Scientific Report No. 66

SELECTED TOPICS IN VECTOR CALCULUS

by

S.W. Maley

August 1981

Electromagnetics Laboratory
Department of Electrical Engineering
University of Colorado
Boulder, Colorado  80309
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Differentiation and Integration of Vector Fields</td>
<td>1</td>
</tr>
<tr>
<td>Definition of the Gradient and Divergence Operators in Terms of Closed Line Integrals</td>
<td>8</td>
</tr>
<tr>
<td>Definition of Curl and Gradient Operators in Terms of Closed Surface Integrals</td>
<td>11</td>
</tr>
<tr>
<td>Differential Identities</td>
<td>13</td>
</tr>
<tr>
<td>Vector Integration</td>
<td>21</td>
</tr>
<tr>
<td>Mean Value Theorems for Vector Integrals</td>
<td>24</td>
</tr>
<tr>
<td>Integral Identities</td>
<td>27</td>
</tr>
<tr>
<td>Differentiation of Line and Surface Integrals</td>
<td>29</td>
</tr>
<tr>
<td>Appendix 1, Inverse of the Gradient Operator</td>
<td>31</td>
</tr>
<tr>
<td>Appendix 2, Inverse of the Divergence and Curl Operators</td>
<td>32</td>
</tr>
<tr>
<td>Appendix 3, Expression of a Vector in Terms of its Vector and Scalar Products with a Known Vector</td>
<td>37</td>
</tr>
<tr>
<td>Appendix 4, Expression of a Vector in Terms of its Vector Products with a Set of Mutually Orthogonal Unit Vectors</td>
<td>40</td>
</tr>
<tr>
<td>Appendix 5, Example of the Determination of a Vector Field from its Divergence and Curl</td>
<td>42</td>
</tr>
<tr>
<td>Appendix 6, List of Formulas</td>
<td>63</td>
</tr>
</tbody>
</table>
SELECTED TOPICS IN VECTOR CALCULUS

Introduction

Vector calculus is extensively used in the formulation of theory relating to physical phenomena. This report is concerned with certain topics that have received little or no attention in the commonly available references. Some of these results are of use in the presentation of elementary theory; some are useful in more advanced applications; and some are of interest primarily for their elucidation of basic concepts.

Differentiation and Integration of Vector Fields

The concept of a derivative in vector calculus is an extension of that in ordinary calculus. Consider a scalar function defined in three-dimensional space. Denote that function \( f(x,y,z) \) where \( x, y \) and \( z \) are rectangular coordinates. Ordinary calculus defines the directional derivative in the direction defined by angles \( \theta_x, \theta_y \) and \( \theta_z \) measured from the \( x, y \)- and \( z \)-axes respectively, as

\[
\frac{\partial f}{\partial z} = \cos \theta_x \frac{\partial f}{\partial x} + \cos \theta_y \frac{\partial f}{\partial y} + \cos \theta_z \frac{\partial f}{\partial z}.
\]

The symbol \( \ell \) may be regarded as distance along a straight line in the direction specified.

In vector calculus, the gradient of function \( f \) is a vector denoted \( \nabla f \) and given by

\[
\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}
\]
where \( \overrightarrow{a}_x, \overrightarrow{a}_y \) and \( \overrightarrow{a}_z \) are unit vectors in the \( x \), \( y \) and \( z \) directions.

The directional derivative is then given by

\[
\frac{\partial f}{\partial \overrightarrow{a}} = \overrightarrow{a}_\xi \cdot \nabla f
\]

where \( \overrightarrow{a}_\xi \) is a unit vector in the specified direction; in terms of \( \theta_x \), \( \theta_y \) and \( \theta_z \) it is given by

\[
\overrightarrow{a}_\xi = \overrightarrow{a}_x \cos \theta_x + \overrightarrow{a}_y \cos \theta_y + \overrightarrow{a}_z \cos \theta_z.
\]

It is apparent that the vector \( \nabla f \) has a magnitude equal to the maximum value that the directional derivative can take (The maximization is with respect to direction.); and the direction of \( \nabla f \) is the direction in which that maximum occurs.

Notationally the operator \( \nabla \) can be regarded as

\[
\nabla = \overrightarrow{a}_x \frac{\partial}{\partial x} + \overrightarrow{a}_y \frac{\partial}{\partial y} + \overrightarrow{a}_z \frac{\partial}{\partial z};
\]

then the operator \( \frac{\partial}{\partial \overrightarrow{a}} \) is

\[
\frac{\partial}{\partial \overrightarrow{a}} = \overrightarrow{a}_\xi \cdot \nabla = \cos \theta_x \frac{\partial}{\partial x} + \cos \theta_y \frac{\partial}{\partial y} + \cos \theta_z \frac{\partial}{\partial z}
\]

and

\[
\frac{\partial f}{\partial \overrightarrow{a}} = (\overrightarrow{a}_\xi \cdot \nabla)f.
\]

In this discussion, the directional derivative is given by either \( \overrightarrow{a}_\xi \cdot (\nabla f) \) or \( (\overrightarrow{a}_\xi \cdot \nabla)f \) so the simpler notation \( \overrightarrow{a}_\xi \cdot \nabla f \), without parentheses, can be used. However, in some applications of the "del" operator, \( \nabla \), it is necessary to use the parentheses.
The inverse of the gradient operator is well known and it takes the form of a line integral. Suppose that $\vec{G}$ is the gradient of $f$; in symbols $\vec{G} = \nabla f$. The field, $f$, at point $p_2$ can be denoted $f(p_2)$ and it can be expressed as

$$f(p_2) = f(p_1) + \int_{p_1}^{p_2} \vec{G} \cdot \text{d}\ell$$

where contour $C$ extends from a reference point $p_1$ to the observation point $p_2$.

This expression gives the field $f$ at point $p_2$ in terms of the field at a reference point and a line integral along any contour $C$ from the reference point $p_1$ to point $p_2$.

The line integral thus determines the field $f$ to within a constant. This is to be expected since any constant present in $f$ vanishes in the determination of $\vec{G} = \nabla f$. Such constants, therefore, cannot be recovered from $\vec{G}$. This result is further discussed in App. 1.
Although a rectangular coordinate system has been used in this discussion, the gradient of a function (or field), defined in a physical space, is independent of the coordinate system; furthermore this is true of all derivatives and integrals used in vector calculus.

One of the three simple types derivatives, the gradient, has been briefly discussed; it operates on a scalar field and produces a vector field as a result. The other two types, the curl and the divergence, operate on vector fields and produce a vector field, in the case of the curl, and a scalar field in the case of the divergence. These other types of derivatives will now be discussed.

The derivative of a vector field with respect to distance in three-dimensional space can be categorized as either of two types depending upon the direction along which the distance is measured. The distance may be in the direction of the vector being differentiated or it may be transverse to that vector. The transverse case will be considered first. Consider a vector \( \bar{F} = a_x F_x + a_y F_y + a_z F_z \) in rectangular coordinates. The transverse type of derivatives involve the directional derivatives of components in directions perpendicular to the direction of the components themselves; or, in other words, the functions \( \frac{\partial F_x}{\partial y}, \frac{\partial F_y}{\partial z}, \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x}, \frac{\partial F_z}{\partial y}, \frac{\partial F_x}{\partial z} \). These, individually, are not readily useable in the formulation of theory of physical phenomena because they are dependent upon the choice of coordinate system.

Experience has shown, however, that the vector combination of these functions,

\[
\bar{a}_x \left( \frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} \right) + \bar{a}_y \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) + \bar{a}_z \left( \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right),
\]

is physically meaningful and independent of choice of coordinate system.
It regularly occurs in physical theory relating to "transverse" derivatives. It is denoted curl $\vec{F}$ or rot $\vec{F}$ (rotation of $\vec{F}$) or, using the operator, $\nabla$, it is written $\nabla \times \vec{F}$ and read curl $\vec{F}$, rotation $\vec{F}$ or simply del-cross-$\vec{F}$.

Although other types of transverse derivatives could be defined, this dominant one is used in the study of physical theory.

The other one of the two types of derivatives of a vector field involves the directional derivatives of components in the direction of the components themselves. In terms of the vector defined above these are $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$. These functions individually, like the individual transverse derivatives discussed above, are not readily useable in formulation of theory of physical phenomena because they are dependent on the choice of coordinate system. In this case experience has shown that the scalar sum

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$$

is physically meaningful and independent of choice of coordinate system. This combination regularly occurs in physical theory relating to the derivative of a vector in the direction of the vector. It is called the divergence of $\vec{F}$ and is written div $\vec{F}$; or, in terms of the operator $\nabla$, it is written $\nabla \cdot \vec{F}$ and called divergence of $\vec{F}$ or simply del-dot-$\vec{F}$.

Other types of derivatives of vectors in the direction of the vectors could be defined, but this is the dominant one used in the study of physical theory.

At this stage of discussion of vector differentiation and integration, the three important types of derivative, gradient, curl and divergence have been defined and discussed and the inverse of the gradient, which is a line integral, has been discussed. To complete the discussion, the
inverses of the curl and divergence operations need to be considered. It is appropriate, at this stage of the discussion to briefly consider a simpler problem so that analogies can be introduced. Suppose that \( \vec{A} \) is an unknown vector and that
\[
\vec{A} \times \vec{B} = \vec{C}
\]
where \( \vec{B} \) and \( \vec{C} \) are known vectors. This can be considered as a vector equation in one unknown, \( \vec{A} \). Since this is a vector equation, its components can be written as three scalar equations. Furthermore, since \( \vec{A} \) has three components it would seem, at first thought, that the three equations could be solved for the three unknown components of \( \vec{A} \). If this procedure is attempted, it will be found that the three equations are dependent; therefore solution for \( \vec{A} \) is not possible. One additional independent equation involving the components of \( \vec{A} \) is needed. That equation could be selected as
\[
\vec{A} \cdot \vec{B} = D
\]
where \( D \) is a known scalar.

The set of two equations
\[
\vec{A} \times \vec{B} = \vec{C}
\]
and
\[
\vec{A} \cdot \vec{B} = D
\]
can be solved for \( \vec{A} \). The solution can be written (Ref. App. 3)
\[
\vec{A} = \frac{\vec{B} \times \vec{C} + DB}{B \cdot \vec{B}}.
\]
It may be said that both the vector and scalar products of an unknown vector and a known vector must be known to determine the unknown vector.

Return now to the task of finding the inverse of the curl and divergence operations. Suppose that \( \bar{A} \) is an unknown vector and

\[
\nabla \times \bar{A} = \bar{B}
\]

where \( \bar{B} \) is known. As in the case of the algebraic equation \( \bar{A} \times \bar{B} = \bar{C} \), it seems, at first thought, that this can be expressed as three equations in the three unknown components of \( \bar{A} \). In this case the equations are partial differential equations rather than algebraic; but, if they are independent, a solution is possible. However, in this case, just as in the case of the algebraic equations, they are not independent; and one additional equation is required for a solution. The additional equation can be chosen as

\[
\nabla \cdot \bar{A} = D;
\]

then the solution for \( \bar{A} \) can be written (Ref. App. 2)

\[
\bar{A} = \int_{V} \frac{\nabla \times \bar{B} - \nabla D}{4\pi R} \, dv
\]

where \( R \) is the distance from the point of integration to the observation point at which \( \bar{A} \) is to be determined. This integral operator can be considered as the inverse of both the curl and divergence operations.

Most references on applications of vector calculus explain that both the curl and divergence of a vector field must be known to determine the field but do not give an explicit expression for such determination. Examples of the use of the above result are given in App. 5.
DEFINITION OF THE GRADIENT AND DIVERGENCE
OPERATORS IN TERMS OF CLOSED
LINE INTEGRALS

The identity
\[
\oint_C \mathbf{F} \cdot d\mathbf{\xi} = - \int_S \nabla \mathbf{F} \times d\mathbf{S}
\]
leads to a definition of \( \nabla \mathbf{F} \) in terms of a closed line integral.

Let the surface, \( S \), become very small, say \( \Delta S \). Then \( d\mathbf{S} \approx \mathbf{a}_n \Delta S \) where \( \mathbf{a}_n \) is the unit vector normal to \( \Delta S \). Then
\[
\oint_C \mathbf{F} \cdot d\mathbf{\xi} \approx - (\nabla \mathbf{F}) \times \mathbf{a}_n \Delta S
\]
and
\[
\nabla \mathbf{F} \times \mathbf{a}_n = \lim_{\Delta S \to 0} \frac{- \oint_C \mathbf{F} \cdot d\mathbf{\xi}}{\Delta S}
\]

This can be considered a fundamental definition of \( \nabla \mathbf{F} \times \mathbf{a}_n \). Let \( \mathbf{a}_1, \mathbf{a}_2 \) and \( \mathbf{a}_3 \) be mutually orthogonal unit vectors; let \( \Delta \mathbf{S}_1 = \mathbf{a}_1 \Delta S_1, \Delta \mathbf{S}_2 = \mathbf{a}_2 \Delta S_2 \) and \( \Delta \mathbf{S}_3 = \mathbf{a}_3 \Delta S_3 \) be small vector areas; and let \( C_1, C_2 \) and \( C_3 \) be the bounding contours for \( S_1, S_2 \) and \( S_3 \). Then \( \nabla \mathbf{F} \) can be expressed as,
\[
\nabla \mathbf{F} = -\frac{1}{2} \left[ \lim_{\Delta \mathbf{S}_1 \to 0} \frac{\mathbf{a}_1 \times \oint_{C_1} \mathbf{F} \cdot d\mathbf{\xi}}{\Delta \mathbf{S}_1} + \lim_{\Delta \mathbf{S}_2 \to 0} \frac{\mathbf{a}_2 \times \oint_{C_2} \mathbf{F} \cdot d\mathbf{\xi}}{\Delta \mathbf{S}_2} + \lim_{\Delta \mathbf{S}_3 \to 0} \frac{\mathbf{a}_3 \times \oint_{C_3} \mathbf{F} \cdot d\mathbf{\xi}}{\Delta \mathbf{S}_3} \right]
\]
where App. 4 has been used.
The identity,
\[ \oint_{\gamma} \vec{a}_n \cdot \vec{F} \times d\vec{x} = - \int_{\Sigma} \nabla \cdot [\vec{a}_n \times (\vec{a}_n \times \vec{F})] \, dS, \]
where \( \vec{a}_n \) is a unit vector normal to surface \( \Sigma \), leads to a definition of \( \nabla \cdot \vec{F} \) in terms of closed line integrals. Let \( C_1 \) be a contour in a plane normal to the unit vector \( \vec{a}_1 \) and similarly define \( C_2 \) and \( C_3 \) where the unit vectors \( \vec{a}_1, \vec{a}_2 \) and \( \vec{a}_3 \) are mutually orthogonal. Also define \( \Delta S_1, \Delta S_2 \) and \( \Delta S_3 \) to be areas bounded by \( C_1, C_2 \) and \( C_3 \) respectively. Then it is easily shown that

\[
\nabla \cdot \vec{F} = \frac{1}{2} \left[ \lim_{\Delta S_1 \to 0} \frac{\oint_{C_1} \vec{a}_1 \cdot \vec{F} \times d\vec{x}}{\Delta S_1} + \lim_{\Delta S_2 \to 0} \frac{\oint_{C_2} \vec{a}_2 \cdot \vec{F} \times d\vec{x}}{\Delta S_2} + \lim_{\Delta S_3 \to 0} \frac{\oint_{C_3} \vec{a}_3 \cdot \vec{F} \times d\vec{x}}{\Delta S_3} \right]
\]

or

\[
\nabla \cdot \vec{F} = \frac{1}{2} \left[ \lim_{\Delta S_1 \to 0} \frac{\oint_{C_1} \vec{a}_1 \times \vec{F} \cdot d\vec{x}}{\Delta S_1} + \lim_{\Delta S_2 \to 0} \frac{\oint_{C_2} \vec{a}_2 \times \vec{F} \cdot d\vec{x}}{\Delta S_2} + \lim_{\Delta S_3 \to 0} \frac{\oint_{C_3} \vec{a}_3 \times \vec{F} \cdot d\vec{x}}{\Delta S_3} \right]
\]

A similar sort of characterization of the curl, \( \nabla \times \vec{F} \) of a vector field is well known; it is

\[
(\nabla \times \vec{F}) \cdot \vec{a}_n = \lim_{\Delta S \to 0} \frac{\oint_{C} \vec{F} \cdot d\vec{x}}{\Delta S}
\]
where $C$ is the bounding contour of area $A S$. In this case, if
$(V \times F) \cdot \vec{a}_n$ is known for three different unit vectors $\vec{a}_n$, $V \times F$ can be
determined. More specifically let $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ be mutually orthogonal
unit vectors, let $\Delta S_1 = \vec{a}_1 \Delta S_1$, $\Delta S_2 = \vec{a}_2 \Delta S_2$ and $\Delta S_3 = \vec{a}_3 \Delta S_3$ be small
vector areas; and let $C_1, C_2$ and $C_3$ be bounding contours of $\Delta S_1$, $\Delta S_2$
and $\Delta S_3$. Then $V \times F$ can be expressed as

$$V \times F = \lim_{\Delta S_1 \to 0} \frac{\vec{a}_1 \oint_{C_1} F \cdot d\vec{a}}{\Delta S_1} + \lim_{\Delta S_2 \to 0} \frac{\vec{a}_2 \oint_{C_2} F \cdot d\vec{a}}{\Delta S_2} + \lim_{\Delta S_3 \to 0} \frac{\vec{a}_3 \oint_{C_3} F \cdot d\vec{a}}{\Delta S_3}.$$
DEFINITION OF CURL AND GRADIENT
OPERATORS IN TERMS OF CLOSED
SURFACE INTEGRALS

The identity
\[ \oint_S \mathbf{F} \times d\mathbf{s} = - \int_V (\nabla \times \mathbf{F}) \, dv \]
leads to a definition of \( \nabla \times \mathbf{F} \) in terms of a closed surface integral. Let the volume, \( V \), enclosed by surface \( S \) become very small, say \( \Delta v \), then
\[ \oint_S \mathbf{F} \times d\mathbf{s} \approx - \Delta v (\nabla \times \mathbf{F}) \]
and
\[ \nabla \times \mathbf{F} = \lim_{\Delta v \to 0} \frac{- \oint_S \mathbf{F} \times d\mathbf{s}}{\Delta v} . \]

This can be considered a fundamental definition of \( \nabla \times \mathbf{F} \).

The identity
\[ \oint_S F \, dS = \int_V (\nabla F) \, dv \]
leads to a definition of \( \nabla F \) in terms of a closed surface integral. Let the volume, \( V \), enclosed by surface, \( S \), become very small, say \( \Delta v \), then
\[ \int \mathbf{F} \cdot d\mathbf{S} = \Delta v(\nabla \mathbf{F}) \]

and

\[ \nabla \mathbf{F} = \lim_{\Delta v \to 0} \frac{\int \mathbf{F} \cdot d\mathbf{S}}{\Delta v} \]

This can be considered a fundamental definition of \( \nabla \mathbf{F} \).

A similar sort of characterization of the divergence, \( \nabla \cdot \mathbf{F} \) is well known; it is

\[ \nabla \cdot \mathbf{F} = \lim_{\Delta v \to 0} \frac{\int_{\partial \mathbf{F} \cdot d\mathbf{S}}}{\Delta v} \]
DIFFERENTIAL IDENTITIES

In the application of vector calculus to the theory of physical phenomena, it is sometimes necessary to apply the differentiation operator, \( \nabla \), to various sorts of products of scalar and vector fields. A number of identities relating to such operations are well known and widely used; for example
\[
\nabla (fg) = f \nabla g + g \nabla f \quad \text{and} \quad \nabla \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G}).
\]
Nothing more will be said of these known identities except to note two conspicuous omissions from the list. There is no identity giving the expansion of the operation \( \nabla (\vec{G} \cdot \vec{F}) \) nor the operation \( \nabla \times (\vec{G} \times \vec{F}) \).

The reason, perhaps, is related to the need for two slightly more involved types of differential operators. An example of one of these types is \( (\vec{F} \cdot \nabla) \) and an example of the other if \( \vec{F} \times \nabla \). The operator \( (\vec{F} \cdot \nabla) \) is a scalar operator which, when applied to a vector field \( \vec{G} \) (i.e. \( (\vec{F} \cdot \nabla)\vec{G} \)) gives the product of the magnitude of \( \vec{F} \) and the directional derivative of \( \vec{G} \) in the direction of \( \vec{F} \). For illustrative purposes consider rectangular coordinates. \( (\vec{F} \cdot \nabla) \) is given by
\[
(\vec{F} \cdot \nabla) = \left( \frac{\partial}{\partial x} \vec{F}_x + \frac{\partial}{\partial y} \vec{F}_y + \frac{\partial}{\partial z} \vec{F}_z \right) \cdot \left( \frac{\partial}{\partial x} \vec{G}_x + \frac{\partial}{\partial y} \vec{G}_y + \frac{\partial}{\partial z} \vec{G}_z \right)
\]
\[
= F_x \frac{\partial}{\partial x} + F_y \frac{\partial}{\partial y} + F_z \frac{\partial}{\partial z} \vec{G}.
\]
From this, it follows that
\[
(\vec{F} \cdot \nabla)\vec{G} = F_x \frac{\partial\vec{G}}{\partial x} + F_y \frac{\partial\vec{G}}{\partial y} + F_z \frac{\partial\vec{G}}{\partial z}.
\]
The directional derivative of $\vec{G}$ in the direction of $\vec{F}$ is

$$
\left( \frac{F_x}{|\vec{F}|} \frac{\partial}{\partial x} + \frac{F_y}{|\vec{F}|} \frac{\partial}{\partial y} + \frac{F_z}{|\vec{F}|} \frac{\partial}{\partial z} \right) \vec{G}
$$

$$
= \frac{1}{|\vec{F}|} \left( F_x \frac{\partial \vec{G}}{\partial x} + F_y \frac{\partial \vec{G}}{\partial y} + F_z \frac{\partial \vec{G}}{\partial z} \right).
$$

and the product of this with the magnitude of $\vec{F}$ is

$$
|\vec{F}| \left( \frac{F_x}{|\vec{F}|} \frac{\partial}{\partial x} + \frac{F_y}{|\vec{F}|} \frac{\partial}{\partial y} + \frac{F_z}{|\vec{F}|} \frac{\partial}{\partial z} \right) \vec{G} = F_x \frac{\partial \vec{G}}{\partial x} + F_y \frac{\partial \vec{G}}{\partial y} + F_z \frac{\partial \vec{G}}{\partial z}
$$

which is the same result as obtained for $(\vec{F} \cdot \nabla)\vec{G}$. Expansion in terms of the components of $\vec{G}$ gives

$$
(F \cdot \nabla)\vec{G} = \vec{a}_x \left( F_x \frac{\partial \vec{G}}{\partial x} + F_y \frac{\partial \vec{G}}{\partial y} + F_z \frac{\partial \vec{G}}{\partial z} \right)
$$

$$
+ \vec{a}_y \left( F_x \frac{\partial \vec{G}}{\partial x} + F_y \frac{\partial \vec{G}}{\partial y} + F_z \frac{\partial \vec{G}}{\partial z} \right)
$$

$$
+ \vec{a}_z \left( F_x \frac{\partial \vec{G}}{\partial x} + F_y \frac{\partial \vec{G}}{\partial y} + F_z \frac{\partial \vec{G}}{\partial z} \right).
$$

It is apparent that the scalar operator $(\vec{F} \cdot \nabla)$ can also be applied to a scalar field, $g$. In this case it is easily seen that

$$
(\vec{F} \cdot \nabla)g = \vec{F} \cdot \nabla g.
$$

The other type of operator needed is illustrated by $(\vec{F} \times \nabla)$. This is a vector operator; it is not so easily described in words as was true of the scalar operator $(\vec{F} \cdot \nabla)$. Again consider rectangular coordinates; then $(\vec{F} \times \nabla)$ is given by
\( (F \times \nabla) = (\overline{a}_x F_x + \overline{a}_y F_y + \overline{a}_z F_z) \times (\overline{a}_x \frac{\partial}{\partial x} + \overline{a}_y \frac{\partial}{\partial y} + \overline{a}_z \frac{\partial}{\partial z}) \)

\[ = \overline{a}_x (F_y \frac{\partial}{\partial z} - F_z \frac{\partial}{\partial y}) + \overline{a}_y (F_z \frac{\partial}{\partial x} - F_x \frac{\partial}{\partial z}) + \overline{a}_z (F_x \frac{\partial}{\partial y} - F_y \frac{\partial}{\partial x}); \]

and \((F \times \nabla) \times \overline{G}\) is

\[ (F \times \nabla) \times \overline{G} = [\overline{a}_x (F_y \frac{\partial}{\partial z} - F_z \frac{\partial}{\partial y}) + \overline{a}_y (F_z \frac{\partial}{\partial x} - F_x \frac{\partial}{\partial z}) + \overline{a}_z (F_x \frac{\partial}{\partial y} - F_y \frac{\partial}{\partial x})] \times (\overline{a}_x G_x + \overline{a}_y G_y + \overline{a}_z G_z) \]

\[ = \overline{a}_x (F_z \frac{\partial G_y}{\partial x} - F_y \frac{\partial G_z}{\partial x} - F_x \frac{\partial G_y}{\partial y} + F_y \frac{\partial G_x}{\partial x}) \]

\[ + \overline{a}_y (F_x \frac{\partial G_z}{\partial y} - F_z \frac{\partial G_x}{\partial y} - F_y \frac{\partial G_z}{\partial z} + F_z \frac{\partial G_y}{\partial z}) \]

\[ + \overline{a}_z (F_y \frac{\partial G_x}{\partial z} - F_x \frac{\partial G_y}{\partial z} - F_z \frac{\partial G_x}{\partial x} + F_x \frac{\partial G_z}{\partial z}) \] .

Although the operators \((\overline{F} \cdot \nabla)\) and \((\overline{F} \times \nabla)\) were expanded in rectangular coordinates in the above illustrations, any orthogonal coordinate system can be used.

Using operators of the types \((\overline{F} \cdot \nabla)\) and \((\overline{F} \times \nabla)\), a number of useful identities can be written; some of these are listed below and in App. 6.

\[ \nabla(\overline{G} \cdot \overline{F}) = (\overline{F} \times \nabla) \overline{G} + (\overline{G} \times \nabla) \overline{F} + (\nabla \cdot \overline{F}) \overline{G} + (\nabla \cdot \overline{G}) \overline{F} \]

\[ \nabla(\overline{G} \cdot \overline{F}) = (\overline{G} \cdot \nabla) \overline{F} + (\overline{F} \cdot \nabla) \overline{G} - (\nabla \times \overline{F}) \overline{G} - (\nabla \times \overline{G}) \overline{F} \]

\[ \nabla \times (\overline{G} \times \overline{F}) = (\nabla \times \overline{G}) \overline{F} - (\nabla \times \overline{F}) \overline{G} + (\overline{F} \times \nabla) \overline{G} - (\overline{G} \times \nabla) \overline{F} \]

\[ \nabla \times (\overline{G} \times \overline{F}) = (\nabla \cdot \overline{F}) \overline{G} + (\overline{F} \cdot \nabla) \overline{G} - (\nabla \cdot \overline{G}) \overline{F} - (\overline{G} \cdot \nabla) \overline{F} \]
\[(\vec{G} \times \nabla) \times \vec{F} = \frac{1}{2} [\nabla (\vec{F} \cdot \vec{G}) - \nabla \times (\vec{G} \times \vec{F}) - (\nabla \cdot \vec{F}) \vec{G} + (\nabla \times \vec{G}) \times \vec{F} - (\nabla \times \vec{F}) \times \vec{G}]\]

\[(\vec{F} \cdot \nabla) \vec{G} = (\nabla \cdot \vec{G}) \vec{F} + (\nabla \times \vec{G}) \times \vec{F} + (\vec{F} \times \nabla) \times \vec{G}\]

\[(\vec{F} \cdot \nabla) \vec{G} = \frac{1}{2} [\nabla (\vec{F} \cdot \vec{G}) - \nabla \times (\vec{G} \times \vec{F}) + \nabla (\vec{F} \times \vec{G}) + (\nabla \times \vec{G}) \times \vec{F} + (\nabla \times \vec{F}) \times \vec{G}]\]

\[(\vec{F} \times \nabla) \times \vec{G} + (\vec{G} \times \nabla) \times \vec{F} = \nabla (\vec{F} \cdot \vec{G}) - (\nabla \cdot \vec{F}) \vec{G} - (\nabla \cdot \vec{G}) \vec{F}\]

\[(\vec{F} \times \nabla) \times \vec{G} - (\vec{G} \times \nabla) \times \vec{F} = \nabla \times (\vec{G} \times \vec{F}) + (\nabla \times \vec{F}) \times \vec{G} - (\nabla \times \vec{G}) \times \vec{F}\]

\[(\nabla \times \vec{F}) \times \vec{G} + (\nabla \times \vec{G}) \times \vec{F} = (\vec{G} \cdot \nabla) \vec{F} + (\vec{F} \cdot \nabla) \vec{G} - \nabla (\vec{G} \times \vec{F})\]

\[(\nabla \times \vec{F}) \times \vec{G} - (\nabla \times \vec{G}) \times \vec{F} = (\vec{F} \times \nabla) \times \vec{G} - (\vec{G} \times \nabla) \times \vec{F} - \nabla \times (\vec{G} \times \vec{F})\]

\[(\vec{F} \cdot \nabla) \vec{G} + (\vec{G} \cdot \nabla) \vec{F} = \nabla (\vec{F} \cdot \vec{G}) + (\nabla \times \vec{F}) \times \vec{G} + (\nabla \times \vec{G}) \times \vec{F}\]

\[(\vec{F} \cdot \nabla) \vec{G} - (\vec{G} \cdot \nabla) \vec{F} = \nabla \times (\vec{G} \times \vec{F}) + (\nabla \times \vec{F}) \vec{G} - (\nabla \cdot \vec{F}) \vec{G}\]

\[(\nabla \cdot \vec{F}) \vec{G} + (\nabla \cdot \vec{G}) \vec{F} = \nabla (\vec{G} \cdot \vec{F}) - (\nabla \times \vec{G}) \times \vec{F} - (\nabla \times \vec{F}) \times \vec{G}\]

\[(\nabla \cdot \vec{F}) \vec{G} - (\nabla \cdot \vec{G}) \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \nabla) \vec{G} + (\vec{G} \cdot \nabla) \vec{F}\]

These identities can be verified by expansion of both sides in any orthogonal coordinate system.
EXAMPLES

Let
\[ \vec{F} = a_r \ r \]
\[ \vec{G} = a_r r \theta + a_\phi \theta \]

Then
\[ \vec{F} \cdot \vec{G} = r^2 \theta \]
\[ \vec{G} \times \vec{F} = a_\theta r \theta \]

and
\[ \nabla(\vec{F} \cdot \vec{G}) = a_r 2r \theta + a_\theta r \]

and
\[ \nabla \times (\vec{G} \times \vec{F}) = a_\phi 2 \theta . \]

To illustrate the above identities, the following expressions are needed:

\[ \nabla \cdot \vec{F} = 3 \]
\[ \nabla \cdot \vec{G} = 3 \theta \]
\[ (\nabla \cdot \vec{F}) \vec{G} = a_r 3r \theta + a_\phi 3 \theta \]
\[ (\nabla \cdot \vec{G}) \vec{F} = a_r 3r \theta \]
\[ \nabla \times \vec{G} = a_r (r \sin \theta + \theta r \cos \phi) - \frac{a_\theta}{r} \theta - a_\phi \]
\[ \nabla \times \vec{F} = 0 \]
\[ (\nabla \times \vec{G}) \times \vec{F} = -a_\theta r + a_\phi \theta \]
\[ (\nabla \times \vec{F}) \times \vec{G} = 0 \]
The scalar operators $\mathbf{G} \cdot \nabla$ and $\mathbf{F} \cdot \nabla$ are also needed; these are

$$
\mathbf{G} \cdot \nabla = (\bar{a}_r \frac{\partial}{\partial r} + \bar{a}_\theta \frac{\partial}{\partial \theta} + \bar{a}_\phi \frac{\partial}{\partial \phi}) \cdot (\bar{a}_r \frac{\partial}{\partial r} + \bar{a}_\theta \frac{\partial}{\partial \theta} + \bar{a}_\phi \frac{\partial}{\partial \phi})
$$

$$
= r \frac{\partial}{\partial r} + \frac{\theta}{r \sin \theta} \frac{\partial}{\partial \phi}
$$

and

$$
\mathbf{F} \cdot \nabla = \bar{a}_r r \cdot (\bar{a}_r \frac{\partial}{\partial r} + \bar{a}_\theta \frac{\partial}{\partial \theta} + \bar{a}_\phi \frac{\partial}{\partial \phi})
$$

$$
= r \frac{\partial}{\partial r} .
$$

From these operators

$$
(\mathbf{G} \cdot \nabla)\mathbf{F} = \left( r \frac{\partial}{\partial r} + \frac{\theta}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \bar{a}_r \mathbf{r}
$$

$$
= r \bar{a}_r + r^2 \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right) + \frac{\theta}{r \sin \theta} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi}
$$

$$
= r \bar{a}_r + 0 + \frac{\theta}{\sin \theta} \frac{\partial}{\partial \phi}
$$

$$
= \bar{a}_r r \theta + \bar{a}_\theta
$$

and

$$
(\mathbf{F} \cdot \nabla)\mathbf{G} = r \frac{\partial}{\partial r} \left( \bar{a}_r r \theta + \bar{a}_\phi \right)
$$

$$
= r \bar{a}_r r \theta + r^2 \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right) + r \theta \frac{\partial}{\partial r} \frac{\partial}{\partial \phi}
$$

$$
= \bar{a}_r r \theta + 0 + r \theta (0)
$$

$$
= \bar{a}_r r \theta .
$$

The vector operators $\mathbf{G} \times \nabla$ and $\mathbf{F} \times \nabla$ are also needed; they are
\[ \vec{G} \times \vec{V} = \begin{vmatrix} \vec{a}_r & \vec{a}_\theta & \vec{a}_\phi \\ r \partial r & \partial r & r \partial \phi \\ \partial r & \frac{1}{r} \partial \theta & \frac{1}{r \sin \theta} \partial \phi \end{vmatrix} = -\vec{a}_r \frac{\partial}{\partial r} + \vec{a}_\theta \left( \frac{\partial}{\partial r} - \frac{\partial}{\sin \theta} \frac{\partial}{\partial \phi} \right) + \vec{a}_\phi \frac{\partial}{\partial \theta} \]

and

\[ \vec{F} \times \vec{V} = \begin{vmatrix} \vec{a}_r & \vec{a}_\theta & \vec{a}_\phi \\ r \partial r & \partial r & r \partial \phi \\ \frac{1}{r} \partial \theta & \frac{1}{r \sin \theta} \partial \phi \end{vmatrix} = -\vec{a}_\theta \frac{\partial}{\sin \theta} + \vec{a}_\phi \frac{\partial}{\partial \theta} \]

From these vector operators it follows that

\[
(\vec{G} \times \vec{V}) \times \vec{F} = -\vec{a}_r \frac{\partial}{\partial r} \times (\vec{a}_r \vec{r}) + \vec{a}_\theta \left( \frac{\partial}{\partial r} - \frac{\partial}{\sin \theta} \frac{\partial}{\partial \phi} \right) \times (\vec{a}_r \vec{r}) + \vec{a}_\phi \frac{\partial}{\partial \theta} \times (\vec{a}_r \vec{r})
\]

\[
= -\theta \vec{a}_r \times \frac{3 \vec{a}_r}{\partial \theta} + \theta \vec{a}_\theta \times \vec{a}_r - \frac{\partial r}{\sin \theta} \vec{a}_\phi \times \frac{\partial \vec{a}_r}{\partial \phi} + \theta \vec{a}_\phi \times \vec{a}_r
\]

\[
= -\vec{a}_r \vec{a}_r - \vec{a}_\phi - \vec{a}_r \vec{r} \theta - \vec{a}_r \theta \vec{r} = -\vec{a}_r 2r \theta - \vec{a}_\phi 2 \theta
\]

and

\[
(\vec{F} \times \vec{V}) \times \vec{G} = -\vec{a}_\theta \frac{\partial}{\sin \theta} \times (\vec{a}_r \vec{r} \theta + \vec{a}_\phi \vec{r} \phi) + \vec{a}_\phi \frac{\partial}{\partial \theta} \times (\vec{a}_r \vec{r} \theta + \vec{a}_\phi \vec{r} \phi)
\]

\[
= -\frac{\partial \vec{r}}{\sin \theta} \vec{a}_\theta \times \frac{\partial \vec{a}_r}{\partial \theta} - \frac{\partial \vec{a}_\phi}{\sin \theta} \vec{a}_\theta \times \frac{\partial \vec{a}_r}{\partial \phi} + \vec{r} \vec{a}_\phi \times \vec{a}_r + \vec{r} \vec{a}_\phi \times \frac{\partial \vec{a}_r}{\partial \phi} + \vec{a}_\phi \times \frac{\partial \vec{a}_r}{\partial \phi} + \theta \vec{a}_\phi \times \frac{\partial \vec{a}_r}{\partial \theta}
\]
\[= - \frac{r\theta}{\sin \theta} \bar{a}_\theta \times \bar{a}_\phi \sin \theta - \frac{\theta}{\sin \theta} \bar{a}_\theta \times (\bar{a}_r \sin \theta - \bar{a}_\theta \cos \theta) + r\bar{a}_\theta + r\theta \bar{a}_\phi \times \bar{a}_\theta + 0 + 0\]

\[= -r\theta \bar{a}_r - \theta \bar{a}_\phi + r\bar{a}_\theta - r\theta \bar{a}_r\]

\[= -\bar{a}_r 2r\theta + \bar{a}_\theta r - \bar{a}_\phi \theta . \]

Using these results, each of the above identities is easily shown to be valid for this example.
VECTOR INTEGRATION

Vector integration in three dimensional space can be categorized as line integration, surface integration or volume integration. A line integral is a one-dimensional integral along a line in three dimensional space; for example

\[ G = \oint \mathbf{F} \cdot d\mathbf{x} \]

where \( \mathbf{F} \) is a vector field and \( d\mathbf{x} \) is the vector differential of length along the line (or contour) \( C \). \( d\mathbf{x} \) can be expressed as \( d\mathbf{x} = \mathbf{a}_t \, dl \) where \( dl \) is the scalar differential of length and \( \mathbf{a}_t \) is a unit vector tangent to contour \( C \) in the direction of integration, at the point of integration. The contour, \( C \), and its end points, must be precisely defined to make the line integral meaningful; it is usually defined as the intersection of two surfaces, each, in turn, defined by an equation. If contour \( C \) is closed, the modified integral sign \( \oint \) will be used. In the above example, the line integral involved the scalar product of a vector field and a vector differential of length. Other combinations are possible, some involving vector products and some involving ordinary products; for example,

\[ H = \oint \mathbf{F} \times d\mathbf{x} , \]

\[ J = \oint \mathbf{F} \, dl , \]

\[ K = \oint f \, d\mathbf{x} \]
and

\[ L = \int_{\mathcal{C}} f \, d\mathbf{\ell} \]

A surface integral is a two-dimensional integral over a surface defined in the three-dimensional space, for example,

\[ M = \int_{\mathcal{S}} \mathbf{F} \cdot \overline{d\mathbf{S}} \]

where \( \mathbf{F} \) is a vector field and \( \overline{d\mathbf{S}} \) is the vector differential of area on surface \( \mathcal{S} \). In this presentation a single integral sign, with subscript \( \mathcal{S} \), denotes a surface integral despite the fact that such an integral is two-dimensional. If surface \( \mathcal{S} \) is closed, the modified integral sign \( \int_{\mathcal{S}} \) will be used. This notation is common although the use of a double integral sign is also widespread. The vector differential of area, \( \overline{d\mathbf{S}} \), can be expressed as \( \overline{d\mathbf{S}} = \mathbf{a}_n \, dS \) where \( dS \) is the scalar differential of area and \( \mathbf{a}_n \) is the unit vector normal to the surface, on the positive side of the surface, at the point of integration. The surface \( \mathcal{S} \) must be precisely defined to make the integral meaningful. As in the case of the line integral, other sorts of integrands are possible; for example

\[ \overline{\mathbf{N}} = \int_{\mathcal{S}} \mathbf{F} \times \overline{d\mathbf{S}} \]

\[ \overline{P} = \int_{\mathcal{S}} \mathbf{F} \, d\mathbf{S} \]

\[ \overline{Q} = \int_{\mathcal{S}} f \, \overline{d\mathbf{S}} \]

and

\[ \overline{R} = \int_{\mathcal{S}} f \, d\mathbf{S} \]
A volume integral is a three-dimensional integral in three-dimensional space; for example

\[ S = \int_V f \, dv \]

or

\[ \bar{T} = \int_V \bar{F} \, dv \]

where \( dv \) is the scalar differential of volume and \( V \) is the volume throughout which integration occurs. In this discussion a single integral sign, with subscript \( V \), denotes a volume integral despite the fact that such an integral is three-dimensional. This notation is common although the use of a triple integral sign also is common.
MEAN VALUE THEOREMS FOR VECTOR INTEGRALS

For functions of a single variable there is a theorem relating to evaluation of an integral in terms of the product of the length of the interval and the value of the function at some point in the interval. In symbols

$$\int_{a}^{b} f(x)dx = f(\xi)(b-a)$$

for $a \leq \xi \leq b$ if the function is continuous in the interval $a \leq x \leq b$.

Similar theorems for vector integrals can be found using the same procedures. Some such theorems are mentioned below. Suppose $\overline{F}$ is a real function.

$$\int_{C} \overline{F} \cdot d\overline{x} = \overline{F}(p_{c}) \cdot \overline{a}_{tc} \ell$$

where $p_{c}$ is a point on contour $C$, $\overline{a}_{tc}$ is the unit vector tangent to $C$ at that point and $\ell$ is the length of the contour. Similarly

$$\int_{S} \overline{F} \cdot d\overline{s} = \overline{F}(p_{s}) \cdot \overline{a}_{ns} S$$

where $p_{s}$ is a point on surface $S$, $\overline{a}_{ns}$ is the unit vector normal to $S$ at that point and $S$ is the area of the surface.

If $\overline{F}$ is complex then
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F}_r(p_{cl}) \cdot \mathbf{a}_{tc} \cdot r + i \mathbf{F}_i(p_{c2}) \cdot \mathbf{a}_{tc} \cdot r \]

where \( \mathbf{F} = \mathbf{F}_r + i \mathbf{F}_i \) and where points \( p_{cl} \) and \( p_{c2} \) are both on \( C \) but need not be the same point. Similarly

\[ \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F}_r(p_{s1}) \cdot \mathbf{a}_{ns1} \cdot S + 2 \mathbf{F}_i(p_{s2}) \cdot \mathbf{a}_{ns2} \cdot S \]

where points \( p_{s1} \) and \( p_{s2} \) are both on surface \( S \) but need not be the same point.

The case in which \( \mathbf{F} \) is real but the integral is a vector results in somewhat more involved expressions; for example

\[ \int_C \mathbf{F} \times d\mathbf{r} = \int_C \mathbf{a}_x \cdot (\mathbf{F}(p_{cx}) \times \mathbf{a}_{tcx}) \]

\[ + \mathbf{a}_y \cdot (\mathbf{F}(p_{cy}) \times \mathbf{a}_{tcy}) \]

\[ + \mathbf{a}_z \cdot (\mathbf{F}(p_{cz}) \times \mathbf{a}_{tcz}) \]

where \( p_c \) and a unit tangent vector, \( \mathbf{a}_{tc} \), for each of the coordinate directions. Rectangular coordinates are assumed here but the relationship is valid for any orthogonal coordinate system. Similarly

\[ \int_S \mathbf{F} \times d\mathbf{S} = \int_S \mathbf{a}_x \cdot (\mathbf{F}(p_{sx}) \times \mathbf{a}_{nsx}) \]

\[ + \mathbf{a}_y \cdot (\mathbf{F}(p_{sy}) \times \mathbf{a}_{nsy}) \]

\[ + \mathbf{a}_z \cdot (\mathbf{F}(p_{sz}) \times \mathbf{a}_{nsz}) \]
where there is a point $p_s$ and a unit normal vector, $\bar{a}_{ns}$, for each of the coordinate directions. As before if $\bar{F}$ is complex, each of the terms on the right of these last two results becomes two terms for the real and imaginary parts. This is a straightforward extension and will not be discussed further.

Similar sorts of mean value theorems can be found for other types of vector integrals.
INTEGRAL IDENTITIES

The use of vector calculus in the formulation of physical theory is greatly facilitated by use of two well-known theorems, Stokes theorem and the divergence theorem. Stokes theorem is

$$\oint_C \mathbf{F} \cdot d\mathbf{x} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where $S$ is any surface bounded by contour $C$. The most significant feature of this identity is the fact that it relates a closed line integral to a surface integral. There are two other relationships of this type which are sometimes useful but which are not easily found in the literature; they are

$$\oint_C \mathbf{F} \times d\mathbf{x} = -\int_S (d\mathbf{S} \times \nabla) \times \mathbf{F}$$

$$= \int_S (\nabla \cdot \mathbf{F}) dS + \int_S (\nabla \times \mathbf{F}) \times d\mathbf{S} - \int_S (d\mathbf{S} \cdot \nabla) \mathbf{F}$$

and

$$\oint_C F \, d\mathbf{x} = -\int_S \nabla F \times d\mathbf{S}.$$  

A number of other similar relations involving more complex integrands are given in the list of formulas in the Appendix.

The divergence theorem is well known and is given by

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \, dv$$
where \( V \) is the volume bounded by surface \( S \). The most significant feature of this identity is the fact that it relates a closed surface integral to a volume integral. There are two other relationships of this type which are sometimes useful but not easily found in the references; they are

\[
\oint_S \mathbf{F} \cdot d\mathbf{S} = -\int_V (\nabla \times \mathbf{F}) \, dv
\]

and

\[
\oint_S F \, dS = \int_V (\nabla F) \, dv
\]
DIFFERENTIATION OF LINE AND SURFACE INTEGRALS

The following theorems as well as related results are discussed in detail in #60 of this series of reports.

**Theorem 1**

Let \( Q \) be the line integral

\[
Q = \int_{P_1}^{P_2} \overline{\vec{F}} \cdot \overline{d\vec{x}}
\]

along contour, \( C \), from point \( P_1 \) to point \( P_2 \). Assume that the vector function, \( \overline{\vec{F}} \), is a function of time, \( t \). Also assume the contour, \( C \), is in motion with respect to the frame of reference with respect to which \( \overline{\vec{F}} \) is defined. Further assume the end points \( P_1 \) and \( P_2 \) are in motion. Let the motion, with respect to the frame of reference, be defined by velocity, \( \overline{\vec{v}} \), which is a function of position along contour \( C \). The derivative of \( Q \) with respect to \( t \) can be expressed as

\[
\frac{dQ}{dt} = \overline{\vec{F}}(P_2) \cdot \overline{\vec{v}}_2 - \overline{\vec{F}}(P_1) \cdot \overline{\vec{v}}_1 + \int_{P_1}^{P_2} (\nabla \times \overline{\vec{F}}) \times \overline{\vec{v}} \cdot \overline{d\vec{x}} + \int_{P_1}^{P_2} \frac{\partial \overline{\vec{F}}}{\partial t} \cdot \overline{d\vec{x}}
\]

where \( \overline{\vec{v}}_2 \) is the velocity \( \overline{\vec{v}} \) at end point 2 and \( \overline{\vec{v}}_1 \) is the velocity at end point 1.

It is apparent that, if contour \( C \) is closed, the first two terms cancel and the result is expressed in terms of closed line integrals only.
Theorem 2

Let \( Q \) be the surface integral

\[
Q = \int_S \mathbf{F} \cdot d\mathbf{S}
\]

over the surface \( S \) which is bounded by the contour \( C \). Assume that the vector function, \( \mathbf{F} \), is a function of time, \( t \). Also assume the surface \( S \), is in motion with respect to the frame of reference with respect to which \( \mathbf{F} \) is defined. Further assume the contour, \( C \), bounding surface \( S \) is in motion. Let the motion, with respect to the frame of reference, be defined by velocity, \( \mathbf{v} \), which is a function of position on surface, \( S \). The derivative of \( Q \) with respect to \( t \) can be expressed as

\[
\frac{dQ}{dt} = \int_C \mathbf{F} \times \mathbf{v} \cdot d\mathbf{z} + \int_S \frac{\partial \mathbf{F}}{\partial t} \cdot d\mathbf{S} + \int_S (\nabla \cdot \mathbf{F}) \mathbf{v} \cdot d\mathbf{S}.
\]

It is apparent that, if surface \( S \) is closed, the first term on the right hand side, that is the line integral along closed contour \( C \), vanishes.
APPENDIX 1

INVERSE OF THE GRADIENT OPERATOR

The gradient \( \vec{G} \) of a scalar field, \( F \), is given symbolically by

\[ \vec{G} = \nabla F, \]

and the operation needed to determine \( \vec{G} \) from \( F \) is well known. The inverse operation, the determination of \( F \) from \( \vec{G} \) is also well known; it is found from the line integral

\[ F(P_2) - F(P_1) = \int_{P_1}^{P_2} \vec{G} \cdot d\ell. \]

The field \( F \) at point \( P_2 \) is given by

\[ F(P_2) = F(P_1) + \int_{P_1}^{P_2} \vec{G} \cdot d\ell. \]

This expresses the field at point \( P_2 \) in terms of the field at a reference point \( P_1 \) and a line integral along a contour \( C \) from \( P_1 \) to \( P_2 \). The choice of the contour between points \( P_1 \) and \( P_2 \) is arbitrary, because, as is shown in elementary texts, the line integral in independent of the path of integration. If point \( P_1 \) is considered fixed and \( P_2 \) is variable then the above integral relation gives the field \( F \) as a function of position, \( P_2 \). It is in the form of a constant, \( F(P_1) \), plus a line integral. The line integral thus determines \( F \) to within a constant. This is to be expected since any constants present in \( F \) vanish in the determination of \( \vec{G} \) from \( \vec{G} = \nabla F \). Such constants therefore cannot be recovered from \( \vec{G} \).
APPENDIX 2

INVERSE OF THE DIVERGENCE

AND CURL OPERATIONS

Consider the relation \( \overrightarrow{B} = \nabla \times \overrightarrow{A} \). If \( \overrightarrow{A} \) is known, \( \overrightarrow{B} \) can be found by straightforward procedures. The inverse problem of finding \( \overrightarrow{A} \) from \( \overrightarrow{B} \) is somewhat more involved. Since \( \nabla \) is a differential operator, it may be expected that \( \overrightarrow{A} \) can be expressed in terms of an integral operator operating on \( \overrightarrow{B} \). This, in fact, is so as will be shown. Consider rectangular coordinates; the relation \( \overrightarrow{B} = \nabla \times \overrightarrow{A} \) can be written in terms of its components.

\[
B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}
\]

\[
B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}
\]

\[
B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}
\]

At first thought it would seem these three equations could be solved for the three unknowns \( A_x \), \( A_y \) and \( A_z \). This, however, is not so because of a dependency relationship among the three equations. To demonstrate this, differentiate the first equation with respect to \( x \), the second with respect to \( y \) and the third with respect to \( z \). The results are

\[
\frac{\partial B_x}{\partial x} = \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_y}{\partial x \partial y}
\]
\[
\frac{\partial B_y}{\partial y} = \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x}
\]

\[
\frac{\partial B_z}{\partial z} = \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}
\]

The sum of the first two of these equations is

\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_y}{\partial x \partial z}
\]

Comparing this with the third relation it is seen that the right hand sides are the same (except for a factor of -1); thus the third equation is the negative of the sum of the first two demonstrating that only two of the three original equations are independent. It may be further observed that the left hand sides of the equations must satisfy

\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -\frac{\partial B_y}{\partial z}
\]

or

\[
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0
\]

This result could have been obtained directly from the well-known identity 
\( \nabla \cdot \nabla \times \mathbf{A} = 0 \) thereby demonstrating the dependence in a somewhat more straightforward manner. It is apparent that the equation needed, to complete the set of three independent equations, must involve \( \frac{\partial A_x}{\partial x} \), \( \frac{\partial A_y}{\partial y} \) and \( \frac{\partial A_z}{\partial z} \), which do not appear in any of the three original equations.

This requirement can be satisfied by specifying the divergence \( D = \nabla \cdot \mathbf{A} \), of \( \mathbf{A} \). Thus let the additional equation be
\[ D = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]

The problem now is the solution of the system of equations

\[ B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \]
\[ B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \]
\[ B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \]
\[ D = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \]

Only two of the first three equations are needed but for the technique used it is convenient to have all three.

Differentiate the first equation with respect to \( z \), the third with respect to \( x \) and the fourth with respect to \( y \). The results are

\[ \frac{\partial B_x}{\partial z} = \frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial z^2} \]
\[ \frac{\partial B_z}{\partial x} = \frac{\partial^2 A_y}{\partial x \partial y} - \frac{\partial^2 A_x}{\partial x \partial y} \]
\[ \frac{\partial D}{\partial y} = \frac{\partial^2 A_x}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z} \]

The first two of these equations may be solved for \( \frac{\partial^2 A_z}{\partial z \partial y} \) and \( \frac{\partial^2 A_x}{\partial x \partial y} \); these results are substituted into the last equation giving

\[ \frac{\partial D}{\partial y} = \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial B_x}{\partial x} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial B_x}{\partial z} + \frac{\partial^2 A_y}{\partial z^2} \]

or
\[ \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} = \frac{\partial D}{\partial y} + \frac{\partial B_z}{\partial x} - \frac{\partial B_z}{\partial z}. \]

In a similar manner, the following equations are obtained:

\[ \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = \frac{\partial D}{\partial x} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}. \]

\[ \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} = \frac{\partial D}{\partial z} + \frac{\partial B_x}{\partial z} - \frac{\partial B_y}{\partial z}. \]

The last three equations can be assembled into a vector equation as follows:

\[ \vec{a}_x \left( \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} \right) + \vec{a}_y \left( \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2} \right) + \vec{a}_z \left( \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \right) \]

\[ = \vec{a}_x \left( \frac{\partial D}{\partial x} + \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} \right) + \vec{a}_y \left( \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial y} \right) + \vec{a}_z \left( \frac{\partial D}{\partial z} + \frac{\partial B_x}{\partial z} - \frac{\partial B_y}{\partial x} \right). \]

In vector notation this may be written

\[ \nabla^2 \vec{A} = \nabla D - \nabla \times \vec{B}. \]

Since \( D = \nabla \cdot \vec{A} \) and \( \vec{B} = \nabla \times \vec{A} \) this could be written \( \nabla^2 \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A}) \) which is a well known relation. Indeed, it could serve as the starting point for this analysis. The solution of the above equation is well known; it may be written

\[ \vec{A} = \int_V \frac{\nabla \times \vec{B} - \nabla D}{4\pi R} \, dv \]

where \( R \) is the distance from the integration point to the observation point. Alternatively using \( D = \nabla \cdot \vec{A} \) and \( \vec{B} = \nabla \times \vec{A} \), \( \vec{A} \) is given by
\[ \bar{A} = \int_\Omega \frac{\nabla \times (\nabla \times \bar{A}) - \nabla (\nabla \cdot \bar{A})}{4\pi R} \, dV \]

It is apparent that if \( \bar{A} \) contains constant or linear terms that are lost in the evaluation of \( \nabla \times (\nabla \times \bar{A}) - \nabla (\nabla \cdot \bar{A}) \), such terms cannot be recovered by the volume integration operation given above; this however is the only limitation in the use of the above integral for \( \bar{A} \).
APPENDIX 3

EXPRESSION OF A VECTOR IN TERMS OF ITS VECTOR AND SCALAR PRODUCTS WITH A KNOWN VECTOR

Suppose \( \vec{A} \) is an unknown vector but its vector and scalar products with a known vector, \( \vec{B} \), are known. More specifically

\[
\vec{A} \times \vec{B} = \vec{C}
\]

and

\[
\vec{A} \cdot \vec{B} = D
\]

where \( \vec{B}, \vec{C} \) and \( D \) are known.

The first relation \( \vec{A} \times \vec{B} = \vec{C} \) determines only the component of \( \vec{A} \) that is normal to \( \vec{B} \). The second relation \( \vec{A} \cdot \vec{B} = D \) determines only the component of \( \vec{A} \) that is collinear with \( \vec{B} \). Thus neither of the above relations can completely determine \( \vec{A} \) but the two of them together do completely determine \( \vec{A} \).

Let \( \vec{A} = \vec{A}_n + \vec{A}_t \) where \( \vec{A}_n \) is normal to \( \vec{B} \) and \( \vec{A}_t \) is collinear with \( \vec{B} \). Then

\[
\vec{A} \times \vec{B} = (\vec{A}_n + \vec{A}_t) \times \vec{B}
\]

\[
= \vec{A}_n \times \vec{B} + \vec{A}_t \times \vec{B}
\]

Since \( \vec{A}_t \times \vec{B} = 0 \)

\[
\vec{A} \times \vec{B} = \vec{A}_n \times \vec{B} = \vec{C}
\]
\( \overrightarrow{A}_n \), \( \overrightarrow{B} \) and \( \overrightarrow{C} \) are mutually perpendicular so \( \overrightarrow{A}_n \) is in the direction of \( \overrightarrow{B} \times \overrightarrow{C} \) and can be expressed as

\[
\overrightarrow{A}_n = \alpha \overrightarrow{B} \times \overrightarrow{C}
\]

where \( \alpha \) is a function to be determined.

The component \( \overrightarrow{A}_t \) is colinear with \( \overrightarrow{B} \) and can be expressed

\[
\overrightarrow{A}_t = \beta \overrightarrow{B}
\]

where \( \beta \) is a function to be determined. Thus \( \overrightarrow{A} \) can be expressed as

\[
\overrightarrow{A} = \alpha (\overrightarrow{B} \times \overrightarrow{C}) + \beta \overrightarrow{B}
\]

The functions \( \alpha \) and \( \beta \) can be determined by application of the conditions

\[
\overrightarrow{A} \times \overrightarrow{B} = \overrightarrow{C}
\]

and

\[
\overrightarrow{A} \cdot \overrightarrow{B} = D.
\]

The first condition gives

\[
\overrightarrow{A} \times \overrightarrow{B} = [\alpha (\overrightarrow{B} \times \overrightarrow{C}) + \beta \overrightarrow{B}] \times \overrightarrow{B}
\]

\[
= \alpha (\overrightarrow{B} \times \overrightarrow{C}) \times \overrightarrow{B} + \overrightarrow{B} \times \overrightarrow{B}
\]

\[
= \alpha (\overrightarrow{B} \times \overrightarrow{C}) \times \overrightarrow{B} + 0 = \overrightarrow{C};
\]

but

\[
(\overrightarrow{B} \times \overrightarrow{C}) \times \overrightarrow{B} = \overrightarrow{C}(\overrightarrow{B} \cdot \overrightarrow{B}) - \overrightarrow{B}(\overrightarrow{B} \cdot \overrightarrow{C});
\]

and \( \overrightarrow{B} \) and \( \overrightarrow{C} \) are mutually perpendicular; so

\[
(\overrightarrow{B} \times \overrightarrow{C}) \times \overrightarrow{B} = \overrightarrow{C}(\overrightarrow{B} \cdot \overrightarrow{B}).
\]
Substitution into the above relation gives

\[ \alpha \overline{C} (\overline{B} \cdot \overline{B}) = \overline{C} \]

and from this

\[ \alpha = \frac{1}{\overline{B} \cdot \overline{B}} . \]

The second condition \( \overline{A} \cdot \overline{B} = D \) gives

\[ \overline{A} \cdot \overline{B} = (\alpha(\overline{B} \times \overline{C}) + \beta \overline{B}) \cdot \overline{B} \]

\[ = \alpha[(\overline{B} \times \overline{C}) \cdot \overline{B}] + \beta \overline{B} \cdot \overline{B} = D \]

Since \( (\overline{B} \times \overline{C}) \cdot \overline{B} = 0 \)

\[ \beta = \frac{D}{\overline{B} \cdot \overline{B}} . \]

Collecting results it is seen that

\[ \overline{A} = \frac{\overline{B} \times \overline{C} + \overline{B} \overline{B} \beta}{\overline{B} \cdot \overline{B}} \]

or since \( \overline{C} = \overline{A} \times \overline{B} \) and \( D = \overline{A} \cdot \overline{B} \)

\[ \overline{A} = \frac{\overline{B} \times (\overline{A} \times \overline{B}) + \overline{B} (\overline{A} \cdot \overline{B})}{\overline{B} \cdot \overline{B}} \]

This is the desired result, an expression for a vector in terms of its vector and scalar products with a known vector.
APPENDIX 4

EXPRESSION OF A VECTOR IN TERMS OF ITS VECTOR PRODUCTS WITH A SET OF MUTUALLY ORTHOGONAL UNIT VECTORS

Let \( \vec{a}_1, \vec{a}_2 \) and \( \vec{a}_3 \) be mutually orthogonal unit vectors and let

\[
\vec{A} \times \vec{a}_1 = \vec{B}_1
\]

\[
\vec{A} \times \vec{a}_2 = \vec{B}_2
\]

and

\[
\vec{A} \times \vec{a}_3 = \vec{B}_3.
\]

The vector \( \vec{A} \) can be expressed in terms of \( \vec{B}_1, \vec{B}_2, \vec{B}_3, \vec{a}_1, \vec{a}_2 \) and \( \vec{a}_3 \) as

\[
\vec{A} = \frac{1}{2} [\vec{a}_1 \times \vec{B}_1 + \vec{a}_2 \times \vec{B}_2 + \vec{a}_3 \times \vec{B}_3].
\]

This identity is easily proven by expressing \( \vec{A} \) in terms of its components in the directions of \( \vec{a}_1, \vec{a}_2 \) and \( \vec{a}_3 \). Thus

\[
\vec{A} = \vec{a}_1 A_1 + \vec{a}_2 A_2 + \vec{a}_3 A_3
\]

where

\[
A_1 = \vec{a}_1 \cdot \vec{A}
\]
\[
A_2 = \vec{a}_2 \cdot \vec{A}
\]

and

\[
A_3 = \vec{a}_3 \cdot \vec{A}.
\]

Then

\[
\vec{B}_1 = \vec{A} \times \vec{a}_1 = (\vec{a}_1 A_1 + \vec{a}_2 A_2 + \vec{a}_3 A_3) \times \vec{a}_1.
\]
Suppose \( \vec{a}_3 = \vec{a}_1 \times \vec{a}_2, \quad \vec{a}_1 = \vec{a}_2 \times \vec{a}_3, \quad \vec{a}_2 = \vec{a}_3 \times \vec{a}_1 \). 

Since \( \vec{a}_1, \vec{a}_2 \) and \( \vec{a}_3 \) are mutually orthogonal either these relationships are true or, alternatively, the relationships, \( \vec{a}_3 = -\vec{a}_1 \times \vec{a}_2, \quad \vec{a}_1 = -\vec{a}_2 \times \vec{a}_3, \quad \vec{a}_2 = -\vec{a}_3 \times \vec{a}_1 \), are true. Either choice leads to the proof of the expression for \( \vec{A} \) in terms of \( \vec{B}_1, \vec{B}_2, \vec{B}_3, \vec{a}_1, \vec{a}_2 \) and \( \vec{a}_3 \). \( \vec{B}_1 \) can thus be expressed

\[
\vec{B}_1 = (\vec{a}_2 \times \vec{a}_1)A_2 + (\vec{a}_3 \times \vec{a}_1)A_3 \\
= -\vec{a}_3A_2 + \vec{a}_2A_3.
\]

Similarly

\[
\vec{B}_2 = \vec{A} \times \vec{a}_2 \\
= \vec{a}_3A_1 - \vec{a}_1A_3
\]

and

\[
\vec{B}_3 = \vec{A} \times \vec{a}_3 \\
= -\vec{a}_2A_1 + \vec{a}_1A_2.
\]

Thus

\[
\vec{A} = \frac{1}{2}[\vec{a}_1 \times \vec{B}_1 + \vec{a}_2 \times \vec{B}_2 + \vec{a}_3 \times \vec{B}_3] \\
= \frac{1}{2}[\vec{a}_1 \times (-\vec{a}_3A_2 + \vec{a}_2A_3) + \vec{a}_2 \times (\vec{a}_3A_1 - \vec{a}_1A_3) + \vec{a}_3 \times (-\vec{a}_2A_1 + \vec{a}_1A_2)] \\
= \frac{1}{2}[\vec{a}_2A_2 + \vec{a}_3A_3 + \vec{a}_1A_1 + \vec{a}_3A_3 + \vec{a}_1A_1 + \vec{a}_2A_2] \\
= \vec{a}_1A_1 + \vec{a}_2A_2 + \vec{a}_3A_3.
\]

This is an identity, verifying the correctness of

\[
\vec{A} = \frac{1}{2}[\vec{a}_1 \times \vec{B}_1 + \vec{a}_2 \times \vec{B}_2 + \vec{a}_3 \times \vec{B}_3].
\]
APPENDIX 5

EXAMPLE OF THE DETERMINATION OF $\bar{D}$
FROM $\nabla \cdot \bar{D}$ AND $\nabla \times \bar{D}$

Define $\bar{D}$ by

$$
\bar{D} = \begin{cases} 
\bar{a} \rho^2 & \rho < a \\
-\frac{\bar{a}}{3} & \rho = a \\
-\bar{a} \frac{3}{3\rho} & \rho > a 
\end{cases}
$$

in cylindrical coordinates.

Physically, this is the electrostatic flux density due to a charge distribution $\rho_v$ given by

$$
\rho_v = \begin{cases} 
\rho & \rho < a \\
0 & \rho = a \\
0 & \rho > a 
\end{cases}
$$

The divergence and curl of $\bar{D}$ are given by

$$
\nabla \cdot \bar{D} = \begin{cases} 
\rho & \rho < a \\
0 & \rho = a \\
0 & \rho > a 
\end{cases}
$$

$$
\nabla \times \bar{D} = 0
$$

The vector field $\bar{D}$ can be found from $\nabla \cdot \bar{D}$ and $\nabla \times \bar{D}$ using the identity

$$
\bar{D} = \int_V \left( \nabla \times (\nabla \times \bar{D}) - \nabla (\nabla \cdot \bar{D}) \right) \frac{dv}{4\pi R}.
$$

It is seen that

$$
\nabla \times (\nabla \times \bar{D}) = 0
$$

and
\[ \nabla(\nabla \cdot \mathbf{D}) = \begin{cases} -\overline{a}_\rho & \rho < a \\ 0 & a < \rho \end{cases} \]

Thus
\[ \mathbf{D} = -\int \frac{-\overline{a}_\rho \, dv}{4\pi R} \]

where \( \overline{a}_\rho \) is the \( \rho \)-directed unit vector at the point of integration.

Due to symmetry, \( \mathbf{D} \) has only a \( \rho \)-directed component, \( D_\rho \), which can be expressed as
\[
4\pi D_\rho = -4 \int_{\rho'=0}^{a} \int_{\phi'=0}^{\pi} \int_{z=0}^{\infty} \frac{\cos \phi' \, d\phi' \, d\rho' \, dz}{\rho^2 + (\rho')^2 + z^2 - 2\rho \rho' \cos \phi} \]

where the factor 4 on the right results from limiting the integration over \( z \) to the range \( 0 < z < \infty \), and from limiting the integration over \( \phi \) to the range \( 0 < \phi < \pi \).

This integral converges if the integration over \( \phi \) precedes the integration over \( z \). This procedure, however, results in expressions that are not easily simplified to a form permitting verification of the result.

To avoid this problem, first consider \( D_{\rho_0} \) which is simply \( D_\rho \) evaluated at \( \rho = 0 \). It is given by
\[
4\pi D_{\rho_0} = -4 \int_{\rho'=0}^{a} \int_{\phi'=0}^{\pi} \int_{z=0}^{\infty} \frac{\cos \phi' \, d\phi' \, d\rho' \, dz}{(\rho')^2 + z^2} \]

In this special case the integration over \( \phi \) can be done first and the result is zero. So \( D_{\rho_0} = 0 \). Next consider
\[ 4\pi(D_\rho - D_{\rho_0}) = -4 \int_{\rho' = 0}^{a} \int_{\phi = 0}^{\pi} \int_{z = 0}^{\infty} \cos \phi \rho' \frac{1}{\sqrt{\rho'^2 + (\rho')^2 + z^2 - 2\rho\rho' \cos \phi}} \left( \frac{1}{\sqrt{(\rho')^2 + z^2}} \right) \, dz \, d\phi \, d\rho'. \]

Since \( D_{\rho_0} = 0 \), \( D_\rho = D_\rho - D_{\rho_0} \) but the integrand in this expression can be integrated over \( z \). The integral needed is

\[ \int_{z = 0}^{\infty} \left[ \frac{1}{\sqrt{x^2 + y^2}} - \frac{1}{\sqrt{x^2 + q^2}} \right] \, dx = \ln \frac{q}{y}. \]

Using this

\[ 4\pi(D_\rho - D_{\rho_0}) = -4 \int_{\rho' = 0}^{a} \int_{\phi = 0}^{\pi} \cos \phi \rho' \ln \frac{\rho'}{\sqrt{\rho'^2 + (\rho')^2 - 2\rho\rho' \cos \phi}} \, d\rho' \, d\phi. \]

Since

\[ \int_{0}^{\pi} \cos \phi \ln \rho' \, d\phi = 0, \]

this can be written

\[ 4\pi(D_\rho - D_{\rho_0}) = 4 \int_{\rho' = 0}^{a} \int_{\phi = 0}^{\pi} \cos \phi \rho' \ln \sqrt{\rho'^2 + (\rho')^2 - 2\rho' \rho \cos \phi} \, d\phi \, d\rho'. \]

The evaluation of this integral proceeds differently for the case \( \rho > a \) than for the case \( \rho < a \). Consider first the case \( \rho > a \). The integral over \( \rho' \) can be written

\[ \frac{1}{2} \int_{\rho' = 0}^{a} \rho' \ln(\rho'^2 + (\rho')^2 - 2\rho' \rho \cos \phi) \, d\rho'. \]
\[
\frac{a}{2} \int_{\rho' = 0}^{a} \rho'^{2} \ln[(\rho' - \rho \cos \phi)^2 + \rho^2(1 - \cos^2 \phi)] d\rho'
\]

After changing the variable of integration to \( \rho'' = \rho' - \rho \cos \phi \) this becomes

\[
\frac{a - \rho \cos \phi}{2} \int_{\rho'' = -\rho \cos \phi}^{a - \rho \cos \phi} (\rho'' + \rho \cos \phi) \ln[(\rho'')^2 + \rho^2 \sin^2 \phi] d\rho''
\]

Application of integrals #1 and #2, Sec. 2.733, p. 219, of the Integral Table by Gradshteyn & Ryzhik results in

\[
\frac{1}{2} \left\{ \frac{1}{2} \left[ (\rho'')^2 + \rho^2 \sin^2 \phi \right] \ln[(\rho'')^2 + \rho^2 \sin^2 \phi] - (\rho'')^2 \right\}^{a - \rho \cos \phi}_{-\rho \cos \phi} + \rho \cos \phi \left[ \rho'' \ln((\rho'')^2 + \rho^2 \sin^2 \phi) - 2\rho'' + 2\rho \sin \phi \tan^{-1} \frac{\rho''}{\rho \sin \phi} \right]^{a - \rho \cos \phi}_{-\rho \cos \phi}
\]

\[
= \frac{1}{2} \left\{ \left( \frac{a^2 + \rho^2}{2} - \rho^2 \cos^2 \phi \right) \ln(a^2 + \rho^2 - 2a \rho \cos \phi) - \frac{a^2}{2} - \frac{\rho^2}{2} \ln \rho^2 - \rho \cos \phi + \rho^2 \cos^2 \phi \ln \rho^2 + 2\rho^2 \cos \phi \sin \phi \left[ \tan^{-1} \frac{a - \rho \cos \phi}{\rho \sin \phi} - \tan^{-1} \frac{-\rho \cos \phi}{\rho \sin \phi} \right] \right\}
\]

Using the addition formula

\[
\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\]

the tangent of the angle in the brackets, in the last term, is seen to be

\[
\frac{a - \rho \cos \phi}{\rho \sin \phi} - \frac{-\rho \cos \phi}{\rho \sin \phi}
\]

\[
\frac{1}{1 + (a - \rho \cos \phi)(-\rho \cos \phi)} (\rho \sin \phi)^2
\]

\[
= \frac{a}{\rho \sin \phi} - \frac{a}{\rho \sin \phi} = \frac{2 \rho^2 \sin^2 \phi - a \rho \cos \phi + \rho^2 \cos^2 \phi}{\rho^2 \sin^2 \phi} = \frac{a \rho \sin \phi}{\rho^2 \sin^2 \phi} = \frac{a \rho \sin \phi}{1 - a \rho \cos \phi}
\]
Using these results, it is seen that

\[
4\pi(D_\rho - D_{\rho_0}) = 4\left(\frac{1}{2}\right)\int_0^\pi \left(\frac{a^2 + \rho^2}{2} - \rho^2 \cos^2 \phi \right) \cos \phi \ln (a^2 + \rho^2 - 2a\rho \cos \phi) d\phi \\
- \frac{1}{2}(a^2 + \rho^2 \ln \rho^2) \int_0^\pi \cos \phi d\phi - a\int_0^\pi \cos^2 \phi d\phi + \rho^2 \ln \rho^2 \int_0^\pi \cos^3 \phi d\phi \\
+ 2\rho^2 \int_0^\pi \cos^2 \phi \sin \phi \tan^{-1} \left(\frac{a}{\rho} \frac{\sin \phi}{1 - \frac{a}{\rho} \cos \phi}\right) d\phi.
\]

The second and fourth integrals in this expression are zero, and the third is

\[
\int_0^\pi \cos^2 \phi d\phi = \frac{\pi}{2}.
\]

The first and last integrals are somewhat more difficult to evaluate.

Consider the first one; it may be written

\[
\int_0^\pi \left(\frac{a^2 + \rho^2}{2} - \rho^2 \cos^2 \phi \right) \cos \phi \ln \rho^2 + \ln \left(1 - 2 \frac{a}{\rho} \cos \phi + \left(\frac{a}{\rho}\right)^2\right) d\phi \\
- \int_0^\pi \frac{a^2 + \rho^2}{2} \cos \phi \ln \rho^2 d\phi - \int_0^\pi \rho^2 \cos^2 \phi \cos \phi \ln \rho^2 d\phi \\
+ \int_0^\pi \frac{a^2 + \rho^2}{2} \cos \phi \ln \left(1 - 2 \frac{a}{\rho} \cos \phi + \left(\frac{a}{\rho}\right)^2\right) d\phi \\
- \int_0^\pi \rho^2 \cos^2 \phi \cos \phi \ln \left(1 - 2 \frac{a}{\rho} \cos \phi + \left(\frac{a}{\rho}\right)^2\right) d\phi
\]

The first two of the four integrals in this expression are zero. The third can be evaluated using \#6, Sec. 4.397, p. 607 of the Integral Table by Gradshteyn & Ryzhik, and the last can be evaluated using \#6 and \#8 of the same section. The results are

\[
\int_0^\pi \frac{a^2 + \rho^2}{2} \cos \phi \ln (1 - 2 \frac{a}{\rho} \cos \phi + \left(\frac{a}{\rho}\right)^2) d\phi = -\frac{\pi}{2} \left(\frac{a^3}{\rho} + a\rho\right)
\]

and using \(2 \cos^2 \phi = 1 + \cos 2\phi\)
\[
\int_0^{\pi} \rho^2 \cos^2 \phi \cos \phi \ln(1 - 2 \frac{a}{\rho} \cos \phi + (\frac{a}{\rho})^2) d\phi
\]

\[
= \frac{\rho^2}{2} \int_0^{\pi} (1 + \cos 2\phi) \cos \phi \ln(1 - 2 \frac{a}{\rho} \cos \phi + (\frac{a}{\rho})^2) d\phi
\]

\[
= \frac{\rho^2}{2} (-\pi \frac{a}{\rho} + \frac{\rho^2}{2} \left[ - \frac{\pi}{2} \left( \frac{a^3}{\rho^3} + \frac{a}{\rho} \right) \right]
\]

\[
= - \frac{\pi}{12} \left( \frac{a^3}{\rho} + 9a\rho \right).
\]

Collecting these results

\[
\int_0^{\pi} \left( \frac{a^2 + \rho^2}{2} - \rho^2 \cos^2 \phi \right) \cos \phi \ln(a^2 + \rho^2 - 2a\rho \cos \phi) d\phi
\]

\[
= - \frac{\pi}{2} \left( \frac{a^3}{\rho} + a\rho \right) + \frac{\pi}{12} \left( \frac{a^3}{\rho} + 9a\rho \right) = - \frac{\pi}{12} \left( 5 \frac{a^3}{\rho} - 3a\rho \right).
\]

Next consider the last integral in the above expression for \( D_\rho - D_{\rho_0} \); it is

\[
2\rho^2 \int_0^{\pi} \cos^2 \phi \sin \phi \tan^{-1} \frac{\frac{a}{\rho} \sin \phi}{1 - \frac{a}{\rho} \cos \phi} d\phi.
\]

Using \( 2 \cos^2 \phi = 1 + \cos 2\phi \) this may be written

\[
\rho^2 \int_0^{\pi} \sin \phi \tan^{-1} \frac{\frac{a}{\rho} \sin \phi}{1 - \frac{a}{\rho} \cos \phi} d\phi + \rho^2 \int_0^{\pi} \cos 2\phi \sin \phi \tan^{-1} \frac{\frac{a}{\rho} \sin \phi}{1 - \frac{a}{\rho} \cos \phi} d\phi.
\]

These integrals may be evaluated using #1 and #3, Sec. 4.575, p. 627 of the integral table by Gradshteyn & Ryzhik. The result is

\[
\rho^2 \frac{\pi}{2} \left( \frac{a}{\rho} \right) + \rho^2 \frac{\pi}{4} \left( \frac{a^3}{\rho^3} - \frac{a}{\rho} \right) = \frac{\pi}{12} \left( \frac{a^3}{\rho} + 3a\rho \right).
\]
Thus
\[ 2\pi^2 \int_0^\pi \cos^2 \phi \sin \phi \tan^{-1} \frac{a \sin \phi}{1 - \frac{a}{\rho} \cos \phi} \, d\phi = \frac{\pi}{12} \left( \frac{a^3}{\rho} + 3a\rho \right). \]

The expression for \( D_\rho - D_{\rho_0} \) may now be written
\[
4\pi(D_\rho - D_{\rho_0}) = 4\left(\frac{1}{2}\right) \left\{ -\frac{\pi}{12} \left( 5 \frac{a^3}{\rho} - 3 \frac{a}{\rho} \right) - \frac{\pi}{2} a\rho + \frac{\pi}{12} \left( \frac{a^3}{\rho} + 3a\rho \right) \right\}
\]
\[ = 4\left(\frac{1}{2}\right) \left( -\frac{\pi}{3} \frac{a^3}{\rho} \right). \]

This, at first thought, seems to complete the evaluation of the integral for \( D_\rho - D_{\rho_0} \) but such is not the case. The function \( \nabla \cdot \vec{D} \) is discontinuous at \( \rho = a \); therefore \( \nabla(\nabla \cdot \vec{D}) \) is singular at that surface. The singularity can be handled by assuming \( \nabla \cdot \vec{D} \) decreases from \( a \) to \( 0 \) linearly as \( \rho \) increases from \( a \) to \( a + b \). \( \nabla(\nabla \cdot \vec{D}) \) then takes the value \(-\frac{a}{b} \frac{a}{\rho} \) within that interval and the contribution to the volume integral for that interval can be calculated. The limiting value of the contribution as \( b \to 0 \) is the required correction term to account for the singularity. The correction is
\[
\lim_{b \to 0} 4\left(\frac{1}{2}\right) \int_{\rho = a}^{a+b} \int_0^\pi \cos \phi \rho' \ln(\rho'^2 + (\rho')^2 - 2\rho'\rho \cos \phi)(-\frac{a}{b})d\rho'd\phi
\]
\[ = -4\left(\frac{1}{2}\right) a^2 \int_0^\pi \cos \phi \ln(\rho^2 + a^2 - 2a\rho \cos \phi) \, d\phi \]
\[ = -4\left(\frac{1}{2}\right) a^2 \int_0^\pi \cos \phi \left[ \ln \rho^2 + \ln(1 - 2 \frac{a}{\rho} \cos \phi + \left(\frac{a}{\rho}\right)^2) \right] \, d\phi \]
\[ = -4\left(\frac{1}{2}\right) a^2 \int_0^\pi \cos \phi \ln \rho^2 \, d\phi - 4\left(\frac{1}{2}\right) a^2 \int_0^\pi \cos \phi \ln(1 - 2 \frac{a}{\rho} \cos \phi + \left(\frac{a}{\rho}\right)^2) \, d\phi \]
The first of these integrals is zero, and the second is the same as one evaluated above; the result is

\[-4\left(\frac{1}{2}\right)a^2(-\frac{\pi}{\rho}) = 4\left(\frac{1}{2}\right)\pi \frac{a^3}{\rho}.\]

Adding this correction term gives the final result

\[4\pi(D_\rho - D_{\rho_0}) = 4\left(\frac{1}{2}\right)(-\frac{\pi}{3} \frac{a^3}{\rho} + \frac{\pi}{3} \frac{a^3}{\rho}) = 4\left(\frac{1}{2}\right)\frac{2\pi}{3} \frac{a^3}{\rho}\]

or

\[D_\rho - D_{\rho_0} = \frac{1}{3} \frac{a^3}{\rho}.\]

Since \(D_{\rho_0} = 0\)

\[D_\rho = \frac{1}{3} \frac{a^3}{\rho}.\]

This is the correct result for \(\rho > a\). The case \(\rho < a\) will now be considered.

The starting point for this case is

\[4\pi(D_\rho - D_{\rho_0}) = 4 \int_{\rho'=0}^{\rho} \int_{\phi'=0}^{\pi} \cos \phi \rho' \ln \sqrt{\rho^2 + (\rho')^2 - 2\rho \rho' \cos \phi} \, d\phi d\rho'.\]

The integration can be in terms of two integrals, one involving the volume out to \(\rho' = \rho\) and the other involving the volume from \(\rho' = \rho\) to \(\rho' = a\). Thus

\[4\pi(D_\rho - D_{\rho_0}) = 4 \int_{\rho'=0}^{\rho} \int_{\phi'=0}^{\pi} \cos \phi \rho' \ln \sqrt{\rho^2 + (\rho')^2 - 2\rho \rho' \cos \phi} \, d\phi d\rho' + 4 \int_{\rho'=\rho}^{a} \int_{\phi'=0}^{\pi} \cos \phi \rho' \ln \sqrt{\rho^2 + (\rho')^2 - 2\rho \rho' \cos \phi} \, d\phi d\rho'.\]
The first of these integrals can be obtained as the limit of a previously result. It was previously shown that for $\rho > a$

$$4 \int_{\rho = 0}^{a} \int_{\phi = 0}^{\pi} \cos \phi \rho' \ln \sqrt{\rho^2 + (\rho')^2} - 2\rho \rho' \cos \phi \, d\phi d\rho'$$

$$= 4 \left( \frac{1}{2} \right) \left( -\frac{a}{3} \right).$$

Using this, it is seen that

$$4 \int_{\rho = 0}^{\rho} \int_{\phi = 0}^{\pi} \cos \phi \rho' \ln \sqrt{\rho^2 + (\rho')^2} - 2\rho \rho' \cos \phi \, d\phi d\rho'$$

$$= \lim_{a \to \rho} \left[ 4 \left( \frac{1}{2} \right) \left( -\frac{a^3}{3} \rho \right) \right] = -4 \left( \frac{1}{2} \right) \left( \frac{\pi}{3} \rho^2 \right).$$

The integration over $\rho'$ in the second of the integrals involves the same integrand as for the case $\rho > a$ but the lower limit is different. Using those results it is seen that

$$\int_{\rho}^{a} \rho' \ln \sqrt{\rho^2 + (\rho')^2} - 2\rho \rho' \cos \phi \, d\rho'$$

$$= \frac{1}{2} \int_{\rho}^{a} \rho' \ln(\rho^2 + (\rho')^2) - 2\rho \rho' \cos \phi \, d\rho'$$

$$= \frac{1}{2} \left[ \frac{1}{2} \left( \rho'^2 + \rho^2 \sin^2 \phi \right) \ln \left( \rho'^2 + \rho^2 \sin^2 \phi \right) - \left( \rho'' \right)^2 \right]_{\rho}^{a} - \rho \cos \phi$$

$$+ \rho \cos \phi \left[ \rho'' \ln \left( \rho'^2 + \rho^2 \sin^2 \phi \right) - 2\rho'' + 2\rho \sin \phi \tan^{-1} \frac{\rho''}{\rho \sin \phi} \right]_{\rho}^{a} - \rho \cos \phi$$

$$= \frac{1}{2} \left[ \left( \frac{a^2 + \rho^2}{2} - \rho^2 \cos^2 \phi \right) \ln \left( \frac{a^2 + \rho^2}{2} - 2\rho \cos \phi \right) - \frac{a^2 - \rho^2}{2} - a \rho \cos \phi$$

$$- \rho^2 (1 - \cos^2 \phi) \ln \left( \rho^2 - 2\rho \cos \phi + \rho^2 \right) + \rho^2 \cos \phi$$

$$+ 2\rho^2 \cos \phi \sin \phi \left[ \tan^{-1} \frac{a - \rho \cos \phi}{\rho \sin \phi} - \tan^{-1} \frac{\rho - \rho \cos \phi}{\rho \sin \phi} \right].$$
Using this result and the identity
\[ \tan^{-1} \frac{\alpha}{\beta} = \frac{\pi}{2} - \tan^{-1} \frac{\beta}{\alpha} \]
it is seen that
\[
4 \int_{\rho'=0}^{a} \int_{\phi'=0}^{\pi} \cos \phi \rho' \ln \sqrt{\rho'^2 + (\rho')^2 - 2\rho' \rho \cos \phi} \, d\phi d\rho'
\]
\[
= 4 \left( \frac{1}{2} \right) \left\{ \int_{0}^{\pi} \left( \frac{a^2 + \rho^2}{2} - \rho^2 \cos^2 \phi \right) \cos \phi \ln(a^2 + \rho^2 - 2a \rho \cos \phi) \, d\phi \right. \\
- \left. \frac{a^2 - \rho^2}{2} \int_{0}^{\pi} \cos \phi \, d\phi + \rho(\rho-a) \int_{0}^{\pi} \cos^2 \phi \, d\phi \right. \\
\left. - \rho^2 \int_{0}^{\pi} (1 - \cos^2 \phi) \cos \phi \ln(\rho^2 - 2\rho \cos \phi + \rho^2) \, d\phi \right. \\
+ 2\rho^2 \int_{0}^{\pi} \cos^2 \phi \sin \phi \left[ \tan^{-1} \frac{\sin \phi}{1 - \cos \phi} - \tan^{-1} \frac{\rho}{a} \frac{\sin \phi}{1 - \frac{\rho}{a} \cos \phi} \right] \, d\phi \right\}.
\]
The second of the above integrals is zero and the third is given by
\[ \int_{0}^{\pi} \cos^2 \phi \, d\phi = \frac{\pi}{2}. \]
The remaining integrals are similar to integrals previously considered but the evaluation of some of them is slightly different because \( \rho < a \).
Consider the first integral; it can be written
\[
\int_{0}^{\pi} \left( \frac{a^2 + \rho^2}{2} - \rho^2 \cos^2 \phi \right) \cos \phi \ln \left[ \ln a^2 + \ln \left( 1 - 2a \frac{\rho}{a} \cos \phi + \left( \frac{\rho}{a} \right)^2 \right) \right] \, d\phi
\]
\[
= \int_{0}^{\pi} \frac{a^2 + \rho^2}{2} \cos \phi \ln a^2 \, d\phi - \int_{0}^{\pi} \rho^2 \cos^3 \phi \ln a^2 \, d\phi
\]
\[ + \int_0^\pi \frac{a^2 + \rho^2}{2} \cos \phi \ln(1 - 2 \frac{\rho}{a} \cos \phi + \left(\frac{\rho}{a}\right)^2) \, d\phi \]

\[ - \int_0^\pi \rho^2 \cos^3 \phi \ln(1 - 2 \frac{\rho}{a} \cos \phi + \left(\frac{\rho}{a}\right)^2) \, d\phi \]

The first two integrals on the right hand side of this expression are both zero. The third can be evaluated using integral #6, Sec. 4.397, p. 507 of the Integral Table of Gradshteyn & Ryzhik. Thus

\[ \int_0^\pi \frac{a^2 + \rho^2}{2} \cos \phi \ln\left(1 - 2 \frac{\rho}{a} \cos \phi + \left(\frac{\rho}{a}\right)^2\right) \, d\phi = \frac{a^2 + \rho^2}{2} \left(-\pi \frac{\rho}{a}\right) \]

\[ = - \frac{\pi}{2} \left(\frac{\rho^3}{a} + \alpha \rho\right) \]

Integral #8 of the same section can be employed in the evaluation of the last term on the right hand side.

\[ -\rho^2 \int_0^\pi \cos^3 \phi \ln\left(1 - 2 \frac{\rho}{a} \cos \phi + \left(\frac{\rho}{a}\right)^2\right) \, d\phi \]

\[ = - \frac{\rho^2}{2} \int_0^\pi (1 + \cos 2\phi) \cos \phi \ln\left(1 - 2 \frac{\rho}{a} \cos \phi + \left(\frac{\rho}{a}\right)^2\right) \, d\phi \]

\[ = - \frac{\rho^2}{2} \int_0^\pi \cos \phi \ln\left(1 - 2 \frac{\rho}{a} \cos \phi + \left(\frac{\rho}{a}\right)^2\right) \, d\phi \]

\[ - \frac{\rho^2}{2} \int_0^\pi \cos 2\phi \cos \phi \ln\left(1 - 2 \frac{\rho}{a} \cos \phi + \left(\frac{\rho}{a}\right)^2\right) \, d\phi \]

\[ = - \frac{\rho^2}{2} \left(-\pi \frac{\rho}{a}\right) - \frac{\rho^2}{2} \left(- \pi \left(\frac{\rho}{2a} + \frac{\rho}{a}\right)\right) = \frac{\pi}{12} \left(\frac{\rho^5}{a^3} + 9 \frac{\rho^3}{a}\right). \]
Collecting these results it is seen that

\[ \int_{0}^{\pi} \left( \frac{a^2 + \rho^2}{2} - \rho^2 \cos^2 \phi \right) \cos \phi \ln(a^2 + \rho^2 - 2a \rho \cos \phi) d\phi \]

\[ = -\frac{\pi}{2} \left( \frac{\rho^3}{a} + a \rho \right) + \frac{\pi}{12} \left( \frac{\rho^5}{a^3} + 9 \frac{\rho^3}{a} \right) \]

\[ = \frac{\pi}{12} \left( \frac{\rho^5}{a^3} + 3 \frac{\rho^3}{a} - 6a \rho \right). \]

Next consider the integral

\[ -\rho^2 \int_{0}^{\pi} (1 - \cos^2 \phi) \cos \phi \ln(\rho^2 - 2\rho^2 \cos \phi + \rho^2) d\phi. \]

This can be written

\[ -\rho^2 \int_{0}^{\pi} \sin^2 \phi \cos \phi [\ln(\rho^2 + \ln(1 - 2 \cos \phi + 1)) d\phi \]

\[ = -\rho^2 \int_{0}^{\pi} \sin^2 \phi \cos \phi \ln(1 - 2 \cos \phi + 1) d\phi. \]

The first of these integrals is zero; the second can be transformed as follows:

\[ -\rho^2 \int_{0}^{\pi} \sin^2 \phi \cos \phi \ln(1 - 2 \cos \phi + 1) d\phi = -\rho^2 \int_{0}^{\pi} \frac{1 - \cos 2\phi}{2} \cos \phi \ln(1 - 2 \cos \phi + 1) d\phi \]

\[ = -\frac{\rho^2}{2} \int_{0}^{\pi} \cos \phi \ln(1 - 2 \cos \phi + 1) d\phi + \frac{\rho^2}{2} \int_{0}^{\pi} \cos 2\phi \cos \phi \ln(1 - 2 \cos \phi + 1) d\phi. \]

The same integrals as employed above can be used, in limiting form, in this case.

\[ \int_{0}^{\pi} \cos \phi \ln(1 - 2 \cos \phi + 1) d\phi = \lim_{\alpha \to 1} (-\alpha) = -\pi \]
and
\[
\int_0^\pi \cos 2\phi \cos \phi \sin(1 - 2\cos \phi + 1) d\phi = \lim_{\alpha \to 1} \left[- \frac{\pi}{2} \left(\frac{\alpha^3}{3} + \alpha\right)\right] = - \frac{\pi}{2} \left(\frac{1}{3} + 1\right) = - \frac{2\pi}{3}.
\]

Thus
\[
-\rho^2 \int_0^\pi (1 - \cos^2 \phi) \cos \phi \sin(\rho^2 - 2\rho^2 \cos \phi + \rho^2) d\phi = - \frac{\rho^2}{2} (-\pi) + \frac{\rho^2}{2} (-\frac{2\pi}{3}) = \frac{\pi \rho^2}{3}.
\]

Finally, consider the integral
\[
2\rho^2 \int_0^\pi \cos^2 \phi \sin \phi \left[\tan^{-1} \frac{\sin \phi}{1 - \cos \phi} - \tan^{-1} \frac{\rho}{a} \frac{\sin \phi}{1 - \rho \cos \phi}\right] d\phi
\]
\[
= \rho^2 \int_0^\pi (1 + \cos 2\phi) \sin \phi \left[\tan^{-1} \frac{\sin \phi}{1 - \cos \phi} - \tan^{-1} \frac{\rho}{a} \frac{\sin \phi}{1 - \rho \cos \phi}\right] d\phi
\]
\[
= \rho^2 \int_0^\pi \sin \phi \tan^{-1} \frac{\sin \phi}{1 - \cos \phi} d\phi + \rho^2 \int_0^\pi \cos 2\phi \sin \phi \tan^{-1} \frac{\rho}{a} \frac{\sin \phi}{1 - \rho \cos \phi} d\phi
\]
\[
- \rho^2 \int_0^\pi \sin \phi \tan^{-1} \frac{\rho}{a} \frac{\sin \phi}{1 - \rho \cos \phi} d\phi - \rho^2 \int_0^\pi \cos 2\phi \sin \phi \tan^{-1} \frac{\rho}{a} \frac{\sin \phi}{1 - \rho \cos \phi} d\phi.
\]

The last two of the four integrals in this expression can be evaluated using integrals #1 and #3, Sec. 4.575, p. 627 of the Integral Table by Gradshteyn & Ryzhik; the first two can be evaluated using limiting forms of these. Thus
\[2\rho^2 \int_0^\pi \cos^2 \phi \sin \phi \left[ \tan^{-1} \frac{\rho \sin \phi}{1 - \rho \cos \phi} - \tan^{-1} \frac{\rho}{a} \right] d\phi = \rho^2 \lim_{\alpha \to 1} \left( \frac{\pi}{2} \alpha + \frac{\pi}{4} \left( \frac{\alpha^3}{3} - \alpha \right) \right) = \rho^2 \left( \frac{\pi}{2} \frac{\rho}{a} \right) - \rho^2 \left( \frac{\pi}{4} \frac{\rho}{a} - \frac{\rho}{a} \right) = \frac{\pi}{2} \rho^2 - \frac{\pi}{6} \rho^2 - \frac{\pi}{2} \frac{\rho^3}{a} - \frac{\pi}{12} \frac{\rho^5}{a} + \frac{\pi}{4} \frac{\rho^3}{a} = - \frac{\pi}{12} \left( \frac{\rho^5}{a} + 3 \frac{\rho^3}{a} - 4\rho^2 \right). \]

Collecting results it is now seen that
\[
4 \int_0^a \int_0^\pi \cos \phi \rho' \ln \sqrt{\rho^2 + (\rho')^2} - 2\rho \rho' \cos \phi \ d\phi \ d\rho' =
\]
\[
= 4(\frac{1}{2}) \left\{ \frac{\pi}{12} \left( \frac{5}{a^3} + 3 \frac{3}{a} - 6\rho \right) + \frac{\pi}{2} \rho(\rho - a) + \frac{\pi}{6} \rho^2 \right. 
\]
\[
- \frac{\pi}{12} \left( \frac{5}{a^3} + 3 \frac{3}{a} - 4\rho^2 \right) \right\}
\]
\[
= 4(\frac{1}{2}) \left\{ - \pi a \rho + \pi \rho^2 \right\}.
\]

Next, the effect of the singularity in the integrand at \( \rho = a \) must be evaluated. Proceeding in the same way as for the case \( \rho > a \), the correction term, \( C \), is
\[
C = \lim_{b \to 0} - \frac{a+b}{b} \int_0^\pi \cos \phi \rho' \ln(\rho^2 + (\rho')^2) - 2\rho \rho' \cos \phi \left( - \frac{a}{b} \right) d\rho' d\phi
\]
\[
= 4(\frac{1}{2}) a^2 \int_0^\pi \cos \phi \ln(\rho^2 + a^2 - 2\rho a \cos \phi) d\phi
\]
\[
= 4(\frac{1}{2}) a^2 \int_0^\pi \cos \phi \left[ \ln a^2 + \ln(1 - 2 \frac{\rho}{a} \cos \phi + (\frac{\rho}{a})^2) \right] d\phi.
\]
\[ = 4 \left( \frac{1}{2} \right) \int_0^\pi \cos \phi \ln a^2 \, d\phi + a^2 \int_0^\pi \cos \phi \ln \left( 1 - \frac{\rho}{a} \cos \phi + \left( \frac{\rho}{a} \right)^2 \right) d\phi \]

The first of these integrals is zero; the second is the same as one of those evaluated above. Thus the correction term is

\[ C = 4 \left( \frac{1}{2} \right) \left[ a^2 \left( - \pi \frac{\rho}{a} \right) \right] = 4 \left( \frac{1}{2} \right) (-\pi \rho) \]

The final result for the case \( \rho < a \) can now be written

\[ 4\pi (D_{\rho} - D_{\rho_0}) = 4 \int_{\rho' = 0}^a \int_0^\pi \cos \phi \rho' \ln \sqrt{\rho^2 + (\rho')^2 - 2\rho \rho' \cos \phi} \, d\phi d\rho' \]

\[ = 4 \int_{\rho' = 0}^a \int_0^\pi \cos \phi \rho' \ln \sqrt{\rho^2 + (\rho')^2 - 2\rho \rho' \cos \phi} \, d\phi d\rho' + C \]

Substituting for the terms on the right

\[ 4\pi (D_{\rho} - D_{\rho_0}) = -4 \left( \frac{1}{2} \right) \left( \frac{\pi}{3} \right) \rho^2 + 4 \left( \frac{1}{2} \right) (-\pi \rho + \pi \rho^2) + 4 \left( \frac{1}{2} \right) (-\pi \rho) \]

\[ = 4 \left( \frac{1}{2} \right) \frac{2\pi}{3} \rho^2 \]

so

\[ D_{\rho} - D_{\rho_0} = \frac{\rho^2}{3} \]

Finally, since \( D_{\rho_0} = 0 \)

\[ D_{\rho} = \frac{\rho^2}{3} \]
The vector $\vec{D}$ has not been determined everywhere it is given by

$$\vec{D} = \vec{a}_\rho \begin{cases} \frac{\rho}{3} & \rho < a \\ \frac{a^3}{3} & a < \rho \end{cases}$$

This result agrees with the expression specified for $\vec{D}$ at the beginning of this example and demonstrates that $\vec{D}$ can be found if its curl and divergence are known.
APPENDIX 5 (continued)

EXAMPLE OF THE DETERMINATION
OF $\vec{A}$ FROM $\nabla \times \vec{A}$ AND $\nabla \cdot \vec{A}$

Define $\vec{A}$ by

\[
\vec{A} = \begin{cases} 
- \frac{1}{a_z} \frac{\mu I \rho^2}{4\pi a^2}, & \rho < a \\
- \frac{1}{a_z} \frac{\mu I}{2\pi} \ln \rho, & a < \rho
\end{cases}
\]

in cylindrical coordinates. The curl and divergence of $\vec{A}$ are

\[
\nabla \times \vec{A} = \begin{cases} 
a_{\phi} \frac{\mu I \rho}{2\pi a^2}, & \rho < a \\
a_{\phi} \frac{\mu I}{2\pi a}, & a < \rho
\end{cases}
\]

and

\[
\nabla \cdot \vec{A} = 0.
\]

The vector field $\vec{A}$ can be determined by use of the identity

\[
\vec{A} = \int \frac{\nabla \times (\nabla \times \vec{A}) - \nabla (\nabla \cdot \vec{A})}{4\pi R} \, dv.
\]

It is first noted that

\[
\nabla \times (\nabla \times \vec{A}) = \begin{cases} 
- \frac{1}{a_z} \frac{\mu I}{\pi a^2}, & \rho < a \\
0 & a < \rho
\end{cases}
\]

First consider the case $\rho > a$; $\vec{A}$ is given by
\[ A = \frac{\mu I}{\pi a^2} \int_{\rho' = 0}^{a} \int_{\phi = 0}^{\pi} \int_{z = 0}^{\infty} \frac{\rho'\,d\phi\,d\rho'\,dz}{\sqrt{\rho^2 + (\rho')^2 + z^2 - 2\rho\rho'\cos \phi}} \]

The factor 4 preceding the integral is a consequence of the limitation of the range of integration to \(0 < \phi < \pi\) and to \(0 < z < \infty\).

The integral for \(A\) does not converge. However, a useful result can be obtained by evaluating \(A - A_a\), where \(A_a\) is \(A\) evaluated for \(\rho = a\). The expression for this is

\[
A - A_a = \frac{\mu I}{\pi a^2} \frac{1}{4\pi} \int_{\rho' = 0}^{a} \int_{\phi = 0}^{\pi} \int_{z = 0}^{\infty} \left[ \frac{\rho'}{\sqrt{\rho^2 + (\rho')^2 + z^2 - 2\rho\rho'\cos \phi}} - \frac{\rho'}{\sqrt{a^2 + (\rho')^2 + z^2 - 2a\rho'\cos \phi}} \right] d\rho'\,d\phi\,dz.
\]

This integral can be evaluated. Integration with respect to \(z\) gives

\[
A = A_a = \frac{\mu I a_z}{2\pi a^2} \int_{\rho' = 0}^{a} \int_{\phi = 0}^{\pi} \rho' \ln \left( \frac{\sqrt{a^2 + (\rho')^2 - 2a\rho'\cos \phi}}{\sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos \phi}} \right) d\phi d\rho'.
\]

\[
= \frac{\mu I a_z}{2\pi a^2} \left[ \int_{\rho' = 0}^{a} \int_{\phi = 0}^{\pi} \rho' \ln(a^2 + (\rho')^2 - 2a\rho'\cos \phi) d\phi d\rho' \right] - \int_{\rho' = 0}^{a} \int_{\phi = 0}^{\pi} \rho' \ln(\rho^2 + (\rho')^2 - 2\rho\rho'\cos \phi) d\phi d\rho'.
\]
Using #9, Sec. 4.224, p. 541 of the Integral Table by Gradshteyn and Ryzhik and noting that \( \rho' \leq a \), it is seen that

\[
\pi \int_{\phi=0}^{\pi} \ln(a^2 + (\rho')^2 - 2a\rho'\cos \phi) \, d\phi = \pi \ln a^2
\]

and for the case \( \rho > a \)

\[
\pi \int_{\phi=0}^{\pi} \ln(\rho^2 + (\rho')^2 - 2\rho\rho'\cos \phi) \, d\phi = \pi \ln \rho^2.
\]

Using these results, \( \overline{A} - \overline{A}_a \), for the case \( \rho > a \), is given by

\[
\overline{A} - \overline{A}_a = \frac{\mu I \overline{a}}{2\pi a} \int_{\rho'=0}^{\infty} \rho' \ln \frac{\rho^2}{\rho'^2} \, d\rho' = \frac{\mu I \overline{a}}{2\pi a} \left( \frac{a^2}{2} \right) \ln \frac{a^2}{\rho^2}
\]

or

\[
\overline{A} - \overline{A}_a = \frac{-\mu I \overline{a}}{2\pi} \left( \ln \rho = \ln a \right).
\]

\( \overline{A} \) can be expressed as

\[
\overline{A} = \left( \overline{A}_a + \frac{\mu I}{2\pi} \ln a \right) - \overline{a} \frac{\mu I}{2\pi} \ln \rho.
\]

The quantity in the parenthesis on the right hand side of this expression is a constant, and the second term on the right is the expression initially specified for \( \overline{A} \) for the case \( \rho > a \). This result illustrates the obvious fact that the expression

\[
\overline{A} = \int_{V} \frac{\nabla \times (\nabla \times \overline{A}) - \nabla(\nabla \cdot \overline{A})}{4\pi R} \, dv
\]

for
gives an expression for $\bar{A}$ that is correct except for constants and linear terms that would be lost in the evaluation of $\nabla \times (\nabla \times \bar{A}) - \nabla (\nabla \cdot \bar{A})$.

To complete this example, the case $\rho < a$ must be considered.

For $\rho > \rho'$

$$\int_0^\pi \ln(\rho^2 + (\rho')^2 - 2\rho\rho' \cos \phi) d\phi = \pi \ln \rho^2$$

as found above. However for $\rho < \rho'$

$$\int_0^\pi \ln(\rho^2 + (\rho')^2 - 2\rho\rho' \cos \phi) d\phi = \pi \ln (\rho')^2,$$

The integration over $\rho'$ must be expressed as the sum of two integrals, one over the interval $0 < \rho' < \rho$ and the other over the interval $\rho < \rho' < a$. Thus, for the case $\rho < a$,

$$\bar{A} - \bar{A}_a = \frac{\mu I_a}{2\pi a^2} \left[ \int_0^\rho \rho' \ln \frac{a^2}{\rho^2} d\rho' + \int_{\rho}^a \rho' \ln \frac{a^2}{(\rho')^2} d\rho' \right]$$

or

$$\bar{A} - \bar{A}_a = \frac{\mu I_a}{2\pi a^2} \left[ \ln \frac{a^2}{\rho^2} \int_0^\rho \rho' d\rho' + \ln a^2 \int_\rho^a \rho' d\rho' - \int_{\rho}^a \rho' \ln (\rho')^2 d\rho' \right]$$

$$= \frac{\mu I_a}{2\pi a^2} \left\{ \ln \frac{a^2}{\rho^2} \left[ \frac{(\rho')^2}{2} \right]_0^\rho + \ln a^2 \left[ \frac{(\rho')^2}{2} \right]_0^a - \left[ \frac{(\rho')^2}{2} \ln (\rho')^2 - \frac{(\rho')^2}{2} \right]_0^\rho \right\}$$

$$= \frac{\mu I_a}{2\pi a^2} \left\{ \ln \frac{a^2}{\rho^2} \left( \frac{\rho^2}{2} \right) + \ln a^2 \left( \frac{a^2}{2} - \frac{\rho^2}{2} \right) - \frac{a^2}{2} \ln \rho^2 + \frac{a^2}{2} \ln \rho^2 + \frac{a^2}{2} + \frac{\rho^2}{2} \ln \rho^2 - \frac{\rho^2}{2} \right\}$$

$$= \frac{\mu I_a}{2\pi a^2} \left( \frac{a^2}{2} - \frac{\rho^2}{2} \right).$$
$\overline{A}$ can be written as

$$\overline{A} = (\overline{A}_a + \frac{\mu I \overline{a}}{4\pi} z) - \overline{a}_z \frac{\mu I}{4\pi} \left( \frac{\rho^2}{a^2} \right).$$

The term in parentheses, on the right hand side, is a constant and the last term, on the right, is the expression initially specified for $\overline{A}$ for the case $\rho < a$. This result again illustrates that the integral expression for $\overline{A}$ gives the correct result except for constant terms and linear terms.
APPENDIX 6

LIST OF FORMULAS

Notation

\[ \int_{C_{P_1}}^{P_2} \] - line integral along contour \( C \) from point \( P_1 \) to point \( P_2 \)

\[ \oint_{C} \] - line integral along closed contour \( C \)

\( d\xi \) - scalar differential of length

\( d\mathbf{\xi} \) - vector differential of length, \( d\mathbf{\xi} = \mathbf{a}_t \, d\xi \)

\[ \int_{S} \] - surface integral over surface \( S \) bounded by contour \( C \)

\[ \oint_{S} \] - surface integral over closed surface \( S \)

\( dS \) - scalar differential of area

\( d\mathbf{S} \) - vector differential of area, \( d\mathbf{S} = \mathbf{a}_n \, dS \)

\[ \int_{V} \] - volume integral over volume \( V \) bounded by closed surface \( S \)

\( dv \) - differential volume

\( \mathbf{v} \) - velocity of a point on a line or surface

\( \mathbf{v}_k \) - \( \mathbf{v} \) at point \( p_k \)

\( \mathbf{a}_t \) - unit vector tangent to contour \( C \) in the direction of integration

\( \mathbf{a}_n \) - unit vector normal to surface \( S \) on the positive side of \( S \)
\( \vec{a}_1, \vec{a}_2, \vec{a}_3 \) - a mutually orthogonal set of unit vectors oriented such that \( \vec{a}_3 = \vec{a}_1 \times \vec{a}_2 \)

\( \Delta S_k \) - magnitude of vector area \( \Delta \vec{S}_k = \Delta S_k \vec{a}_k \), oriented in the direction of \( \vec{a}_k \)

\( \vec{A}, \vec{B}, \vec{F}, \vec{G} \) - arbitrary continuous vector fields

\( F \) - arbitrary continuous scalar field

\( t \) - time

Formulas

\[
\begin{align*}
\oint\left\{ \vec{a}_n \cdot \vec{F} \times d\ell \right\} &= - \oint\left\{ \nabla \cdot [\vec{a}_n \times (\vec{a}_n \times \vec{F})] dS \right\} \\
\oint\left\{ \vec{a}_n \times \vec{F} \cdot d\ell \right\} &= \int_{S} \nabla \cdot \left( \vec{a}_n \times (\vec{a}_n \times \vec{F}) \right) dS
\end{align*}
\]

\( \vec{a} = \frac{d\vec{V}}{dt} = (\vec{V} \cdot \nabla) \vec{V} \)

\( \vec{A} = \frac{1}{\vec{B} \cdot \vec{B}} \left[ \vec{B} \times (\vec{A} \times \vec{B}) + \vec{B}(\vec{A} \cdot \vec{B}) \right] \)

\( \vec{A} = \frac{1}{2}[\vec{a}_1 \times (\vec{A} \times \vec{a}_1) + \vec{a}_2 \times (\vec{A} \times \vec{a}_2) + \vec{a}_3 \times (\vec{A} \times \vec{a}_3)] \)

\( \nabla (\vec{G} \cdot \vec{F}) = (\vec{F} \cdot \nabla) \vec{G} + (\vec{G} \times \nabla) \vec{F} + (\nabla \cdot \vec{F})\vec{G} + (\nabla \cdot \vec{G})\vec{F} \)

\( \nabla (\vec{G} \cdot \vec{F}) = (\vec{G} \cdot \nabla)\vec{F} + (\vec{F} \cdot \nabla)\vec{G} - (\nabla \times \vec{F}) \times \vec{G} - (\nabla \times \vec{G}) \times \vec{F} \)
\[ \nabla \times (\vec{G} \times \vec{F}) = (\nabla \times \vec{G}) \times \vec{F} - (\nabla \times \vec{F}) \times \vec{G} + (\vec{F} \times \nabla) \times \vec{G} - (\vec{G} \times \nabla) \times \vec{F} \]

\[ \nabla \times (\vec{G} \times \vec{F}) = (\vec{G} \cdot \nabla)\vec{F} + (\vec{F} \cdot \nabla)\vec{G} - (\nabla \cdot \vec{G}) \vec{F} - (\nabla \cdot \vec{F}) \vec{G} \]

\[ (\vec{G} \times \nabla) \times \vec{F} = \frac{1}{2} \left[ \nabla \times (\vec{F} \times \vec{G}) - (\nabla \times \vec{G}) \times \vec{F} - (\vec{G} \times \nabla) \times \vec{F} - (\vec{G} \times \vec{F}) \times \nabla \right] \]

\[ (\vec{G} \cdot \nabla) \vec{F} = (\vec{G} \cdot \nabla)\vec{F} + (\vec{F} \times \nabla) \times \vec{G} - (\vec{F} \times \vec{G}) \times \nabla \]

\[ (\vec{F} \cdot \nabla) \vec{G} = \frac{1}{2} \left[ (\vec{F} \times \vec{G}) \times \vec{F} - (\vec{G} \times \vec{F}) \times \vec{G} - (\vec{F} \times \vec{G}) \times \vec{G} + (\vec{G} \times \vec{F}) \times \vec{F} - (\vec{G} \times \vec{F}) \times \vec{F} \right] \]

\[ (\vec{F} \times \vec{G}) \times \vec{G} + (\vec{G} \times \vec{F}) \times \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \vec{G}) \times \vec{F} - (\vec{G} \times \vec{F}) \times \vec{F} \]

\[ (\vec{F} \times \vec{G}) \times \vec{F} - (\vec{G} \times \vec{F}) \times \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \vec{G}) \times \vec{F} - (\vec{G} \times \vec{F}) \times \vec{F} \]

\[ (\vec{G} \times \vec{F}) = (\vec{G} \cdot \nabla)\vec{F} + (\vec{F} \times \nabla) \times \vec{G} - (\vec{F} \times \vec{G}) \times \nabla \]

\[ (\vec{F} \cdot \nabla) \vec{G} + (\vec{G} \cdot \nabla) \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \vec{G}) \times \vec{G} - (\vec{G} \times \vec{G}) \times \vec{F} \]

\[ (\vec{F} \cdot \nabla) \vec{G} + (\vec{G} \cdot \nabla) \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \vec{G}) \times \vec{G} - (\vec{G} \times \vec{G}) \times \vec{F} \]

\[ (\vec{F} \cdot \nabla) \vec{G} - (\vec{G} \cdot \nabla) \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \vec{G}) \times \vec{G} - (\vec{G} \times \vec{G}) \times \vec{F} \]

\[ (\vec{F} \cdot \nabla) \vec{G} - (\vec{G} \cdot \nabla) \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \vec{G}) \times \vec{G} - (\vec{G} \times \vec{G}) \times \vec{F} \]

\[ (\vec{F} \cdot \nabla) \vec{G} + (\vec{G} \cdot \nabla) \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \vec{G}) \times \vec{G} - (\vec{G} \times \vec{G}) \times \vec{F} \]

\[ (\vec{F} \cdot \nabla) \vec{G} - (\vec{G} \cdot \nabla) \vec{F} = \nabla \times (\vec{G} \times \vec{F}) - (\vec{F} \times \vec{G}) \times \vec{G} - (\vec{G} \times \vec{G}) \times \vec{F} \]

\[ \vec{A} = \left\{ \begin{array}{l} \frac{\nabla \times (\nabla \times \vec{A}) - \nabla (\nabla \cdot \vec{A})}{4\pi R} \end{array} \right\} \]

\[ \vec{a}_1 \int_{C_1} \vec{F} \cdot d\vec{\ell} \quad \vec{a}_2 \int_{C_2} \vec{F} \cdot d\vec{\ell} \quad \vec{a}_3 \int_{C_3} \vec{F} \cdot d\vec{\ell} \]

\[ \nabla \times \vec{F} = \lim_{\Delta S_1 \to 0} \frac{\int_{C_1} \vec{F} \cdot d\vec{\ell}}{\Delta S_1} + \lim_{\Delta S_2 \to 0} \frac{\int_{C_2} \vec{F} \cdot d\vec{\ell}}{\Delta S_2} + \lim_{\Delta S_3 \to 0} \frac{\int_{C_3} \vec{F} \cdot d\vec{\ell}}{\Delta S_3} \]

\[ \nabla \vec{F} = \frac{1}{2} \left[ \begin{array}{l} \lim_{\Delta S_1 \to 0} \frac{\int_{C_1} \vec{F} \cdot d\vec{\ell}}{\Delta S_1} + \lim_{\Delta S_2 \to 0} \frac{\int_{C_2} \vec{F} \cdot d\vec{\ell}}{\Delta S_2} + \lim_{\Delta S_3 \to 0} \frac{\int_{C_3} \vec{F} \cdot d\vec{\ell}}{\Delta S_3} \end{array} \right] \]
\[ \nabla \cdot \vec{F} = \frac{1}{2} \left[ \lim_{\Delta S_1 \to 0} \frac{\oint_{C_1} \vec{a}_1 \cdot \vec{F} \times d\ell}{\Delta S_1} + \lim_{\Delta S_2 \to 0} \frac{\oint_{C_2} \vec{a}_2 \cdot \vec{F} \times d\ell}{\Delta S_2} + \lim_{\Delta S_3 \to 0} \frac{\oint_{C_3} \vec{a}_3 \cdot \vec{F} \times d\ell}{\Delta S_3} \right] \]

\[ \nabla \times \vec{F} = \lim_{\Delta S \to 0} \frac{\oint_{C} \vec{F} \times d\ell}{\Delta S} \]

\[ \nabla \times \vec{F} = \lim_{\Delta S \to 0} \frac{\oint_{C} \vec{F} \times d\ell}{\Delta S} \]

\[ \nabla \vec{F} = \lim_{\Delta S \to 0} \frac{\oint_{C} \vec{F} \cdot d\vec{S}}{\Delta S} \]

\[ \frac{d}{dt} \int_{C} \oint_{P_1} \vec{F} \cdot d\ell = \vec{F}(P_2) \cdot \vec{v}_2 - \vec{F}(P_1) \cdot \vec{v}_1 + \int_{C} \oint_{P_1} \nabla \times \vec{F} \times \vec{v} \cdot d\ell + \int_{C} \oint_{P_1} \frac{\partial \vec{F}}{\partial t} \cdot d\ell \]

\[ \frac{d}{dt} \int_{S} \oint_{P_1} \vec{F} \cdot d\vec{S} = \oint_{S} \oint_{P_1} \vec{F} \times \vec{v} \cdot d\ell + \int_{S} \oint_{P_1} \frac{\partial \vec{F}}{\partial t} \cdot d\vec{S} + \int_{S} \oint_{P_1} (\nabla \cdot \vec{F}) \vec{v} \cdot d\vec{S} \]