Spectra as Locally Finite Z-Groupoids

by

Paul Roy Lessard

B.S., The Evergreen State College, 2008

M.S., The University of Vermont, 2013

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics
2019

This thesis entitled:
Spectra as Locally Finite **Z**-Groupoids
written by Paul Roy Lessard
has been approved for the Department of Mathematics

Prof. Jonathan Wise

Prof. Agnés Beaudry

Date

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Lessard, Paul Roy (Ph.D., Mathematics)

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Thesis directed by Prof. Jonathan Wise

Since 1983, Grothendieck's suggestion that:

> "...the study of homotopical n-types should be essentially equivalent to the study of so-called n-groupoids..."

has gone from suggestion in [26], to conjecture, to theorem in [33], to counter-example in [46], and finally to abiding definition. Through a remarkable instance of Lakatos', "method of proofs and refutations," weak ω-groupoid, is now taken as synonymous with spaces by many.

As for analytic models of ω-groupoids perhaps the most intuitive, although certainly not the most widely known, is made possible by the category Θ. If $\triangle$ is the category of composition data for compositions of morphisms in a 1-category, then Θ is the category of composition data for compositions of morphisms in an ω-category[1] . There is a Cisinski model category structure on $\widehat{\Theta}$ equivalence to space, first constructed in [9] and then developed by alternative techniques in [19].

In [31] where the category Θ is first suggested, and indeed first defined as dual to the category of combinatorial disks, it is noted that the dimensional shift on Θ suggests an elegant presentation of the unreduced suspension on cellular sets. In this thesis we follow that thread.

We discover that stabilizing Θ at this dimensional shift provides a category on which may be written a sketch of an essentially algebraic theory for strict **Z**-categories. This natural notion is analogous to strict ω-categories but in place of the objects and $\mathbf{N}_{\geq 1}$-sorts of morphisms of an ω-category, a **Z**-category has only **Z**-sorts of morphisms with every $(z+1)$-morphism being a morphism between some z-morphisms.

Finally, we prove that the category of pointed, locally finite, weak **Z**-groupoids admit a model structure Quillen equivalent to the Hovey structure on $\mathsf{Sp}^{\mathbf{N}}\left(\widehat{\Theta}_{\bullet}, \Sigma_J\right)$; we provide a new naive weak stable homotopy hypothesis.

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[1] This is the notion of Θ defining a cellular nerve for higher categories put forth in [9].

Dedication

This work is dedicated to my mother, Pauline Olsen, who died peacefully at home with my father, after 25 years of Parkinson's disease, during the revision of this thesis.

> "There were only two books I ever stopped reading to you as a baby. One was: "All's Quiet on the Western Front," because I didn't remember quite how gory it was, and the other: "One Thousand and One Arabian Nights," because I didn't think that you, as a baby, would understand why someone would kill for love."

Pauline Marie Olsen, 29 May 1944 - 22 June 2019

Acknowledgements

I am grateful to Mike Shulman whose suggestion long ago that I look at the nLab entry for Kan's combinatorial spectra led to this work. I am grateful to my friend John Willis with whom I shared an office and many coffees and conversations. I am grateful to my advisor Jonathan Wise for his willingness to try and supervise a thesis so far from his area of expertise; it seems we have both come out alive. I am also grateful to my committee for their willingness to support my excitement with a topic so far from their expertise. Of them, I am particularly grateful to Agnés Beaudry who kindly made me aware of how much work had to be done before this thesis reached the form it now takes.

I am also grateful to Felix Wellen and Steve Awodey whose invitation to present at CMU in the spring of 2019 and funding of a weeklong visit afterwards gave me the freedom and inspiration both to finish and continue. As part of this I am grateful to all of the HoTT community at CMU, who made it clear to me that I had not been working in a vacuum, I had only been far away. I am grateful to Harry Gindi who pointed out an error in the presentation of the Eckmann-Hilton degeneracy which motivated both correcting the proof and a renovation of much of the work.

I wish to acknowledge too the excellent teaching I received before CU Boulder, in particular I would not have been in position to write this thesis, or dream of mathematics, were it not for Richard Foote and Peter Dodds of UVM and Rachel Hastings and David McAvity of the Evergreen State College.

The support of my family has also been invaluable. Most importantly however, I am grateful to Rachel Kennedy, for a list of reasons far larger than any cardinal.

Contents

Chapter 1

On Notation

Given any small category A we denote by $\widehat{A}$ the category of presheaves on A and by $\widehat{A}_\bullet$ the category of presheaves of pointed sets on A. The disjoint base-point functor $\widehat{A} \longrightarrow \widehat{A}_\bullet$ we denote by $(\_)_+$. When we make it explicit, the Yoneda embedding will be denoted by Yon. The presheaf on A represented by some object a therein will be denoted A^a. In the category $\widehat{A}$ the empty presheaf will be denoted $\varnothing_A$ or $\varnothing$ and the single point presheaf will be denoted by $\bullet_A$ or $\bullet$ in both $\widehat{A}$ and $\widehat{A}_\bullet$.

Chapter 2

Essentially Algebraic Theories

2.1 Essentially algebraic theories and essentially algebraic categories

Definition 2.1.1. An **essentially algebraic theory** is a category with all finite limits. Given an essentially algebraic theory $\mathscr{T}$, a **model** of $\mathscr{T}$ is a functor $\mathscr{T} \longrightarrow \mathsf{Set}$ which is left exact, meaning it preserves finite limits. For suitable categories $\mathscr{C}$ and $\mathscr{D}$ we will denote by $\mathsf{Lex}\,(\mathscr{C}, \mathscr{D})$ the full subcategory of the functor category $\mathsf{Fun}\,(\mathscr{C}, \mathscr{D})$ subtended by the left exact functors. An **essentially algebraic category** $\mathscr{C}$ is a category for which there exists a finite limit complete category $\mathscr{T}$ and an equivalence of categories $\mathsf{Lex}\,(\mathscr{T}, \mathsf{Set}) \xrightarrow{\sim} \mathscr{C}$.

An extraordinarily large class of categories admit such a presentation.

Theorem 2.1.2. *(Adamek-Rosicky) There exists an equivalence of categories between the category of locally finitely presentable categories and the category of essentially algebraic categories. In particular, every locally finitely presentable category may be presented as a category of set valued left exact functors on an essentially algebraic theory.*

Proof. This is Corollary 1.52 of [2]. □

2.2 Sketches for essentially algebraic theories

When a category does not possess all finite limits, we may still use that category to describe an essentially algebraic theory hence an essentially algebraic category, however we must be a bit more careful. In this section we'll provide a minimal introduction to the theory of sketches for an essentially algebraic theory, which will allow us to use categories which are not essentially algebraic theories to define essentially algebraic categories.

Definition 2.2.1. Given any category $\mathscr{C}$ and a functor $F : \mathscr{C} \longrightarrow \mathsf{Set}$, then F is **flat** if the category $(\mathscr{C} \downarrow F)^{\mathsf{op}}$ is filtered.

Lemma 2.2.2. *Given an essentially algebraic theory* $\mathscr{C}$*, a functor* $F : \mathscr{C} \longrightarrow \mathsf{Set}$ *is flat if and only if is left exact.*

However, it is not the case that if $\mathscr{C}$ is not possessed of all finite limits that these two notions are coincident. While flat functors preserve all the finite limits present, not all functors which preserve all the finite limits present are flat. It is into this milieu that we may appeal to a notion and a theorem of Ehresman and Kennison.

Definition 2.2.3. Let $\mathscr{C}$ be a small category. Let e be an object of $\mathscr{C}$. A **sieve on** e is a subfunctor of

$$\mathscr{C}^e = \mathsf{Hom}\left(\_, e\right),$$

i.e. it is a functor $X : \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ together with a natural monomorphism $X \longrightarrow \mathscr{C}^e$.

Remark 2.2.4. A sieve X on $e \in \mathsf{Ob}\left(\mathscr{C}\right)$ is then comprised of a set of maps into e which satisfies the condition: if $\overline{f} \in X\left(d\right)$, i.e. $f : d \longrightarrow e$, then for any $g : c \longrightarrow d$ of $\mathscr{C}$, $\overline{f \circ g} \in X\left(c\right)$[1] .

A sieve may be specified by a set of generalized elements of the functor $\mathscr{C}^e$. Indeed, a set

$$\left\{\overline{f}_i : d_i \longrightarrow e\right\}_{i \in \mathbf{I}}$$

may be said to specify the sieve

$$\bigcup_{i \in \mathbf{I}} \mathscr{C}^{d_i} \overset{\cup \overline{f_i}}{\hookrightarrow} \mathscr{C}^e.$$

Definition 2.2.5. A **sketch** on $\mathscr{C}$ is a pair $(\mathscr{C}, \mathsf{C})$ where C is a family of finitely generated sieves[2] on the objects of $\mathscr{C}$. A **model** for $(\mathscr{C}, \mathsf{C})$ is a functor $X : \mathscr{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ such that, for each sieve

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[1] The metaphor is that an $\overline{f} \in X\left(d\right)$ is a hole through which $\overline{f \circ g}$ fits.

[2] by finitely generated sieve we mean a sieve specified by a finite collection of maps.

$V \subset \mathscr{C}^e$ in C, the canonical map

$$X(e) \longrightarrow \lim_{\substack{\longleftarrow \\ (d \to e) \in V}} X(d)$$

is an isomorphism.

These are exactly the presheaves which "preserve" a chosen collection of sieves as colimiting sieves.

Definition 2.2.6. Let $\mathsf{Mod}(\mathscr{C}, \mathsf{C})$ denote the full subcategory of $\mathsf{Fun}(\mathscr{C}^{\mathsf{op}}, \mathsf{Set})$ subtended by those functors which are models in the sense just defined.

Theorem 2.2.7. *(Ehresman, Kennison, Kelly,...) Given a sketch $(\mathscr{C}, \mathsf{C})$ there exists a universal morphism $\mathscr{C} \longrightarrow \mathsf{LE}(\mathscr{C})$, where $\mathsf{LE}(\mathscr{C})$ is an essentially algebraic theory, such that the precomposition functor*

$$\mathsf{Lex}(\mathsf{LE}(\mathscr{C}^{\mathsf{op}}, \mathsf{C}), \mathsf{Set}) \longrightarrow \mathsf{Mod}(\mathscr{C}, \mathsf{C})$$

is an equivalence of categories.

Proof. See Theorem 4.2.2 [8]. ☐

These notions should be understood to generalize both the notion of site originally due to Grothendieck and the notion of algebraic theory due to Lawvere, both notions can be gotten as instances of this theory.

2.3 Example: sketches for an essentially algebraic theory of categories

Definition 2.3.1. For each $n \in \mathbf{N}$, let

$$[n] = \left\langle \underbrace{\bullet \to \bullet \to \cdots \to \bullet \to \bullet}_{n \ arrows} \right\rangle$$

and let the simplex category $\triangle$ be the full subcategory of Cat subtended by the objects of that form. We'll refer to the presheaves of (pointed) sets on $\triangle$ as (pointed) simplicial sets.

This subcategory admits a familiar presentation, which we record as a lemma and corollary without proof.

Lemma 2.3.2. *The category $\triangle$ is generated by the morphisms*

$$\begin{tikzcd}
d^i : [n-1] \arrow[r] & {[n]} \\
0 \arrow[r, mapsto] & 0 \\
\vdots & \vdots \\
i-1 \arrow[r, mapsto] & i-1 \\
{} \arrow[dr, mapsto] & i \\
\vdots & i+1 \\
n-1 \arrow[dr, mapsto] & \vdots \\
 & n
\end{tikzcd}
\qquad
\begin{tikzcd}
s^i : [n+1] \arrow[r] & {[n]} \\
0 \arrow[r, mapsto] & 0 \\
\vdots & \vdots \\
i-1 \arrow[r, mapsto] & i-1 \\
i \arrow[r, mapsto] & i \\
i+1 \arrow[ur, mapsto] & \vdots \\
\vdots & n \\
n+1 \arrow[ur, mapsto] &
\end{tikzcd}$$

where $0 \leq i \leq n$.

Notation 2.3.3. We may also denote a map $[n] \longrightarrow [m]$ by a non-decreasing list $\{i_1, \ldots, i_n\} \subset [m]$.

Corollary 2.3.4. *The morphisms of the simplex category $\triangle$ satisfy the* ***co-simplicial identities***

$$\begin{cases}
d^j d^i = d^i d^{j-1} & \textit{if } i < j \\
s^j d^i = d^i s^{j-1} & \textit{if } i < j \\
s^j d^j = id = s^j d^{j+1} & \\
s^j d^i = d^{i+1} s^j & \textit{if } i \geq j \\
s^j s^i = s^i s^{j+1} & \textit{if } i \leq j
\end{cases}$$

Definition 2.3.5. Let $\triangle^+$ be the wide subcategory of $\triangle$ generated by the morphisms d^i.

Perhaps the most important example of an essentially algebraic category is the category of small categories and in this section we will explore how the category $\triangle$ may be invoked to sketch an essentially algebraic theory of categories.

A small category is first a directed, reflexive graph, so any sketch for an essentially algebraic theory of Cat must admit an embedding of $\triangle_{\leq 1}$, the subcategory of $\triangle$ diagrammed below.

$$\begin{tikzcd}
{[0]} \arrow[rr, bend left, "d^1"] \arrow[rr, bend right, "d^0"'] & & {[1]} \arrow[ll, "s^0"']
\end{tikzcd}$$

The reason is that a directed reflexive graph is comprised of a set of edges, a set of vertices, functions identifying source and target vertices for each edge, and a function from the set of vertices to the set of edges specifying a set of self-edges. These data are precisely the data of a functor $X : \triangle_{\leq 1}^{\mathsf{op}} \longrightarrow \mathsf{Set}$.

Now, a category is a directed reflexive graph with a composition law, so in order that a sketch encode categories it must also have an object which corresponds to pairs of composable 1-morphisms; any sketch for an essentially algebraic theory of categories must extend $\triangle_{\leq 1}$ to include the object

$$\varinjlim \left\{ \begin{tikzcd} {[1]} & & {[1]} \\ & {[0]} \arrow[ul, "d^0"'] \arrow[ur, "d^1"] & \end{tikzcd} \right\}.$$

Lemma 2.3.6. *The object* $[2]$ *in* $\triangle$ *is the colimit*

$$\varinjlim \left\{ \begin{tikzcd} {[1]} & & {[1]} \\ & {[0]} \arrow[ul, "d^0"'] \arrow[ur, "d^1"] & \end{tikzcd} \right\}$$

taken in $\triangle$*.*

It is then that **any** sketch for an essentially algebraic theory of categories must contain the following subcategory of $\triangle$.

$$\begin{tikzcd} {[0]} \arrow[r, bend left, "d^1"] \arrow[r, bend right, "d^0"'] & {[1]} \arrow[l, "s^0"'] \arrow[r, bend left=60, "d^2"] \arrow[r, bend right=60, "d^0"'] & {[2]} \arrow[l, bend right, "s^1"'] \arrow[l, bend left, "s^0"] \end{tikzcd}$$

Since for any category we must also have a composition law, not merely a notion of consecutive arrows, we need the map $d^1 : [1] \longrightarrow [2]$ of $\triangle$ too; we need all of $\triangle_{\leq 2}$.

In order that the sketch encode the associativity of the composition we must have an object corresponding to composable triples, i.e. a colimit of the diagram

$$\varinjlim \left\{ \begin{tikzcd} {[1]} & & {[1]} & & {[1]} \\ & {[0]} \arrow[ul, "d^0"'] \arrow[ur, "d^1"] & & {[0]} \arrow[ul, "d^0"'] \arrow[ur, "d^1"] & \end{tikzcd} \right\}.$$

Lemma 2.3.7. *The object* $[3]$ *in* $\triangle$ *is the colimit*

$$\varinjlim \left\{ \begin{tikzcd} {[1]} & & {[1]} & & {[1]} \\ & {[0]} \arrow[ul, "d^0"'] \arrow[ur, "d^1"] & & {[0]} \arrow[ul, "d^0"'] \arrow[ur, "d^1"] & \end{tikzcd} \right\}$$

taken in $\triangle$.

As it turns out, the sieves suggested by these presentations are exactly those which need be preserved for a model to present a category.

Definition 2.3.8. The embedding

$$i : \triangle \longrightarrow \mathsf{Cat}$$

induces an adjunction

$$\mathfrak{R} : \rightleftarrows \mathsf{Cat} : \mathscr{N}$$

with, for any category X,

$$\mathscr{N}\left(X\right) = \mathsf{Hom}_{\mathsf{Cat}}\left(i\left(\_\right), X\right)$$

and for any simplicial set X,

$$\mathfrak{R}\left(X\right) = \varinjlim_{([n]\to X)\in\triangle\downarrow X} \left[n\right].$$

This adjunction restricts along the inclusions $\triangle_{\leq 3} \longrightarrow \triangle$ to an adjunction which we'll use to reveal our categorial universal algebraic description of categories.

Remark 2.3.9. In the following lemma, by an abuse of notation, we will let $\mathfrak{R} : \widehat{\triangle_{\leq 3}} \longrightarrow \mathsf{Cat}$ denote the functor

$$\mathfrak{R}\left(X\right) = \varinjlim_{([n]\to X)\in\triangle_{\leq 3}\downarrow X} \left[n\right].$$

Lemma 2.3.10. *Let* $\mathsf{C}_{\mathsf{Cat}}$ *be the set of sieves on* $\triangle_{\leq 3}$ *generated by the finite families of maps*

$$\begin{tikzcd}
{[1]} \arrow[dr, "d^2"] & & {[1]} \arrow[dl, "d^0"'] \\
& {[2]} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[1]} \arrow[dr, "\{0{,}1\}"'] & {[1]} \arrow[d, "\{1{,}2\}"] & {[1]} \arrow[dl, "\{2{,}3\}"] \\
& {[3]} &
\end{tikzcd} .$$

Then, the composition

$$\mathsf{Mod}\left(\triangle_{\leq 3}, \mathsf{C}_{\mathsf{Cat}}\right) \longrightarrow \widehat{\triangle_{\leq 3}} \xrightarrow{\mathfrak{R}} \mathsf{Cat}$$

is an equivalence of categories. More, for each model

$$X : \triangle_{\leq 3}^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

the category $\mathfrak{R}(X)$ *may be given thus:*

$$\mathsf{Ob}\left(\mathfrak{R}(X)\right) = X\left([0]\right),$$

$$\mathsf{Mor}\left(\mathfrak{R}(X)\right) = X\left([1]\right),$$

and the composition law

$$\mathsf{Mor}\left(\mathfrak{R}(X)\right) \underset{\mathsf{Ob}(\mathfrak{R}(X))}{\times} \mathsf{Mor}\left(\mathfrak{R}(X)\right) \longrightarrow \mathsf{Mor}\left(\mathfrak{R}(X)\right)$$

being the composition $X([1]) \underset{X([0])}{\times} X([1]) \xrightarrow{\sim} X([2]) \xrightarrow{d_1} X([1])$.

To extend this equivalence of $\mathsf{Mod}\left(\triangle_{\leq 3}, \mathsf{C}_{\mathsf{Cat}}\right)$ and Cat along the functor

$$\widehat{\triangle} \longrightarrow \widehat{\triangle_{\leq 3}}$$

we need to expand the set of sieves which must be preserved.

Definition 2.3.11. Given $[n]$ and $[m]$ in $\triangle$, let the 0-**globular sum** of $[n]$ and $[m]$, denoted $[n] \underset{0}{\oplus} [m]$, be defined to be the colimit

$$\varinjlim \left\{ \begin{tikzcd} {[n]} & & {[m]} \\ & {[0]} \arrow[ul] \arrow[ur] & \end{tikzcd} \right\}$$

where the maps are as the final and initial 0-simplices of $[n]$ and $[m]$ respectively.

Lemma 2.3.12. *The objects of the category $\triangle$ are generated under 0-globular sums by objects in $\triangle_{\leq 1}$, and the category is generated by $\triangle_{\leq 2}$, hence by $\triangle_{\leq 3}$.*

Proof. Given the generation of the maps of $\triangle$ by the face and degeneracy maps it suffices to describe those maps as arising from 0-globular sums. The proof proceeds by induction. Suppose that all $\triangle^{+}$ maps into $[n]$ have been realized as finite pushouts of maps from $\triangle_{\leq 2}$. Consider the coface maps

$$d^i : [n] \longrightarrow [n+1]\,.$$

If $i = 0$ then d^i is the pushout

$$\begin{tikzcd} {[0]} \arrow[r, "d^0"] \arrow[d, "\{0\}"] & {[1]} \arrow[d] \\ {[n]} \arrow[r, "d^0"] \arrow[ur, phantom, "\lrcorner", very near end] & {[n+1]} \end{tikzcd}$$

where by $\{0\}$ we mean the map $[0] \longrightarrow [n]$ which assigns the element $0 \in [0]$ to the element $0 \in [n]$. If $i = n+1$ then d^i is the pushout

$$\begin{tikzcd} {[0]} \arrow[r, "d^1"] \arrow[d, "\{n\}"] & {[1]} \arrow[d] \\ {[n]} \arrow[r, "d^{n+1}"] & {[n+1]} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} .$$

Lastly, if $0 < i < n+1$ then d^i can be got by the pushout

$$\begin{tikzcd} {[i]} \arrow[r, "d^i"] \arrow[d, "\{0{,}1{,}\ldots{,}i\}"] & {[i+1]} \arrow[d] \\ {[n]} \arrow[r, "d^i"] & {[n+1]} \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

which suffices by the induction hypothesis. Similar arguments demonstrate that the codegeneracy maps are likewise generated. ☐

Remark 2.3.13. Note that $d^1 : [1] \longrightarrow [2]$ is actually the only map $[1] \longrightarrow [2]$ which is not a pushout of a map $[0] \longrightarrow [1]$.

Definition 2.3.14. Given $n \in \mathbf{N}$, let $V^n \subset \triangle^n$ be the sieve generated the family of maps

$$\begin{tikzcd}
{[1]} \arrow[dr, "\{0{,}1\}"'] & \cdots & {[1]} \arrow[dl, "\{n-1{,}n\}"] \\
 & {[n]} &
\end{tikzcd}.$$

Let $\mathsf{V}_1 = \{V^n \subset \triangle^n\}_{n \in \mathbf{N}}$.

Remark 2.3.15. The sieves of V_1 are exactly the sieves which correspond to the 0-globular presentation of each $[n]$, i.e. is the canonical isomorphisms $[1] \underset{0}{\oplus} \cdots \underset{0}{\oplus} [1] \xrightarrow{\sim} [n]$.

Lemma 2.3.16. *The inclusion* $\triangle_{\leq 3} \longrightarrow \triangle$ *induces an equivalence of categories*

$$\mathsf{Mod}\,(\triangle, \mathsf{V}_1) \xrightarrow{\sim} \mathsf{Mod}\,(\triangle_{\leq 3}, \mathsf{C}_{\mathsf{Cat}}),$$

whence

$$\mathsf{Mod}\,(\triangle, \mathsf{V}_1) \xrightarrow{\sim} \mathsf{Cat}.$$

Remark 2.3.17. After we've introduced the technology of Cisinski model category theory, we'll use this to explain why, from a categorial model theoretic position, quasi-categories present $(\infty, 1)$-categories.

Chapter 3

The Categorical Wreath Product and the Cell Categories Θ_n and Θ

In this section we provide a development of Joyal's categories Θ_n and Θ by way of Berger's wreath product of categories. The presentation here is adapted from [19]. In the previous chapter we developed the theory of essentially algebraic theories. In particular, we developed $(\triangle, \mathsf{V}_1)$ as a sketch for the essentially algebraic theory of small categories. In this chapter we'll continue this practice and develop sketches for the essentially algebraic theory of strict n-categories and strict ω-categories.

3.1 Segal's category Γ

Segal's Γ category is a presentation of the opposite category of the category of finite pointed sets. As such, the objects of Γ will be choices of finite pointed sets while the morphisms of Γ will be given as parameterizations of the fibers of a function in the other direction. Of course, we will abstract the properties of the parametrization of subsets which correspond to the fibers of a map, in place of making reference to any function in the opposite direction.

Definition 3.1.1. Let Γ, **Segal's gamma category**, be the category specified thus: let

$$\mathsf{Ob}\left(\Gamma\right) = \left\{ \langle k \rangle = \{1, \ldots, k\} \middle| \, k \geq 1 \right\} \cup \left\{ \langle 0 \rangle = \varnothing \right\},$$

and let $\mathsf{Hom}_{\Gamma}\left(\langle n \rangle, \langle m \rangle\right)$ be defined by the expression

$$\mathsf{Hom}_{\Gamma}\left(\langle n \rangle, \langle m \rangle\right) = \left\{ \varphi : \langle n \rangle \longrightarrow \mathsf{Sub}_{\mathsf{Set}}\left(\langle m \rangle\right) \middle| \, \forall i \neq j \in \langle m \rangle \, , \; \varphi\left(i\right) \cap \varphi\left(j\right) = \varnothing \right\}$$

where, for any category A and object a thereof, $\mathsf{Sub}_A(a)$ is the category of subobjects of a. Define the composition of morphisms in Γ by setting

$$\langle \ell \rangle \xrightarrow{\varphi} \langle m \rangle \xrightarrow{\sigma} \langle n \rangle$$

to be the map

$$\sigma \circ \varphi : i \longmapsto \bigcup_{j \in \varphi(i)} \sigma(j).$$

Lemma 3.1.2. *Let H be the functor*

$$\begin{aligned} H : \mathsf{FinSet}_\bullet^{\mathsf{op}} &\longrightarrow \Gamma \\ \langle m \rangle_+ &\longmapsto \langle m \rangle \\ \left(f : \langle m \rangle_+ \to \langle n \rangle_+\right) &\longmapsto H(f) : \langle n \rangle \to \langle m \rangle : i \mapsto f^{-1}(i) \end{aligned}$$

where $\langle m \rangle_+$ denotes the application of the disjoint basepoint functor of the set $\langle m \rangle$. Let G be the functor

$$\begin{aligned} G : \Gamma^{\mathsf{op}} &\longrightarrow \mathsf{FinSet}_\bullet \\ \langle m \rangle &\longmapsto \langle m \rangle_+ \end{aligned}$$

with, for any $f : \langle m \rangle \longrightarrow \langle n \rangle$, the function $G(f) : \langle n \rangle_+ \longrightarrow \langle m \rangle_+$ being given by the formula

$$G(f)(i) = \begin{cases} k & \text{if } i \in f(k) \\ \bullet & \text{o.w.} \end{cases}.$$

Then the functors H and G comprise an equivalence of categories.

Proof. The proof is formal and left to the reader. $\square$

This lemma is often given as the definition of Γ with our presentation given as a lemma, as in [5]. It should also be noted that one may restrict this equivalence of categories to an isomorphism of categories by taking a skeleton of $\mathsf{FinSet}_\bullet$, i.e. taking the full subcategory subtended by, for each isomorphism class of objects in $\mathsf{FinSet}_\bullet$, a choice of object. Implicitly we have done so in the manner we have written the functor H.

We now recall without proof a well known fact.

Lemma 3.1.3. *The category* $\mathsf{FinSet}_\bullet$ *has all finite limits and all finite colimits, and the coproduct is described as follows.*

$$\langle m\rangle_+ \coprod \langle n\rangle_+ = \langle m\rangle_+ \vee \langle n\rangle_+ = \langle m+n\rangle_+$$

Not only does $\mathsf{FinSet}_\bullet$ have all finite colimits, but there is a well known formula for them. For any finite diagram $X : D \longrightarrow \mathsf{FinSet}_\bullet$ we have an isomorphism

$$\varinjlim X \xrightarrow{\sim} \bigvee\nolimits_{d\in\mathsf{Ob}(D)} X(d)/\sim$$

where $\sim$ is the equivalence relation on $\bigvee_{d\in\mathsf{Ob}(D)} X(d)$ generated by the identification of elements and their images. More explicitly, we set $x \in X(i) \sim y \in X(j)$ if there there exist objects $b_1, \ldots, b_n, t_1, \ldots, t_{n-1} \in \mathsf{Ob}(D)$, and morphisms $\ell_1, \ldots, \ell_n, s_1, \ldots, s_n$ as in the diagram

$$\begin{tikzcd}
X_i & & X(t_1) & & X(t_{n-1}) & & X_n \\
& X(b_1) \arrow[ul, "X(\ell_1)"] \arrow[ur, "X(s_1)"'] & & \cdots \arrow[ul] \arrow[ur] & & X(b_n) \arrow[ul, "X(\ell_n)"] \arrow[ur, "X(s_n)"'] &
\end{tikzcd}$$

as well as elements $w_1 \in X(b_1), \ldots, w_n \in X(b_n)$ such that

$$\begin{tikzcd}
x & & X(s_1)(w_1) = X(\ell_2)(w_2) & & X(s_{n-1})(w_{n-1}) = X(\ell_n)(w_n) & & y \\
& w_1 \arrow[ul, mapsto, "X(\ell_1)"] \arrow[ur, mapsto, "X(s_1)"'] & & \cdots \arrow[ul, mapsto] \arrow[ur, mapsto] & & w_n \arrow[ul, mapsto] \arrow[ur, mapsto] &
\end{tikzcd}.$$

This formula translates easily across the equivalence of categories $H : \mathsf{FinSet}_\bullet^{\mathsf{op}} \longrightarrow \Gamma$; in Γ, finite limits may be computed as the subobjects of products satisfying a condition. First however, we must be clear what subobjects in Γ are.

It is immediate from the equivalence of categories of Lemma 3.1.2 that the monomorphisms of Γ are precisely the maps of the form $H(f) : \langle m\rangle \longrightarrow \langle n\rangle$ where $f : \langle n\rangle_+ \longrightarrow \langle m\rangle_+$ is an epimorphism of $\mathsf{FinSet}_\bullet$. This property can then be stated entirely in terms of Γ as in the following lemma.

Lemma 3.1.4. *A morphism* $f : \langle m\rangle \longrightarrow \langle n\rangle$ *of* Γ *is a monomorphism if and only if*

$$\bigcup_{i\in\langle m\rangle} f(i) = \langle n\rangle .$$

As such we may identify subobjects and disjoint coverings. The coverings of interest for computing finite limits are dual to the equivalence relation which identify an element and its image.

Lemma 3.1.5. *Given a finite diagram*

$$\begin{tikzcd}
X : D \arrow[r] & \Gamma \\
d \arrow[r, mapsto] & \langle m_d \rangle \\
(f : d \to e) \arrow[r, mapsto] & (X(f) : \langle m_d \rangle \to \langle m_e \rangle)
\end{tikzcd}$$

then the canonical map

$$\varprojlim X \longrightarrow \prod_{d \in \mathsf{Ob}(D)} \langle m_d \rangle = \left\langle \sum_{d \in \mathsf{Ob}(D)} m_d \right\rangle$$

is the subobject corresponding to the covering of $\left\langle \sum_{d \in \mathsf{Ob}(D)} m_d \right\rangle$ *by the minimal non-empty subsets*

$$U \subset \left\langle \sum_{d \in \mathsf{Ob}(D)} m_d \right\rangle$$

which enjoy the following closure properties:

- *for all* $x \in U$ *and for all* $f \in \mathsf{Mor}(D)$, *if* $x \in X(f)(y)$, *then* $y \in U$; *and*
- *if* $x \in U$, *then for all* $f \in \mathsf{Mor}(D)$, $X(f)(x) \subset U$.

Remark 3.1.6. The careful reader will note that we make no hypotheses such as $y \in \mathsf{Dom}(X(f))$ in the statement of the closure properties above. The reason we may forgo such qualifications is that $X(f) : \mathsf{Dom}(X(f)) \longrightarrow \mathsf{Cod}(X(f))$ naturally extends along the map $\mathsf{Dom}(X(f)) \longrightarrow \left\langle \sum_{d \in \mathsf{Ob}(D)} m_d \right\rangle$ since

$$\mathsf{Dom}(X(f)) \longrightarrow \left\langle \sum_{d \in \mathsf{Ob}(D)} m_d \right\rangle \longrightarrow \mathsf{Dom}(X(f))$$

composes to the identity.

Proof. This follows from the colimit formula for $\mathsf{FinSet}_\bullet$ and the fact that, for $f : \langle m \rangle_+ \longrightarrow \langle n \rangle_+$, $x \in \langle m \rangle \subset \langle m \rangle_+$, and $y \in \langle n \rangle \subset \langle n \rangle_+$, $f(x) = y$ if and only if $x \in f^{-1}(y) = H(f)(y)$. □

3.2 The Categorical wreath product

Definition 3.2.1. Let A and B be small categories. Given a functor $G : B \longrightarrow \Gamma$, we define $B \int_G A = B \int A$, with the second notation suppressing the functor G when the meaning is clear, to be the category whose objects are pairs

$$[b; (a_1, \ldots, a_m)]$$

where b is an object of B, $G(b) = \langle m \rangle$, and $(a_1, \ldots, a_m)$ describes a function $G(b) \longrightarrow \mathsf{Ob}(A)$. The morphisms of $B \int A$, denoted

$$[g; \mathbf{f}] : \left[b; (a_i)_{i \in G(b)}\right] \longrightarrow \left[d; (c_i)_{i \in G(d)}\right]$$

are comprised of a morphism

$$g : b \longrightarrow d$$

of B and a morphism of $\widehat{A}$,

$$\mathbf{f} = \left((f_{ji} : a_i \to c_j)_{j \in G(g)(i)}\right)_{i \in G(b)} : \prod_{i \in G(b)} \left(A^{a_i} \longrightarrow \prod_{j \in G(g)(i)} A^{c_j}\right).$$

The composition

$$\left[b; (a_i)_{i \in G(b)}\right] \xrightarrow{[g;\mathbf{f}]} \left[d; (c_i)_{i \in G(d)}\right] \xrightarrow{[r;\mathbf{q}]} \left[\ell; (k_i)_{i \in G(\ell)}\right]$$

is denoted $[r \circ g; \mathbf{q} \circ \mathbf{f}]$ where the meaning of $r \circ g$ is clear and

$$\mathbf{q} \circ \mathbf{f} = \left((q_{jk} \circ f_{ki})_{j \in G(r \circ g)(i)}\right)_{i \in G(b)}$$

with the values for $k \in G(d)$ being those unique k in $G(g)(i)$ such that $j \in G(r)(k)$.

Example 3.2.2. We define a functor $F : \triangle \longrightarrow \Gamma$ by setting

$$F([n]) = \langle n \rangle$$

and setting for each $\varphi : [m] \longrightarrow [n]$,

$$F(\varphi) : \langle m \rangle \longrightarrow \langle n \rangle$$

to be the function

$$F(\varphi) : \langle m \rangle \longrightarrow \mathsf{Sub}_{\mathsf{Set}}(\langle n \rangle)$$

given thus:

$$F(\varphi)(i) = \{ j |\, \varphi(i-1) < j \leq \varphi(i) \} .$$

Consider then the category $\triangle \int_F \triangle = \triangle \int \triangle$ and observe that we may sketch the object $[[1];[0]]$ as

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

and the object [[1];[[1];[0]]] as

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

with the morphisms between them being those we expect from any definition of higher categories.

Definition 3.2.3. The wreath construction is functorial in both variables, that is, we may define

$$(\_) \int (\_) : (\mathsf{Cat} \downarrow \Gamma) \times \mathsf{Cat} \longrightarrow \mathsf{Cat}$$

to be the functor taking a pair $(X \longrightarrow \Gamma, A)$ to $X \int A$ and sending a pair

$$\left(\begin{tikzcd} X \arrow[rr, "H"] \arrow[dr, "F"'] & & Y \arrow[dl, "G"] \\ & \Gamma & \end{tikzcd}, P : A \longrightarrow B \right)$$

to the functor

$$H \int P : X \int A \longrightarrow Y \int B$$

which sends an object $\left[x; (a_i)_{i \in F(x)}\right]$ to the object $\left[H(x); (P(a_i))_{i \in F(x) = G \circ H(x)}\right]$.

Now, given the definition of the wreath product $B \int A$ by way of $\widehat{A}$, it follows that the functor

$$\mathrm{id}_B \int \mathsf{Yon} : B \int A \longrightarrow B \int \widehat{A}$$

enjoys some of the nice properties of the Yoneda embedding.

Lemma 3.2.4. *Given $G : B \longrightarrow \Gamma$ a functor and A a small category, then the functor*

$$id_B \int \mathsf{Yon} : B \int A \longrightarrow B \int \widehat{A}$$

is a limit preserving full and faithful embedding.

Proof. To prove that id $\int$ Yon is a fully faithful embedding see that the set

$$\mathsf{Hom}_{B \int A}\left(\left[b; (a_i)_{i \in G(b)}\right], \left[d; (c_k)_{k \in G(d)}\right]\right)$$

is by definition isomorphic to the set

$$\coprod_{f \in \mathsf{Hom}_B(b,d)} \coprod_{i \in G(b)} \mathsf{Hom}_{\widehat{A}}\left(A^{a_i}, \prod_{j \in G(f)(i)} A^{c_j}\right)$$

which is isomorphic to

$$\mathsf{Hom}_{B \int A}\left(\left[b; (A^{a_i})_{i \in G(b)}\right], \left[d; (A^{c_k})_{k \in G(d)}\right]\right).$$

For the promised limit preservation, suppose

$$\left[b; (a_i)_{i \in G(b)}\right] \xrightarrow{\sim} \varprojlim \left[d^j; \left(c_k^j\right)_{k \in G(d^j)}\right]$$

for some diagram

$$\left[d^{(\ )}; \left(c_k^{(\_)}\right)_{k \in G\left(d^{(\_)}\right)}\right] : J \longrightarrow B \int A.$$

Then, by the isomorphism discovered above, we find that

$$\begin{aligned}
\mathsf{Hom}_{B \int \widehat{A}}(\_, [b; (A^{a_i})]) &\xrightarrow{\sim} \\
\mathsf{Hom}_{B \int A}(\_, [b; (a_i)]) &\xrightarrow{\sim} \mathsf{Hom}_{B \int A}\left(\_, \varprojlim \left[d^j; \left(c_k^j\right)_{k \in G(d^j)}\right]\right) \\
&\xrightarrow{\sim} \varprojlim \mathsf{Hom}_{B \int A}\left(\_, \left[d^j; \left(c_k^j\right)_{k \in G(d^j)}\right]\right) \\
&\xrightarrow{\sim} \varprojlim \mathsf{Hom}_{B \int \widehat{A}}\left(\_, \left[d^j; \left(A^{c_k^j}\right)_{k \in G(d^j)}\right]\right) \\
&\xrightarrow{\sim} \mathsf{Hom}_{B \int \widehat{A}}\left(\_, \varprojlim \left[d^j; \left(A^{c_k^j}\right)_{k \in G(d^j)}\right]\right)
\end{aligned}$$

and id $\int$ Yon is found to preserve limits. □

Remark 3.2.5. While we have shown $\mathrm{id} \int \mathsf{Yon} : B \int A \longrightarrow B \int \widehat{A}$ to be a limit preserving fully faithful embedding, unlike the Yoneda embedding, this functor does not in general land in a category with all limits and colimits. Indeed, the absence of products in $\triangle$ implies that, at least in general, $\triangle \int \widehat{A}$ does not have products. However, the inclusion $\triangle \int \widehat{A}$ does add some interesting limits, in particular, the equalizer of $d^0 : [0] \longrightarrow \left[[1] \,; \triangle^0\right]$ and $d^1 : [0] \longrightarrow \left[[1] \,; \triangle^0\right]$ exists in $\triangle \int \widehat{\triangle}$, it is $[[1] \,; \varnothing_{\triangle}]$. We'll consider questions related to this in great detail in the appendix.

3.3 The Categories Θ_n

Definition 3.3.1. Let

$$\gamma : \triangle \longrightarrow \triangle \int \triangle$$

be the functor extending the assignment $\gamma\left([n]\right) = [[n] \,; [0] \cdots [0]]$. Note that this functor is an embedding. We may then define the categories Θ_n to be the n^{th} wreath product of $\triangle$ with itself,

$$\Theta_n = \underbrace{\triangle \wr (\cdots \wr \triangle)}_{n}.$$

We also set Θ equal to the colimit

$$\varinjlim \left\{ \triangle \xrightarrow{\gamma} \triangle \wr \triangle \xrightarrow{\gamma} \cdots \right\}.$$

The category Θ then also admits a filtration:

$$\triangle = \Theta_1 \hookrightarrow \triangle \int \triangle = \Theta_2 \hookrightarrow \cdots \Theta.$$

It should also be noted that

$$\Theta \xrightarrow{\sim} \triangle \int \Theta \xrightarrow{\sim} \triangle \int \triangle \int \Theta \xrightarrow{\sim} \cdots$$

so we may denote cells, the objects of Θ, in many compatible ways. For example for any T a cell of Θ we may also write $T = [[n] \,; T_1, \ldots, T_n]$ for some unique $n \in \mathbf{N}$ and unique $T_1, \ldots, T_n$ cells of Θ.

Definition 3.3.2. Let $\overline{n}$ be the object of Θ

$$\underbrace{[[1] : [[1] \,; \cdots [[1] \,; [0]] \cdots]]}_{n \text{ ones}}$$

let $s : \overline{n} \longrightarrow \overline{n+1}$ be the morphism, $\big[\mathrm{id}; \big[\mathrm{id}; \cdots \big[d^1\big]\big]\big]$, let $t : \overline{n} \longrightarrow \overline{n+1}$ be the morphism, $\big[\mathrm{id}; \big[\mathrm{id}; \cdots \big[d^0\big]\big]\big]$, and let $i : \overline{n+1} \to \overline{n}$ be the morphism $\big[\mathrm{id}; \big[\mathrm{id}; \cdots \big[s^0;\big]\big]\big]$. Let $\mathbf{G}$ be the full subcategory of Θ subtended by the objects of the form $\overline{n}$.

With notation for the globes in hand, the recursive decomposition suggested by the observation that for any T a cell of Θ_ℓ we may also write $T = [[n]\,; T_1, \ldots, T_n]$ for some unique $n \in \mathbf{N}$ and unique $T_1, \ldots, T_n$ cells of $\Theta_{\ell-1}$, can be carried out to provide a useful canonical representation of a cell T in terms of colimits computed in Θ.

Lemma 3.3.3. *Given any object T of Θ, there exists a unique list of non-negative integers,*

$$n_0, m_1, n_1, \ldots, n_{\ell-1}, m_{\ell-1}, n_\ell$$

with each

$$m_i \leq n_{i-1}, n_i,$$

such that

$$\varinjlim \left\{ \begin{tikzcd} \overline{n_0} & & \overline{n_1} & & \overline{n_{\ell-1}} & & \overline{n_\ell} \\ & \overline{m_1} \arrow[ul, "s^{n_1-m_1}"] \arrow[ur, "t^{n_0-m_1}"'] & & \cdots \arrow[ul] \arrow[ur] & & \overline{m_{\ell-1}} \arrow[ul, "s^{n_\ell-m_{\ell-1}}"] \arrow[ur, "t^{n_{\ell-1}-m_{\ell-1}}"'] & \end{tikzcd} \right\} \xrightarrow{\sim} T.$$

Remark 3.3.4. It is important to note that the colimit in this lemma is taken in Θ and **not** in $\widehat{\Theta}$. In work of Ara this description of a cell is known as the **globular sum** presentation. It is from careful consideration of this presentation of the cells of Θ that Ara proves the universality of Θ among categories wherein we may **compose** globes.

Example 3.3.5. As an easy example, see that in $\triangle \hookrightarrow \Theta$ the globular sum presentation of $[n]$ is as

$$\varinjlim \left\{ \begin{tikzcd} \overline{1} & & \overline{1} & & \overline{1} & & \overline{1} \\ & \overline{0} \arrow[ul] \arrow[ur] & & \cdots \arrow[ul] \arrow[ur] & & \overline{0} \arrow[ul] \arrow[ur] & \end{tikzcd} \right\} \xrightarrow{\sim} [n]\,.$$

For an example from Θ_2 consider $[[2]\,;[1]\,[0]]$ which has the globular sum presentation

$$\varinjlim \left\{ \begin{tikzcd} \overline{2} & & \overline{1} \\ & \overline{0} \arrow[ul] \arrow[ur] & \end{tikzcd} \right\} \xrightarrow{\sim} [[2]\,;[1]\,[0]]\,.$$

Proof. First, we provide a function A from the set

$$\{\,(n_0, m_1, \dots, m_{\ell-1}, n_\ell) |\, n_i, m_j \in \mathbf{N}, \forall 1 \le i \le \ell,\ \ m_i \le n_{i-1}, n_i\}$$

to the set of objects of Θ and then prove that the cells $A\,(n_0, \dots, n_\ell)$ together with natural inclusions enjoy the universal property of the requisite colimit.

The function A is defined recursively. Let

$$Z = Z\,(n_0, m_1, \dots, m_{\ell-1}, n_\ell)$$

be the ordered set of indices

$$[i_1 < i_2 < \cdots < i_k] = [1 \le i \le \ell - 1 |\, m_i = 0]\,.$$

Then $A\,(n_0, m_1, \dots, m_{\ell-1}, n_\ell)$ is the tree

$$\left[\begin{array}{ll} & A\,(n_0 - 1, m_1 - 1, \dots, m_{i_1 - 1} - 1, n_{i_1} - 1) \\ [1 + |Z|]\,; & \vdots \\ & A\,(n_{i_k + 1} - 1, m_{i_k + 1} - 1, \dots, m_{\ell - 1} - 1, n_\ell - 1) \end{array} \right]$$

where the right hand side is interpreted in $\triangle \int \Theta \xrightarrow{\sim} \Theta$.

To verify that A indeed computes the requisite colimits observe that

$$A\,(n_0, 0, n_1) = \left[[1+1]\,; \overline{n_0 - 1}\ \overline{n_1 - 1}\right]$$

which enjoys the universal property of the colimit

$$\varinjlim \left\{ \begin{tikzcd} \overline{n_0} & & \overline{n_1} \\ & \overline{0} \arrow[ul, "\sigma^{n_0}"'] \arrow[ur, "\tau^{n_1}"'] & \end{tikzcd} \right\}.$$

It then follows by recursion that A computes the colimit correctly. $\square$

3.4 Sketches for essentially algebraic theories of strict n-categories and strict ω-categories

In section 2.3 we developed the sketch $(\triangle, \mathsf{V}_1)$ for the theory of strict 1-categories. The sieves generated by the families $\coprod_n [1] \longrightarrow [n]$ encoded the notion that an n-simplex ought be specified uniquely by a composable string of n-one simplices. We studied how this notion corresponds to the associative composition law for categories. In strict higher categories there are of course higher compositions and higher associativities to be encoded, e.g. the whiskering of a 1-morphism by a 2-morphism or Godemont's middle four interchange law. As it happens, the categories Θ and the globular presentations are specifically adapted to this purpose.

3.4.1 Strict higher categories

Definition 3.4.1. We will define the notions of strict n-category and strict n-functor by induction on n. Let a **strict n-category** be comprised of:

- a set of objects $\mathsf{Ob}\,(X)$;
- for each pair $(x, y) \in \mathsf{Ob}\,(X)^2$ of objects of X, an $(n-1)$-category $\mathsf{Map}_X\,(x, y)$;
- for each triple $(x, y, z) \in \mathsf{Ob}\,(X)^3$ of objects of X, a strict $(n-1)$-functor

  $$\circ_0 : \mathsf{Map}_X\,(y, z) \times \mathsf{Map}_X\,(x, y) \longrightarrow \mathsf{Map}_X\,(x, z)\,;$$

  and
- for each object $x \in \mathsf{Ob}\,(X)$, a distinguished object $\mathrm{id}_x \in \mathsf{Map}_X\,(x, x)$;

which satisfy the expected associativity and unit axioms.

Given two strict n-categories X and Y, a **strict n-functor** $F : X \longrightarrow Y$ is comprised of:

a function $F_0 : \mathsf{Ob}\,(X) \longrightarrow \mathsf{Ob}\,(Y)$; and

for every pair $(x, y) \in \mathsf{Ob}\,(X)^2$, a strict $(n-1)$-functor

$$F_{x,y} : \mathsf{Map}_X\,(x, y) \longrightarrow \mathsf{Map}_Y\,(F_0\,(x)\,, F_0\,(y))\,;$$

which satisfy the expected composition and unit preservation axioms. Denote the category of strict n-categories and strict n-functors by $n - \mathsf{Cat}$.

3.4.2 Cellular nerves and wreath products

Recall that in section 2.3 we used the (defining) embedding $\triangle \longrightarrow \mathsf{Cat}$ to define the nerve realization adjunction

$$\mathfrak{R} : \rightleftarrows \mathsf{Cat} : \mathscr{N} .$$

The wreath product and that same embedding $\triangle \longrightarrow \mathsf{Cat}$ will allow us to define embeddings

$$\Theta_n \longrightarrow n - \mathsf{Cat}.$$

Suppose A to be a cartesian monoidal category. Then, there is a functor

$$i : \triangle \int A \longrightarrow \mathsf{Cat}^A$$

where Cat^A is the category of A-enriched categories. This functor assigns to $[[n] \, ; (a_1, \ldots, a_n)]$ the A-category $i\left([[n] \, ; (a_1, \ldots, a_n)]\right)$ with

$$\mathsf{Ob}\left(i\left([[n] \, ; (a_1, \ldots, a_n)]\right)\right) = [n]$$

and

$$\mathsf{Hom}_{i([[n];(a_1,\ldots,a_n)])}(j,k) = \begin{cases} \prod_{j<\ell\leq k} a_\ell & j \leq k \\ \varnothing & j \not\leq k \end{cases} .$$

Lemma 3.4.2. *The functor*

$$i : \triangle \int A \longrightarrow \mathsf{Cat}^A$$

is a full and faithful embedding.

Proof. We'll provide a sketch of the proof. It is easily seen that the functions from

$$\mathsf{Hom}_{\triangle \int A}\left([[n] \, ; (a_1, \ldots, a_n)] \, , [[m] \, ; (b_1, \ldots, b_m)]\right)$$

to

$$\mathsf{Hom}_{\mathsf{Cat}^A}\left(i\left([[n]\,;(a_1,\ldots,a_n)]\right),i\left([[m]\,;(b_1,\ldots,b_m)]\right)\right),$$

which comprise the morphism part of the functor i are injective; we'll argue that in fact they are surjective too.

See that an A-functor

$$G : i\left([[n]\,;(a_1,\ldots,a_n)]\right) \longrightarrow i\left([[m]\,;(b_1,\ldots,b_m)]\right)$$

subsumes the data of a function of sets

$$G_0 : [n] \longrightarrow [m]\,.$$

But, since $\mathsf{Hom}_{i([[m];(b_1,\ldots,b_m)])}(j,k) = \varnothing$ whenever $j \nleq k$, in fact G_0 is a morphism of $\triangle$, and what's more, it then follows that the A-parts of G are exactly the data of a tuple of maps of A,

$$\left(a_i \longrightarrow \prod_{j\in F(G_0)(i)}\right)_{i\in\langle n\rangle}.$$

□

Definition 3.4.3. Let

$$\mathfrak{R}_n : \widehat{\Theta_n} \rightleftarrows n - \mathsf{Cat} : \mathscr{N}_n$$

be the adjunctions, where for any strict n-category X, the Θ_n-set $\mathscr{N}_n(X)$ is defined as

$$\mathscr{N}_n(X) = \mathsf{Hom}_{n-\mathsf{Cat}}\left(i\left(\_\right),X\right)$$

and for any Θ_n-set X, the strict n-category $\mathfrak{R}_\mathfrak{n}(\mathfrak{X})$ is defined as

$$\mathfrak{R}_n(X) = \varinjlim_{(\Theta^T\to X)\in\Theta_n\downarrow X} i(T)\,.$$

Let

$$\mathfrak{R}_\omega : \widehat{\Theta} \rightleftarrows \omega - \mathsf{Cat} : \mathscr{N}_\omega$$

be the adjunction defined likewise between $\widehat{\Theta}$ and $\omega - \mathsf{Cat}$.

Then, just as we did with simplicial sets and categories, we may ask which n-cellular sets arise as the nerves of strict n-categories.

Definition 3.4.4. Let

$$\mathsf{V}_n = \left\{V^T \subset \Theta_n^T\right\}_{T \in \mathsf{Ob}(\Theta_n)}$$

be the sets of sieves associated to the globular presentations of the objects of Θ_n, and likewise let

$$\mathsf{V} = \left\{V^T \subset \Theta^T\right\}_{T \in \mathsf{Ob}(\Theta)}.$$

Proposition 3.4.5. *(Ara, Berger) For each $n \in \mathbf{N}$, the nerve functors*

$$\mathscr{N}_n : n - \mathsf{Cat} \longrightarrow \widehat{\Theta_n}$$

induce equivalences of categories

$$n - \mathsf{Cat} \xrightarrow{\sim} \mathsf{Mod}\left(\Theta_n, \mathsf{V}_n\right).$$

More, the ω-nerve

$$\mathscr{N}_\omega : \omega - \mathsf{Cat} \longrightarrow \widehat{\Theta}$$

induces an equivalence of categories

$$\omega - \mathsf{Cat} \xrightarrow{\sim} \mathsf{Mod}\left(\Theta, \mathsf{V}\right).$$

Proof. Though the presentation is different, the proof in the ω-case is to be found as Proposition 3.14 of [3] and the other cases fall out m.m. □

Chapter 4

Model Category Theory

In this chapter we will develop the theory of model categories. Originally found in [43], we will develop this theory by way of a more modern presentation found in [45].

4.1 Lifting Problems

Definition 4.1.1. A **lifting problem** in a category $\mathscr{C}$ is a commutative square.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\ell"'] & B \arrow[d, "r"] \\
C \arrow[r, "v"] & D
\end{tikzcd}$$

A **solution to a lifting problem** is a diagonal morphism $s : C \longrightarrow D$ which fits into the lifting problem resulting in a commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\ell"'] & B \arrow[d, "r"] \\
C \arrow[r, "v"] \arrow[ur, "s"] & D
\end{tikzcd}$$

If a solution to the lifting problem exists, we'll say that u **extends along** ℓ and v **lifts along** r. Given fixed ℓ and r there may be many possible lifting problems, if for all lifting problems with fixed ℓ and r there exists a solution then we'll say ℓ **enjoys the left lifting property with respect to** r and similarly we'll sat the r **enjoys the right lifting property with respect to** ℓ. We'll denote that situation by $\ell \pitchfork r$.

Quillen's great realization about homotopy theory that is the heart of his development of model category theory is that all of homotopy theory can be understood by way of lifting problems.

Definition 4.1.2. Given a class of morphism A of a category $\mathscr{C}$ we'll denote by ${}^{\pitchfork}A$, the A-**projectives**, the class of morphisms of $\mathscr{C}$ which enjoy the left lifting property with respect to all morphisms in A and by $A^{\pitchfork}$, the A-**injectives**, the class of all morphisms of $\mathscr{C}$ which enjoy the right lifting property with respect to all morphisms in A. If for two class L and R of morphisms of $\mathscr{C}$ we have that all morphisms in L enjoy the left lifting property with respect to morphisms in R and equivalently all morphisms in R enjoy the right lifting property with respect to all of the morphisms in L then we'll write $L \pitchfork R$.

4.2 Weak Factorization Systems

Definition 4.2.1. A **weak factorization system** $(\mathscr{L}, \mathscr{R})$ on a category $\mathscr{C}$ consists of two classes of morphisms of $\mathscr{C}$, $\mathscr{L}$ and $\mathscr{R}$, such that:

WFS1 for each $f : A \longrightarrow C$ of $\mathscr{C}$ there exists an $(\mathscr{L}, \mathscr{R})$ **factorization**, i.e. a factorization

$$\begin{tikzcd} A \arrow[r, "\ell"] \arrow[rr, bend right, "f"'] & B \arrow[r, "r"] & C \end{tikzcd}$$

with $\ell \in \mathscr{L}$ and $r \in \mathscr{R}$;

WFS2 $\mathscr{L} \pitchfork \mathscr{R}$; and

WFS3 $\mathscr{L}$ and $\mathscr{R}$ are closed under retracts.

Example 4.2.2. In Set and in fact in every topos, the pair $(\mathsf{Epi}, \mathsf{Mono})$ comprises a weak factorization system. More, these factorization systems are actually orthogonal factorization systems, weak factorization systems in which the promised lifts are unique. A shockingly large body of mathematics may be phrased as either as orthogonal factorization systems. For example, the theory of sheaves for example may be phrased as an orthogonal factorization system, and as we'll see, homotopy theory as model category theory, may be phrased as a pair of compatible weak factorization systems.

4.3 Model Categories

Definition 4.3.1. A **model category structure** on a category $\mathscr{C}$, $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ is comprised of:

RMC0 a category $\mathscr{C}$ with all limits and colimits;

and three classes of morphisms of $\mathscr{C}$, Cof, W, and Fib. These three classes of morphism are referred to as **cofibrations**, **weak equivalences**, **and fibrations** respectively. They are required to satisfy:

RMC1 W enjoys the 2-out-of-3 property; and

RMC2 $(\mathsf{Cof} \cap \mathsf{W}, \mathsf{Fib})$ and $(\mathsf{Cof}, \mathsf{W} \cap \mathsf{Fib})$ are weak factorization systems on $\mathscr{C}$.

In the context of a model category well denote a morphism $f : X \longrightarrow Y$ of class Cof by $f : X \hookrightarrow Y$, of class W by $f : X \xrightarrow{\sim} Y$, and of class Fib by $f : X \twoheadrightarrow Y$.

Given two model categories $(\mathscr{C}, \mathsf{Cof}_{\mathscr{C}}, \mathsf{W}_{\mathscr{C}}, \mathsf{Fib}_{\mathscr{C}})$ and $(\mathscr{D}, \mathsf{Cof}_{\mathscr{D}}, \mathsf{W}_{\mathscr{D}}, \mathsf{Fib}_{\mathscr{D}})$, an adjunction

$$\begin{tikzcd} L : \mathscr{C} \arrow[r, shift left] & \mathscr{D} : R \arrow[l, shift left] \end{tikzcd}$$

is said to be a **Quillen adjunction** if any of the following equivalent conditions are satisfied:

- $L\left(\mathsf{Cof}_{\mathscr{C}}\right) \subset \mathsf{Cof}_{\mathscr{D}}$ and $L\left(\mathsf{Cof}_{\mathscr{C}} \cap \mathsf{W}\right) \subset \mathsf{Cof}_{\mathscr{D}} \cap \mathsf{W}$;
- $R\left(\mathsf{Fib}_{\mathscr{D}}\right) \subset \mathsf{Fib}_{\mathscr{D}}$ and $R\left(\mathsf{W}_{\mathscr{D}} \cap \mathsf{Fib}_{\mathscr{D}}\right) \subset \mathsf{W}_{\mathscr{D}} \cap \mathsf{Fib}_{\mathscr{D}}$;
- $L\left(\mathsf{Cof}_{\mathscr{C}}\right) \subset \mathsf{Cof}_{\mathscr{D}}$ and $R\left(\mathsf{Fib}_{\mathscr{D}}\right) \subset \mathsf{Fib}_{\mathscr{C}}$; or
- $L\left(\mathsf{Cof}_{\mathscr{C}} \cap \mathsf{W}_{\mathscr{C}}\right) \subset \mathsf{Cof}_{\mathscr{D}} \cap \mathsf{W}_{\mathscr{D}}$ and $R\left(\mathsf{W}_{\mathscr{D}} \cap \mathsf{Fib}_{\mathscr{D}}\right) \subset \mathsf{W}_{\mathscr{C}} \cap \mathsf{Fib}_{\mathscr{C}}$.

The left adjoint functors in such an entity are known as **left Quillen** functors and the right adjoints are likewise known as **right Quillen** functors.

Example 4.3.2. The **Quillen-Serre model structure on** CW, the category of CW complexes together with all continuous maps, recovers the usual homotopy theory of spaces. Let W_{QS} be the class of homotopy equivalences, that is morphisms $f : X \longrightarrow Y$ of CW such that for all $x \in X$, the induced morphism of groups

$$\prod_{n\geq 1} \pi_n\left(X, x\right) \longrightarrow \prod_{n\geq 1} \pi_n\left(Y, f\left(x\right)\right),$$

and the morphism $\pi_0\left(X.x\right) \longrightarrow \pi_0\left(Y, f\left(x\right)\right)$ of pointed sets are isomorphisms. Let $\mathsf{Fib}_{\mathsf{QS}}$ be the set of **Serre fibrations,** that is the class of maps which enjoy the right lifting property with respect to the set of maps

$$\left\{D^n \overset{\mathrm{id}\times 0}{\longrightarrow} D^n \times [0,1]\right\}_{n\in\mathbf{N}}.$$

Let $\mathsf{Cof}_{\mathsf{QS}}$ be the set of maps which are retracts of cellular inclusions.

Example 4.3.3. Consider the category of connective chain complexes valued in an abelian category $\mathscr{A}$, $\mathsf{Ch}^+\left(\mathscr{A}\right)$. There are two important model structures on $\mathsf{Ch}^+\left(\mathscr{A}\right)$, the so called projective and injective model structures.

In both of these model structures, the class W is comprised of the quasi-isomorphisms. In the **projective model structure** we:

- set the fibrations to be the degree-wise epimorphisms; and
- set the cofibrations to be the degree-wise monomorphisms with projective co-kernel.

In the **injective model structure** we:

- set the cofibration to be the degree-wise monomorphisms; and
- set the fibrations to be the degree wise epimorphisms with injective kernel.

Remark 4.3.4. Chain complexes valued in an abelian category are equivalent to the category of reduced functors, functor which preserve zero objects, from a category $\mathscr{S}_{\mathsf{Ch}^+}$. These model structures turn out to be instances of model structures on general functor categories with the target being a model category.

Definition 4.3.5. Since the underlying category of all model categories are possessed of all limits and colimits, there is an initial object $\varnothing_{\mathscr{C}}$ and a final object $\bullet_{\mathscr{C}}$ in each. For each object X of $\mathscr{C}$ then we've canonical morphisms $\varnothing_{\mathscr{C}} \longrightarrow X$ and $X \longrightarrow \bullet_{\mathscr{C}}$.

An object X is said to be **cofibrant** if the canonical map $\varnothing_{\mathscr{C}} \longrightarrow X$ is a cofibration and **fibrant** if the canonical map $X \longrightarrow \bullet_{\mathscr{C}}$ is a fibration. A $(\mathsf{Cof}, \mathsf{W} \cap \mathsf{Fib})$ factorization of $\varnothing_{\mathscr{C}} \longrightarrow X$, $\varnothing_{\mathscr{C}} \hookrightarrow X^{\mathsf{Cof}} \overset{\sim}{\twoheadrightarrow} X$ displays some object X^{Cof} as a cofibrant object weakly equivalent to X. Such an object is called a **cofibrant replacement** of X. Dually a $(\mathsf{Cof} \cap \mathsf{W}, \mathsf{Fib})$ factorization of the canonical map $X \longrightarrow \bullet_{\mathscr{C}}$, $X \overset{\sim}{\hookrightarrow} X^{\mathsf{Fib}} \twoheadrightarrow \bullet_{\mathscr{C}}$, yields a fibrant object X^{Fib} weakly equivalent to X, a so called **fibrant replacement** for X. An object is said to be **bi-fibrant** if it is at once fibrant and co-fibrant.

Example 4.3.6. In the projective model structure on $\mathsf{Ch}^{+}(\mathscr{A})$, cofibrant replacements are projective resolutions. Dually, in the injective model structure on $\mathsf{Ch}^{+}(\mathscr{A})$, fibrant replacements are injective resolutions.

Remark 4.3.7. An elegant enhancement of this material, so called algebraic model category theory, has cofibrant objects as co-algebras for a co-monad and dually the fibrant objects as algebras for a monad.

Except for the presence of a class of morphisms called weak equivalences it is not yet clear precisely how our conception of homotopy theory, developed from experience with spaces, squares with this abstract formalism. For instance, precisely what is **a homotopy?**

Definition 4.3.8. Given an object X of a model category $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$, a $(\mathsf{Cof} \cap \mathsf{W}, \mathsf{Fib})$ factorization of the diagonal map $\Delta : X \longrightarrow X \times X$,

$$X \overset{\sim}{\hookrightarrow} X^{\mathbf{I}} \twoheadrightarrow X \times X$$

yields an object $X^{\mathbf{I}}$, a **path object** for X. Similarly, a $(\mathsf{Cof}, \mathsf{W} \cap \mathsf{Fib})$ factorization of the co-diagonal $\nabla : X \coprod X \longrightarrow X$,

$$X \coprod X \hookrightarrow \mathsf{Cyl}(X) \overset{\sim}{\twoheadrightarrow} X$$

yields a **cylinder object,** $\mathsf{Cyl}\left(X\right)$, for X.

A left homotopy $f \rightsquigarrow g$ of maps $f, g : X \longrightarrow Y$ is a map $h : \mathsf{Cyl}\left(X\right) \longrightarrow Y$ such that the pre-compositions, $\mathsf{in}_{\mathsf{l}}^{\star}$ and $\mathsf{in}_{\mathsf{r}}^{\star}$, yield f and g respectively. Dually, a right homotopy $f \rightsquigarrow g$ is a map $h : X \longrightarrow Y^{\mathbf{I}}$ such that the two post-composition $\mathsf{pr}_{\mathsf{l}\star}$ and $\mathsf{pr}_{\mathsf{r}\star}$ yield f and g respectively.

Let $\mathscr{C}^{\circ}$ full subcategory of $\mathscr{C}$ subtended by the bi-fibrant objects. Define the **homotopy category of** $\mathscr{C}$, denoted $\mathsf{Ho}\left(\mathscr{C}\right)$, to be the quotient of $\mathscr{C}^{\circ}$ by the equivalence relation induced by left homotopy. We denote those quotients of $\mathsf{Hom}_{\mathscr{C}}\left(X, Y\right)$, $\mathsf{Hom}_{\mathsf{Ho}(\mathscr{C})}\left(X, Y\right)$, which are the sets of homotopy classes of maps by $[X, Y]_{\mathscr{C}}$.

Remark 4.3.9. We could well have chosen right homotopy in the definition above. Doing so yields an equivalent category.

Definition 4.3.10. A Quillen adjunction

$$\begin{tikzcd} L : \mathscr{C} \arrow[r, shift left] & \mathscr{D} : R \arrow[l, shift left] \end{tikzcd}$$

is said to be a **Quillen equivalence** if the induced maps $\mathsf{Ho}\left(L\right) : \mathsf{Ho}\left(\mathscr{C}\right) \to \mathsf{Ho}\left(\mathscr{D}\right)$ and $\mathsf{Ho}\left(R\right) : \mathsf{Ho}\left(\mathscr{D}\right) \longrightarrow \mathsf{Ho}\left(\mathscr{C}\right)$ comprise the functors of an equivalence of categories.

4.4 Pointed Model Categories

Given a model category $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ the category $\bullet \downarrow \mathscr{C}$ admits a model structure by way of the forgetful right adjoint functor $U : \bullet \downarrow \mathscr{C} \longrightarrow \mathscr{C}$.

Lemma 4.4.1. *The data* $\left(\bullet \downarrow \mathscr{C}, U^{-1}\left(\mathsf{Cof}\right), U^{-1}\left(\mathsf{W}\right), U^{-1}\left(\mathsf{Fib}\right)\right)$ *comprise a model category.*

4.5 The Small Object Argument

Until this point, weak factorization systems have not been assumed to be functorial.As it so happens however, in most cases, the pairs of coupled weak factorization systems which define model categories can be made functorial. In this section we'll cover sufficient conditions for the existence of functorial weak factorization systems.

The composition operation on the arrows of a category $\mathscr{C}$ may be written as a functor $\mathscr{C}^{[1]} \times_{\mathscr{C}^{[0]}} \mathscr{C}^{[1]} \longrightarrow \mathscr{C}^{[1]}$, the interpretation in $\mathscr{C}$ of the map d^1. A section of that map is then a functorial factorization, a functorial replacement of one arrow by a pair of composable arrows which compose to the original.

Definition 4.5.1. Let I be a set of morphisms of a small category $\mathscr{C}$. Then an **I-cell complex** is a transfinite composition of pushouts of the morphisms of $\mathscr{C}$ in I. We denote this class of morphisms $\mathsf{Cell}\,(I)$.

Theorem 4.5.2. *(The small object argument) Suppose $\mathscr{C}$ to be a category possessed of all small colimits and suppose I to be a set of morphisms of $\mathscr{C}$. Then there exists a functorial factorization on $\mathscr{C}$, that is to say that there exists a section*

$$\gamma \times \delta : \mathscr{C}^{\triangle^1} \longrightarrow \mathscr{C}^{\triangle^1} \underset{\mathscr{C}}{\times} \mathscr{C}^{\triangle^1}$$

of the composition functor with $\gamma\,(f)$ in $I^{\pitchfork}$ and $\delta\,(f)$ of class $\mathsf{Cell}\,(I)$.

Proof. See 2.1.4 of [29] $\square$

While it is beyond the scope of this thesis to repeat a proof of this theorem a short discussion is in order. Given a morphism $f : X \longrightarrow Y$ we produce a λ-sequence, for carefully chosen ordinal λ,

$$X \longrightarrow Z_1^f \longrightarrow Z_2^f \longrightarrow \cdots$$

together with morphisms $Z_i^f \longrightarrow Y$, natural in their index, factoring f as $X \longrightarrow Z_\alpha^f \longrightarrow Y$. The desired factorization is then $X \longrightarrow \varinjlim Z_\alpha^f \longrightarrow Y$.

The production of the factorizations $X \longrightarrow Z_\alpha^f \longrightarrow Y$ begins by setting $X = Z_0^f$ and then, at each successor stage of the induction, that is for each $\alpha < \alpha + 1 \leq \lambda$, we let S_α be the set of lifting problems of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Z_\alpha^f \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

and then define $Z^f_{\alpha+1}$ to be the indicated pushout,

$$\begin{tikzcd}
\coprod_{s\in S_\alpha} A_s \arrow[r] \arrow[d] & Z^f_\alpha \arrow[d] \\
\coprod_{s\in S_\alpha} B_s \arrow[r] & Z^f_{\alpha+1} \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd}$$

with the next factorization of f coming from the universal property of the pushout. For limit ordinals the construction is of course the colimit over all previously constructed factorizations.

4.5.1 Smallness

The choice of λ in the small argument is done precisely so that each lifting problem against a map of I cannot see the whole sequence, and thereby exhaust the solutions to the lifting problems against I maps which have added.

Definition 4.5.3. Let $\mathscr{C}$ be a category with all colimits. Let $\mathscr{D}$ be a subcategory of $\mathscr{C}$. Let κ be a cardinal. We say that W, an object of $\mathscr{C}$, is **κ-small relative to $\mathscr{D}$** if

$$\mathsf{Hom}_{\mathscr{C}}(W, \_) : \mathscr{C} \longrightarrow \mathsf{Set}$$

preserves λ-sequential colimits from $\mathscr{D}$. That is to say, for every regular cardinal $\lambda \geq \kappa$, and every λ-sequence $X_0 \to X_1 \to \cdots \to X_{\beta<\lambda} \to \cdots$ in $\mathscr{D} \hookrightarrow \mathscr{C}$, the canonical morphism

$$\varinjlim \mathsf{Hom}_{\mathscr{C}}(W, X_\beta) \longrightarrow \mathsf{Hom}_{\mathscr{C}}\left(W, \varinjlim X_\beta\right)$$

is an isomorphism. An object W is said to be **small relative to $\mathscr{D} \hookrightarrow \mathscr{C}$** if there exists some cardinal κ for which W is κ-small relative to $\mathscr{D} \hookrightarrow \mathscr{C}$. Let I be a set of maps of $\mathscr{C}$. If κ is a cardinal, then an object X of $\mathscr{C}$ is said to be κ-small relative to I if it is κ-small relative to the subcategory of relative I-cell complexes.

Remark 4.5.4. In the small object argument we simply choose λ a κ-filtered ordinal where κ is some cardinal such that all the domains of the maps in I are all κ-small relative to $\mathsf{Cell}\,(I)$.

4.5.2 Saturation, Relative Cell Complexes, and Retracts

The small object argument with then allow us to begin with a set I of maps in $\mathscr{C}$, and define a factorization system $\left(\mathsf{Cell}\left(I\right), I^{\pitchfork}\right)$, and a factorization system $\left({}^{\pitchfork}\left(I^{\pitchfork}\right), I^{\pitchfork}\right)$ provided that $\mathsf{Cell}\left(I\right) \subset {}^{\pitchfork}\left(I^{\pitchfork}\right)$.

Lemma 4.5.5. *Given a small category $\mathscr{C}$ with all colimits and a class of morphisms W of $\mathscr{C}$. Then the class ${}^{\pitchfork}\left(\mathsf{W}^{\pitchfork}\right)$ is closed under:*

(1) pushouts;

(2) transfinite composition; and

(3) retracts.

Proof. See appendix B.2.1. ☐

4.5.3 Applicability of the argument

Definition 4.5.6. Let $\mathscr{C}$ be a category and let I be a set of maps of $\mathscr{C}$. We say that I **permits the small object argument** if the targets of all the maps which are elements of I are small relative to I.

A model category $(\mathscr{C}, \mathsf{Fib}, \mathsf{Cof}, \mathsf{W})$ is said to be **cofibrantly generated** if there is a set I of cofibrations and a set J of acyclic cofibrations such that:

- I permits the small object argument and such that $\mathsf{Fib} = I^{\pitchfork}$; and
- J permits the small object argument and such that $\mathsf{Fib} \cap \mathsf{W} = J^{\pitchfork}$.

Such a model category is said to be **cellular** if the relative cell complexes are well behaved, arising from the satisfaction of the further axioms:

- sources and targets of the maps which are the elements of I are compact;
- the targets of the maps which are elements of J are small relative to I;
- the cofibrations are effective monomorphisms.

Chapter 5

Grothendieck's Homotopy Theory

In this chapter we'll recover the treatment in [40] of Grothendieck's entirely algebraic description of **the** homotopy category. This presentation makes no reference to point set topology or model categories.

5.1 The Homotopy category Hot

Definition 5.1.1. let A be a category. Well say W, a class of morphisms of A, is **weakly saturated** if it satisfies the following conditions:

WS1 The identities are of class W;

WS2 if two out of three morphisms which comprise a commuting triangle are of class W then so too the third; and

WS3 if for any morphism $f : x \longrightarrow y$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}"'] & y \arrow[r, "g"] \arrow[rr, bend left, "f\circ g"] & x \arrow[r, "f"] & y\ . \end{tikzcd}$$

commutes and $f \circ g$ is of class W, then so too is f (whence by **WS1** and **WS2**, so too will g be).

Let $\mathsf{WSC}\,(A)$ be the full subcategory of the category of classes of morphisms of A subtended by the weakly saturated classes.

Lemma 5.1.2. *The category* $\mathsf{WSC}\,(A)$ *is closed under the formation of limits.*

Definition 5.1.3. A class of morphisms of the category Cat is said to be a **weak fundamental localizer** if it satisfies the following conditions:

WFL1 W is weakly saturated;

WFL2 the functors $A \longrightarrow \bullet$ are of class W for any category A with a terminal object; and

WFL3 if $f : A \longrightarrow B$ is a functor such that, for all $b \in \mathsf{Ob}\,(B)$, the induced functors $f \downarrow b : A \downarrow b \longrightarrow B \downarrow b$ are of class W, then so too is f.

Let WFL be the full subcategory of $\mathsf{WSC}\,(\mathsf{Cat})$ subtended by the weak fundamental localizers.

Lemma 5.1.4. *The category* WFL *of weak fundamental localizers is closed under the formation of limits.*

Definition 5.1.5. Let W_∞ be the initial weak fundamental localizer. Define the **category of homotopy types** by the formula

$$\mathsf{Hot} = \mathsf{W}_\infty^{-1}\mathsf{Cat}.$$

Theorem 5.1.6. *(Grothendieck) The category* Hot *is equivalent to the homotopy category of spaces.*

For those familiar with other formulations of abstract homotopy theory the axioms **WFL2** and **WFL3** may be weirdly familiar.

If W is a weak fundamental localizer, then axiom **WFL2** is the requirement that the notion of homotopy equivalence induced by W extends the notion that contractible categories are essentially trivial as they are contractible. See that for any category A with a terminal object, ∞, there is a canonically given extension

$$\begin{tikzcd}
A \coprod A \arrow[r, "\mathrm{id}_A \coprod \infty"] \arrow[d] & A \\
A \times [1] \arrow[ru, dashed] &
\end{tikzcd}$$

which is a contraction of A onto the terminal object ∞.

For a vision of axiom **WFL3**, again suppose W to be a weak fundamental localizer, and suppose $f : A \longrightarrow B$ to be a morphism of Cat. Note first that for each $b \in \mathsf{Ob}(B)$, $B \downarrow b$ admits a terminal object, $b \overset{\mathrm{id}}{\to} b$, so seeing as W is a weak fundamental localizer and it therefore satisfies **WFL2**, the canonical map $B \downarrow b \longrightarrow \bullet$ is of class W. Interpreting morphisms as directed paths, we see that $A \downarrow b$ is the homotopy fiber of f at b, and thus **WFL3** is the requirement that for all points of B, the homotopy fiber is W-equivalent to the point; **WFL3** is the requirement that any notion of homotopy equivalence encoded by a weak fundamental localizer generalize the following fact about morphisms of sets: a function $\varphi : A \longrightarrow B$ is an isomorphism if and only $\varphi^{-1}(b)$ is a singleton for all $b \in B$.

5.2 Test Categories

Proposition 5.2.1. *Let A be a small category. Then the Yoneda embedding enjoys the universal property of the* ***free colimit completion*** *of A, i.e. for all functors $f : A \longrightarrow B$ where B is endowed with all colimits, then there exists a unique extension as in the diagram below.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathsf{Yon}"'] & B \\
\widehat{A} \arrow[ur, dashed, "\widehat{f}"'] &
\end{tikzcd}$$

Proof. Proposition 4.8 of [1]. ☐

Presheaves however also describe **categories**.

Definition 5.2.2. Let A be a small category. Let

$$i_A = A \downarrow (\_) : \widehat{A} \longrightarrow \mathsf{Cat}$$

be the functor associating to each presheaf its category of elements. Let

$$i_A^\star : \mathsf{Cat} \longrightarrow \widehat{A}$$

be the functor which associates to each category C, the presheaf on A whose sets of sections are given thus:

$$i_A^{\star} C(a) = \mathsf{Hom}_{\mathsf{Cat}}(A \downarrow a, C).$$

Lemma 5.2.3. *There is an adjunction* $i_A \dashv i_A^{\star}$.

As a consequence then, we know that i_A preserves colimits. But then, given any category A, since $\mathsf{Yon} : A \longrightarrow \widehat{A}$ enjoys the universal property of the free colimit completion, the subcategory $\mathrm{im}\left(\widehat{A}\right) \hookrightarrow \mathsf{Cat}$ is generated under colimits by the set of objects $\{A \downarrow a\}_{a \in \mathsf{Ob}(A)}$.

Definition 5.2.4. Given any small category A, set $\mathsf{W}_A = i_A^{-1}(\mathsf{W}_\infty)$.

It is then by the universal property of localization that the composition of i_A with the canonical localization $\mathsf{Cat} \longrightarrow \mathsf{Hot}$, which we denote by $|\_|_A$ descends by an abuse of notation to $|\_|_A : \mathsf{W}_A^{-1}\widehat{A} \longrightarrow \mathsf{Hot}$.

It is by way of $|\_|_A$ that we may say that $\widehat{A}$ models homotopy types; we begin with presheaf X on A, consider the category $A \downarrow X$, and then consider the image thereof in Hot. The questions of the manner and quality of this modeling may then be realized as characterizations of the functor $|\_|_A$.

Definition 5.2.5. We'll call A a **weak test category** if:

WTC1 $\mathsf{W}_\infty = (i_A^{\star})^{-1}(\mathsf{W}_A)$;

WTC2 the induced functor

$$\overline{i_A} : \mathsf{W}_A^{-1}\widehat{A} \longrightarrow \mathsf{W}_\infty^{-1}\mathsf{Cat}$$

and the induced functor

$$\overline{i_A^{\star}} : \mathsf{W}_\infty^{-1}\mathsf{Cat} \longrightarrow \mathsf{W}_A^{-1}\widehat{A}$$

comprise the two functors of an equivalence of categories.

A weak test category then is a category for which the functor $|\_|_A$ realizes all homotopy types and all homotopy classes of maps, i.e. a category A such that $|\_|_A$ is full and essentially surjective.

Now, one property of the presheaf topoi $\widehat{A}$ is that they are categories enriched over themselves.

Definition 5.2.6. Let X, Y be presheaves on A and set $\mathsf{Map}(X,Y)$ to be the functor

$$\begin{tikzcd}\mathsf{Map}(X,Y) : A^{\mathsf{op}} \arrow[r] & \mathsf{Set} \\ a \arrow[r, mapsto] & \mathsf{Hom}(X \times A^{a}, Y) .\end{tikzcd}$$

Since this enrichment comes by way of the cartesian product, the compatibility of the enrichment and the localization can be phrased in those terms.

Definition 5.2.7. A small category A is said to be a **strict test category** if A is a weak test category and the canonical localization functor $\widehat{A} \longrightarrow \mathsf{W}_A^{-1}\widehat{A}$ preserves finite products.

We'll end this section with the description of some of the most important examples of these concepts.

Proposition 5.2.8. *The category $\triangle^{+}$ is a weak test category but not a strict test category. The category $\triangle$ is a strict test category.*

The proof that each is as we claim however will wait until the next section and the development of more theory.

5.3 Decalage

Recall that the weak fundamental localizer W_∞ is the smallest weakly saturated class of functors between small categories which:

- puts the right cones as W_∞-equivalent to the trivial category $\bullet_{\mathsf{Cat}}$; and
- obeys the principle that maps which are W_∞-essentially monomorphic and W_∞-essentially surjective are W_∞-equivalences.

As discussed in section 5.1, the first point is but a rephrasing of **WFL2** whereas this second point is less obvious and bears explication. Recall that a morphism in set is an isomorphism if and only if it is epic and monic and a morphism in set is epic and monic if and only if the fiber over every point is a singleton. The satisfaction of the axiom **WFL3** is the requirement that this notion hold up to the relation of W_∞ equivalence.

Thus, it should come as no surprise that the existence and behavior of colimit preserving cone constructions for the categories $\widehat{A}$ can be leveraged to prove that a small category A is a weak test category or a strict test category.

Example 5.3.1. Consider Kan's cone construction for simplicial sets. Let K be the endofunctor of $\triangle$ defined as follows.

$$\begin{tikzcd}
K : \triangle \arrow[r] & \triangle \\
{[n]} \arrow[r, mapsto] & {[n+1]} \\
\left(d^i : [n-1] \to [n]\right) \arrow[r, mapsto] & \left(d^i : [n] \to [n+1]\right) \\
\left(s^i : [n+1] \to [n]\right) \arrow[r, mapsto] & \left(s^i : [n+2] \to [n+1]\right)
\end{tikzcd}$$

Let $\widehat{K}$ be the left Kan extension of the composition $\triangle \xrightarrow{K} \triangle \xrightarrow{\mathsf{Yon}} \widehat{\triangle}$ along the Yoneda embedding. The application of this universal colimit preserving functor $\widehat{K} : \widehat{\triangle} \longrightarrow \widehat{\triangle}$ to a simplicial set X can be computed as a colimit.

$$\widehat{K}(X) = \varinjlim_{x:\triangle^n \to X \in \triangle \downarrow X} K(\triangle^n)$$

Note that for any presheaf on $\triangle$, X, there is a monomorphism

$$X \coprod \triangle^0 \longrightarrow \widehat{K}(X).$$

This morphism assigns to each n-simplex

$$x \in X([n]) \leftrightarrow \left(\triangle^n \xrightarrow{x} X\right)$$

the n-simplex

$$d_{n+1}\left(\widehat{K}(x)\right) \in \widehat{K}(X)([n]) \leftrightarrow \left(\triangle^n \xrightarrow{d^{n+1}} \triangle^{n+1} \xrightarrow{\widehat{K}(x)} \widehat{K}(X)\right),$$

and to the unique 0-simplex of $\triangle^0$ the natural transformation assigns the common 0-simplex of all the simplices $x \in \widehat{K}(X)$,

$$\triangle^0 \xrightarrow{\left(d^0\right)^{n+1}} \triangle^{n+1} \xrightarrow{x} \widehat{K}(X) \cdot$$

What's more is that this monomorphism explicitly defines a nullhomotopy of X in $\widehat{K}(X)$. Indeed, see that there exists a unique extension fitting into the diagram below.

$$\begin{tikzcd}
X \coprod X \arrow[r] \arrow[d] & X \coprod \triangle^0 \arrow[r] & \widehat{K}(X) \\
X \times \triangle^1 \arrow[urr, dashed] & &
\end{tikzcd}.$$

This example of colimit preserving cone construction on a presheaf category is abstracted in [19] into the notion of decalage[1] .

Definition 5.3.2. Given a category A a **decalage** on A consists of the following data:

(1) an endofunctor $L : A \longrightarrow A$;

(2) an object a_0 of A; and

(3) a pair of natural transformations

$$\mathrm{id}_A \xrightarrow{\gamma} L \xleftarrow{\gamma'} a_0$$

where a_0 is the constant functor returning a_0.

We'll denote a decalage by the quintuple $(A, L, a_0, \gamma, \gamma')$. Given a pair of decalages $(\mathscr{C}, L, c_0, \gamma, \gamma')$ and $(\mathscr{D}, M, d_0, \delta, \delta')$ a morphism

$$\left(\mathscr{C}, L, c_0, \gamma, \gamma'\right) \longrightarrow \left(\mathscr{D}, M, d_0, \delta, \delta'\right)$$

is a functor $U : \mathscr{C} \longrightarrow \mathscr{D}$ satisfying the following conditions:

MD1 the diagram $\begin{tikzcd} \mathscr{C} \arrow[r, "L"] \arrow[d, "U"'] & \mathscr{C} \arrow[d, "U"] \\ \mathscr{D} \arrow[r, "M"] & \mathscr{D} \end{tikzcd}$ commutes;

MD2 $U(c_0) = d_0$;

MD3 the natural transformation

$$\delta \star U : \mathrm{id}_{\mathscr{D}} \circ U \Rightarrow M \circ U$$

---

[1] It's also worth noting that a co-monadic treatment of decalage is found in Verity's works on complicial sets

and the natural transformation

$$U \star \gamma = U \circ \mathrm{id}_{\mathscr{C}} \Rightarrow U \circ L$$

define the same natural transformation

$$(\mathrm{id}_{\mathscr{D}} \circ U = U \circ \mathrm{id}_{\mathscr{C}} = U) \Rightarrow \underbrace{(M \circ U = U \circ L)}_{\textbf{MD1}};$$

and

MD4 the natural transformation

$$\delta' \star U : d_0 \circ U \Rightarrow M \circ U$$

and the natural transformation

$$U \star \gamma' : U \circ c_0 \Rightarrow U \circ L$$

define the same natural transformation

$$\underbrace{(U \circ c_0 = d_0 = d_0 \circ U)}_{\textbf{MD2}} \Rightarrow \underbrace{(U \circ L = M \circ U)}_{\textbf{MD1}}.$$

Beyond the data of a decalage, Kan's construction enjoys certain desirable properties which also merit abstraction. First, we abstract the organization of $\widehat{K}(X)$ over $\triangle^1$.

Definition 5.3.3. A **separating decalage** is comprised of a decalage $(\mathscr{C}, L, c_0, \gamma, \gamma')$ such that:

DS1 the components of the natural transformation $\gamma' : c_0 \Rightarrow L$ are monomorphisms;

DS2 for all morphisms $f : b \longrightarrow c$ of $\mathscr{C}$, the square

$$\begin{tikzcd}
b \arrow[r, "\gamma'_c"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & L(b) \arrow[d] \\
c \arrow[r, "\gamma'_c"] & L(c)
\end{tikzcd}$$

is cartesian as indicated; and

DS3 for any object c of $\mathscr{C}$, the fiber product

$$\varprojlim \left\{ \begin{tikzcd} & \mathscr{C}^{c_0} \arrow[d] \\ \mathscr{C}^{c} \arrow[r, "\gamma'_c"] & \mathscr{C}^{L(c)} \end{tikzcd} \right\}$$

of presheaves on $\mathscr{C}$ is the initial presheaf $\varnothing_{\mathscr{C}}$.

Proposition 5.3.4. *(Cisinski-Maltsiniotis) Let A be a small category. If A admits a separating decalage, then A is a weak test category.*

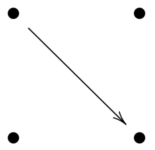
Proof. Proposition 3.11 of [19]. $\square$

Example 5.3.5. The category $\triangle^{+}$ and thus $\triangle$ are weak test categories; consider that the decalage on $\triangle$ implicit in example 5.3.1 is separating.

Indeed, consider that for all $[n]$ we have that $D\left([n]\right) = [n+1]$ and the squares

$$\begin{tikzcd} \varnothing_{\triangle} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \triangle^{[0]} \arrow[d] \\ \triangle^{[n]} \arrow[r, "d^{n+1}"] & \triangle^{[n+1]} \end{tikzcd}$$

are cartesian as indicated. The category $\triangle^{+}$ however is not a strict test category; it can be readily seen that $|\_|_{\triangle^{+}}$ does not preserve products. By direct computation we find that $\left(\triangle^{+}\right)^{1} \times \left(\triangle^{+}\right)^{1}$ is pictured as



whence we see that the realization $\left|\left(\triangle^{+}\right)^{1} \times \left(\triangle^{+}\right)^{1}\right|_{\triangle^{+}}$ is not contractible, whereas the object of Hot, $\left|\left(\triangle^{+}\right)^{1}\right|_{\triangle^{+}} \times \left|\left(\triangle^{+}\right)^{1}\right|_{\triangle^{+}}$ **is** contractible.

Those already familiar with this introductory topology exercise [2] will recognize that this deficiency arises from missing degeneracies. As it turns out however, not many are really necessary.

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[2] see ex. 3 of HW 7 from Math 6210 Fall 2012 CU Boulder

Definition 5.3.6. A **splitting** of a decalage $(A, L, a_0, \gamma, \gamma')$ is a family of retractions of the components of γ,

$$\{\rho_d : L(a) \longrightarrow a\}_{a \in \mathsf{Ob}(A)} .$$

Importantly, this family need **not** be natural. A decalage which admits a splitting is said to be **splittable**. A morphism of **split decalages**,

$$\big(A, L, a_0, \gamma, \gamma', \rho\big) \longrightarrow \big(B, M, b_0, \delta, \delta', \mu\big)$$

is a morphism of the underlying decalages $U : A \longrightarrow B$ which moreover satisfies:

MDS1 $\mu_{U(c)} = U(\rho_c)$ for all objects c of A.

Proposition 5.3.7. *(Cisinski-Maltsiniotis) Let A be a small category with a splittable separating decalage, then A is a strict test category.*

Proof. Corollary 3.12 of [19]. ☐

Example 5.3.8. The category $\triangle$ is a strict test category. Indeed for each simplex

$$x \in X([n]),$$

or equivalently each

$$\triangle^n \xrightarrow{x} X$$

we have that

$$d_{n+1} \circ s_n \left(\widehat{K}(x)\right) = \widehat{K}(x) \in \widehat{K}(X)([n+1]),$$

or equivalently

$$\begin{tikzcd}
\triangle^{n+1} \arrow[r, "s^n"] \arrow[rr, bend right, "\mathrm{id}"'] & \triangle^n \arrow[r, "d^{n+1}"] & \triangle^{n+1} \arrow[r, "\widehat{K}(x)"] & \widehat{K}(X)
\end{tikzcd} .$$

In fact, this notion of decalage and the wreath product of categories from section 3.2 were synthesized in [19] and used to prove that the categories Θ_n for $n \geq 0$ or $n = \infty$ are strict test categories.

Proposition 5.3.9. *The Kan decalage*

$$\left(\triangle, K, d^{n+1} : [n] \longrightarrow [n+1], \{n+1\} : [0] \longrightarrow [n]\right)$$

on $\triangle$ induces split and separating decalages on the categories categories Θ_n for $n \geq 0$ or $n = \omega$.

Proof. This is an iterated application of Proposition 5.4 of [19]. We'll briefly describe the functor part of the decalage in case where $n = 2$ and leave the rest of the construction and induction to the reader; for the decalage on $\triangle \int \triangle$, let

$$K_2 : \triangle \int \triangle \longrightarrow \triangle \int \triangle$$

be the endofunctor which acts on objects

$$[[n]\,;[a_1]\,,\ldots,[a_n]] \longmapsto [[n+1]\,;[a_1+1]\,,\ldots,[a_n+1]\,,[0]]$$

and on morphisms in the obvious way. $\square$

Corollary 5.3.10. *The categories Θ_n for $n \geq 0$ or $n = \omega$ are strict test categories.*

Chapter 6

Cisinski's Theory

In [18] that author relates Grothendieck's homotopy theory to Quillen's; Cisinski proves that whenever A is a local test category, there exists a model category structure on $\widehat{A}$ which is Quillen equivalent to the Kan structure on $\widehat{\triangle}$ by way of the Quillen equivalence between Cat with the Thomason model structure and $\widehat{\triangle}$ with the Kan structure. In that book however, Cisinski does a great deal more. In particular Cisinski develops the theory of what are now known as Cisinski model categories. In this section we'll recover the portion of that theory which we require.

6.1 Cisinski model categories

Definition 6.1.1. Given a small category A, a Cisinski model category on $\widehat{A}$ is a model structure on $\widehat{A}$ wherein the cofibrations are the monomorphisms.

Remark 6.1.2. It's worth remembering that the monomorphisms of $\widehat{A}$ are the object-wise injective natural transformations.

An important property of Cisinski model categories is that all of them are cofibrantly generated. In fact, Cisinski's theory is **the** theory of cofibrantly generated model categories on $\widehat{A}$ wherein the cofibrations are the monomorphisms.

Definition 6.1.3. We'll say that a small category A is a **local test category** if for all objects a of A, the categories $A \downarrow a$ are weak test categories.

Theorem 6.1.4. *(Grothendieck-Cisinski) Let A be a small category. If A is a local test category then*

$$\left(\widehat{A}, \mathsf{Mono}, \mathsf{W}_A, (\mathsf{Mono} \cap \mathsf{W}_A)^{\pitchfork}\right)$$

comprises a model category structure on $\widehat{A}$ which is Quillen equivalent to spaces by way of the canonical functor. What's more, if A is a strict test category, then the cartesian product of weak equivalences is again a weak equivalence.

Proof. Theorem 2.10 of [19]. □

There are however a great deal more examples of Cisinski model categories than those which present spaces. More, as it so happens, Cisinski model categories are cellular model categories, that is cofibrantly generated model categories with well behaved cell complexes.

6.2 Cellular Models and Skeletal Categories

Definition 6.2.1. A cellular model for a small category A is a set of cofibrations $\mathscr{M}_A$ such that ${}^{\pitchfork}\left(\mathscr{M}^{\pitchfork}\right)$ is the entire class of monomorphisms.

Proposition 6.2.2. *For any small category A there exists a cellular model consisting of all monomorphisms into quotients of representable presheaves.*

Proof. See Proposition 1.2.27 of [18]. □

However we've often better cellular models. For instance we've the following well known result.

Proposition 6.2.3. *(Gabriel-Zissmann) The set*

$$\mathscr{M}_{\triangle} = \{\partial \triangle^n \longrightarrow \triangle^n\}_{n \in \mathbf{N}}$$

is a cellular model for $\widehat{\triangle}$.

Proof. Suppose $f : X \longrightarrow Y$ to be a monomorphism. We may identify f as specifying a subfunctor of Y. Since the formation of pushouts in preserves monomorphisms, it suffices to observe that every subfunctor $X \subset Y$ admits a construction by iterated pushouts of monomorphisms. Consider then the pushout squares

$$\begin{tikzcd}
\coprod_{E_X(n)} \partial\triangle^n \arrow[r] \arrow[d] & X \bigcup \mathrm{Sk}^{n-1} Y \arrow[d] \\
\coprod_{E_X(n)} \triangle^n \arrow[r] & X \bigcup \mathrm{Sk}^{n} Y \arrow[ul, phantom, "\lrcorner", very near start]
\end{tikzcd} .$$

where $E_X(n)$ is, for each n, the set of non-degenerate simplices of Y which are not simplices of X. By observing the isomorphisms on either end below we see f as got from pushouts of $\mathscr{M}_{\triangle}$.

$$X \overset{\sim}{\to} X \bigcup \mathrm{Sk}^{-1} Y \longrightarrow X \bigcup \mathrm{Sk}^{0} Y \longrightarrow \cdots \longrightarrow \varinjlim \left\{ X \bigcup \mathrm{Sk}^{n} Y \right\} \overset{\sim}{\to} Y$$

□

The aspects of the category $\triangle$ required for this proof have been abstracted into the following notion.

Definition 6.2.4. A small category A together with two subcategories A^+ and A^- and a function $\lambda_A : \mathsf{Ob}(A) \longrightarrow \mathbf{N}$ comprise a **skeletal category** provided the following axioms are satisfied.

Sk0 All isomorphism of A are in both A^+ and A^- and λ_A is invariant on isomorphism classes;

Sk1 If $a \longrightarrow b$ is a morphism of A^+ (respectively of A^-) which is not an isomorphism, then $\lambda_A(a) \lneqq \lambda_A(b)$ (respectively $\lambda_A(a) \gneqq \lambda_b(b)$);

Sk2 every morphism of A may be factored into a morphism of A^- followed by a morphism of A^+ in an essentially unique way, meaning that there exists an initial such factorization; and

Sk3 if $f : a \longrightarrow b$ is an A^- map, then the set

$$\mathsf{Sect}(f) = \{ g : b \longrightarrow a | \, f \circ g = \mathrm{id}_b \}$$

is non-empty, and if $f, f' : a \longrightarrow b$ are two A^- maps, then they are equal if and only if $\mathsf{Sect}(f) = \mathsf{Sect}(f')$.

Remark 6.2.5. While in our discussion here the important of **Sk3** will not likely become clear, this axiom turns out to force the well definition of the notion of the minimal non-degenerate cell through which a given cell factors. This is the notion of the so called Eilenberg-Zilber decomposition.

Definition 6.2.6. Given a skeletal category A and an object $a \in \mathsf{Ob}\,(A)$, let

$$\mathrm{Sk}^n A^a = \varinjlim_{A^+_{<n}\downarrow a} A^b$$

where $A^+_{\leq n}$ is the full subcategory of A^+ on the objects with degree less than or equal to n. Given an object $a \in \mathsf{Ob}\,(A)$, we define $\partial A^a = \mathrm{Sk}^{\lambda(a)-1} A^a$ where $\lambda\,(a)$ is the degree of a.

With the addition a single hypothesis beyond that of a skeletal category (one that we will note is enjoyed by Reedy categories, see section 8.1.1) the hypothesis that no object admits a non-trivial automorphism, the Eilenberg-Zilber cellular model for $\triangle$ can be seen to generalize.

Proposition 6.2.7. *(Cisinski) If A is a skeletal category then the set*

$$\mathscr{M}_A = \{\partial A^a \longrightarrow A^a\}$$

generates the class of ***normal monomorphisms*** $\mathsf{Cell}\,(\mathscr{M}_A) =^{\pitchfork} \left(\mathscr{M}_A^{\pitchfork}\right)$. *Moreover, if A admits no non-trivial automorphisms, then* $\mathsf{Mono}\left(\widehat{A}\right) = \mathsf{Cell}\,(\mathscr{M}_A) =^{\pitchfork} \left(\mathscr{M}_A^{\pitchfork}\right)$.

Proof. Proposition 8.1.37 of [18]. □

6.3 Anodyne Extensions

Definition 6.3.1. Suppose A to be a small category. We define the class of morphisms $\mathsf{TrivFib}$ to be the class of morphisms which enjoy the right lifting property with respect to the monomorphisms; more succinctly we set $\mathsf{TrivFib} = \mathsf{Mono}^{\pitchfork}$.

A **functorial cylinder** $\mathbf{I} = \left(I, \partial^0, \partial^1, \sigma\right)$ on A is comprised of a diagram of natural transformations of endofunctors of $\widehat{A}$

$$\begin{tikzcd}
\mathrm{id}_{\widehat{A}} \arrow[d, "\partial^1"] \arrow[dr] & \\
I \arrow[r, "\sigma"] & \mathrm{id}_{\widehat{A}} \\
\mathrm{id}_{\widehat{A}} \arrow[u, "\partial^0"] \arrow[ur] &
\end{tikzcd}$$

such that the natural transformation

$$\partial^1 \coprod \partial^0 : \mathrm{id}_{\widehat{A}} \coprod \mathrm{id}_{\widehat{A}} \longrightarrow I$$

is a natural monomorphism. Such a diagram can be specified by a segment. A **segment** for A is an object I of $\widehat{A}$ together with sections $\partial^1, \partial^0 : \bullet_{\widehat{A}} \longrightarrow I$ such that the diagram below is a pullback square.

$$\begin{tikzcd}
\varnothing_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \bullet_{\widehat{A}} \arrow[d, "\partial^0"] \\
\bullet_{\widehat{A}} \arrow[r, "\partial^1"] & I
\end{tikzcd}$$

Lemma 6.3.2. *Given a segment for A, $\left(I, \partial^1, \partial^0\right)$, then*

$$\left(id_{\widehat{A}} \times I, id_{\widehat{A}} \times \partial^1, id_{\widehat{A}} \times \partial^0, \mathsf{pr}_1\right)$$

comprises a functorial cylinder on A.

Proof. The proof is purely formal and left to the reader. $\square$

Remark 6.3.3. In light of the lemma above and the fact that the majority of functorial cylinders are given by segments in this fashion, in place of $I\left(\_\right)$, we will often denote a functorial cylinder by $\otimes I$, and we will denote the natural monomorphism

$$\partial^1 \coprod \partial^0 : \mathrm{id}_{\widehat{A}} \coprod \mathrm{id}_{\widehat{A}} \longrightarrow I$$

by $\left(\_\right) \otimes \partial I \longrightarrow \left(\_\right) \otimes I$.

Definition 6.3.4. An **elementary homotopy datum $\mathfrak{I}$** on A is comprised of functorial cylinder $\left(I, \partial^1, \partial^0, \sigma\right)$ satisfying the two axioms:

DH1 the functor I preserves all small limits; and

DH2 for all monomorphisms $f : X \longrightarrow Y$ of $\widehat{A}$ the squares

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\partial^{\varepsilon}"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "\partial^{\varepsilon}"] \\
X \otimes I \arrow[r, "f \otimes I"] & Y \otimes I
\end{tikzcd}$$

for $\varepsilon = 0, 1$ are cartesian as indicated.

Remark 6.3.5. The preservation of small limits implies that I in the definition above preserves monomorphisms.

Corollary 6.3.6. *Suppose* $\left(I, \partial^1, \partial^0\right)$ *to be a segment for* A. *Then, the functorial cylinder*

$$\left(id_{\widehat{A}} \times I, id_{\widehat{A}} \times \partial^1, id_{\widehat{A}} \times \partial^0, \mathsf{pr}_1\right)$$

is an elementary homotopy datum.

Proof. The proof is purely formal and left to the reader. □

An elementary homotopy datum defines an elementary notion of homotopy as the language suggests.

Definition 6.3.7. Given an elementary homotopy datum $\mathfrak{I} = \left(I, \partial^0, \partial^1, \sigma\right)$, a elementary $\mathfrak{I}$-homotopy $f \rightsquigarrow g : X \longrightarrow Y$ is comprised of a morphism $h : X \otimes I \longrightarrow Y$ with $h \circ \partial^1 = f$ and $h \circ \partial^0$. We say $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ are $\mathfrak{I}$-homotopic if they are related by the congruence relation generated by elementary $\mathfrak{I}$-homotopy. Given an elementary homotopy datum $\mathfrak{I}$ on a small category A, denote the quotient category $\widehat{A}/\sim_{\mathfrak{I}}$, where $\sim \mathfrak{I}$ denotes $\mathfrak{I}$-homotopy, by $h_{\mathfrak{I}}\left(A\right)$.

One is tempted to think that only particularly nice categories admit an elementary homotopy datum. Instead however, for all small categories A there is a well behaved one, Lawvere's interval, the subobject classifier.

Lemma 6.3.8. *The subobject classifier* Ω *for the presheaf topos* $\widehat{A}$*,*

$$\begin{tikzcd} \Omega : A^{\mathsf{op}} \arrow[r] & \mathsf{Set} \\ a \arrow[r, mapsto] & \{X \subset A^a\} \end{tikzcd}$$

together with the natural transformations

$$\lambda^1 : \bullet_A \longrightarrow \Omega$$

identifying the empty subfunctor everywhere and the natural transformation

$$\lambda^0 : \bullet_A \longrightarrow \Omega$$

identifying the maximal sieves $A^a \subset A^a$ *at each* a *serves as a segment for* A*.*

Proof. The proof if purely formal and left to the reader. □

Definition 6.3.9. A class of anodyne extensions An relative to an elementary homotopy datum $\mathfrak{I}$ is a class of morphisms of $\widehat{A}$ satisfying:

An0 there exists a set Λ of monomorphisms of $\widehat{A}$ such that $\mathsf{An} = {}^{\pitchfork}\left(\Lambda^{\pitchfork}\right)$;

An1 if $f : X \longrightarrow Y$ is a monomorphism of $\widehat{A}$ then

$$\varinjlim \left\{ \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\partial^{\varepsilon}"] & Y \\ X \otimes I & \end{tikzcd} \right\} \longrightarrow Y \otimes I$$

is of class An for $\varepsilon = 0, 1$; and

An2 if $f : X \longrightarrow Y$ is of class An then so too the morphism

$$\varinjlim \left\{ \begin{tikzcd} X \otimes \partial I \arrow[r, "f \otimes \partial I"] \arrow[d] & Y \otimes \partial I \\ X \otimes I & \end{tikzcd} \right\} \longrightarrow Y \otimes I.$$

We've now developed all the requisite abstraction to lay down Cisinski's central theorem.

Lemma 6.3.10. *Given a category A, an elementary homotopy datum $\mathfrak{I} = \left(I, \partial^0, \partial^1\right)$ thereupon, a cellular model $\mathscr{M}_A$ for A and a class S of monomorphisms of A, then there is a smallest class of monomorphisms of $\widehat{A}$ containing S, call it $\mathsf{An}_{\mathfrak{I}}(S)$. More, this class $\mathsf{An}_{\mathfrak{I}}(S)$ is generated by a set $\Lambda_{\mathfrak{I}}(S, \mathscr{M}_A)$ to be defined below, in the sense that*

$$\mathsf{An}_{\mathfrak{I}}(S) = {}^{\pitchfork}\left(\left(\Lambda_{\mathfrak{I}}(S, \mathscr{M}_A)\right)^{\pitchfork}\right).$$

We define $\Lambda_{\mathfrak{I}}(S, \mathscr{M}_A)$ to be the infinite union

$$\bigcup_{n \geq 0} \Lambda^n_{\mathfrak{I}}(S, \mathscr{M}_A)$$

where the summands are defined inductively,

$$\Lambda^{n+1}_{\mathfrak{I}}(S, \mathscr{M}_A) = \Lambda\left(\Lambda^n_{\mathfrak{I}}(S, \mathscr{M}_A)\right)$$

with

$$\Lambda^0_{\mathfrak{I}}(S, \mathscr{M}_A) = S \cup \{ I \otimes K \cup \{\varepsilon\} \otimes L \longrightarrow I \otimes L |\, K \longrightarrow L \in \mathscr{M}_A,\ \varepsilon = 0, 1\}$$

and for any set T of monomorphisms of $\widehat{A}$,

$$\Lambda(T) = \{ I \otimes K \cup \partial I \otimes L \longrightarrow I \otimes L |\, K \longrightarrow L \in T\}.$$

Theorem 6.3.11. *Let a morphism of $\widehat{A}$ be a* ***cofibration*** *if and only if it is a monomorphism; we set*

$$\mathsf{Cof} = \mathsf{Mono}\left(\widehat{A}\right).$$

A ***trivial fibration*** *is a morphism of presheaves enjoying the right lifting property with respect to the monomorphisms; we denote the class of trivial fibrations* $\mathsf{TrivFib}$.

A ***naive fibration*** *is a morphism of presheaves which enjoys the right lifting property with respect to the anodyne extensions; we denote the class of naive fibrations* FibN.

An object X of $\widehat{A}$ is said to be ***fibrant*** *if the canonical morphism $X \longrightarrow \bullet_A$ is a naive fibration.*

A morphism of $\widehat{A}$, $f : X \longrightarrow Y$, is said to be of class W *or a* ***weak equivalence*** *if*

$$f^{\star} : \mathsf{Hom}_{h_{\mathfrak{J}}(A)}(Y, T) \longrightarrow \mathsf{Hom}_{h_{\mathfrak{J}}(A)}(X, T)$$

is bijective.

Lastly, a ***fibration*** *is a morphism of $\widehat{A}$ which enjoys the right lifting property with respect to the trivial cofibrations and we denote the class of such morphisms* Fib. *With the definitions above,* $\left(\widehat{A}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib}\right)$ *comprises a cofibrantly generated closed model category.*

Remark 6.3.12. All Cisinski model categories are nice enough for the enhanced formalism of algebraic model categories as described in [45] and in a more restricted sense, we've access directly to the fibrant objects as algebras for the fibrant replacement endofunctor as described in [41].

Given the defining role played by a class of anodyne extensions in the model categories arising from the previous theorem we are wont to ask the relation between the class of trivial cofibration $\mathsf{Cof} \cap \mathsf{W}$ and the class of anodyne extensions; in general, they are not the same and the containments $\Lambda \subset \mathsf{An}(\Lambda) \subset \mathsf{Cof} \cap \mathsf{W}$ are both proper. For more detail see Remark 1.3.46 of [18] and Remark X.2.4 of [25]. However, in many instances the notions coincide

Proposition 6.3.13. *Suppose A to be a small category, $\mathfrak{J}$ to be an elementary homotopy datum, and* An *to be a class of anodyne extensions relative to $\mathfrak{J}$. Then the following are equivalent:*

(1) all trivial cofibrations are anodyne extensions;

(2) all naive fibrations are fibrations;

(3) all naive fibrations which are also weak equivalences are trivial fibrations; and

(4) all naive fibrations may be factored as an anodyne extension followed by a fibration.

Proof. See Proposition 1.3.47 of [18]. □

Remark 6.3.14. While in some sense disconcerting, the fact that in general anodyne extensions are not coincident with trivial cofibrations should not menace as these conditions are regularly satisfied.

6.4 Cellular models, Monomorphism Preservation, and Colimit Preserving functors

Definition 6.4.1. Given a set I of morphisms in a category $\mathscr{C}$, let $\mathsf{Cell}\,(I)$ be the set of all transfinite compositions of pushouts of coproducts of morphisms in I. The morphisms in $\mathsf{Cell}\,(I)$ are said to be the $I-$**relative cell complexes**. An object X of $\mathscr{C}$ is said to be a **cell complex** if the canonical morphisms from the initial object into X is a relative cell complex. A **cellular model** for a category A is a set of morphisms of $\widehat{A}$, $\mathscr{M}$, such that $\mathsf{Cell}\,(\mathscr{M})$ is the set of monomorphisms of $\widehat{A}$.

One property of the category Set shared with all topoi, and many more categories, is the property that all pushouts of monomorphisms exist and are again monomorphisms.

Definition 6.4.2. Given a category A, we'll say that A is **weakly adhesive** if, for all monomorphisms $f : x \longrightarrow y$ of A, and all morphisms $g : x \longrightarrow z$, then the pushout

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "g"'] & y \arrow[d, "\overline{g}"] \\
z \arrow[r, "\overline{f}"] \arrow[ur, phantom, "\lrcorner", very near end] & w
\end{tikzcd}$$

exists and $\overline{f}$ is a monomorphism.

Remark 6.4.3. We have chosen to call the property here weakly adhesive as it is a consequence of a much more sophisticated property put forth in [34]. More, this notion of weakly adhesive is specific to this document and we make no claim for the utility of the notion beyond our purposes here.

Corollary 6.4.4. *Given a colimit preserving functor $F : \widehat{A} \longrightarrow B$ where B is a weakly adhesive category, then F preserves monomorphisms if, for any cellular model $\mathscr{M}$ for $\widehat{A}$, $F\,(\mathscr{M}) \subset \mathsf{Mono}\,(B)$.*

Proof. Follows from colimit preservation, the definition of weakly adhesive, and the definition of cellular model. ☐

This is one of the incredible strengths of Cisinski model category theory. The model category theoretic property of cofibration preservation is rendered a purely category theoretic one: monomorphism preservation.

What's more, we recall for many skeletal categories A, the set $\mathscr{M}_A = \{\partial A^a \longrightarrow A^a |\, a \in A\}$ comprises a cellular model. These cellular models are defined by colimits. Indeed, recall that

$$\varinjlim_{(f:b\to a)\in A^+\downarrow a-\{\mathrm{id}_a\}} A^b = \partial A^a.$$

As such, we get a refinement of the criterion for monomorphism preservation given above.

Corollary 6.4.5. *Suppose A to be a regular skeletal category, B to be a weakly adhesive category, and $F : \widehat{A} \longrightarrow B$ to be a colimit preserving functor. Then, F preserves monomorphisms if and only if, for all $a \in \mathsf{Ob}(A)$, the maps*

$$\varinjlim_{(f:b\to a)\in A^+\downarrow a-\{id_a\}} F\left(A^b\right) \longrightarrow F\left(A^a\right)$$

are monomorphisms.

Proof. The proof follows from colimit preservation, Cisinski's proof that $\{\partial A^a \longrightarrow A^a\}_{a\in\mathsf{Ob}(A)}$ is a cellular model for $\widehat{A}$, and Corollary 6.4.4. ☐

A proof that

$$\varinjlim_{(f:b\to a)\in A^+\downarrow a-\{\mathrm{id}_a\}} F\left(A^b\right) \longrightarrow F\left(A^a\right)$$

is a monomorphism however can be quite complex. These arbitrary colimits can be replaced with nicely stated coequalizers provided A enjoys a property we'll call incrementality.

Definition 6.4.6. Given a skeletal category A, we'll say that A is **incremental** if all A^+ morphisms factor as a sequence of A^+ morphisms of degree one.

Lemma 6.4.7. *Suppose A to be an incremental skeletal category. Then for every a of A the canonical map*

$$\varinjlim \left\{ \coprod_{(f:b\to a,g:c\to a)\in X^2} A^b \times_{A^a} A^c \rightrightarrows \coprod_{(f:b\to a)\in X} A^b \right\} \xrightarrow{\sim} \partial A^a$$

where $X = \{f : b \to a |\, f \in \mathsf{Mor}(A_+), \lambda_A(f) = 1\}$ is an isomorphism.

Proof. Let a be an object of A. Since

$$\partial A^a = \varinjlim_{(f:b\to a)\in A^+\downarrow a-\{\mathrm{id}_a\}} A^b$$

the incremental hypothesis has it that every A^+ maps factors through some coface. Thus the canonical map

$$\coprod_{(f:b\to a)\in X} A^b \longrightarrow \partial A^a$$

is an epimorphism, whence the canonical map

$$\varinjlim \left\{ \textstyle\coprod_{(f:b\to a, g:c\to a)\in X^2} A^b \times_{A^a} A^c \rightrightarrows \coprod_{(f:b\to a)\in X} A^b \right\} \xrightarrow{\sim} \partial A^a$$

is an isomorphism as claimed. □

Remark 6.4.8. We've chosen cofaces but a more sophisticated treatment could use maximal non-degenerate non-identity cells. Our choice is adapted to $\triangle$ and the categories Θ_n and the category Θ.

As a corollary to this presentation of boundaries in incremental skeletal categories we get a more easily checked criterion for monomorphism preservation.

Proposition 6.4.9. *Let A be an incremental regular skeletal category, let B be a a weakly adhesive category, and let*

$$F : \widehat{A} \longrightarrow B$$

be a colimit preserving functor. Then F preserves monomorphisms if and only if:

(1) $F(f)$ is a monomorphism for every coface $f : b \to a$ in A^+; and

(2) for all $a \in \mathsf{Ob}(A)$, and any pair of cofaces $f : b \to a$ and $g : c \to a$, the canonical comparison map

$$F\left(A^b \times_{A^a} A^c\right) \longrightarrow F\left(A^b\right) \times_{F(A^a)} F\left(A^c\right)$$

is an epimorphism.

Proof. It is by Proposition 6.4.7 that the cellular model $\{\partial A^a \longrightarrow A^a\}_{a\in A}$ admit the co-equalizer presentation:

$$\left\{\varinjlim\left\{\coprod A^b \times_{A^a} A^c \rightrightarrows \coprod A^b\right\} \longrightarrow A^a\right\}_{a\in \mathsf{Ob}(A)}.$$

By Corollary 6.4.4 and colimit preservation it then suffices to show that the maps

$$\varinjlim\left\{\coprod F\left(A^b \times_{A^a} A^c\right) \rightrightarrows \coprod F\left(A^b\right)\right\} \longrightarrow F\left(A^a\right)$$

are monomorphism for all $a \in \mathsf{Ob}(A)$. What'e more, since $F\left(A^b\right) \longrightarrow F\left(A^a\right)$ is by hypothesis a monomorphism for each coface $f : b \to a$, it then suffices to prove that the canonical comparison maps

$$F\left(A^b \underset{A^a}{\times} A^c\right) \longrightarrow F\left(A^b\right) \underset{F(A^a)}{\times} F\left(A^c\right)$$

are epimorphisms. □

Remark 6.4.10. Morally speaking, the last piece of the argument is that the canonical comparison maps being epimorphisms puts it that, wherever $F(b)$ and $F(c)$ intersect in $F(a)$, b and c already did so in a.

The Lemma 2.1.10 of [18] which inspired this treatment can now be had as a corollary.

Definition 6.4.11. A square

$$\begin{tikzcd} a \arrow[r, "i"] \arrow[d, "j"'] & b \arrow[d, "l"] \\ c \arrow[r, "k"'] & d \end{tikzcd}$$

in a category $\mathscr{C}$ is said to be **absolutely cartesian** if, for any functor $F : C \longrightarrow D$, the image of that square is again cartesian.

The purpose of making this definition is that in $\triangle$, for all $n \geq 2$, and any cofaces of $[n]$, d^i and d^j, the pullback squares

$$\begin{tikzcd} {[n-2]} \arrow[r] \arrow[d] & {[n-1]} \arrow[d, "d^j"] \\ {[n-1]} \arrow[r, "d^i"] & {[n]} \end{tikzcd}$$

exist in $\triangle$ and are moreover absolute in the sense of the definition above[1] .

Remark 6.4.12. While $\triangle$ has very few finite limits, those it does have are very rigid.

As a consequence, the application of our Proposition 6.4.9 to $\widehat{\triangle}$ is almost impossibly elegant.

Corollary 6.4.13. *(Cisinski) Let A be a small category and let $F : \longrightarrow \widehat{A}$ preserve small colimits. Then F preserves monomorphisms if and only if the morphism*

$$F\left(\triangle^{d^1 \amalg d^0}\right) : F\triangle^0 \coprod F\triangle^0 \longrightarrow F\triangle^1$$

is a monomorphism of $\widehat{A}$.

Proof. It is immediate that a monomorphisms preserving functor will preserve $\triangle^{d^1 \amalg d^0} : \partial\triangle^1 \longrightarrow \triangle^1$ as a monomorphism; the content is the converse.

Now, coface pair fibered products are absolute for $[n]$ with $n \geq 2$, whence the canonical comparison maps are isomorphisms. It then suffices to prove the claim for the objects [0] and [1]. Since $\partial\triangle^0 = \varnothing$, any colimit preserving functor into B will preserve $\partial\triangle^0 \longrightarrow \triangle^0$ as a monomorphism. So we need only check that $F\left(\triangle^{d^1 \amalg d^0}\right) : F\triangle^0 \coprod F\triangle^0 \longrightarrow F\triangle^1$ is a monomorphism. ☐

We should also note that going from presheaves $\widehat{A}$ to pointed presheaves $\widehat{A}_\bullet$ does not seriously alter the Lemma. Indeed, we record the following as a corollary.

Corollary 6.4.14. *Given an incremental regular skeletal category A, a weakly adhesive category B, and a colimit preserving functor*

$$F : \widehat{A}_\bullet \longrightarrow B$$

then F preserves monomorphisms if and only if:

(1) $F(f)$ is a monomorphism for every coface $f : b \to a$ in A^+; and

(2) for all $a \in \mathsf{Ob}(A)$, and any pair of cofaces $f : b \to a$ and $g : c \to a$, the canonical comparison map

$$F\left(A^b \times_{A^a} A^c\right) \longrightarrow F\left(A^b\right) \times_{F(A^a)} F\left(A^c\right)$$

---

[1] This is proved in [18]

is an epimorphism.

Proof. Since $(\_)_+$ preserves the property that $\mathscr{M}_A$ is a cellular model, i.e.

$$(\mathscr{M}_A)_+ = \left\{\partial A^a_+ \longrightarrow A^a_+\right\}_{a \in \mathsf{Ob}(A)}$$

is a cellular model for $\widehat{A}_\bullet$, the corollary follows mutatis mutandis. □

6.5 A-Localizers

Given a small category A we may in fact classify all of the Cisinski model category structures which may be put upon it by their classes of weak equivalences.

Definition 6.5.1. An A **localizer** is is a class of morphisms W of $\widehat{A}$ satisfying the three following conditions:

L1 if any two morphisms comprising a commutative triangle in $\widehat{A}$ are of class W, then so too the third;

L2 all trivial fibrations of $\widehat{A}$ are of class W; and

L3 the class comprised of morphisms which are both monic and of class W is closed under pushouts and transfinite composition.

Given an A-localizer we will refer to the morphisms of that class as $\mathsf{W}-$equivalences.

Given a small category A, the A-localizers comprise a category, with morphisms being the obvious containments, and this category possesses all limits.

Let A be a small category. Given a class of morphisms S of $\widehat{A}$, we set $\mathsf{W}(S)$ to be the smallest A-localizer which contains S. An A-localizer is said to be **accessible** if it is generated in this sense by a set of morphisms of A.

Theorem 6.5.2. *Given A a small category and W a class of morphisms of $\widehat{A}$, the following are equivalent:*

1. *the class* W *is an accessible* A*-localizer;*

2. *there exists a set of monomorphisms of* $\widehat{A}$ *such that* W *is the class of weak equivalences of a closed model category generated by Lawvere's cylinder and a set* S*.*

3. *There exists a model category structure on* $\widehat{A}$ *with* W *the class of weak equivalences and the cofibrations being the monomorphisms of the category* $\widehat{A}$*.*

Proof. Theorem 1.4.3 of [18]. □

We will use this machinery to provide a small but compelling zoo of examples in the next chapter.

Chapter 7

Simplicial Zoology

Recall from section 2.3 that we defined $\triangle$ as the full subcategory of Cat subtended by the totally ordered finite categories. Using that embedding we developed an adjunction

$$\mathfrak{R} : \rightleftarrows \mathsf{Cat} : \mathscr{N}$$

and using that adjunction developed an equivalence of categories

$$\mathsf{Mod}\left(\triangle, \mathsf{V}_1\right) \xrightarrow{\sim} \mathsf{Cat}.$$

The two most important examples of Cisinski model structures on $\widehat{\triangle}$ derive their meaning and utility from that adjunction and equivalence.

7.1 Quasi-categories

Amongst all of the models for $(\infty, 1)$-categories, e.g. quasi-categories, complete Segal spaces, simplicial categories etc., the most popular model is without a doubt the presentation of $(\infty, 1)$ $-$categories as quasi-categories.

Definition 7.1.1. Given $n \in \mathbf{N}$ and $i \in [n]$, define Λ^n_i, the i^{th}**-horn of the** n**-simplex**, to be the simplicial set

$$\Lambda^n_i = \varinjlim_{([m]\to[n])\in\triangle^+\downarrow[n]-\left\{\mathrm{id}_{[n]},d^i\right\}} \triangle^m.$$

If $i = 0, n$ then we'll call the horn Λ^n_i **outer** and if $0 < i < n$ we'll call the horn Λ^n_i **inner**.

Given a simplicial set X, we'll say that X is a **quasi-category** if, for all $n \in \mathbf{N}$, all $0 < i < n$, and all maps $f : \Lambda^n_i \longrightarrow X$, there exists a lift $\widetilde{f} : \triangle^n \longrightarrow X$ as in the diagram below.

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d] & X \\
\triangle^n \arrow[ur, dashed, "\widetilde{f}"] &
\end{tikzcd}$$

Importantly, no hypothesis on the uniqueness of these lifts is made in the definition of quasi-categories. When one is imposed, we recover another essentially algebraic treatment of categories.

Lemma 7.1.2. *A simplicial set X is the nerve $\mathscr{N}(\mathscr{C})$ of a category $\mathscr{C}$ if and only if, for all $n \in \mathbf{N}$, for all $0 < i < n$, and all maps $f : \Lambda^n_i \longrightarrow X$, there exist unique lifts $\widetilde{f} : \triangle^n \longrightarrow X$ as in the diagram below.*

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d] & X \\
\triangle^n \arrow[ur, dashed, "\widetilde{f}"] &
\end{tikzcd}$$

Proof. See Proposition 1.1.2.2. of [36]. □

Two question abide however. The first is moral: why do quasi-categories present $(\infty, 1)$-categories? The second is more technical: can quasi-categories be got as the fibrant objects of a Cisinski model structure on $\widehat{\triangle}$. As we'll see, the famous theorem of Joyal which paved the way for the **doing** of category theory in quasi-categories, e.g. [36, 38], answers both questions.

Theorem 7.1.3. *(Joyal) The fibrant objects of the Cisinski model structure specified by the $\triangle$-localizer*

$$\mathsf{W}\left(\mathsf{V}_1 = \left\{V^{[n]} \longrightarrow \triangle^n\right\}_{n \in \mathbf{N}}\right)$$

are the quasi-categories.

Proof. See Proposition 5.20 of [4]. □

Quasi-categories then are precisely those presheaves which are models for the sketch $(\triangle, \mathsf{V}_1)$ up to weak equivalence. Indeed, quasi-categories simply **are** weak 1-categories.

Remark 7.1.4. It's important to note that while quasi-categories are detectable by way of their enjoyment of the right lifting property with respect to all inner horns, the inner horns are **not** enough to detect all fibrations. Instead, right lifting against $\mathsf{Cof} \cap \mathsf{W}(\mathsf{V}_1)$, or another choice of generating set of trivial cofibrations need be checked.

7.2 Kan complexes

Definition 7.2.1. A simplicial set X is said to be a **Kan complex** if, for all $n \in \mathbf{N}$, all $i \in [n]$,and all $f : \Lambda^n_i \longrightarrow X$, there exists a lift $\widetilde{f} : \triangle^n \longrightarrow X$ as in the diagram below.

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d] & X \\
\triangle^n \arrow[ur, dashed, "\widetilde{f}"] &
\end{tikzcd}$$

Just as in the case of quasi-categories, if we impose a uniqueness condition on these lifts we recover a strict algebraic notion.

Lemma 7.2.2. *A simplicial set X is the nerve, $\mathscr{N}(\mathscr{C})$, of a groupoid $\mathscr{C}$ if and only if for all $n \in \mathbf{N}$, all $i \in [n]$,and all $f : \Lambda^n_i \longrightarrow X$, there exists a unique lift $\widetilde{f} : \triangle^n \longrightarrow X$ as in the diagram below.*

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d] & X \\
\triangle^n \arrow[ur, dashed, "\widetilde{f}"] &
\end{tikzcd}$$

Proof. From Lemma 7.1.2 we have that X enjoys the unique inner horn lifting property then $X = \mathscr{N}(\mathscr{C})$ for some category $\mathscr{C}$. It then falls to us only to prove that if $X = \mathscr{N}(\mathscr{C})$, then X enjoys the outer horn lifting property if and only if $\mathscr{C}$ is a groupoid. We'll prove that if X enjoys the unique outer horn lifting property, then $\mathscr{C}$ is a groupoid. The converse is left to the literature, e.g. [36].

Suppose $X = \mathscr{N}(\mathscr{C})$ enjoys the unique outer horn lifting property. Then in particular, X enjoys the unique lifting property with respect to the outer horn inclusions $\Lambda^2_0, \Lambda^2_2 \longrightarrow \triangle^2$. But, if

$X = \mathscr{N}(\mathscr{C})$, then a map $\Lambda^2_2 \longrightarrow X$ classifies a pair of maps

$$\begin{tikzcd} & y \arrow[dr, "f"] & \\ x \arrow[rr, "g"] & & z \end{tikzcd}$$

in $\mathscr{C}$. But, amongst such classifications we find those of the form

$$\begin{tikzcd} & y \arrow[dr, "f"] & \\ z \arrow[rr, "\mathrm{id}"] & & z \end{tikzcd}$$

and lifts of these to $\triangle^2$ classify inverses to f

$$\begin{tikzcd} & y \arrow[dr, "f"] & \\ z \arrow[ur, "f^{-1}"] \arrow[rr, "\mathrm{id}"] & & z \end{tikzcd} .$$

Since lifts must exist for all

$$\begin{tikzcd} & y \arrow[dr, "f"] & \\ z \arrow[rr, "\mathrm{id}"] & & z \end{tikzcd}$$

in $\mathscr{C}$ then $\mathscr{C}$ is a groupoid. □

Unlike in the case of quasi-categories, the set of all horns,

$$\{\Lambda^n_i \longrightarrow \triangle^n\}_{n \in \mathbf{N}, i \in [n]} ,$$

does serve as a set of generating trivial cofibrations for a model structure on $\widehat{\triangle}$. Indeed, it coincides with the test model structure we get from the split separating Kan decalage on $\triangle$.

Definition 7.2.3. Let the **Kan model structure** on $\widehat{\triangle}$ be cofibrantly generated model structure on $\widehat{\triangle}$ with generating cofibrations

$$\mathscr{M}_{\triangle} = \{\partial\triangle^n \longrightarrow \triangle^n\}_{n \in \mathbf{N}}$$

and generating trivial cofibrations

$$\Lambda_{\mathsf{Kan}} = \{\Lambda^n_i \longrightarrow \triangle^n\}_{n \in \mathbf{N}, i \in [n]} .$$

Lemma 7.2.4. *The Kan model structure and the test model structure on $\widehat{\triangle}$ are the same.*

Proof. Both structures are Cisinski model structures on $\widehat{\triangle}$ with realization weak equivalences as weak equivalences; in the case of the test structure this is definitional, in the case of the Kan structure this is Theorem 11.2. of [25]. □

But this model structure is also a localization of the Cisinski model structure specified by the localizer $\mathsf{W}(\mathsf{V}_1)$.

Corollary 7.2.5. *The Kan model structure on $\widehat{\triangle}$ is the Cisinski model structure specified by the $\triangle$-localizer*

$$\mathsf{W}\left(\mathsf{V}_1 \bigcup \{\Lambda^n_0 \longrightarrow \triangle^n, \Lambda^n_n \longrightarrow \triangle^n\}_{n\in\mathbf{N}}\right).$$

Proof. By Joyal's theorem, here Theorem 7.1.3, we know that

$$\{\Lambda^n_i \longrightarrow \triangle^n\}_{n\in\mathbf{N},0<i<n} \subset \mathsf{W}(\mathsf{V}_1)$$

so it follows that the generating trivial cofibrations of the Kan structure, the set of all horns Λ_{Kan}, is contained in $\mathsf{W}\left(\mathsf{V}_1 \bigcup \{\Lambda^n_0 \longrightarrow \triangle^n, \Lambda^n_n \longrightarrow \triangle^n\}_{n\in\mathbf{N}}\right)$. So the

$$\mathsf{W}\left(\mathsf{V}_1 \bigcup \{\Lambda^n_0 \longrightarrow \triangle^n, \Lambda^n_n \longrightarrow \triangle^n\}_{n\in\mathbf{N}}\right)$$

model structure is a localization of the Kan structure.

Conversely, it is easy to see that the spine inclusions are cofibrations and weak equivalences in the Kan model structure, and likewise the outer horns are trivial cofibration, whence

$$\mathsf{W}\left(\mathsf{V}_1 \bigcup \{\Lambda^n_0 \longrightarrow \triangle^n, \Lambda^n_n \longrightarrow \triangle^n\}_{n\in\mathbf{N}}\right) \subset \mathsf{W}(\Lambda_{\mathsf{Kan}}).$$

□

This last description is of particular interest; Kan complexes are the $(\infty,1)$-groupoids[1] in quasi-categories presentation of $(\infty,1)$-categories.

---

[1] see the proof of Lemma 7.2.2 for the moral.

Chapter 8

(Multi)-Reedy Category Theory

8.1 Reedy and multi-Reedy categories

Suppose $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ to be a model category and suppose $\mathscr{D}$ to be a category. It is almost never the case that $\mathsf{Fun}\left(\mathscr{D}, \mathscr{C}\right)$ with cofibrations, weak equivalences, and fibrations defined object-wise comprises a model category. The problem is that the lifts constructed are not necessarily natural; the morphisms of the diagram impose compatibility conditions amongst the lifts. We can however ask under what hypotheses on $\mathscr{D}$ may we at least be granted the existence of model category structure on $\mathsf{Fun}\left(\mathscr{D}, \mathscr{C}\right)$ wherein the weak equivalences are defined object-wise.

8.1.1 Reedy categories

Definition 8.1.1. A category C together with two wide subcategories C^- and C^+ along with a degree function $\lambda_C : \mathsf{Ob}\left(C\right) \longrightarrow \mathbf{N}$ is said to be a **Reedy category**, if these data satisfy the following hypotheses:

factorization every morphism $f : a \to b$ in C factors uniquely as $a \xrightarrow{f^-} c \xrightarrow{f^+} b$ with f^- a morphism of C^- and f^+ a morphism of C^+; and

degree every morphism $f : a \to b$ of C^- has $\lambda_C\left(f\right) = \lambda_C\left(b\right) - \lambda_C\left(a\right) \leq 0$ and every morphism $f : a \to b$ of C^+ has $\lambda_C\left(f\right) = \lambda_C\left(b\right) - \lambda_C\left(a\right) \geq 0$, moreover if in either subcategory we have $\lambda_C\left(f\right) = 0$, then f is an identity morphism.

Remark 8.1.2. In particular, the reader should note that Reedy categories admit no non-identity isomorphisms. As a consequence, Reedy categories are "evil" in the lingua franca of nLab, meaning that it is not a notion invariant under equivalence. The notion of skeletal category, definition 6.2.4, is one of two prominent attempts to generalize Reedy categories into an invariant notion. A competing notion is due to Berger and Moerdijk. We present Reedy categories here due to the importance of the notion of multi-Reedy, covered later in the section, as all other utilities of Reedy categories are enjoyed by skeletal categories as well.

Example 8.1.3. The most familiar example is that of $\triangle$ for which the degree function acts as $\lambda([n]) = n$ and $\triangle^+$ consists of the monomorphisms and $\triangle^-$ consists of the epimorphisms. It's worth noting that $\triangle^{\mathsf{op}}$ is also Reedy with the same degree function, $(\triangle^{\mathsf{op}})^+ = (\triangle^-)^{\mathsf{op}}$, and $(\triangle^{\mathsf{op}})^- = (\triangle^+)^{\mathsf{op}}$.

Definition 8.1.4. Suppose $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ to be a model category and suppose $(\mathscr{D}, \mathscr{D}^-, \mathscr{D}^+)$ to be a Reedy category. Given a functor $X : \mathscr{D} \longrightarrow \mathscr{C}$ and an object r of $\mathscr{D}$:

- let $L_r X$, the r^{th} **latching object** of X be defined as the colimit
$$\varinjlim_{(s\to r)\in\mathscr{D}^+\downarrow r-\{\mathrm{id}\}} X(s)$$
where $\mathscr{D}^+ \downarrow r - \{\mathrm{id}\}$ is the full subcategory of the comma category $\mathscr{D}^+ \downarrow r$ subtended by all objects except $\mathrm{id} : r \longrightarrow r$; and
- let $M_r X$, the r^{th} **matching object** of X be defined as the limit
$$\varprojlim_{(r\to s)\in r\downarrow\mathscr{D}^- -\{\mathrm{id}\}} X(s).$$

Theorem 8.1.5. *(Reedy) Let* $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ *be a model category and let* $(\mathscr{D}, \mathscr{D}^-, \mathscr{D}^+)$ *be Reedy category. Let* $\mathsf{W}_{\mathsf{Reedy}}$ *be the class of natural transformations in* $\mathsf{Fun}(\mathscr{D}, \mathscr{C})$ *which are object-wise weak equivalences of* $\mathscr{C}$*, let* $\mathsf{Cof}_{\mathsf{Reedy}}$ *be the class of natural transformations* $X \longrightarrow Y$ *in* $\mathsf{Fun}(\mathscr{D}, \mathscr{C})$ *for which the maps*

$$X(r) \coprod_{L_r X} L_r Y \longrightarrow Y(r)$$

are cofibrations of $\mathscr{C}$ *for all* r*, and let* $\mathsf{Fib}_{\mathsf{Reedy}}$ *be the class of natural transformations* $X \longrightarrow Y$ *in* $\mathsf{Fun}(\mathscr{D},\mathscr{C})$ *for which the maps*

$$X(r) \longrightarrow M_r X \underset{M_r Y}{\times} Y(r)$$

are fibrations of $\mathscr{C}$ *for all objects of* $\mathscr{D}$*. Then the data* $(\mathsf{Fun}(\mathscr{D},\mathscr{C}), \mathsf{Cof}_{\mathsf{Reedy}}, \mathsf{W}_{\mathsf{Reedy}}, \mathsf{Fib}_{\mathsf{Reedy}})$ *comprises a model category.*

Proof. See Proposition A.2.9.19 of [36] □

8.1.2 Multi-Reedy

The categories Θ and Θ_n for all $n \geq 0$ are Reedy too, however the proof of this which we'll present requires more technology. We present this material, found in [13] in detail, while omitting proofs, as the technology developed is critical for some of the technical aspects of this thesis.

Definition 8.1.6. Let C be a small category, we define $C(\star)$, the **product multi-category**[1] of C, as follows.

For each $c \in \mathsf{Ob}(C)$, $m \in \mathbf{N}$ and $(d_1, \ldots, d_m) \in \mathsf{Ob}(C)^m$, define the sets

$$\mathsf{MHom}(c, (d_1, \ldots, d_m)) = \prod_{i=1,\ldots,m} \mathsf{Hom}(c, d_i).$$

Then, given any $c \in \mathsf{Ob}(C)$, $m \in \mathbf{N}$, $(d_1, \ldots, d_m) \in \mathsf{Ob}(C)^m$, and for each $i \in \langle m \rangle$, an $n_i \in \mathbf{N}$ and $\left(e^i_1, \ldots, e^i_{n_i}\right)^{n_i}$, there is a composition function

$$\mathsf{MHom}(c, (d_1, \ldots, d_m)) \times \prod \mathsf{MHom}\left(d_i, \left(e^i_1, \ldots, e^i_{n_i}\right)\right) \longrightarrow \mathsf{MHom}\left(c, \left(e^i_j\right)_{i\in\langle m\rangle, j\in\langle n_i\rangle}\right)$$

which sends a pair

$$\left((c \to d_i)_{i\in\langle m\rangle}, \left(\left(d_i \to e^i_j\right)_{j\in\langle n_i\rangle}\right)_{i\in\langle m\rangle}\right)$$

to their composition

$$\left(c \to d_i \to e^i_j\right)_{i\in\langle m\rangle, j\in\langle n_i\rangle}.$$

---

[1] the notion we present here is an instance of what is usually referred to as a co-multi-category, however, following [13], we avoid one more prefix here for simplicity.

Remark 8.1.7. It is left to the reader to verify that this structure is unital, and associative, and comprises a co-multi-category in the language of [6].

The notion of multi-Reedy category is the natural extension of the notion of Reedy category to these product multi-categories.

Definition 8.1.8. A **multi-Reedy** structure on C is comprised of:

- a wide subcategory C^{-} of C;
- a wide sub-multi-category of $C(*)$, $C(*)^{+}$; and
- a degree function $\lambda_{C(*)} : \mathsf{Ob}(C) \longrightarrow \mathbf{N}$;

satisfying the two axioms:

factorization Every multimorphism

$$(\alpha_s)_{s=1,\ldots,m} \in \mathsf{MHom}(c,(d_1,\ldots,d_m))$$

admits a unique factorization $\alpha^{+} \circ \alpha^{-}$ where $\alpha^{-} : c \longrightarrow x$ is of C^{-} and

$$\alpha^{+} \in \mathsf{MHom}(x,(d_1,\ldots,d_m))$$

is a morphism of $C(*)^{+}$.

degree For every multimorphism

$$(\alpha_s)_{s=1,\ldots,m} \in \mathsf{MHom}(c,(d_1,\ldots,d_m))$$

in $C(*)^{+}$ we have that $\lambda_C((\alpha_s)) = \left(\sum_{s=1,\ldots,m} \lambda_C(d_s)\right) - \lambda_C(c) \geq 0$. For every multimorphism (α_s) which lies in the embedding $C \hookrightarrow C(*)$, $\lambda_c(\alpha) = 0$ if and only if α is an identity. For every $f : a \to b$ of C^{-} we have $\lambda_C(f) \leq 0$ and equality is attained if and only if f is an identity morphism.

Example 8.1.9. There is a multi-reedy structure on $\triangle$. Let $\triangle^-$ be the same $\triangle^-$ as in the Reedy structure on $\triangle$ and let $\triangle(*)^+$ be comprised of all **joint monomorphisms**, that is families of maps $f_1, \ldots, f_n$ such that $g = h$ if and only if

$$f_1 \circ g = f_1 \circ h, \ldots, f_n \circ g = f_n \circ h.$$

See that $\triangle \cap \triangle(*)^+ = \triangle^+$ from the usual Reedy structure on $\triangle$.

8.1.3 Multi-Reedy Categories and the Wreath Product of Categories

Proposition 8.1.10. *(Bergner-Rezk) If C admits a multi-Reedy structure $\left(C, C^-, C(*)^+, \lambda_C\right)$, then*

$$\left(C, C^-, C(*)^+ \cap C, \lambda_C\right)$$

is a Reedy structure on C.

Proof. Proposition 2.5 of [13]. □

Theorem 8.1.11. *(Bergner-Rezk) If C is equipped with the structure of a multi-reedy category and functor $H : C \longrightarrow \Gamma$, then the following declarations comprise a multi-reedy structure on $\triangle \int C$:*

- let $\left(\triangle \int C\right)^-$ be the wide subcategory of $\triangle \int C$ having as morphisms all those morphisms of $\triangle \int C$,

  $$[x; \mathbf{y}] : [[m]; c_1, \ldots, c_m] \longrightarrow [[n]; e_1, \ldots, e_n],$$

  for which $x : [m] \longrightarrow [n]$ is of $\triangle^-$ and each $y_{h,j}$ appearing in some

  $$\left(y_{i,k}\right)_{k \in F(x)(i)} : c_i \longrightarrow \prod_{j \in F(x)(i)} e_j$$

  is of C^- (where $F : \triangle \longrightarrow \Gamma : [m] \mapsto \langle m \rangle$ is the functor permitting the definition of $\triangle \int C$, so $F(x)(i) \subset \langle n \rangle$, see section 3.2).

- let $\triangle \int C(*)^+$ be the wide sub-multi-category of $\triangle \int C(*)$ with multimorphisms

  $$([x^s; \mathbf{y}^s]) : [[m]; c_1, \ldots, c_m] \longrightarrow \prod_{s \in \{1, \ldots, u\}} [[n^s]; e_1^s, \ldots, e_{n^s}^s],$$

  such that:

- $*$ the $\triangle(\star)$ multimorphism $(x^s):[m]\longrightarrow\prod_{s\in\{1,\dots,u\}}[n^s]$ is of $\triangle(*)^+$; and
- $*$ for each $i\in\{1,\dots,m\}$ the multimorphism

$$\left(y^s_{i,j}\right):c_i\longrightarrow\prod_{s\in\{1,\dots,u\}}\prod_{j\in F(x^s)(i)}e^s_j$$

is a $C(*)^+$ multimorphism.

Proof. Proposition 2.11 of [13]. $\square$

Example 8.1.12. For $n\geq 0$ or $n=\omega$, the category Θ_n is multi-reedy by the theorem above and therefore Reedy by the doubly prior proposition.

The multi-Reedy structure on $\triangle\wr A$ which is the topic of this subsection is not only compatible with the notion of 0-globular sums in $\triangle\wr A$ but in fact, could have been defined by way of the associated decompositions.

Example 8.1.13. Recall that for any object $[[n];a_1\dots a_n]$ of $\triangle\wr A$, we have that

$$\varinjlim\left\{\begin{tikzcd} {[[1];a_1]} & & \cdots & & {[[1];a]} \\ & {[0]} \arrow[ul] \arrow[ur] & & {[0]} \arrow[ul] \arrow[ur] & \end{tikzcd}\right\}\xrightarrow{\sim}[[n];a_1\dots a_n].$$

Thus, given two morphisms

$$[f;\mathbf{g}]:[[n];a_1\dots a_n]\longrightarrow[[m];b_1\dots b_m]$$

and

$$[p;\mathbf{q}]:[[j];c_1\dots c_j]\longrightarrow[[k];d_1\dots d_k],$$

their 0-globular sum,

$$[f;\mathbf{g}]\bigoplus_0[p;\mathbf{q}]=[f+p;(\mathbf{g},\mathbf{q})]:[[n+j];a_1\dots a_nc_1\dots c_j]\longrightarrow[[m+k];b_1\dots b_md_1\dots d_k],$$

exists provided that $f(n)=m+p(0)$.

When it exists, this sum of maps may be factored as $\mathrm{id} \underset{0}{\oplus} [p; \mathbf{q}] \circ [f; \mathbf{g}] \underset{0}{\oplus} \mathrm{id}$ with

$$[f; \mathbf{g}] \oplus \mathrm{id} : [[n]\, ; a_1 \ldots a_n] \underset{0}{\oplus} [[j]\, ; c_1 \ldots c_m] \longrightarrow [[m]\, ; b_1 \ldots b_m] \underset{0}{\oplus} [[j]\, ; c_1 \ldots c_j]$$

and

$$\mathrm{id} \oplus [p; \mathbf{q}] : [[m]\, ; b_1 \ldots b_m] \underset{0}{\oplus} [[j]\, ; c_1 \ldots c_j] \longrightarrow [[m]\, ; b_1 \ldots b_m] \underset{0}{\oplus} [[k]\, ; d_1 \ldots d_k]\ .$$

This observation may of course be finitely extended. It then follows that every morphism of $\triangle \wr A$,

$$[f; \mathbf{g}] : [[n]\, ; a_1 \ldots a_n] \longrightarrow [[m]\, ; b_1 \ldots b_m]\, ,$$

is of the form

$$\left[f^1; \mathbf{g}^1\right] \underset{0}{\oplus} \cdots \underset{0}{\oplus} [f^n; \mathbf{g}^n]\, ,$$

where for each $i \in \langle n \rangle$ we have

$$\left[f^i; \mathbf{g}^i\right] : [[1]\, ; a_i] \longrightarrow \left[[f\,(i) - f\,(0)]\, ; b_{f(i-1)+1} \ldots b_{f(i)}\right].$$

Thus, we may always factor a map $\left[f^1; \mathbf{g}^1\right] \underset{0}{\oplus} \cdots \underset{0}{\oplus} [f^n; \mathbf{g}^n]$ as

$$\mathrm{id} \underset{0}{\oplus} \cdots \underset{0}{\oplus} \mathrm{id} \oplus [f^n; \mathbf{g}^n] \circ \cdots \circ \left[f^1; \mathbf{g}^n\right] \underset{0}{\oplus} \mathrm{id} \underset{0}{\oplus} \cdots \underset{0}{\oplus} \mathrm{id}.$$

This relationship of this decomposition and factorization to the Reedy and multi-Reedy structure on $\triangle \wr A$ are given in the following lemmata.

Lemma 8.1.14. *Let A be a multi-Reedy category. Then, given two morphisms*

$$[f; \mathbf{g}] : [[n]\, ; a_1 \ldots a_n] \longrightarrow [[m]\, ; b_1 \ldots b_m]$$

and

$$[p; \mathbf{q}] : [[j]\, ; c_1 \ldots c_j] \longrightarrow [[k]\, ; d_1 \ldots d_k]\, ,$$

the morphism

$$[f; \mathbf{g}] \underset{0}{\oplus} [p; \mathbf{q}] = [f + p; (\mathbf{g}, \mathbf{q})] : [[n + j]\, ; a_1 \ldots a_n c_1 \ldots c_j] \longrightarrow [[m + k]\, ; b_1 \ldots b_m d_1 \ldots d_k]$$

is of $(\triangle \wr A)^+$ if and only if $[f; \mathbf{g}]$ and $[p; \mathbf{q}]$ are of $(\triangle \wr A)^+$.

Proof. It is clear that

$$f + p : [n + j] \longrightarrow [m + k]$$

is of $\triangle^+$ if and only if f and p are so it remains to shown only that the conditions on the constituent multimorphisms are the same for both proposition. Then as the individual multimorphisms indexed by f and p or $f + p$ are the same, it is clear that the conditions in either configuration are the same for the proposition that $[f; \mathbf{g}]$ and $[p; \mathbf{q}]$ are of $(\triangle \wr A)^+$ or that $[f + p; (\mathbf{g}, \mathbf{q})]$ is of $(\triangle \wr A)^+$. More formally, if

$$\mathbf{g} = \Big((g_{ji} : a_i \longrightarrow b_j)_{j \in \{f(i-1)+1, \ldots, f\}} \Big)_{i \in \langle n \rangle}$$

and

$$\mathbf{q} = \Big((q_{ji} : c_i \longrightarrow d_j)_{j \in \{p(i-1)+1, \ldots, p(i)\}} \Big)_{i \in \langle m \rangle},$$

then the condition:

- for all $i \in \langle n \rangle$, $(g_{ji} : a_i \longrightarrow b_j)_{j \in \{f(i-1)+1, \ldots, f\}}$ is in $A^+(\star)$ and for all $i \in \langle m \rangle$,

  $$(q_{ji} : c_i \longrightarrow d_j)_{j \in \{p(i-1)+1, \ldots, p(i)\}}$$

  is in $A^+(\star)$;

is obviously equivalent to the condition:

- for each

  $$x \in \{(g_{j1} : a_1 \longrightarrow b_j), \ldots, (g_{jn} : a_n \longrightarrow b_j), (q_{j1} : c_1 \longrightarrow d_j), \ldots, (q_{jm} : c_m \longrightarrow d_j)\},$$

  x is in $A^+(\star)$.

□

As a corollary then we've also that the factorization made possible by the globular decomposition is compatible with Reedy structure on $\triangle \wr A$.

Corollary 8.1.15. *If A is a Reedy category then given a morphism*

$$\left[f^1;\mathbf{g}^1\right]\underset{0}{\oplus}\cdots\underset{0}{\oplus}\left[f^n;\mathbf{g}^n\right]$$

of $(\triangle\wr A)^+$, then for each $i\in\langle n\rangle$, the morphisms

$$id\underset{0}{\oplus}\cdots\underset{0}{\oplus}id\underset{0}{\oplus}\left[f^i;\mathbf{g}^i\right]\underset{0}{\oplus}id\underset{0}{\oplus}\cdots\underset{0}{\oplus}id$$

which factor $\left[f^1;\mathbf{g}^1\right]\underset{0}{\oplus}\cdots\underset{0}{\oplus}\left[f^n;\mathbf{g}^n\right]$ as in the lemma are of $(\triangle\wr A)^+$.

Chapter 9

Cellular Biology

In section 3.4 we extended the adjunction

$$\mathfrak{R} : \rightleftarrows \mathsf{Cat} : \mathscr{N} \ .$$

To a family of adjunctions

$$\mathfrak{R}_n : \widehat{\Theta_n} \rightleftarrows n - \mathsf{Cat} : \mathscr{N}_n$$

and

$$\mathfrak{R}_\omega : \widehat{\Theta} \rightleftarrows \omega - \mathsf{Cat} : \mathscr{N}_\omega \ .$$

We also extended the equivalence

$$\mathsf{Mod}\,(\triangle, \mathsf{V}_1) \xrightarrow{\sim} \mathsf{Cat}$$

to a family of equivalences

$$\mathsf{Mod}\,(\Theta_n, \mathsf{V}_n) \xrightarrow{\sim} n - \mathsf{Cat}$$

and

$$\mathsf{Mod}\,(\Theta, \mathsf{V}) \xrightarrow{\sim} \omega - \mathsf{Cat}.$$

Then, in chapter 7 we softened the equivalence of $\mathsf{Mod}\,(\triangle, \mathsf{V}_1) \xrightarrow{\sim} \mathsf{Cat}$ to discover quasi-categories to be weak 1-categories and find the Kan complexes to be the weak 1-groupoids. In this section, we'll extend this softening to the equivalences $\mathsf{Mod}\,(\Theta_n, \mathsf{V}_n) \xrightarrow{\sim} n - \mathsf{Cat}$.

9.1 Ara's cellular (∞, n)-categories

Seeing as we found the Joyal model structure to be defined as the minimal one localizing the spine inclusions, Cisinski and Joyal conjectured the following.

Conjecture 9.1.1. *(Cisinski-Joyal) The model structure on* $\widehat{\Theta}_n$ *induced by* $\mathsf{W}(\mathsf{V}_n)$*, where* V_n *is the set of monomorphisms associated to the globular presentations of the objects of* Θ_n*, presents* (∞, n)*-categories.*

However, as first discovered by Gindi, and by Ara independently, this is not quite right. We've already proven that the nerves of strict 1-categories are quasi-categories. It's not much more work to prove that the nerves of equivalences of categories are $\mathsf{W}(\mathsf{V}_1)-$equivalences, that is to say they are weak equivalences in the Joyal model structure. However, even in the first higher dimension case $n = 2$ this breaks down for the model structure of the conjecture.

While it can easily be seen the nerves of strict $2-$categories are fibrant objects with respect to the Cisinski model structure specified by $\mathsf{W}(\mathsf{V}_2)$, it is not the case that every equivalence of strict 2-categories passes under the nerve functor to a $\mathsf{W}(\mathsf{V}_2)$-equivalence.

Example 9.1.2. (Gindi, Ara) Let $\langle \lozenge \cong \blacklozenge \rangle$ denote the strict 1-category on two objects with all hom-sets being singletons. Then, the embedding $e : \lozenge \longrightarrow \langle \lozenge \cong \blacklozenge \rangle$ is an equivalence of 1-categories. However,

$$J(e) : J([0]) = J(\mathscr{N}_1(\lozenge)) \longrightarrow J(\mathscr{N}_1(\langle \lozenge \cong \blacklozenge \rangle))$$

is not a $\mathsf{W}(\mathsf{V}_2)$-equivalence.

Indeed, in [4] this result is generalized.

Proposition. *Suppose* $n \geq 2$*. Then, for all* $k < n$*,* $J^k(\mathscr{N}(e))$ *is not an* $\mathsf{W}(\mathsf{V}_n)$*-equivalence in* $\widehat{\Theta_n}$*.*

Proof. See Corollary 6.21 of [4]. □

As it turns out however, these missing equivalences generate all of the missing equivalences.

Definition 9.1.3. Let $\mathsf{E}_n = \left\{ J^k\left(\mathscr{N}(e)\right) \middle| \, 0 < k < n \right\}$. Let the Cisinski model structure on $\widehat{\Theta_n}$ specified by $\mathsf{W}\left(\mathsf{V}_n \cup E_n\right)$ be referred to as the n^{th} **Ara model structure**, and define **n-quasi-categories** to be the fibrant objects of the n^{th} Ara model structure.

Now, by definition, for each $n > 1$, the elements of the set E_n are weak equivalences with respect to the n^{th} Ara model structure. What's more, Ara proves that the n^{th} Ara model structure presents (∞, n)-categories as it is Quillen equivalent to Rezk's Θ_n spaces, which is the simplicial localization of the the essentially algebraic theory of n-categories.

Theorem 9.1.4. *There is a Quillen equivalence*

$$\begin{tikzcd}\widehat{\Theta_n} \arrow[r, shift left] & \widehat{\Theta_n \times \triangle} \arrow[l, shift left]\end{tikzcd}$$

between the n^{th} Ara model structure and the model structure for Θ_n-spaces.

Proof. See Theorem 8.4 of [4] □

9.2 The Test model structure on $\widehat{\Theta_n}$ and $\widehat{\Theta}$

We've already seen that by way of Cisinski and Maltsiniotis theory of decalage that there are model structures on $\widehat{\Theta_n}$ and $\widehat{\Theta}$ which are Quillen equivalent to the Kan structure on $\widehat{\triangle}$ or equivalently, the Thomason structure on Cat. We are moreover granted a combinatorial description of these model structures as those generated by the trivial cofibrations $\Lambda(\varnothing)$ and the cellular model $\mathscr{M}_{\Theta_n}$ since the categories Θ_n and Θ are Reedy and therefore skeletal. Earlier work of Berger, see [9], provides another presentation of that model category which echoes the description of the Kan model structure on $\widehat{\triangle}$ and provides a generating set of trivial cofibrations without recourse to any inductive definition.

Definition 9.2.1. Let T be an object of Θ_n (respectively Θ), and let $k : S \longrightarrow T$ be a coface of T, that is let k be a codimension 1 Θ_n^+-map (respectively Θ^+ map). Define

$$\Lambda^k \longrightarrow \Theta_n^T$$

to be the colimits of Θ_n-sets,

$$\varinjlim_{(S\to T)\in\Theta_n^+\downarrow T-\{\mathrm{id},k\}} \Theta_n^S.$$

Theorem 9.2.2. *(Berger) The cofibrantly generated model structure on* $\widehat{\Theta_n}$ *(respectively* $\widehat{\Theta}$*) specified by the cellular model* $\mathscr{M}_{\Theta_n}$ *(respectively* $\mathscr{M}_\Theta$*) and the set*

$$\Lambda_{\mathsf{Ber}} = \left\{\Lambda^k \longrightarrow \Theta_n^T\right\}_{T\in\mathsf{Ob}(\Theta_n),k\in\mathsf{Co-face}(T)}$$

(respectively the set $\Lambda_{\mathsf{Ber}} = \left\{\Lambda^k \longrightarrow \Theta^T\right\}_{T\in\mathsf{Ob}(\Theta),k\in\mathsf{Co-face}(T)}$*) is the same as the test model structure on* $\widehat{\Theta_n}$ *(respectively* $\widehat{\Theta}$*).*

Proof. The test model structure is the Cisinski structure specified by the realization weak equivalences. Proposition 3.9 of [9] proves that Berger's cofibrantly generated structure also has the realization weak equivalences as the weak equivalences. □

Remark 9.2.3. While we won't invoke them here and therefore do not define the notions, Berger develops a notion of inner and outer horns of $\widehat{\Theta_n}$, which agrees in the case $n = 1$ with the simplicial inner and outer horns. More, Berger proves that a Θ_n-set is the nerve of a strict n-category if and only if it enjoys the unique inner cellular horn lifting property. The status of model structures on $\widehat{\Theta_n}$ defined by inner horns is an area of active research.

Chapter 10

(Reedy) Homotopy (Co)-Limits and Simplicial Model Categories

10.1 (Reedy) Homotopy limits and colimits and derived functors

Consider the Quillen-Serre model structure on CW and consider within CW the two diagrams

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}.$$

and

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & D^1 \\ D^1 & \end{tikzcd}.$$

While there is an obvious natural transformation of these diagrams, the components of which are $D^1 \longrightarrow \bullet$, id_{S^0}, and $D^1 \longrightarrow \bullet$, and each of those maps is weak equivalence, it is not the case that the induced map

$$\varinjlim \left\{ \begin{tikzcd} S^0 \arrow[r] \arrow[d] & D^1 \\ D^1 & \end{tikzcd} \right\} \longrightarrow \varinjlim \left\{ \begin{tikzcd} S^0 \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \right\}.$$

is a weak equivalence. The colimit on the left is the space S^1 whereas the colimit on the right is the trivial space $\bullet$.

The issue is of course that while the purely categorical notion of colimit is by definition isomorphism invariant, it is not invariant with respect to the softer notion of weak equivalence. The notion of homotopy limits and colimits are the universal weak equivalence invariant replacements for purely categorical notions of limit and colimit. These notions are defined by way of the theory of derived functors.

Definition 10.1.1. Given a left Quillen functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ we may set the **left derived functor of** F, denoted $\mathbb{L}F$, to be the pre-composition of F with a cofibrant replacement functor. Dually, given a right Quillen functor $G : \mathscr{D} \longrightarrow \mathscr{C}$ set the **right derived functor of** G, denoted $\mathbb{R}G$, to be the pre-composition of G with a fibrant replacement functor.

Recall that, given any two categories $\mathscr{C}$ and $\mathscr{D}$, the diagonal map $\mathscr{C} \longrightarrow \mathsf{Fun}\,(\mathscr{D}, \mathscr{C})$, which assigns to an object c of $\mathscr{C}$ the constant functor $c : \mathscr{D} \longrightarrow \mathscr{C}$, admits a left adjoint functor, $\varinjlim$, and a right adjoint functor, $\varprojlim$. It is this definition of limit and colimit which we will replace with a homotopy invariant notion.

Definition 10.1.2. Suppose $\mathscr{D}$ to be a Reedy category and suppose $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ to be a model category. The **(Reedy) homotopy colimit functor**

$$\mathrm{ho}\varinjlim : \mathsf{Fun}\,(\mathscr{D}, \mathscr{C}) \longrightarrow \mathscr{C}$$

is the left derived functor $\mathbb{L}\varinjlim$ and the **(Reedy) homotopy limit functor**

$$\mathrm{ho}\varprojlim : \mathsf{Fun}\,(\mathscr{D}, \mathscr{C}) \longrightarrow \mathscr{C}$$

is the right derived functor $\mathbb{R}\varprojlim$.

Then, an important result of basic Reedy theory shows that these definition exhibit the property we described as desirable at the beginning of the chapter.

Lemma 10.1.3. *Let $\mathscr{D}$ be a Reedy category and let $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ be a model category. Suppose $F : X \Longrightarrow Y : \mathscr{D} \longrightarrow \mathscr{C}$ to be a natural transformation of diagrams in $\mathscr{C}$. Suppose further that the components of F are weak equivalences of $\mathscr{C}$. Then:*

- *if X and Y are Reedy cofibrant, the induced map $\varinjlim X \longrightarrow \varinjlim Y$ is a weak equivalence; and*
- *if X and Y are Reedy fibrant, the induced map $\varprojlim X \longrightarrow \varprojlim Y$ is a weak equivalence.*

Proof. See appendix A.2.9. of [36]. $\square$

The question of model structures on categories of diagrams allowing us to compute, or at least define the universal property of, homotopy limits and colimits, is of course much broader than the Reedy case considered above. However, many of the most important diagram shapes are Reedy, including pushouts and pullback. Indeed for many diagram shapes this theory is enough.

Example 10.1.4. Suppose $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ to be a pointed model category. Then we define

$$\Sigma : \mathscr{C} \longrightarrow \mathscr{C}$$

to be the functor

$$X \longmapsto \mathbb{L}\underrightarrow{\lim}\left\{\begin{tikzcd} X \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}\right\}$$

and we define

$$\Omega : \mathscr{C} \longrightarrow \mathscr{C}$$

to be the functor

$$X \longmapsto \mathbb{R}\underleftarrow{\lim}\left\{\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & X \end{tikzcd}\right\}.$$

These are the suspension and loop-space functors defined by way of their correct homotopical universal property.

10.2 Simplicial model categories and homotopy (co)-limits

Computing Reedy fibrant and Reedy cofibrant replacements of a diagram however remains a difficult problem. While the criteria are easily checked, the production of satisfactory replacement diagrams is not. With a little extra structure on the target model category however there is an elegant formula, see [20].

Definition 10.2.1. A category $\mathscr{C}$ is said to be a **simplicial category** if there exists a bifunctor

$$\mathsf{Map}_{\mathscr{C}}(\_, \_) : \mathscr{C}^{\mathsf{op}} \times \mathscr{C} \longrightarrow \widehat{\triangle}$$

with following properties.

(1) (enrichment in $\widehat{\triangle}$) the evaluation at [0] maps

$$\mathsf{ev}_{[0]} : \mathsf{Map}_{\mathscr{C}}(a, b) \longrightarrow \mathsf{Hom}_{\mathscr{C}}$$

are isomorphisms natural in a and b;

(2) (tensored in $\widehat{\triangle}$) the functors

$$\mathsf{Map}_{\mathscr{C}}(a, \_) : \mathscr{C} \longrightarrow \widehat{\triangle}$$

have left adjuncts, $a \otimes (\_) : \widehat{\triangle} \longrightarrow \mathscr{C}$ which are associative in the sense that there are isomorphisms

$$a \otimes (K \times L) \overset{\sim}{\to} (a \otimes K) \otimes L$$

natural in objects a of $\mathscr{C}$ and simplicial sets K and L; and

(3) (co-tensored in $\widehat{\triangle}$) the functors

$$\mathsf{Map}_{\mathscr{C}}(\_, b) : \mathscr{C}^{\mathsf{op}} \longrightarrow$$

have left adjuncts

$$\mathsf{map}_{\mathscr{C}}(\_, b) : \widehat{\triangle} \longrightarrow \mathscr{C}^{\mathsf{op}}.$$

A priori, there need be no compatibility between this structure and any of the model structures on simplicial sets, usually the Kan, so such compatibility is a further axiom.

Definition 10.2.2. A model category $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ on a simplicial category $(\mathscr{C}, \mathsf{Map}_{\mathscr{C}}, \otimes, \mathsf{map}_{\mathscr{C}})$ is said to be a **simplicial model category** if it satisfies the axiom.

SM7 If $j : a \longrightarrow b$ to be a cofibration of $\mathscr{C}$, and $q : c \longrightarrow d$ is a fibration of $\mathscr{C}$, then the induced map

$$\mathsf{Map}(b, c) \overset{(j^*, q_*)}{\longrightarrow} \mathsf{Map}(a, c) \underset{\mathsf{Map}(a,d)}{\times} \mathsf{Map}(b, d)$$

is a fibration of simplicial sets, which is trivial if either j or q were trivial.

While a full treatment of the theory of homotopy limits and colimits in simplicial model categories is beyond the scope of this document, there is one important instance we'll recall as an unproven lemma.

Lemma 10.2.3. *For any pointed simplicial model category, the functor* $\otimes S^1$ *is naturally weakly equivalent to* Σ*.*

Chapter 11

Spectra

If algebraic topology is the study of algebra valued homotopy invariants, i.e. functors $\mathsf{Hot} \longrightarrow \mathscr{C}$ where $\mathscr{C}$ is (essentially) algebraic, then one thorny problem sits at the very heart of the topic. Spaces are naturally specified as colimits, and in fact usually as iterated pushouts (hence the unreasonable effectiveness of the Grothendieck, Maltsiniotis, and Cisinski treatment we've summarized here). In (essentially) algebraic categories on the other hand, colimits in general, and even pushouts, are not easy to compute, in particular they do **not** agree with the colimits of the underlying diagrams of sets. The situation is just as bad when instead of set valued algebraic categories, we are concerned with space valued ones.

To simplify the problem and render the theory more computationally tractable, we may restrict ourselves to the class of algebra valued homotopy invariants which take pushouts to pullbacks; spectra are the homotopical entities which represent such invariants[1] . We'll provide a model category theoretic treatment of sequential spectra valued in any cellular model category, due to Hovey, and finally we'll come to a compelling purely combinatorial treatment of spectra given by Kan.

11.1 Sequential Spectra

In this section we recover the theory of sequential spectra from [30]. Throughout this chapter, fix $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ a cellular model category, fix $S : \mathscr{C} \longrightarrow \mathscr{C}$ a left Quillen endofunctor, and fix $L : \mathscr{C} \longrightarrow \mathscr{C}$ to be the functor right adjoint to S.

---

[1] It should be noted that this justification for spectra is really a justification for the linear stage of the Goodwillie calculus.

Definition 11.1.1. Let $\mathsf{Sp}\left(\mathscr{C}, S\right)$ be the category defined as follows. Let $\mathsf{Ob}\left(\mathsf{Sp}\left(\mathscr{C}, S\right)\right)$ be the collection of $\mathbf{N}$-indexed sets of objects of $\mathscr{C}$ $\left(X_n\right)_{n\in\mathbf{N}}$ together with indexed sets of maps

$$\left(\varphi_n : SX_n \longrightarrow X_{n+1}\right)_{n\in\mathbf{N}}.$$

Denote these objects by $\left(\left\{X_n\right\}_{n\in\mathbf{N}}, \left\{\varphi_n\right\}_{n\in\mathbf{N}}\right)$, or when convenient and not obscure, by $\left(X_n\right)_{n\in\mathbf{N}}$. Given $\left(\left\{X_n\right\}_{n\in\mathbf{N}}, \left\{\varphi_n\right\}_{n\in\mathbf{N}}\right)$ and $\left(\left\{Y_n\right\}_{n\in\mathbf{N}}, \left\{\psi_n\right\}_{n\in\mathbf{N}}\right)$ objects of $\mathsf{Sp}\left(\mathscr{C}, S\right)$ let

$$\mathsf{Hom}\left(\left(\left(X_n\right)_{n\in\mathbf{N}}, \left(\varphi_n\right)_{n\in\mathbf{N}}\right), \left(\left(Y_n\right)_{n\in\mathbf{N}}, \left(\psi_n\right)_{n\in\mathbf{N}}\right)\right)$$

be the set

$$\left\{ \left(f_n : X_n \longrightarrow Y_n\right)_{n\in\mathbf{N}} \;\middle|\; \begin{tikzcd} SX_n \arrow[r, "\varphi_n"] \arrow[d, "\Sigma_J f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\ SY_n \arrow[r, "\psi_n"] & Y_{n+1} \end{tikzcd} \right\}.$$

Lemma 11.1.2. *Limits and colimits in* $\mathsf{Sp}\left(\mathscr{C}, S\right)$ *are computed index-wise.*

Proof. The proof is formal and left to the reader. ☐

Definition 11.1.3. We define the **evaluation at n functors**,

$$\begin{aligned} \mathsf{Ev}_n : \mathsf{Sp}\left(\mathscr{C}, S\right) &\longrightarrow \mathscr{C} \\ \left(\left(X_m\right)_{m\in\mathbf{N}}, \left(\varphi_m\right)_{m\in\mathbf{N}}\right) &\longmapsto X_n \end{aligned}$$

Corollary 11.1.4. *The functors* $\mathsf{Ev}_n : \mathsf{Sp}\left(\mathscr{C}, S\right) \longrightarrow \mathscr{C}$ *admit both left adjoint functors,* $F_n : \mathscr{C} \longrightarrow \mathsf{Sp}\left(\mathscr{C}, S\right)$ *and right adjoint functors* $R_n : \mathscr{C} \longrightarrow \mathsf{Sp}\left(\mathscr{C}, S\right)$ *respectively.*

Proof. This follows from the adjoint functor theorem and the prior lemma. ☐

Lemma 11.1.5. *The functors* $F_n : \mathscr{C} \longrightarrow \mathsf{Sp}\left(\mathscr{C}, S\right)$ *defined by universal property above admit explicit description; for an object* X *of* $\mathscr{C}$ *let* F_nX *be the object of* $\mathsf{Sp}\left(\mathscr{C}, S\right)$ *whose spaces* $\left(F_nX\right)_m$ *are specified thus*

$$\left(F_nX\right)_m = \begin{cases} \bullet & m < n \\ S^{m-n}X & m \geq n \end{cases}$$

and whose structure maps

$$S\left(F_nX\right)_m \longrightarrow \left(F_nX\right)_{m+1}$$

are either the identity or the canonical inclusion depending on m and n.

Definition 11.1.6. Given an endofunctor $H : \mathscr{C} \longrightarrow \mathscr{C}$ together with a natural transformation $\tau : SF \longrightarrow FS$ we define the **prolongation of H to $\mathsf{Sp}\left(\mathscr{C}, S\right)$**, denoted also by H with the dependence on τ implicit, by

$$\begin{tikzcd} H : \mathsf{Sp}\left(\mathscr{C}, S\right) \arrow[r] & \mathsf{Sp}\left(\mathscr{C}, S\right) \\ \left(\left(X_m\right)_{m\in\mathbf{N}}, \left(\varphi_m\right)_{m\in\mathbf{N}}\right) \arrow[r, mapsto] & \left(\left(HX_m\right)_{m\in\mathbf{N}}, \left(SHX_m \xrightarrow{\tau} HSX_m \xrightarrow{H\varphi_m} HX_{m+1}\right)_{m\in\mathbf{N}}\right) \end{tikzcd}$$

with the action of H on morphisms index-wise.

Lemma 11.1.7. *(Hovey) The adjunction $S \dashv L$ of endofunctors on $\mathscr{C}$ prolongs to an adjunction $S \dashv L$ of endofunctors on $\mathsf{Sp}\left(\mathscr{C}, S\right)$.*

Proof. For the prolongation it suffices to produce natural transformations $SS \longrightarrow SS$ and $SL \longrightarrow LS$. For the first we chose

$$\mathrm{id} : SS \longrightarrow S$$

and for the second the second we chose the natural transformation

$$SL \xrightarrow{\varepsilon} \mathrm{id} \xrightarrow{\eta} LS.$$

As in [30], the remainder of the proof that the prolongations are again adjoint to each other, is formal and left to the reader. □

Theorem 11.1.8. *(Hovey) Let I be the set of generating cofibrations of $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ and let J be the set of generating acyclic cofibrations. Let*

$$I_S = \bigcup_{n\in\mathbf{N}} F_nI$$

and let

$$J_S = \bigcup_{n\in\mathbf{N}} F_nJ.$$

These two sets define a cofibrantly generated model category structure on $\mathsf{Sp}\left(\mathscr{C}, S\right)$.

Proof. See Theorem 1.14 of [30]. □

Definition 11.1.9. Let the model category structure on $\mathsf{Sp}\left(\mathscr{C}, S\right)$ described above be called the **level model structure** on $\mathsf{Sp}\left(\mathscr{C}, S\right)$.

The level model structure on $\mathsf{Sp}\left(\mathscr{C}, S\right)$ however may not be the correct one. In particular, if we are interested in the stable homotopy theory associated to a model category of spaces $\mathscr{C}$ it probably is not. The correct model structure is a left Bousfield localization of the level structure on $\mathscr{C}$.

Definition 11.1.10. Let

$$\mathcal{S} = \left\{ F_{n+1}SQX \xrightarrow{\sigma_n^{QX}} F_nQX \middle| n \in \mathbf{N}, X \in \bigcup_{f \in I} \left\{ \mathsf{Cod}\left(f\right), \mathsf{Dom}\left(f\right) \right\} \right\}$$

where Q denotes a cofibrant replacement functor and the maps σ_n^{QX} are the maps adjoint to the identities

$$\mathrm{id}_{SQX} : SQX \longrightarrow SQX = \mathsf{Ev}_{n+1}F_nQX.$$

In definition 3.3 of [30] it is stated that we may localize $\mathsf{Sp}\left(\mathscr{C}, S\right)$ with the level model category structure at the set $\mathcal{S}$ provided $\mathscr{C}$ is endowed with a left proper cellular model category.

In particular, it should be noted that all Cisinski model structures, both pointed and unpointed, satisfy these hypotheses. Lastly, we see that this localization provides the desideratum.

Theorem 11.1.11. *If $\mathscr{C}$ is a Cisinski model category, then fibrant objects of $\mathsf{Sp}\left(\mathscr{C}, S\right)$ with respect to the stable model category structure are the level-wise fibrant L-spectra. What's more, the maps $\sigma_n^A : F_{n+1}SA \longrightarrow F_nA$ are stable weak equivalences.*

Proof. See Theorem 3.4 of [30]. □

11.2 Kan Spectra

Kan presents a model of spectra which turns on the observation that on the set of cells of a CW-spectrum the suspension introduces an equivalence relation; we may identify an m-cell of the

n^{th} space in a CW-spectrum $\psi : D^m \longrightarrow X_n$ with an $m+1$-cell $\psi' : D^{m+1} \longrightarrow X_{n+1}$ of the $n+1^{st}$ if ψ' factors through $\varphi_n \circ \Sigma\psi$. In this way spectra can be seen to be made up of so called stable cells. Kan realized that if a suspension functor for simplicial sets such that a simplex suspends to another simplex could be had we could model spectra much as we model spaces by simplicial sets.

Definition. Let $K : \triangle \longrightarrow \triangle$ be the functor which assigns $[n] \longmapsto [n+1]$, $d^i \longmapsto d^i$, and $s^j \longmapsto s^j$. Let Σ_K be the left Kan extension along the composition $\triangle \xrightarrow{\mathsf{Yon}} \widehat{\triangle} \xrightarrow{(\_)_+} \widehat{\triangle}_\bullet$ of the functor $\triangle \longrightarrow \widehat{\triangle}_\bullet$, which assigns

$$[n] \longmapsto \triangle^{K([n])}/_{\triangle^n_+ \vee \triangle^0_+},$$

where the inclusion $\triangle^n \longrightarrow \triangle^{K([n])}$ is the map d^{n+1} and the inclusion of the point is opposite that face. Let Ω_K denote the right adjoint to Σ_K.

Note then that $\Sigma_K \triangle^n$ has exactly $d^0, \ldots, d^n : \triangle^n \longrightarrow \Sigma_K \triangle^n$ as non trivial faces and $d^{n+1} = \bullet$. For any $\ell \in \mathbf{N}$ then, $\Sigma_K^\ell \triangle^n_+$ is an ℓ sphere with n-many non-degenerate sides and in the same configuration as those of an n-simplex. The non-trivial aspect of the combinatorics is dimension invariant.

In order to construct spectra then we can either then stabilize simplicial sets at Σ_K by taking sequential spectra or we can first stabilize $\triangle$ at K.

Definition. Let $\triangle_{\mathsf{st}}$ be the strict colimit in Cat of the diagram

$$\triangle \xrightarrow{K} \triangle \xrightarrow{K} \cdots .$$

This category is isomorphic to the category whose set of objects is $\mathbf{Z}$ with morphisms generated by coface maps $d^i : z \longrightarrow z+1$ for each $i \in \mathbf{N}$ and codegeneracy maps $s^j : z+1 \longrightarrow z$ for each $j \in \mathbf{N}$ subject to the co-simplicial relations.

Definition. Let $\mathsf{K{-}Sp}$ be the full subcategory of the category of presheaves of pointed sets, $\widehat{\triangle_{\mathsf{st}\bullet}}$, subtended by those presheaves X such that for all $z \in \mathbf{Z}$, and $x \in X(z)$, there exists some $m \in \mathbf{N}$ such that $d^{m+i}(x) = \bullet$ for all $i \in \mathbf{N}$. We'll refer to this vanishing property as **local finiteness**.

The presentation above of Kan's model, found in [17] then culminates in the proposition which follows.

Proposition. *(Kan) Let* $\Omega\mathsf{Sp}\left(\widehat{\triangle}_{\bullet}, \Sigma_K\right)$ *denote the full subcategory of* $\mathsf{Sp}\left(\widehat{\triangle}_{\bullet}, \Sigma_K\right)$ *subtended by the objects*

$$\left((X_i)_{i\in\mathbf{N}}, (\varphi_i : \Sigma_K X_i \longrightarrow X_{i+1})\right)$$

for which the adjoints $\overline{\varphi_i} : X_i \longrightarrow \Omega_K X_{i+1}$ *are isomorphisms. Then, the category* $\mathsf{K}{-}\mathsf{Sp}$ *is equivalent to the sub-category*

$$\Omega\mathsf{Sp}\left(\widehat{\triangle}_{\bullet}, \Sigma_K\right) \hookrightarrow \mathsf{Sp}\left(\widehat{\triangle}_{\bullet}, \Sigma_K\right)$$

of sequential spectra.

Chapter 12

Z-categories: Strict and Weak

In section 3.4 we discussed how the categories Θ_n and Θ serve as the underlying categories for sketches for the essentially algebraic theory of strict n-categories and strict ω-categories. More formally, we recalled the following definition and proposition which we credit to Ara and Berger.

Definition 12.0.1. Let

$$\mathsf{V}_n = \left\{ V^T \subset \Theta_n^T \right\}_{T \in \mathsf{Ob}(\Theta_n)}$$

be the sets of sieves associated to the globular presentations of the objects of Θ_n, and likewise let

$$\mathsf{V} = \left\{ V^T \subset \Theta^T \right\}_{T \in \mathsf{Ob}(\Theta)} .$$

Proposition 12.0.2. *(Ara, Berger) For each $n \in \mathbf{N}$, the nerve functors*

$$\mathscr{N}_n : n - \mathsf{Cat} \longrightarrow \widehat{\Theta_n}$$

induce equivalences of categories

$$n - \mathsf{Cat} \xrightarrow{\sim} \mathsf{Mod}\left(\Theta_n, \mathsf{V}_n\right) .$$

More, the ω-nerve

$$\mathscr{N}_\omega : \omega - \mathsf{Cat} \longrightarrow \widehat{\Theta}$$

induces an equivalence of categories

$$\omega - \mathsf{Cat} \xrightarrow{\sim} \mathsf{Mod}\left(\Theta, \mathsf{V}\right) .$$

In that development we used one of two obvious functors $\triangle \longrightarrow \triangle \wr \triangle$. We used the functor $[n] \longmapsto [[n]\,;([0]\,,\dots,[0])]$ to define Θ as the colimit

$$\varinjlim \left\{ \triangle \xrightarrow{\gamma} \triangle \wr \triangle \xrightarrow{\triangle \wr \gamma} \triangle \wr (\triangle \wr \triangle) \longrightarrow \dots \right\}.$$

There is however another obvious functor $\triangle \longrightarrow \triangle \wr \triangle$, the functor which maps the object $[n]$ to the object $[[1]\,;[n]]$. This functor in turn begets an endofunctor $J : \Theta \longrightarrow \Theta$. In this chapter we will formally invert J and thereby discover an elegant definition for the abiding concept of $\mathbf{Z}$-category, a notion much like categories but for which everything is a morphism of some degree $z \in \mathbf{Z}$ with source and target morphisms of degree $z - 1$.

12.1 The Shift $J : \Theta \longrightarrow \Theta$

Definition 12.1.1. Define the functor J by the following formula.

$$\begin{tikzcd}
J : \Theta \arrow[r] & \Theta \\
T \arrow[r, mapsto] & {[[1]\,;T]} \\
(f : S \to T) \arrow[r, mapsto] & {\left[\mathrm{id}_{[1]}; f\right]}
\end{tikzcd}$$

Notation 12.1.2. A suggestive notation for J which we will often make use of is to let $J\,(T) = T + 1$. The purpose of the two notations is clarity, as depending on context one or the other is simpler.

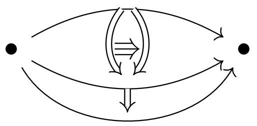
This functor J is a two point suspension functor for pasting diagrams. Indeed,

$$J\,(\overline{n}) = \overline{n} + 1 = \overline{n+1}$$

and more, a pasting diagram T of $0 \leq i \leq n$ dimensional arrows is assigned to a pasting diagram of $1 \leq i \leq n+1$ dimensional arrows stretched out between two new zero-cells of $J\,(T)$. For example, see that the whiskering datum

$$\begin{tikzcd}
\bullet \arrow[d, bend right=60, ""{name=L}] \arrow[d, bend left=60, ""'{name=R}] \arrow[from=L, to=R, Rightarrow, shorten=2pt] \\
\bullet \arrow[d] \\
\bullet
\end{tikzcd}$$

is sent by J to the higher composition



This intuition is also manifest in the following lemma regarding the non-degenerate cells of an object $J(T)$ of Θ.

Lemma 12.1.3. *Given an object $T \in \mathsf{Ob}(\Theta)$, then*

$$\mathsf{Hom}_{\Theta^+}(S, T+1) = \begin{cases} \{d^1, d^0\} & S = [0] \\ \mathsf{Hom}_{\Theta^+}(S', T) & S = S' + 1 \\ \varnothing & S \neq [0], S' + 1 \end{cases}$$

where Θ^+ is the direct subcategory of the Reedy structure on Θ.

Proof. Indeed, we may observe that

$$\begin{aligned} \mathsf{Hom}_{\Theta^+}([0], [[1]; T]) &\overset{\sim}{\to} \mathsf{Hom}_{\triangle}([0], [1]) \\ &= \{d^1, d^0\}, \end{aligned}$$

$$\begin{aligned} \mathsf{Hom}_{\Theta^+}\left(\left[[1], S'\right], [[1]; T]\right) &= \left\{[\mathrm{id}; \varphi] \mid \varphi : S' \longrightarrow T \text{ in } \Theta^+\right\} \\ &\overset{\sim}{\to} \mathsf{Hom}_{\Theta^+}\left(S', T\right), \end{aligned}$$

and if $S = [[n]; S_1, \ldots, S_n]$ with $n \geq 2$ then $\mathsf{Hom}_{\triangle^+}([n], [1]) = \varnothing$ so $\mathsf{Hom}_{\Theta^+}(S, T+1) = \varnothing$ for $S \neq [0], S' + 1$. ☐

Since Θ is not possessed of many limits and in particular lacks any products but the trivial one, the lemma above is as close as we'll get to finding a functor which is right adjoint to J. The functor J however does preserve colimits and in particular J preserves the globular presentations of the cells of Θ.

Lemma 12.1.4. *The functor $J : \Theta \longrightarrow \Theta$ preserves globular sum decomposition of cells of Θ.*

Proof. Observe that $J\left(A\left(n_0, m_1, \ldots, m_\ell, n_\ell\right)\right) = A\left(n_0+1, m_1+1, \ldots, m_\ell+1, n_\ell+1\right)$. $\square$

In fact, as a consequence, all colimits in Θ are preserved by J.

Corollary 12.1.5. *The functor $J : \Theta \longrightarrow \Theta$ preserves all colimits in Θ.*

Proof. Since any colimit which exists in Θ may be refined into a globular sum presentation, this follows from the previous lemma. $\square$

As a result, the sieves which define the sketch (Θ, V), are also preserved under J in the following sense. Indeed, recall that

$$\mathsf{V} = \left\{V^T \hookrightarrow \Theta^T\right\}_{T \in \mathsf{Ob}(\Theta)}$$

where for each $T = A\left(n_0, m_1, \ldots, m_\ell, n_\ell\right)$, we set $V^T = \bigcup_{i \in \langle \ell \rangle} \Theta^{\overline{n_i}}$ taken as a union of subfunctors of Θ^T. It is then easy to see that the following corollary holds.

Corollary 12.1.6. *Let*

$$J : \widehat{\Theta} \longrightarrow \widehat{\Theta}$$

denote by an abuse of notation the left Kan extension of J along the Yoneda embedding. Then, for any $T \in \mathsf{Ob}(\Theta)$, we have that $V^{J(T)} \xrightarrow{\sim} J\left(V^T\right)$.

Proof. The proof is a formal consequence of the preceding corollary. $\square$

12.2 Formally inverting J

Definition 12.2.1. Let Θ_{st} be the colimit of the sequence

$$\Theta \xrightarrow{J} \Theta \xrightarrow{J} \cdots .$$

taken in Cat and let, for each $n \geq 0$,

$$\Phi^{\infty - n} : \Theta \longrightarrow \Theta_{\mathsf{st}}$$

denote the canonical functor into the n^{th}- copy of Θ in the sequence $\Theta \xrightarrow{J} \Theta \xrightarrow{J} \cdots$.

Observe that for all $n \geq 0$, the diagrams

$$\begin{tikzcd}
\Theta \arrow[rr, "J"] \arrow[dr, "\Phi^{\infty-n}"'] & & \Theta \arrow[dl, "\Phi^{\infty-n-1}"] \\
& \Theta_{\mathsf{st}} &
\end{tikzcd}$$

commute. As such, the $\mathbf{N}$-indexed family of functors, $\{\Phi^{\infty-n}\}_{n\in\mathbf{N}}$, naturally extends to a $\mathbf{Z}$-indexed one.

Definition 12.2.2. Let $n > 0$, and let $\Phi^{\infty+n} : \Theta \longrightarrow \Theta_{\mathsf{st}}$ be defined by the formula

$$\Phi^{\infty+n} = \Phi^{\infty-0} \circ J^n.$$

Now, for all $z \in \mathbf{Z}$, the diagrams

$$\begin{tikzcd}
\Theta \arrow[rr, "J"] \arrow[dr, "\Phi^{\infty-z}"'] & & \Theta \arrow[dl, "\Phi^{\infty-(z+1)}"] \\
& \Theta_{\mathsf{st}} &
\end{tikzcd}$$

commute.

Notation 12.2.3. When it is easier to do so, we will denote $\Phi^{\infty-z}(T) \in \mathsf{Ob}(\Theta_{\mathsf{st}})$ by the shorter T_{-z}. Note that the negative sign in T_{-z} will not appear in general, it is only necessary in relation to $\Phi^{\infty-n}$.

Remark 12.2.4. In the more compact notation, the commutation of this last triangle may now be phrased as

$$T_z = (T+1)_{(z-1)}.$$

This simple equality will feature prominently in many computations going forward.

12.3 The reflexive Z-globular category $\mathbf{G}_{\mathbf{Z}}$

Observe that the functor $J : \Theta \longrightarrow \Theta$ restricts along the embedding $\mathbf{G} \longrightarrow \Theta$ to a functor $\mathbf{G} \longrightarrow \mathbf{G}$ which, by abuse of notation, we also denote by J. Thus, the commutative diagram

$$\begin{tikzcd}
\mathbf{G} \arrow[r, "J"] \arrow[d] & \mathbf{G} \arrow[r, "J"] \arrow[d] & \mathbf{G} \arrow[r, "J"] \arrow[d] & \cdots \\
\Theta \arrow[r, "J"] & \Theta \arrow[r, "J"] & \Theta \arrow[r] & \cdots
\end{tikzcd}$$

provides us a full and faithful embedding

$$\varinjlim \left\{ \mathbf{G} \xrightarrow{J} \mathbf{G} \xrightarrow{J} \cdots \right\} = \mathbf{G}_{\mathsf{st}} \longrightarrow \Theta_{\mathsf{st}}.$$

But the formal inversion of J restricted to $\mathbf{G}$ admits a more elegant description.

Definition 12.3.1. Let $\mathbf{G}_{\mathbf{Z}}$ be the category with set of objects $\{\overline{z} |\, z \in \mathbf{Z}\}$ and morphisms generated by the diagram

$$\begin{tikzcd}
\vdots \arrow[d, "i"] \\
\overline{z+1} \arrow[u, bend left, "s"] \arrow[u, bend right, "t"'] \arrow[d, "i"] \\
\overline{z} \arrow[u, bend left, "s"] \arrow[u, bend right, "t"'] \arrow[d, "i"] \\
\overline{z-1} \arrow[u, bend left, "s"] \arrow[u, bend right, "t"'] \arrow[d, "i"] \\
\vdots \arrow[u, bend left, "s"] \arrow[u, bend right, "t"']
\end{tikzcd}$$

subject to the reflexive globular identities,

$$\begin{aligned}
s \circ t &= s \circ s \\
t \circ t &= t \circ s \\
i \circ s &= i \circ t.
\end{aligned}$$

This category $\mathbf{G}_{\mathbf{Z}}$ can be seen isomorphic to the colimit $\mathbf{G}_{\mathsf{st}}$.

Proposition 12.3.2. *The categories* $\mathbf{G}_{\mathbf{Z}}$ *and* $\mathbf{G}_{\mathsf{st}}$ *are isomorphic.*

Proof. For each $n \geq 0$, $\zeta^{\infty-n} : \mathbf{G} \longrightarrow \mathbf{G}_{\mathbf{Z}}$, denote the embeddings $\overline{m} \longmapsto \overline{m-n}$. These embeddings are compatible with J in the sense that the triangles

$$\begin{tikzcd}
\mathbf{G} \arrow[rr, "J"] \arrow[dr, "\zeta^{\infty-z}"'] & & \mathbf{G} \arrow[dl, "\zeta^{\infty-(z+1)}"] \\
& \mathbf{G}_{\mathbf{Z}} &
\end{tikzcd}$$

commute. As such, we get a functor $\mathbf{G}_{\mathsf{st}} \longrightarrow \mathbf{G}_{\mathbf{Z}}$, and it is purely formal to see that this functor is an equivalence of categories. □

12.4 A Sketch for the essentially algebraic theory of Z-categories

In Lemma 3.3.3 and its corollaries we worked with the canonical presentation of any object T of Θ as the colimit

$$\varinjlim \left\{ \begin{tikzcd}
\overline{n_0} & & \cdots & & \overline{n_\ell} \\
& \overline{m_1} \arrow[ul] \arrow[ur] & & \overline{m_\ell} \arrow[ul] \arrow[ur] &
\end{tikzcd} \right\}$$

for some $n_0, m_1, \ldots, m_\ell, n_\ell \in \mathbf{N}$. As a consequence, we saw that if

$$V^T = \bigcup_{i \in [\ell]} \Theta^{\overline{n_i}}$$

then $J\left(V^T\right) = \bigcup_{i \in [\ell]} \Theta^{\overline{n_i+1}} \subset \Theta^{J(T)}$. This endows Θ_{st} with a generation property with respect to the embedding $\mathbf{G}_{\mathbf{Z}} \longrightarrow \Theta_{\mathsf{st}}$ much like that of Θ with respect to $\mathbf{G} \longrightarrow \Theta$.

Lemma 12.4.1. *Given any object* T_z *of* Θ_{st}*, there exists a unique list of integers,*

$$n_0, m_1, n_1, \ldots, n_{\ell-1}, m_{\ell-1}, n_\ell$$

with each

$$m_i \leq n_{i-1}, n_i,$$

such that

$$\varinjlim \left\{ \begin{tikzcd} \overline{n_0} & & \overline{n_1} & & \overline{n_{\ell-1}} & & \overline{n_\ell} \\ & \overline{m_1} \arrow[ul, "s^{n_1-m_1}"] \arrow[ur, "t^{n_0-m_1}"'] & & \cdots \arrow[ul] \arrow[ur] & & \overline{m_{\ell-1}} \arrow[ul, "s^{n_\ell-m_{\ell-1}}"] \arrow[ur, "t^{n_{\ell-1}-m_{\ell-1}}"'] & \end{tikzcd} \right\} \xrightarrow{\sim} T_z.$$

Proof. It follows immediately from Lemma 3.3.3 and the definition of Θ_{st} that such a sequence of integers exists. The uniqueness follows formally from consideration of the definition of Θ_{st}. □

As such, the notion of the spine inclusions also stabilizes.

Definition 12.4.2. Given T_z an object of Θ_{st}, let $V^{T_z} = \bigcup \Theta_{\mathsf{st}}^{\overline{n_i}} \hookrightarrow \Theta_{\mathsf{st}}^{T_z}$ where T_z is given as in the lemma above by the sequence of integers $n_0, m_1, \ldots, m_\ell, n_\ell$. Let V be the set $\left\{ V^{T_z} \hookrightarrow \Theta_{\mathsf{st}}^{T_z} \right\}_{T_z \in \mathsf{Ob}(\Theta_{\mathsf{st}})}$. We define a **strict Z-category** to be a model for $(\Theta_{\mathsf{st}}, \mathsf{V})$, that is a presheaf X on Θ_{st} for which the canonical map $X(T_z) \longrightarrow X\left(V^{T_z}\right)$ is an isomorphism.

Example 12.4.3. (Lumsdaine-Shulman) Recall that every set S is classified by a functor $\overline{S} : \bullet \longrightarrow \mathsf{Set}$. This trivial observation can be formalized as a natural transformation of models for (Θ, V), equivalently strict ω-categories, as the obvious natural monomorphism

$$J\left(\mathscr{N}_\omega(\mathsf{Set})\right) \longrightarrow \mathscr{N}_\omega(\mathsf{Cat})$$

where $\mathscr{N}_\omega$ is the cellular nerve functor $\omega - \mathsf{Cat} \longrightarrow \widehat{\Theta}$. But every category $\mathscr{A}$ is classified by a 2-functor

$$\overline{\mathscr{A}} : \bullet \longrightarrow \mathsf{Cat},$$

equivalently a natural monomorphism $J\left(\mathscr{N}_\omega(\mathsf{Cat})\right) \longrightarrow \mathscr{N}_\omega(2 - \mathsf{Cat})$. Thus, we get a sequence of natural monomorphisms

$$J^2\left(\mathscr{N}_\omega(\mathsf{Set})\right) \longrightarrow J\left(\mathscr{N}_\omega(\mathsf{Cat})\right) \longrightarrow \mathscr{N}_\omega(2 - \mathsf{Cat}).$$

This process is infinitely iterable and in fact defines a **Z**-category as follows.

Letting $\Phi^{\infty - n} : \widehat{\Theta} \longrightarrow \widehat{\Theta_{\mathsf{st}}}$ denote, by an abuse of notation, the left Kan extensions of the functors $\Phi^{\infty-n}$ of section 12.2 along the Yoneda embedding, we may define the **Z**-category which encodes the notion described above as the following colimit.

$$\varinjlim \left\{ \Phi^{\infty - 0} \left(\mathscr{N}_{\omega} \left(\mathsf{Set} \right) \right) \longrightarrow \cdots \longrightarrow \Phi^{\infty - n} \left(\mathscr{N}_{\omega} \left(n - \mathsf{Cat} \right) \right) \longrightarrow \cdots \right\}$$

More interesting examples of the notion of **Z**-category do exist but require a weakening of the laws. In particular, we may quote Lurie, in [37] in the assertion that:

> “You can define a monoidal (∞, n)-category to be an $(\infty, n+1)$-category with a specified object, such that all other objects are isomorphic to it (in the complete Segal space model, this means that the space of objects should be connected). Similarly you can define a braided monoidal (∞, n)-category to be an $(\infty, n+2)$-category equipped with a distinguished object satisfying a simple connectivity condition, and so on and so forth. You get to the symmetric monoidal case by taking the homotopy inverse limit (that is, a symmetric monoidal (∞, n)-category is a collection of pointed $(\infty, n+k)$-categories, each of which is obtained by "looping" the next one”

Pointed weak **Z**-categories then should be to symmetric monoidal higher categories as spectra are to connective spectra. In the long view it can be hoped that that a synthesis of stable homotopy theory and monoidal category theory might be possible.

Chapter 13

Model Category Theory of the Functors J and Σ_J

In the previous chapter we developed an essentially algebraic theory of **Z**-categories by formally inverting Θ at the shift functor J which sent a cell T of Θ to the cell $[[1];T]$ of Θ. In this chapter we'll explore the model category theoretic properties of J and its relatives. We'll prove that J is left Quillen with respect to the test model structure on $\widehat{\Theta}$ and we'll develop a functor Σ_J, naturally specified by J, and show that Σ_J presents the reduced suspension of pointed cellular sets.

13.1 The Functor $J : \widehat{\Theta} \longrightarrow \widehat{\Theta}$ is Left Quillen

Recall that in Lemma 12.1.3 we proved that

$$\mathsf{Hom}_{\Theta^+}(S, T+1) = \begin{cases} \{d^1, d^0\} & S = [0] \\ \mathsf{Hom}_{\Theta^+}(S', T) & S = S' + 1 \\ \varnothing & S \neq [0], S' + 1 \end{cases} .$$

As a consequence, we find that while $J : \widehat{\Theta} \longrightarrow \widehat{\Theta}$ does not preserve boundaries, it comes very close to doing so.

Example 13.1.1. We'll compute the boundaries of $\Theta^{\overline{0}}$ and $\Theta^{\overline{1}}$. It is by definition that

$$\partial\Theta^{\overline{0}} = \varinjlim_{(S \to \overline{0}) \in \Theta^+ \downarrow \overline{0}} \Theta^S = \varnothing$$

as the category $\Theta^+ \downarrow \overline{0}$ is empty. But, since this is the empty colimit of presheaves, and J preserves colimits, then $J\left(\partial\Theta^{\overline{0}}\right) = \varnothing$. However, $\partial\Theta^{J(\overline{0})=\overline{1}}$is the coproduct $\Theta^{\overline{0}} \coprod \Theta^{\overline{0}}$ so J does not preserve the boundary of the 0-cell.

Nor as we'll show does J preserve the boundary of the $\overline{1}$ cell. Indeed, see that by definition we have the following equality.

$$\partial\Theta^{\overline{1}} = \varinjlim_{\left(S\to\overline{1}\right)\in\Theta^{+}\downarrow\overline{1}-\left\{\mathrm{id}_{\overline{1}}\right\}} \Theta^{S} = \Theta^{\overline{0}}\coprod\Theta^{\overline{0}}$$

So applying J to $\partial\Theta^{\overline{1}}$ yields

$$J\left(\partial\Theta^{\overline{1}}\right) = \Theta^{\overline{1}}\coprod\Theta^{\overline{1}}.$$

That cellular set however is not the boundary $\partial\Theta^{J\left(\overline{1}\right)}$. Instead, the boundary $\partial\Theta^{\overline{2}}$ is the colimit over the diagram

$$\varinjlim\left\{\begin{tikzcd}\Theta^{\overline{0}} \arrow[r, "d^{1}"] \arrow[dr, "d^{1}"] & \Theta^{\overline{1}} \\ \Theta^{\overline{0}} \arrow[r, "d^{0}"] \arrow[ur, "d^{0}"] & \Theta^{\overline{1}}\end{tikzcd}\right\}$$

As it turns out however, these are the only cases in which the boundary functor and J do not commute.

Proposition 13.1.2. *Let T be an object of $\Theta-\left\{\overline{0},\overline{1}\right\}$. Then*

$$J\left(\partial\Theta^{T}\right)\xrightarrow{\sim}\partial\Theta^{J(T)}.$$

We'll prove this proposition by way of the following lemma.

Lemma 13.1.3. *Let $F : A\to B$ be a functor and let D be a category with all colimits. Then for all diagram $G : B\longrightarrow D$, we have that $\varinjlim G\circ F=\varinjlim G$ if and only if, for all objects b of B, the categories $b\downarrow F$ satisfy the conditions:*

non-emptiness: the category $b\downarrow B$ is non-empty;

connectedness: for any pair of objects, $\left(f : b\to F\left(a\right)\right),\left(f' : b\to F\left(a'\right)\right)$

$$f\to x_{1}\leftarrow\cdots\to x_{n}\leftarrow f'$$

of morphisms connecting f and f'.

Proof. The lemma is an immediate consequence of Lemma 2.8 of [1]. ☐

We may now attend to the proof of our proposition.

Proof. See that the functor J induces a functor

$$J' : \left(\Theta^+ \downarrow T - \{\mathrm{id}_T\}\right) \longrightarrow \left(\Theta^+ \downarrow J(T) - \left\{\mathrm{id}_{J(T)}\right\}\right)$$

which sends an object $(f : S \to T)$ to the object $(J(f) : J(S) \to J(T))$. So, by the lemma, it suffices to show that for all objects $(f : S \longrightarrow J(T))$ of $\left(\Theta^+ \downarrow J(T) - \left\{\mathrm{id}_{J(T)}\right\}\right)$ we have that the category $(f : S \to J(T)) \downarrow J'$ is non-empty and connected.

Now, for any $(f : S \longrightarrow J(T))$, the category $(f : S \to J(T)) \downarrow J'$ is the category the objects of which are commutative diagrams

$$\begin{tikzcd}
S \arrow[rr, "h"] \arrow[dr, "f"'] & & J(R) \arrow[dl, "J(g)"] \\
& J(T) &
\end{tikzcd}$$

and the morphisms of which are commutative diagrams

$$\begin{tikzcd}
& S \arrow[dl, "h"'] \arrow[d, "f"] \arrow[dr, "h'"] & \\
J(R) \arrow[r, "J(g)"] \arrow[rr, bend right, "J(p)"'] & J(T) & J(R') \arrow[l, "J(g')"']
\end{tikzcd}$$

We first consider the non-emptiness. Since the only objects of $\left(\Theta^+ \downarrow J(T) - \left\{\mathrm{id}_{J(T)}\right\}\right)$ not of the form $J(f) : J(S) \longrightarrow J(T)$ are d^0, d^1, it suffices to show that, for all $T \neq \overline{0}, \overline{1}$:

- the category $(\Theta^+ \downarrow T - \{\mathrm{id}_T\})$ is non-empty; and
- that the categories

  $$\left(d^1 : \overline{0} \to J(T)\right) \downarrow J'$$

  and

  $$\left(d^0 : \overline{1} \longrightarrow J(T)\right) \downarrow J'$$

are non-empty.

So long as $T \neq \overline{0}$ the first condition holds, and we'll show that so long as $T \neq \overline{1}$, the second condition holds. Indeed, as a corollary to Theorem A.4.1, we have that any Θ^+ map of codimension $n \geq 2$, factors as a sequence of n-many Θ^+ maps of codimension 1, so as long as $T \neq \overline{1}$, it follows that the second condition holds.

Provided they are non-empty, all of the categories $(f : S \to J(T)) \downarrow J'$ are connected as by Theorem A.4.1 we may assume any two objects

$$\begin{tikzcd}
S \arrow[rr, "h"] \arrow[dr, "f"'] & & J(R) \arrow[dl, "J(g)"] \\
& J(T) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
S \arrow[rr, "h'"] \arrow[dr, "f"'] & & J(R') \arrow[dl, "J(g')"] \\
& J(T) &
\end{tikzcd}$$

are such that $J(g)$, m.m. $J(g')$, are cofaces, and for all $T \neq \overline{0}, \overline{1}$, the fibered product of co-faces is non-empty by Lemma A.5.4. □

An immediate consequence of this quasi-preservation of boundaries is that J does not preserve monomorphisms whence it cannot serve as a left Quillen functor for any Cisinski model category structure on $\widehat{\Theta}$.

Remark 13.1.4. We can eliminate this problem if, morally speaking, we take the quotient of J by this concern. In a precise sense, this will be done in the next section.

While J does not preserve boundaries, J does preserve Berger's horns, defined in section 9.2.

Proposition 13.1.5. *Let $T \neq \overline{0}$ be a cell of Θ and let $\kappa : S \longrightarrow T$ be a coface of T, then the canonical map $J(\Lambda^{\kappa}) \longrightarrow \Lambda^{J(\kappa)}$ induces an isomorphism*

$$J\left(\Lambda^{\kappa} \longrightarrow \Theta^{T}\right) \xrightarrow{\sim} \Lambda^{J(\kappa)} \longrightarrow \Theta^{J(T)}.$$

Proof. The proof is almost identical to that of Proposition 13.1.2. □

13.2 Defining $\Sigma_J : \widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta}_\bullet$

For any cell T there is a canonical monomorphism $d^1 \coprod d^0 : \partial\Theta^{\overline{1}} \longrightarrow \Theta^{J(T)}$. Taking the quotient of that target by that monomorphism gives us a reduced suspension functor on the image of $\Theta \hookrightarrow \widehat{\Theta}$.

Definition 13.2.1. Let $P : \partial\Theta^{\overline{1}} \longrightarrow J$ be the natural transformation which is component-wise the monomorphism $d^1 \coprod d^0 : \partial\Theta^{\overline{1}} \longrightarrow \Theta^{J(T)}$. Define the functor

$$\Sigma_J : \widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta}_\bullet$$

to be the left Kan extension along the composition of $\Theta \xrightarrow{\mathsf{Yon}} \widehat{\Theta} \xrightarrow{(\_)_+} \widehat{\Theta}_\bullet$ of the functor

$$\Theta \longrightarrow \widehat{\Theta}_\bullet : T \longmapsto {}^{\Theta^{J(T)}}\!/_{P_T}$$

where the base point is the unique 0-cell.

Since we defined Σ_J by way of colimit preservation and $\widehat{\Theta}_\bullet$ is locally presentable, then Σ_J admits a right adjoint.

Lemma 13.2.2. *The functor*

$$\begin{tikzcd} \Omega_J : \widehat{\Theta}_\bullet \arrow[rr] & & \widehat{\Theta}_\bullet \\ X \arrow[r, mapsto] & \Omega X : T \mapsto \mathsf{Hom}\left(\Sigma_J\left(T\right), X\right) & \end{tikzcd}$$

is right adjoint to Σ_J.

Proof. The proof is purely formal and left to the reader. □

The sections that follow are devoted to proving that Σ_J preserves monomorphisms and that Σ_J is naturally weakly equivalent to $(\_) \wedge S^1$. It is then as a corollary that we will find Σ_J to be a left Quillen suspension functor.

13.3 The functor Σ_J is left Quillen

The proof that Σ_J preserves is left Quillen with respect to the pointing of the test model structure, the transport of the test model structure on $\widehat{\Theta}$ to $\widehat{\Theta}_\bullet$, follows quickly from the proof that $J : \widehat{\Theta} \longrightarrow \widehat{\Theta}$ almost preserves boundaries.

Corollary 13.3.1. *The functor* $\Sigma_J : \widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta}_\bullet$ *is left Quillen with respect to the pointed test model structure on* $\widehat{\Theta}_\bullet$*.*

Proof. Recall that the set

$$(\mathscr{M}_\Theta)_+ = \left\{\partial\Theta^T_+ \longrightarrow \Theta^T_+\right\}_{T\in\mathsf{Ob}(\Theta)}$$

comprises a cellular model for $\widehat{\Theta}_\bullet$ and that the set

$$(\Lambda_{\mathsf{Ber}})_+ = \left\{\Lambda^\kappa_+ \longrightarrow \Theta^T_+\right\}_{T\in\mathsf{Ob}(\Theta),(\kappa:S\to T)\in\mathsf{CoFace}(T)}$$

serves as a set of generating acyclic cofibrations of the pointed test structure. Since Σ_J is definitionally colimit preserving and $\widehat{\Theta}_\bullet$ is weakly adhesive, to prove monomorphism preservation, it suffices to prove that

$$\Sigma_J\left((\mathscr{M}_\Theta)_+\right) \subset \mathsf{Mono}\left(\widehat{\Theta}_\bullet\right).$$

This follows quickly.

See that for all $T \neq \overline{0}, \overline{1}$, the maps $\Sigma_J\partial\Theta^T_+ \longrightarrow \Sigma_J\Theta^T_+$ are pushouts of the monomorphisms $\partial\Theta^{J(T)}_+ \longrightarrow \Theta^{J(T)}_+$ along the quotient map $J\left(\partial\Theta^T_+\right) = \partial\Theta^{J(T)}_+ \longrightarrow \Sigma_J\Theta^T_+$. For $T = \overline{1}$ see that the map

$$\Sigma_J\left(\partial\Theta^{\overline{1}}_+ \longrightarrow \Theta^{\overline{1}}_+\right)$$

is the monomorphism

$$S^1 \vee S^1 \longrightarrow \Sigma_J\Theta^{\overline{1}}_+$$

and see that $\Sigma_J\partial\Theta^{\overline{0}} = \bullet$ which emits canonical monomorphisms.

Likewise, to prove the preservation of weak equivalences by Σ_J, it suffices to prove that

$$\Sigma_J\left(\Lambda_{\mathsf{Ber}}\right) \subset \mathsf{TrivCof} \subset \mathsf{Mor}\left(\widehat{\Theta}_\bullet\right).$$

In that case however, a similar argument holds. Since $\mathsf{TrivCof}$ is closed under pushouts, and $\Sigma_J \Lambda^{\kappa}_{+} \longrightarrow \Sigma_J \Theta^{T}_{+}$ is the pushout of the generating acyclic cofibration $J\left(\Lambda^{\kappa}_{+} \longrightarrow \Theta^{T}_{+}\right) = \Lambda^{J(\kappa)}_{+} \longrightarrow \Theta^{J(T)}_{+}$ along the quotient map $\Lambda^{J(\kappa)}_{+} \longrightarrow \Sigma_J \Lambda^{\kappa}_{+}$, then $\Sigma_J\left(\left(\Lambda_{\mathsf{Ber}}\right)_{+}\right) \subset \mathsf{TrivCof} \subset \mathsf{Mor}\left(\widehat{\Theta_{\bullet}}\right)$.

Lastly, in the previous section we proved $\Sigma_J \dashv \Omega_J$. □

13.4 The Eckmann-Hilton degeneracies

Before we can prove that Σ_J is a reduced suspension functor, we require the definition of a family of maps we call the Eckmann-Hilton Degeneracies.

Definition 13.4.1. Then, given a cell T of Θ we define the **Eckmann-Hilton** degeneracy

$$E_T : J\left(T\right) \longrightarrow T$$

by recursion. Given a cell T of Θ we have that

$$T = \left[\left[k\right]; T_1 \cdots T_k\right]$$

for some $k \geq 0$ and cells $T_1, \ldots, T_k$ of Θ. In light of the isomorphism

$$\varinjlim \left\{ \begin{tikzcd} & T_1 + 1 \\ \overline{0} \arrow[ur] \arrow[dr] & \\ & \vdots \\ \overline{0} \arrow[ur] \arrow[dr] & \\ & T_k + 1 \end{tikzcd} \right\} \xrightarrow{\sim} T,$$

a commutative diagram

$$\begin{tikzcd} & J^2\left(T_1\right) \arrow[ddr, "\varphi_1"] & \\ \overline{1} \arrow[ur] \arrow[dr] & & \\ & \vdots & T \\ \overline{1} \arrow[ur] \arrow[dr] & & \\ & J^2\left(T_k\right) \arrow[uur, "\varphi_k"] & \end{tikzcd}$$

defines a morphism $J(T) \longrightarrow T$.

See then that setting, for each $i \in \langle k \rangle$, the maps

$$\varphi_i : J^2(T_i) = [[1]; J(T)] \longrightarrow [[k]; T_1 \cdots T_k]$$

to be the maps

$$[\{0,k\}; (\mathsf{in}_+, \ldots, \mathsf{in}_+, E_{T_i}, \mathsf{in}_-, \ldots, \mathsf{in}_-)],$$

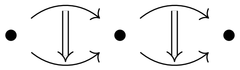
where by in_+ and in_- we mean the compositions of the canonical maps $J(T_i) \longrightarrow [0]$ and right-most (left most) maps $[0] \longrightarrow T_k$ and setting $E_{\overline{0}} : \overline{1} \longrightarrow \overline{0}$ to be the canonical map defines just such a commutative diagram; let E_T be the map $J(T) \longrightarrow T$ induced by that diagram.

Remark 13.4.2. An important fact to observe about the maps E_T is that $\mathsf{in}_- \circ E_T = \mathsf{in}_-$ and likewise $\mathsf{in}_+ \circ E_T = \mathsf{in}_+$.

Lemma 13.4.3. *The maps E_T comprise the components of a natural transformation*

$$E : J \longrightarrow id_\Theta.$$

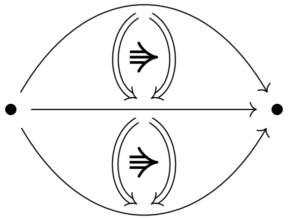
Example 13.4.4. A simple example of a component of the natural transformation E is given by setting $T = [[2]; [1]\,[1]]$, so T is the strict 2-category generated by the following pasting diagram.



Then

$$T + 1 = [[1]; [[2]; [1]\,[1]]]$$

and $J(T)$ is thus the strict 3-category generated by the following pasting diagram.



In the notation of the construction this puts $T = [[2]\,; T_1, T_2]$ where $T_1 = T_2 = \overline{1}$.

Our construction instructs us to consider $T+1$ as the 1-globular sum, $\overline{3} \underset{1}{\oplus} \overline{3}$, that is to say, the colimit

$$\varinjlim \left\{ \begin{tikzcd} & \overline{3} = J^2\left(T_1\right) \\ \overline{1} \arrow[ur] \arrow[dr] & \\ & \overline{3} = J^2\left(T_2\right) \end{tikzcd} \right\},$$

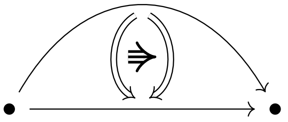
and to define the map $E_T : J\left(T\right) \longrightarrow T$ by way of this decomposition as $\varphi_1 \underset{1}{\oplus} \varphi_2$. Now, the formula for the maps evaluates here to

$$\varphi_1 = \left[\{0,2\}\,; \left(E_{\overline{0}_i}, \mathsf{in}_-\right)\right]$$

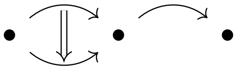
and

$$\varphi_1 = \left[\{0,2\}\,; \left(\mathsf{in}_+, E_{\overline{0}}\right)\right].$$

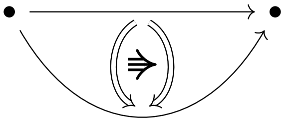
In terms of pasting diagrams the map φ_1 is the obvious degeneracy of the 3-cell



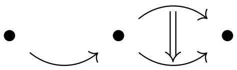
onto the whiskered 2-cell



and the map φ_2 is the obvious degeneracy of



onto the whiskered 2-cell



13.5 The Functor Σ_J is weakly equivalent to $\left(\_\right) \wedge S^1$

In this section we'll use the Eckmann-Hilton degeneracies of the previous section together with the shuffle decomposition of prisms $\Theta^T \times \Theta^S$ (see appendix A) to describe a natural weak equivalence $\widetilde{E} : \Sigma_J \Longrightarrow (\_) \wedge S^1$. We'll first present a component of the natural transformation $\widetilde{E}$ and then generalize that description.

Example 13.5.1. Suppose T to be a cell of Θ of the form

$$T = \left[[2]\,; L', R'\right] = L \underset{0}{\oplus} R$$

where L and R are given

$$L = J\left(L'\right) = \left[[1]\,; L'\right]$$

and

$$R = J\left(R\right) = \left[[1] : R'\right],$$

so

$$J\left(T\right) = J\left(L\right) \underset{1}{\oplus} J\left(R\right).$$

In such case we note that the shuffle decomposition of products for cellular sets, See appendix A.6 or [9], puts

$$\varinjlim \left\{ \begin{tikzcd}
 & \Theta^{[[3];[0],L',R']} \\
\Theta^{[[2];L',R']} \arrow[ur, "{[d^1;(\bullet,\mathrm{id}_{L'}),\mathrm{id}_{R'}]}" description] \arrow[dr, "{[d^1;(\mathrm{id}_{L'},\bullet),\mathrm{id}_{R'}]}" description] & \\
 & \Theta^{[[3];L',[0],R']} \\
\Theta^{[[2];L',R']} \arrow[ur, "{[d^2;\mathrm{id}_{L'},(\bullet,\mathrm{id}_{R'})]}" description] \arrow[dr, "{[d^2;\mathrm{id}_{L'},(\mathrm{id}_{R'},\bullet)]}" description] & \\
 & \Theta^{[[3];L',R',[0]]}
\end{tikzcd} \right\}_+ \xrightarrow{\sim} \Theta^T_+ \wedge \Theta^{\bar{1}}_+$$

We'll define a map

$$\Sigma_J \Theta_+^{[[2];L',R']} \longrightarrow \Theta_+^{[[2];L',R']} \wedge \Theta_+^{\bar{1}}$$

as a pair of maps

$$\Theta^{J(L)}, \Theta^{J(R)} \longrightarrow \Theta_+^{[[2];L',R']} \wedge \Theta_+^{\bar{1}}$$

described in terms of the colimit above which agree on their common 1-cell and thus define a map

$$\Theta^{J(L)\underset{1}{\oplus}J(R)} \longrightarrow \Theta_+^{[[2];L',R']} \wedge \Theta_+^{\bar{1}}$$

which pushes out along the quotient

$$\Theta_+^{J(L)\underset{1}{\oplus}J(R)} \longrightarrow \Sigma_J \Theta_+^{L\underset{0}{\oplus}R}$$

to a commuting square

$$\begin{tikzcd}
\Theta_+^{J(L)\underset{1}{\oplus}J(R)} \arrow[r] \arrow[d] & \Theta_+^{[[2];L',R']} \wedge \Theta_+^{\bar{1}} \arrow[d] \\
\Sigma_J \Theta_+^{L\underset{0}{\oplus}R} \arrow[r] & \Theta_+^{[[2];L',R']} \wedge S^1
\end{tikzcd} .$$

For the promised pair of maps we'll use

$$[\{0,3\}\,;(E_{L'},\bullet,\mathsf{in}_-)] : J\,(L) = [[1]\,;L] \longrightarrow \left[[3]\,;L',[0]\,,R'\right]$$

and

$$[\{0,3\}\,;(\mathsf{in}_+,E_{R-1},\bullet)] : J\,(R) = [[1]\,;R] \longrightarrow \left[[3]\,;L',R',[0]\right] .$$

We may then observe that the diagram

$$\begin{tikzcd}
 & {[[1]\,;L]} \arrow[r, "{[\{0,3\};(E_{L'},\bullet,\mathsf{in}_-)]}"] & {[[3]\,;L',[0]\,,R']} \\
{[1]} \arrow[ur, "{\left[\mathrm{id}_{[1]};\mathsf{in}_+\right]}"] \arrow[rr, "{[\{0,2\};\mathsf{in}_+,\mathsf{in}_-]}"] \arrow[dr, "{\left[\mathrm{id}_{[1]};\mathsf{in}_-\right]}"'] & & {[[2]\,;L',R']} \arrow[u, "{\left[d^2;\mathrm{id}_{L'},(\bullet,\mathrm{id}_{R'})\right]}"'] \arrow[d, "{\left[d^2;\mathrm{id}_{L'},(\mathrm{id}_{R'},\bullet)\right]}"] \\
 & {[[1]\,;R]} \arrow[r, "{[\{0,3\};(\mathsf{in}_+,E_{R'},\bullet)]}"'] & {[[3]\,;L',R',[0]]}
\end{tikzcd}$$

commutes. Indeed, see that the commutativity of the top squares is granted by the equality of the multimorphism

$$\begin{tikzcd}
 & & L' \\
{[0]} \arrow[r, "\mathsf{in}_+"] & L \arrow[ur, "E_{L'}"] \arrow[r, "\bullet"] \arrow[dr, "\mathsf{in}_-"] & {[0]} \\
 & & R'
\end{tikzcd}$$

and the multimorphism

$$\begin{tikzcd}
 & L' \arrow[r, "\mathrm{id}"] & L' \\
{[0]} \arrow[ru, "\mathsf{in}_+"] \arrow[r, "\mathsf{in}_-"] & R' \arrow[r, "\bullet"] \arrow[rd, "\mathrm{id}"'] & {[0]} \\
 & & R'
\end{tikzcd}$$

as $E_{L'} \circ \mathsf{in}_+ = \mathsf{in}_+$, $\bullet \circ \mathsf{in}_+ = \bullet \circ \mathsf{in}_-$, and the fact that since $\mathsf{in}_- : L \longrightarrow R'$ is the composition $L \longrightarrow [0] \xrightarrow{\mathsf{in}_-} R'$ we have that $\mathsf{in}_- \circ \mathsf{in}_+ = \mathsf{in}_-$. Similarly, the commutativity of the bottom square is granted by the equality of the multimorphism

$$\begin{tikzcd}
 & L' \arrow[r, "\mathrm{id}"] & L' \\
{[0]} \arrow[ru, "\mathsf{in}_+"] \arrow[r, "\mathsf{in}_-"] & R' \arrow[r, "\bullet"] \arrow[rd, "\mathrm{id}"'] & {[0]} \\
 & & R'
\end{tikzcd}$$

and the multimorphism

$$\begin{tikzcd}
 & & L' \ . \\
{[0]} \arrow[r, "\mathsf{in}_-"] & R \arrow[ru, "\mathsf{in}_+"] \arrow[r, "\bullet"] \arrow[rd, "E_{R'}"'] & {[0]} \\
 & & R'
\end{tikzcd}$$

Thus we are granted the promised morphism

$$\Theta^{J(L)\underset{1}{\oplus}J(R)} \longrightarrow \Theta_+^{[[2];L',R']} \wedge \Theta_+^{\bar{1}}$$

which is easily seen to pushout along the quotient

$$\Theta_+^{J(L)\underset{1}{\oplus}J(R)} \longrightarrow \Sigma_J \Theta_+^{L\underset{0}{\oplus}R}$$

to define a commuting square

$$\begin{tikzcd}
\Theta_+^{J(L)\underset{1}{\oplus}J(R)} \arrow[r] \arrow[d] & \Theta_+^{[[2];L',R']} \wedge \Theta_+^{\bar{1}} \ . \arrow[d] \\
\Sigma_J \Theta_+^{L\underset{0}{\oplus}R} \arrow[r, "\widetilde{E}_{L\underset{0}{\oplus}R}"] & \Theta_+^{[[2];L',R']} \wedge S^1
\end{tikzcd}$$

This example assembles all of the ingredients necessary to define $\widetilde{E}$ in full generality.

Definition 13.5.2. Given an object T of Θ, let

$$\widetilde{E}_T : \Sigma_J \Theta_+^T \longrightarrow \Theta_+^T \wedge S^1$$

be the pushout along the quotient

$$\Theta_+^{J(T)} \longrightarrow \Sigma_J \Theta_+^T$$

of the map

$$\Theta^{J(T)} \longrightarrow \Theta_+^T \wedge \Theta^{\overline{1}}$$

defined as follows.

Suppose

$$T = \left[[\ell]\,; A_1', \ldots, A_\ell'\right] = A_1 \underset{0}{\oplus} \cdots \underset{0}{\oplus} A_\ell$$

where for each $i \in \langle \ell \rangle$, $J\left(A_i'\right) = A_\ell$, so

$$J\left(T\right) = [[1]\,; [[\ell]\,; A_1 \ldots A_\ell]] = J\left(A_1\right) \underset{1}{\oplus} \cdots \underset{1}{\oplus} J\left(A_\ell\right).$$

Then that the shuffle decomposition of prisms provides the presentation

$$\varinjlim \left\{ \begin{tikzcd} & X_0 \\ \Theta^T \arrow[ur] \arrow[dr] & \\ & X_1 \\ \vdots \arrow[ur] \arrow[dr] & \\ & X_{\ell-1} \\ \Theta^T \arrow[ur] \arrow[dr] & \\ & X_\ell \end{tikzcd} \right\}_+ \xrightarrow{\sim} \Theta_+^T \wedge \Theta_+^{[1]}$$

where for each $j \in [\ell]$, $X_j = \Theta^{\left[[\ell+1];A_1',\ldots,A_j',[0],A_{j+1}',\ldots,A_\ell'\right]}$. We may then define

$$\widetilde{E}_T : \Sigma_J \Theta_+^T \longrightarrow \Theta_+^T \wedge S^1$$

to be the pushout along $\Theta_+^{J(T)} \longrightarrow \Sigma_J \Theta_+^T$ of the 1-globular sum

$$\left[\{0, \ell+1\}\,; \left(E_{A_1'}, \bullet\right), \mathsf{in}_-, \ldots, \mathsf{in}_-\right] \underset{1}{\oplus} \cdots \underset{1}{\oplus} \left[\{0, \ell+1\}\,; \mathsf{in}_+, \ldots, \mathsf{in}_+ \left(E_{A_\ell'}, \bullet\right)\right].$$

See that for each $0 \leq j < \ell$, the diagrams

$$\begin{tikzcd}
. & {[[1]\,;A_j]} \arrow[r] & {\left[[\ell+1]\,;A'_1,\dots,A'_j,[0]\,,A'_{j+1},\dots,A'_\ell\right]} \\
{[1]} \arrow[ur, "{\left[\mathrm{id}_{[1]};\mathsf{in}_+\right]}"] \arrow[rr] \arrow[dr, "{\left[\mathrm{id}_{[1]};\mathsf{in}_-\right]}"'] & & {[[\ell]\,;A'_1,\dots,A'_\ell]} \arrow[u, "{\left[d^{j+1};\mathrm{id},\dots,\mathrm{id},(\bullet,\mathrm{id}),\mathrm{id},\dots,\mathrm{id}\right]}"'] \arrow[d, "{\left[d^{j+1};\mathrm{id},\dots,\mathrm{id},(\mathrm{id},\bullet),\mathrm{id},\dots,\mathrm{id}\right]}"] \\
& {[[1]\,;A_{j+1}]} \arrow[r] & {\left[[\ell+1]\,;A'_1,\dots,A'_{j+1},[0]\,,A'_{j+2},\dots,A'_\ell\right]}
\end{tikzcd}$$

commute, where the top unlabelled morphism in the diagram is the map

$$\left[\{0,\ell+1\}\,;\left(\mathsf{in}_+,\dots,\mathsf{in}_+,\left(E_{A'_j},\bullet\right),\mathsf{in}_-,\dots,\mathsf{in}_-\right)\right]$$

the middle unlabeled morphism in the diagram is the map

$$\left[\{0,\ell\}\,;\left(\begin{cases}\mathsf{in}_+ : [0] \to A_i & i \leq j \\ \\ \mathsf{in}_- : [0] \to A_i & i > j\end{cases}\right)_{i\in\langle\ell\rangle}\right]$$

and the bottom unlabeled morphism in the diagram is the map

$$\left[\{0,\ell+1\}\,;\left(\mathsf{in}_+,\dots,\mathsf{in}_+,\left(E_{A'_{j+1}},\bullet\right),\mathsf{in}_-,\dots,\mathsf{in}_-\right)\right].$$

Indeed, the commutativity of the top square is granted by the equality of multimorphisms

$$\begin{tikzcd}
& & A'_1 & & & A'_1 \arrow[r, "\mathrm{id}"] & A'_1 \\
& & \vdots & & & \vdots & \vdots \\
& & A'_{j-1} & & & A'_{j-1} \arrow[r, "\mathrm{id}"] & A'_{j-1} \\
{[0]} \arrow[r, "\mathsf{in}_+"] & A_j \arrow[uuur, "\mathsf{in}_+"] \arrow[ur, "\mathsf{in}_+"] \arrow[r, "E_{A'_j}"] \arrow[dr, "\bullet"] \arrow[ddr, "\mathsf{in}_-"] \arrow[ddddr, "\mathsf{in}_-"] & A'_j & = & {[0]} \arrow[uuur, "\mathsf{in}_+"] \arrow[ur, "\mathsf{in}_+"] \arrow[r, "\mathsf{in}_+"] \arrow[ddr, "\mathsf{in}_-"] \arrow[ddddr, "\mathsf{in}_-"] & A'_j \arrow[r, "\mathrm{id}"] & A'_j \\
& & {[0]} & & & & {[0]} \\
& & A'_{j+1} & & & A'_{j+1} \arrow[ur, "\bullet"] \arrow[r, "\mathrm{id}"] & A'_{j+1} \\
& & \vdots & & & \vdots & \vdots \\
& & A'_\ell & & & A'_\ell \arrow[r, "\mathrm{id}"] & A'_\ell
\end{tikzcd}$$

and the commutativity of the bottom square follows from the equality of multimorphisms

$$\begin{tikzcd}
 & A'_1 \arrow[r, "\mathrm{id}"] & A'_1 & & & & A'_1 \\
 & \vdots & \vdots & & & & \vdots \\
 & A'_j \arrow[r, "\mathrm{id}"] & A'_j & & & & A'_j \\
{[0]} \arrow[uuur, "\mathsf{in}_+"] \arrow[ur, "\mathsf{in}_+"] \arrow[r, "\mathsf{in}_+"] \arrow[ddr, "\mathsf{in}_-"] \arrow[ddddr, "\mathsf{in}_-"] & A'_{j+1} \arrow[r, "\mathrm{id}"] \arrow[dr] \arrow[r, phantom, "\bullet"'] & A'_{j+1} & = & {[0]} \arrow[r, "\mathsf{in}_+"] & A_{j+1} \arrow[uuur, "\mathsf{in}_+"] \arrow[ur, "\mathsf{in}_+"] \arrow[r, "E_{A'_{j+1}}"] \arrow[dr] \arrow[r, phantom, "\bullet"'] \arrow[ddr, "\mathsf{in}_-"] \arrow[ddddr, "\mathsf{in}_-"] & A'_{j+1} \\
 & & {[0]} & & & & {[0]} \\
 & A'_{j+2} \arrow[r, "\mathrm{id}"] & A'_{j+2} & & & & A'_{j+2} \\
 & \vdots & \vdots & & & & \vdots \\
 & A'_\ell \arrow[r, "\mathrm{id"] & A'_\ell & & & & A'_\ell
\end{tikzcd}$$

Lemma 13.5.3. *For each cell T of Θ, the maps*

$$\widetilde{E}_T : \Sigma_J \Theta_+^T \longrightarrow \Theta_+^T \wedge S^1$$

are weak equivalences with respect to the test model structure.

Proof. It is purely formal to see that there are deformation retractions

$$S^1 \longrightarrow \Sigma_J \Theta_+^T \longrightarrow S^1$$

and

$$S^1 \longrightarrow \Theta_+^T \wedge S^1 \longrightarrow S^1$$

where the maps $\Sigma_J \Theta_+^T \longrightarrow S^1$ and $\Theta_+^T \wedge S^1 \longrightarrow S^1$ are the obvious weak equivalences. Along these, $\widetilde{E}$ retracts to the identity on S^1, whence $\widetilde{E}$ is natural weak equivalence. □

It is then that we may present the theorem which is the purpose of this section. The natural transformation

$$\widetilde{E} : \Sigma_J \Longrightarrow (\_) \wedge S^1$$

of functors

$$\Theta \longrightarrow \widehat{\Theta}_\bullet$$

extends to one of their extensions

$$\Sigma_J \Longrightarrow (\_) \wedge S^1 : \widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta}_\bullet.$$

We've already seen that, on Θ, $\widetilde{E}$ is a natural weak equivalence. As it so happens, the theory of Reedy (or skeletal) categories allows us to extend this result to the extension $\widetilde{E} : \Sigma_J \Longrightarrow (\_) \wedge S^1 : \widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta}_\bullet$

Theorem 13.5.4. *The left Kan extension of*

$$\widetilde{E} : \Sigma_J \Longrightarrow (\_) \wedge S^1 : \Theta \longrightarrow \widehat{\Theta}_\bullet$$

to a natural transformation of functors $\widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta}_\bullet$ *is a natural weak equivalence with respect to the test model structure.*

Proof. Since Θ is Reedy, it follows that $\Theta \downarrow X$ is Reedy for any (pointed) cellular set X. Then, as colimits of Reedy cofibrant diagrams are homotopy colimits, natural weak equivalences between cofibrant diagrams

$$\Theta \downarrow X \longrightarrow \widehat{\Theta}_{(\bullet)}$$

pass under $\varinjlim$ to weak equivalences in $\widehat{\Theta}_{(\bullet)}$.

See then that for any pointed cellular set X, it is by definition that

$$\Sigma_J X = \varinjlim_{(\Theta^T \to X) \in \Theta \downarrow X} \Sigma_J \Theta^T_+$$

and it is by colimit preservation that

$$X \wedge S^1 = \varinjlim_{(\Theta^T \to X) \in \Theta \downarrow X} \Theta^T_+ \wedge S^1$$

so $\widetilde{E}$ comprises a natural weak equivalence between the diagrams

$$\begin{tikzcd}
\Sigma_J : \Theta \downarrow X \arrow[r] & \widehat{\Theta}_\bullet \\
(\Theta^T \to X) \arrow[r, mapsto] & \Sigma_J \Theta^T_+
\end{tikzcd}$$

and

$$\begin{tikzcd}(\_) \wedge S^1 : \Theta \downarrow X \arrow[r] & \widehat{\Theta}_{\bullet} & . \\ \left(\Theta^T \to X\right) \arrow[r, mapsto] & \Theta^T_+ \wedge S^1\end{tikzcd}$$

To prove the theorem then it suffices to prove those diagrams to be Reedy cofibrant.

Now, for a diagram $F : \Theta \downarrow X \longrightarrow \widehat{\Theta}_{\bullet}$, the criterion of Reedy cofibrancy is that for all $\left(\Theta^T \to X\right)$ of $\Theta \downarrow X$, the latching maps $L_{(\Theta^T \to X)}F \longrightarrow F\left(\Theta^T \to X\right)$ are monomorphisms. However, we note that the maps

$$L_{(\Theta^T \to X)}F \quad = \quad \varinjlim_{(\Theta^S \to \Theta^T) \in (\Theta \downarrow X)^+ \downarrow (\Theta^T \to X) - \{\mathrm{id}\}} F\left(\Theta^S \to X\right) \longrightarrow F\left(\Theta^T \to X\right),$$

if F preserves colimits, are the maps

$$F\left(\partial\Theta^T \to \Theta^T \to X\right) \to F\left(\Theta^T \to X\right).$$

Then, since the diagrams we've called Σ_J and $(\_) \wedge S^1$ are known to be left Quillen the theorem is proved, as those functors preserve colimits and they preserve monomorphisms. $\square$

Of course, since $(\_) \wedge S^1$ presents the reduced suspension, so too does Σ_J.

Corollary 13.5.5. *The functor Σ_J is a reduced suspension functor for $\widehat{\Theta}_{\bullet}$ with the test model structure.*

Chapter 14

Spectra as Locally Finite Z-Groupoids

We closed the doubly previous chapter with the remark that weak **Z**-categories should be to symmetric monoidal higher categories, e.g. (∞, n)-categories, as spectra are to connective spectra. Then, in the previous chapter we proved that Σ_J serves as a reduced suspension functor. In this chapter we synthesize these developments into a new treatment of spectra as locally finite **Z**-groupoids.

14.1 Sequential Spectra: $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ and $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, (\_) \wedge S^1\right)$

Recall that in section 11.1 we recovered Hovey's theory of sequential spectra with respect to a left Quillen endofunctor valued in a cofibrantly generated model category. Since Σ_J and $(\_) \wedge S^1$ are both left Quillen endofunctors this theory defines both level-wise and stable model category structures on the categories $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ and $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, (\_) \wedge S^1\right)$. What's more, in Theorem 13.5.4, we proved that Σ_J and $(\_) \wedge S^1$ were weakly equivalent; this weak equivalence induces a Quillen adjunction.

Proposition 14.1.1. *The trivial Quillen equivalence*

$$id : \widehat{\Theta}_\bullet \rightleftarrows \widehat{\Theta}_\bullet : id$$

and the natural weak equivalence[1]

$$\widetilde{E} : \Sigma_J \Longrightarrow (\_) \wedge S^1$$

---

[1] see section 13.5

define a Quillen equivalence between $\mathsf{Sp}\left(\widehat{\Theta}_{\bullet},\Sigma_J\right)$ *and* $\mathsf{Sp}\left(\widehat{\Theta}_{\bullet},(\_)\wedge S^1\right)$ *with either the stable or level model structures.*

Proof. The inducement of a Quillen adjunction between $\mathsf{Sp}\left(\widehat{\Theta}_{\bullet},\Sigma_J\right)$ and $\mathsf{Sp}\left(\widehat{\Theta}_{\bullet},(\_)\wedge S^1\right)$ by the trivial adjunction together with the natural weak equivalence $\widetilde{E}$ is demonstrated in Proposition 5.3 of [30]. Theorem 5.5 of [30] then puts that Quillen adjunction an equivalence as $\widetilde{E}$ is a natural weak equivalence for all objects. □

14.2 The Adjunction $\Phi_{\bullet}\dashv\Psi_{\bullet}$

Recall that in section 11.2 we recorded Kan's observation that, given the reduced suspension functor Σ_K induced by the endomorphism $K:\triangle\longrightarrow\triangle$, there were two natural ways to present the stabilization of pointed simplicial sets at Σ_K. We may either stabilize $\widehat{\triangle}_{\bullet}$ at Σ_K by the usual sequential spectrum machine, that is we construct the category $\mathsf{Sp}\left(\widehat{\triangle}_{\bullet},\Sigma_K\right)$, or we may stabilize the simplex category $\triangle$ at K first yielding

$$\triangle_{\mathsf{st}}=\varinjlim\left\{\triangle\xrightarrow{K}\triangle\longrightarrow\cdots\right\}$$

and then consider the category of pointed presheaves $\widehat{\triangle_{\mathsf{st}}}_{\bullet}$.

While we did not recover the details in section 11.2, Kan then compares the two categories $\mathsf{Sp}\left(\widehat{\triangle}_{\bullet},\Sigma_K\right)$ and $\widehat{\triangle_{\mathsf{st}}}_{\bullet}$ by way of an adjunction and finds that this adjunction restricts to an adjoint equivalence of categories between the subcategories

$$\Omega\mathsf{Sp}\left(\widehat{\triangle}_{\bullet},\Sigma_K\right)\longrightarrow\mathsf{Sp}\left(\widehat{\triangle}_{\bullet},\Sigma_K\right)$$

and

$$K-\mathsf{Sp}\longrightarrow\widehat{\triangle_{\mathsf{st}}}_{\bullet},$$

where $\Omega\mathsf{Sp}\left(\widehat{\triangle}_{\bullet},\Sigma_K\right)$ is the full subcategory subtended by the objects

$$\left(\{X_i\},\{\varphi_i:\Sigma_K X_i\longrightarrow X_{i+1}\}\right)$$

for which the adjoint maps $\overline{\varphi_i}:X_i\longrightarrow\Omega_K X_{i+1}$ are isomorphisms, and $K-\mathsf{Sp}$ is the full subcategory subtended by the objects satisfying the vanishing condition we called local finiteness. In this section we'll repeat this development with $\widehat{\Theta}_\bullet$ in place of $\widehat{\triangle}_\bullet$, with Σ_J in place of Σ_K, and our category Θ_{st}, which we used to present the essentially algebraic theory of **Z**-categories, taking the place of $\triangle_{\mathsf{st}}$.

14.2.1 The Adjunctions $\Phi_\bullet^{\infty+z} \dashv \Psi_\bullet^{\infty+z}$

Recall that in section 12.2 we defined the **Z**-indexed family of functors $\Phi^{\infty+z} : \Theta \longrightarrow \Theta_{\mathsf{st}}$ as follows. When z is negative, $\Phi^{\infty+z}$ is the inclusions into the $-z^{th}$ copy of Θ in the defining colimit $\Theta_{\mathsf{st}} = \varinjlim \left\{ \Theta \xrightarrow{J} \Theta \xrightarrow{J} \cdots \right\}$. When z is positive, we defined $\Phi^{\infty+z} = \Phi^{\infty-0} \circ J^z$. In an abuse of notation, we denoted by those same symbols the left Kan extensions of those functors to functors $\widehat{\Theta} \longrightarrow \widehat{\Theta_{\mathsf{st}}}$. By a further abuse of notation we may denote the left Kan extension of the functors

$$\begin{tikzcd}
\Theta \arrow[r] & \widehat{\Theta_{\mathsf{st}}}_\bullet \\
T \arrow[r, mapsto] & \Phi^{\infty+z}\left(T\right)_+
\end{tikzcd}$$

along the functor $\Theta \xrightarrow{\mathsf{Yon}} \widehat{\Theta} \xrightarrow{(\_)_+} \widehat{\Theta}_\bullet$ by the same $\Phi^{\infty+z}$.

The functors thus defined

$$\Phi^{\infty+z} : \widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta_{\mathsf{st}}}_\bullet$$

freely add source and target cells to all cell at or below level z. However, there is another natural choice of a family functors $\widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta_{\mathsf{st}}}_\bullet$ which identifies all those freely added cells with the basepoint.

Definition 14.2.1. Given $z \in \mathbf{Z}$, let $\Phi_\bullet^{\infty+z}$ be the functor

$$\begin{tikzcd}
\Theta \arrow[r] & \widehat{\Theta_{\mathsf{st}}}_\bullet \\
T \arrow[r, mapsto] & \Phi^{\infty+z-1}\Sigma_J\left(\Theta^{T}_{+}\right)
\end{tikzcd}$$

and by a now familiar abuse of notation, let $\Phi_\bullet^{\infty+z} : \widehat{\Theta}_\bullet \longrightarrow \widehat{\Theta_{\mathsf{st}}}_\bullet$ denote the left Kan extension of $\Phi_\bullet^{\infty+z}$ along $\Theta \xrightarrow{\mathsf{Yon}} \widehat{\Theta} \xrightarrow{(\_)_+} \widehat{\Theta}_\bullet$.

Example 14.2.2. See that $\Phi_\bullet^{\infty+0}\Theta^{\overline{0}}_{+} = \Phi^{\infty-1}S^1$, a connected 0-sphere.

Now, by the familiar nerve/realization machinery, there are right adjoints $\Psi^{\infty+z}$ and $\Psi_\bullet^{\infty+z}$.

Definition 14.2.3. Let, for any $z \in \mathbf{Z}$, the functor $\Psi^{\infty+z}$ be the functor $\widehat{\Theta_{\mathsf{st}}} \longrightarrow \widehat{\Theta}$ which sends a stable cellular set $X : \Theta_{\mathsf{st}}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ to the cellular set

$$\begin{tikzcd}
\Psi^{\infty+z}(X) : \Theta^{\mathsf{op}} \arrow[r] & \mathsf{Set} \\
T \arrow[r, mapsto] & x \in \mathsf{Hom}\left(\Phi^{\infty+z}(T), X\right) = X(T_z).
\end{tikzcd}$$

Similarly, for any $z \in \mathbf{Z}$, let $\Psi_{\bullet}^{\infty+z}$ be the functor $\widehat{\Theta_{\mathsf{st}\bullet}} \longrightarrow \widehat{\Theta}_{\bullet}$, which sends a pointed stable cellular set $X : \Theta_{\mathsf{st}}^{\mathsf{op}} \longrightarrow \mathsf{Set}_{\bullet}$ to the pointed cellular set

$$\begin{tikzcd}
\Psi_{\bullet}^{\infty+z}(X) : \Theta^{\mathsf{op}} \arrow[r] & \mathsf{Set}_{\bullet} \\
T \arrow[r, mapsto] & x \in \mathsf{Hom}\left(\Phi_{\bullet}^{\infty+z}(T), X\right) \subset X(T_z).
\end{tikzcd}$$

Lemma 14.2.4. *For each $z \in \mathbf{Z}$, the functor $\Psi^{\infty+z} : \widehat{\Theta_{\mathsf{st}}} \longrightarrow \widehat{\Theta}$ is right adjoint to $\Phi^{\infty+z}$, and $\Psi_{\bullet}^{\infty+z} : \widehat{\Theta_{\mathsf{st}\bullet}} \longrightarrow \widehat{\Theta}_{\bullet}$ is right adjoint to the functor $\Phi_{\bullet}^{\infty+z}$.*

Much as for all $z \in \mathbf{Z}$, the functors $\Phi^{\infty+z}$ are compatible with the functor J in the sense that

$$\Phi^{\infty+(z-1)}(J(T)) = \Phi^{\infty+z}(T)$$

the functors $\Phi_{\bullet}^{\infty+z}$ are compatible with Σ_J in the sense that

$$\Phi_{\bullet}^{\infty+(z-1)}\left(\Sigma_J \Theta_{+}^{T}\right) = \Phi_{\bullet}^{\infty+z}\left(\Theta_{+}^{T}\right).$$

Similarly, for each $z \in \mathbf{Z}$,the functors $\Psi^{\infty+z} : \widehat{\Theta_{\mathsf{st}}} \longrightarrow \widehat{\Theta}$ and $\Psi_{\bullet}^{\infty+z} : \widehat{\Theta_{\mathsf{st}\bullet}} \longrightarrow \widehat{\Theta}_{\bullet}$ enjoy compatibilities with the functor J^{-1} , the yet unnamed right adjoint to J, and the functor Ω_J right adjoint to Σ_J.

Lemma 14.2.5. *There are natural isomorphisms*

$$J^{-1} \circ \Psi^{\infty+z} \overset{\sim}{\Longrightarrow} \Psi^{\infty+z+1}$$

and

$$\Omega_J \circ \Psi_{\bullet}^{\infty+z} \overset{\sim}{\Longrightarrow} \Psi_{\bullet}^{\infty+z+1}.$$

Proof. The proof is straightforward and formal. We'll prove construct the first natural isomorphism and the second will follow m.m. See that for any $X \in \mathsf{Ob}\left(\widehat{\Theta_{\mathsf{st}}}\right)$ and any $T \in \mathsf{Ob}\left(\Theta\right)$,

$$\begin{aligned} J^{-1} \circ \Psi^{\infty+z}\left(X\right)\left(T\right) &= \mathsf{Hom}_{\widehat{\Theta}}\left(J\left(\Theta^{T}\right), \Psi^{\infty+z}\left(X\right)\right) \\ &= \mathsf{Hom}_{\widehat{\Theta}}\left(\Phi^{\infty+z} \circ J\left(\Theta^{T}\right), X\right) \\ &\overset{\sim}{\to} \mathsf{Hom}_{\widehat{\Theta}}\left(\Phi^{\infty+z+1}\left(\Theta^{T}\right), X\right) \\ &= \Psi^{\infty+z+1}\left(X\right)\left(T\right). \end{aligned}$$

□

14.2.2 The Adjunction $\Phi_\bullet \dashv \Psi_\bullet$

Over $n \in \mathbf{N}$, the family of adjunctions

$$\Phi_\bullet^{\infty-n} \dashv \Psi_\bullet^{\infty-n}$$

may be combined to define an adjunction

$$\begin{tikzcd} \Phi_\bullet : \mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right) \arrow[r, shift left] & \widehat{\Theta_{\mathsf{st}}}_\bullet : \Psi_\bullet \arrow[l, shift left] \end{tikzcd}.$$

Definition 14.2.6. Let

$$\Phi_\bullet : \mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right) \longrightarrow \widehat{\Theta_{\mathsf{st}}}_\bullet$$

be the functor which sends a sequential spectrum $\left(\left\{X_n\right\}, \left\{\varphi_n : \Sigma_J X_n \longrightarrow X_{n+1}\right\}\right)$ to the colimit, taken in $\widehat{\Theta_{\mathsf{st}}}_\bullet$,

$$\varinjlim \left\{ \begin{tikzcd} \Phi_\bullet^{\infty-0}\left(X_0\right) \arrow[d, equal] & & \Phi_\bullet^{\infty-1}\left(X_1\right) \arrow[d, equal] & & \cdots \\ \Phi_\bullet^{\infty-1}\left(\Sigma_J X_0\right) \arrow[urr, "\Phi_\bullet^{\infty-1}(\varphi_0)"] & & \Phi_\bullet^{\infty-2}\left(\Sigma_J X_0\right) \arrow[urr, "\Phi_\bullet^{\infty-2}(\varphi_1)"] & & \end{tikzcd} \right\}.$$

Let $\Psi_\bullet$ be the functor

$$\begin{tikzcd} \Psi : \widehat{\Theta_{\mathsf{st}}}_\bullet \arrow[r] & \mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right) \\ X \arrow[r, mapsto] & \left\{\Psi_\bullet^{\infty-n} X\right\}_{n \in \mathbf{N}} \end{tikzcd}$$

with the implicit structure maps being the adjoints to the isomorphisms described in lemma 14.2.5.

As an immediate corollary, we've the following.

Corollary 14.2.7. *The functor* $\Psi_\bullet : \widehat{\Theta_{\mathsf{st}\bullet}} \longrightarrow \mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ *lands in* $\Omega\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$.

As it happens, the functors $\Phi_\bullet$ and $\Psi_\bullet$ are adjoint functors, and more they restrict to a categorical equivalence on subcategories of their respective domains. First, we'll assemble the adjunction.

14.2.2.1 The composition $\Phi_\bullet \circ \Psi_\bullet$ and the co-unit $\varepsilon : \Phi_\bullet \circ \Psi_\bullet \Longrightarrow \mathbf{id}_{\widehat{\Theta_{\mathsf{st}\bullet}}}$

For each $n \in \mathbf{N}$ and $X \in \mathsf{Ob}\left(\widehat{\Theta_{\mathsf{st}\bullet}}\right)$, see that we've isomorphisms

$$\begin{aligned} \Phi_\bullet^{\infty-n} \circ \Psi_\bullet^{\infty-n}(X) &= \Phi_\bullet^{\infty-n}\left(\mathsf{Hom}_{\widehat{\Theta_{\mathsf{st}\bullet}}}\left(\Phi_\bullet^{\infty-n}(\_), X\right)\right) \\ &\overset{\sim}{\leftarrow} \varinjlim_{\left(f:\Phi_\bullet^{\infty-n}(T)\to X\right)\in\left(\Phi_\bullet^{\infty-n}\downarrow X\right)} \Phi_\bullet^{\infty-n}(T) \end{aligned}$$

so for each $n \in \mathbf{N}$,$\Phi_\bullet^{\infty-n}\left(\Psi_\bullet^{\infty-n}(X)\right)$ is the subfunctor of X comprised of all the cells of X which are trivial below level $-n$. More, since $\Phi_\bullet^{\infty-n} = \Phi_\bullet^{\infty-n-1} \circ \Sigma_J$ we get functors

$$\left(\Phi_\bullet^{\infty-n} \downarrow X\right) \longrightarrow \left(\Phi_\bullet^{\infty-n-1} \downarrow X\right)$$

so, setting

$$\left(\Phi_\bullet^{\infty-\mathbf{N}} \downarrow X\right) = \varinjlim\left\{ \left(\Phi_\bullet^{\infty-0} \downarrow X\right) \to \left(\Phi_\bullet^{\infty-1} \downarrow X\right) \to \left(\Phi_\bullet^{\infty-2} \downarrow X\right) \to \cdots \right\}$$

we find that for all $X \in \widehat{\Theta_{\mathsf{st}\bullet}}$,

$$\Phi_\bullet \circ \Psi_\bullet(X) \overset{\sim}{\leftarrow} \varinjlim_{\left[f:\Phi_\bullet^{\infty-n}(T)\to X\right]\in\left(\Phi_\bullet^{\infty-\mathbf{N}}\downarrow X\right)} \Phi^{\infty-n}(T).$$

What's more, this object is the subfunctor of X comprised of all cells of X which are trivial below some level. From this observation it follows formally that there is a natural monomorphism $\Phi_\bullet \circ \Psi_\bullet \Longrightarrow \mathrm{id}_{\widehat{\Theta}_\bullet}$; it is this natural monomorphism which is to be the co-unit ε of the putative adjunction $\Phi_\bullet \dashv \Psi_\bullet$.

Definition 14.2.8. Let $J-\mathsf{Sp}$ be the full subcategory of $\widehat{\Theta_{\mathsf{st}\bullet}}$ subtended by those objects X such that for all T_z of Θ_{st} and $x \in X(T_z)$ there exists an $n \in \mathbf{N}$ such that for any sequence $\varphi_0, \varphi_1, \ldots, \varphi_n$

of monomorphisms of Θ_{st}, $\varphi_n \circ \cdots \circ \varphi_0 (x) = \bullet$. We refer to this condition as the **local finiteness of** X as it corresponds to the requirement that only finitely many faces of the cells of X are not the base-point.

Remark 14.2.9. Local finiteness and the property of triviality below some level are equivalent.

Definition 14.2.10. We note that the natural transformation ε may now be interpreted as the inclusion of the maximal locally finite subfunctor. More, we've an immediate corollary.

Lemma 14.2.11. *The natural transformation* $\varepsilon : \Phi_\bullet \circ \Psi_\bullet \Longrightarrow id_{\widehat{\Theta_{\mathsf{st}}}_\bullet}$ *restricts along* $J - \mathsf{Sp} \longrightarrow \widehat{\Theta_{\mathsf{st}}}_\bullet$ *to the identity.*

14.2.2.2 The composition $\Psi_\bullet \circ \Phi_\bullet$ and the unit $\eta : \mathbf{id}_{\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)} \Longrightarrow \Phi_\bullet \circ \Psi_\bullet$

Given a sequential spectrum $\{X_n\}$, see that, for each $i \in \mathbf{N}$,

$$\Psi_\bullet \circ \Phi_\bullet \left(\{X_n\}\right)_i = \Psi_\bullet^{\infty-i} \circ \Phi_\bullet \left(\{X_n\}\right)$$

and that second pointed cellular may be written

$$\mathsf{Hom}_{\widehat{\Theta_{\mathsf{st}}}_\bullet}\left(\Phi_\bullet^{\infty-i}(\_)\,,\ \begin{tikzcd} \Phi_\bullet^{\infty-0}(X_0) \arrow[d, equal] & \Phi_\bullet^{\infty-1}(X_1) \arrow[d, equal] & \cdots \\ \Phi_\bullet^{\infty-1}(\Sigma_J X_0) \arrow[ru] & \Phi_\bullet^{\infty-2}(\Sigma_J X_1) \arrow[ru] & \end{tikzcd}\right).$$

Now, since for all cells T of Θ, the pointed stable cellular set $\Phi_\bullet^{\infty-i}(T)$ is finite, so $\mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\_\right)$ preserves sequential colimits in the second argument and the chain of equalities extends by an isomorphism to the colimit

$$\varinjlim\left\{\begin{tikzcd} \mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\Phi_\bullet^{\infty-0}(X_0)\right) \arrow[d, equal] & \mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\Phi_\bullet^{\infty-1}(X_1)\right) \arrow[d, equal] & \cdots \\ \mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\Phi_\bullet^{\infty-1}(\Sigma_J X_0)\right) \arrow[ru] & \mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\Phi_\bullet^{\infty-2}(\Sigma_J X_1)\right) \arrow[ru] & \end{tikzcd}\right\}.$$

What's more, since the colimit is sequential, that cellular set is isomorphic to the colimit

$$\varinjlim\left\{\begin{tikzcd} \mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\Phi_\bullet^{\infty-i}(X_i)\right) \arrow[d, equal] & \mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\Phi_\bullet^{\infty-i-1}(X_{i+1})\right) \arrow[d, equal] & \cdots \\ \mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\Phi_\bullet^{\infty-i-1}(\Sigma_J X_i)\right) \arrow[ru] & \mathsf{Hom}\left(\Phi_\bullet^{\infty-i}(\_)\,,\Phi_\bullet^{\infty-i-2}(\Sigma_J X_{i+1})\right) \arrow[ru] & \end{tikzcd}\right\}.$$

But

$$\mathsf{Hom}\left(\Phi_{\bullet}^{\infty-i}\left(\_\right), \Phi^{\infty-i-n}\left(X_{i+n}\right)\right) = \mathsf{Hom}\left(\Phi_{\bullet}^{\infty-i-n}\circ\Sigma_J^n, \Phi^{\infty-i-n}\left(X_{i+n}\right)\right)$$

so by definition of Θ_{st}, that last colimit is isomorphic to the colimit

$$\varinjlim \left\{ \begin{tikzcd} X_i \arrow[d, equal] & \Omega_J\left(X_{i+1}\right) \arrow[d, equal] & \cdots \\ \Omega_J\left(\Sigma_J X_i\right) \arrow[ru] & \Omega_J^2\left(\Sigma_J X_{i+1}\right) \arrow[ru] & \end{tikzcd} \right\}.$$

The functor $\Psi_\bullet \circ \Phi_\bullet$ is the Ω-fication functor and the unit $\eta : \mathrm{id}_{\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)} \Longrightarrow \Psi_\bullet \circ \Phi_\bullet$ is index-wise the inclusion into the first object in those colimits. As a purely formal lemma then, we have the following.

Lemma 14.2.12. *The natural transformation*

$$\Phi_\bullet \circ \eta : \Phi_\bullet \Longrightarrow \Phi_\bullet \circ \Psi_\bullet \circ \Phi_\bullet$$

is the identity.

Proof. The lemma is a purely formal consequence of the commutation of colimits and colimits. □

14.2.2.3 The triangle identities

The composition

$$\Phi_\bullet \overset{\Phi_\bullet \circ \eta}{\Longrightarrow} \Phi_\bullet \circ \Psi_\bullet \circ \Phi_\bullet \overset{\eta \circ \Phi_\bullet}{\Longrightarrow} \Phi_\bullet$$

is the identity as the first natural transformation is the identity by Lemma 14.2.12 and as $\Phi_\bullet$ lands in $J - \mathsf{Sp}$, Lemma 14.2.11 provides that the second natural transformation is identity too.

The proof that the composite natural transformation is the identity $\Psi_\bullet \overset{\eta\circ\Psi_\bullet}{\Longrightarrow} \Psi_\bullet \circ \Phi_\bullet \circ \Psi_\bullet \overset{\Psi_\bullet\circ\varepsilon}{\Longrightarrow} \Psi_\bullet$ follows similarly. Since $\Psi_\bullet$ lands in $\Omega\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ the natural transformation $\eta \circ \Psi_\bullet$ is the identity and since $\Psi_\bullet$ only has access to the locally finite cells of X, the natural transformation $\Psi_\bullet \circ \varepsilon$ is the identity.

Importantly, from the discussion above it is clear that the doubly restricted adjunction

$$\Phi_\bullet : \Omega\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right) \begin{tikzcd} {} \arrow[r, shift left] & {} \arrow[l, shift left] \end{tikzcd} J - \mathsf{Sp} : \Psi_\bullet$$

is an adjoint isomorphism, since on those domains $\Phi_\bullet \circ \Psi_\bullet$ and $\Psi_\bullet \circ \Phi_\bullet$ are the identity functors.

14.2.3 The Adjunction $\Phi_\bullet \dashv \Psi_\bullet$ is Quillen

Having constructed the adjunction $\Phi_\bullet : \mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right) \rightleftarrows J - \mathsf{Sp} : \Psi_\bullet$ we wish to use this adjunction to impose a model structure on $\widehat{\Theta_{\mathsf{st}\bullet}}$ such that the adjunction may be promoted to a Quillen adjunction. If we restrict this adjunction along $J - \mathsf{Sp} \longrightarrow \widehat{\Theta_{\mathsf{st}\bullet}}$ this can be done by right transfer of a model category structure.

Theorem 14.2.13. *(Crans) Suppose* $(\mathscr{C}, \mathsf{Cof}, \mathsf{W}, \mathsf{Fib})$ *to be model category cofibrantly generated by sets* $I_{\mathscr{C}}$ *and* $J_{\mathscr{C}}$*. Suppose* $G \dashv D$ *to be an adjunction with* $G : \mathscr{C} \longrightarrow \mathscr{D}$ *the left adjoint functor. If:*

- $\mathscr{D}$ *is complete under the formation of small limits and colimits;*
- $GI_{\mathscr{C}}$ *and* $GJ_{\mathscr{C}}$ *admit the small object argument; and*
- $D\left({}^{\pitchfork}\left((GJ_{\mathscr{C}})^{\pitchfork}\right)\right) \subset \mathsf{W}$*;*

then, letting $\mathsf{Fib}_{\mathscr{D}} = D^{-1}(\mathsf{Fib})$*,* $\mathsf{W}_{\mathscr{D}} = D^{-1}(\mathsf{W})$*, and* $\mathsf{Cof}_{\mathscr{D}} = {}^{\pitchfork}(\mathsf{Fib}_{\mathscr{D}} \cap \mathsf{W}_{\mathscr{D}})$*, the data*

$$(\mathscr{D}, \mathsf{Cof}_{\mathscr{D}}, \mathsf{W}_{\mathscr{D}}, \mathsf{Fib}_{\mathscr{D}})$$

comprise a model category cofibrantly generated by the sets $GI_{\mathscr{C}}$ *and* $GJ_{\mathscr{C}}$ *and the adjunction* $G \dashv D$ *is promoted to a Quillen adjunction.*

We may immediately apply this theorem to the adjunction

$$\Phi_\bullet : \mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right) \rightleftarrows J - \mathsf{Sp} : \Psi_\bullet \ .$$

Corollary 14.2.14. *The data*

$$\left(J - \mathsf{Sp}, {}^{\pitchfork}\left(\Psi_\bullet^{-1}\left(\mathsf{W}_{\mathsf{level}} \cap \mathsf{Fib}_{\mathsf{level}}\right)\right), \Psi_\bullet^{-1}\left(\mathsf{W}_{\mathsf{level}}\right), \Psi_\bullet^{-1}\left(\mathsf{Fib}_{\mathsf{level}}\right)\right)$$

comprise a model category, cofibrantly generated by the cofibrations $\Phi_\bullet\left(\bigcup_{n\in\mathbf{N}} F_n\left(\mathscr{M}_{\Theta+}\right)\right)$ *and trivial cofibrations* $\Phi_\bullet\left(\bigcup_{n\in\mathbf{N}} F_n\left(\Lambda_{\mathsf{Ber}+}\right)\right)$*. More, this model structure is Quillen adjoint by* $\Phi_\bullet \dashv \Psi_\bullet$ *to*

$$\left(\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right), \mathsf{Cof}_{\mathsf{level}}, \mathsf{W}_{\mathsf{level}}, \mathsf{Fib}_{\mathsf{level}}\right).$$

Proof. For the first condition, it is evident that $\Omega\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ whence $J-\mathsf{Sp}$ comprise a category complete under the formation of all small limits and colimits. For the second see that since for each n the composition $\Phi_\bullet \circ F_n$ preserves the finiteness of the sources and targets of the maps of $\mathscr{M}_{\Theta+}$ and Λ_{Ber}, $\Phi_\bullet\left(\bigcup_{n\in\mathbf{N}} F_n\left(\mathscr{M}_{\Theta+}\right)\right)$ and $\Phi_\bullet\left(\bigcup_{n\in\mathbf{N}} F_n\left(\Lambda_{\mathsf{Ber}+}\right)\right)$ admit the small object argument. For the third condition, the condition that

$$\Psi_\bullet\left({}^{\pitchfork}\left(\Phi_\bullet\left(\bigcup_{n\in\mathbf{N}} F_n J\right)^{\pitchfork}\right)\right) \subset \mathsf{W}_{\mathsf{level}},$$

one need only see that by the isomorphism between $J-\mathsf{Sp}$ and $\Omega\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$, the class of morphisms $\Psi_\bullet\left({}^{\pitchfork}\left(\Phi_\bullet\left(\bigcup_{n\in\mathbf{N}} F_n\left(\Lambda_{\mathsf{Ber}+}\right)\right)^{\pitchfork}\right)\right)$ is precisely the class of level acyclic fibrations between objects of $\Omega\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$. □

It is then by anther lemma of Crans that this Quillen adjunction can be seen to be a Quillen equivalence when we equip $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ with the stable model structure.

Lemma 14.2.15. *(Crans) Given a Quillen adjunction $G \dashv D$ with $G : \mathscr{C} \longrightarrow \mathscr{D}$ then $G \dashv D$ is a Quillen equivalence if and only if, for every co-fibrant object C of $\mathscr{C}$ and every fibrant object F of $\mathscr{D}$, a map $G(C) \longrightarrow F$ is of $\mathsf{W}_{\mathscr{D}}$ if and only if the reflected map $C \longrightarrow D(F)$ is of $\mathsf{W}_{\mathscr{C}}$.*

Corollary 14.2.16. *The adjunction $\Phi_\bullet \dashv \Psi_\bullet$ between $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ with the stable model structure and $J-\mathsf{Sp}$ with the structure discovered above is a Quillen equivalence*

Proof. Since level fibrations between objects of $\Omega\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ are fibrations in the level structure, and likewise level acyclic fibrations between objects of $\Omega\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$, we find that $\Psi_\bullet$ satisfies the hypotheses necessary for $\Phi_\bullet \dashv \Psi_\bullet$ to be a Quillen adjunction with respect to the stable model structure on $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$. For the equivalence condition, suppose

$$\varphi : \Phi_\bullet(C) \longrightarrow F$$

to be a weak equivalence of $J-\mathsf{Sp}$. Since $\Phi_\bullet$ is both full and surjective on objects then any such morphism is of the form

$$\Phi_\bullet(\varphi') : \Phi_\bullet(C) \longrightarrow \Phi_\bullet(F')$$

for some φ' and F'. But this map is weak equivalence in $J - \mathsf{Sp}$ if and only if $\Psi_\bullet \circ \Phi_\bullet (\varphi')$ is.

If $\Psi_\bullet \circ \Phi_\bullet (\varphi')$ is a weak equivalence of $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ then since the component of the unit

$$\eta_C : C \longrightarrow \Psi_\bullet \circ \Phi_\bullet (C)$$

is a stable equivalence the composite is. Conversely, if $C \longrightarrow \Psi_\bullet \circ \Phi_\bullet (F')$ is a weak equivalence, since $\eta_C : C \longrightarrow \Psi_\bullet \circ \Phi_\bullet (C)$ is, the two out of three property puts $\Psi_\bullet \circ \Phi_\bullet (F')$ as one too. ☐

14.3 The Category β and an isomorphism $\widehat{\beta}_\bullet \xrightarrow{\sim} J - \mathsf{Sp}$

In the last section we found that the stable model structure on $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ induced an equivalent model structure on $J - \mathsf{Sp}$ which is moreover cofibrantly generated by the sets $\Phi_\bullet \left(\bigcup_{n \in \mathbf{N}} F_n I_\Theta\right)$ and $\Phi_\bullet \left(\bigcup_{n \in \mathbf{N}} F_n \Lambda_{\mathsf{Ber}+}\right)$. However, we can present $J - \mathsf{Sp}$ as a category of pointed presheaves and demystify this model structure; in this section we develop a category β such that $\widehat{\beta}_\bullet \xrightarrow{\sim} J - \mathsf{Sp}$ and find that the model structure of the previous section is in fact a pointed Cisinski model structure and is easily interpreted as presenting the weak-$\mathbf{Z}$-groupoids long promised.

Definition 14.3.1. Given$z \in \mathbf{Z}$ and $T \in \mathsf{Ob}(\Theta)$, let

$$[\bullet_z; T] \in \mathsf{Ob}(J - \mathsf{Sp}) \subset \mathsf{Ob}\left(\widehat{\Theta_{\mathsf{st}\bullet}}\right)$$

be the quotient ${}^{\Theta_{\mathsf{st}}^{Tz}}/{}_{\partial\Theta^{\overline{z}}}$. Let $\beta \hookrightarrow \widehat{\Theta_{\mathsf{st}\bullet}}$ be the full subcategory of the target subtended by the objects of the form $[\bullet_z; T]$.

Lemma 14.3.2. *The category $\widehat{\beta}_\bullet$ is canonically isomorphic to the category $J - \mathsf{Sp}$.*

Proof. See that the local finiteness condition on those pointed presheaves found in $J - \mathsf{Sp}$ implies that all cells of any $X \in \mathsf{Ob}(J - \mathsf{Sp})$ factor through some cell of the form $[\bullet; T]$ and thus, by the universal property of the pointed Yoneda embedding, the left Kan extension $\widehat{\beta}_\bullet \longrightarrow J - \mathsf{Sp}$ is an isomorphism. ☐

While β is not skeletal, both the notions of horns and the notion of boundary defined for such categories make sense for β as they make sense for Θ_{st}. Indeed, since Θ_{st} is the colimit over a sequence of embeddings of skeletal categories, the notion of boundary is preserved. Given $z \in \mathbf{Z}$ and $T \in \mathsf{Ob}(\Theta)$, see that

$$\partial\Theta^{T_z} = \varinjlim_{R \longrightarrow T \in \Theta \downarrow \partial T} \Theta_{\mathsf{st}}^{R_z}$$

is well defined in that if $T_z = T'_{z'}$ then the two colimits $\partial\Theta^{T_z}$ and $\partial\Theta^{T'_{z'}}$ coincide.

Then, since every monomorphism of Θ_{st} arises as one of Θ it follows that

$$\left\{\partial\Theta_{\mathsf{st}}^{T_z} \longrightarrow \Theta_{\mathsf{st}}^{T_z}\right\}_{z \in \mathbf{Z}, T \in \mathsf{Ob}(\Theta)}$$

is a cellular model for Θ_{st}. Letting

$$\partial\beta^{[\bullet_z;T]} = \varinjlim_{R \longrightarrow T \in \Theta \downarrow \partial T} \beta^{[\bullet_z;R]}$$

we may describe the set

$$\left\{\partial\beta^{[\bullet_z;T]} \longrightarrow \beta^{[\bullet_z;T]}\right\}_{z \in \mathbf{Z}, T \in \mathsf{Ob}(\Theta)}$$

which serves as cellular model for $\widehat{\beta}_\bullet$ by the local finiteness hypothesis.

Lemma 14.3.3. *The saturated class generated by* $\Phi_\bullet\left(\bigcup_{n \in \mathbf{N}} F_n \mathscr{M}_{\Theta+}\right)$ *is the same as the saturated class generated by*

$$\left\{\partial\beta^{[\bullet_z;T]} \longrightarrow \beta^{[\bullet_z;T]}\right\}_{z \in \mathbf{Z}, T \in \mathsf{Ob}(\Theta)}.$$

Proof. First, see that for any $n \in \mathbf{N}$, and $T \in \mathsf{Ob}(\Theta)$, the image under $\Phi_\bullet$ of the canonical inclusion

$$F_n\left(\partial\Theta_+^T \longrightarrow \Theta_+^T\right)$$

is the canonical inclusion

$$\partial\beta^{[\bullet_{-n};T]} \longrightarrow \beta^{[\bullet_{-n};T]}.$$

Thus,

$$\Phi_\bullet\left(\bigcup_{n \in \mathbf{N}} F_n I_\Theta\right) \subset \left\{\partial\beta^{[\bullet_z;T]} \longrightarrow \beta^{[\bullet_z;T]}\right\}_{z \in \mathbf{Z}, T \in \mathsf{Ob}(\Theta)}$$

so

$${}^{\pitchfork}\left(\Phi_\bullet\left(\bigcup_{n \in \mathbf{N}} F_n I_\Theta\right)^{\pitchfork}\right) \subset^{\pitchfork} \left(\left\{\partial\beta^{[\bullet_z;T]} \longrightarrow \beta^{[\bullet_z;T]}\right\}^{\pitchfork}\right).$$

For the reverse containment observe that every morphism in $\left\{\partial\beta^{[\bullet_z;T]} \longrightarrow \beta^{[\bullet_z;T]}\right\}$ is a quotient of a morphism in $\Phi_\bullet\left(\bigcup_{n\in\mathbf{N}} F_n\mathscr{M}_{\Theta+}\right)$ so the closure under pushouts of saturated classes finishes the proof. $\square$

Corollary 14.3.4. *The model structure on* $\widehat{\beta}_\bullet$ *Quillen equivalent to* $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ *with respect to the stable model structure is a pointed Cisinski model structure, meaning the cofibrations are the monomorphisms and vice-versa.*

Berger's horn, the elements of the set Λ_{Ber}, generalize likewise.

Definition 14.3.5. Let

$$\Lambda_\kappa\beta^{[\bullet_z;T]} = \varinjlim_{(R\longrightarrow T)\in\Theta\downarrow\partial T-\{\kappa\}} \beta^{[\bullet_z;R]}.$$

Lemma 14.3.6. *The saturated class generated by* $\Phi_\bullet\left(\bigcup_{n\in\mathbf{N}} F_n J_\Theta\right)$ *is the same as the saturated class generated by*

$$\left\{\Lambda_\kappa\beta^{[\bullet_z;T]} \longrightarrow \beta^{[\bullet_z;T]}\right\}_{z\in\mathbf{Z},T\in\mathsf{Ob}(\Theta),\kappa\in\mathsf{Coface}(T)}.$$

Corollary 14.3.7. *The model category structure on* $\widehat{\beta}_\bullet$ *Quillen equivalent to* $\mathsf{Sp}\left(\widehat{\Theta}_\bullet, \Sigma_J\right)$ *with the stable model structure is cofibrantly generated by the sets*

$$I_\beta = \left\{\partial\beta^{[\bullet_z;T]} \longrightarrow \beta^{[\bullet_z;T]}\right\}_{z\in\mathbf{Z},T\in\mathsf{Ob}(\Theta)}$$

and

$$J_\beta = \left\{\Lambda_\kappa\beta^{[\bullet_z;T]} \longrightarrow \beta^{[\bullet_z;T]}\right\}_{z\in\mathbf{Z},T\in\mathsf{Ob}(\Theta),\kappa\in\mathsf{Coface}(T)}.$$

It is this presentation that fully justifies the claim that spectra may be presented as locally finite weak-**Z**-groupoids. Just as Berger's horns encode an invertible theory of composition in ω degrees, these horns encode an invertible theory of composition in **Z** degrees.

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Appendix A

The Anatomy of Categories $B \int A$

Given a category A, recall that a morphism in $\triangle \wr A$ of the form

$$\left[\left(d^1\right)^{n-1} : \mathbf{g}\right] : [[1]\,; a] \longrightarrow [[n]\,; b_1, \ldots, b_n]$$

is comprised of the simplicial map

$$\begin{tikzcd}
\left(d^1\right)^{n-1} : [1] \arrow[r] & {[n]} \\
0 \arrow[r, mapsto] & 0 \\
1 \arrow[r, mapsto] & n
\end{tikzcd}$$

and a multimorphism

$$\begin{tikzcd}
& b_1 \;. \\
a \arrow[ur, "g_1"] \arrow[dr, "g_n"] & \vdots \\
& b_n
\end{tikzcd}$$

When A has finite products such a multimorphism is of course equivalent to a map $a \longrightarrow \prod b_i$. While $\triangle$ does not have finite products it naturally embeds into a category that does.

Definition A.0.1. Let FinPos denote the full subcategory of the category of partially ordered sets subtended by the finite posets.

Of course, the category FinPos is possessed not merely of finite products, but in fact all finite limits. What's more, there is an obvious full and faithful embedding $\triangle \longrightarrow \mathsf{FinPos}$ which factors the Yoneda embedding. It is thus reasonable to ask if the embedding $\triangle \longrightarrow \mathsf{FinPos}$ preserves finite limits[1] as the Yoneda embedding does.

---

[1] Of course, the Yoneda embedding preserves all limits, but $\triangle$ is only endowed with a very few non-trivial finite limits, and no non-trivial infinite limits.

Now, recall that the simplicial nerve functor $\mathscr{N} : \mathsf{Cat} \longrightarrow$ is a right adjoint functor, hence preserves limits. Note too, that the functor $\mathsf{FinPos} \longrightarrow$ factors as $\mathsf{FinPos} \longrightarrow \mathsf{Cat} \xrightarrow{\mathscr{N}}$. Thus, to show that the functor $\mathsf{FinPos} \longrightarrow$ preserves finite limits, is suffices to observe that $\mathsf{FinPos} \longrightarrow \mathsf{Cat}$ does. But this is clear and $\triangle \longrightarrow \mathsf{FinPos}$ must preserve finite limits too.

Remark A.0.2. The shuffle decomposition of products of simplices (see [7]), a classical result whose importance cannot be overstated, may now be formulated as a natural bijection of bifunctors.

Lemma A.0.3. *The embedding* $\mathsf{FinPos} \longrightarrow$ *induces natural bijections*

$$\mathsf{Hom}_{\mathsf{FinPos}}\left(\_\,,\_ \times \cdots \times \_\right) \xrightarrow{\sim} \mathsf{Hom}_{\widehat{\triangle}}\left(\triangle^{(\_)}, \triangle^{(\_)} \times \cdots \times \triangle^{(\_)}\right) : \triangle^{\mathsf{op}} \times (\triangle)^n \longrightarrow \mathsf{Set}.$$

Proof. See that we have natural isomorphisms

$$\begin{aligned} \mathsf{Hom}_{\mathsf{FinPos}}\left([n]\,,[\ell_1] \times \cdots \times [\ell_m]\right) &\xrightarrow{\sim} \prod_{i \in \langle m \rangle} \mathsf{Hom}_{\mathsf{FinPos}}\left([n]\,,[\ell_i]\right) \\ &\xrightarrow{\sim} \prod_{i \in \langle m \rangle} \mathsf{Hom}_{\triangle}\left([n]\,,[\ell_i]\right) \\ &\xrightarrow{\sim} \mathsf{Hom}_{\triangle}\left(\triangle^n, \triangle^{\ell_1} \times \cdots \times \triangle^{\ell_m}\right). \end{aligned}$$

□

The multi-Reedy structure on $\triangle$, as is observed in [13] example 2.7 and repeated here, is intimately related to this category.

Corollary A.0.4. *Let* n, m *and* $\ell_1, \ldots, \ell_m$ *be non-negative integers. Then, the* $\triangle^+(\star)$ *multimorphisms*

$$[n] \longrightarrow \prod_{i \in \langle m \rangle} [\ell_i]$$

of $\triangle^+(\star)$ *are in natural bijection with the non-degenerate simplices of the prism* $[\ell_1] \times \cdots \times [\ell_m]$.

As a consequence of this, we find that any $\triangle^+(\star)$ multimorphism

$$\begin{tikzcd} & {[m_1]} \\ {[n]} \arrow[ur, "f_1"] \arrow[dr, "f_n"'] & \vdots \\ & {[m_\ell]} \end{tikzcd}$$

admits a factorization[2]

$$\begin{tikzcd}
 & & {[m_1]} \\
{[n]} \arrow[r, "f^+"] & \left[\sum_{i\in\langle\ell\rangle} m_i\right] \arrow[ru, "\mathsf{hr}_1"] \arrow[rd, "\mathsf{hr}_n"] & \vdots \\
 & & {[m_\ell]}
\end{tikzcd}$$

with f^+ a morphism of $\triangle^+$ and the multimorphism $\mathbf{hr} = (\mathsf{hr}_i)_{i\in\langle\ell\rangle}$ is a degree zero $\triangle^+(\star)$ multimorphism. Importantly, while the object $\left[\sum_{i\in\langle\ell\rangle} m_i\right]$ is uniquely quantified, the maps hr_k are not. Not only do they depend on the multimorphism $\mathbf{f}$, but they are uniquely determined by $\mathbf{f}$ only when $n = \sum_{i\in\langle\ell\rangle} m_i$ which is the case precisely when $\mathbf{f}$ corresponds to a non-degenerate top dimensional simplex of the prism $\prod_{i\in\langle\ell\rangle} \triangle^{m_i}$.

What's more, given one of the degree zero $\triangle^+(\star)$ multimorphisms

$$\begin{tikzcd}
 & {[m_1]} \\
\left[\sum_{i\in\langle\ell\rangle} m_i\right] \arrow[ru, "\mathsf{hr}_1"] \arrow[rd, "\mathsf{hr}_\ell"] & \vdots \\
 & {[m_\ell]}
\end{tikzcd}$$

which correspond to the non-degenerate top dimensional simplices of the prism $\prod_{i\in\langle\ell\rangle} \triangle^{m_i}$, the associativity of the cartesian product is at least partly manifest as a factorization property.

For every $k \in \langle\ell\rangle$, there exists a factorization

$$\begin{tikzcd}
 & & {[m_1]} \\
 & \left[\sum_{i\leq k} m_i\right] \arrow[ru, "\mathsf{hr}_1"] \arrow[rd, "\mathsf{hr}_k"'] & \vdots \\
 & & {[m_k]} \\
\left[\sum_{i\in\langle\ell\rangle} m_i\right] \arrow[ruu, "\mathsf{hr}_{i\leq k}"] \arrow[rdd, "\mathsf{hr}_{k<i}"] & & \\
 & & {[m_{k+1}]} \\
 & \left[\sum_{k<i} m_i\right] \arrow[ru, "\mathsf{hr}_{k+1}"] \arrow[rd, "\mathsf{hr}_\ell"'] & \vdots \\
 & & {[m_\ell]}
\end{tikzcd}$$

with $(\mathsf{hr}_{i\leq k}, \mathsf{hr}_{k<i})$ a degree zero $\triangle^+(\star)$ multimorphism, a top-dimensional non-degenerate simplex of the prism $\triangle^{\sum_{i\leq k} m_i} \times \triangle^{\sum_{k<i} m_i}$, and both $(\mathsf{hr}_1, \ldots, \mathsf{hr}_k)$ and $(\mathsf{hr}_{k+1}, \ldots, \mathsf{hr}_\ell)$ being degree zero

---

[2] The choice of hr for the components of the multimorphism is from Czech, a language with two words for prism, where the word hranol is the word used for the mathematical notion of prism.

$\triangle^+$ $(\star)$ multimorphisms, top-dimensional non-degenerate simplices of the prisms $\triangle^{[m_1]} \times \cdots \times \triangle^{[m_k]}$ and $\triangle^{[m_{k+1}]} \times \cdots \times \triangle^{[m_\ell]}$ respectively.

Remark A.0.5. The reader may be wont to ask why we have specified factorizations only for the partitions of $\langle \ell \rangle$ into an upper and lower half. While for multimorphisms generally, order may not matter, in the manner in which multimorphisms appear in wreath products order **does** matter. Where history will fall on this definition is not yet known, nor do we have a strong opinion; we have made a choice apt for our purposes here, but it may not be the only or best choice. It's also worth noting that the degree to which order matters is considered very carefully in [10] with the purpose there being the modeling of finite loop spaces, in which order matters in large doses, but not in small ones.

A.1 Prismatic Multi-Reedy Categories

In this section we will abstract the content of the discussion with which we began the chapter[3]

.

Definition A.1.1. A **prismatic multi-Reedy category** A, is a multi-Reedy category A which satisfies the following conditions.

PMR1 Given any A^+ $(\star)$ multimorphism

$$\begin{tikzcd} & b_1 \\ a \arrow[ur, "f_1"] \arrow[dr, "f_\ell"] & \vdots \\ & b_\ell \end{tikzcd}$$

admits a factorization[4]

$$\begin{tikzcd} & & b_1 \\ a \arrow[r, "f^+"] & \sum_{i \in \langle \ell \rangle} b_i \arrow[ur, "\mathsf{hr}_1"] \arrow[dr, "\mathsf{hr}_n"] & \vdots \\ & & b_\ell \end{tikzcd}$$

---

[3] It is worth mentioning again that this is only one possible abstraction of the Eilenberg-Zilber decomposition of products of simplices.

[4] The choice of hr for the components of the multimorphism is from Czech, a language with two words for prism, where the word hranol is the word used for the mathematical notion of prism.

with f^+ a morphism of A^+ and the multimorphism $\mathbf{hr} = (\mathsf{hr}_i)_{i\in\langle\ell\rangle}$ is a degree zero $A^+\,(\star)$ multimorphism; and

PMR2 Given any degree zero $A^+\,(\star)$ multimorphism

$$\begin{tikzcd}
 & b_1 , \\
\sum_{i\in\langle\ell\rangle} b_i \arrow[ur, "\mathsf{hr}_1"] \arrow[dr, "\mathsf{hr}_\ell"] & \vdots \\
 & b_\ell
\end{tikzcd}$$

for every $k \in \langle\ell\rangle$, there exists a factorization

$$\begin{tikzcd}
 & & b_1 \\
 & \sum_{i\leq k} b_i \arrow[ur, "\mathsf{hr}_1"] \arrow[dr, "\mathsf{hr}_k"'] & \vdots \\
 & & b_k \\
\sum_{i\in\langle\ell\rangle} b_i \arrow[uur, "\mathsf{hr}_{i\leq k}"] \arrow[ddr, "\mathsf{hr}_{k<i}"] & & \\
 & & b_{k+1} \\
 & \sum_{k<i} b_i \arrow[ur, "\mathsf{hr}_{k+1}"] \arrow[dr, "\mathsf{hr}_\ell"'] & \vdots \\
 & & b_\ell
\end{tikzcd}$$

of the multimorphism $\mathbf{hr} = (\mathsf{hr}_1, \ldots, \mathsf{hr}_\ell)$, with $(\mathsf{hr}_{i\leq k}, \mathsf{hr}_{k<i})$ a degree zero $A^+\,(\star)$ multimorphism and the multimorphisms $\mathbf{hr}_{i\leq k} = (\mathsf{hr}_1, \ldots, \mathsf{hr}_k)$ and $\mathbf{hr}_{k<i}\,(\mathsf{hr}_{k+1}, \ldots, \mathsf{hr}_\ell)$ being degree zero $A^+\,(\star)$ multimorphisms.

Example A.1.2. The multi-Reedy structure on $\triangle$ is prismatic.

Remark A.1.3. In the case of $\triangle$, the property **PMR1** held in a stronger sense than is abstracted here. While in $\triangle$, the object $\sum_{i\in\langle\ell\rangle} b_i$ was unique while the maps hr_i might not be, no uniqueness of the object is here supposed. What's more, this cannot be more strict and still apply in categories of the form $\triangle \int A$. This is not to say that **no** uniqueness abides; indeed if the multimorphism $(f_1, \ldots, f_\ell) : a \longrightarrow \prod b_i$ is degree zero, then the Reedy structure puts $a = \sum_{i\in\langle\ell\rangle} b_i$.

Remark A.1.4. As noted in the introduction to the chapter and in the acknowledgements, the author has been made aware of a similar notion due to Gindi and Cisinski of regular cartesian reedy category, found in [24]. Our notion is stricter than Gindi's; while he abstract all non-degenerate cells of a prism having dimension bounded above by the sum of a prism's dimension, we abstract all top-dimensional cells being of the same dimension.

A.2 Wreath Product of a Prismatic Multi-Reedy Category is Prismatic Multi-Reedy

As one might hope, the property of being a prismatic multi-Reedy category is preserved under wreath products. Indeed, in this section we will prove the following theorem.

Theorem. *Let A be a prismatic multi-Reedy category. Then $\triangle \wr A$ with the multi-Reedy structure of Theorem 8.1.11 is a prismatic multi-Reedy category.*

For simplicity and clarity we'll break the proof of this theorem into three lemmata.

Lemma A.2.1. *Let A be a prismatic multi-Reedy category. Then any $\triangle \wr A^+$ multimorphism*

$$\begin{tikzcd}
 & \left[\left[m^1\right]; b^1_1, \ldots, b^1_{m^1}\right] \\
\left[\left[n\right]; a_1, \ldots, a_n\right] \arrow[ur, "{[f^1;\mathbf{g}^1]}"] \arrow[dr, "{[f^\ell;\mathbf{g}^\ell]}"'] & \vdots \\
 & \left[\left[m^\ell\right]; b^\ell_1, \ldots, b^\ell_{m^\ell}\right]
\end{tikzcd}$$

factors as

$$\begin{tikzcd}
 & & \left[\left[m^1\right]; b^1_1, \ldots, b^1_{m^1}\right] \\
\left[\left[n\right]; a_1, \ldots, a_n\right] \arrow[r] & \left[\left[1+n+1\right]; c_-, a_1, \ldots, a_n, c_+\right] \arrow[ur, "{[p^1;\mathbf{q}^1]}"] \arrow[dr, "{[p^\ell;\mathbf{q}^\ell]}"'] & \vdots \\
 & & \left[\left[m^\ell\right]; b^\ell_1, \ldots, b^\ell_{m^\ell}\right]
\end{tikzcd}$$

for some objects c_- and c_+, with the morphism

$$\left[\left[n\right]; a_1, \ldots, a_n\right] \longrightarrow \left[\left[1+n+1\right]; c_-, a_1, \ldots, a_n, c_+\right]$$

being of $\triangle \wr A^+$, and the multimorphism

$$\begin{tikzcd}
 & \left[\left[m^1\right]; b^1_1, \ldots, b^1_{m^1}\right] \\
\left[\left[1+n+1\right]; c_-, a_1, \ldots, a_n, c_+\right] \arrow[ur, "{[p^1;\mathbf{q}^1]}"] \arrow[dr, "{[p^\ell;\mathbf{q}^\ell]}"'] & \vdots \\
 & \left[\left[m^\ell\right]; b^\ell_1, \ldots, b^\ell_{m^\ell}\right]
\end{tikzcd}$$

being of $\triangle \wr A^+ (\star)$ with the further property that for all $i \in \langle \ell \rangle$, $p^i\left(0\right) = 0$ and $p^i\left(1+n+1\right) = 0$.

Proof. The proof of the lemma is entirely formal. Let

$$c_- = \sum_{j \in \langle \ell \rangle} \sum_{0 < k \leq f^j(0)} b^j_k,$$

$$c_+ = \sum_{j \in \langle \ell \rangle} \sum_{f^j(n) < k \leq m^j} b^j_k,$$

and the components

$$\left[p^i; \mathbf{q}^i\right] : \left[[1+n+1]; c_-, a_1, \ldots, a_n, c_+\right] \longrightarrow \left[\left[m^\ell\right]; b^\ell_1, \ldots, b^\ell_{m^1}\right]$$

be the 0-globular sums

$$\left[\left[0 \mapsto 0, 1 \mapsto f^i(0)\right]; \mathbf{hr} \circ \mathsf{hr}^i\right] \underset{0}{\oplus} \left[f^i; \mathbf{g}^i\right] \underset{0}{\oplus} \left[\left[0 \mapsto f^i(n), 1 \mapsto m^i\right]; \mathbf{hr} \circ \mathsf{hr}^i\right]$$

where the left-most instance of $\mathbf{hr} \circ \mathsf{hr}^i$ is the composite multimorphism

$$\begin{tikzcd}
 & & b^i_0 \\
\sum_{j \in \langle \ell \rangle} \sum_{0 < k \leq f^j(0)} b^j_k \arrow[r, "\mathsf{hr}^i"] & \sum_{0 < k \leq f^i(0)} b^i_k \arrow[ur, "\mathsf{hr}^i_0"] \arrow[dr, "\mathsf{hr}^i_{f^i(0)}"] & \vdots \\
 & & b^i_{f^i(0)}
\end{tikzcd}$$

and similarly the right-most instance of $\mathbf{hr} \circ \mathsf{hr}^i$ is the composite multimorphism

$$\begin{tikzcd}
 & & b^i_{f^i(n)+1} . \\
\sum_{j \in \langle \ell \rangle} \sum_{f^j(n) < k \leq m^i} b^j_k \arrow[r, "\mathsf{hr}^i"] & \sum_{f^i(n) < k \leq m^i} b^i_k \arrow[ur, "\mathsf{hr}^i_0"] \arrow[dr, "\mathsf{hr}^i_{f^i(0)}"] & \vdots \\
 & & b^i_{m^i}
\end{tikzcd}$$

The claim that the morphism $\mathsf{in}_+ \underset{0}{\oplus} \mathrm{id} \underset{0}{\oplus} \mathsf{in}_-$ is of $\triangle \wr A^+$ and that the multimorphism

$$\left(\left[p^1; \mathbf{q}^1\right], \ldots, \left[p^\ell; \mathbf{q}^\ell\right]\right)$$

is of $\triangle \wr A^+ (\star)$ then follow from Corollary 8.1.15 and Theorem 8.1.11. $\square$

The purpose of this lemma is that for an arbitrary $\triangle \wr A^+ (\star)$ multimorphism

$$\begin{tikzcd}
 & \left[\left[m^1\right]; b^1_1, \ldots, b^1_{m^1}\right] \\
\left[[n]; a_1, \ldots, a_n\right] \arrow[ur, "\left[f^1; \mathbf{g}^1\right]"] \arrow[dr, "\left[f^\ell; \mathbf{g}^\ell\right]"'] & \vdots \\
 & \left[\left[m^\ell\right]; b^\ell_1, \ldots, b^\ell_{m^\ell}\right]
\end{tikzcd}$$

we may often assume that $f^i(0) = 0$ and $f^i(n) = m^i$ for all $i \in \langle \ell \rangle$ which, as we'll see, can greatly simplify manipulation.

Lemma A.2.2. *Let A be a prismatic multi-Reedy category. Then $\triangle \wr A$ enjoys the property **PMR1**.*

Proof. Throughout the proof we will fix the following quantification of a $\triangle \wr A^+(\star)$ multimorphism: let

$$\begin{tikzcd}
 & \left[[m^1];b^1_1,\ldots,b^1_{m^1}\right] \\
\left[[n];a_1,\ldots,a_n\right] \arrow[ur, "{[f^1;\mathbf{g}^1]}"] \arrow[dr, "{[f^\ell;\mathbf{g}^\ell]}"'] & \vdots \\
 & \left[[m^\ell];b^\ell_1,\ldots,b^\ell_{m^\ell}\right]
\end{tikzcd}$$

be an $\triangle \wr A^+(\star)$ multimorphism. In light of the prior lemma we may assume that $f^i(0) = 0$ and $f^i(n) = m^i$ for all $i \in \langle \ell \rangle$. We will factor the original multimorphism $\left(\left[f^1;\mathbf{g}^1\right],\ldots,\left[f^\ell;\mathbf{g}^\ell\right]\right)$ as $\mathbf{t} \circ s \circ r$ where r and s are both $\triangle \wr A^+$ morphisms and $\mathbf{t}$ is a $\triangle \wr A^+(\star)$ multimorphism of degree zero.

We construct r by a 0-globular sum. The maps

$$[[1];a_i] \longrightarrow \left[[1];\sum_{j\in\langle\ell\rangle}\sum_{k\in F(f^j)(i)} b^j_k\right]$$

provided by the prismatic hypothesis on A, which we note are of $\triangle \wr A^+$, assemble into an $\triangle \wr A^+$ morphism

$$[[n];a_1,\ldots,a_n] \longrightarrow \left[[n];\sum_{j\in\langle\ell\rangle}\sum_{k\in F(f^j)(1)} b^j_k,\ldots,\sum_{j\in\langle\ell\rangle}\sum_{k\in F(f^j)(n)} b^j_k\right].$$

It is this morphism we call r.

Now see too that for each $i \in \langle n \rangle$, the prismatic structure on A gives us an $\triangle \wr A^+$ morphism

$$\left[[1];\sum_{j\in\langle\ell\rangle}\sum_{k\in F(f^j)(i)} b^j_k\right] \longrightarrow \left[\left[\sum_{j\in\langle\ell\rangle}\left|F\left(f^j\right)(i)\right|\right];b^1_{f^1(i-1)+1},\ldots,b^1_{f^1(i)},\ldots,b^\ell_{f^\ell(i-1)+1},\ldots,b^\ell_{f^\ell(i)}\right].$$

It is their 0-globular sum, another $\triangle \wr A^+$ morphism,

$$\left[[n];\sum_{j\in\langle\ell\rangle}\sum_{k\in F(f^j)(1)} b^j_k,\ldots,\sum_{j\in\langle\ell\rangle}\sum_{k\in F(f^j)(n)} b^j_k\right] \longrightarrow X$$

where

$$X = \left[\left[\sum_{i\in\langle n\rangle} \sum_{j\in\langle \ell\rangle} \left| F\left(f^j\right)(i)\right| \right]; \begin{array}{c} b^1_1, \ldots, b^1_{f^1(1)}, \ldots, b^\ell_1, \ldots, b^\ell_{f^\ell(1)}, \\ \vdots, \\ b^1_{f^1(n-1)+1}, \ldots, b^1_{m^1}, \ldots, b^\ell_{f^\ell(n-1)}, \ldots, b^\ell_{f^\ell(n)} \end{array} \right]$$

which we denote by s.

The notationally cumbersome object X of $\triangle \wr A$ emits an $\triangle \wr A^+$ multimorphism $\mathbf{t}$, the components of which are the obvious morphisms

$$\left[\left[\sum_{i\in\langle n\rangle} \sum_{j\in\langle \ell\rangle} \left| F\left(f^j\right)(i)\right| \right]; \begin{array}{c} b^1_1, \ldots, b^1_{f^1(1)}, \ldots, b^\ell_1, \ldots, b^\ell_{f^\ell(1)}, \\ \vdots, \\ b^1_{f^1(n-1)+1}, \ldots, b^1_{m^1}, \ldots, b^\ell_{f^\ell(n-1)}, \ldots, b^\ell_{f^\ell(n)} \end{array} \right] \xrightarrow{t^j} \left[\left[m^j\right]; b^j_1, \ldots, b^j_{m^j}\right].$$

This multimorphism $\mathbf{t}$ is of $\triangle \wr A^+ (*)$ by inspection and is of degree zero as

$$\sum_{i\in\langle n\rangle} \sum_{j\in\langle \ell\rangle} \left| F\left(f^j\right)(i)\right| = \sum_{i\in\langle \ell\rangle} m^i$$

since our hypothesis put $f^j(0) = 0$ and $f^j(n) = m^j$ for each $j \in \langle \ell \rangle$. It is then formal to check that

$$\begin{tikzcd} & \left[\left[m^1\right]; b^1_1, \ldots, b^1_{m^1}\right] \\ {[[n]; a_1, \ldots, a_n]} \arrow[ur, "{[f^1; \mathbf{g}^1]}"] \arrow[dr, "{[f^\ell; \mathbf{g}^\ell]}"'] & \vdots \\ & \left[\left[m^\ell\right]; b^\ell_1, \ldots, b^\ell_{m^\ell}\right] \end{tikzcd}$$

factors as $\mathbf{t} \circ s \circ r$. Since $s \circ r$ is of $\triangle \wr A^+$ and $\mathbf{t}$ is a degree zero multimorphism of $\triangle \wr A^+ (\star)$ we've shown that $\triangle \wr A$ enjoys **PMR1**. $\square$

Implicit in the proof of Lemma A.2.2 is a formula for the top dimensional cells of a prism of $\triangle \wr A$. Indeed, suppose

$$\begin{tikzcd} & \left[\left[m^1\right]; b^1_1, \ldots, b^1_{m^1}\right] \\ {[[n]; a_1, \ldots, a_n]} \arrow[ur, "{[f^1; \mathbf{g}^1]}"] \arrow[dr, "{[f^\ell; \mathbf{g}^\ell]}"'] & \vdots \\ & \left[\left[m^\ell\right]; b^\ell_1, \ldots, b^\ell_{m^\ell}\right] \end{tikzcd}$$

to be a degree 0 $\triangle \wr A^{+}\left(\star\right)$ multimorphism. Then by the previous lemma we may factor that multimorphism through the degree zero $\triangle \wr A^{+}$ morphism

$$s \circ r : \left[\left[n\right]; a_1, \dots, a_n\right] \longrightarrow \left[\left[\sum_{i \in \langle n \rangle} \sum_{j \in \langle \ell \rangle} \left|F\left(f^j\right)(i)\right|\right]; \begin{matrix} b^1_1, \dots, b^1_{f^1(1)}, \dots, b^\ell_1, \dots, b^\ell_{f^\ell(1)}, \\ \vdots, \\ b^1_{f^1(n-1)+1}, \dots, b^1_{m^1}, \dots, b^\ell_{f^\ell(n-1)}, \dots, b^\ell_{m^\ell} \end{matrix}\right].$$

But then, since $\triangle \wr A$ is Reedy, a degree zero $\triangle \wr A^{+}$ morphism is the identity, and

$$n = \sum_{i \in \langle \ell \rangle} m^i = \sum_{i \in \langle n \rangle} \sum_{j \in \langle \ell \rangle} \left|F\left(f^j\right)(i)\right|$$

and $a_1 = b^1_1, \dots, a_n = b^\ell_{m^\ell}$. What's more, the maps to the objects $\left[\left[m^i\right]; b^i_1, \dots, b^i_{m^i}\right]$ are implicit in that formula (they are implicit in the ordering of the b^j_k) thus the following lemma follows by inspection.

Lemma A.2.3. *Let A be a prismatic multi-Reedy category. Then $\triangle \wr A$ enjoys the property* ***PMR2****.*

We've thus assembled a proof the theorem with which we began the section.

Theorem A.2.4. *Let A be a prismatic multi-Reedy category. Then $\triangle \wr A$ with the multi-Reedy structure of Theorem 8.1.11 is a prismatic multi-Reedy category.*

Proof. The lemmata above provide a proof that $\triangle \int A$ is prismatic and the multi-Reedy structure is from Theorem 8.1.11 as stated. □

Remark A.2.5. The description of the top-dimensional cells of a prism of objects in $\triangle \wr A$ generalizes and makes more explicit a formula for the shuffled bouquets of [9].

As a corollary to that theorem we find that the categories Θ_n and Θ are prismatic multi-Reedy categories.

Corollary A.2.6. *The multi-Reedy categories Θ_n, for $n \in \mathbf{N}$, are prismatic and the multi-Reedy category Θ is prismatic.*

Proof. Since $\triangle$ is prismatic, then Θ_2 is prismatic, and by induction, the categories Θ_n are prismatic. Then since Θ is a colimit over embeddings, $\Theta = \varinjlim \left\{\triangle \longrightarrow \Theta_2 \longrightarrow \cdots\right\}$, we find that any multimorphism lies in some $\Theta_n \longrightarrow \Theta$, whence as each Θ_n is prismatic, then so too is Θ. □

A.2.1 Verticalizers

An important class of maps in $\triangle \int A$ which arise from the multimorphisms of the form

$$\begin{tikzcd}
 & a_1 \\
\sum_{i\in\langle\ell\rangle} a_i \arrow[ur, "\mathsf{hr}_1"] \arrow[dr, "\mathsf{hr}_\ell"] & \vdots \\
 & a_\ell
\end{tikzcd}$$

are the maps we'll call verticalizers.

Definition A.2.7. Given a top-dimensional cell of a prism

$$\begin{tikzcd}
 & a_1 \\
\mathbf{hr} : \sum_{i\in\langle\ell\rangle} a_i \arrow[ur, "\mathsf{hr}_1"] \arrow[dr, "\mathsf{hr}_\ell"] & \vdots \\
 & a_\ell
\end{tikzcd}$$

define the verticalizer $v_{\sum a_i}$ to be the map

$$\left[\begin{pmatrix} 0 \mapsto 0 \\ \\ 1 \mapsto \ell \end{pmatrix} ; \mathbf{hr}\right] : \left[[1] ; \sum_{i\in\langle\ell\rangle} a_i\right] \longrightarrow [[\ell] ; (a_1, \ldots, a_\ell)]$$

Example A.2.8. The name verticalizer is derived from the case where $A = \triangle$. Consider the multimorphism

$$\begin{tikzcd}
 & {[1]\ ,} \\
{[\ell]} \arrow[ur, "(0|1{,}\ldots\ell)"] \arrow[dr, "(0\ldots\ell-1|\ell)"'] & \vdots \\
 & {[1]}
\end{tikzcd}$$

Which corresponds to the first ℓ-simplex of the prism $\left(\triangle^1\right)^\ell$. This morphism wreaths with the simplicial map $\begin{pmatrix} 0 \mapsto 0 \\ \\ 1 \mapsto \ell \end{pmatrix}$ to define the verticalizer

$$\left[\begin{pmatrix} 0 \mapsto 0 \\ \\ 1 \mapsto \ell \end{pmatrix} ; (0|1, \ldots \ell) , \ldots , (0 \ldots \ell - 1|\ell)\right] : [[1] ; [\ell]] \longrightarrow [[\ell] ; [1]$$

For a visual presentation of this claim, consider the case where $\ell = 2$. And see that the verticalizer here is the 2-functor from the 2-category on the pasting diagram

$$\begin{tikzcd}
\bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M, above}, ""{name=N, below}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet
\arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=N, to=D]
\end{tikzcd}$$

to the 2-category on the pasting diagram

$$\begin{tikzcd}
\bullet \arrow[r, bend left, ""{name=A, below}] \arrow[r, bend right, ""{name=B, above}] & \bullet \arrow[r, bend left, ""{name=C, below}] \arrow[r, bend right, ""{name=D, above}] & \bullet
\arrow[Rightarrow, from=A, to=B] \arrow[Rightarrow, from=C, to=D]
\end{tikzcd}$$

which is given by sending the topmost 2-cell of the source to the whiskered 2-cell

$$\begin{tikzcd}
\bullet \arrow[r, bend left, ""{name=A, below}] \arrow[r, bend right, ""{name=B, above}] & \bullet \arrow[r, bend left] & \bullet
\arrow[Rightarrow, from=A, to=B]
\end{tikzcd}$$

and the bottom 2-cell to the whiskered 2-cell below.

$$\begin{tikzcd}
\bullet \arrow[r, bend right] & \bullet \arrow[r, bend left, ""{name=C, below}] \arrow[r, bend right, ""{name=D, above}] & \bullet
\arrow[Rightarrow, from=C, to=D]
\end{tikzcd}$$

The Eckmann-Hilton degeneracies of chapter 13 which we use to prove the central theorem of this thesis are an important instance of degeneracies of verticalizers.

A.3 0-cells

Another seemingly innocuous property of $\triangle$ which is in fact fundamental is that $\triangle$ has an object $[0]$ of degree 0 and for all objects $[n]$ of $\triangle$, there exist maps $[0] \longrightarrow [n]$.

Definition A.3.1. A Reedy category A is said to **have a 0-cell** if:

- there exists an object $0_A \in \mathsf{Ob}(A)$ of degree zero, and
- for all objects b of A, there exists a map $0_A \longrightarrow b$.

An object 0_A as above is referred to as a **0-cell** of A.

The Reedy structure on A shows that this notion is **much** more structured than it may at first appear.

Lemma A.3.2. *Suppose A to be a Reedy category with a 0-cell 0_A. Then the following hold:*

- *the zero cell 0_A is unique up to isomorphism; and*
- *any map $0_A \longrightarrow a$ is of A^+.*

Proof. Let 0_A and $0'_A$ be zero cells of A. There exists a map $0_A \longrightarrow 0'_A$ by definition. This map admits an A^- then A^+ factorization. However, since $\lambda(0_A) = 0$, then the first leg of the factorization is an identity and so it the second. A similar factorization argument proves the second claim. □

One final triviality which we wish to note is the following.

Lemma A.3.3. *For any small category A, the Reedy category $\triangle \int A$ has a 0-cell.*

Proof. The object $[0]$ is the 0-cell of $\triangle \int A$. □

A.4 Coface Factorization and the Incremental Skeletality Theorem

In this section we'll develop conditions on a multi-Reedy category A such that $\triangle \int A$ is an incremental Reedy category.

Theorem A.4.1. *Let A be a prismatic multi-Reedy category. If A has a 0-cell and A is incremental, then $\triangle \int A$ is incremental.*

We'll do this by proving for an exhaustive categorization of the forms of $\triangle \int A^+$ maps, that each sort admits a coface factorization.

Definition A.4.2. Let A be a Reedy category and let $f : a \longrightarrow b$ be an A^+ morphism. A **coface** factorization of f is a factorization of f, $f'' \circ f' = f$, where f' and f'' are both A^+ morphisms and the degree of f' is one, i.e. f' is a coface.

A.4.1 Character of cofaces in $\triangle \int A$

Before we prove the main theorem of the chapter, we may already completely characterize the cofaces of the Reedy categories $\triangle \int A$. Since the degree of objects is additive over 0-globular sums, it is clear that any coface in $\triangle \int A$ is a o-globular sum of a simpler coface and identities. In fact, we need only characterize the cofaces for which either the object $[0]$ or objects of the form $[[1]\,;a]$ are the source.

Lemma A.4.3. *Suppose A to be a multi-Reedy category. Then, in the category $\triangle \int A$, the only cofaces with source $[0]$ are $\mathsf{in}_-, \mathsf{in}_+ : [0] \longrightarrow [[1]\,;a]$ where a is of degree zero.*

Proof. Clearly a must be of degree zero, and $\mathsf{in}_-, \mathsf{in}_+$ exhaust the maps between $[0]$ and any object $[[1]\,;a]$. ☐

Remark A.4.4. Of course, when A has a zero cell, there are then exactly two cofaces of which $[0]$ is the source: $\mathsf{in}_-, \mathsf{in}_+ : [0] \longrightarrow [[1]\,;0_A]$.

Lemma A.4.5. *Suppose A to be a multi-Reedy category. Then in the category $\triangle \int A$, the only cofaces of $\triangle \int A$ with source $[[1]\,;a]$ are:*

$$[id; f] : [[1]\,;a] \longrightarrow [[1]\,;b]$$

where f is a coface of A, or

$$\left[d^1; \mathbf{f}\right] : [[1]\,;a] \longrightarrow [[2]\,;(b,c)]$$

where $\mathbf{f}$ is a degree zero $A^+(\star)$ multimorphism.

Proof. The proof is formal and left to the reader. ☐

Remark A.4.6. When A is prismatic, or at least enjoys the property **PMR1**, then $a = b + c$ and $\mathbf{f} = \mathbf{hr}$ for some top-dimensional cell of the prism $A^b \times A^c$.

It is then as a corollary that all cofaces may de described.

Corollary A.4.7. *Let A be a multi-Reedy category. Then every coface $[[n]\,;(a_1,\ldots,a_n)] \longrightarrow [[m]\,;(b_1,\ldots,b_m)]$ of $\triangle\int A$ is a $0-$globular sum of identities and exactly one map of a form:*

$$\mathsf{in}_-,\mathsf{in}_+ : [0] \longrightarrow [[1]\,;a]\,,$$

$$[id; f] : [[1]\,;a] \longrightarrow [[1]\,;b]$$

where f is a coface of A, or

$$\left[d^1;\mathbf{f}\right] : [[1]\,;a] \longrightarrow [[2]\,;(b,c)]$$

where $\mathbf{f}$ is a degree zero $A^+(\star)$ multimorphism.

Proof. The proof is left to the reader. □

A.4.2 Incrementality by coface factorizations

It will now be simple to prove that any $\triangle\int A^+$ map factors through a coface. As mentioned in the previous subsection, we know that any $\triangle\int A^+$ morphism is globular sum of more basic $\triangle\int A^+$ morphisms. Indeed, every $\triangle\int A^+$ morphism is a 0-globular sum of maps of the forms:

$$\mathsf{in}_-,\mathsf{in}_+ : [0] \longrightarrow [[n]\,;(a_1,\ldots,a_n)]$$

and

$$\left[\begin{pmatrix} 0\mapsto 0 \\ 1\mapsto n \end{pmatrix};\mathbf{f}\right] : [[1]\,;a] \longrightarrow [[n]\,;(b_1,\ldots,b_n)]\,,$$

where $\mathbf{f}$ is an A^+ multimorphism. What's more, is that in Lemma 8.1.14 we proved that this presentation induced a factorization of any $\triangle\int A^+$ map into a sequence of $\triangle\int A^+$ maps of the form $\mathrm{id} \underset{0}{\oplus} [f;\mathbf{g}] \underset{0}{\oplus} \mathrm{id}$ where $[f;\mathbf{g}]$ is of one of the forms enumerated above. It therefore suffices to provide an coface factorization of maps of the form $\mathsf{in}_-,\mathsf{in}_+$ or $\left[\begin{pmatrix} 0\mapsto 0 \\ 1\mapsto n \end{pmatrix};\mathbf{f}\right]$. In subsequent lemmata: Lemma A.4.8, Lemma A.4.9, and Lemma A.4.10; we will do precisely that.

Lemma A.4.8. *Given a multi-Reedy category with a* 0*-cell,* 0_A*. Then, for any object*

$$[[n]\,;(a_1,\ldots,a_n)]$$

of $\triangle\int A$*, the* $\triangle\int A^+$ *map*

$$\mathsf{in}_-,\mathsf{in}_+ : [0] \longrightarrow [[n]\,;(a_1,\ldots,a_n)]$$

admit coface factorizations.

Proof. See that in either case there must be at least one coface factorization through the object $[[1]\,;0_A]$ since 0_A is a 0-cell for A. ☐

Now, when A is a prismatic multi-Reedy category, the prismatic axiom **PMR1** induces $\triangle\int A^+$ factorizations of maps of the form $\left[\begin{pmatrix} 0\mapsto 0 \\ 1\mapsto n \end{pmatrix};\mathbf{f}\right]$,

$$\begin{tikzcd}
{[[1]\,;a]} \arrow[r, "{[\mathrm{id};f^+]}"] \arrow[rr, bend right, swap, "{\left[\begin{pmatrix} 0\mapsto 0 \\ 1\mapsto n \end{pmatrix};\mathbf{f}\right]}"] & {\left[[1]\,;\sum_{i\in\langle\ell\rangle} a_i\right]} \arrow[r, "{\left[\begin{pmatrix} 0\mapsto 0 \\ 1\mapsto \ell \end{pmatrix};\mathbf{hr}\right]}"] & {[[\ell]\,;(a_1,\ldots,a_\ell)]}
\end{tikzcd}$$

where f^+ is a A^+ map and $\mathbf{hr}$ is a degree zero $A^+\,(\star)$ multimorphism. As such we need only prove the existence of coface factorization for $\triangle\int A^+$ maps of the form $[\mathrm{id};f]$ or $\left[\begin{pmatrix} 0\mapsto 0 \\ 1\mapsto \ell \end{pmatrix};\mathbf{hr}\right]$.

Lemma A.4.9. *Let* A *be an incremental multi-Reedy category. Then,* $\triangle\int A^+$ *maps of the form*

$$[[1]\,;f] : [[1]\,;a] \longrightarrow [[1]\,;b]$$

admit coface factorizations.

Proof. This is immediate from the incremental hypothesis on A. ☐

Lemma A.4.10. *Let A be a prismatic multi-Reedy category. Then, the $\triangle \int A^+$ morphisms of the form*

$$\left[\begin{pmatrix} 0 \mapsto 0 \\ \\ 1 \mapsto \ell \end{pmatrix} ; \mathbf{hr}\right] : \left[[1] \,; \sum_{i \in \langle \ell \rangle} b_i\right] \longrightarrow [[n] \,; (b_1, \ldots, b_n)]$$

admit coface factorizations.

Proof. It is almost immediate that axiom **PMR2** provides such a factorization, Indeed, for example consider the factorization

$$\left[[1] \,; \sum_{i \in \langle \ell \rangle} b_i\right] \longrightarrow \left[[2] \,; b_1, \sum_{i \in \langle \ell \rangle - \{1\}} b_i\right] \longrightarrow [[n] \,; b_1, \ldots, b_n]$$

of $\left[\begin{pmatrix} 0 \mapsto 0 \\ \\ 1 \mapsto \ell \end{pmatrix} ; \mathbf{hr}\right]$. ☐

Proof. (of Theorem A.4.1) The discussion and lemmata above constitute a proof. ☐

It is then as a corollary to Theorem A.4.1 that we may prove the categories Θ_n for $n \in \mathbf{N}$ and Θ are incremental Reedy categories.

Corollary A.4.11. *The categories Θ_n for $n \in \mathbf{N}$ and the category Θ are incremental Reedy.*

Proof. Since $\triangle$ incremental multi-reedy and has a 0-cell, Theorem A.4.1 applies, and the categories Θ_n are incremental since each admits a $0-$cell.

For the proof that Θ is incremental it suffices to observe that Θ is a filtered colimit of categories and embeddings, thus every morphism of Θ is a morphism of Θ_n for some $n \in \mathbf{N}$. ☐

A.5 Intersection of cofaces in $\triangle \int A$

Recall that in Corollary A.4.7 we characterized all cofaces in a category $\triangle \int A$ where A is multi-Reedy as 0-globular sums of identities with a basic library of forms. This characterization and the arguments that follow here will allow us to provide a formula for most coface pair fibered products in $\triangle \int A$, or at least in $\triangle \int \widehat{A}$.

Lemma A.5.1. *Let A be a multi-Reedy category and let $[[\ell]\,;(c_1,\dots,c_\ell)]$ be an object of $\triangle \int A$ with degree $\lambda\left([[\ell]\,;(c_1,\dots,c_\ell)]\right) \geq 2$. Then, for any cofaces*

$$[f;\mathbf{g}] : [[n]\,;(a_1,\dots,a_n)] \longrightarrow [[\ell]\,;(c_1,\dots,c_\ell)]$$

and

$$[p;\mathbf{q}] : [[m]\,;(b_1,\dots,b_m)] \longrightarrow [[\ell]\,;(c_1,\dots,c_\ell)]$$

of $\triangle \int A$, the limit over the underlying co-span in $\triangle$,

$$\begin{tikzcd} & {[n]} \arrow[d, "f"] \\ {[m]} \arrow[r, "p"] & {[\ell]} \end{tikzcd}$$

exists in $\triangle$.

Proof. Since the degree of $[[\ell]\,;(c_1,\dots,c_\ell)]$ is at least 2, then $\ell \geq 1$.

If $\ell = 1$, then the degree of $c_1 = c$ must be at least one, so any coface of $[[\ell]\,;(c_1,\dots,c_\ell)] = [[1]\,;c]$ is a coface $f : a \longrightarrow c$ wreathed with the identity. Thus, for any pair of cofaces, the underlying simplicial co-span is

$$\begin{tikzcd} & {[1]} \arrow[d, "\mathrm{id}"] \\ {[1]} \arrow[r, "\mathrm{id}"] & {[1]} \end{tikzcd}$$

which has a limit in $\triangle$.

If $\ell \geq 2$, then Corollary A.4.7 provides that $f, p \in \left\{d^0,\dots,d^\ell,\mathrm{id}\right\}$ whence the limit over the co-span

$$\begin{tikzcd} & {[n]} \arrow[d, "f"] \\ {[m]} \arrow[r, "p"] & {[\ell]} \end{tikzcd}$$

exists as it is either:

- a pullback along the identities;

- an equalizer of the same monomorphism; or

- one of Cisisnki's absolutely cartesian squares in $\triangle$.

$\square$

As it turns out, since $\widehat{A}$ is endowed with all limits, the existence of the underlying simplicial limit is the sole obstruction to the existence of the limits

$$\varprojlim \left\{ \begin{tikzcd} & {[[n]\,;(a_1,\ldots,a_n)]} \arrow[d, "{[f;\mathbf{q}]}"] \\ {[m]\,;(b_1,\ldots,b_m)} \arrow[r, "{[p;\mathbf{q}]}"] & {[[\ell]\,;(c_1,\ldots,c_\ell)]} \end{tikzcd} \right\}$$

in $\triangle \int \widehat{A}$.

Definition A.5.2. Let

$$\begin{tikzcd} & {[[n]\,;(a_1,\ldots,a_n)]} \arrow[d, "{[f;\mathbf{q}]}"] \\ {[m]\,;(b_1,\ldots,b_m)} \arrow[r, "{[p;\mathbf{q}]}"] & {[[\ell]\,;(c_1,\ldots,c_\ell)]} \end{tikzcd}$$

be a co-span in $\triangle \int A$ for which the fibered product $[m] \underset{[\ell]}{\times} [n]$ of the underlying simplicial morphisms exists. Then, for each $k \in F\left([m] \underset{[\ell]}{\times} [n]\right)$, let D_k be the diagram in A on the set of objects

$$\{b_r\}_{r\in F(\mathsf{pr}_1)(k)} \cup \{c_t\}_{t\in F(f\circ\mathsf{pr}_2=p\circ\mathsf{pr}_1)(k)} \cup \{a_s\}_{s\in F(\mathsf{pr}_2)(k)}$$

with morphisms

$$b_r \xrightarrow{q_t^r} c_t$$

for each $t \in F(p)(r)$ for some $r \in F(\mathsf{pr}_1)(k)$, and

$$a_s \xrightarrow{g_t^s} c_t$$

for each $t \in F(f)(s)$ for some $s \in F(\mathsf{pr}_2)(k)$.

Working in $\widehat{A}$, for each such k, let

$$\mathbf{pr}_b^k : \varprojlim D_k \longrightarrow \prod_{i\in F(\mathsf{pr}_1)(k)} A^{b_i}$$

and

$$\mathbf{pr}_c^k : \varprojlim D_k \longrightarrow \prod_{j\in F(\mathsf{pr}_2)(k)} A^{a_j}$$

denote the products of the canonical projections. Likewise, let

$$\mathbf{pr}_b : \prod_{k\in[m]\underset{[\ell]}{\times}[n]} \left(\varprojlim D_k \longrightarrow \prod_{i\in F(\mathsf{pr}_1)(k)} A^{b_i}\right)$$

and

$$\mathbf{pr}_c : \prod_{k\in[m]\underset{[\ell]}{\times}[n]} \left(\varprojlim D_k \longrightarrow \prod_{j\in F(\mathsf{pr}_2)(k)} A^{a_j}\right)$$

denote the products of those products of canonical projections.

Remark A.5.3. Note that

$$[\mathsf{pr}_1;\mathbf{pr}_b] : \left[[m]\underset{[\ell]}{\times}[n]\,;\left(\varprojlim D_k\right)_{k\in F\left([m]\underset{[\ell]}{\times}[n]\right)}\right] \longrightarrow \left[[m]\,;(b_i)_{i\in F([m])}\right]$$

and

$$[\mathsf{pr}_2;\mathbf{pr}_a] : \left[[m]\underset{[\ell]}{\times}[n]\,;\left(\varprojlim D_k\right)_{k\in F\left([m]\underset{[\ell]}{\times}[n]\right)}\right] \longrightarrow \left[[n]\,;(a_j)_{j\in F([n])}\right]$$

comprise well formed morphisms of $\triangle \int \widehat{A}$.

Lemma A.5.4. *Let*

$$\begin{tikzcd}
& {[[n]\,;(a_1,\dots,a_n)]} \arrow[d, "{[f;\mathbf{q}]}"] \\
{[m]\,;(b_1,\dots,b_m)} \arrow[r, "{[p;\mathbf{q}]}"] & {[[\ell]\,;(c_1,\dots,c_\ell)]}
\end{tikzcd}$$

be a co-space in $\triangle \int A$ *for which the fibered product* $[m]\underset{[\ell]}{\times}[n]$ *of the underlying simplicial morphisms exists. Then, the span*

$$\begin{tikzcd}
{\left[[m]\underset{[\ell]}{\times}[n]\,;\left(\varprojlim D_k\right)_{k\in F\left([m]\underset{[\ell]}{\times}[n]\right)}\right]} \arrow[r, "{[\mathsf{pr}_2;\mathbf{pr}_a]}"] \arrow[d, "{[\mathsf{pr}_1;\mathbf{pr}_b]}"] & {[[n]\,;(a_1,\dots,a_n)]} \\
{[m]\,;(b_1,\dots,b_m)} &
\end{tikzcd}$$

enjoys the universal property of the fibered product of $[f;\mathbf{g}]$ *and* $[p;\mathbf{q}]$ *in* $\triangle \int \widehat{A}$.

Proof. We'll provide an inverse to the morphism of presheaves induced by the span

$$\left(\left[\mathsf{pr}_1;\mathbf{pr}_b\right],\left[\mathsf{pr}_2;\mathbf{pr}_a\right]\right)$$

.

Suppose

$$\begin{tikzcd}
\left[[z];(x_1,\dots,x_z)\right] \arrow[r, "{[t,\mathbf{u}]}"] \arrow[d, "{[r;\mathbf{s}]}"] & \left[[n];(a_1,\dots,a_n)\right] \\
{[m];(b_1,\dots,b_m)} &
\end{tikzcd}$$

to be a pair of maps such that $[p;\mathbf{q}]\circ[r;\mathbf{s}]=[f;\mathbf{g}]\circ[t;\mathbf{u}]$. These data subsume a pair of simplicial maps (r,t) such that $p\circ r=f\circ t$, whence we get a factorization of the pair (r,t) as

$$\begin{tikzcd}
 & & {[m]} \\
{[z]} \arrow[r, "\overline{(r,t)}"] & {[m]\underset{[\ell]}{\times}[n]} \arrow[ur, "\mathsf{pr}_1"] \arrow[dr, "\mathsf{pr}_2"'] & \\
 & & {[n]}
\end{tikzcd}\quad .$$

It is then by hypothesis that we have an equality of maps in $\widehat{A}$; the product of maps

$$\prod_{k\in F([z])}\left(x_k \xrightarrow{\mathbf{s}^k} \prod_{i\in F(t)(k)}\left(b_i \xrightarrow{\mathbf{q}^i} \prod_{j\in F(p)(i)} c_j\right)\right)$$

is equal to the product of maps

$$\prod_{k\in F([z])}\left(x_k \xrightarrow{\mathbf{u}^k} \prod_{i\in F(t)(k)}\left(a_i \xrightarrow{\mathbf{g}^i} \prod_{j\in F(p)(i)} c_j\right)\right).$$

But then both maps factor through

$$\overline{(\mathbf{s},\mathbf{u})}:\prod_{k\in F([z])}\left(x_k \xrightarrow{\overline{(\mathbf{s}^k,\mathbf{u}^k)}} \prod_{i\in F\left(\overline{(r,s)}\right)(k)}\left(\varprojlim D_i\right)\right).$$

The proof that the map $([r;\mathbf{s}],[t;\mathbf{u}])\longmapsto\left(\overline{(r,s)},\overline{(\mathbf{s},\mathbf{u})}\right)$ comprises an inverse to the map of presheaves induced by $\left(\left[\mathsf{pr}_1;\mathbf{pr}_b\right],\left[\mathsf{pr}_2;\mathbf{pr}_a\right]\right)$ is purely formal and left to the reader. □

A.6 On products of cells and simplices

In [9] we find a description of the product of two cells of Θ as a union of “shuffled bouquets”. In describing the weak equivalence

$$\Sigma_J \Theta_+^T \longrightarrow \Theta_+^T \wedge \Theta_+^{\overline{1}}$$

we’ve already made much use of the presentation of $\Theta^T \times \Theta^{\overline{1}}$, with $T = [[\ell]\,; A_1, \ldots, A_\ell]$, as the co-limit

$$\varinjlim \left\{ \begin{tikzcd} & X_0 \\ \Theta^T \arrow[ur] \arrow[dr] & \\ & X_1 \\ \vdots & \\ & X_{\ell-1} \\ \Theta^T \arrow[ur] \arrow[dr] & \\ & X_\ell \end{tikzcd} \right\} \xrightarrow{\sim} \Theta^T \times \Theta^{[1]}$$

where for $j \in [\ell]$, $X_j = \Theta^{[[\ell+1];A_1,\ldots,A_j,[0],A_{j+1},\ldots,A_\ell]}$. For the simplicial enrichment of $\widehat{\beta}_\bullet$ we’ll need not only the product with the interval, but the product of a cell T with the n-simplex. What follows is a technical lemma which presents $\Theta^T \times \Theta^{[n]}$ as a co-equalizer, extending the description of $\Theta^T \times \Theta^{[1]}$.

For the remainder of this section we will assume the following declarations, however they do not apply to the document globally.

Definition A.6.1. Let $T = [[\ell] : A_1, \ldots, A_\ell]$ be a cell of Θ and let $n \in \mathbf{N}$ be given. Let

$$N = \{\, (i_1, \ldots, i_n) |\, 0 \leq i_1 < i_2 < \cdots < i_n \leq \ell\}$$

and observe that there is a partial order on N, $(i_1, \ldots, i_n) \leq (j_1, \ldots, j_n)$ if for all $k \in \{1, 2, \ldots, n\}$,$i_k \leq j_k$. For each $(i_1, \ldots, i_n) \in N$, let

$$X_{(i_1,i_2,\ldots,i_n)} = \left[[\ell+n]\,; A_0, \ldots, A_{i_1}, \overline{0}, A_{i_1+1}, \ldots, A_{i_2}, \overline{0}, A_{i_2+1}, \ldots, A_{i_n}, \overline{0}, A_{i_n+1}, \ldots, A_\ell\right]$$

and for each $k \in \{1, \ldots, n\}$, let

$$Y_{\left(i_1,\ldots,\widehat{i_k},\ldots,i_n\right)} = \left[[\ell+n-1]\,; A_1, \ldots, A_{i_1}, \overline{0}, A_{i_1+1}, \ldots, A_{i_k}, A_{i_k+1}, \ldots, A_{i_n}, \overline{0}, A_{i_n+1}, \ldots, A_\ell\right].$$

Let

$$d^{(i_1,i_2,\dots,i_n)} : [\ell] \longrightarrow [\ell+n]$$

and

$$S^{(i_1,\dots,i_n)} : [\ell+n] \longrightarrow [\ell]$$

be the morphism of $\triangle$ given below.

$$\begin{tikzcd}
{[\ell]} \arrow[r, "d^{(i_1,i_2,\dots,i_n)}"] & {[\ell+n]} \arrow[r, "S^{(i_1,\dots,i_n)}"] & {[\ell]} \\
0 \arrow[r, mapsto] & 0 \arrow[r, mapsto] & 0 \\
1 \arrow[r, mapsto] & 1 \arrow[r, mapsto] & 1 \\
\vdots & \vdots & \vdots \\
i_1-1 \arrow[r, mapsto] & i_1-1 \arrow[r, mapsto] & i_1-1 \\
i_1 \arrow[dr, mapsto] & i_1 \arrow[ur, mapsto] & i_1 \\
\vdots & i_1+1 \arrow[ur, mapsto] & \vdots \\
 & \vdots & \\
i_2-1 \arrow[dr, mapsto] & & i_2-1 \\
i_2 \arrow[ddr, mapsto] & i_2 \arrow[ur, mapsto] & i_2 \\
\vdots & i_2+1 \arrow[uur, mapsto] & \vdots \\
i_n-1 \arrow[ddr, mapsto] & i_2+2 \arrow[uur, mapsto] & i_n-1 \\
i_n \arrow[dddr, mapsto] & \vdots & i_n \\
\vdots & i_n+n-2 \arrow[uur, mapsto] & \\
\ell \arrow[dddr, mapsto] & i_n+n-1 \arrow[uuur, mapsto] & \ell \\
 & i_n+n \arrow[uuur, mapsto] & \\
 & \vdots & \\
 & \ell+n \arrow[uuur, mapsto] &
\end{tikzcd}$$

Let $s^{\overline{(i_1,\ldots,i_n)}} : [\ell+n] \longrightarrow [n]$ be the morphism

$$\begin{tikzcd}
{[\ell+n]} \arrow[r, "s^{\overline{(i_1,\ldots,i_n)}}"] & {[n]} \\
0 \arrow[r, mapsto] & 0 \\
1 \arrow[ur, mapsto] & 1 \\
\vdots & \vdots \\
i_1-1 \arrow[uuur, mapsto] & \vdots \\
i_1 \arrow[uuuur, mapsto] & \vdots \\
i_1+1 \arrow[uuuur, mapsto] & \vdots \\
\vdots & \vdots \\
i_n-1 \arrow[dddr, mapsto] & \vdots \\
i_n \arrow[ddr, mapsto] & \vdots \\
i_n+1 \arrow[ddr, mapsto] & \vdots \\
\vdots & n-1 \\
\ell+n \arrow[r, mapsto] & n
\end{tikzcd}$$

Remark A.6.2. Note that both $S^{(i_1,\ldots,i_n)}$ and $S^{(i_1+1,\ldots,i_n+1)}$ are both retractions of $d^{(i_1,\ldots,i_n)}$. This familiar aspect of $\triangle$, albeit with non-standard notation in this case, is essential to the proof.

Lemma A.6.3. *There are canonical morphisms*

$$\left[d^{i_k+1}; id\vee\bullet\right] : \left[[\ell+n-1]\,; A_1,\ldots,A_{i_1},\overline{0},A_{i_1+1},\ldots,A_{i_k},A_{i_k+1},\ldots,A_{i_n},\overline{0},A_{i_n+1},\ldots,A_\ell\right] \longrightarrow X_{(i_1,\ldots,i_n)}$$

and

$$\left[d^{i_k+1}; id\vee\bullet\right] : \left[[\ell+n-1]\,; A_1,\ldots,A_{i_1},A_{i_1+1},\overline{0},\ldots,A_{i_k},A_{i_k+1},\ldots,A_{i_n},\overline{0},A_{i_n+1},\ldots,A_\ell\right] \longrightarrow X_{(i_1,\ldots,i_n)}$$

which wreath the map $d^{i_k} : [\ell+n-1] \longrightarrow [\ell+n]$ *with the identities and canonical maps into* $\overline{0}$. *There is a canonical morphism*

$$\left[S^{(i_1,\ldots,i_n)}; id\right] : X_{(i_1,\ldots,i_n)} \longrightarrow [[\ell]\,; A_1,\ldots,A_n]\,.$$

Lemma A.6.4. *Let* $k \in \{1,\ldots,n\}$ *and let* $(i_1,\ldots,i_n) \in N$ *be such that* $(i_1,\ldots,i_k+1,\ldots,i_n) \in N$.

Then the squares

$$\begin{tikzcd}
Y_{\left(i_1,\ldots,\widehat{i_k},\ldots,i_n\right)} \arrow[r, "\left[d^{i_k+1}\right]"] \arrow[d, "\left[d^{i_k+1};id\vee\bullet\right]"'] & X_{(i_1,\ldots,i_k+1,\ldots,i_n)} \arrow[d, "\left[s^{(i_1,\ldots i_k+1,\ldots,i_n)};id\right]\times\left[s^{\overline{(i_1,\ldots i_k+1,\ldots,i_n)}};\bullet\right]"] \\
X_{(i_1,\ldots,i_k,\ldots,i_n)} \arrow[r, "\left[s^{(i_1,\ldots,i_n)};id\right]\times\left[s^{\overline{(i_1,\ldots,i_n)}};\bullet\right]"] & {[[\ell]\,;A_1,\ldots,A_n]\times[n]}
\end{tikzcd}$$

are cartesian.

Proof. Since the category Θ is generated under pushouts by objects of the form $[0]$ and $[[1]\,;S]$ it suffices to check that the square enjoys the correct the universal property against such objects. Considering maps from $[0]$ into the co-span shows that $[\ell+n-1]$ must be the simplicial aspect of the fiber product. Consideration of objects of the form $[[1]\,;S]$ then proves the rest. $\square$

Proposition A.6.5. *Let $B\,(T,n)\hookrightarrow\Theta\hookrightarrow\widehat{\Theta}$ be the diagram whose set of objects is*

$$\left\{X_{(i_1,\ldots,i_n)}\middle|\,0\le i_1<\cdots<i_n\le\ell\right\}\bigcup\left\{Y_{\left(i_1,\ldots,\widehat{i_k},\ldots,i_n\right)}\middle|\,0\le i_1<\cdots<i_n\le\ell,k\in\{1,\ldots,n\}\right\}$$

and all morphisms of the form

$$\left[d^{i_k+1};id\vee\bullet\right]:Y_{\left(i_1,\ldots,\widehat{i_k},\ldots,i_n\right)}\longrightarrow X_{(i_1,\ldots,i_n)}.$$

Then $\varinjlim B\,(T,n)\xrightarrow{\sim}\Theta^{T}\times\Theta^{[n]}$.

Proof. This follows from the prior lemmata. $\square$

Appendix B

Technical lemmata

B.1 Absolutely cartesian squares

Lemma B.1.1. *Let $\mathscr{C}$ be a category, and suppose*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "l"] \\ C \arrow[r, "k"'] & D \end{tikzcd} \tag{B.1}$$

to be a commutative square in $\mathscr{C}$. Suppose further that the morphisms j, k, and ℓ admit retractions r, q, and p respectively. That is suppose we've maps r, q, and p such that the diagrams which follow commute.

$$\begin{tikzcd} C \arrow[r, "r"] \arrow[rr, bend left, "id_C"] & A \arrow[r, "j"] & C \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[rr, bend left, "id_D"] & C \arrow[r, "k"] & D \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[r, "p"] \arrow[rr, bend left, "id_D"] & B \arrow[r, "l"] & D \end{tikzcd}$$

Lastly, suppose that the the square

$$\begin{tikzcd} A \arrow[r, "i"] & B \\ C \arrow[u, "r"'] \arrow[r, "k"'] & D \arrow[u, "p"'] \end{tikzcd}$$

also commutes. Then the square given in equation B.1.1 is absolutely cartesian.

Proof. See [18]. □

B.2 Cell complex lemma

Lemma B.2.1. *Given a small category $\mathscr{C}$ and a class of morphisms W of $\mathscr{C}$. Then the class ${}^{\pitchfork}\left(\mathsf{W}^{\pitchfork}\right)$ is closed under:*

(1) pushouts;

(2) transfinite composition; and

(3) retracts.

Proof. (1) For closure under pushouts we need only observe that if $f : X \longrightarrow Y$ is a pushout of some map $\tilde{f}$, then each lifting problem which must have a a solution for f defines one for $\tilde{f}$. The lifting problem for $\tilde{f}$ has a solution by hypothesis, and then this solution extends to one for f by the universal property of the pushout.

(2) For the closure of ${}^{\pitchfork}\left(\mathsf{W}^{\pitchfork}\right)$ under transfinite composition let κ be some ordinal, and let $w_1, w_2, \ldots, w_\kappa$ be morphisms of class ${}^{\pitchfork}\left(\mathsf{W}^{\pitchfork}\right)$ with each $w_i : X_{i-1} \longrightarrow X_i$. To show $\varinjlim w_i$ to again be of class ${}^{\pitchfork}\left(\mathsf{W}^{\pitchfork}\right)$ we need only construct a lift in the commutative diagrams

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "w_1"] & Y \arrow[dddd, "f"] \\
X_1 \arrow[d, "w_2"] & \\
X_2 \arrow[d] & \\
\vdots \arrow[d] & \\
X_\kappa \arrow[r] & Z
\end{tikzcd}$$

whenever f is of class $\mathsf{W}^{\pitchfork}$. In such case we may do so inductively, extending the top morphism first along w_1, then that extension may be extended along w_2 etc. producing the requisite lift.

(3) For the closure of ${}^{\pitchfork}\left(\mathsf{W}^{\pitchfork}\right)$ under retracts, consider that a lifting problem

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "g"] & A \arrow[d] \\
Z \arrow[r] & B
\end{tikzcd}$$

can be completed under the hypothesis that g is a retract of a morphism of class ${}^{\pitchfork}\left(\mathsf{W}^{\pitchfork}\right)$ to a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "g"] & Y \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d, "g"] & A \arrow[d] \\
Z \arrow[r] & W \arrow[r] \arrow[urr, dashed] & Z \arrow[r] & B
\end{tikzcd}$$

which admits the indicated solution which solves the original lifting problem since the compositions $X \longrightarrow Y \longrightarrow X$ and $Z \longrightarrow W \longrightarrow Z$ are the respective identity morphisms.