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# Short Communication: American Student Loans: Repayment and Valuation*

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**Abstract.** American student loans are fixed-rate debt contracts that give borrowers the option to repay their balances in full by a fixed maturity, or to enroll in income-based schemes, whereby payments are proportional to their income above subsistence, and any balance remaining after several years of payments is forgiven but taxed as income. For a small loan, the cost-minimizing repayment strategy dictates maximum payments until full repayment, forgoing both income-based schemes and forgiveness. For a large loan, income-based repayment is optimal, either immediately or after a period of maximum payments. The critical balance depends on the loan rate, the tax rate, and the forgiveness horizon. Overall, income-based repayment significantly benefits large but not small borrowers.

**Key words.** student loans, repayment, valuation, loan forgiveness

**AMS subject classifications.** 91G10, 91G80

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## 1. Introduction.

American student loans, once a minor share of household debt, today account for over \$1.5 trillion, surpassing both auto loans and credit cards. As this trend continues unabated, the growing burden of student loans on household finances has become a controversial policy issue and a source of public concern. The bulk of such debt (\$1.4 trillion) is in federal student loans, sophisticated financial contracts with unique covenants of income-based repayment, consolidation, deferral, forbearance, and forgiveness. These features make student loans extremely hard products for individual borrowers to manage and for governments and financial institutions to value.

Federal loans make funds available to students to cover their tuition and living expenses. A few months after graduation or unenrollment, former students are responsible for repaying their loans, which grow at a national fixed rate. If they enroll in income-based schemes, monthly payments are due only if their income is above a certain subsistence threshold, and are proportional to the amount by which it exceeds such threshold. After a number of years of qualifying payments (usually 20–25), the remaining balance is forgiven but taxed as ordinary income and hence forgiven only in the amount that it exceeds the ensuing taxes.

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This paper finds the optimal repayment strategy for a borrower who wishes to minimize the loan's cost, i.e., the present value of future loan payments. This strategy in turn determines the loan's implied value to the issuer under the student's optimal behavior. Central to cost minimization is the tension between forgiveness and compounding: On one hand, a borrower is tempted to delay repayments until the loan is forgiven and only taxes on the forgiven balance are due. On the other hand, the loan rate is much higher than the borrower's discount rate; hence the cost of delaying payments increases exponentially with the forgiveness horizon—potentially offsetting its ostensible savings. Such a tension suggests a simple heuristic: repay a loan as slowly as possible if the forgiveness horizon is short, in that the savings from forgiveness override the cost of compounding; otherwise, repay the loan as quickly as possible. Yet, such a heuristic neglects the monthly minimum payments required to qualify for forgiveness, which are in turn proportional to income, forcing most of the balance of a small loan to be repaid, and leaving little, if any, amount to be forgiven. Vice versa, minimum payments have little effect on large loans, as the bulk of their balance matures to the forgiveness horizon.

We prove that the cost-minimizing repayment strategy is of two types: if the balance is small enough, one should pay as much as possible until the loan is paid off. Otherwise, one should do so up to a *critical horizon*, and then enroll in the income-based repayment scheme, paying the required minimum until the loan is forgiven. The critical horizon is the time at which the benefits of forgiveness match the costs of compounding, i.e.,

$$t_c := \left(T + \frac{\log \omega}{\beta}\right)^+ \in [0, T), \tag{1.1}$$

where $T > 0$ is the forgiveness maturity in years, $\omega \in (0, 1)$ is the tax rate (varying by individual circumstances, but usually from 30% to 50%), and $\beta > 0$ is the spread between the borrower's discount rate and the student loan rate. There are currently three main rates for federal student loans: one for undergraduate loans, with a borrowing limit of \$57,500, one for graduate loans, with a limit of \$138,500, and one for Direct PLUS loans, available to either parents or graduate students for additional borrowing up to the cost of attendance.[1] Because PLUS loans do not have a fixed limit, they are prevalent in large balances, for which the repayment problem is most pressing and our analysis most consequential.

Despite the popularity of student loans, the problem of finding the cost-minimizing repayment strategy does not seem to have been considered and solved in detail before. Although it is common knowledge that borrowers with large balances and a relatively low tax rate are better off enrolling in income-based repayment plans, thereby paying the minimum required by the scheme, typical student resources tend to recommend immediate enrollment in such schemes, rather than considering the possibility of later enrollment preceded by high repayments, which we find to be optimal. The savings from high initial payments and later enrollment in income-based schemes can be substantial, especially for large loans with a high interest rate, which are common for professional degrees.

Consider, for example, a graduate from a dental school with a balance of \$300,000 in

[1] For disbursement dates before July 1, 2020, the rate is 4.53% for undergraduate loans, 6.08% for graduate loans, and 7.08% for Direct PLUS loans. See https://studentaid.gov/understand-aid/types/loans/subsidized-unsubsidized.

Direct PLUS loans carrying an interest rate of 7.08%.[2] With a starting salary of $100,000 that grows at an annual rate of 4%, suppose that the graduate can afford to repay at most 30% of income above subsistence without carrying a credit card balance. Keeping such maximal payments, the loan would be repaid in less than 20 years and its cost would be about $512,000 (here and henceforth, figures are rounded to the nearest $1,000) assuming a discount rate of 1.5%. Alternatively, the graduate could immediately enroll in the income-based repayment scheme, thereby paying only 10% of income above subsistence.[3] Such a strategy would leave after 25 years a balance of $1,053,000 forgiven and hence taxed as income, thereby generating a liability of about $421,000 assuming a tax rate of 40%.[4] Overall, the cost of the loan under immediate enrollment would be $524,000, which is $12,000 higher than with the full repayment option.

Instead, a graduate who followed the strategy identified in this paper would repay 30% of income above subsistence for the first 8.5 years, switching thereafter to an income-based scheme that would cap payments at 10% of income above subsistence. Such a strategy would leave a balance of $462,000 to be forgiven after 25 years, generating a tax liability of $184,000. The total cost of the loan would be $490,000, which is $21,000 lower than the full-repayment cost, and $34,000 lower than the immediate-enrollment cost.

As this numerical example shows, income-based repayments are a double-edged sword: while they enable a borrower to minimize immediate payments and benefit from loan forgiveness, they also have the potential to increase the overall costs of a loan through a larger tax liability on the final forgiven balance. For this reason, it can be optimal to delay enrollment after a few years of high payments so as to curb the subsequent growth of the balance during the years of minimum payments. Figure 1 demonstrates the relevance of income-based repayments and later enrollments for large loan balances, long forgiveness horizons, and high interest rates. For undergraduate loans carrying a 4.53% interest rate, income-based repayments are optimal for large balances, but enrollment is immediate (light shaded area), because the critical horizon $t_{\mathrm{c}}$ is zero. For graduate (6.08%) and PLUS (7.08%) loans, the critical horizon is positive, hence later enrollment (dark shaded area) arises with a sufficiently long forgiveness horizon.

In the United States, student loans were first introduced in the postwar period, and their growth accelerated after the establishment of Sallie Mae in 1973. Student loans are now a mainstream scheme to finance higher education, and the Department of Education estimates that there are nearly 45 million student debt holders in the US, with 2.5 million borrowers owing more than $100,000 each.[5] While student loans have the merit to expand access to higher education, research in the past decade has also brought potential demerits to light. Recent

[2]According to the American Dental Education Association, 83% of dental school graduates have student loan debt, with an average balance of $292,169; therefore $300,000 is a typical figure for such graduates.

[3]We assume a household of three people, which would imply a subsistence income, defined as 150% of the poverty level, of $32,000. The calculation assumes that the poverty rate grows at an annual 4%, a figure that is close to the historical average of 3.6% per year from 1959 (when the poverty level was first established) to 2019. See https://www.census.gov/data/tables/time-series/demo/income-poverty/historical-poverty-people.html.

[4]A graduate loan borrower without partial financial hardship (PFH) is only eligible for income-driven repayment plans with a 25-year term; see http://www.ibrinfo.org/existingidr.vp.html.

[5]And each of the top hundred borrowers owes more than $1 million; see https://www.wsj.com/articles/mike-meru-has-1-million-in-student-loans-how-did-that-happen-1527252975.

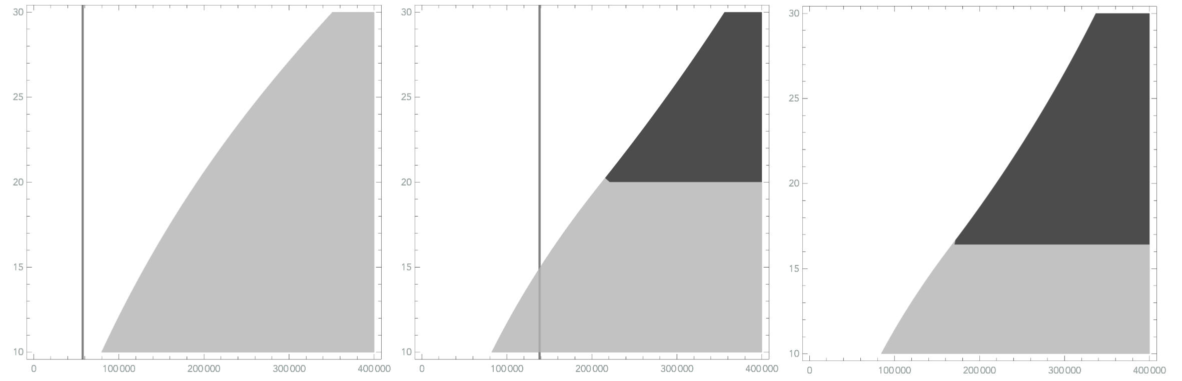


**Figure 1.** *Combinations of loan balance (horizontal, in dollars) and forgiveness horizon (vertical, in years) for which income-based repayments are cost-minimizing (shaded), for direct undergraduate (left,* 4.53% *rate), graduate (center,* 6.08%*), and PLUS loans (right,* 7.08%*). The light shaded area denotes immediate enrollment, and the dark shaded area denotes later enrollment. In the nonshaded area, the minimal cost is achieved by maximizing loan payments. Vertical lines denote maximum amounts for undergraduate and graduate loans (PLUS loans have no maximum). Parameters: Forgiveness horizon* $T = 25$*, annual growth of income and poverty level* $g = 4\%$*, discount rate* $r = 1.5\%$*, tax rate* $\omega = 40\%$*. Minimum and maximum payments are* 10% *and* 30% *of income above subsistence of* \$32,000.

empirical work finds that higher balances of student loans contribute to reducing home ownership [11], inhibiting propensity to entrepreneurship [7] and public sector employment [13], delaying marriage [5], postponing parenthood [14] and enrollment in graduate or professional degrees [10, 15], and increasing the cohabitation with parents [1, 4]. Also controversial is the interaction between student loans and tuition: empirical work [9] suggests that an increase in the subsidized loan maximum leads to a sticker-price increase in tuition of about 60 cents on the dollar, thereby suggesting that colleges (rather than students) may be the beneficiaries of a large fraction of government loan subsidies. (This is the so-called "Bennett hypothesis," named after William Bennet, who publicly formulated the link between student loan availability and tuition fees as Secretary of Education in 1987.)

This paper contributes to the understanding of student loans by identifying the cheapest repayment strategy in the presence of an income-based scheme, which is the most distinctive feature of these loans. We focus on the objective of minimizing the cost of loan repayments because, a priori, it is consistent with the maximization of the net worth of a household. A posteriori, the cost-minimizing strategy also offers significant protection to negative shocks through income-based repayment.

In principle, deviating from cost minimization would be justified when an alternative strategy would offer lower risk, but our results suggest that the potential improvements in this regard may be rather limited. Indeed, the central risk reduction that can be achieved in student loans is through income-based repayment, which allows monthly payments to be proportional to income above subsistence, thereby partially hedging income fluctuations. However, our results show that enrollment in such schemes is already optimal for large loan balances (for which the potential risk reduction is largest), for the purpose of minimizing costs—even neglecting their hedging potential. Put differently, income-based repayment reduces both cost and risk, which means that the minimization of these two quantities is largely aligned. A partial tradeoff between cost and risk is present when the cheapest strategy entails later enrollment in the income-based scheme. In this case, the borrower could anticipate enrollment, at the price of increased repayment costs for the duration of the loan, while only reducing risk for the period by which enrollment is anticipated.

Our model is intentionally deterministic, for two reasons: First, and foremost, deterministic analysis is more accessible and is in fact central to students' decision making, as attested by the typical online comparison tools, which do not entertain randomness.[6] Second, the scarcity of literature on optimal repayment of student loans suggests that the first step towards solving this problem is to consider its specific tradeoffs in the simplest possible setting. Taking such a first step is the goal of this paper.

Our results are also relevant to the valuation of student loans, as the federal government receives both the students' repayments and most of the final tax liability. (The forgiven amount may also incur state taxes, but they are significantly lower than federal taxes.) Absent defaults, the value of a student loan for the lender is significantly higher than the loan balance, due to the significant spread between student loan rates and risk-free rates, even after accounting for the rebates embedded in income-based repayments. For undergraduate and graduate loans, this result is driven by the suboptimality of income-based repayments for small balances, implying that the balance threshold for which forgiveness benefits are large enough to yield a net subsidy is typically higher than the maximum loan amount. Vice versa, for Direct PLUS loans, which are limited only by the cost of attendance, the rate spread and forgiveness horizon are so large that subsidies never materialize.[7]

The issue of default deserves some discussion. Student loans, unlike other unsecured debt such as credit card balances, cannot be discharged in bankruptcy except in very rare circumstances [12], while borrowers' wages can be garnished for life. As a result, delinquency on student loans does not reduce the borrower's liabilities: instead, it adds collection fees to the loan's balance and significantly reduces access to credit by impairing the debtor's credit score. In addition, a borrower with subsistence income (or no income at all) can remain in good standing without making payments by enrolling in income-based repayment schemes, thereby avoiding delinquency at no cost. Empirical work confirms that student loan defaults are difficult to reconcile with borrowers' optimal choices, and may be due to borrowers' insufficient information about their options [3]. Using individually identifiable information on student loan borrowers, Cornaggia and Xia [2] find that "the majority of distressed student borrowers have their loans in disadvantageous repayment plans even when eligible for more advantageous options." Furthermore, Looney and Yannelis [8] find that over 30% of student loans of \$5,000 or *less* are in default, even though they would be paid in full in ten years with monthly payments below \$100 (and without income-based repayment). Also, delinquencies

[6]See, among others, https://studentloanhero.com/calculators/, https://smartasset.com/student-loans/student-loan-calculator, and https://www.calculator.net/student-loan-calculator.html.

[7]Using fair-value accounting, the Congressional Budget Office estimates that all loan categories except Parent PLUS entail net subsidies. See Table 5 in https://www.cbo.gov/system/files/2020-03/51310-2020-03-studentloan.pdf. A 2020 report commissioned by the Department of Education projected an overall cost of \$435 billion for the program. See https://www.wsj.com/articles/student-loan-losses-seen-costing-u-s-more-than-400-billion-11605963600.

tend to decrease as loan balances increase, contrary to the incentives of strategic default. For these reasons, the model in this paper does not entertain delinquency, as its aim is identifying optimal repayment strategies rather than explaining observed defaults. Put differently, our focus is normative rather than positive.

## 2. Model and main result.

A student graduates with a loan balance of $x > 0$ and seeks the repayment strategy $\alpha$ that minimizes the present value of future payments, discounted at some rate $r > 0$, which represents the opportunity cost of money, i.e., the alternative safe return that could be obtained on any dollar used to pay off the loan. For example, a household with a mortgage may ponder whether to increase mortgage or student loan payments; hence the mortgage rate is a close approximation to the household's discount rate. For a household without other debt, the discount rate represents the return on a safe investment.

The loan carries an interest rate of $r+\beta$, higher than the discount rate (i.e., $\beta > 0$), which means that paying off the loan earlier entails lower compounding costs.[8] Thus, denoting by $\alpha_t$ the chosen repayment rate at time $t$, the loan balance $b_t^\alpha$ evolves over time according to the dynamics

$$db_t^\alpha = (r+\beta)b_t^\alpha dt - \alpha_t dt, \quad b_0 = x > 0. \tag{2.1}$$

The student loan also includes a forgiveness provision, whereby at some future horizon $T > 0$ the remaining balance $b_T^\alpha$ of the loan is forgiven, but then taxed at rate $\omega \in (0,1)$, whence a payment of $\omega b_T^\alpha$ is due at time $T$. Such a provision encourages delay of payments as the forgiveness horizon approaches, thereby countering the compounding motive.

The payment rate at time $t$ is constrained to the range $m(t)$ to $M(t)$, which depends on the former student's income, with $m(t)$ reflecting the minimum payment due under income-based repayments, and $M(t)$ the maximum payment that accommodates other living expenses without incurring debt, such as credit card balances, which carry a higher rate than that of student loans. In particular, the borrower can afford to repay the loan under its original term and possibly make additional payments up to $M(t)$. (Otherwise, immediate enrollment in income-based repayment is inevitable, excluding costlier choices such as forbearance and default.) Because, a posteriori, it is optimal to either maximize or minimize payments, the original payment rate does not enter the solution other than through the condition that it is less than or equal to $M(t)$.

Specifically, for any $x > 0$ and Lebesgue measurable $\alpha : [0,T] \to [0,\infty)$, the present value of future payments is $J(x,\alpha) := \int_0^\tau e^{-rt}\alpha_t dt + e^{-r\tau}\omega b_\tau$, where

$$\tau := \inf\{t \geq 0 : b_t = 0\} \wedge T \tag{2.2}$$

is the time when the loan is either paid in full or forgiven. The goal is to minimize the present value of future payments, i.e., $v(x) := \inf_{\alpha\in\mathcal{A}} J(x,\alpha)$, where the set of feasible repayment strategies is defined as

$$\mathcal{A} := \{\alpha : t \mapsto \alpha_t \text{ is Lebesgue measurable with } m(t) \leq \alpha_t \leq M(t) \text{ for } 0 \leq t \leq \tau\}$$

[8]The case of a household with debt that carries a higher interest than student loans, such as credit card debt, is somewhat trivial, as the borrower's optimal policy is to pay off such debt first. Thus, we focus on the case of a positive spread $\beta$.

for some Lebesgue integrable $m, M : [0,T] \to (0,\infty)$ satisfying $m(t) < M(t)$ for all $t \in [0,T]$.[9] The main result describes the optimal repayment strategy in relation to the loan's parameters.

Theorem 2.1. *Let $x^* := \int_0^{t^*} e^{-(r+\beta)s} M(s)ds > 0$, where the time $t^* \in (t_c, T)$ is the unique solution to*

$$\int_{t_c}^{t^*} e^{-rs} M(s)(1-\omega e^{\beta(T-s)})ds = \int_{t_c}^{T} e^{-rs} m(s)(1-\omega e^{\beta(T-s)})ds, \tag{2.3}$$

*with $t_c \in [0,T)$ defined as in (1.1). Then, for any $x > 0$, the strategy $\alpha^* \in \mathcal{A}$ defined as*

$$\alpha_t^* := \begin{cases} M(t)1_{[0,t_c]}(t) + m(t)1_{(t_c,T]}(t), & t \in [0,T], \quad \text{if } x > x^*, \quad \text{(max-min)} \\ M(t), & t \in [0,T], \quad \text{if } x \le x^*, \quad \text{(max)} \end{cases}$$

*attains the minimum loan value. Also, $v(x) = v_1(x)$ for $x > x^*$ and $v(x) = v_2(x)$ for $x \le x^*$, where*

$$v_1(x) := \int_0^{t_c} e^{-rs} M(s)ds + \int_{t_c}^{T} e^{-rs} m(s)ds + \omega e^{\beta T}\left(x - \int_0^{t_c} e^{-(r+\beta)s} M(s)ds - \int_{t_c}^{T} e^{-(r+\beta)s} m(s)ds\right), \tag{2.4}$$

$$v_2(x) := \int_0^{t_M} e^{-rs} M(s)ds, \quad \text{where } t_M > 0 \text{ satisfies } x = \int_0^{t_M} e^{-(r+\beta)s} M(s)ds. \tag{2.5}$$

The message of this result is straightforward: the cheapest repayment strategy mandates maximum payments when the initial balance is sufficiently low ($x < x^*$, "max" strategy). Otherwise ($x > x^*$, "max-min" strategy), maximum payments are in order before the critical horizon $t_c$ in (1.1), at which point enrollment in the income-based repayment occurs, implying minimum payments thereafter. If the critical horizon is zero (for example, if either the tax rate or the interest rate spread is very low), then enrollment is immediate, and minimum payments span the entire life of the loan (i.e., the "max-min" boils down to "min"). The *critical balance* $x^*$ that separates these two regimes is precisely the unique balance which yields the same repayment cost under both strategies.

The critical balance $x^*$ is particularly sensitive to the discount rate $r$, with low rates making it optimal to repay large balances early, and high discount rates encouraging deferral. The intuition is clear: a borrower with a higher opportunity-cost of capital has a stronger preference for later rather than earlier payments because the latter entail a higher sacrifice in return. Note also that the ostensible complexity of calculating the critical balance $x^*$ is not a significant barrier for a borrower who wishes to choose the cheapest repayment strategy: in practice, the borrower only needs to compare the cost of the max and max-min strategies, choosing the cheaper of the two.

[9]Minimizing the present value of debt servicing until repayment is mathematically similar to maximizing the present value of future dividends until bankruptcy, as in [6]. The main difference is that the present model is deterministic, but it includes both minimum required payments and forgiveness at a finite horizon, substantially changing the structure of the solution. We are grateful to an anonymous referee for pointing out this analogy.

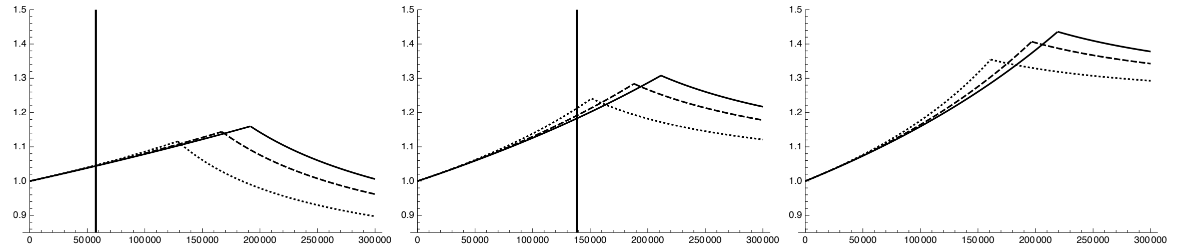

**Figure 2.** *Cost-to-balance ratio (vertical) against loan balance (horizontal) for undergraduate (left,* 4.53% *rate), graduate (center,* 6.08%*), and PLUS loans (right,* 7.08%*), with discount rate of* 1.5% *(solid),* 3% *(dashed), and* 6% *(dotted). Vertical lines denote maximum amounts for undergraduate and graduate loans (PLUS loans have no maximum). Parameters: Forgiveness horizon* $T = 20$*, annual growth of income and poverty level* $g = 4\%$*, tax rate* $\omega = 40\%$*. Minimum and maximum payments are* 10% *and* 30% *of income above subsistence of* \$32,000.

An important corollary of this result is that only large loan balances, those above $x^*$, benefit from income-based repayment schemes. Instead, smaller balances should be paid off as early as possible through maximum payments. To better understand this issue, Figure 2 plots the cost-to-balance ratio for the three main types of loans: for discount rates 1.5%, representative of the public lender or a borrower without other debt, 3%, indicative of a borrower with good credit score, and 6%, typical of a borrower with poor credit score.

For balances below the maximum limits, the solid lines (which reflect valuation at government-borrowing rates) are firmly above one, meaning that income-based schemes do not offer any net subsidies. In fact, discount rates have a minor impact for the valuation of small loan balances, leading to noticeable differences only after enrollment in income-based schemes becomes optimal. Indeed, a higher discount rate lowers the enrollment threshold $x^*$, significantly decreasing the borrowing cost per unit of balance. Once this threshold is exceeded, the marginal cost of any additional borrowed dollar is exactly $\omega e^{\beta T}$, and the additional balance affects payments neither in the "max" nor in the "min" periods of the loan. Additional balance increases lead the overall cost-to-balance ratio to converge to the marginal ratio $\omega e^{\beta T}$.

In summary, the marginal cost of borrowing increases with the balance until enrollment in income-based repayment becomes optimal. At that point, the marginal cost of additional borrowing drops to the constant $\omega e^{\beta T}$, as the average cost of borrowing gradually declines to the same constant. Thus, an implication of income-based repayment is that the average unit cost of borrowing is higher for medium balances than it is for very high balances.

## 3. Proofs.

This section contains the proof of the main result Theorem 2.1, which identifies the cheapest repayment strategy in relation to the initial balance. First, Lemma 3.2 reduces the search for the optimal strategy to the class of strategies with maximum, followed by minimum payments (with the latter possibly absent). Next, Propositions 3.3, 3.4, and 3.5 together demonstrate that the optimal strategy must be either $\alpha^1_t := M(t)1_{[0,t_c]}(t) + m(t)1_{(t_c,T]}(t)$ or $\alpha^2_t := M(t)$. Finally, Lemma 3.6 compares the costs of $\alpha^1$ and $\alpha^2$, establishing Theorem 2.1 at the end of this section. The discussion begins by observing a simple expression for the remaining balance (2.1) in terms of the initial balance and the discounted value of repayments.

Remark 3.1. *For any measurable $\alpha : [0,T] \to [0,\infty)$, the unique solution to* (2.1) *is*

$$b_t = e^{(r+\beta)t}\left(x - \int_0^t e^{-(r+\beta)s}\alpha_s ds\right), \quad t \geq 0. \tag{3.1}$$

*Indeed, the claim follows by integrating the equality*

$$d(e^{-(r+\beta)s}b_s) = e^{-(r+\beta)s}\{-(r+\beta)b_s ds + db_s\} = -e^{-(r+\beta)s}\alpha_s ds.$$

The next result shows that it is sufficient to consider repayment strategies of a very specific form: repaying first at the maximum rate $M(t)$ and then at the minimum rate $m(t)$.

Lemma 3.2. *For any $x > 0$, $v(x) = \inf_{\alpha\in\mathcal{B}} J(x,\alpha)$, where*

$$\mathcal{B} := \{\alpha \in \mathcal{A} : \exists\ t_0 \geq 0 \textit{ s.t. } \alpha_t = M(t)1_{[0,t_0]}(t) + m(t)1_{(t_0,T]}(t) \textit{ for a.e. } t \in [0,T]\}. \tag{3.2}$$

*Proof.* Fix $x > 0$. First, observe that $\inf_{\alpha\in\mathcal{B}} J(x,\alpha) = \inf_{\alpha\in\mathcal{B}'} J(x,\alpha)$, where

$$\mathcal{B}' := \{\alpha \in \mathcal{A} : \exists\ t_0 \geq 0 \text{ s.t. } \alpha_t = M(t)1_{[0,t_0]}(t) + m(t)1_{(t_0,\tau]}(t) \text{ for a.e. } t \in [0,\tau]\}.$$

Indeed, for any $\alpha \in \mathcal{B}$, the truncated strategy $\alpha'$, defined by $\alpha'_t := \alpha_t 1_{[0,\tau]}(t)$, belongs to $\mathcal{B}'$ and satisfies $J(x,\alpha') = J(x,\alpha)$; conversely, for any $\alpha' \in \mathcal{B}'$, the extended strategy $\alpha$, defined by $\alpha_t := \alpha'_t 1_{[0,\tau]}(t) + m1_{(\tau,T]}(t)$, belongs to $\mathcal{A}$ and satisfies $J(x,\alpha) = J(x,\alpha')$. For this reason, the remaining proof focuses on establishing $v(x) = \inf_{\alpha\in\mathcal{B}'} J(x,\alpha)$. To this end, it remains to show that for any $\alpha \in \mathcal{A}\setminus\mathcal{B}'$, there exists $\bar\alpha \in \mathcal{B}'$ such that $J(x,\bar\alpha) < J(x,\alpha)$, i.e.,

$$\int_0^{\tau(\bar\alpha)} e^{-rt}\bar\alpha_t dt + \omega e^{-r\tau(\bar\alpha)} b^{\bar\alpha}_{\tau(\bar\alpha)} < \int_0^{\tau(\alpha)} e^{-rt}\alpha_t dt + \omega e^{-r\tau(\alpha)} b^{\alpha}_{\tau(\alpha)}, \tag{3.3}$$

where $\tau$ in (2.2) is denoted as $\tau(\bar\alpha)$ or $\tau(\alpha)$, and $b$ in (2.1) is denoted as $b^{\bar\alpha}$ or $b^{\alpha}$, to emphasize their dependence on the chosen repayment strategy.

For any $\alpha \in \mathcal{A}\setminus\mathcal{B}'$, the first claim is that there exists $0 < t_0 < \tau(\alpha)$ such that

$$\int_0^{t_0} (M(t) - \alpha_t)e^{-(r+\beta)t} dt = \int_{t_0}^{\tau(\alpha)} (\alpha_t - m(t))e^{-(r+\beta)t} dt. \tag{3.4}$$

Define $f : [0,\tau(\alpha)] \to \mathbb{R}$ by $f(t) := \int_0^t (M(s) - \alpha_s)e^{-(r+\beta)s} ds - \int_t^{\tau(\alpha)} (\alpha_s - m(s))e^{-(r+\beta)s} ds$. As $\alpha, M, m$ are all Lebesgue integrable, $f$ is by definition continuous. Also, $\alpha \notin \mathcal{B}'$ implies that $\alpha_t$ cannot be equal to $m(t)$ for a.e. $t \in [0,\tau(\alpha)]$, whence $f(0) = -\int_0^{\tau(\alpha)} (\alpha_s - m(s))e^{-(r+\beta)s} ds < 0$. Likewise, $\alpha_t$ cannot be equal to $M(t)$ for a.e. $t \in [0,\tau(\alpha)]$, implying $f(\tau(\alpha)) = \int_0^{\tau(\alpha)} (M(s) - \alpha_s)e^{-(r+\beta)s} ds > 0$. The continuity of $f$ thus ensures the existence of $0 < t_0 < \tau(\alpha)$ such that $f(t_0) = 0$, i.e., (3.4) holds. Now, define $\bar\alpha : [0,T] \to [0,\infty)$ by

$$\bar\alpha_t := M(t)1_{[0,t_0]}(t) + m(t)1_{(t_0,\tau(\alpha)]}(t), \quad 0 \leq t \leq T. \tag{3.5}$$

Observe that $\tau(\bar\alpha) = \tau(\alpha)$: indeed, by (3.4),

$$\begin{aligned}\int_0^{\tau(\alpha)} e^{-(r+\beta)t}\bar\alpha_t dt = \int_0^{\tau(\alpha)} e^{-(r+\beta)t}\alpha_t dt + \int_0^{t_0} (M(t) - \alpha_t)e^{-(r+\beta)t} dt \\ - \int_{t_0}^{\tau(\alpha)} (\alpha_t - m(t))e^{-(r+\beta)t} dt = \int_0^{\tau(\alpha)} e^{-(r+\beta)t}\alpha_t dt.\end{aligned}$$

This fact, together with Remark 3.1, implies $b^{\bar{\alpha}}_{\tau(\alpha)} = b^{\alpha}_{\tau(\alpha)}$. The case $b^{\alpha}_{\tau(\alpha)} > 0$ leads to $\tau(\alpha) = T$ and thus $b^{\bar{\alpha}}_T = b^{\bar{\alpha}}_{\tau(\alpha)} > 0$, which readily implies $\tau(\bar{\alpha}) = T = \tau(\alpha)$. If $b^{\alpha}_{\tau(\alpha)} = 0$, then $b^{\bar{\alpha}}_{\tau(\alpha)} = 0$ and thus $\tau(\bar{\alpha}) \le \tau(\alpha)$, thanks to the definition of $\tau$ in (2.2). If $\tau(\bar{\alpha}) < \tau(\alpha) \le T$, then $b^{\bar{\alpha}}_{\tau(\bar{\alpha})} = 0$, again by (2.2). It then follows from the definition of $\bar{\alpha}$ and the formula of $b^{\bar{\alpha}}$ in (3.1) that $b^{\bar{\alpha}}_{\tau(\alpha)} < b^{\bar{\alpha}}_{\tau(\bar{\alpha})} = 0$, a contradiction. Thus, $\tau(\bar{\alpha}) = \tau(\alpha)$, as required, which implies $\bar{\alpha} \in \mathcal{B}$.

It remains to show (3.3). As a consequence of (3.4),

$$
\begin{aligned}
e^{-\beta t_0}\int_0^{t_0} e^{-rt}(M(t)-\alpha_t)dt &< \int_0^{t_0} e^{-(r+\beta)t}(M(t)-\alpha_t)dt \\
&= \int_{t_0}^{\tau} e^{-(r+\beta)t}(\alpha_t - m(t))dt < e^{-\beta t_0}\int_{t_0}^{\tau} e^{-rt}(\alpha_t - m(t))dt. \qquad (3.6)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\int_0^{\tau(\bar{\alpha})} e^{-rt}\bar{\alpha}_t dt + \omega e^{-r\tau(\bar{\alpha})} b^{\bar{\alpha}}_{\tau(\bar{\alpha})} &= \int_0^{\tau(\alpha)} e^{-rt}\alpha_t dt + \int_0^{t_0} e^{-rt}(M(t)-\alpha_t)dt \\
&\quad - \int_{t_0}^{\tau(\alpha)} e^{-rt}(\alpha_t - m(t))dt + \omega e^{-r\tau(\alpha)} b^{\bar{\alpha}}_{\tau(\alpha)} < \int_0^{\tau(\alpha)} e^{-rt}\alpha_t dt + \omega e^{-r\tau(\alpha)} b^{\alpha}_{\tau(\alpha)},
\end{aligned}
$$

where the equality follows from $\tau(\bar{\alpha}) = \tau(\alpha)$ and the definition of $\bar{\alpha}$ in (3.5), and the inequality is due to (3.6) and $b^{\bar{\alpha}}_{\tau(\alpha)} = b^{\alpha}_{\tau(\alpha)}$. That is, (3.3) is established. ∎

### 3.1. Three cases.

The following analysis distinguishes three cases, depending on how large the initial balance of the loan is. Consider the two useful thresholds

$$
\underline{x} := \int_0^T e^{-(r+\beta)s} m(s)ds \quad \text{and} \quad \overline{x} := \int_0^T e^{-(r+\beta)s} M(s)ds. \qquad (3.7)
$$

The first case is that of an initial balance $x > 0$ of the loan so large that even maximum payments cannot pay it off by time $T$.

Proposition 3.3. *Fix $x > \overline{x}$ and recall $t_c \in [0, T)$ defined in (1.1). Then, $\alpha^* \in \mathcal{A}$ defined by*

$$
\alpha^*_t := M(t)1_{[0,t_c]}(t) + m(t)1_{(t_c,T]}(t), \quad 0 \le t \le T, \qquad (3.8)
$$

*is an optimal control. Moreover, $\tau(\alpha^*) = T$ and $v(x) = v_1(x)$, with $v_1$ defined as in (2.4).*

*Proof.* Note that with $x > \overline{x}$ even maximum payments, i.e., $\tilde{\alpha}_t := M(t)$ for all $0 \le t \le T$, cannot pay off the debt by time $T$. Indeed, $b^{\tilde{\alpha}}_T = e^{(r+\beta)T}(x - \int_0^T e^{-(r+\beta)s}M(s)ds) > 0$ by (3.1), whence $b^{\alpha}_T > 0$ and $\tau(\alpha) = T$ for all $\alpha \in \mathcal{A}$. Thus, a strategy of the form

$$
\alpha^*_t := M(t)1_{[0,t_0]}(t) + m(t)1_{(t_0,T]}(t), \quad 0 \le t \le T, \ 0 \le t_0 \le T, \qquad (3.9)
$$

satisfies

$$
J(x, \alpha^*) = \int_0^T e^{-rt}\alpha^*_t dt + \omega e^{-rT} b^{\alpha^*}_T = f(t_0), \quad \text{where} \qquad (3.10)
$$

$$
f(t) := \int_0^t e^{-rs}M(s)ds + \int_t^T e^{-rs}m(s)ds + \omega e^{\beta T}\left(x - \int_0^t e^{-(r+\beta)s}M(s)ds - \int_t^T e^{-(r+\beta)s}m(s)ds\right). \qquad (3.11)
$$

Note that the second equality in (3.10) follows from (3.9) and Remark 3.1. By direct calculation, $f'(t) = e^{-rt}(M(t) - m(t))\left(1 - \omega e^{\beta(T-t)}\right)$, which shows that $f$ is strictly decreasing for $t < T + \frac{\ln \omega}{\beta}$ and strictly increasing for $t > T + \frac{\ln \omega}{\beta}$. It then follows from (3.10) that by taking $t_0 = t_c$ in (1.1), $\alpha^*$ in (3.9) attains $\inf_{\alpha \in \mathcal{B}} J(x, \alpha) = v(x)$, where the equality follows from Lemma 3.2. ∎

Next, consider the case where the initial balance $x > 0$ of the loan is so small that even minimum payments can pay it off by time $T$.

Proposition 3.4. *Fix $0 < x \le \underline{x}$. Consider the unique $t_M \in (0, T]$ such that*

$$x = \int_0^{t_M} e^{-(r+\beta)s} M(s) ds. \tag{3.12}$$

*Then, $\alpha^* \in \mathcal{A}$ defined by $\alpha^*_t = M(t)$, $0 \le t \le T$, is an optimal control. Moreover, $\tau(\alpha^*) = t_M < T$ and $v(x) = v_2(x)$, with $v_2$ defined as in (2.5).*

*Proof.* As $0 < x \le \underline{x}$, even minimum payments ($\tilde{\alpha}_t := m(t)$ for all $0 \le t \le T$) pay off the debt by time $T$. Indeed, $b^{\tilde{\alpha}}_T = e^{(r+\beta)T}(x - \int_0^T e^{-(r+\beta)s} m(s) ds) \le 0$ by (3.1), whence

$$b^\alpha_\tau = 0 \quad \text{and} \quad \tau(\alpha) \le T \qquad \text{for all } \alpha \in \mathcal{A}. \tag{3.13}$$

Also, observe that $0 < x \le \underline{x}$ and $0 < m(t) < M(t)$ readily imply the existence of a unique $t_M \in (0, T]$ such that (3.12) holds.

Now, focus on strategies $\alpha^*$ as in (3.9), with $0 \le t_0 \le T$. For each $0 \le t_0 \le T$, since $b^{\alpha^*}_\tau = 0$ by (3.13), it follows that $x = \int_0^\tau e^{-(r+\beta)s} \alpha^*_s ds$, in view of (3.1). This fact, together with (3.9) and (3.12), implies

$$x = \int_0^{t_0} e^{-(r+\beta)s} M(s) ds + \int_{t_0}^{\tau} e^{-(r+\beta)s} m(s) ds \quad \text{for } 0 \le t_0 \le t_M. \tag{3.14}$$

Thus, $\tau$ is a function $t_0 \mapsto \tau(t_0)$, $0 \le t_0 \le t_M$. By the strict positivity of $M$ and $m$, (3.14) indicates that $\tau(t_0) = t_0 + \eta(t_0)$, where

$$t_0 \mapsto \eta(t_0) \ \text{ is strictly decreasing on } [0, t_M] \text{ with } \eta(t_M) = 0. \tag{3.15}$$

It then follows from the decreasing property of $\eta$ and (3.14) that $t_0 \mapsto \tau(t_0)$ is differentiable a.e. Indeed, given $0 \le t_0 \le t_M$, for any $h \in \mathbb{R}$ such that $0 < t_0 + h < t_M$, (3.14) entails

$$\int_0^{t_0} e^{-(r+\beta)s} M(s) ds + \int_{t_0}^{\tau(t_0)} e^{-(r+\beta)s} m(s) ds = \int_0^{t_0+h} e^{-(r+\beta)s} M(s) ds + \int_{t_0+h}^{\tau(t_0+h)} e^{-(r+\beta)s} m(s) ds,$$

which reduces to

$$\int_{\tau(t_0+h)}^{\tau(t_0)} e^{-(r+\beta)s} m(s) ds = \int_{t_0}^{t_0+h} e^{-(r+\beta)s} (M(s) - m(s)) ds. \tag{3.16}$$

Thus, the right-hand side above vanishes as $h \to 0$; hence $\tau(t_0+h) \to \tau(t_0)$; i.e., $t_0 \mapsto \tau(t_0)$ is continuous, and so is $t_0 \mapsto \eta(t_0)$. By the continuity of $\eta$, the Lebesgue differentiation theorem implies that, for a.e. $t_0 \in [0, t_M]$,

$$\lim_{h\to 0} \frac{1}{\tau(t_0)-\tau(t_0+h)} \int_{\tau(t_0+h)}^{\tau(t_0)} e^{-(r+\beta)s} m(s)ds = e^{-(r+\beta)\tau(t_0)} m(\tau(t_0)),$$
$$\lim_{h\to 0} \frac{1}{h} \int_{t_0}^{t_0+h} e^{-(r+\beta)s} (M(s)-m(s))ds = e^{-(r+\beta)t_0} (M(t_0)-m(t_0)).$$

Dividing the second equality above by the first one and recalling (3.16) yields

$$\tau'(t_0) = \lim_{h\to\ 0} \frac{\tau(t_0+h)-\tau(t_0)}{h} = -e^{(r+\beta)(\tau(t_0)-t_0)} \frac{M(t_0)-m(t_0)}{m(\tau(t_0))}. \tag{3.17}$$

Thanks to (3.9) and (3.13),

$$J(x,\alpha^*) = \int_0^\tau e^{-rt}\alpha_t^* dt + \omega e^{-r\tau} b_\tau^{\alpha^*} = \int_0^\tau e^{-rt}\alpha_t^* dt = g(t_0), \tag{3.18}$$

where $g : [0, t_M] \to \mathbb{R}$ is defined as

$$g(t_0) := \int_0^{t_0} e^{-rs} M(s)ds + \int_{t_0}^{\tau(t_0)} e^{-rs} m(s)ds. \tag{3.19}$$

By direct calculation,

$$\begin{aligned} g'(t_0) &= e^{-rt_0}(M(t_0)-m(t_0)) + e^{-r\tau(t_0)} m(\tau(t_0))\tau'(t_0) \\ &= e^{-rt_0}(M(t_0)-m(t_0)) \left(1-e^{\beta(\tau(t_0)-t_0)}\right) < 0 \quad \text{for a.e. } 0 \le t_0 < t_M, \end{aligned} \tag{3.20}$$

where the second line follows from (3.17) and the inequality is due to (3.15). This shows that $g(t_0)$, $0 \le t_0 \le t_M$, has a global minimum at $t_0 = t_M$. Thus, it follows from (3.18) that by taking $t_0 = t_M$, $\alpha^*$ in (3.9) attains $\inf_{\alpha\in\mathcal{B}} J(x,\alpha) = v(x)$, where the equality follows from Lemma 3.2. Consequently, $v(x) = J(x,\alpha^*) = g(t_M) = \int_0^{t_M} e^{-rs}M(s)ds$, where the last equality follows from (3.19) and (3.15). Finally, simply because $\tau(t_M) = t_M$ (again, by (3.15)), one can without loss of generality take $\alpha_t^* = M(t)$ for all $0 \le t \le T$. ■

Finally, consider the intermediate case of an initial balance $x > 0$ small enough that maximum payments can pay off the debt by time $T$, but also large enough that minimum payments cannot pay it off by time $T$.

Proposition 3.5. *Let $\underline{x} < x \le \overline{x}$, $t_c \in [0, T)$, as in (1.1), and define $x_c \in [\underline{x}, \overline{x})$ as*

$$x_c := \int_0^{t_c} e^{-(r+\beta)s} M(s)ds + \int_{t_c}^{T} e^{-(r+\beta)s} m(s)ds. \tag{3.21}$$

(i) *If $x > x_c$, then*

$$v(x) = v_1(x) \wedge v_2(x),$$

*where $v_1$ and $v_2$ are defined as in (2.4) and (2.5), respectively. Furthermore, if $v_1(x) < v_2(x)$, $\alpha^* \in \mathcal{A}$ defined in (3.8) is an optimal control; otherwise, $\alpha^* \in \mathcal{A}$ defined by $\alpha_t^* = M(t)$, $0 \le t \le T$, is an optimal control.*

(ii) *If $x \le x_c$, then $v(x) = v_2(x)$ with $v_2$ defined as in (2.5). Moreover, $\alpha^* \in \mathcal{A}$ defined by $\alpha^*_t = M(t)$, $0 \le t \le T$, is an optimal control.*

*Proof.* As $\underline{x} < x \le \overline{x}$ and $0 < m(t) < M(t)$, there exists a unique $\tilde{t} \in (0, T]$ such that

$$x = \int_0^{\tilde{t}} e^{-(r+\beta)s} M(s) ds + \int_{\tilde{t}}^T e^{-(r+\beta)s} m(s) ds. \tag{3.22}$$

Thus, $\tilde{t} \le t_M$ by (3.22) and the definition of $t_M > 0$ in (2.5). Now, decompose $\mathcal{B}$ in (3.2) into $\mathcal{B}_1 := \{\alpha \in \mathcal{B} : b^\alpha_T > 0\}$ and $\mathcal{B}_2 := \{\alpha \in \mathcal{B} : b^\alpha_T \le 0\}$. In view of Remark 3.1 and (3.22),

$$\mathcal{B}_1 = \{\alpha \in \mathcal{B} : 0 \le t_0 < \tilde{t}\} \quad \text{and} \quad \mathcal{B}_2 = \{\alpha \in \mathcal{B} : \tilde{t} \le t_0 \le T\}. \tag{3.23}$$

For any $\alpha \in \mathcal{B}_1$, argue as in (3.10) to obtain that $J(x, \alpha) = f(t_0)$, where $f : \mathbb{R} \to \mathbb{R}$ is defined as in (3.11). As shown after (3.11), $f(t)$ is strictly decreasing for $t < T + \frac{\ln \omega}{\beta}$ and strictly increasing for $t > T + \frac{\ln \omega}{\beta}$. Thus, (3.23) implies that

$$\inf_{\alpha \in \mathcal{B}_1} J(x, \alpha) = f\left(t_c \wedge \tilde{t}\right), \tag{3.24}$$

where $t_c \in [0, T)$ is defined as in (1.1). For any $\alpha \in \mathcal{B}_2$, argue as in (3.18) to obtain $J(x, \alpha) = g(t_0 \wedge t_M)$, where $g : [0, t_M] \to \mathbb{R}$ is defined as in (3.19). As shown below (3.19), $g(t)$ is strictly decreasing for $t < t_M$. Thus, (3.23) and $\tilde{t} \le t_M$ imply that

$$\inf_{\alpha \in \mathcal{B}_2} J(x, \alpha) = g(t_M). \tag{3.25}$$

In view of $\tilde{t} \le t_M$, note also that $g(t_M) \le g(\tilde{t}) = f(\tilde{t})$, where the equality follows from the definitions of $f$ and $g$ ((3.11) and (3.19)) and (3.22). Now, by Lemma 3.2, (3.24), and (3.25),

$$v(x) = \inf_{\alpha \in \mathcal{B}} J(x, \alpha) = f\left(t_c \wedge \tilde{t}\right) \wedge g(t_M) = \begin{cases} f(t_c) \wedge g(t_M) = v_1(x) \wedge v_2(x) & \text{if } t_c < \tilde{t}, \\ g(t_M) = v_2(x) & \text{if } t_c \ge \tilde{t}, \end{cases}$$

where the third equality exploits $g(t_M) \le f(\tilde{t})$, and $v_1$ and $v_2$ are defined as in (2.4) and (2.5), respectively. Note from (3.22) and (3.21) that $t_c < \tilde{t}$ if and only if $x_c < x$. The desired result thus follows from the previous equality. ∎

In summary, Propositions 3.3, 3.4, and 3.5 together demonstrate that the optimal strategy must be either $\alpha^1_t := M(t)1_{[0,t_c]}(t) + m(t)1_{(t_c,T]}(t)$ or $\alpha^2_t := M(t)$. Specifically, (i) if $\alpha^1$ can pay off the balance $x$ by time $T$ (i.e., $x \le x_c$), Propositions 3.4 and 3.5(ii) state that it is best to pay off the debt as soon as possible, i.e., $\alpha^2$ is optimal; (ii) if $\alpha^1$ cannot pay off the balance $x$ by time $T$ but $\alpha^2$ can (i.e., $x_c < x \le \overline{x}$), Proposition 3.5(i) states that one needs to compare the costs $v_1(x)$ and $v_2(x)$ to determine which one of $\alpha^1$ and $\alpha^2$ is optimal; (iii) if $\alpha^2$ cannot pay off the balance $x$ by time $T$ (i.e., $x > \overline{x}$), Proposition 3.3 states that $\alpha^1$ is optimal. Thus, for the case $x_c < x \le \overline{x}$, it is necessary to analyze $v_1(x)$ and $v_2(x)$ further to determine the optimal strategy.

Lemma 3.6. (i) *$v_1(x)-v_2(x)$ is strictly decreasing on $[\hat{x},\overline{x}]$, where*

$$\hat{x} := \int_0^{t_c} e^{-(r+\beta)s} M(s)ds \in [0,\overline{x}). \tag{3.26}$$

(ii) *There exists a unique $x^* \in (\hat{x},\overline{x})$ such that $v_1(x^*)=v_2(x^*)$. Hence, $v_1(x)>v_2(x)$ for $x\in[\hat{x},x^*)$ and $v_1(x)<v_2(x)$ for $x\in(x^*,\overline{x}]$. Moreover, $x^*$ is identified as in Theorem 2.1 and satisfies $x^*>x_c$, with $x_c$ defined as in (3.21).*

*Proof.* In view of (2.5), $t_M>0$ is in fact a function of $x$ and is strictly increasing by definition. Henceforth, denote $t_M$ as $t_M(x)$ for clarity. As $t_M(\hat{x})=t_c$ by construction, it follows that $t_M(x)>t_c$ for all $\hat{x}<x\le\overline{x}$. Thanks to the definitions of $v_1$ and $v_2$ in (2.4) and (2.5), as well as the relation (3.12), for any $\hat{x}\le x\le\overline{x}$,

$$\begin{aligned}
v_1(x)-v_2(x) &= -\int_{t_c}^{t_M(x)} e^{-rs}M(s)ds + \int_{t_c}^{T} e^{-rs}m(s)ds\\
&\quad + \omega e^{\beta T}\left(\int_{t_c}^{t_M(x)} e^{-(r+\beta)s}M(s)ds - \int_{t_c}^{T} e^{-(r+\beta)s}m(s)ds\right)\\
&= -\int_{t_c}^{t_M(x)} e^{-rs}M(s)\left(1-\omega e^{\beta(T-s)}\right)ds + \int_{t_c}^{T} e^{-rs}m(s)\left(1-\omega e^{\beta(T-s)}\right)ds.
\end{aligned} \tag{3.27}$$

Note that $s>t_c=(T+\frac{\ln w}{\beta})^+$ if and only if $1-\omega e^{\beta(T-s)}>0$. Hence, because $M(s)$ and $1-\omega e^{\beta(T-s)}$ are strictly positive and $t_M(x)$ is strictly increasing, (3.27) implies that $v_1(x)-v_2(x)$ is strictly decreasing on $[\hat{x},\overline{x}]$. Now, observe from (3.12) that $t_M(\overline{x})=T$ and from (3.27) that

$$v_1(\overline{x})-v_2(\overline{x}) = -\int_{t_c}^{T} e^{-rs}(M(s)-m(s))\left(1-\omega e^{\beta(T-s)}\right)ds<0,$$

because $M(s)>m(s)$ and $1-\omega e^{\beta(T-s)}>0$. On the other hand, by $t_M(\hat{x})=t_c$, (3.27) yields $v_1(\hat{x})-v_2(\hat{x}) = \int_{t_c}^{T} e^{-rs}m(s)\left(1-\omega e^{\beta(T-s)}\right)ds>0$, again because $1-\omega e^{\beta(T-s)}>0$. As $v_1(x)-v_2(x)$ is strictly decreasing on $[\hat{x},\overline{x}]$, there must exist $x^*\in(\hat{x},\overline{x})$ such that $v_1(x^*)-v_2(x^*)=0$. Note that $x^*$ is identified by setting the right-hand side of (3.27) to be zero, which leads to the characterization in Theorem 2.1. Now, in view of (3.26), (3.21), and (3.7), $\hat{x}<x_c<\overline{x}$ by definition. Observe from (2.4), (3.21), and (3.19) that

$$v_1(x_c) = \int_0^{t_c} e^{-rs}M(s)ds + \int_{t_c}^{T} e^{-rs}m(s)ds = g(t_c) > g(t_M(x_c)) = v_2(x_c),$$

where the inequality follows from the fact that $g:[0,t_M(x_c)]\to\mathbb{R}$ is minimized at $t_M(x_c)$ (recall (3.20)) and the last equality is due to (3.19) and (2.5). This fact readily implies $x_c<x^*$. ∎

*Proof of Theorem* 2.1. By Lemma 3.6, $x^*>x_c$, $v_1(x)>v_2(x)$ for $x\in[\hat{x},x^*)$, and $v_1(x)<v_2(x)$ for $x\in(x^*,\overline{x}]$. The results of Proposition 3.5 are then simplified to $v(x)=v_1(x)$ for $x\in(x^*,\overline{x}]$ and $v(x)=v_2(x)$ for $x\in(\overline{x},x^*]$. Combining this fact with Propositions 3.3 and 3.4 yields the claim. ∎

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