapsto [\beta \mapsto u]_{\mathcal{U}}$.

Definition 2.3.3 (Collapse Map, C)**.** Let A be any set and $\mathcal{B}(A)$ the bounded ultrapower of $\mathbb{U}(A)$ using the ultrafilter $\mathcal{U}$ over the cardinal α. Denote the elements of $\mathcal{B}(A)$ of rank n over A by $\mathcal{B}_n(A)$. Then the **collapse map**

$$C : \mathcal{B}(A) \to \mathbb{U}(\mathcal{B}_0(A))$$

is defined recursively so that $C \restriction \mathcal{B}_0(A) = \mathrm{id}_{\mathcal{B}_0(A)}$ and $C(b) = \{C(b') : b' \in b\}$ for $b \in \mathcal{B}_n(A)$ with $n > 0$.

Remark 2.3.4. This map is called the collapse map because it is an analog of the Mostowski Collapse to a set theory with atoms other than $\emptyset$. The following lemma makes the connection to the Mostowski Collapse a little more clear. In fact, Keisler has called a version of this theorem and lemma the Mostowski Collapse Theorem (see [Keisler(2010), Theorem 7.3]).

Lemma 2.3.5. If $\mathcal{B}(A)$ is considered as a substructure of

$$(\mathbb{U}(A); \in, \{c_u : u \in \mathbb{U}(A)\})^{\mathcal{U}}$$

and we interpret $\in$ in $\mathbb{U}(\mathcal{B}_0(A))$ as the restriction of the usual $\in$ and c_u as $C(\Delta(u)) = C\left(c_u^{\mathcal{B}(A)}\right)$ for all $u \in \mathbb{U}(A)$, then the collapse map C is an isomorphism of $\mathcal{B}(A)$ onto its image.

Proof. That C is a (strong) homomorphism of structures is essentially by definition and the atomicity of A in $\mathbb{U}(A)$. As C is a strong homomorphism, we need only show that C is injective to be an isomorphism onto its image. That C restricted to $\mathcal{B}_0(A)$ is injective follows from the definition of C. We also get that

$$C([\beta \mapsto \emptyset]_{\mathcal{U}}) = C\left(c_{\emptyset}^{\mathcal{B}_0(A)}\right) = \emptyset$$

from the definition of C. For elements non-atomic in $\mathcal{B}(A)$, we have that

$$C(b) = C(\widetilde{b}) \iff \{C(b') : b' \in b\} = \{C(b') : b' \in \widetilde{b}\}$$

implying that $\mathrm{rk}_A(b) = \mathrm{rk}_A(\widetilde{b})$, and so induction on rank shows that C is injective. This can be thought of as a consequence of the fact that extensionality still holds internally in both $\mathcal{B}(A)$ and $\mathbb{U}(\mathcal{B}_0(A))$ when restricted to non-atomic elements. □

Theorem 2.3.6 (Existence of Enlargements)**.** Let A be any set and $\mathcal{B}(A)$ the bounded ultrapower of $\mathbb{U}(A)$ using the non-principal ultrafilter $\mathcal{U}$ over the cardinal α. Then the composition

$$*\colon \mathbb{U}(A) \xrightarrow{\Delta} \mathcal{B}(A) \xrightarrow{C} \mathbb{U}(\mathcal{B}_0(A))$$

is a nonstandard framework. Moreover, if $\alpha \geq |\mathbb{U}(A)|$ and $\mathcal{U}$ is regular, then $*$ is an enlargement of $\mathbb{U}(A)$.

Proof. We note that both Δ and C are injective functions and that ${}^*a = C(\Delta(a)) \in \mathcal{B}_0(A)$ for all $a \in A$ (as $[\beta \mapsto a]_{\mathcal{U}}]$ has rank 0 over A) so $(* \restriction A)$ is an injective map to $\mathcal{B}_0(A)$ as desired. Furthermore,

$${}^*\emptyset = C(\Delta(\emptyset)) = C\left(c_{\emptyset}^{\mathcal{B}_0(A)}\right) = \emptyset.$$

It remains to check that transfer holds to show that $*$ is a nonstandard framework. We claim that ${}^*\mathbb{U}(A)$ is the range of C.

Suppose that $[a_\beta]_{\mathcal{U}} \in \mathcal{B}_0(A)$ is an element of $\mathcal{B}(A)$ and $(a_\beta)_{\beta\in\alpha}$ has all of its elements of rank at most n. Then the set $D := \{a_\beta : \beta \in \alpha\}$ is in $\mathbb{U}(A)$ as D has rank at most $n + 1$. By Łoś's Theorem we have that $[a_\beta]_{\mathcal{U}} \in^{\mathcal{B}(A)} \Delta(D)$. As C is an isomorphism, we have that

$$C([a_\beta]_{\beta\in\alpha}) \in C(\Delta(D)) = {}^*D$$

and hence $C[\mathcal{B}(A)] \subseteq {}^*\mathbb{U}(A)$. For the other inclusion, suppose that $D \in {}^*\mathbb{U}(A)$. That is, $D \in {}^*B$ for some $B \in \mathbb{U}(A)$. By definition of $*$ and C, we have that

$$D \in {}^*B = C(\Delta(B)) = \{C(b) : b \in^{\mathcal{B}(A)} \Delta(B)\}$$

and hence ${}^*\mathbb{U}(A) \subseteq C[\mathcal{B}(A)]$.

As C is an isomorphism, we need only check that Δ preserves bounded sentences in the language $\mathcal{L}_A$. Suppose that φ is a bounded sentence in prenex normal form (with the bounds attached to the quantifiers). That is φ is either

$$\forall x, (x \in v \to \psi(x)) \quad \text{or} \quad \exists x, (x \in v \wedge \psi(x))$$

where v is a constant in $\mathbb{U}(A)$. In the first case, any counterexample to φ must have rank bounded by $\mathrm{rk}_A(v)$ and in the second example a witness must have rank bounded by $\mathrm{rk}_A(v)$. Thus, if a counterexample to φ exists in the full ultrapower, it must also exist in the bounded ultrapower. Similarly, if a witness to φ exists in the full ultrapower, it must also exist in the bounded ultrapower. As the full ultrapower has a counterexample or witness to φ if and only if $\mathbb{U}(A)$ does and C is an isomorphism onto its image, we get that $* : \mathbb{U}(A) \to {}^*\mathbb{U}(A)$ preserves and reflects bounded sentences.

For the enlargement part of the theorem, we note that if D is a collection with the FIP in $\mathbb{U}(A)$ then $\bigcap_{d \in D} \Delta(d)$ is inhabited in the ultraproduct (an element of this intersection is a realization of a type with at most $|\mathbb{U}(A)|$ parameters, all of which come from the base structure, and all such types are realized in a regular ultrapower with index set $\mathbb{U}(A)$). Pick an element of this intersection and call it b, noting that $\mathrm{rk}_A(b) < \mathrm{rk}_A(d)$ for all $d \in D$, and so $b \in \mathcal{B}(A)$. As C is an isomorphism onto its image, we have that

$$C(b) \in \bigcap_{d \in D} C(\Delta(d)) = \bigcap_{d \in D} {}^*d$$

as desired. □

Remark 2.3.7. Although the definition of a bounded ultrapower can be formulated in a way to make sense for a universe up to any transfinite rank, the map C does not make sense in this case as the $\in$ relation interpreted in $(\mathbb{U}(A); \in)^{\mathcal{U}}$ would no longer be well-founded, and so the Mostowski Collapse no longer works (nor would any potential collapsing function to the real $\in$ relation in an ambient set theory satisfying the axiom of foundation).

We note that if we take $\mathcal{U}$ to be good as well as regular we can get the stronger condition that for any α sized collection of sets from ${}^*\mathbb{U}(A)$ with the FIP there is an element of ${}^*\mathbb{U}(A)$ that is in the intersection of this collection. A proof of this fact will be given in Theorem 2.5.16.

2.4 Ultrafilters from Enlargements

Enlargements are intimately connected with ultrafilters. Not only are ultraproducts a key part in the proof of the existence of enlargements, but enlargements can be used to shed light on the properties of ultrafilters. In essence, we will be using the existence of regular ultrafilters on relatively large indexing sets to investigate the properties of ultrafilters on smaller indexing sets.

For this section, the set that we will want an enlargement over will usually be a set containing the set of indexes for an ultrafilter. Our first task for this section will be to show that there is a correspondence between ultrafilters on I and elements of ${}^*I$ when $*$ is an enlargement. The basic idea will be to notice that ultrafilters $\mathcal{U}$ have the FIP, and so the collection $*[\mathcal{U}]$ has an inhabited intersection in ${}^*\mathbb{U}(A)$. If $i \in {}^*I$, we will show that the collection of sets $X \in \mathbb{U}(A)$ having the property that $i \in {}^*X$ form an ultrafilter, and that this ultrafilter will behave as though it is the principal ultrafilter generated by i. For more information on this construction, see [Di Nasso(2015), Section 11.2].

2.4.1 Correspondence Between ${}^*I$ and βI

Ultrafilters $\mathcal{U}$ on I are often broken into two classes: the principal ultrafilters and the non-principal ultrafilters, with the distinguishing characteristic being whether or not there exists a single element $i \in I$ for which, given an subset $A \subseteq I$, the question of whether A is in $\mathcal{U}$ is the same as the question of whether $i \in A$. If there is such an $i \in I$, we say that $\mathcal{U}$ is principal and that $\mathcal{U}$ is generated by i. Stated in a way that makes the connections to enlargements more apparent, the ultrafilter $\mathcal{U}$ is **principal and generated by** i if and only if

$$\bigcap \mathcal{U} = \{i\}$$

and $\mathcal{U}$ is **non-principal** if and only if

$$\bigcap \mathcal{U} = \emptyset.$$

However, since ultrafilters have the finite intersection property by definition, we will have that

$$\bigcap_{A \in \mathcal{U}} {}^*A \neq \emptyset$$

in an enlargement, even if $\mathcal{U}$ is non-principal. The following lemmas allow us to interpret elements of this intersection as generators of $\mathcal{U}$.

Lemma 2.4.1. Let I be a set. Fix a nonstandard framework ${}^*\mathbb{U}(I)$. If $i \in {}^*I$ and $\mathcal{U}_i$ is the principal ultrafilter over ${}^*I$ generated by i then

$$u(i) := \{A \subseteq I : {}^*A \in \mathcal{U}_i\} = \{A \subseteq I : i \in {}^*A\}$$

is an ultrafilter over I.

Proof. We note that ${}^*A \in \mathcal{U}_i$ is equivalent to saying that $i \in {}^*A$ by the definition of a principal ultrafilter. As $i \in {}^*I$, we have that $I \in u(i)$. Similarly, as ${}^*\emptyset = \emptyset$, we have that $\emptyset \notin u(i)$. Closure under supersets follows from transfer of the sentence $\forall x, (x \in c_A \rightarrow x \in c_B)$ whenever $A \subseteq B$ in $\mathbb{U}(I)$. This is the same as saying that $x \in {}^*A \rightarrow x \in {}^*B$ whenever $A \subseteq B$ in $\mathbb{U}(I)$ and x is any element of ${}^*\mathbb{U}(I)$. Closure under binary/finite intersections follows from the fact that the transfer of the sentence $\exists x, (x \in c_A \wedge x \in c_B)$ is witnessed by i for all $A, B \in u(i)$. Thus $u(i)$ is a filter.

Suppose that $A \subseteq I$. Then, by transfer of the true sentence $\forall x, (x \in c_I \rightarrow (x \in c_A \vee x \in c_{I \setminus A}))$ applied to $x = i$, we have that either $A \in u(i)$ or $(I \setminus A) \in u(i)$, so $u(i)$ is an ultrafilter. □

After this proof, we will be less formal about exactly which sentences are being transferred unless it is unclear from context what is meant.

Lemma 2.4.2. If ${}^*\mathbb{U}(I)$ is an enlargement of I then every ultrafilter $\mathcal{U}$ over I is equal to $u(i)$ for some $i \in {}^*I$.

Proof. Since $\mathcal{U} \in \mathbb{U}(I)$ has the FIP, there is some $i \in \bigcap_{A \in \mathcal{U}} {}^*A \subseteq {}^*I$. Then $\mathcal{U} \subseteq u(i)$ and the maximality of $\mathcal{U}$ as a filter gives $\mathcal{U} = u(i)$. □

Theorem 2.4.3. If ${}^*\mathbb{U}(I)$ is an enlargement then the map $u\colon {}^*I \to \beta I$ is surjective, where βI is the set of all ultrafilters on I. Moreover, if ${}^*I$ is given the topology with a base of open sets given by $B = \{{}^*X : X \subseteq I\}$ and βI is given the topology of the Stone-Čech compactification of I as a discrete space then u is a quotient map of topological spaces.

Proof. That u is well-defined and surjective follows from Lemmas 2.4.1 and 2.4.2. That B is the base of a topology follows from the fact that ${}^*\emptyset = \emptyset$, ${}^*I = {}^*I$ and that

$$ {}^*(A_1 \cap A_2 \cap \cdots \cap A_n) = {}^*A_1 \cap {}^*A_2 \cap \cdots \cap {}^*A_n $$

by transfer of the sentence

$$ \forall x,\ (x \in c_I \to (x \in c_{A_1} \wedge x \in c_{A_2} \wedge \cdots \wedge x \in c_{A_n} \longleftrightarrow x \in c_{A_1 \cap A_2 \cap \cdots \cap A_n})). $$

We also have that $B' = \{\{\mathcal{U} \in \beta I : X \in \mathcal{U}\} : X \subseteq I\}$ is a base for the topology on βI. We claim that the map u is the natural quotient map ${}^*I \to {}^*I/{\sim_u}$ where $\sim_u$ is the equivalence relation where $i \sim_u j$ if and only if $u(i) = u(j)$. This construction guarantees that the space ${}^*I/{\sim_u}$ is in bijection with βI by the mapping $u'\colon [i]_{\sim_u} \mapsto u(i)$. It remains to show that u' is open and continuous to get that u' is a homeomorphism. For continuity, we note that for a fixed $X \subseteq I$ we have that

$$ (u')^{-1}(\{\mathcal{U} : X \in \mathcal{U}\}) = {}^*X/{\sim_u} $$

because $X \in u(i)$ if and only if $i \in {}^*X$. Similarly, for openness of u' we have that

$$ u'[{}^*X/{\sim_u}] = \{\mathcal{U} : X \in \mathcal{U}\}. $$

□

Remarks 2.4.4. These comments are not strictly necessary for what we will be doing, but they help to understand why we have been careful about certain things.

(1) If I is an infinite set, then the principal ultrafilter $\mathcal{U}_i$ generated by i over ${}^*I$ is **not the same** as ${}^*u(i)$ for any $i \in {}^*I$. In fact, this latter set is not even an ultrafilter over ${}^*I$, despite satisfying the transfer of the ultrafilter axioms. The reason for this is that $\mathcal{P}({}^*I) \supsetneq {}^*\mathcal{P}({}^*I) = {}^*(\mathcal{P}(I))$. In fact, $\mathcal{P}({}^*I)$ is not in the underlying set of ${}^*\mathbb{U}(I)$.

(2) The base B is **not** a topology on ${}^*I$ as the $*$ map does not, in general, commute with infinite unions. For example, it can be shown that $I \notin {}^*\mathbb{U}(I)$ but $I = \bigcup_{i \in I} {}^*\{i\}$ is open in the topology given on ${}^*I$.

2.5 Properties of $u(i)$ from Properties of $i \in {}^*I$

One nice way of using the correspondence between ${}^*I$ and βI is that the properties of $u(i)$ and the properties of ultrapowers over $u(i)$ can be deduced from the properties of $i \in {}^*I$. Typically, these properties in ${}^*I$ will correspond to some intuitive of idea of what the ultrafilter $u(i)$ is "meant" to do for ultrapowers and will often require looking at additional structure put on i to match this intuitive idea of what the ultrafilter is for. For the rest of this section we will have the standing assumption that ${}^*\mathbb{U}(I)$ is an enlargement of I and that $u\colon {}^*I \to \beta I$ is the map described in the previous section.

Many of the results obtained here are already known, for example Theorems 11.4.3 and 11.4.6 of [Di Nasso(2015)] are roughly the same results as Theorems 2.5.8 and 2.5.16 here, although Corollary 2.5.9 obtains a slightly stronger property for hyperfinite generators of regular ultrafilters.

The first property of ultrafilters that we will investigate is the important but simple property of being non-principal.

Theorem 2.5.1 (Characterization of Non-principal Ultrafilters)**.** The ultrafilter $u(i)$ is non-principal if and only if $i \in {}^*I \setminus I$ (where we identify I with $*[I] = \{{}^*i : i \in I\}$).

Proof. ($\Rightarrow$) For the contrapositive, suppose $i \in I$ then ${}^*i \in {}^*X \iff i \in X$ by transfer of the sentence $c_i \in c_X$. Thus $u({}^*i)$ is principal and generated by i.

($\Leftarrow$) Now suppose that $i \in {}^*I \setminus I$. Then, for all $j \in I$, we have that ${}^*\{j\} = \{{}^*j\}$ by transfer of the sentence $\forall x, (x \in c_{\{j\}} \to x = c_j)$, and so $i \notin {}^*\{j\}$. Thus $\{j\} \notin u(i)$ for any $j \in I$ and $u(i)$ is non-principal. □

Theorem 2.5.1 tells us that non-principal ultrafilters are "generated" by elements of ${}^*I$ that did not already exist in I and that u behaves as we would hope on the elements of $*[I]$: $u({}^*i)$ is the principal ultrafilter generated by $i \in I$. This gives some credence to the interpretation of u as giving the ultrafilter generated by the input and allows us to view non-principal ultrafilters as being principally generated by a non-standard element, which will play a key role in the discussion of ultraproducts.

Next we take a step up in complexity and find a characterization of elements in ${}^*I$ that generate uniform ultrafilters.

Definition 2.5.2 (Uniform Ultrafilter)**.** An ultrafilter $\mathcal{U}$ on I is **uniform** if every element of $\mathcal{U}$ is the same cardinality, which is necessarily the cardinality of I.

Definition 2.5.3 (Above the Diagonal Embedding)**.** We say that $i \in {}^*I$ is **above the diagonal embedding** if for every ordering $\leq$ on I so that $(I, <) \cong (|I|, \in)$ we have that ${}^*j \; {}^*\!< i$ for all $j \in I$.

Theorem 2.5.4 (Characterization of Uniform Ultrafilters)**.** The ultrafilter $u(i)$ is uniform if and only if i is above the diagonal embedding. Moreover, if $|I|$ is regular, then $u(i)$ is uniform if and only if there is **some** ordering $\leq$ on I with $(I, <) \cong (|I|, \in)$ where ${}^*j \; {}^*\!< i$ for all $j \in I$.

Proof. ($\Rightarrow$): Suppose that $u(i)$ is uniform and $\leq$ is an ordering on I so that $(I, <) \cong (|I|, \in)$. Then all the $<$-intervals $(j, \infty) := \{k \in I : k > j\}$ are in $u(i)$ as the complement of (j, ∞) in I has strictly smaller cardinality than I. Thus

$$i \in \bigcap_{j \in I} {}^*(j, \infty)$$

by the definition of $u(i)$. By transfer of the definition of (j, ∞), we have that $i \in {}^*(j, \infty) \leftrightarrow {}^*j \; {}^*\!< i$. Therefore ${}^*j \; {}^*\!< i$ for all $j \in I$.

($\Leftarrow$): Suppose that i is above the diagonal embedding for all orderings $\leq$ on I that make $(I,<) \cong (|I|,\in)$ and suppose that X is a subset of I with $|X| < |I|$. We need only show that $X \notin u(i)$. As $|X| < |I|$, we may create an ordering $\leq$ on I so that X is an initial segment of $(I,<) \cong (|I|,\in)$. Let $j \in I \setminus X$. Then $k < j$ for all $k \in X$, so, by transfer, we have that $k \ {}^*\!< {}^*j$ for all $k \in {}^*X$. In particular, if $X \in u(i)$, this would imply that $i \ {}^*\!< {}^*j$, a contradiction.

For the "moreover," notice that any $X \subseteq I$ with $|X| < |I|$ is bounded in any $(I,<) \cong (|I|,\in)$ if $|I|$ is regular. Thus a j bounding X can be found in any such $(I,<)$ and the ($\Leftarrow$) direction can be carried out using this bound. □

Theorem 2.5.4 can be thought of as saying that hyperprincipal generators for uniform ultrafilters are "large" with respect to any (standard) well-ordering that can be put on the indexing set I.

In order to find the characterization of regular ultrafilters we will need to know a little bit about how ultrafilters interact with functions between index sets.

Lemma 2.5.5. Let I and J be sets and $f\colon I \to J$ any function. Fix an ultrafilter $\mathcal{U}$ over I. Then

$$f(\mathcal{U}) := \{B \subseteq J : f^{-1}(B) \in \mathcal{U}\}$$

is an ultrafilter over J. Moreover, if $f(\mathcal{U})$ is α-regular for some cardinal α then so is $\mathcal{U}$.

Proof. That $\emptyset \notin f(\mathcal{U})$ and $J \in f(\mathcal{U})$ are quickly checked. Closure under supersets follows from $B \subseteq A \implies f^{-1}(B) \subseteq f^{-1}(A)$. Closure under binary/finite intersections follows from the fact that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. If $A \notin f(\mathcal{U})$, then $f^{-1}(A) \cup f^{-1}(J \setminus A) = I \in \mathcal{U}$, and so $J \setminus A \in f(\mathcal{U})$, giving that $f(\mathcal{U})$ is an ultrafilter.

For the moreover, suppose that $(X_\beta)_{\beta<\alpha}$ is a regularizing sequence in $f(\mathcal{U})$. Consider the sequence $(f^{-1}(X_\beta))_{\beta<\alpha}$ in $\mathcal{U}$. This sequence is α-regularizing because infinite intersections are preserved by inverse images. □

Lemma 2.5.6. If $A, B \in \mathbb{U}(I)$ and f is a function $f\colon A \to B$, then $f \in \mathbb{U}(I)$ and ${}^*f$ is a function ${}^*f\colon {}^*A \to {}^*B$.

Proof. The statement that f is function from A to B can be written as a first order statement in the universal language for I, and the transfer of such a statement is that ${}^*f$ is a function from ${}^*A$ to ${}^*B$. □

Lemma 2.5.7. Let I and J be sets in $\mathbb{U}(I)$ and $f\colon I \to J$ any function. If $i \in {}^*I$ then

$$f(u(i)) = u({}^*f(i)).$$

Proof. With f and i as described above, $f(u(i))$ is the set of $B \subseteq J$ so that $f^{-1}(B) \in u(i)$. That is, $B \in f(u(i))$ if and only if $i \in {}^*(f^{-1}(B))$. By transfer of the sentence

$$\forall x,\ (x \in c_{f^{-1}(B)} \leftrightarrow \exists y,\ ((x, y) \in c_f \wedge y \in c_B))$$

we have that $B \in f(u(i))$ if and only if $(i, y) \in {}^*f$ for some $y \in {}^*\mathbb{U}(I)$ with $y \in {}^*B$ (note that we are using (x, y) as the usual abbreviation in first-order sentences in the language of set theory). By transfer of the definition of a function, $y = {}^*f(i)$. Thus $B \in f(u(i))$ if and only if ${}^*f(i) \in {}^*B$. This is equivalent to saying that $f(u(i)) = u({}^*f(i))$. □

Theorem 2.5.8 (Characterization of α-Regular Ultrafilters)**.** For an ultrafilter $u(i)$ the following are equivalent:

(1) $u(i)$ is α-regular.

(2) There exists a map $f\colon I \to \mathcal{P}_\omega(\alpha)$ and some $X \subseteq \alpha$ so that $|X| = \alpha$ and $*[X] \subseteq {}^*f(i)$.

(3) There exists a map $f\colon I \to \mathcal{P}_\omega(\alpha)$ such that for every $X \subseteq \alpha$ with $|X| = \alpha$ we have that $*[X] \subseteq {}^*f(i)$.

Proof. (3) ⇒ (1): Let $f\colon I \to \mathcal{P}_\omega(\alpha)$ and $i \in {}^*I$ be as described above and assume $X \subseteq \alpha$ has $|X| = \alpha$. By the three lemmas above, it will be enough to show that $u({}^*f(i))$ is α-regular. For every $\Delta \in \mathcal{P}_\omega(X)$, define

$$A_\Delta := \{A \in \mathcal{P}_\omega(\alpha) : \Delta \subseteq A\}.$$

By transfer of the definition of A_Δ, we have that ${}^*f(i)$ is in ${}^*A_\Delta$, because

$$ {}^*\Delta = \{{}^*\delta : \delta \in \Delta\} \subseteq *[X] \subseteq {}^*f(i) $$

where the first equality follows from the fact that Δ is finite by using transfer. That is, the ultrafilter $f(u(i))$ over $\mathcal{P}_\omega(\alpha)$ contains A_Δ for each $\Delta \in \mathcal{P}_\omega(\alpha)$. Just as in the usual construction of regular ultrafilters, we have that the A_Δ form an α-regularizing set in $f(u(i))$.

(1) $\Rightarrow$ (2): Suppose that $u(i)$ is α-regular and $(X_\beta)_{\beta<\alpha}$ is an α-regularizing sequence in $u(i)$. Let $f\colon I \to \mathcal{P}_\omega(\alpha)$ where $j \mapsto \{\beta : j \in X_\beta\} =: S_j$. Note that S_j is finite for all j precisely because the X_β form a regularizing sequence. Let $X = \alpha$. It remains to check that $*[\alpha] \subseteq {}^*f(i)$. We have that

$$ f^{-1}[A_{S_j}] = \bigcap_{\beta \in S_j} X_\beta \in u(i) $$

so $A_{S_j} \in f(u(i))$. By Lemma 2.5.7, this is equivalent to ${}^*A_{S_j} \in u({}^*f(i))$, and so ${}^*f(i) \in {}^*A_{S_j}$. By transfer of the definition of A_{S_j}, this guarantees that ${}^*S_j \subseteq {}^*f(i)$ for all $j \in I$. Since $\bigcup {}^*S_j = *[\alpha]$, we get that $*[\alpha] \subseteq {}^*f(i)$.

(2) $\Rightarrow$ (3): Pick f and X satisfying (2). Let $g\colon X \to \alpha$ be a surjective mapping. Define $G\colon \mathcal{P}_\omega(\alpha) \to \mathcal{P}_\omega(\alpha)$ by $G(\Delta) = g[\Delta \cap X]$. We claim that the mapping $G \circ f$ has the desired property. Let $Y \subseteq \alpha$ with $|Y| = \alpha$. We wish to show that $*[Y] \subseteq {}^*G({}^*f(i))$ (note that transfer guarantees that $*$ distributes over composition of functions). By transfer of the definition of G, we have that ${}^*G({}^*f(i)) = {}^*g[{}^*f(i) \cap {}^*X]$. Using the fact that $*[X] \subseteq {}^*f(i) \cap {}^*X$ yields

$$ {}^*g[*[X]] = *[\alpha] \subseteq {}^*(G \circ f)(i). $$

As $Y \subseteq \alpha$, we must also have $*[Y] \subseteq *[\alpha]$, finishing the proof. □

Theorem 2.5.8 can be thought of as saying that hyperprincipal generators for a regular ultrafilter are so large that it makes sets of size α or less appear to be finite.

Corollary 2.5.9. Suppose that $i \in {}^*I$. Then the following are equivalent.

(1) The ultrafilter $u(i)$ is an α-regular ultrafilter.

(2) Whenever $X \subseteq {}^*\alpha$ is such that $|X| \leq \alpha$ and every element of X has the form ${}^*g(i)$ for some function $g\colon I \to \alpha$, there is a function $f\colon I \to \mathcal{P}_\omega(\alpha)$ with $X \subseteq {}^*f(i)$.

Proof. (1) $\Rightarrow$ (2): For each $x \in X$, choose a function $g_x\colon I \to \alpha$ such that ${}^*g_x(i) = x$ and define $G = \{g_x : x \in X\}$. Since $|G| \leq \alpha$, Theorem 2.5.8 guarantees the existence of a function $h\colon I \to \mathcal{P}_\omega(\alpha^I)$ with $*[G] \subseteq {}^*h(i)$ (compose the function guaranteed by part (3) of Theorem 2.5.8 with any function $\alpha \to \alpha^I$ whose range includes G). Define the function $\bigcup h\colon I \to \mathcal{P}_\omega(\alpha)$ by $j \mapsto \bigcup(e_j[h(j)])$ where $e_j\colon \alpha^I \to \alpha$ is the evaluation function $f \mapsto f(j)$. We claim that $X \subseteq {}^*\bigcup h(i)$.

By transfer, if an element $a \in {}^*\alpha$ is equal to $f(i)$ for some $f \in {}^*h(i)$, then $a \in {}^*\bigcup h(i)$. In particular, since $x = {}^*g_x(i)$ and ${}^*g_x \in *[G] \subseteq {}^*h(i)$, we get $X \subseteq {}^*\bigcup h(i)$.

(2) $\Rightarrow$ (1): Note that $*[\alpha] \subseteq {}^*\alpha$ has cardinality α, so the function guaranteed in (2) shows that $u(i)$ is regular by Theorem 2.5.8. □

At this point we are able to mention how a couple of well-known but nontrivial relationships can be deduced between the different types of ultrafilters using the map u.

Remark 2.5.10. Every uniform ultrafilter is non-principal as no $i \in I$ can be strictly above the diagonal embedding in any linear order $<$ on I.

Corollary 2.5.11. Every regular ultrafilter (i.e. $|I|$-regular ultrafilter) is uniform.

Proof. Let $f\colon I \to \mathcal{P}_\omega(|I|)$ be such that $*[|I|] \subseteq {}^*f(i)$. Choose an ordering $<$ on I so that $(I, <) \cong (|I|, \in)$. Suppose towards a contradiction that $i \; {}^*\!\!< {}^*j$. Then $i \in {}^*[0, j)$ and so ${}^*f(i) \in {}^*f[{}^*[0, j)]$. As $|[0, j)| < |I|$, there are elements of $|I|$ that are not elements of $f(k)$ for any $k \in [0, j)$. By transferring this statement for a particular witness β in $|I|$, we see that that ${}^*\beta \notin {}^*f(i)$. □

Definition 2.5.12 (Hyperfinite)**.** An element $D \in {}^*\mathbb{U}(A)$ is called **hyperfinite** if there is a set $N \in \mathbb{U}(A)$ for which $D \in {}^*\mathcal{P}_\omega(N)$.

We finish our characterizations of the hyperprincipal generators of different types of ultrafilters with the hyperprincipal generators of regular and good ultrafilters, but first we need to define good ultrafilters.

Definition 2.5.13 (Monotone)**.** A function $d\colon \mathcal{P}_\omega(I) \to \mathcal{U}$ is said to be **monotone** if $\Delta, \Gamma \in \mathcal{P}_\omega(I)$ and $\Delta \subseteq \Gamma$ implies $d(\Gamma) \subseteq d(\Delta)$.

Definition 2.5.14 (Multiplicative)**.** A function $d\colon \mathcal{P}_\omega(I) \to \mathcal{U}$ is said to be **multiplicative** if $\Delta, \Gamma \in \mathcal{P}_\omega(I)$ imply $d(\Delta \cup \Gamma) = d(\Delta) \cap d(\Gamma)$.

Definition 2.5.15 (Good Ultrafilter)**.** An ultrafilter $\mathcal{U}$ is said to be **good** if whenever $d\colon \mathcal{P}_\omega(I) \to \mathcal{U}$ is a monotone function then there is a multiplicative function $d'\colon \mathcal{P}_\omega(I) \to \mathcal{U}$ such that $d'(\Delta) \subseteq d(\Delta)$ for all $\Delta \in \mathcal{P}_\omega(I)$.

Theorem 2.5.16 (Characterization of Regular and Good Ultrafilters)**.** The ultrafilter $u(i)$ is both regular and good if and only if every collection B of functions $f\colon I \to \mathcal{P}(I)$ such that $|B| \leq |I|$ and $*[B](i) := \{{}^*f(i) : f \in B\}$ has the FIP, also has the property that $\bigcap(*[B](i))$ has an element of the form ${}^*g(i)$ for some function $g\colon I \to I$.

Proof. Suppose that $u(i)$ is regular and good. Let $B = \{f_\alpha : \alpha \in I\}$ be an enumeration of B, possibly with repetition. Define a monotone function $k\colon \mathcal{P}_\omega(I) \to \mathcal{P}(I)$ by

$$k(\Delta) = \left\{\alpha \in I : \bigcap_{\delta \in \Delta} f_\delta(\alpha) \neq \emptyset\right\}.$$

Note that $i \in {}^*k(\Delta)$ for all $\Delta \in \mathcal{P}_\omega(I)$ since $\bigcap_{\delta\in\Delta} {}^*f_\delta(i) \neq \emptyset$ by assumption. In particular, $k(\Delta) \in u(i)$ for all $\Delta \in \mathcal{P}_\omega(I)$. Furthermore, if $\Delta \subseteq \Gamma$ then $k(\Gamma) \subseteq k(\Delta)$. That is, k is a monotone function $\mathcal{P}_\omega(I) \to u(i)$.

By the goodness of $u(i)$, there is a multiplicative function $h \leq k$. By the regularity of $u(i)$, there is a function $c\colon I \to \mathcal{P}_\omega(|I|)$ such that $*[I] \subseteq {}^*c(i)$. Let $C\colon I \to \mathcal{P}_\omega(\mathcal{P}(I))$ be the function $\alpha \mapsto \{f_\delta : \delta \in c(\alpha)\}$. In particular, ${}^*C(i) \in {}^*\mathcal{P}_\omega(\mathcal{P}(I))$ and $*[B](i) \subseteq {}^*C(i)$. Define

$$D = \{\delta \in {}^*I : i \in {}^*h(\{\delta\})\} \text{ and } f_D = \{{}^*f_\delta(i) : \delta \in D\}$$

and consider the set $E = {}^*C(i) \cap {}^*f_D(i)$. Note that $*[B](i) \subseteq {}^*f_D(i)$ since $h(\{\delta\}) \in u(i)$ and i is an element of every set in $*[u(i)]$ by definition, so $*[B](i) \subseteq E$. Furthermore, E is hyperfinite (being a subset of ${}^*C(i)$) and internal as E is definable in terms of the internal sets ${}^*C(i)$ and ${}^*f_D(i)$. Let

$F = \{\delta \in {}^*I : {}^*f_\delta(i) \in E\}$. We claim that $i \in {}^*h(F)$. We know that $h(\Delta) = \bigcap_{\delta\in\Delta} h(\{\delta\})$ by the multiplicativity of h, so, by transfer, ${}^*h(F) = \bigcap_{\delta\in F} {}^*h(\{\delta\})$. Since $F \subseteq D$, we have that $i \in {}^*h(\delta)$ for all $\delta \in F$. In particular, we have that $i \in {}^*k(F)$, so, by transfer of the definition of k, we get $\bigcap_{\delta\in F} {}^*f_\delta(i) \neq \emptyset$. As the collection of sets ${}^*f_\delta(i)$ is internal, the intersection $\bigcap_{\delta\in F} {}^*f_\delta(i)$ must also be internal. As elements of internal sets are internal, any element of $\bigcap_{\delta\in F} {}^*f_\delta(i)$ will be an internal witness to the intersection being inhabited.

We now prove the converse. Assume that $u(i)$ is an ultrafilter satisfying the condition that every collection B of functions $f\colon I \to \mathcal{P}(I)$ such that $|B| \leq |I|$ and $*[B](i)$ has the finite intersection property also has the property that $\bigcap *[B](i)$ contains an internal element. We want to prove that $u(i)$ must be regular and good.

For proving regularity, define for each $\Delta \in \mathcal{P}_\omega(I)$ the function $f_\Delta\colon I \to \mathcal{P}(\mathcal{P}_\omega(I))$ by the rule $j \mapsto \Delta_\uparrow$ where

$$\Delta_\uparrow := \{\Gamma \in \mathcal{P}_\omega(I) : \Gamma \supseteq \Delta\}.$$

Consider the collection of functions $B = \{(j \mapsto \Delta_\uparrow) : \Delta \in \mathcal{P}_\omega(I)\}$.

$*[B](i)$ clearly has the finite intersection property and $|B| = |I|$, so there is a function $g\colon I \to \mathcal{P}_\omega(I)$ such that ${}^*g(i) \in \bigcap(*[B](i))$ (note that $\mathcal{P}_\omega(I)$ is in bijection with I, so we may apply our assumption about $u(i)$ to B). That is ${}^*g(i)$ is a hyperfinite set containing $*[I]$, as ${}^*g(i) \supseteq {}^*\Delta = *[\Delta]$ for each $\Delta \in \mathcal{P}_\omega(I)$ by transfer. The existence of such a function g is equivalent to $u(i)$ being regular by the condition in Theorem 2.5.8 (2).

For proving goodness, let $f\colon \mathcal{P}_\omega(I) \to u(i)$ be a monotone function. For each $\delta \in I$ we define the function n_δ on I as follows:

$$n_\delta(j) = \{\Gamma \in \{\delta\}_\uparrow : j \in f(\Gamma)\}.$$

From this definition and the monotonicity of f, we see that for any $\Delta \in \mathcal{P}_\omega(I)$ the intersection $\bigcap_{\delta\in\Delta} n_\delta(j)$ is nonempty if and only if $j \in f(\Delta)$ if and only if $\Delta \in n_\delta(j)$ for all $\delta \in \Delta$.

In particular, if $\Delta \in \mathcal{P}_\omega(I)$, then ${}^*\Delta \in \bigcap_{\delta\in\Delta} {}^*n_\delta(i)$, showing that the collection $\{{}^*n_\delta(i) : \delta \in I\}$ has the finite intersection property. Furthermore, each n_δ is a function $I \to \mathcal{P}_\omega(I)$, so the assumptions on i guarantee the existence of a function $\widehat{g}\colon I \to \mathcal{P}_\omega(I)$ such that

$$^*\widehat{g}(i) \in \bigcap_{\delta \in *[\mathcal{P}_\omega(I)]} {}^*n_\delta(i).$$

Define

$$g(\Delta) := \left\{ j \in I : \widehat{g}(j) \in \bigcap_{\delta \in \Delta} n_\delta(j) \right\}.$$

We claim that g is a multiplicative refinement of f.

We see that g refines f from the fact that $j \in g(\Delta)$ implies that $\bigcap_{\delta\in\Delta} n_\delta(j) \neq \emptyset$ or, equivalently, that $j \in f(\Delta)$. For multiplicativity, we have that

$$\begin{aligned} g(\Delta \cup \Gamma) &= \left\{ j \in I : \widehat{g}(j) \in \bigcap_{\delta \in \Delta} n_\delta(j) \cap \bigcap_{\gamma\in\Gamma} n_\gamma(j) \right\} \\ &= \left\{ j \in I : \widehat{g}(j) \in \bigcap_{\delta \in \Delta} n_\delta(j) \right\} \cap \left\{ j \in I : \widehat{g}(j) \in \bigcap_{\gamma \in \Gamma} n_\gamma(j) \right\} \\ &= g(\Delta) \cap g(\Gamma). \end{aligned}$$

Furthermore, $i \in {}^*g(\Delta)$ for all $\Delta \in \mathcal{P}_\omega(I)$ since ${}^*\widehat{g}(i) \in \bigcap_{\delta\in\Delta} {}^*n_\delta(i)$ by the definition of $\widehat{g}$. So $g(\Delta) \in u(i)$, as desired. □

Theorem 2.5.16 says that any small collection of (i, I)-internal sets with the finite intersection property have an (i, I)-internal element in their intersection (see Definition 2.6.4 for the definition of (i, I)-internal and the following Theorem 2.6.6 for the relationship between (i, I)-internal elements and elements of $u(i)$-ultrapowers).In particular, Theorem 2.5.16 can be thought of as saying that if B is a small collection of sets of the form $\prod_{j\in\mathcal{U}} E_j$ and B has the FIP then the intersection $\bigcap_{E\in B} E$ is nonempty.

2.6 Nonstandard Model Theory and the Keisler Order

As Model Theory studies the interaction between structures and theories, it would be useful to be able to discuss the language of our structures within a nonstandard framework. The easiest way to achieve this is to add symbols φ to the base set of our universe for each formula in the language we are working in. Although it is possible to encode these formulas in the universe without expanding the base set in a variety of ways (assuming that the language is no larger than the index set and model we wish to work with), adding in the new symbols makes working with the formulas less tedious.

One benefit of the nonstandard approach to Model Theory is that, given a nonstandard framework $*$ on $\mathbb{U}(\mathcal{M} \cup I \cup \mathcal{L})$ (with $\mathcal{L}$ representing the set of formulas in the language), we can study $\mathcal{L}$ by studying ${}^*\mathcal{L}$ instead, which includes nonstandard formulas that can have infinitely many terms or variables. In particular, for the context of Keisler's Order, we can take a "small" set of formulas $\Phi(x)$ with parameters in $\mathcal{M}^{\mathcal{U}}$ (small here means $|\Phi| < \lambda$ where $\mathcal{U}$ is a λ-regular ultrafilter on I) and extend Φ to a hyperfinite collection of formulas, Γ. This allows us to change the question of whether Φ has a realization in $\mathcal{M}^{\mathcal{U}}$ to whether there is a hyperfinite $\widehat{\Phi}$ extending Φ that is consistent (since $\widehat{\Phi}$ is hyperfinite, Łoś's theorem tells us that $\widehat{\Phi}$ being consistent is the same as having a realization). Since consistency is a property of the theory and not of the particular model, this idea may be used to understand Keisler's result that saturation by a regular ultrafilter is independent of the particular model studied.

2.6.1 Enlarging with the Language

For this section, $\mathcal{M}$ will be a structure in the countable language $\mathcal{L}$ and I will continue to be an indexing set. Let A be any set containing the disjoint union of M, I, and the set of all formulas and terms in the language with a specified countable set of variables. Throughout this section, $*$ will be an enlargement of $\mathbb{U}(A)$.

Definition 2.6.1.

(1) Let $\mathcal{L}(M)$ be the set of pairs (φ, f) where $\varphi \in \mathcal{L}$ and f is function from the set of free variables in φ to M with the property that for each free variable x occurring in φ either $f(x) \in M$ or $f(x) = x$. We will frequently write $\varphi(f(x_1), \ldots, f(x_n))$ or $\varphi(f(\bar{x}))$ instead of (φ, f). The pairs (φ, f) are intended to represent formulas in $\mathcal{L}$, possibly with parameters

from M and possibly with some variables left free.

(2) If $\gamma \in \mathcal{L}(\mathcal{M})$ is (φ, f), we define $\gamma_1 = \varphi$ and $\gamma_2 = f$. Similarly, if $\Gamma \subseteq \mathcal{L}(\mathcal{M})$ we define $\Gamma_1 = \{\gamma_1 : \gamma \in \Gamma\}$ and $\Gamma_2 = \{\gamma_2 : \gamma \in \Gamma\}$.

(3) We say that x is free in $(\varphi, f) \in \mathcal{L}(\mathcal{M})$ if x is free in φ and $f(x) = x$. We also say that x is free in $\Gamma \subseteq \mathcal{L}(\mathcal{M})$ if x is free in at least one element of Γ.

(4) We will use the symbol $\vDash_{\mathcal{M}}$ to be the predicate on $\mathcal{L}(M)$ determined by

$$\vDash_{\mathcal{M}} (\varphi, f) \iff \mathcal{M} \vDash \varphi(f(\bar{x})).$$

We do not require that (φ, f) has no variables free in this definition. For an example of defining satisfaction in this way, see [Chang and Keisler(2012)] 1.3.17.

It is worth noting that $\mathcal{L}(M)$ and $\vDash_{\mathcal{M}}$ are elements of $\mathbb{U}(A)$ and so can be analyzed by working with ${}^*\mathcal{L}(M)$ and ${}^*{\vDash_{\mathcal{M}}}$ instead.

In the following Lemma 2.6.2, we will use a function $f\colon I \times \{x_1, \ldots, x_n\} \to M$ to represent the tuple of parameters coming from $\mathcal{M}^{\mathcal{U}}$ for a formula φ. In this context, we will use $[f(-, x_i)]_{u(i)}$ with a fixed x_i to represent an individual element of the tuple where the $-$ represents the independent variable of the function. Similarly, ${}^*f(i, -)$ will be used to represent a function from the independent variables of ${}^*\varphi$ to ${}^*M$ so that $({}^*\varphi, {}^*f(i, -))$ is an element of ${}^*\mathcal{L}(M)$.

Lemma 2.6.2. Suppose $\varphi(x_1, \ldots, x_n) \in \mathcal{L}$, $f\colon I \times \{x_1, \ldots, x_n\} \to M$, and $i \in {}^*I$. Then

$${}^*{\vDash_{\mathcal{M}}} ({}^*\varphi, {}^*f(i, -)) \iff \mathcal{M}^{u(i)} \vDash \varphi\big([f(-, x_1)]_{u(i)}, \ldots, [f(-, x_n)]_{u(i)}\big)$$

where $[g]$ is the equivalence class of $g\colon I \to M$ in $\mathcal{M}^{u(i)}$.

Proof. By Łoś's theorem, definitions, and transfer, we have that

$$\begin{aligned} \mathcal{M}^{u(i)} \vDash \varphi\big([f(-, x_1)]_{u(i)}, \ldots, [f(-, x_n)]_{u(i)}\big) &\iff i \in {}^*\{j \in I : \mathcal{M} \vDash \varphi(f(j, x_1), \ldots, f(j, x_n))\} \\ &\iff i \in \{j \in {}^*I : {}^*{\vDash_{\mathcal{M}}} ({}^*\varphi, {}^*f(j, -))\} \\ &\iff {}^*{\vDash_{\mathcal{M}}} ({}^*\varphi, {}^*f(i, -)). \end{aligned}$$

□

One of the key ideas that we will be using to talk about the Keisler Order using Nonstandard Model Theory is that, after taking an ultrapower using a regular ultrafilter, all small enough sets can be extended to a set having all of the first-order properties of a finite set. In particular, given a small enough set of formulas, we may embed this set into a hyperfinite set of formulas. As we will see later on, the question of whether the original set of formulas free only in the variable x has a realization in an ultrapower will be the same as whether there is a hyperfinite set of formula $\Phi(x)$ extending the original set such that ${}^*\exists x,\ {}^*\bigwedge \Phi(x)$ is satisfied in the enlargement. The following lemma is a key step in the analysis leading to this conclusion.

Lemma 2.6.3. Let $\bigwedge \colon P_\omega(\mathcal{L}(M)) \to \mathcal{L}(M)$ be the function defined by

$$\{\varphi_0(f_0(\bar{x})), \ldots, \varphi_n(f_n(\bar{x}))\} \mapsto \bigwedge_{i=0}^{n} \varphi_i(f_i(\bar{x})),$$

and $\exists x \colon \mathcal{L}(M) \to \mathcal{L}(M)$ by

$$(\varphi, f) \mapsto \exists x,\ \varphi(f(\bar{y})).$$

The functions ${}^*\bigwedge$ and ${}^*\exists x$ have the following property: If $\Gamma \in {}^*\mathcal{P}_\omega(\mathcal{L}(M))$ has only x free then ${}^*\models_{\mathcal{M}} ({}^*\exists x,\ {}^*\bigwedge \Gamma) = {}^*\models_{\mathcal{M}} ({}^*\exists x({}^*\bigwedge(\Gamma)))$ if and only if there is an $m \in {}^*M$ such that ${}^*\models_{\mathcal{M}} \varphi(m, f(\bar{y}))$ for all $\varphi(x, f(\bar{y})) \in \Gamma$.

Proof. Follows from applying transfer to the standard case. $\square$

We will also need the ability to talk about the elements of the ultrapower $\mathcal{M}^{\mathcal{U}}$ within the enlargement. As the next definitions and theorem will show, there is an isomorphism between the collection of elements of the enlargement having the form ${}^*f(i)$ for a standard function $f \colon I \to M$ after being given an appropriate natural structure, and the ultrapower $\mathcal{M}^{u(i)}$.

Definition 2.6.4 ((i, I)-internal)**.** An element $b \in {}^*\mathbb{U}(A)$ is (i, I)**-internal** if there is a function $f \colon I \to \mathbb{U}(A)$ with $f \in \mathbb{U}(A)$ such that $b = {}^*f(i)$.

Definition 2.6.5 ((i, I)-internal Ultrapower)**.** If $\mathcal{M}$ is a first-order structure in the language $\mathcal{L}$ with underlying set $M \in \mathbb{U}(A)$, we say that the (i, I)**-internal ultrapower** of $\mathcal{M}$ is the structure with underlying set $\{{}^*f(i) : f \in \mathbb{U}(A),\ f \colon I \to \mathbb{U}(A)\}$ and given the following structure:

(1) For every function symbol F in the language $\mathcal{L}$, we interpret F so that

$$F({}^*f_0(i), \ldots, {}^*f_n(i)) = {}^*g(i) \iff {}^*\models_{\mathcal{M}} (F({}^*f_0(i), \ldots, {}^*f_n(i)) = {}^*g(i)).$$

(2) For every relation symbol R in the language $\mathcal{L}$, we interpret R so that

$$R({}^*f_0(i), \ldots, {}^*f_n(i)) \iff {}^*\models_{\mathcal{M}} R({}^*f_0(i), \ldots, {}^*f_n(i)).$$

(3) For every constant symbol c in the language $\mathcal{L}$, we interpret c so that

$$c = {}^*f(i) \iff {}^*\models_{\mathcal{M}} (c = {}^*f(i))$$

Theorem 2.6.6. The (i, I)-internal ultrapower of a structure $\mathcal{M}$ with underlying set $M \in \mathbb{U}(A)$ is well-defined and isomorphic to $\mathcal{M}^{u(i)}$.

Proof. We start by giving a bijection from the (i, I)-internal ultrapower of $\mathcal{M}$ to $\mathcal{M}^{u(i)}$. This bijection is defined by ${}^*f(i) \mapsto [f]_{u(i)}$ where $[f]_{u(i)}$ is the equivalence class of f. If the mapping is well-defined, then it is clearly surjective. For injectivity and well-definedness, suppose that ${}^*f(i)$ and ${}^*g(i)$ are elements of the (i, I)-internal ultrapower of $\mathcal{M}$. Then by transfer of the fact

$$\forall x, y, \, (x, y \in M) \to (\models_{\mathcal{M}} (x = y) \leftrightarrow x = y)$$

we get that ${}^*\models_{\mathcal{M}} ({}^*f(i) = {}^*g(i)) \iff {}^*f(i) = {}^*g(i)$. From Theorem 2.6.2, these statements are equivalent to $[f]_{u(i)} = [g]_{u(i)}$. Thus the proposed mapping is a well-defined bijection as desired.

Using Theorem 2.6.2 again shows that this bijection preserves the meanings of every symbol in the language $\mathcal{L}$, so the well-definedness of the structure on the usual ultrapower $\mathcal{M}^{u(i)}$ implies the well-definedness of the structure on the (i, I)-internal ultrapower of $\mathcal{M}$, and, moreover, that the given mapping is an isomorphism. $\square$

In light of the result of Theorem 2.6.6, we will not make a distinction between $\mathcal{M}^{u(i)}$ and the (i, I)-internal ultrapower of $\mathcal{M}$.

Lemma 2.6.7. Suppose that $J \subseteq I$, $\theta(x, y)$ a bounded quantifier formula in the language of set theory possibly with parameters from $\mathbb{U}(A)$, and $B \in \mathbb{U}(A)$. If for all $j \in {}^*J$ there is a $b \in {}^*B$ such that ${}^*\mathbb{U}(A) \models \theta(j, b)$ then there is a function $g\colon I \to B$ such that ${}^*\mathbb{U}(A) \models \theta(j, {}^*g(j))$ for all $j \in {}^*J$.

Proof. By transfer, for every $j \in J$ there is a $b \in B$ such that $\mathbb{U}(A) \models \theta(j, b)$. We may then define g so that $\theta(j, g(j))$ is true for all $j \in J$ and define $g(j)$ arbitrarily on $I \setminus J$. By transfer, ${}^*\mathbb{U}(A) \models \theta(j, {}^*g(j))$ for all $j \in {}^*J$. □

Corollary 2.6.8. If $a_1, \ldots, a_n \in {}^*\mathbb{U}(A)$ are (i, I)-internal and $b \in {}^*\mathbb{U}(A)$ is definable in the language of set theory in terms of $a_1, \ldots, a_n$ then b is (i, I)-internal.

Proof. Let $f_1, \ldots, f_n$ be witnesses for $a_1, \ldots, a_n$ being (i, I)-internal and $\varphi(x, \bar{y})$ a formula in the language of set theory such that b is the unique element of ${}^*\mathbb{U}(A)$ with ${}^*\mathbb{U}(A) \models \varphi(b, \bar{a})$. Define $\theta(x, y)$ to be the formula with parameters from $\mathbb{U}(A)$ given by $\varphi(y, \bar{f}(x))$. Using the fact that every element b of ${}^*\mathbb{U}(A)$ is contained in some standard set ${}^*B$, Lemma 2.6.7 implies that there is a function $g : I \to B$ such that $\theta(i, {}^*g(i)) \equiv \varphi({}^*g(i), \bar{a})$ holds. In particular, the uniqueness of b requires ${}^*g(i) = b$, so g witnesses b being (i, I)-internal. □

Since we will need to talk about types in ultrapowers $\mathcal{M}^{u(i)}$, it will be useful to us to have a way of talking about these types as if they were subsets of ${}^*\mathcal{L}(\mathcal{M})$. Note that, in general, types of $\mathcal{M}^{u(i)}$ will not be able to be thought of as elements of ${}^*\mathbb{U}(A)$.

Definition 2.6.9 (Nonstandardization of a Formula)**.** Let $\varphi(x_0, \ldots, x_n, y_0, \ldots, y_m) = \varphi(\bar{x}, \bar{y})$ be a formula and consider the formula with parameters $\varphi(x_0, \ldots, x_n, [f_0]_{u(i)}, \ldots, [f_m]_{u(i)})$ where each f_j is a function $I \to M$ so that the f_j are parameters from $\mathcal{M}^{u(i)}$. Then the **nonstandardization of** $\varphi(x_0, \ldots, x_n, [f_0]_{u(i)}, \ldots, [f_m]_{u(i)})$ is the function $N(\varphi(\bar{x}, [\bar{f}]_{u(i)}))\colon I \to \mathcal{L}(\mathcal{M})$ defined by

$$N(\varphi(\bar{x}, [\bar{f}]_{u(i)}))(j) := \varphi(\bar{x}, \bar{f}(j))$$

where $\varphi(\bar{x}, \bar{f}(j))$ is the pair (φ, g) and

$$g(z) := \begin{cases} z, & \text{if } z \in \{x_0, \ldots, x_n\}. \\ f_k(j), & \text{if } z = y_k. \end{cases}$$

Remark 2.6.10. Although N depends on the choice of representatives picked for the parameters of the formula $\varphi(\bar{x})$, we will assume that this choice is made in the background and will not explicitly state this choice in most cases. This will not cause problems for us as we will be most concerned with ${}^*N(\varphi(\bar{x}))(i)$, which does not depend on the choice of representatives picked.

Lemma 2.6.11. If $\varphi(x, [\bar{f}]_{u(i)})$ is a formula in the ultrapower $\mathcal{M}^{u(i)}$, then there is an (i, I)-internal witness to ${}^*\exists x,\ {}^*N(\varphi(x, [\bar{f}]_{u(i)}))(i)$ if and only if $\exists x,\ \varphi(x, [\bar{f}]_{u(i)})$ holds in $\mathcal{M}^{u(i)}$. Moreover, these witnesses are in bijection via the isomorphism in Theorem 2.6.6.

Proof. This is a special case of Lemma 2.6.2 once we note that

$$\exists x,\ N(\varphi(x, [\bar{f}]_{u(i)}))(j) = N(\exists x,\ \varphi(x, [\bar{f}]_{u(i)}))(j)$$

for all $j \in I$. □

Theorem 2.6.12. Given a set of formulas $\Phi(x)$ with only x free and having parameters from $\mathcal{M}^{u(i)}$, the collection of formulas $\{{}^*N(\varphi(x))(i) : \varphi(x) \in \Phi(x)\}$ has an (i, I)-internal realization if and only if $\Phi(x)$ has a realization in $\mathcal{M}^{u(i)}$. Moreover, if $\Psi(x)$ is a collection of (i, I)-internal elements of ${}^*\mathcal{L}(\mathcal{M})$ such that $\Psi_1(x) \subseteq *[\mathcal{L}(\mathcal{M})_1]$ (recall Definition 2.6.1 (2)) and x is the only variable free in $\Psi(x)$, then there is a collection of formulas $\widehat{\Psi}(x)$ with parameters from $\mathcal{M}^{u(i)}$ such that $\{{}^*N(\psi(x))(i) : \psi(x) \in \widehat{\Psi}(x)\} = \Psi(x)$.

Proof. The first statement follows from Lemma 2.6.11, in particular the fact that realizations of nonstandardizations and the original formula are in bijection via the isomorphism between the (i, I)-internal ultrapower of $\mathcal{M}$ and the usual ultrapower $\mathcal{M}^{u(i)}$. For the moreover, it will be enough to show that if (φ, f) is an (i, I)-internal element of ${}^*\mathcal{L}(\mathcal{M})$ with $\varphi \in *[\mathcal{L}(\mathcal{M})_1]$, then (φ, f) is in the range of N.

Suppose that $k \colon I \to \mathcal{L}(\mathcal{M})$ is such that ${}^*k(i) = (\varphi, f)$. Note that we may assume the $(k(j))_1$ is φ regardless of j; if not, we may construct $\widetilde{k} \colon I \to \mathcal{L}(\mathcal{M})$ so that

$$\widetilde{k}(j) = \begin{cases} k(j), & \text{if } (k(j))_1 = \varphi \\ (\varphi, \mathrm{id}), & \text{otherwise} \end{cases}.$$

By transfer of the definition of $\widetilde{k}$, ${}^*\widetilde{k}(i) = {}^*k(i)$. Similarly, we may assume that for all $j \in I$ we have that $(k(j))_2 = f_j$ is such that $f_j(x) = x$ and $f_j(y) \in M$ for all other variables y that are free in φ. We then have that the formula $\psi(x) = \varphi(x, [f_j]_{u(i)})$ is such that ${}^*N(\psi(x))(i) = {}^*k(i)$. $\square$

Due to the result of Theorem 2.6.12 we will interchangeably think of types of the ultrapower $\mathcal{M}^{u(i)}$ as consisting of formulas in the usual sense and as subsets of ${}^*\mathcal{L}(\mathcal{M})$ consisting of (i, I)-internal elements whose formulas come from $*[\mathcal{L}(\mathcal{M})_1]$.

Corollary 2.6.13. Suppose that $\Gamma \in {}^*\mathcal{P}_\omega(\mathcal{L}(\mathcal{M}))$ is (i, I)-internal with only x free and ${}^*\models_{\mathcal{M}}$ $({}^*\exists x,\ {}^*\bigwedge \Gamma)$. If $\Phi \subseteq \Gamma$ with $\Phi_1 \subseteq *[\mathcal{L}(\mathcal{M})_1]$, then $\Phi(x)$ is a realized type in $\mathcal{M}^{u(i)}$.

Proof. By Theorem 2.6.12, $\Phi(x)$ thought of as a type of $\mathcal{M}^{u(i)}$ has a realization if and only if $\Phi(x)$ has an (i, I)-internal realization. Such an (i, I)-internal realization of $\Phi(x)$ is guaranteed to exist because we know that ${}^*\models_{\mathcal{M}}$ $({}^*\exists x,\ {}^*\bigwedge \Gamma)$ holds. $\square$

2.6.2 Example Applications to the Keisler Order

One possible application of nonstandard model theory is to give a nonstandard way of understanding the Keisler Order. We give here an alternative proof to Keisler's theorem that regular ultrapowers either saturate all the first-order structures modeling a countable complete theory T or none of them.

Theorem 2.6.14 (Keisler's Order is Well-Defined [Keisler(1967)])**.** Suppose that T is a countable complete first order theory, $\mathcal{M}, \mathcal{N} \models T$, and $\mathcal{U}$ a λ-regular ultrafilter on the set I. Then $\mathcal{M}^{\mathcal{U}}$ is λ^+-saturated if and only if $\mathcal{N}^{\mathcal{U}}$ is λ^+-saturated.

Proof. Let $*$ be a nonstandard framework for $\mathbb{U}(M \cup N \cup \mathcal{L} \cup I)$ (where the union is assumed to be disjoint) and let $i \in {}^*I$ be such that $u(i) = \mathcal{U}$. Suppose that $\mathcal{M}^{\mathcal{U}}$ is λ^+-saturated and let $\Phi(x)$ be a type in $\mathcal{N}^{\mathcal{U}}$ where we are thinking of $\Phi(x)$ as a subset of ${}^*\mathcal{L}(M)$ as in Theorem 2.6.12 and further suppose $|\Phi(x)| \leq \lambda$.

From the characterization of λ-regular ultrafilters given in Corollary 2.5.9 there is an (i, I)-internal $D \in {}^*\mathcal{P}_\omega(\mathcal{L}(N))$ such that $\Phi(x) \subseteq D$ and D only has x free.

Because $\mathcal{M} \equiv \mathcal{N}$, for every $\Delta \in \mathcal{P}_\omega(\mathcal{L}(N))$ there is a mapping $g\colon \Delta \to \mathcal{L}(M)$ such that for any $\Gamma \subseteq \Delta$ we have

$$\vDash_{\mathcal{N}} \left(\exists x, \bigwedge \Gamma\right) \iff \vDash_{\mathcal{M}} \left(\exists x, \bigwedge g[\Gamma]\right)$$

and that $g(\varphi, f) = (\varphi, \widehat{f})$ for some partial function $\widehat{f}$ from the free variables of φ to M such that $\widehat{f}$ leaves at most x free. Since the two properties above are definable, we obtain an (i, I)-internal mapping $g\colon D \to {}^*\mathcal{L}(M)$ having the same properties by using Lemma 2.6.7. As g is (i, I)-internal, $g[\Phi(x)]$ is a collection of (i, I)-internal formulas, which must have the form (φ, f) where $\varphi \in *[\mathcal{L}(M)_1]$. Moreover,

$$^*\vDash_{\mathcal{M}} \left({}^*\exists x,\ {}^*\bigwedge g[\Gamma]\right) \iff {}^*\vDash_{\mathcal{N}} \left({}^*\exists x,\ {}^*\bigwedge \Gamma\right) \iff \top$$

for any (truly) finite $\Gamma \subseteq \Phi(x)$ by the definition of g and $\Phi(x)$ being a type. That is, $\widehat{\Phi} := g[\Phi(x)]$ is a type in $\mathcal{M}^{u(i)}$, and the λ^+-saturation of $\mathcal{M}^{u(i)}$ gives an $m \in {}^*M$ so that ${}^*\vDash_{\mathcal{M}} (g(\varphi, f)(m))$ for all $(\varphi, f) \in \Phi(x)$. Define $\Gamma_m \subseteq \Gamma$ by

$$\Gamma_m := \{(\varphi, f) \in \Gamma : {}^*\vDash_{\mathcal{M}} (g(\varphi, f)(m))\}$$

and note that $\widehat{\Phi}(x) \subseteq \Gamma_m$. By Lemma 2.6.3, ${}^*\vDash_{\mathcal{M}} ({}^*\exists x, {}^*\bigwedge \Gamma_m)$ and Γ_m is (i, I)-internal as it is defined in terms of (i, I)-internal elements (Corollary 2.6.8). Therefore $g^{-1}[\Gamma_m]$ is an (i, I)-internal extension of $\Phi(x)$ and ${}^*\vDash_{\mathcal{N}} \left({}^*\exists x, {}^*\bigwedge g^{-1}[\Gamma_m]\right)$ by the properties of g. By Corollary 2.6.13, $\Phi(x)$ has a realization in $\mathcal{N}^{u(i)}$. □

The same techniques can also be used to prove that regular and good ultrapowers saturate every countable first order theory, giving another alternate proof for a result originally proved in [Keisler(1967)].

Corollary 2.6.15. If $u(i)$ is regular and good, then $\mathcal{M}^{u(i)}$ is $|I|^+$-saturated for all structures $\mathcal{M}$ in a countable language.

Proof. Let $\Phi(x) \subseteq {}^*\mathcal{L}(M)$ correspond to a type of $\mathcal{M}^{u(i)}$ with cardinality at most $|I|$. For each

$\Delta \in \mathcal{P}_\omega(\Phi(x))$ let

$$b_\Delta = \left\{ m \in {}^*M : {}^*\!\models_{\mathcal{M}} \left(\bigwedge_{(\varphi,f)\in\Delta}^{*} \big(\varphi, f \cup \{(x,m)\}\big) \right) \right\}.$$

The collection $B = \{b_\Delta : \Delta \in \mathcal{P}_\omega(\Phi(x))\}$ is then a collection with the finite intersection property since $\Phi(x)$ is finitely satisfiable and each b_Δ is (i, I)-internal. By Theorem 2.5.16, there is an (i, I)-internal element m of $\bigcap B$. By the definition of B and the fact that m is (i, I)-internal, m realizes the type $\Phi(x)$ in $\mathcal{M}^{u(i)}$. $\square$

Chapter 3

Theories with SOP_2 in Keisler's Order: A Nonstandard Perspective

The first (model theoretically defined) collection of theories known to be part of the maximal class were those theories with the Strict Order Property [Shelah(1990)]. This result, for example, showed that any theory with a model having an infinite collection of tuples that is linearly ordered by a definable relation is in the maximal Keisler class. A few years later, it was discovered that having the Stict Order Property of order 3 (SOP_3) is enough to be Keisler maximal, as shown in [Shelah(1996)]. Note that the SOP_n are defined to be weakenings of the Strict Order Property in [Shelah(1996)]. Although an exact model theoretic characterization of the maximal class in Keisler's order is not known, it is conjectured in [Malliaris and Shelah(2016)] that the maximal class is the class of theories having Strict Order Property 2 (SOP_2) as it is shown in the same paper that having SOP_2 is enough to guarantee that a theory is in the maximal Keisler class (note that $\mathrm{SOP}_3 \to \mathrm{SOP}_2$ [Shelah(1996)]).

In the paper [Malliaris and Shelah(2016)], specific types called SOP_2-types are studied and used to prove that theories with SOP_2 are maximal in Keisler's order. We give here some relevant definitions that are equivalent to those used by Malliaris and Shelah but that are phrased so as to be easier to use for our purposes.

Definition 3.0.1 (φ-type)**.** A type $\Phi(\bar{x})$ of $\mathcal{M}$ is a φ-type if every element of $\Phi(\bar{x})$ has the form $\varphi(\bar{x}, \bar{a})$ for a tuple of variables $\bar{x}$ and a tuple of elements $\bar{a}$ from M. Moreover, $\bar{x}$ is the same for all elements of $\Phi(\bar{x})$.

Definition 3.0.2 (SOP_2-Trees and SOP_2-Types, e.g. [Malliaris and Shelah(2016)])**.**

(1) We say that a structure $\mathcal{M}$ **has an** SOP_2**-tree** if there is a formula $\varphi(\bar{x};\bar{y})$ and a sequence of tuples $(\bar{a}_\eta)_{\eta\in 2^{<\omega}}$ from $\mathcal{M}$ all the same length as $\bar{y}$ and for all $\Gamma \subseteq 2^{<\omega}$ the sets

$$\{\varphi(\bar{x};\bar{a}_\eta) : \eta \in \Gamma\}$$

are consistent if and only if Γ is a chain in $2^{<\omega}$. We say that the pair $(\varphi(\bar{x};\bar{y}), (\bar{a}_\eta)_{\eta\in 2^{<\omega}})$ is an SOP_2**-tree in** $\mathcal{M}$.

(2) A first-order complete theory T is said to have SOP_2 if there is some $\mathcal{M} \models T$ such that $\mathcal{M}$ has an SOP_2-tree.

(3) A φ-type $\Phi(\bar{x})$ is an SOP_2**-type** if there is a sequence $(P_j)_{j\in I}$ of sets of tuples from $\mathcal{M}$ such that

   (a) the parameters occurring in $\Phi(\bar{x})$ are from the ultraproduct $\prod_{j\in\mathcal{U}} P_j$ and

   (b) for each $j \in I$ there is an SOP_2-tree $(\varphi(\bar{x};\bar{y}), (\bar{a}_\eta)_{\eta\in 2^{<\omega}})$, possibly different for each j, with $P_j \subseteq \{\bar{a}_\eta : \eta \in 2^{<\omega}\}$.

In this chapter, we will give an alternative proof of the maximality of theories with SOP_2 within Keisler's order by using the methods of nonstandard analysis developed in Chapter 2 along with some of the machinery developed in [Ulrich(2018)] as part of an alternative proof of $\mathfrak{p} = \mathfrak{t}$, and then we will study the structure of SOP_2-types using the machinery of characteristic sequences developed in [Malliaris(2010)] which will lead to a simple description of a small class of distributions that must have multiplicative refinements for a regular ultrafilter to be good (see Section 3.2 for relevant definitions). We also apply this method to study the sorts of distributions that occur from types corresponding to cuts in infinite linear orders as these are the types used in the proof that theories with the Strict Order Property are Keisler maximal.

3.1 Maximality of Theories with SOP_2 in Keisler's Order

It is already known that theories with the SOP_2 property are maximal in Keisler's order on the complete first-order theories in a countable language. The original proof of this fact, in

[Malliaris and Shelah(2016)], uses much of the full machinery of Cofinality Spectrum Problems.

Ulrich has shown that the proof of $\mathfrak{p} = \mathfrak{t}$ found in [Malliaris and Shelah(2016)] can be somewhat simplified to work only with the cofinality spectra of the internally defined natural numbers in models V of ZFC which contain nonstandard natural numbers. Here, we extend the approach used in [Ulrich(2018)] to give an alternative proof to the fact that theories with the SOP_2 property are maximal in Keisler's order.

As we will frequently want to talk about cuts and pre-cuts and their types and representations, we give a definition of these important concepts here. Note that the terms "gap" and "pre-gap" are frequently used in the set theory literature instead of the terms "cut" and "pre-cut". We also give Ulrich's definitions for the cardinals $\mathfrak{p}_{\widehat{V}}$ and $\mathfrak{t}_{\widehat{V}}$.

Definition 3.1.1.

(1) A **pre-cut** in a linear order $\mathbb{L} = (L, <)$ is a pair of nonempty subsets (A, B) of L such that $a < b$ for all $a \in A$ and $b \in B$ and such that A is down-closed and B is up-closed with respect to $<$.

(2) A **cut** in $\mathbb{L}$ is a pre-cut (A, B) of $\mathbb{L}$ such that the set of formula

$$p(x) = \{a < x < b : (a, b) \in A \times B\}$$

is not realized in $\mathbb{L}$. Note that if $(A, <)$ is unbounded above and $(B, <)$ is unbounded below, then $p(x)$ is a type of $\mathbb{L}$ by compactness.

(3) If (A, B) is a pre-cut of $\mathbb{L}$, then the **type** of (A, B) is the pair of (regular) cardinals (κ, λ) where κ is the cofinality of $(A, <)$ and λ is the cofinality of $(B, >)$ (or, the coinitiality of $(B, <)$). We will mostly deal with pre-cuts of type (κ, λ) where $\kappa, \lambda \geq \omega$.

(4) If (A, B) is a pre-cut of $\mathbb{L}$ of type (κ, λ) then a **representation** of (A, B) is a pair of sequences $(a_\alpha)_{\alpha<\kappa}$ and $(b_\beta)_{\beta<\lambda}$ where the sequence $(a_\alpha)_{\alpha<\kappa}$ is increasing and cofinal in A and the sequence $(b_\beta)_{\beta<\lambda}$ is decreasing and coinitial in B.

Definition 3.1.2 ([Ulrich(2018)])**.**

(1) If $\widehat{V}$ is a set model of ZFC, we say that $\widehat{V}$ has **nonstandard natural numbers** if the order type of the internally defined natural numbers of $\widehat{V}$ have an order type in the ambient set theory other than ω. We will use $\widehat{\omega}$ to mean the internally defined natural numbers of $\widehat{V}$.

(2) If $\widehat{V}$ has nonstandard natural numbers, we will define the cardinals

$$\mathfrak{p}_{\widehat{V}} := \min\{\kappa + \lambda : \widehat{\omega} \text{ has a cut of type } (\kappa, \lambda)\}$$

and

$$\mathfrak{t}_{\widehat{V}} := \min\{\kappa : \exists \widehat{n} \in \widehat{\omega}, (a_\alpha)_{\alpha<\kappa} \in \widehat{n}^{<\widehat{n}},\ (a_\alpha)_{\alpha<\kappa} \text{ is increasing and unbounded in } \widehat{n}^{<\widehat{n}}\}.$$

That is, $\mathfrak{p}_{\widehat{V}}$ is the least cardinality needed to represent a (unfilled) cut in $\widehat{\omega}$, and $\mathfrak{t}_{\widehat{V}}$ is the least cardinality of an increasing unbounded chain occurring in the trees $\widehat{n}^{<\widehat{n}}$ where $\widehat{n} \in \widehat{\omega}$. An important note about $\mathfrak{t}_{\widehat{V}}$ is that the sequences $(a_\alpha)_{\alpha<\kappa}$ are external to $\widehat{V}$ and so the internal finiteness of $\widehat{n}$ does not prevent such sequences from occurring.

An important part of our proof will be [Ulrich(2018), Theorem 3.1] and a key step in proving $\mathfrak{p} = \mathfrak{t}$: that $\mathfrak{p}_{\widehat{V}} = \mathfrak{t}_{\widehat{V}}$. We define a cardinal invariant for ultrafilters $\mathcal{U}$ as follows.

Definition 3.1.3. An ultrafilter $\mathcal{U}$ is countably incomplete if there is an $X \in [\mathcal{U}]^{\aleph_0}$ such that $\bigcap X \notin \mathcal{U}$.

Definition 3.1.4. Suppose that $\mathcal{U}$ is a countably incomplete ultrafilter. We define the following cardinals:

(1) $\mathfrak{p}_{\mathcal{U}} := \min\{\kappa + \lambda : (\omega, <)^{\mathcal{U}} \text{ has a cut of type } (\kappa, \lambda)\}$ and

(2) $\mathfrak{t}_{\mathcal{U}} := \min\{\kappa : \exists n \in (\omega, <)^{\mathcal{U}}, (a_\alpha)_{\alpha<\kappa} \in n^{<n},\ (a_\alpha)_{\alpha<\kappa} \text{ is increasing and unbounded in } n^{<n}\}$ or ∞ if the set of such κ is empty (it will turn out that ∞ cannot happen, see Corollary 3.1.7).

Note that we require that $\mathcal{U}$ is countably incomplete so that the ultrapower $(\omega, <)^{\mathcal{U}}$ is not isomorphic to $(\omega, <)$.

Our proof of the maximality of theories with SOP_2 in the Keisler Order will be based on the following theorem characterizing regular and good ultrafilters based on the size of $\mathfrak{p}_{\mathcal{U}}$, the proof of which will be delayed to the end of this section.

Theorem 3.1.5. A regular ultrafilter $\mathcal{U}$ is α^+-good if and only if $\mathfrak{p}_{\mathcal{U}} > \alpha$.

For those familiar with the Strict Order Property (SOP), we note that Theorem 3.1.5 is equivalent to a theorem of [Shelah(1990)] that theories with the SOP are in the maximal class of Keisler's Order, as the theory of any infinite linear order has SOP. However, we present here an alternative proof based on the equality of $\mathfrak{p}_{\widehat{V}}$ and $\mathfrak{t}_{\widehat{V}}$ and [Ulrich(2018), Theorem 2.3] showing that $\mathfrak{p}_{\widehat{V}}$ is related to the ability to use recursion on sequences external to $\widehat{V}$.

We will first translate [Ulrich(2018), Theorem 3.1] that $\mathfrak{p}_{\widehat{V}} = \mathfrak{t}_{\widehat{V}}$ to a statement that is more easily applicable to the context in which we are working.

Lemma 3.1.6. Assuming the consistency of ZFC, for every structure $\mathcal{M}$ in the language $\mathcal{L}$ there is a model V of ZFC containing $\mathcal{M}$, the interpretation in $\mathcal{M}$ of every symbol in $\mathcal{L}$, and the ambiently defined ω. Furthermore, V can be taken to have the same theory as the ambient set theory. For such a V and a countably incomplete ultrafilter $\mathcal{U}$, the ultrapower $\widehat{V} = V^{\mathcal{U}}$ is such that $\mathfrak{p}_{\mathcal{U}} = \mathfrak{p}_{\widehat{V}}$ and $\mathfrak{t}_{\mathcal{U}} = \mathfrak{t}_{\widehat{V}}$.

Proof. It will be enough to show that there is a model V of ZFC containing $\mathbb{U}(M \cup \omega)$ since $\mathbb{U}(M \cup \omega)$ contains M, every possible interpretation of a symbol in $\mathcal{L}$ in M, and the ambient ω. We can achieve this by using the method of the Downward Löwenheim-Skolem theorem. We start with $V_0 = \mathbb{U}(M \cup \omega)$. For each $n \in \omega$, let V_{n+1} be defined by adding a witness to V_n for statements of the form $\exists x, \varphi(x)$ that are satisfied by the ambient set theory and where φ is a formula in the language of set theory with parameters from V_n. Taking $V = \bigcup_{n\in\omega} V_n$ and using the Tarski-Vaught test gives us our desired model of set theory.

Let $\widehat{V} = V^{\mathcal{U}}$. As $\omega \subseteq V$, we have that $(\omega, <)^{\mathcal{U}}$ embeds in $\widehat{V}$ and the image of this embedding is the internally defined natural numbers of $\widehat{V}$. By definition of $\mathfrak{p}_{\mathcal{U}}$ and $\mathfrak{t}_{\mathcal{U}}$ we have that $\mathfrak{p}_{\widehat{V}} = \mathfrak{p}_{\mathcal{U}}$ and $\mathfrak{t}_{\mathcal{U}} = \mathfrak{t}_{\widehat{V}}$. □

Corollary 3.1.7. For any countably incomplete ultrafilter $\mathfrak{p}_{\mathcal{U}} = \mathfrak{t}_{\mathcal{U}}$.

Proof. By Lemma 3.1.6, $\mathfrak{p}_{\mathcal{U}}$ and $\mathfrak{t}_{\mathcal{U}}$ are the same as $\mathfrak{p}_{\widehat{V}}$ and $\mathfrak{t}_{\widehat{V}}$ for some model $\widehat{V}$ of ZFC with nonstandard natural numbers. By [Ulrich(2018), Theorem 3.1] $\mathfrak{p}_{\widehat{V}} = \mathfrak{t}_{\widehat{V}}$, therefore

$$\mathfrak{p}_{\mathcal{U}} = \mathfrak{p}_{\widehat{V}} = \mathfrak{t}_{\widehat{V}} = \mathfrak{t}_{\mathcal{U}}.$$ □

We next use [Ulrich(2018), Theorem 2.4] to obtain a condition under which a type $p(x)$ is guaranteed to be realized in the ultrapower $\mathcal{M}^{\mathcal{U}}$. Note that Ulrich uses ZFC^- to mean ZFC without the powerset axiom.

Theorem 3.1.8 ([Ulrich(2018), Theorem 2.4])**.** Suppose $\widehat{V} \vDash \mathrm{ZFC}^-$ is ω-nonstandard. Suppose $p(x) = \{\varphi_\alpha(x, a_\alpha) : \alpha < \lambda\}$ is a partial type over $\widehat{V}$ of cardinality $\lambda < \mathfrak{p}_{\widehat{V}}$. Suppose $X \in \widehat{V}$ is pseudofinite, and $\varphi_0(x)$ is $x \in \widehat{X}$. Then $p(x)$ is realized in $\widehat{V}$.

Theorem 3.1.9. If $\mathcal{U} = u(i)$ is countably incomplete, and $p(x)$ is a type of $\mathcal{M}^{\mathcal{U}}$ with $|p(x)| < \mathfrak{p}_{\mathcal{U}}$, and there is a hyperfinite subset X of $M^{\mathcal{U}}$ such that $p(x) \cup \{x \in X\}$ is consistent in $(\mathbb{U}(M \cup \omega))^{\mathcal{U}}$ with $p(x)$ interpreted as a type in the language of set theory, then $p(x)$ is realized in $\mathcal{M}^{\mathcal{U}}$.

Proof. Let $\widehat{V}$ be as in Lemma 3.1.6. $M^{\mathcal{U}}$ is a subset of $\widehat{V}$ and the interpretations of all the symbols of $\mathcal{L}$ in $\mathcal{M}^{\mathcal{U}}$ are elements of $\widehat{V}$, so $p(x)$ can be interpreted as a type of $\widehat{V}$. Since $X \in (\mathcal{P}_\omega(M))^{\mathcal{U}}$, X is also a subset of $\widehat{V}$ and is an internally finite subset of $M^{\mathcal{U}}$. By [Ulrich(2018), Theorem 2.4], $p(x)$ is realized in $\widehat{V}$ and hence is realized in $M^{\mathcal{U}}$. □

Recall the definition of $\vDash_{\mathcal{M}}$ and the equivalence between the (i, I)-internal ultrapower of M and $\mathcal{M}^{\mathcal{U}}$ used in Section 2.6. We will use an enlargement $*$ of $\mathbb{U}(M \cup \omega \cup I \cup \mathcal{L})$ here to simplify some ultrapower computations.

Corollary 3.1.10. If $\mathcal{U}$ is an α-regular ultrafilter such that $\mathfrak{p}_{\mathcal{U}} > \alpha$, then $\mathcal{U}$ is α^+-good.

Proof. Suppose that $p(x)$ is a type of the structure $\mathcal{M}^{\mathcal{U}}$ with $|p(x)| < \alpha^+$, so $|p(x)| < \mathfrak{p}_{\mathcal{U}}$. By Theorem 3.1.9, it is enough to show that there is an (i, I)-internal hyperfinite $X \subseteq {}^*M$ such that $\{x \in X\} \cup p(x)$ is consistent.

For each $\Delta \in \mathcal{P}_\omega(p(x))$ there exists an m_Δ in $\mathcal{M}^{\mathcal{U}}$ such that ${}^*\vDash_{\mathcal{M}} ({}^*\bigwedge \Delta(m_\Delta))$. By Corollary 2.5.9 and translating to the language of (i, I)-internal sets, there is an (i, I)-internal hyperfinite set X extending $S = \{m_\Delta : \Delta \in \mathcal{P}_\omega(p(x))\}$. In particular, for any finite $\Delta \subseteq p(x)$, there is an element m_Δ of X satisfying $\bigwedge \Delta(x)$, so $x \in X$ is consistent with $p(x)$. □

Proof of Theorem 3.1.5. Corollary 3.1.10 gives one direction. The other direction follows from cuts in $(\omega, <)^{\mathcal{U}}$ being types and the fact that regular and good ultrapowers $\mathcal{U}$ are guaranteed to realize all types of cardinality less than the cardinality of the index set for $\mathcal{U}$ that are over the ultrapower. □

The following Theorem 3.1.11 completes the argument that theories with SOP_2-trees are maximal in Keisler's order given the fact that a theory T is in the maximum Keisler class if and only if the models of T are only saturated by good ultrafilters [Keisler(1967)].

Theorem 3.1.11 ([Malliaris and Shelah(2016)])**.** If T is a countable first-order theory with SOP_2 and $\mathcal{M} \vDash T$ contains an SOP_2-tree, then a regular ultrafilter $\mathcal{U}$ saturates $\mathcal{M}$ if and only if $\mathcal{U}$ is good.

Proof. If $\mathcal{U}$ is good, then $\mathcal{U}$ saturates every structure in a countable language (recall Definition 1.1.5), so we only need to show the forwards direction. Let α be the cardinality of the indexing set I that $\mathcal{U}$ is an ultrafilter over. Our strategy will be to show that $\mathfrak{p}_{\mathcal{U}} = \mathfrak{t}_{\mathcal{U}} > \alpha$ so that $\mathcal{U}$ is good by Corollary 3.1.10. To prove that $\mathfrak{t}_{\mathcal{U}} > \alpha$, let $\varphi(x; \bar{y})$ be a formula in $\mathcal{L}$ such that there are constant tuples in M of the form $(\bar{a}_\eta)_{\eta \in 2^{<\omega}}$ such that, whenever A is a subset of the given constant tuples, $\{\varphi(x, \bar{a}_\eta) : \bar{a}_\eta \in A\}$ is consistent in $\mathcal{M}$ if and only if $\{\eta : \bar{a}_\eta \in A\}$ is linearly ordered in $2^{<\omega}$. That is, $(\varphi, (\bar{a}_\eta)_{\eta \in 2^{<\omega}})$ is an SOP_2-tree.

Consider an enlargement $*$ on a base set B containing I, M, and $\mathcal{L}$ as subsets. We note that

$$\psi = \forall n \in \omega,\ \exists f \colon n^{<n} \to 2^{<\omega},\ \text{“}f \text{ is an order embedding”}$$

is a true statement in $\mathbb{U}(B)$, so transfer guarantees that such order embeddings also exist for $n \in {}^*\omega$ and ${}^*(2^{<\omega})$. In particular, Lemma 2.6.7 translated to the language of (i, I)-internal sets, guarantees that if $n \in {}^*\omega$ is chosen to be (i, I)-internal, then we may also choose the embedding f to be (i, I)-internal.

Suppose towards a contradiction that $n \in \omega^{\mathcal{U}}$ is such that there is an increasing sequence $b := (b_\beta)_{\beta<\lambda}$ in $n^{<n}$ where $\lambda \leq \alpha$ and there is no upper bound to b in $n^{<n}$. Let f be an (i, I)-internal embedding of $n^{<n}$ in ${}^*(2^{<\omega})$ guaranteed to exist by the transfer of ψ, and consider the sequence $f(b) = (f(b_\beta))_{\beta<\lambda}$. Consider the collection $\Phi(x) := \{\varphi(x; {}^*\bar{a}_{f(b_\beta)}) : \beta < \lambda\}$ where the parameters ${}^*\bar{a}_{f(b_\beta)}$ come from applying $*$ to the function $2^{<\omega} \to M^m$ defined by $\eta \mapsto \bar{a}_\eta$. $\Phi(x)$, thought of as a collection of formula in $\mathcal{M}^{\mathcal{U}}$ using 2.6.11 is finitely satisfiable in $\mathcal{M}^{\mathcal{U}}$ since each finite subset is internal and transfer guarantees that linearly ordered finite subsets of the SOP_2 are consistent, hence realized, in $\mathcal{M}^{\mathcal{U}}$. Since $\mathcal{U}$ saturates $\mathcal{M}$ by assumption, we have that there is some $m \in \mathcal{M}^{\mathcal{U}}$ such that $\mathcal{M}^{\mathcal{U}} \vDash \varphi(m; \bar{a}_{f(b_\beta)})$ for every $\beta < \lambda$. Moreover, we can extend $\{f(b_\beta) : \beta < \lambda\}$ to an internal hyperfinite set C in ${}^*(2^{<\omega})$ using the regularity of $\mathcal{U}$ and Corollary 2.5.9. Define

$$\Psi(x) := \{\varphi(x; \bar{a}_c) : c \in C \text{ and } {}^*\vDash_{\mathcal{M}} \varphi(m, \bar{a}_c)\}.$$

As $\Psi(x)$ is realized in ${}^*M$ by m, it must be that $C' := \{c \in C : \varphi(x; \bar{a}_c) \in \Psi(x)\}$ is linearly ordered in ${}^*(2^{<\omega})$. C' is hyperfinite so there must be a maximum element of $C' \cap \mathrm{im}(f)$, say c. Because f is an order embedding and c is (i, I)-internal (being defined by (i, I)-internal elements), $f^{-1}(c)$ is an element of $n^{<n}$ that is above b_β for all $\beta < \lambda$, contradicting the claim that b is unbounded. □

3.2 The Shape of Types and their Distributions

Given a collection of formulas $\Phi(x)$ in an ultraproduct $\widehat{\mathcal{M}} = \prod_{i \in \mathcal{U}} \mathcal{M}_i$ with parameters, it is natural to ask how to determine, given representatives $f \in \prod_{i \in I} M_i$ for the parameters occurring in $\Phi(x)$, whether $\Phi(x)$ is a type of $\widehat{\mathcal{M}}$ and whether $\Phi(x)$ is realized in $\widehat{\mathcal{M}}$ by looking at the behavior of the formulas $\varphi(x, \bar{f}(i))$ in the structures $\mathcal{M}_i$ for each $\varphi(x, \bar{f}) \in \Phi(x)$. One answer to this question is to look at certain hypergraphs associated to the structures $\mathcal{M}_i$, such as the characteristic sequences defined in [Malliaris(2010)]. The definition given here is a slight generalization of the notion defined by Malliaris.

Definition 3.2.1 (Characteristic Hypergraph)**.** Given a collection of formulas, possibly with parameters, $\Psi(x)$ in a structure $\mathcal{N}$, the **characteristic hypergraph** of $\Psi(x)$ in $\mathcal{N}$ is the structure $\mathcal{G}^{\mathcal{N}}_{\Psi(x)}$ with underlying set $\Psi(x)$ and having relations $P_n(x_0, \dots, x_{n-1})$ for each n defined by

$$P_n(\varphi_0(x), \dots, \varphi_{n-1}(x)) \iff \bigwedge_{(i,k)\in n^2,\, i\neq k} \varphi_i(x) \neq \varphi_k(x) \text{ and } \mathcal{N} \vDash \exists x, \bigwedge_{i=0}^{n-1} \varphi_i(x).$$

Remark 3.2.2. Note that the P_n are symmetric in the sense that any permutation of the inputs does not change the evaluation of P_n. Due to this fact, we will frequently think of P_n as being a predicate on the set $[\Psi(x)]^n$ of n-element subsets of $\Psi(x)$.

If for each $i \in I$ we define $\Phi_i(x)$ to be $\{\varphi(x, \bar{f}(i)) : \varphi(x, \bar{f}) \in \Phi(x)\}$, then we get a characteristic hypergraph $\mathcal{G}^{\mathcal{M}_i}_{\Phi_i(x)}$ for each $i \in I$. Using Łoś's theorem, we can tell whether $\Phi(x)$ is a type of $\widehat{M}$ by looking at the characteristic hypergraphs $\mathcal{G}^{\mathcal{M}_i}_{\Phi_i(x)}$ in the following well-known way.

Lemma 3.2.3. $\Phi(x)$ is a type of $\widehat{\mathcal{M}}$ if and only if for every $\Delta(x) \in \mathcal{P}_\omega(\Phi(x))$ we have that

$$\{i \in I : \mathcal{G}^{\mathcal{M}_i}_{\Phi_i(x)} \vDash P_{|\Delta_i(x)|}(\Delta_i(x))\} \in \mathcal{U}.$$

Proof. By definition, $\mathcal{G}^{\mathcal{M}_i}_{\Phi_i(x)} \vDash P_{|\Delta_i(x)|}(\Delta_i(x))$ is equivalent to $\mathcal{M}_i \vDash \exists x, \bigwedge \Delta_i(x)$. By Łoś's theorem, we have that

$$\{i \in I : \mathcal{M}_i \vDash \exists x, \bigwedge \Delta_i(x)\} \in \mathcal{U} \iff \widehat{\mathcal{M}} \vDash \exists x, \bigwedge \Delta(x).$$

By compactness, $\Phi(x)$ is a type if and only if every $\Delta(x) \in \mathcal{P}_\omega(\Phi(x))$ satisfies $\widehat{\mathcal{M}} \vDash \exists x, \bigwedge \Delta(x)$. $\square$

Continuing to follow the work of Malliaris, we can restate the result of Lemma 3.2.3 as a result about the ultraproduct $\prod_{i\in\mathcal{U}} \mathcal{G}^{\mathcal{M}_i}_{\Phi_i(x)}$.

Theorem 3.2.4. Let $\widehat{\mathcal{G}} = \prod_{i\in\mathcal{U}} \mathcal{G}^{\mathcal{M}_i}_{\Phi_i(x)}$ and let $g_{\varphi(x)} \in \prod_{i\in I} \Phi_i(x)$ be the function $g_{\varphi(x)}(i) = \varphi(x, \bar{f}(i))$ for each $\varphi(x, \bar{f}) \in \Phi(x)$. Then $\Phi(x)$ is a type of $\widehat{\mathcal{M}}$ if and only if

$$[\{[g_{\varphi(x)}]_{\mathcal{U}} : \varphi(x) \in \Phi(x)\}]^n \subseteq P_n^{\widehat{\mathcal{G}}}$$

for all $n \in \omega$.

Proof. We will proceed by showing that $g\colon \Phi(x) \to \prod_{i\in I} \Phi_i(x)$ defined by $\varphi(x) \mapsto [g_{\varphi(x)}]_{\mathcal{U}}$ is an embedding $g\colon \mathcal{G}^{\widehat{M}}_{\Phi(x)} \to \widehat{\mathcal{G}}$. The result then follows from compactness and the definition of the P_n.

g is an injective function: if $\varphi(x,\bar{f})$ and $\psi(x,\bar{h})$ are not equal formulas in $\Phi(x)$, then either $\bar{f} \neq \bar{h}$ as sequences of elements from $\widehat{\mathcal{M}}$ or $\varphi \neq \psi$ as formulas without parameters. In the latter case, $\varphi(x,\bar{f}(i)) \neq \psi(x,\bar{h}(i))$ for any $i \in I$, so Łoś's theorem tells us that $[g_{\varphi(x)}]_{\mathcal{U}} \neq [g_{\psi(x)}]_{\mathcal{U}}$. In the first case, Łoś's theorem tells us that $\bar{f}(i) \neq \bar{h}(i)$ for a set of i in $\mathcal{U}$, so $\varphi(x,\bar{f}(i)) \neq \psi(x,\bar{h}(i))$ for a set of i in $\mathcal{U}$. Applying Łoś's theorem, $[g_{\varphi(x)}]_{\mathcal{U}} \neq [g_{\psi(x)}]_{\mathcal{U}}$ as elements of $\widehat{\mathcal{G}}$.

Suppose that $\mathcal{G}^{\widehat{M}}_{\Phi(x)} \vDash P_n(\Delta(x))$ for some $\Delta(x) \in [\Phi(x)]^n$. By the definition of P_n, this is equivalent to the the statement $\widehat{\mathcal{M}} \vDash \exists x, \bigwedge \Delta(x)$. By Łoś's theorem, equivalently

$$\{i \in I : \mathcal{M}_i \vDash \exists x, \bigwedge \Delta_i(x)\} \in \mathcal{U} \quad \text{or} \quad \{i \in I : \mathcal{G}^{\mathcal{M}_i}_{\Phi_i(x)} \vDash P_n(\Delta_i(x))\} \in \mathcal{U}.$$

Noting that the projection of $g[\Delta]$ to the i-th coordinate is $\Delta_i(x)$ and using Łoś's theorem once again shows that $\widehat{\mathcal{G}} \vDash P_n(g[\Delta])$. As every step was an equivalence, we we also get that $P_n(g[\Delta])$ implies $P_n(\Delta)$, finishing the proof. □

In our current general framework, the finitary nature of the P_n makes it difficult to extract useful information about whether or not a type $\Phi(x)$ is realized in $\widehat{\mathcal{M}}$ from the characteristic hypergraphs $\mathcal{G}^{\mathcal{M}_i}_{\varphi(x)}$. However, if we move to the context of Keisler's order where we only care about regular ultrafilters and sets of formulas having cardinality less than or equal to the cardinality of the index set for $\mathcal{U}$, Corollary 2.5.9 tells us that $\Phi(x)$ can be thought of as being "finite" in a certain sense. The apparent finiteness of $\Phi(x)$ in this context will allow us to gain further information about whether $\Phi(x)$ is realized in $\widehat{\mathcal{M}}$ by looking at the characteristic hypergraphs $\mathcal{G}^{\mathcal{M}_i}_{\Phi_i(x)}$.

Convention 3.2.5. As we now are working in the context of Keisler's order, we refine our previous conventions.

(1) For the rest of the chapter, we will assume, for all $i \in I$, that $\mathcal{M}_i = \mathcal{M}$ for some fixed structure $\mathcal{M}$. We will also use $\widehat{\mathcal{M}}$ to mean $\mathcal{M}^{\mathcal{U}}$ when the ultrafilter $\mathcal{U}$ is clear from context.

(2) For the rest of the chapter, we will assume that $\Phi(x)$ consists of formulas $\varphi(x,\bar{f})$ where φ is a fixed formula and $\bar{f}$ is allowed to vary over elements in $\widehat{\mathcal{M}}$. We will call such sets of formulas **φ-sets**. Furthermore, we assume that $|\Phi(x)| \le |I|$. This convention is justified by the result of [Malliaris(2009), Theorem 12] that any regular ultrapower realizing all such types $\Phi(x)$ must realize all types $\Psi(x)$ where $|\Psi(x)| \le |I|$ even if the formulas in $\Psi(x)$ are allowed to vary.

(3) We will fix an enlargement $*\colon \mathbb{U}(A) \to \mathbb{U}(^*A)$ where A contains the disjoint union of I, M, and $\mathcal{L}$.

The following theorem is an analog of the well-known result that every small enough type in a regular ultrapower $\widehat{\mathcal{M}}$ has a distribution. We will discuss distributions and their correspondence with our current point of view in more detail in Section 3.2.2.

Theorem 3.2.6. If $\mathcal{U} = u(i)$ is a regular ultrafilter on I and $\Phi(x)$ is a type in $\widehat{M}$, then the following are equivalent:

(1) $\Phi(x)$ is realized in $\widehat{\mathcal{M}}$.

(2) There is a function $k\colon I \to \mathcal{P}_\omega(\Phi(x))$ such that

  (a) for all $j \in I$, if $\Delta = k(j)$ then $\mathcal{M} \vDash \exists x, \bigwedge \Delta_j(x)$ and,

  (b) for all $\varphi(x,\bar{f}) \in \Phi(x)$ we have that $\{j \in I : \varphi(x,\bar{f}) \in k(j)\} \in \mathcal{U}$.

(3) There is an (i, I)-internal hyperfinite $H \subseteq {}^*\mathcal{L}(M)$ with only x free such that $H \supset \Phi(x)$ (where $\Phi(x)$ is considered as a subset of ${}^*\mathbb{U}(A)$ as in Theorem 2.6.12) and ${}^*\mathbb{U}(A) \vDash {}^*\!\vDash_{\mathcal{M}}$ $({}^*\exists x,\ {}^*\bigwedge H)$.

Proof. (1) $\Rightarrow$ (2): For this proof, we will add the elements of $|I|$ to our base set for our enlargement. Enumerate the elements of $\Phi(x)$, possibly with duplicates, as $(\varphi(x,\bar{f}_\alpha))_{\alpha\in|I|}$. For each $\alpha \in \Phi(x)$, define $\pi_\alpha\colon I \to \mathcal{L}(x) \times |I|$ by $j \mapsto (\varphi(x,\bar{f}_\alpha(j)),\alpha)$. We then have that

$$\widehat{\Phi} := \{{}^*\pi_\alpha(i) : \alpha \in |I|\} = \{(\varphi(x, {}^*\bar{f}_\alpha(i)), {}^*\alpha) : \alpha \in |I|\}.$$

By Corollary 2.5.9, there is an (i, I)-internal hyperfinite set $\widehat{H}$ containing $\widehat{\Phi}$. By taking intersections, we can assume that $\widehat{H}$ is a subset of ${}^*\mathcal{L}(x) \times {}^*|I|$. Let $m \in \widehat{M}$ be a realization of $\Phi(x)$ and define

$$H = \{(\varphi(x, \bar{a}), \alpha) \in \widehat{H} : {}^*{\models}_{\mathcal{M}} \varphi(m, \bar{a})\}.$$

By Corollary 2.6.8, H is (i, I)-internal. Let $\widehat{k} \colon I \to \mathcal{P}_\omega(\mathcal{L}(x) \times |I|)$ be a witness to H being (i, I)-internal. Then we can define a function $k \colon I \to \mathcal{P}_\omega(\Phi(x))$ by

$$k(j) := \begin{cases} \{\varphi(x, \bar{f}_\alpha) : (\varphi(x, \bar{f}_\alpha(j)), \alpha) \in \widehat{k}(j)\}, & \text{if } \exists x \in M, \bigwedge_{(\varphi(x,\bar{a}),\alpha) \in \widehat{k}(j)} \varphi(x, \bar{a}). \\ \emptyset, & \text{otherwise.} \end{cases}.$$

We claim that k satisfies the conditions (a) and (b). The definition of k directly guarantees that (a) is satisfied. For (b), we know that ${}^*\exists x \in {}^*M,\ {}^*\bigwedge_{(\varphi(x,\bar{a}),\alpha) \in {}^*\widehat{k}(i)} \varphi(x, \bar{a})$ is witnessed by m, so the set of j for which $\exists x \in M, \bigwedge_{(\varphi(x,\bar{a}),\alpha) \in \widehat{k}(j)} \varphi(x, \bar{a})$ holds is in $\mathcal{U} = u(i)$ by the definition of $u(i)$. Moreover, for any given $\alpha \in |I|$, we have that ${}^*\pi_\alpha(i) \in {}^*\widehat{k}(i)$, so the set of j for which $\pi(j) \in \widehat{k}(j)$ is in $\mathcal{U} = u(i)$ by the definition of $u(i)$. Since $\mathcal{U}$ is closed under finite intersections, we have that

$$\left\{ j \in I : \exists x \in M, \bigwedge_{(\varphi(x,\bar{a}),\alpha) \in \widehat{k}(j)} \varphi(x, \bar{a}) \right\} \cap \{j \in I : \pi(j) \in \widehat{k}(j)\} \in \mathcal{U},$$

so $\{j \in I : \varphi(x, \bar{f}_\alpha(j)) \in k(j)\} \in \mathcal{U}$ for each $\alpha \in |I|$, as desired.

(2) $\Rightarrow$ (3): Define $\widetilde{k} \colon I \to \mathcal{P}_\omega(\mathcal{L}(x))$ by $j \mapsto (k(j))_j$ and take $H = {}^*\widetilde{k}(i)$. Transfer applied to property (a) of k gives us that ${}^*\mathbb{U}(A) \models {}^*{\models}_{\mathcal{M}} ({}^*\exists x,\ {}^*\bigwedge H)$. Property (b) of k and the definition of $u(i)$ give that $H \supset \Phi(x)$.

(3) $\Rightarrow$ (1): This is Corollary 2.6.13. □

The equivalence between (1) and (3) in Theorem 3.2.6 has a translation into the language of characteristic hypergraphs in Corollary 3.2.8.

Definition 3.2.7 (Complete Hypergraph)**.** We say that a hypergraph substructure $\mathcal{G}$ of a characteristic hypergraph is **complete** if the underlying set G of $\mathcal{G}$ has the property that $P_n(A)$ holds for every $A \in [G]^n$.

Corollary 3.2.8. The type $\Phi(x)$ in $\widehat{\mathcal{M}}$ has a realization in $\widehat{\mathcal{M}}$ if and only if there exist finite substructures $\mathcal{G}_j \leq \mathcal{G}^{\mathcal{M}}_{\Phi_j(x)}$ for each $j \in I$ such that $\mathcal{G}_j$ is complete and the induced inclusion $\iota\colon \prod_{j\in\mathcal{U}} \mathcal{G}_j \to \widehat{\mathcal{G}}$ has the property that $\{[g_{\varphi(x)}]_{\mathcal{U}} : \varphi(x) \in \Phi(x)\}$ is contained in the image of ι.

Proof. ($\Leftarrow$): Since $\mathcal{G}_j$ is finite and complete, we must have that $P_{|G_j|}(G_j)$ holds. In particular, $\mathcal{M} \vDash \exists x, \bigwedge_{\varphi(x)\in G_j} \varphi(x)$. By transfer, ${}^*\!\vDash_{\mathcal{M}} ({}^*\exists x, {}^*\!\bigwedge_{\varphi\in {}^*G_i} \varphi(x))$, so (3) of Theorem 3.2.6 is satisfied with $\mathcal{H} = {}^*\mathcal{G}_i$ and $\Phi(x)$ is realized.

($\Rightarrow$): If $\Phi(x)$ is realized, then Theorem 3.2.6 (3) guarantees the existence of an (i, I)-internal complete hyperfinite $\mathcal{H}$ containing $\Phi(x)$. Let $K\colon I \to \mathcal{P}_\omega({}^*\mathcal{L}(x))$ witness $\mathcal{H}$ being (i, I)-internal. We may assume that $K(j) \subseteq G^{\mathcal{M}}_j$ for each j (if not, take the intersection of $K(j)$ and $G^{\mathcal{M}}_j$ and note that $\Phi(x) \subseteq {}^*K(i) \cap {}^*\mathcal{G}^{\mathcal{M}}_{\Phi_i(x)}$). Define the function $\widetilde{K}$ by

$$\widetilde{K}(j) := \begin{cases} K(j), & \text{if } |K(j)| \geq 1 \text{ and } \mathcal{M} \vDash \exists x, \bigwedge K(j) \\ \{g_j\}, & \text{otherwise} \end{cases}$$

where g_j is some element of $G^{\mathcal{M}}_{\Phi_j(x)}$. The $\widetilde{K}(j)$ are finite and complete by definition, so it remains to show that $\Phi(x) \subseteq \prod_{j\in\mathcal{U}} \mathcal{G}_j$. This follows from ${}^*\widehat{K}(i) = {}^*K(i)$ (by transfer of the definition of $\widehat{K}$) by the correspondence between $\prod_{j\in\mathcal{U}} K(j)$ and the (i, I)-internal elements of ${}^*K(i)$. □

Combining Theorem 3.2.4 and Corollary 3.2.8 allows us to reframe the question of whether a given structure $\mathcal{M}$ is saturated by a regular ultrafilter $\mathcal{U}$ entirely in terms of the hypergraph theoretic properties of the characteristic hypergraphs that can occur in $\mathcal{M}$.

Theorem 3.2.9. The regular ultrafilter $\mathcal{U}$ saturates $\mathcal{M}$ if and only if whenever $(\Psi_j(x))_{j\in I}$ is a sequence of sets of formula with parameters from $\mathcal{M}$ where each Ψ_j consists only of formulas of the form $\varphi(x, \bar{a})$ where φ is a formula fixed across all of the $j \in I$, and for each $j \in I$ we have that $\mathcal{G}_j$ is a finite substructure of $\mathcal{G}^{\mathcal{M}}_{\Psi_j(x)}$, then every $\mathcal{H}$ that is a complete sub-hypergraph of $\prod_{j\in\mathcal{U}} \mathcal{G}_j$ with $|H| \leq |I|$ is contained in a hyperfinite complete sub-hypergraph of $\prod_{j\in\mathcal{U}} \mathcal{G}_j$.

Proof. ($\Leftarrow$): This is Corollary 3.2.8 along with the result of [Malliaris(2009)] that saturation of all small φ-types is enough to guarantee the saturation of all small types.

($\Rightarrow$): Given a sequence $(\mathcal{G}_j)_{j\in I}$ of hypergraphs as in the statement of the theorem and an $\mathcal{H}$ that is a complete sub-hypergraph of $\prod_{j\in\mathcal{U}}\mathcal{G}_j$ with $|H|\leq|I|$, we will show that H defines a type of $\widehat{\mathcal{M}}$ that is realized if H is contained in a hyperfinite complete sub-hypergraph of $\prod_{j\in\mathcal{U}}\mathcal{G}_j$.

We claim that H is a type of $\widehat{\mathcal{M}}$. Note that every element of $\prod_{j\in\mathcal{U}}\mathcal{G}_j$ is the equivalence class of a function of the form $(\varphi(x,\bar{a}_j))_{j\in I}$ for a fixed formula φ, so we may interpret H as a set of formula in $\widehat{\mathcal{M}}$ via Lemma 2.6.11. We have that $\Delta\subseteq H$ implies ${}^*P_n(\Delta)$, which is equivalent to ${}^*{\models}_{\mathcal{M}}\ ({}^*\exists x,\ {}^*\bigwedge\Delta)$ by transfer of the definition of P_n, so H is finitely realizable and hence a type. For each $h\in H$ and $j\in I$, let $f_h(j)=\bar{a}_j$ where $(\varphi(x,\bar{a}_j))_{j\in I}$ is a fixed representative for h. With these fixed representatives for the parameters occurring in H, the characteristic hypergraphs $\mathcal{G}^{\mathcal{M}}_{H_j}$ are substructures of the hypergraphs $\mathcal{G}_j$, so there is an induced embedding $\iota\colon \prod_{j\in\mathcal{U}}\mathcal{G}^{\mathcal{M}}_{H_j}\to\prod_{j\in\mathcal{U}}\mathcal{G}_j$. Since H is a type of $\widehat{\mathcal{M}}$, we must have that $\widehat{\mathcal{M}}$ realizes H and so H is contained in a hyperfinite complete sub-hypergraph $\mathcal{K}$ of $\prod_{j\in\mathcal{U}}\mathcal{G}^{\mathcal{M}}_{H_j}$. Finally, $\iota[K]$ is a hyperfinite complete sub-hypergraph of $\prod_{j\in\mathcal{U}}\mathcal{G}_j$, as desired. □

From this perspective, it becomes clear that understanding the structure of Keisler's order for all countable complete theories is equivalent to understanding when regular ultraproducts of finite hypergraphs have the sort of complete extension property mentioned in Theorem 3.2.9.

3.2.1 The Shapes of φ-Types

Given Theorem 3.2.9, it would be useful to study which families of finite hypergraphs can arise as substructures of characteristic hypergraphs of types built out of a single formula φ. Not much is known about this problem in general, although some work has been done on φ-types where φ has properties of significance in classification theory, giving some explanation for the apparent connection between complexity of a theory measured in the sense of Keisler's order and complexity of a theory measured as in classification theory.

In particular, in the next few sections we will explore certain families of finite graphs that can arise from theories within the SOP hierarchy, at least down to SOP_2. One important property of the formulas involved in theories with SOP is that their associated characteristic hypergraph sequences have support 2 in the language of [Malliaris(2010)].

Definition 3.2.10 (Support 2)**.** The characteristic hypergraph $\mathcal{G}^{\mathcal{M}}_{\Psi(x)}$ for a φ-type $\Psi(x)$ has **support 2** if for every finite $\Gamma \subseteq \mathcal{G}^{\mathcal{M}}_{\Psi(x)}$ with $|\Gamma| \geq 2$ we have that

$$P_{|\Gamma|}(\Gamma) \iff \forall \Delta \in [\Gamma]^2,\, P_2(\Delta).$$

In particular, if $\mathcal{G}^{\mathcal{M}}_{\Psi(x)}$ has support 2, then P_n for $n > 2$ is determined by P_2 using the rule that $P_n(\Gamma)$ holds if and only if (Γ, P_2) is a complete graph. This allows us to take many of our theorems about characteristic hypergraphs and rephrase them so that we only have to care about the graph reduct of $\mathcal{G}^{\mathcal{M}}_{\Psi(x)}$.

For the particular sorts of types that we will be most interested in, the parameters will be restricted to having representatives coming from particular subsets of the underlying set of $\mathcal{M}$. We give here some terminology and notation for talking about this specific case.

Definition 3.2.11. Let M be the underlying set of a structure $\mathcal{M}$, let V be a subset of M^n, and let $\Psi(x)$ be a collection of formula of the form $\varphi(x, \bar{a})$ where φ is fixed and $\bar{a}$ are elements of M^n.

(1) If all the parameters of $\Psi(x)$ come from V, we say that $\Psi(x)$ is a V**-set**. If $\Psi(x)$ is consistent, we will call $\Psi(x)$ a V**-type**.

(2) For any structure $\mathcal{K}$ in the language consisting of $\{P_n : n \in \omega \setminus \{0\}\}$, let $\mathcal{K}'$ be the reduct of $\mathcal{K}$ to the language consisting of just P_2.

(3) Let $\mathrm{GSh}_{\mathcal{M}}(V)$ be the set

$$\{[\mathcal{G}']_{\cong} : \mathcal{G}' \text{ is a finite substructure of the characteristic hypergraph of a } V\text{-set in } \mathcal{M}\}.$$

That is, $\mathrm{GSh}_{\mathcal{M}}(V)$ is the set of all isomorphism types of graph reducts of finite substructures of the characteristic hypergraphs of V-sets. We will call $\mathrm{GSh}_{\mathcal{M}}(V)$ the **graph shape of** V **in** $\mathcal{M}$.

(4) If $\Phi(x)$ is both a V-set and a φ-set, we will call $\Phi(X)$ a (φ, V)-set.

(5) If all of the characteristic hypergraphs of V-sets that are also φ-sets have support 2, then we will say that V has support 2 with φ.

Remark 3.2.12. In order to keep $[\mathcal{G}']_{\cong}$ a set, we may choose to consider only the members of the isomorphism class that occur as substructures of the characteristic hypergraphs of V-sets. That is, we will think of $[\mathcal{G}']_{\cong}$ as being the class of graphs that are finite substructures of characteristic hypergraphs of a V-set in $\mathcal{M}$ (the graphs that show up in the definition of the graph shape) and which are isomorphic to $\mathcal{G}'$.

One important property of $\mathrm{GSh}_{\mathcal{M}}(V)$ is that it is closed under substructures, or in the language of graph theory, closed under induced sub-graphs. It is well known that collections of finite graphs that are closed under induced sub-graphs can be characterized by the $\leq$-minimal subgraphs not appearing in $\mathrm{GSh}_{\mathcal{M}}(V)$. In particular, if Γ is the collection of minimal graph isomorphism types not appearing in $\mathrm{GSh}_{\mathcal{M}}(V)$, then

$$\mathrm{GSh}_{\mathcal{M}}(V) = \{[\mathcal{G}]_{\cong} : \forall \gamma \in \Gamma,\, \gamma \nleq \mathcal{G}\},$$

where we interpret $\gamma \nleq \mathcal{G}$ as the statement that there is no injective graph homomorphism that preserves non-edges from γ to $\mathcal{G}$.

3.2.2 Distributions and Necessary Sets

In the classical study of Keisler's order, the information about which sets of formulas in a type of $\widehat{\mathcal{M}}$ are consistent in each of the factor structures of $\widehat{\mathcal{M}}$ is played by the role of distributions instead of the characteristic hypergraphs we have been studying.

Definition 3.2.13 (Distributions)**.** Given a type $\Phi(x)$ of the regular ultrapower $\widehat{\mathcal{M}}$ with $|\Phi(x)| \leq |I|$ and fixed representatives $I \to M$ for the parameters occurring in $\Phi(x)$ a function $d \colon \mathcal{P}_\omega(\Phi(x)) \to \mathcal{U}$ such that

(1) for any $F \in [\mathcal{P}_\omega(\Phi(x))]^\omega$ we have that $\bigcap_{\Delta(x)\in F} d(\Delta(x)) = \emptyset$, that is the range of d is a regularizing set in $\mathcal{U}$, and

(2) $j \in d(\Delta(x))$ implies the set of formulas $\{\varphi(x, \bar{f}(j)) : \varphi(x, [\bar{f}]) \in \Delta(x)\}$ is realized in $\mathcal{M}$

is called a **distribution**.

Two types of distribution are of particular interest in the study of Keisler's order. We recall the definitions of these properties from the discussion of good ultrafilters in Chapter 2.

Definition 3.2.14 (Monotone and Multiplicative)**.** A distribution d is **monotone** if whenever $\Delta \subseteq \Gamma$ in $\mathcal{P}_\omega(\Phi(x))$ then $d(\Gamma) \subseteq d(\Delta)$. A distribution d is said to be **multiplicative** if whenever $\Lambda, \Xi \in \mathcal{P}_\omega(\Phi(x))$ then $d(\Lambda \cup \Xi) = d(\Lambda) \cap d(\Xi)$.

The basic idea of a distribution is that d is a record of in which coordinates each finite subset of $\Phi(x)$ is satisfied. In this sense, distributions act as a proof that $\Phi(x)$ is a type of $\widehat{\mathcal{M}}$ via Łoś's theorem. Moreover, Definition 3.2.13 (1) can be thought of as saying that each index in I is only responsible for a finite number of the formulas occurring in $\Phi(x)$, which is only possible by the regularity of $\mathcal{U}$. From this point of view, monotonicity is a consistency condition in that $d(\Gamma) \subseteq d(\Delta)$ when $\Delta \subseteq \Gamma$ is simply saying that whenever $\mathcal{M}$ realizes a set of formulas, then of course $\mathcal{M}$ will realize any subset of those formulas. As any distribution can be refined to a monotone distribution, it is common to work only with monotone distributions.

Multiplicative distributions are important to understanding whether the type $\Phi(x)$ is realized in the ultrapower $\widehat{\mathcal{M}}$. If a distribution d is multiplicative, then $j \in d(\Delta(x))$ if and only if $j \in d(\{\delta(x)\})$ for every $\delta(x) \in \Delta(x)$. In particular, a distribution being multiplicative can be thought of as saying that for each $j \in I$, the finitely many formulas that the j-th index model is responsible for are consistent. In the language of characteristic hypergraphs, this says that, if $\Delta_j(x) = \{\varphi(x, \bar{f}(j)) : j \in d(\{\varphi(x, \bar{f})\})\}$, then $\mathcal{G}^{\mathcal{M}}_{\Delta_j(x)}$ is a complete hypergraph for all $j \in I$. Using the fact that the codomain of d is $\mathcal{U}$ then tells us that $\prod_{j\in\mathcal{U}} \mathcal{G}^{\mathcal{M}}_{\Delta_j(x)}$ is a hyperfinite complete sub-hypergraph of $\mathcal{G}^{\widehat{\mathcal{M}}}_{\Phi(x)}$. Putting all this information together gives us a classical result about when types are realized in a regular ultrapower.

Theorem 3.2.15 ([Keisler(1967)])**.** A type $\Phi(x)$ of the regular ultrapower $\widehat{\mathcal{M}}$ has a realization in $\widehat{\mathcal{M}}$ if and only if $\Phi(x)$ has a multiplicative distribution.

As one might expect from the previous discussion, there is a close relationship between distributions for $\Phi(x)$ and the characteristic hypergraphs $(\mathcal{G}^{\mathcal{M}}_{\Phi_j(x)})_{j\in I}$. This correspondence between the characteristic graphs $\mathcal{G}^{\mathcal{M}}_{\Phi_j(x)}$ and distributions for $\Phi(x)$ along with the fact that $\mathrm{GSh}_{\mathcal{M}}(V)$ is characterized by the minimal graphs not appearing in $\mathrm{GSh}_{\mathcal{M}}(V)$ motivates the following question: how can we use the minimal graphs not appearing in $\mathrm{GSh}_{\mathcal{M}}(V)$ to characterize the sorts of distributions that need to be looked at in order to guarantee that φ-types with parameters coming from $V^{\mathcal{U}}$ are saturated by $\mathcal{U}$? To help to answer this question, we introduce the idea of necessary sets for graphs.

Definition 3.2.16 (Necessary Sets)**.** Let $\mathcal{H} = (H; E)$ be a graph and W a collection of graphs. We say that N **is necessary for** $\mathcal{H}$ **in** W if $N \subseteq [H]^2 \setminus E$ and whenever $\iota\colon \mathcal{H} \to \mathcal{G}$ is an injective graph homomorphism to an element $\mathcal{G} \in W$ then there is some $\{h_1, h_2\}$ in N such that $\mathcal{G} \models \iota(h_1)\ E\ \iota(h_2)$.

Remark 3.2.17. Since the definition of being necessary for $\mathcal{H}$ in W only depends on the isomorphism type of graphs occurring in W, we will sometimes speak of being necessary for collections of isomorphism types of graphs rather than being necessary for collections of actual graphs.

We will be primarily concerned with necessary sets for the collection Γ of minimal graphs not occurring in $\mathrm{GSh}_{\mathcal{M}}(V)$. Since graphs $\mathcal{G} \in \Gamma$ cannot be the induced subgraphs of an element $\mathcal{H}$ of $\mathrm{GSh}_{\mathcal{M}}(V)$, this tells that whenever we see $\mathcal{G}$ occurring as a subgraph of $\mathcal{H}$, then there must be edges in $\mathcal{H}$ that do not occur in $\mathcal{G}$. It is this structure that we will exploit to find a small class of distributions that can detect whether an ultrafilter is good.

Theorem 3.2.18. Suppose that $V \subseteq M^n$ has support 2 with φ. Let X be any set of graphs containing exactly one representative for each minimal isomorphism class not occurring in $\mathrm{GSh}_{\mathcal{M}}(V)$ and no other graphs. For each $\mathcal{H} \in X$ let $N_{\mathcal{H}}$ be a necessary set for $\mathcal{H}$ in $\mathrm{GSh}_{\mathcal{M}}(V)$. Then the following are equivalent:

(1) Every φ-type $\Phi(x)$ in $\widehat{\mathcal{M}}$ for which a sequence $(\mathcal{G}_j)_{j\in I}$ can be found with $\mathcal{G}_j \leq \mathcal{G}^{\mathcal{M}}_{\Phi_j(x)}$, $\Phi(x) \subseteq \prod_{j\in\mathcal{U}} \mathcal{G}_j$, and such that the underlying set of each $\mathcal{G}_j$ is a finite φ-set with parameters from V, is realized in $\widehat{\mathcal{M}}$.

(2) For every sequence $(\mathcal{K}_j)_{j\in I}$ of graphs such that no induced subgraph of $\mathcal{K}_j$ is isomorphic to an element of X, the ultraproduct $\mathcal{K} := \prod_{j\in\mathcal{U}} \mathcal{K}_j$ has the property that every complete subgraph of $\mathcal{K}$ of cardinality at most $|I|$ is contained in a hyperfinite complete subgraph of $\mathcal{K}$.

(3) Every monotone function $d\colon \mathcal{P}_\omega(I) \to \mathcal{U}$ satisfying the properties

   (a) $d(\Gamma) = \bigcap_{\Delta\in[\Gamma]^2} d(\Delta)$ for all $\Gamma \in \mathcal{P}_\omega(I)$ with $|\Gamma| > 2$ and

   (b) for each $\mathcal{H} = (H; E^{\mathcal{H}})$ in X every injective function $\eta\colon H \to I$ satisfies

$$\bigcup_{e\in E^{\mathcal{H}}} d(\eta[e]) \subseteq \bigcap_{k\in N_{\mathcal{H}}} d(\eta[k]),$$

   has a multiplicative refinement.

Proof. (3)⇒(2): Suppose that $\mathcal{N} \leq \mathcal{K}$ is a complete subgraph with $|\mathcal{N}| \leq |I|$. Choose a representative $f_n \in \prod_{j\in I} K_j$ for each $n \in \mathcal{N}$ and choose a surjection $s\colon I \to N$. We define a function $d\colon \mathcal{P}_\omega(I) \to \mathcal{U}$ by

$$d(\Delta) = \{j \in I : \forall\{\alpha,\beta\} \in [\Delta]^2,\, \mathcal{K}_j \vDash f_{s(\alpha)}(j)\ E\ f_{s(\beta)}(j)\}.$$

Łoś's theorem gives us that $d(\Delta) \in \mathcal{U}$ for all $\Delta \in \mathcal{P}_\omega(I)$. That d satisfies (a) follows directly from the definition of d. For (b), suppose that $\eta\colon H \to I$. If $j \in \bigcup_{e\in E^{\mathcal{H}}} d(\eta[e])$, that means that $\mathcal{K}_j \vDash f_{\eta(h_1)}(j)\ E\ f_{\eta(h_2)}$ for each $\{h_1,h_2\} \in E^{\mathcal{H}}$. That is, η defines an injective graph homomorphism $\mathcal{H} \to \mathcal{K}_j$. By the definition of necessary sets, there must be some pair $\{h_3,h_4\} \in N^{\mathcal{H}}$ such that $\mathcal{K}_j \vDash f_{\eta(h_3)}(j)\ E\ f_{\eta(h_4)}(j)$. That is, $j \in d(\eta[\{h_3,h_4\}])$.

Now, by assumption, there is a multiplicative $\widehat{d}\colon \mathcal{P}_\omega(I) \to \mathcal{U}$ refining d. Consider the subgraphs $\widehat{\mathcal{K}}_j \leq \mathcal{K}_j$ with underlying set

$$\widehat{K}_j := \{f_n(j) : j \in \widehat{d}(\{s^{-1}(n)\})\}.$$

Note that some of the $\widehat{K}_j$ may be empty, in which case we will define $\widehat{K}_j$ to be $\{f_n(j)\}$ for some fixed $n \in N$ instead. We claim that the $\widehat{\mathcal{K}}_j$ are all complete finite graphs and that $\mathcal{N}$ is a subgraph of $\prod_{j\in\mathcal{U}} \widehat{\mathcal{K}}_j$. Finiteness follows from the range of d being a regularizing set. For completeness, suppose that $f_{n_1}(j)$ and $f_{n_2}(j)$ are distinct elements of $\widehat{K}_j$. That is,

$$j \in \widehat{d}(\{s^{-1}(n_1)\}) \cap d(\{s^{-1}(n_2)\}) = \widehat{d}(\{s^{-1}(n_1), s^{-1}(n_2)\}).$$

Since $\widehat{d}$ is a refinement of d, it must be that $\mathcal{K}_j \vDash f_{n_1}(j)\ E\ f_{n_2}(j)$ by the definition of d. Thus every distinct pair of elements of $\widehat{\mathcal{K}}_j$ is edge related and $\widehat{\mathcal{K}}_j$ is complete. For each $n \in N$, we have that $\widehat{d}(\{s^{-1}(n)\}) \in \mathcal{U}$, so $f_n(j)$ is in $\mathcal{U}$-many $\widehat{K}_j$. That is, $\prod_{j\in\mathcal{U}} \widehat{\mathcal{K}}_j$ contains every element of $\mathcal{N}$.

(2)⇒(1): Take $\mathcal{K}_j = \mathcal{G}_j$ and then this follows from Corollary 3.2.8.

(1)⇒(3): Given a function d as in (3), we will construct a φ-type $\Phi(x)$ so that d is a distribution for $\Phi(x)$. For each $j \in I$, define $D_j := \{k \in I : j \in d(\{k\})\}$. We define $E_j \subseteq D_j^2$ to be the set $\{(k,\ell) : j \in d(\{k,\ell\})$ and $k \neq \ell\}$. If D_j is empty as currently defined, we will replace D_j with the set $\{j\}$ and E_j will be $\emptyset$. Given these definitions, $\mathcal{D}_j := (D_j, E_j)$ is a graph. We claim that the isomorphism type of $\mathcal{D}_j$ is in $\mathrm{GSh}_{\mathcal{M}}(V)$, or, equivalently, no graph $\mathcal{H} \in X$ admits a non-edge preserving injective graph homomorphism to $\mathcal{D}_j$.

Suppose that $\eta\colon \mathcal{H} \to \mathcal{D}_j$ is an injective graph homomorphism. Then η can be thought of as an injective function $H \to I$, so

$$\bigcup_{e\in E^{\mathcal{H}}} d(\eta[e]) \subseteq \bigcap_{k\in N_{\mathcal{H}}} d(\eta[k]).$$

η being a graph homomorphism implies that $j \in \bigcup_{e\in E^{\mathcal{H}}} d(\eta[e])$, so we must also have that $j \in \bigcap_{k\in N_{\mathcal{H}}} d(\eta[k])$. In particular, $j \in d(\{a,b\})$ for some $(a,b) \notin E^{\mathcal{H}}$. That is, $\neg(a\ E^{\mathcal{H}}\ b)$ but $\eta(a)\ E_j\ \eta(b)$, so η cannot be non-edge preserving.

As the isomorphism type of $\mathcal{D}_j$ occurs in $\mathrm{GSh}_{\mathcal{M}}(V)$, there is a (φ, V)-set $\Psi_j(x)$ with characteristic hypergraph $\mathcal{G}^{\mathcal{M}}_{\Psi_j(x)}$ that has an induced sub-hypergraph $\mathcal{G}_j$ that is isomorphic to $\mathcal{D}_j$. Choose an isomorphism $\iota_j\colon \mathcal{D}_j \to \mathcal{G}_j$ for each $j \in I$. For each $\ell \in I$, define the function $f_\ell\colon I \to M$ by

$$f_\ell(j) := \begin{cases} \iota_j(\ell), & \text{if } j \in d(\{\ell\}) \\ a_j, & \text{otherwise} \end{cases}$$

where a_j is some fixed element of D_j. We claim that $\Phi(x) = \{[f_\ell] : \ell \in I\}$ is a type of $\widehat{\mathcal{M}}$ and that d is a distribution for $\Phi(x)$. Since V has support 2 with φ, $\Phi(x)$ is a type if $\Phi(x)$ is a complete subgraph of $\prod_{j \in \mathcal{U}} \mathcal{G}_j$. We have that $[f_\ell]$ is edge related to $[f_k]$ if and only if $\{j \in I : f_k(j)\ E_j\ f_\ell(j)\} \in \mathcal{U}$ by Łoś's theorem. This is achieved by the fact that

$$\{j \in I : f_k(j)\ E_j\ f_\ell(j)\} \supseteq d(\{k, \ell\}) \in \mathcal{U}$$

by the definition of E_j. By (1), $\Phi(x)$ is a realized type of $\widehat{\mathcal{M}}$ so there is a multiplicative distribution $g\colon \mathcal{P}_\omega(\Phi(x)) \to \mathcal{U}$ for $\Phi(x)$ using the representatives f_ℓ already established for $\Phi(x)$. Let $F\colon I \to \Phi(x)$ be the function $\ell \mapsto [f_\ell]$. We claim that the function $\widehat{d}\colon \mathcal{P}_\omega(\Phi(x)) \to \mathcal{U}$ defined by

$$\Delta \mapsto g(F[\Delta]) \cap d(\Delta)$$

is multiplicative.

It is enough to check that $\widehat{d}(\Delta) = \bigcap_{\delta \in \Delta} \widehat{d}(\{\delta\})$ for $|\Delta| > 1$. We have that

$$\begin{aligned} \widehat{d}(\Delta) &= g(F[\Delta]) \cap d(\Delta) \\ &= \bigcap_{\Gamma \in [\Delta]^2} (g(F[\Gamma]) \cap d(\Gamma)) \end{aligned}$$

so long as $|\Delta| \geq 2$ by the multiplicativity of g and property (3) (a) in the statement of the theorem for d (the result that we want is trivial for $|\Delta| = 1$). It is thus enough to show that

$$g(\{f_\ell, f_k\}) \cap d(\{\ell, k\}) = g(\{f_\ell\}) \cap g(\{f_k\}) \cap d(\{\ell\}) \cap d(\{k\}).$$

As d and g are both monotone functions, we automatically get the inclusion of the left-hand side in the right-hand side.

Suppose that $j \in g(\{f_\ell\}) \cap g(\{f_k\}) \cap d(\{\ell\}) \cap d(\{k\})$. By the multiplicativity of g, we get that $j \in g(\{f_\ell, f_k\})$. As g is a distribution, we must have that $\mathcal{M} \vDash \exists x,\ f_\ell(j) \wedge f_k(j)$ and so there is an edge between $f_\ell(j)$ and $f_k(j)$ in the characteristic hypergraph $\mathcal{G}^{\mathcal{M}}_{\Phi_j(x)}$. We choose θ_j to be an isomorphism of $\mathcal{D}_j$ onto an induced subgraph of $\mathcal{G}^{\mathcal{M}}_{\Phi_j(x)}$ with $\theta_j(\ell) = f_\ell(j)$ and $\theta_j(k) = f_k(j)$, so $\mathcal{D}_j$ must have an edge between ℓ and k. By the definition of E_j, it must be that $j \in d(\{\ell, k\})$, giving us that $j \in g(\{f_\ell, f_k\}) \cap d(\{\ell, k\})$ as desired. □

3.3 Shapes in the SOP Hierarchy

3.3.1 Shapes of SOP_2-types

We know from the definition of SOP_2 (Definition 3.0.2) that whenever $\Phi(x)$ is a φ-set whose set of parameters V form an SOP_2-tree within the structure $\mathcal{M}$ then any $W \subseteq V$ creates a consistent collection of formulas

$$p_W(\bar{x}) = \{\varphi(\bar{x}, \bar{w}) : \bar{w} \in W\}$$

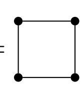
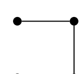
if and only if W is a subset of a branch of the SOP_2-tree whose elements come from V. In other words, $p_W(\bar{x})$ is consistent if and only if every pair $\bar{w}, \bar{w}' \in W$ is comparable in the SOP_2-tree (that is, W is a linear sub-order of the tree). We thus have that V has support 2 with φ (recall Definition 3.2.11(5)) and the graph shape corresponding to V is the collection of all finite graphs $\mathcal{G}$ that can be obtained from taking a finite subset of the full binary tree. There are edges between the two distinct vertices w w' in $\mathcal{G}$ if and only if w and w' are comparable in the full binary tree. Fortunately, most of the combinatorial work of determining the isomorphism types of such graphs has already been done by [Wolk(1962)].

Definition 3.3.1. We say that a tuple (φ, V) **witnesses** SOP_2 **in a structure** $\mathcal{M}$ if there is an SOP_2-tree

$$(\varphi(\bar{x}; \bar{y}), (\bar{v}_\eta)_{\eta \in 2^{<\omega}})$$

in $\mathcal{M}$ such that $V = \{\bar{v}_\eta : \eta \in 2^{<\omega}\}$.

Lemma 3.3.2. Suppose that φ is a formula and the pair (φ, V) witnesses SOP_2 in the structure $\mathcal{M}$. Then V has support 2 with φ and the set Γ of $\leq$-minimal isomorphism types not in $\mathrm{GSh}_{\mathcal{M}}(V)$ are the two isomorphism types:

$C_4 :=$  and $\ell_4 :=$  .

Proof. Because (φ, V) witnesses SOP_2 in $\mathcal{M}$, there is a surjection $h : 2^{<\omega} \to V$ with the property that a subset W of $2^{<\omega}$ is linearly ordered if and only if the collection of formulas

$$p_W(\bar{x}) = \{\varphi(\bar{x}, h(s)) : s \in W\}$$

is consistent in $\mathcal{M}$. If $s \in 2^{<\omega}$, we will denote $h(s)$ by $\bar{v}_s$. Let $\Delta \in \mathcal{P}_\omega(2^{<\omega})$. Then the set of formula $\{\varphi(\bar{x}, \bar{v}_s) : s \in \Delta\}$ is consistent (equivalently, realized) in $\mathcal{M}$ if and only if the set Δ is linearly ordered as a suborder of $2^{<\omega}$ if and only if the elements of Δ are pairwise comparable in the ordering on $2^{<\omega}$. Thus the elements $[\mathcal{G}]_\cong$ of $\mathrm{GSh}(V)$ are such that $\mathcal{G}$ is the graph of the comparability relation for a finite suborder of $2^{<\omega}$ (equivalently, $\mathcal{G}$ is the graph of the comparability relation for a finite forest of subtrees of $2^{<\omega}$; i.e. the connected components of $\mathcal{G}$ correspond to finite subtrees of $2^{<\omega}$). Every finite collection of finite trees can be embedded as a suborder of $2^{<\omega}$, so the shape of V contains every possible disjoint union of finitely many comparability graphs arising from finite trees. We claim that no other isomorphism types appear in $\mathrm{GSh}_\mathcal{M}(V)$. We briefly postpone the proof of the claim to study the structure of these isomorphism types in more detail.

By [Wolk(1962)], a finite connected graph $\mathcal{G} = (G, E)$ is the comparability graph of a tree if and only if for all distinct $x_0, x_1, x_2, x_3 \in G$ the formula

$$x_0 \; E \; x_1 \; E \; x_2 \; E \; x_3 \to (x_0 \; E \; x_2) \vee (x_1 \; E \; x_3)$$

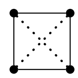
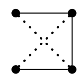
holds (called **the diagonal property**). Clearly, every graph $\mathcal{G}$ with $|G| \leq 3$ satisfies the diagonal property. Suppose that a graph with vertices $\{0, 1, 2, 3\}$ does not satisfy the diagonal property. We may suppose that $0 \; E \; 1 \; E \; 2 \; E \; 3$. If these are the only edges, the graph is ℓ_4 and clearly not diagonal. Now there are only 3 possible edges that can be added, $\{0, 3\}, \{1, 3\}$, or $\{0, 2\}$. If just $\{0, 3\}$ is added, we have a graph isomorphic to C_4, which is not diagonal. We wish to show that the remaining possibilities are all diagonal. The complete graph on 4 elements is the comparability graph of a tree that is a 4 element linear order. Up to isomorphism, the remaining possibilities are the comparability graphs of the two trees

and so are diagonal.

If $\mathcal{G}$ is a non-diagonal graph with more than 4 vertices, there must be a 4-vertex induced subgraph of $\mathcal{G}$ that is not diagonal (pick the four vertices to be a set of vertices witnessing the fact that the graph is not diagonal). Thus a graph $\mathcal{G}$ is diagonal if and only if $\mathcal{G}$ is the disjoint union of a collection of comparability graphs of trees if and only if $C_4 \not\leq \mathcal{G}$ and $\ell_4 \not\leq \mathcal{G}$. □

In order to use the information in Lemma 3.2.18 to characterize the distributions of SOP_2-types, we will need to determine the necessary sets for the two graph isomorphism types in $\mathrm{GSh}_{\mathcal{M}}(V)$.

Lemma 3.3.3. With φ, V, and Γ as in Lemma 3.3.2, necessary sets for the elements of Γ in $\mathrm{GSh}_{\mathcal{M}}(A)$ are given by the edges:

 and 

where the dotted edges represent the elements of the corresponding necessary sets. Moreover, there are no subsets of the given necessary sets that are still necessary.

Proof. By the diagonal characterization of graphs arising from the comparability relation on finite trees in [Wolk(1962)], the analysis in the proof of Lemma 3.3.2, and Lemma 3.2.18, the above sets are necessary for ℓ_4 and C_4. Since both C_4 and ℓ_4 become diagonal when any one of the dotted edges is added, these necessary sets are minimal under $\subseteq$. □

This analysis allows us to give both a characterization of the distributions of SOP_2-types as well as to give a new characterization of good ultrafilters in terms of which complete subgraphs of ultraproducts of graphs with factors coming from $\mathrm{GSh}_{\mathcal{M}}(V)$ can be extended to internal complete subgraphs.

Theorem 3.3.4. Let $\mathcal{U}$ be a regular ultrafilter on the infinite cardinal λ. Then the following are equivalent:

(1) $\mathcal{U}$ is good.

(2) If $\mathcal{M}$ is a structure and (φ, V) witness SOP_2 in $\mathcal{M}$ then every type of cardinality $\leq \lambda$ in $\widehat{\mathcal{M}} := \mathcal{M}^{\mathcal{U}}$ whose parameters come from $V^{\mathcal{U}}$ is realized in $\widehat{\mathcal{M}}$.

(3) For all sequences of graphs $(\mathcal{G}_\alpha)_{\alpha<\lambda}$ such that $C_4 \not\leq \mathcal{G}_\alpha$ and $\ell_4 \not\leq \mathcal{G}_\alpha$ for all $\alpha < \lambda$ the ultraproduct $\mathcal{G} := \prod_{\alpha\in\mathcal{U}} \mathcal{G}_\alpha$ has the property that every complete $H \subseteq \mathcal{G}$ with $|H| = \lambda$ is contained within an internal complete subgraph of $\mathcal{G}$.

(4) Every monotone function $d\colon P_\omega(\lambda) \to \mathcal{U}$ that satisfies

(a) $d(\Gamma) = \bigcap_{\Delta\in[\Gamma]^2} d(\Delta)$ for all $\Gamma \in \mathcal{P}_\omega(I)$ with $|\Gamma| > 2$ and

(b) $d(\{x_0, x_1\}) \cap d(\{x_1, x_2\}) \cap d(\{x_2, x_3\}) \subseteq d(\{x_0, x_2\}) \cup d(\{x_1, x_3\})$ for all distinct $x_0, x_1, x_2, x_3 \in \lambda$,

has a multiplicative refinement.

Proof. (1) $\iff$ (2): This is a restatement of the fact that a regular ultrafilter $\mathcal{U}$ is good if and only if $\mathcal{U}$ realizes all SOP_2-types over sets of size λ, which is proved by Malliaris and Shelah in [Malliaris and Shelah(2016)] (see Conclusion 11.9 and Main Theorem 11.11).

(2) $\iff$ (3): This follows from Theorem 3.2.18, Lemma 3.3.2, and Lemma 3.3.3.

(2) $\iff$ (4): Combining Theorem 3.2.18 and Lemma 3.3.3 we have that every type of $\widehat{\mathcal{M}}$ of cardinality λ with parameters coming from $V^{\mathcal{U}}$ is realized in $\widehat{\mathcal{M}}$ if and only if both

$$d(\{x_0, x_1\}) \cap d(\{x_1, x_2\}) \cap d(\{x_2, x_3\}) \subseteq d(\{x_0, x_2\}) \cup d(\{x_1, x_3\})$$

$$d(\{x_0, x_3\}) \cap d(\{x_0, x_1\}) \cap d(\{x_1, x_2\}) \cap d(\{x_2, x_3\}) \subseteq d(\{x_0, x_2\}) \cup d(\{x_1, x_3\})$$

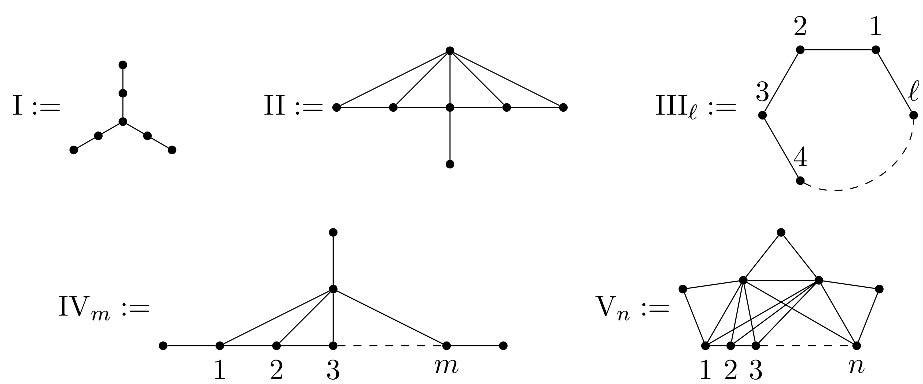
whenever x_0, x_1, x_2, x_3 are distinct elements of λ. As the intersection on the left-hand side of the second condition is always a subset of the left-hand side of the first condition, both conditions are satisfied if and only if the first condition is satisfied. □

3.3.2 Cuts in Linear Orders and SOP

We make an analysis of types that witness a cut in an ultrapower of an infinite linear order being filled. This analysis follows the same basic structure as the analysis of SOP_2-types presented above, but is slightly complicated by the shape corresponding to such types being more complicated.

The types corresponding to cuts can be used to show that SOP is maximal in Keisler's order, although this result can also be obtained from the fact that SOP implies SOP_2 [Shelah(1996)]. We first characterize the graphs $\mathcal{G}$ whose vertices are open intervals in some linear order $\mathbb{L}$ and which have an edge between two intervals I_1 and I_2 if and only if $I_1 \cap I_2$ is nonempty in $\mathbb{L}$. Such graphs will be the elements of the shapes of the types being studied. Conveniently, the isomorphism types of these graphs will not depend on the linear order under consideration (see Lemma 3.3.5 below). Because of this, we will call such graphs interval intersection graphs without any reference to the linear order from which the intervals come from.

Lemma 3.3.5. Suppose that $\mathbb{L} = (L, <)$ is an infinite linear order and that $\varphi(x; a, b)$ is the formula $a < x < b$. Then L^2 has support 2 with φ and the set Γ of $\leq$-minimal graphs not in $\mathrm{GSh}_{\mathcal{M}}(L^2)$ consists exactly of the graphs/graph families:



where $\ell \geq 4$, $m \geq 2$, and $n \geq 1$ (in V_1 the nodes labeled 1 and n are the same node).

Proof. That L^2 has support 2 with φ follows from the fact that $\{\varphi(x; a_i, b_i) : i \in \{0, 1, \ldots, n\}\}$ is consistent if and only if the interval $(\max_i a_i, \min_j b_j)$ is non-empty and that this is equivalent to the intersection $(a_i, b_i) \cap (a_j, b_j)$ being nonempty for all $i, j \in \{0, 1, \ldots, n\}$.

The statement of the lemma is now equivalent to asking that the above graphs are the $\leq$-minimal graphs that cannot arise in the following way: the vertices are a finite collection of non-empty open intervals in $\mathbb{L}$ and there is an edge between two intervals if and only if their intersection is non-empty.

This result is proved in the particular case that $\mathbb{L} = \mathbb{R}$ in [Lekkerkerker and Boland(1962/63)]. Our strategy will be to reduce the general case to the case of distinct intervals in $\mathbb{R}$. That is, we show that $\mathrm{GSh}(\mathbb{L}^2)$ is equal to the set of isomorphism types of interval intersection graphs in $\mathbb{R}$ arising from distinct intervals.

First, we note that we may assume that $\mathbb{L}$ contains $(\omega, <)$ as a suborder (if not, the reverse order on $\mathbb{L}$ works). Suppose that $\mathcal{G}$ is the intersection graph for the collection of open intervals $\{(a_i, b_i) : i \in \{0, 1, \ldots, n\}\}$ where $a_i, b_i \in \mathbb{R}$. We may then construct a set of intervals in ω by stretching out all of the (a_i, b_i) and translating them to the positive half of $\mathbb{R}$ so that the images of (a_i, b_i) and (a_j, b_j) intersect in ω if and only if $(a_i, b_i) \cap (a_j, b_j) \neq \emptyset$ and hence our new intervals have an intersection graph isomorphic to the intersection graph of the (a_i, b_i). Mapping these intervals to intervals of $\mathbb{L}$ via the inclusion $\omega \hookrightarrow \mathbb{L}$ does not change the intersection graph so long as we did not allow intersections to look like $(n, n+1)$ in ω (this can be achieved by choosing how much to stretch out the original intervals in $\mathbb{R}$). Thus the set of intersection graphs in $\mathbb{L}$ contains all of the isomorphism types of intersection graphs in $\mathbb{R}$.

For the other inclusion, suppose that $\mathcal{G} = (\{I_i : i \in \{0, 1, \ldots, n\}\}, E)$ is the intersection graph for the non-empty intervals I_i in $\mathbb{L}$. Let $I_i = (a_i, b_i)$ and, for each $\{I_i, I_j\} \in E$, choose an element $e_{ij} \in I_i \cap I_j$. Define an order $\mathbb{M}$ with underlying set the union of the sets

$$\{a_i : i \in \{0, 1, \ldots, n\}\}$$

$$\{b_i : i \in \{0, 1, \ldots, n\}\}$$

$$\{e_{ij} : i < j, \{I_i, I_j\} \in E\}$$

and with ordering given by $<^{\mathbb{L}}$. $\mathbb{M}$ is finite and so isomorphic to an initial segment of ω. The map $\mathbb{M} \hookrightarrow \omega \hookrightarrow \mathbb{R}$ sends the intervals I_i to intervals of $\mathbb{R}$ having the same intersection graph as the I_i. □

For the next several propositions we fix the following suppositions: $\mathcal{M}$ is a structure with an infinite $L \subseteq M$ (not necessarily definable) and a definable binary relation $<$ on $\mathcal{M}$ such that $<$

restricts to a linear ordering on L that contains a copy of ω (as an ordered set). Let $\varphi(x; y, z)$ be the formula $y < x < z$.

The following fact from [Lekkerkerker and Boland(1962/63), Theorem 3] will be useful for determining minimal necessary sets for the graphs that cannot arise as the intersection graph of a finite collection of intervals.

Definition 3.3.6 (Irreducible Cycle)**.** A graph $\mathbb{G}$ has an **irreducible cycle of length** n if there are n distinct vertices $v_1, \ldots, v_n$ of $\mathbb{G}$ such that $v_1 v_2 \ldots v_{n-1} v_n v_1$ is a path and there is an edge between v_i and v_j if and only if $i \equiv j \pm 1 \pmod{n}$.

Theorem 3.3.7 ([Lekkerkerker and Boland(1962/63)])**.** A finite graph $\mathbb{G}$ cannot be represented as the intersection graph of a finite collection of intervals if and only if it has at least one of the two following properties:

(1) $\mathbb{G}$ has an irreducible cycle of length ≥ 4.

(2) $\mathbb{G}$ has a collection of three vertices that are pairwise not neighbors and, for each of the three vertices v, there is a path P between the other two vertices such that v is not a neighbor of any vertex in P.

Definition 3.3.8 (Asteroidal Graph)**.** A graph $\mathbb{G}$ is called **asteroidal** if it satisfies the condition of Fact 3.3.7(2). If $\mathbb{G}$ is a graph and the vertices $A = \{v_1, v_2, v_3\}$ witness $\mathbb{G}$ being asteroidal, we will say that A is an **asteroidal set in** $\mathbb{G}$.

We will use the notion of asteroidal sets to help us to determine the necessary sets for the minimal asteroidal graphs given in Lemma 3.3.5. In particular, we know that if A is asteroidal in the graph $\mathbb{G}$ and $\mathbb{G}'$ is an interval intersection graph extending $\mathbb{G}$ then there must be some $a \in A$ and $g \in \mathbb{G}$ such that a and g are neighbors in $\mathbb{G}'$ but not in $\mathbb{G}$ (see Lemma 3.3.9). We are not aware of a nice way other than this to determine what the necessary sets are, so the proofs determining the minimal necessary sets will largely proceed by ruling out all of the elements not in the necessary set one at a time.

Lemma 3.3.9. Let $\mathbb{G}$ be an asteroidal graph and $A = \{v_1, v_2, v_3\}$ be asteroidal in $\mathcal{G}$ with paths P_{v_i} between the nodes $A \setminus \{v_i\}$ for each $1 \leq i \leq 3$ witnessing A being asteroidal in $\mathbb{G}$. Then any interval intersection graph $\mathbb{G}'$ having $\mathbb{G}$ as a (necessarily not induced!) subgraph must induce a new edge in $\mathbb{G}$ that is in the set $\bigcup_i (\{v_i\} \times [P_{v_i}])$ where $[P_{v_i}] \subseteq G$ is the set of vertices in the path P_{v_i}.

Proof. If $\mathbb{G}'$ does not induce any new edges between elements of A and the paths P_{v_i}, the P_{v_i} will also witness the fact that A is asteroidal in $\mathbb{G}'$, a contradiction. □

We start by analyzing the class of graphs labeled III_ℓ because knowing the necessary sets for this class of graphs will be useful in finding the minimal necessary sets for all of the other classes.

Lemma 3.3.10. The edges $A_\ell := \{\{1,3\}\} \cup \{\{2,k\} : 4 \leq k \leq \ell\}$, and any translation of A_ℓ by an automorphism of III_ℓ, are $\subseteq$-minimal necessary sets for III_ℓ in L^2.

Proof. That this set is necessary is proven in [Lekkerkerker and Boland(1962/63), Lemma 1] (note that the word **acyclic** in the referenced lemma means that the graph has no irreducible cycles of length **greater than 3**, so every interval intersection graph is acyclic in this sense). It will then be enough to find an interval intersection graph $\mathbb{G}$ for each $e \in A_\ell$ such that $\mathbb{G}$ extends III_ℓ, e is an edge of $\mathbb{G}$, and no other $e' \in A_\ell$ is an edge in $\mathbb{G}$.

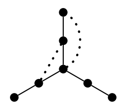
Let $e = \{i, j\} \in A_\ell$ where $i \in \{1, 2\}$ and consider the graph $\mathbb{G}$ that is III_ℓ with all of the edges $\{j, k\}$ for $k \neq j$ added. If $e = \{1, 3\}$, no new edges incident to 2 are added and if $j \neq 3$ then no elements of A_ℓ are added except e. We claim that $\mathbb{G}$ is an interval intersection graph.

Since j is a neighbor of every element of $\mathbb{G}$, we may construct a collection of intervals that have an intersection graph isomorphic to the subgraph of $\mathbb{G}$ induced by $G \setminus \{j\}$ and then take j to be the union of these intervals. The induced subgraph looks like

which is the intersection graph of a collection of intervals that barely overlap at the endpoints.

The definition of necessary sets for $\mathbb{G}$ in S implies that any translation of a necessary set for $\mathbb{G}$ by an automorphism is a necessary set (in particular, composition with an automorphism of $\mathbb{G}$ is a permutation of the injective graph homomorphisms from $\mathbb{G}$ to any other graph). □

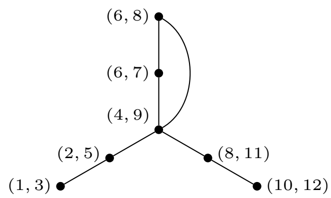
Lemma 3.3.11. The 6 edges (the two dotted edges below and their images under permutations of the 'arms') shown below form a $\subseteq$-minimal necessary set for I in L^2.

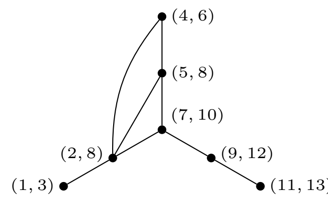


Proof. For $0 \leq n \leq 3$, we will call the vertices along the 'arm' at the polar angle $\pi/2 + 2n\pi/3$ by a_{n1} and a_{n2} where a_{n1} is the neighbor of the center vertex and a_{n2} is the outermost vertex on the given arm. We will denote the center vertex by c. We then have that the set $A = \{a_{n2} : 0 \leq n \leq 2\}$ is asteroidal in I. By Lemma 3.3.9, any intersection graph having I as a subgraph must add an edge incident to one of the elements of A, so we will start our search for necessary edges here.

Let $\mathbb{G}$ be an intersection graph and $i : \mathrm{I} \to \mathbb{G}$ an injective graph homomorphism (we will denote $i(a_{kj})$ by a'_{kj}). We may assume that a'_{02} has an edge that does not occur in $\mathbb{G}$ since any permutation of A can be achieved by a graph automorphism of I. If $a'_{02}\ E^{\mathbb{G}}\ a'_{12}$ then the set of vertices $\{a'_{12}, a'_{02}, a'_{01}, c', a'_{11}\}$ can be taken to form a subgraph of $\mathbb{G}$ isomorphic to III_5 with the nodes labeled in the order given. From the necessary sets found for III_n, one of the edges $e_1 := \{a'_{02}, a'_{11}\}, \{a'_{12}, a'_{01}\}, e_2 := \{a'_{02}, c'\}$ must also appear in $\mathbb{G}$. We note that the first two edges differ by an automorphism of I so we may treat only one of them explicitly (note that the proposed necessary set is stable under automorphism of I). If the edge e_1 is in $\mathbb{G}$, then the set $\{a'_{02}, a'_{11}, c', a'_{01}\}$ forms a subgraph isomorphic to III_4, which means that either e_2 or $e_3 = \{a'_{11}, a'_{01}\}$ appear in $\mathbb{G}$.

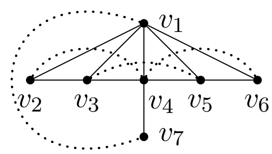
Thus the statement that either e_1 or e_2 appears in $\mathbb{G}$ can be reduced to the statement that either e_2 or e_3 (the dotted edges) appear in $\mathbb{G}$. To show the desired result it will be enough to check that there are two intersection graphs having I as a subgraph with the edge e_2 but not e_3 and vice versa. To do this, we note that the following two graphs are interval intersection graphs in $\mathbb{R}$ (with labels indicating a set of intervals with the given intersection graph):





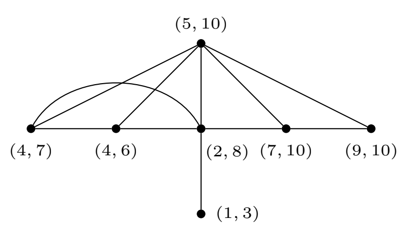
The other edges belong in the $\subseteq$-minimal necessary set by symmetric arguments. □

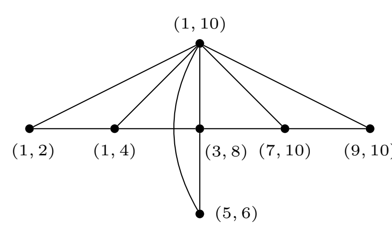
Lemma 3.3.12. The dotted edges in



form a necessary set of minimal cardinality for II in L^2.

Proof. As before, we recognize that the set $A = \{v_2, v_6, v_7\}$ is asteroidal in II and start our analysis with edges that are adjacent to elements of A. We start by noticing that the edges $\{v_2, v_4\}$, $\{v_4, v_6\}$, and $\{v_1, v_7\}$ must be in any necessary set as II becomes an interval intersection graph in $\mathbb{R}$ when any one of these edges is added. For example, the collections of intervals given by:



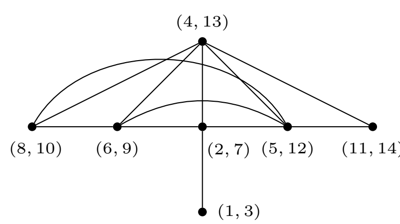


(note that the case of the edge $\{v_4, v_6\}$ being added is symmetric to the case of $\{v_2, v_4\}$).

It remains to find any intersection graphs $\mathbb{G}$ with II as a subgraph and which fail to contain any of the edges so far found. The remaining choices for adding an edge incident to an element of A are (up to symmetry) $e_1 = \{v_2, v_7\}$, $e_2 = \{v_2, v_5\}$, and $e_3 = \{v_2, v_6\}$. Note that the edge $\{v_3, v_7\}$ (or the symmetric $\{v_5, v_7\}$) can be added to II without changing the fact that A is asteroidal. Thus another edge incident to an element of A must be still be added to produce an intersection graph, allowing us to ignore the edge $\{v_3, v_7\}$.

Suppose first that an intersection graph $\mathbb{G}$ extends II and has the edge e_1. Then the vertices $\{v_1, v_4, v_7, v_2\}$ form a cycle of length 4, and so either the edge $\{v_1, v_7\}$ or the edge $\{v_2, v_4\}$ belong to $\mathbb{G}$. Both of these edges are already accounted for, so all intersection graphs with the edge $\{v_2, v_7\}$ are already accounted for.

Now suppose that $\mathbb{G}$ is an intersection graph containing the edge $\{v_2, v_5\}$. But then $\mathbb{G}$ contains the 4-cycle v_2, v_3, v_4, v_5 and so must contain either the edge $\{v_3, v_5\}$ or the edge $\{v_2, v_4\}$. The latter is already covered, so adding $\{v_3, v_5\}$ to the necessary set will cover all intersection graphs with the edge $\{v_2, v_5\}$ (and, by a symmetric argument, all intersection graphs with the edge $\{v_3, v_6\}$). It remains to check that the edge $\{v_3, v_5\}$ occurs in an intersection graph not having any of the other edges in the proposed necessary set. The graph II with the edges $\{v_2, v_5\}$ and $\{v_3, v_5\}$ added accomplishes this goal. In particular, the collection of real intervals

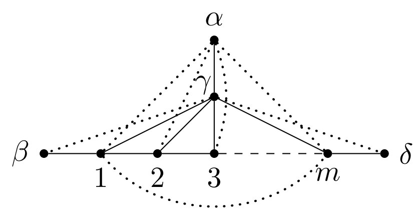


has this intersection graph.

Finally suppose that $\mathbb{G}$ is an intersection graph extending II and containing the edge $\{v_2, v_6\}$. Then the vertices $\{v_2, v_6, v_5, v_4, v_3\}$ form a subgraph isomorphic to III_5 with labels occurring in the given order, so one of the edges $\{v_2, v_5\}$, $\{v_6, v_4\}$, or $\{v_6, v_3\}$ must also be in $\mathbb{G}$. Any graph containing the first two of these edges has already been explicitly covered, and any intersection graph with $\{v_3, v_6\}$ is covered by an argument symmetric to the argument for graphs containing $\{v_2, v_5\}$. Thus every such $\mathbb{G}$ already has an edge in our proposed necessary set.

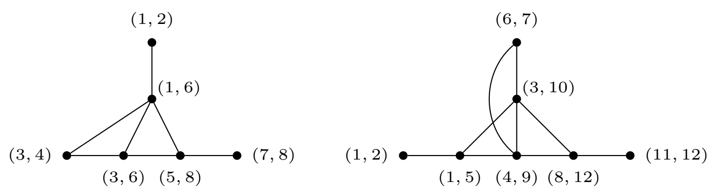
This necessary set is of minimal cardinality because all necessary sets must contain all but one of the proposed edges. $\square$

Lemma 3.3.13. The dotted edges in



form a necessary set of minimal cardinality for IV_m in L^2.

Proof. As before, we start by identifying the set $A = \{\alpha, \beta, \delta\}$ as being asteroidal in IV_m. The edges $\{\beta, \gamma\}$, $\{\gamma, \delta\}$, and $\{\alpha, i\}$ for each $1 \leq i \leq m$ must be in every necessary set for IV_m as IV_m becomes an intersection graph as soon as any one of the above edges is added. Up to symmetry, representative examples of these intersection graphs are:



It remains to show that any intersection graph extending IV_m and not containing any of the aforementioned edges must contain the edge $\{1, m\}$ (and that such intersection graphs do exist).

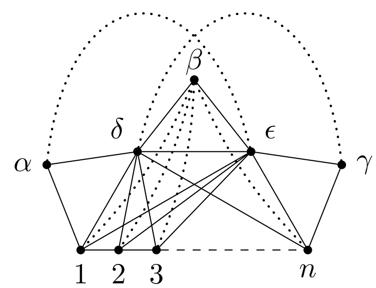
If $\mathbb{G}$ extends IV_m and $\{\alpha, \beta\}$ is in $\mathbb{G}$ then $\{\alpha, \beta, 1, \gamma\}$ forms a 4-cycle and so either $\{\alpha, 1\}$ is in $\mathbb{G}$ or $\{\beta, \gamma\}$ is in $\mathbb{G}$. However, both of these options are already covered, so $\{\alpha, \beta\}$ cannot be in a $\subseteq$-minimal necessary set. A symmetric argument gets the same result for the edge $\{\alpha, \delta\}$.

Now, in order to avoid A being asteroidal, an intersection graph extending IV_m and having none of the previously mentioned edges must have one of the edges $\{\beta, \delta\}$, $\{\beta, m\}$, or $\{\delta, 1\}$. If $\{\beta, \delta\}$ is an edge of $\mathbb{G}$ then the vertices $\{\beta, \delta, m, \gamma, 1\}$ form a 5-cycle so one of the edges $\{1, m\}, \{\gamma, \delta\}$, or $\{\gamma, \beta\}$ must appear in $\mathbb{G}$, but we have already ruled out the latter two, so $\{1, m\}$ must be in $\mathbb{G}$.

If $\{\beta, m\}$ is in $\mathbb{G}$ then the vertices $\{\beta, m, \gamma, 1\}$ form a 4-cycle and so either $\{\beta, \gamma\}$ is in $\mathbb{G}$ or $\{1, m\}$ is in $\mathbb{G}$. We already know that graphs with $\{\beta, \gamma\}$ are covered, so we must have that $\{1, m\}$ is in $\mathbb{G}$. A symmetric argument for $\{\delta, 1\}$ also shows that $\{1, m\}$ must be an edge in $\mathbb{G}$.

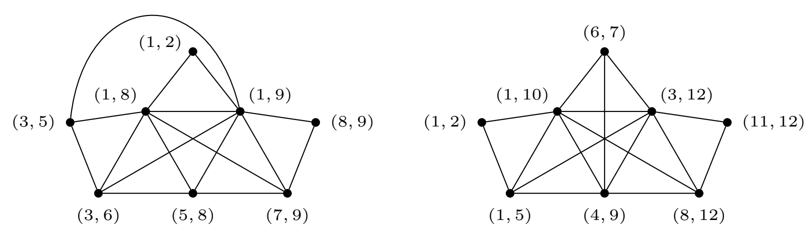
The result now follows from the fact that the extension of IV_m with the vertices $\{\beta, \delta\} \cup \{1, \ldots, m\}$ inducing a complete subgraph and otherwise having only the edges from IV_m is an intersection graph. An example of intervals producing such a graph could be given by setting all of the vertices labelled $1, \ldots, m$ to the same non-empty interval and also identify the vertices labelled β and δ. These identifications lead to a quotient graph of the proposed graph that is isomorphic to a graph that is a single irreducible path, which was shown to be an intersection graph in Lemma 3.3.10. $\square$

Lemma 3.3.14. The dotted edges (i.e. the edges $\{\gamma,\delta\}$, $\{\alpha,\epsilon\}$, and $\{\beta,i\}$ for $1 \le i \le n$) in



form the unique necessary set of minimal cardinality for V_n in L^2.

Proof. We first note that the set $A = \{\alpha,\beta,\gamma\}$ is asteroidal in V_n. We also note that if any one of the edges $\{\gamma,\delta\}$, $\{\alpha,\epsilon\}$, or $\{\beta,i\}$ for some $1 \le i \le n$ are added to V_n, then the resulting graph is an intersection graph, and so every necessary set for V_n contains the edges $\{\alpha,\epsilon\}$, $\{\gamma,\delta\}$, and $\{\beta,i\}$ for each $1 \le i \le n$. Representative examples of such graphs are (up to symmetry):



Suppose that $\mathbb{G}$ is an intersection graph extending V_n and containing the edge $\{\alpha,\beta\}$. Then the vertices $\{\alpha,\beta,\epsilon,1\}$ form a 4-cycle, and so one of the edges $\{\beta,1\}$ or $\{\alpha,\epsilon\}$ must be contained in $\mathbb{G}$ as well. However, we already know that both of these edges must be in the necessary set.

If $\mathbb{G}$ contains the edge $\{\alpha,\gamma\}$ then the vertices $\{\alpha,\gamma,\epsilon,\delta\}$ form a 4-cycle, so one of the edges $\{\alpha,\epsilon\}$ or $\{\gamma,\delta\}$ must be in $\mathbb{G}$. Again, we already know that both of these edges must be in every necessary set.

At this point, no other edges can prevent A from being asteroidal in an extension, so we have covered all the required cases. $\square$

Now that we have computed the minimal necessary sets for each of the minimal graphs that cannot appear as induced subgraphs of interval intersection graphs, we can state a theorem for SOP that is an analogue of Theorem 3.3.4.

Theorem 3.3.15. Let $\mathcal{U}$ be a regular ultrafilter on the infinite cardinal λ. The following are equivalent:

(1) $\mathcal{U}$ is good.

(2) $\mathcal{U}$ saturates some theory with SOP.

(3) If $(\mathbb{G}_\alpha)_{\alpha<\lambda}$ is a sequence of finite graphs such that $\mathrm{I}, \mathrm{II}, \mathrm{III}_\ell, \mathrm{IV}_m, \mathrm{V}_n \not\leq \mathbb{G}_\alpha$ for each α and $\ell \geq 4, m \geq 2$, and $n \geq 1$, then every complete $\mathbb{H} \subseteq \mathbb{G}_\alpha^{\mathcal{U}} =: \mathbb{G}$ with $|\mathbb{H}| \leq \lambda$ has some internal complete subgraph of $\mathbb{G}$ containing it.

(4) Every monotone function $d\colon \mathcal{P}_\omega(\lambda) \to \mathcal{U}$ that satisfies both

 (a) $d(\Gamma) = \bigcap_{\Delta \in [\Gamma]^2} d(\Delta)$ for all $\Gamma \in \mathcal{P}_\omega(I)$ with $|\Gamma| > 2$ and

 (b) the conditions given by specializing the subset relation in Theorem 3.2.18(3)(b) to the necessary sets found for $\mathcal{H} = \mathrm{I}, \mathrm{II}, \mathrm{III}_\ell, \mathrm{IV}_m, \mathrm{V}_n$ in Lemmas 3.3.10–3.3.14,

 has a multiplicative refinement.

Proof. (1) $\iff$ (2): Follows from e.g. [Shelah(1996), Thm. 2.9].

(2) $\iff$ (3) $\iff$ (4): Follows from Theorem 3.2.18 and Lemmas 3.3.10–3.3.14. $\square$

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