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VLF AND ELF PROPAGATION ALONG A HORIZONTAL WIRE LOCATED ABOVE OR BURIED IN THE EARTH†

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ABSTRACT

Propogation constant of an electromagnetic wave supported by a long horizontal thin-wire is determined from a modified modal equation for three different situations: an elevated wire, a buried wire, and a wire located in the air-earth interface. Analytical expressions are derived under the assumption that the height of the wire is much less than the skin-depth of a conducting earth, and that the angular distribution around the wire can be ignored. Although not restrictive in frequency, these conditions are generally satisfied in the VLF and ELF applications. It is shown that the propagation constant of a buried wire is significantly different from an elevated one for almost all heights of practical interest, and that the use of the mean-square-average of the wave numbers in the media is restricted to the case when the wire is in the interface or buried in close proximity of the earth surface.

1. INTRODUCTION

The problem concerned with electromagnetic waves supported by a long, horizontal thin conducting-wire in the presence of a dissipative earth has been of great interest because of its many physical applications. Theoretically the propagation constant of a possible propagating mode can be obtained by formulating a modal equation which satisfies the boundary conditions at the interface as well as on the wire surface.[1] However such a formulation usually involves complete, infinite integrals which have to be evaluated numerically and thus make a general study of the problem rather impractical.[1,2] Up until recently, theoretical investigation of the problem has been mostly restricted to the case where the height of the wire is small compared with the free-space wavelength but large compared with the skin-depth of earth. In this case, a propagating mode of slow-wave nature (with respect to air) can be found.[1,3-6] Such a mode can be shown as identical to the TEM-mode of a two-wire transmission-line when the earth conductivity is infinite, and thus termed a transmission-line mode. Only recently, another mode which exhibits a fast-wave nature is also found.[2] This new mode usually has a lower attenuation rate than the previous one and can be excited more effectively at some higher frequencies. However, none of these results appear to be applicable in the VLF and ELF ranges where the height of the wire is normally much smaller than the skin-depth.

A limiting case which has been studied and is applicable to the VLF and ELF ranges involves a wire of vanishing radius, located exactly in the air-earth interface.[1,7] Because of the geometrical symmetry pertaining to the structure, the propagation constant can be shown as equal to the mean-square-average of the wave numbers in the two media, i.e. $[(k_0{}^2+k_c{}^2)/2]^{\frac{1}{2}}$. However, no assessment has been made to whether the same value can be used when the wire is only slightly elevated (say, a few meters) or buried under the earth.

In this paper an approximate modal equation is derived subject to the condition that the height is very small compared with the skin-depth of earth, and that the non-uniform angular distribution of current density around the wire is neglected. Analytical expressions for the propagation constant are derived based upon observations from two special cases. One corresponds to the situation when the radius of the wire is vanishing but the height is fixed; the other for a vanishingly-small height but a fixed ratio of radius to height. It is shown in our analysis that the propagation constant of a buried wire is very different from that of an elevated one for almost all heights of practical interest. Thus, the use of the mean-square-average of the two wave numbers is strictly restricted to the case when the wire is actually located in the interface, or <u>buried</u> slightly under the interface.

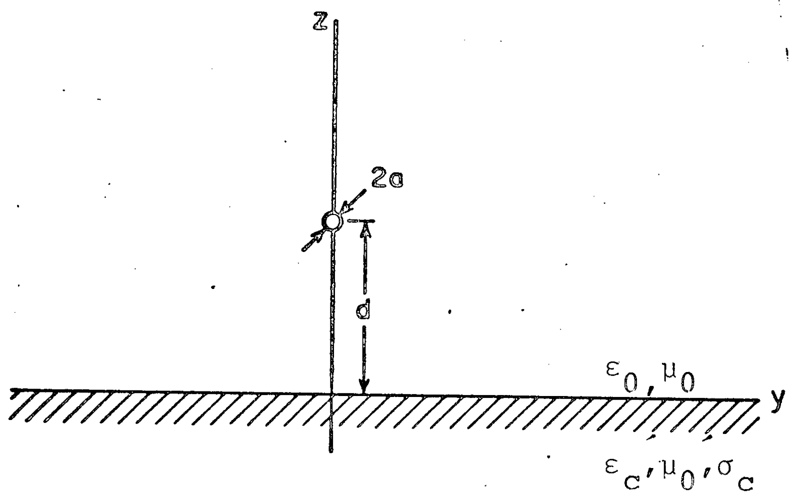

Figure 1

2. MODAL EQUATION

As depicted in Figure 1, we consider the problem of an infinitely-long, horizontal, thin-wire of radius A, located at a height D in air above a homogeneous, plane earth of conductivity σ_c and relative permittivity ε_c. The current on the wire is assumed to be in the form of $\exp(-iwt + ik_0\alpha x)$ where k_0 is the wave number in air and α is the yet-undetermined (complex) propagation constant of a propagating mode; w is the operating angular frequency. As pointed out by Wait,[1] if we can neglect the proximity effect of the current density around the wire under the assumption of A<<D, we can derive a modal equation for determing the value of α by requiring the boundary condition on the wire surface be satisfied. For a perfectly-conducting wire, the condition of a vanishing axial electrical field yields the following modal equation.[1,2]

$$\zeta^2[H_0(a\zeta)J_0(a\zeta) - H_0(2d\zeta)] + P(\alpha) - Q(\alpha) = 0; \qquad (1)$$

$$P(\alpha) = \frac{-i4}{\pi}\int_0^\infty \frac{\exp(-2du_1)}{u_1 + u_2}\, d\lambda , \qquad (2)$$

$$Q(\alpha) = \frac{-i4\alpha^2}{\pi}\int_0^\infty \frac{\exp(-2du_1)}{u_2 + n^2 u_1}\, d\lambda , \qquad (3)$$

where $u_1 = (\lambda^2 - \zeta^2)^{\frac{1}{2}}$ and $u_2 = (\lambda^2 - \zeta_n^2)^{\frac{1}{2}}$,

$\zeta = (1-\alpha^2)^{\frac{1}{2}}$ and $\zeta_n = (n^2-\alpha^2)^{\frac{1}{2}}$ with $0 \leq \arg\zeta$ or $\arg\zeta_n \leq \pi$, and $a = k_0A$, $d = k_0D$; $n = (\varepsilon_c + i\sigma_c/\omega\varepsilon_0)^{\frac{1}{2}}$ is the refractive index of earth; J_0 and H_0 are the Bessel's function and Hankel's function of the first-kind, respectively. In order to allow the convergence of the Sommerfeld integrals $P(\alpha)$ and $Q(\alpha)$, we further require the argument of the radical u_1 and u_2 be restricted in the range from $-\pi/2$ to $\pi/2$. An interested reader is referred to the work by Wait[1], and by Olsen and Chang[2] for a detailed derivation of (1).

We can now start to derive an approximate modal equation under the assumption of

$$|n|d \ll 1 \quad \text{and} \quad |n| \gg 1. \tag{4}$$

Physical significance of the first condition is that the height of the wire has to be much smaller than the skin-depth in earth. Thus, the derivation which follows can not be used for the case of a perfectly-conducting earth where $\sigma_c \to \infty$.

In order to make use of the assumption given in (4), we first note that as d approaches zero, the value of $P(\alpha)$ will increase without bound like a logarithmic function of d for any finite value of α. Thus, an approximate expression of $P(\alpha)$ is readily available if we retain only the leading logarithmic term, plus a constant term independent of d. To achieve this, we recast the expression of $P(\alpha)$ in the form of

$$P(\alpha) = P_1(\alpha) + P_2(\alpha) , \tag{5}$$

$$P_1(\alpha) = \frac{-i2}{\pi} \int_0^\infty \exp(-2du_1)\frac{d\lambda}{u_1} . \tag{6}$$

$$P_2(\alpha) = \frac{-i4}{\pi} \int_0^\infty \left(\frac{1}{u_1+u_2} - \frac{1}{2u_1}\right)\exp(-2du_1)\,d\lambda \tag{7}$$

The first term $P_1(\alpha)$ is immediately known as the integral representation of the Hankel's function $H_0(2d\zeta)$ which contains the desirable logarithmic term, $\ln(2d\zeta)$.[8] The second term $P_2(\alpha)$ however approaches to a constant when d is zero. Thus, with the use of $u_1^2 - u_2^2 = \zeta_n^2 - \zeta^2 = (n^2 - 1)$, we can evaluate $P_2(\alpha)$ approximately as

$$P_2(\alpha) = \frac{-i4}{\pi}\int_0^\infty [\frac{1}{n^2-1} (u_1 - u_2) - \frac{1}{2u_1}] d\lambda$$

$$= \frac{-i2}{\pi}\left(\frac{\zeta_n^2}{n^2-1}\right)\ln \zeta/\zeta_n \ . \tag{8}$$

A subsequent substitution of (8) and (6) into (5) yields

$$P(\alpha) = H_0(2d\zeta) - \frac{i2}{\pi}\left(\frac{\zeta_n^2}{n^2-1}\right)\ln \zeta/\zeta_n \ . \tag{9}$$

To estimate the error involved in the approximate expression, we note from (7) that for a small but non-vanishing d, the exponential factor $\exp(-2du_1)$ in the exact expression can differ from unity only when u_1 is very large, say $u_1 \geq 0.1/d$ or in equivalent $\lambda \geq [\zeta^2 + (0.1/d)^2]^{\frac{1}{2}}$. Since we have $|n|d \ll 1$ from the condition in (4), we can expand the term $u_2 = (u_1^2 + 1 - n^2)^{\frac{1}{2}}$ so that $(u_1 + u_2)^{-1}$ becomes $(2u_1)^{-1}[1 + 0.25(n^2-1)/u_1^2]$. Thus the leading term in the integrand is seen to be proportional to $u_1^{-3}\exp(-2du_1)$ for a large value of u_1. A subsequent asymptotic evaluation of the integral then shows the error in our approximation is of the order $(|n|d)^2$, provided that $(|\zeta|d)^2 \ll 1$. (This approximation would then preclude any possible mode in the range of α where $|\alpha| \geq 1/d$. From a physical viewpoint, a solution in this range would not seem likely, and if it does, would probably have no practical interest.)

Approximation to the other Sommerfeld integral $Q(\alpha)$ can be similarly obtained:

$$Q(\alpha) = Q_1(\alpha) + Q_2(\alpha); \tag{10}$$

$$Q_1(\alpha) = \frac{-i4}{\pi}\left(\frac{\alpha^2}{n^2+1}\right)\int_0^\infty \exp(-2du_1)\frac{d\lambda}{u_1} = \frac{2\alpha^2}{1+n^2}H_0(2d\zeta), \tag{11}$$

$$Q_2(\alpha) = \frac{-i4}{\pi}\left(\frac{\alpha^2}{n^4-1}\right)\int_0^\infty \{(\frac{1}{u_1} - \frac{1}{u_2}) + (\frac{n^2}{1+n^2})\left[\frac{1}{\lambda^2-\lambda_p^2}(\frac{n^2}{u_2} - \frac{1}{u_1})\right]\}d\lambda, \tag{12}$$

where $\lambda_p = [\zeta^2 - (1+n^2)^{-1}]^{\frac{1}{2}}$ is the well-known Sommerfeld pole located in the proper Riemann sheet defined by $-\pi/2 \leq \arg(u_1)$, $\arg(u_2) \leq \pi/2$ in a complex λ-plane.[2] The first term in (12) is readily known as $\ln(\zeta_n/\zeta)$, while the second term has been evaluated previously by Olsen and Chang as [2]

$$\int_0^\infty \frac{d\lambda}{u_1(\lambda^2-\lambda_p^2)} = \frac{i}{\lambda_p(\zeta^2-\lambda_p^2)^{\frac{1}{2}}}\{\frac{i\pi}{2} + \ln[(\zeta^2-\lambda_p^2)^{\frac{1}{2}} - i\lambda_p] - \ln\zeta\}, \tag{13a}$$

$$\int_0^\infty \frac{d\lambda}{u_2(\lambda^2-\lambda_p^2)} = \frac{i}{\lambda_p(\zeta_n^2-\lambda_p^2)^{\frac{1}{2}}}\{\frac{i\pi}{2} + \ln[(\zeta_n^2-\lambda_p^2)^{\frac{1}{2}} - i\lambda_p] - \ln\zeta_n\}. \tag{13b}$$

Principle values of the logarithmic terms in (13a) and (13b) are used. Also, the value of λ_p is chosen as $\operatorname{Im}\lambda_p > 0$.

Substitution of (11), (12), (13a) and (13b) into (10) readily yields

$$Q(\alpha)=(\frac{2\alpha^2}{1+n^2})\,[H_0(2d\zeta)+\frac{in^2}{\pi(n^2-1)}\{\ell n\,\frac{\zeta}{\zeta_n}-\frac{in^2}{\lambda_p(1+n^2)^{\frac{1}{2}}}[\ell n\frac{n^2-i\lambda_p(1+n^2)^{\frac{1}{2}}}{1-i\lambda_p(1+n^2)^{\frac{1}{2}}}+\ell n\frac{\zeta}{\zeta_n}]\}]\,. \tag{14}$$

Thus, an analytical expression of the modal equation is now obtained in the form of (1),(9), and (14). Subject to the assumption that useful solutions are located in the region $|\alpha|^2 d^2 << 1$ in the complex α-plane, further simplification is possible when use is made of the small-argument expansion of the Bessel's and Hankel's functions. After some manipulation, it is then not difficult to show that (1) reduces to

$$-\zeta^2\,\ell n(2d/a)+\frac{1}{1+n^2}\{[\zeta^2\ell n\,\zeta d+\zeta_n^2\,\ell n\,\zeta_n d+(\gamma-i\pi/2)(\zeta^2+\zeta_n^2)]-M(\alpha)\}=0\ , \tag{15}$$

where

$$M(\alpha)=\frac{2n^2}{n^2-1}\{\ell n\,\frac{\zeta}{\zeta_n}-\frac{i\alpha^2}{\lambda_p(1+n^2)^{\frac{1}{2}}}[\ell n\,\frac{n^2-i\lambda_p(1+n^2)^{\frac{1}{2}}}{1-i\lambda_p(1+n^2)^{\frac{1}{2}}}+\ell n\,\frac{\zeta}{\zeta_n}]\}, \tag{16}$$

and $\gamma = 0.577216$, is the Euler's constant. It is of particular interest to note that the modal equation as given in (15), except for the first term, is symmetrical to both sides of the air-earth interface. Therefore, we only need to replace the term $\zeta^2\ell n(2d/a)$ with $(\zeta_n^2/n^2)\ell n(2d/a)$ in order to obtain the corresponding modal equation for a buried wire.

3. APPROXIMATE PROPAGATION CONSTANTS

Although the exact solution of the modal equation as given in (15) appears to be very involved, useful information leading to appropriate approximate solutions can be obtained by considering two special cases: one corresponds to the situation when the height d reduces to zero but the ratio $2d/a$ is finite; the other when the radius of the wire reduces to zero but the height d is finite. In the first case, the leading term of the modal equation (15) is $(\zeta^2 + \zeta_n^2)\, \ell n\, d$. When d approaches to zero, the solution is readily obtained from $\zeta^2 + \zeta_n^2 = 0$, which gives $\alpha = \alpha_0$ where

$$k_0\alpha_0 = k_0[(1+n^2)/2]^{\frac{1}{2}}. \tag{17}$$

This means the propagation constant along the wire is equal to the mean-square-average of the wave numbers in the two media. Our result therefore agrees with the Coleman's approximation for an infinitely-thin wire located exactly in the interface.[1,7] However, as soon as the wire is elevated, the term $\ell n\, d$ becomes a large but finite number which has to be weighted by the factor $1/|n|^2$ in the modal equation (15). Since $|n| >> 1$, we may no longer consider the term $(\zeta^2 + \zeta_n^2)\ell n\, d$ as the only leading term of (17). To examine this possibility, we assume $\alpha = \alpha_0 + \Delta\alpha_0$ and employ a perturbation technique to obtain from (15) a modified modal equation as

$$(1+2\Delta\alpha_0/\alpha_0)\ell n(2d/a) + \frac{1}{n^2}\{-(2\Delta\alpha_0/\alpha_0)[2\ \ell n\ \alpha_0 d - i\pi/2 + 2\gamma+1] - i\pi/2\}$$
$$-2M(\alpha_0)/n^4 = 0\ . \tag{18}$$

The term associated with $M(d_0)$ is of the order $|n|^{-4}$ and therefore can be omitted. The change in α is now obtained as

$$\frac{\Delta\alpha_0}{\alpha_0} = -\frac{1}{2}\{1 - \frac{2}{n^2 \ln(2d/a)} [\ln(dn/\sqrt{2}) - i\pi/4 + \gamma + 1/2]\}^{-1} . \tag{19}$$

which is valid provided that $|\Delta\alpha_0|^2 << |\alpha_0|^2$.

An inspection of (19) clearly indicated that the propagation constant along the wire can be approximated by α_0 only when the condition

$$L_c = |\ln nd|/\ln(2d/a) > |n|^2 \tag{20}$$

holds. To demonstrate the physical significance of this condition in ELF application, we can choose the following set of typical parameters:

$$\begin{cases} \sigma_c = 10^{-3} \text{ mhos/m}, \quad \varepsilon_c = 10 \; ; \quad f = 64 \text{ Hz} \\ n = 3.75 \times 10^2(1+i); \quad \ln 2d/a = 3 \; . \end{cases}$$

Then, for $D = 1$ meter, $|\ln nd| = 8.57$ and for $D = 10^{-2}$ meters, $|\ln nd| = 13.17$. It is apparent from these values that condition (20) seldom can be satisfied for any height of practical interest. Therefore, the use of $\alpha_0 = [(1+n^2)/2]^{\frac{1}{2}}$ is strictly restricted to the case which the actually located in the interface.

Following a similar analysis, we can show that the modified modal equation for a buried wire close to the interface is

$$(-1+2\Delta\alpha_c/\alpha_0)\,\ell n(2d/a) + \{-(2\Delta\alpha_c/\alpha_0)[2\ \ell n\alpha_0 d - i\pi/2 + 2\gamma^2 + 1] - i\pi/2\}$$

$$-2M(\alpha_0)/n^2 = 0\ , \qquad (21)$$

where $\alpha = \alpha_0 + \Delta\alpha_c$. It then follows that the change in the propagation constant is in the form of

$$\frac{\Delta\alpha_c}{\alpha_0} = \frac{1}{2}\{1 - \frac{2}{\ell n(2d/a)+i\pi/2}\ [\ell n(nd/\sqrt{2})+\gamma+\frac{1}{2}]\}^{-1}. \qquad (22)$$

Unlike the case of an elevated wire, the derivation is subject to the condition of

$$L_c = |\ell n\ nd|/\ell n(2d/a) > 1, \qquad (23)$$

instead of $L_c > |n|^2$, and therefore has a much wider range of application. A comparison of (19) and (22) indicates that the change in the propagation constant for a buried wire from the mean-square-average value is much less than that of an elevated wire. It is of further interest to note that the condition given in (23) is equivalent to $d^2 << a\delta$ where $\delta = \sqrt{2}/n$ is the skin-depth of the earth material. As is evident from the previous numerical example, such a condition is usually satisfied in many ELF and VLF applications.

In order to find an appropriate expression for the propagation constant of an elevated wire, we now turn to the other special case when the radius of the wire reduces to zero. From (15), it is seen that the leading term in this case is

ζ^2 $\ln(2d/a)$ which yields immediately a solution of $\alpha = \alpha_1$ where

$$k_0\alpha_1 = k_0 \tag{24}$$

i.e. the same as the wave number in air. Thus, for an infinitely thin wire, the result is independent of both the earth's electric constants and wire height. For a wire of finite size however, the turn $\ln(2d/a)$ is not necessarily very large and hence, can not be considered as the only leading contribution to the modal equation. But since most of the other terms in (15) are weighted by the factor $1/n^2$, we can recast the modal equation into the form of

$$\zeta^2 = \frac{\ln(nd) - i\pi/2 + \gamma + (1/n^2)[n^2\ln(\zeta_n/n) + M(\alpha)]}{\ln(2d/a) - (1/n^2)[\ln(\zeta\zeta_n d^2) - i\pi + 2\gamma]} \tag{25}$$

so that as $a \to 0$, the resultant expression explicitly reduces to that of (24). Provided that the following conditions hold:

$$|n|^2 >> |\ln n| \ ; \quad |n|^2 >> L_c >> 1/|n|^2 \ , \tag{26}$$

we can show without difficulty that the terms in the square brackets of both the numerator and denominator of (2) can be completely ignored. The appropriate solution to the modal equation is therefore given as

$$\zeta^2 = [\ln(nd) - i\pi/2 + \gamma]/\ln(2d/a) \ ,$$

or

$$k_0\alpha_2 = k_0[1 - \frac{\ln(nd) - i\pi/2 + \gamma}{\ln(2d/a)}]^{\frac{1}{2}} \ . \tag{27}$$

Thus, unlike the perturbation result given earlier for a buried wire, the value of α as obtained from (27) can be substantially different from its limiting case when the radius of the wire reduces to zero. This is particularly true when the wire is placed close to the earth surface, as evident from the previous numerical example. We also note that the condition given in (26), which is less restrictive than (23) of the buried case, should be applicable for practically all ELF and VLF problems of interest.

4. CONCLUDING REMARKS

In this paper, we have successfully derived explicit expressions for the propagation constant of current waves propagating along a horizontal wire of finite radius for three different situations: (i) an elevated wire above a conducing earth, as given in (25); (ii) a buried wire as given in (22); (iii) a wire located in the interface as given in (17). Conditions from which these expressions are derived, i.e. (4), (23) and (26) are in general valid for ELF and VLF applications. We have shown in our analysis that the value taken from the mean-square-average of the wave numbers in the two media is strictly valid when the wire is in the interface or buried in close proximity of the interface. We have also shown that the propagation constant of a slightly-elevated wire can vary substantially from the free-space value whenever $|\ln nd|$ is greater than $\ell n(2d/a)$. It should be mentioned that in obtaining the approximate solutions for all the three cases, the value of $M(\alpha)$ is shown to be less than $1/|n|^2$ and hence is neglected. However, as is evident from (16) inherent in the expression of $M(\alpha)$ is a square-root singularity occurred at $\alpha_p = n/(1+n^2)^{\frac{1}{2}}$. At this location, $\zeta = -1/(1+n^2)^{\frac{1}{2}}$ and $\lambda_p = 0$. Thus, additional solutions might exist in the neighborhood of α_p where $M(\alpha)$ is dominant. This corresponds exactly to the fast-wave mode studied earlier for higher frequencies. However from the result given in [2], it is known that such a mode cannot be excited efficiently under the condition given in (4). It also should be mentioned our present analysis has not included the effect of a reflecting ionosphere, or a non-uniform distribution of earth parameter, or an insulation layer surrounding the wire, etc. All these factors could significantly alter the performance of a ELF or VLF system.

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