Information Theoretic Limits of MIMO Interference and Relay Networks

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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

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Information Theoretic Limits of MIMO Interference and Relay Networks

Thesis directed by Prof. Mahesh K. Varanasi

In this thesis, the information theoretic performance limits of two important building blocks of the general multi-user wireless network, namely, the *interference channel* and the *relay channel*, are characterized. We consider both time-invariant and time-varying or fading channel. In the first part, we focus on the 2-user interference channel with time-invariant channel coefficients. First, we characterize the capacity region of a class of MIMO IC called *strong in partial order* ICs. It turns out that for this class of channels decoding both the messages at both the receivers is optimal, i.e., the capacity region is identical to that of the compound multiple access channel (MAC). The defining constraints on the channel coefficients for the class of *strong in partial order* ICs enable us to derive a novel tight upper bound to the sum rate of the channel – a problem that is very difficult for general channel coefficients. To avoid this difficulty for the general IC, we next derive upper and lower bounds which are not identical but are within a constant number of bits to each other which characterizes the capacity region of the 2-user multi-input multi-output (MIMO) Gaussian interference channel (IC) with an arbitrary number of antennas at each node to within a constant gap that is independent of the signal-to-noise ratio (SNR) and all channel parameters. In contrast to an earlier result in [Telatar and Tse, ISIT, 2007], where both the achievable rate region and upper bounds to the capacity region of a general class of interference channels was specified as the union over all possible input distributions here we provide, a simple and an explicit achievable coding scheme for the achievable region and an explicit outer bound. We also illustrate an interesting connection of the simple achievable coding scheme to MMSE estimators at the receivers. A reciprocity result is also proved which is that the capacity of the reciprocal MIMO IC is within a constant gap of the capacity region of the forward MIMO IC.

We also analyze the channel's performance in the high SNR regime, which is obtained from

the explicit expressions of the approximate capacity region and the resulting asymptotic rate region is known as the generalized degrees of freedom (GDoF) region. A close examination of the super position coding scheme which is both GDoF and approximate capacity optimal reveals that joint signal-space and signal-level interference alignment is necessary to achieve the GDoF region of the channel. The admissible DoF-splits between the private and common messages of the HK scheme are also specified. A study of the GDoF region reveals various insights through the joint dependence of optimal interference management techniques (at high SNR) on the SNR exponents and the numbers of antennas at the four terminals. For instance, it reveals that, unlike in the scalar IC, treating interference as noise is not always GDoF-optimal even in the very weak interference regime. Moreover, while the DoF-optimal strategy that relies just on transmit/receive zero-forcing beamforming and time-sharing is not GDoF optimal (and thus has an unbounded gap to capacity), the precise characterization of the very strong interference regime where single-user DoF performance can be achieved simultaneously for both users depends on the relative numbers of antennas at the four terminals and thus deviates from what it is in the SISO case. For asymmetric numbers of antennas at the four nodes the shape of the symmetric GDoF curve can be a distorted W curve to the extent that for certain MIMO ICs it is a V curve.

In the second part of the thesis, we concentrate on time varying fading channels. We first characterize the fundamental diversity-multiplexing tradeoff (DMT) of the quasi-static fading MIMO Z interference channel (ZIC) with channel state information at the transmitters (CSIT) and arbitrary number of antennas at each node. A short-term average power constraint is assumed. It is shown that a variant of the superposition coding scheme described above, where the 2nd transmitter's signal depends on the channel matrix to the first receiver and the 1st user's transmit signal is independent of CSIT, can achieve the full CSIT DMT of the ZIC. We also characterize the achievable DMT of a transmission scheme, which does not utilize any CSIT and show that for some range of multiplexing gains, the full CSIT DMT of the ZIC can be achieved by it. The size of this range of multiplexing gains depends on the system parameters such as the number of antennas at the four nodes (referred to hereafter as "antenna configuration"), signal-to-noise ratios (SNR) and interference-to-noise ratio (INR) of the direct links and cross link, respectively. Interestingly, for certain special cases such as when the interfered receiver has a relatively larger number of antennas than that at the other nodes or when the INR is stronger than the SNRs, the No-CSIT scheme can achieve the F-CSIT DMT for all multiplexing gains. Thus, under these circumstances, the optimal DMT of the MIMO ZIC with F-CSIT is same as the DMT of the corresponding ZIC with No-CSIT. For other channel configurations, the DMT achievable by the No-CSIT scheme serves as a lower bound to the fundamental No-CSIT DMT of the MIMO ZIC.

We also characterize the fundamental diversity-multiplexing tradeoff of the three-node, multi-input, multi-output (MIMO), quasi-static, Rayleigh faded, half-duplex relay channel for an arbitrary number of antennas at each node and in which opportunistic scheduling (or dynamic operation) of the relay is allowed, i.e., the relay can switch between receive and transmit modes at a channel dependent time. In this most general case, the diversity-multiplexing tradeoff is characterized as a solution to a simple, two-variable optimization problem. This problem is then solved in closed form for special classes of channels defined by certain restrictions on the numbers of antennas at the three nodes. The key mathematical tool developed here that enables the explicit characterization of the diversity-multiplexing tradeoff is the joint eigenvalue distribution of three mutually correlated random Wishart matrices. Besides being relevant here, this distribution result is interesting in its own right. Previously, without actually characterizing the diversity-multiplexing tradeoff, the optimality in this tradeoff metric of the dynamic compress-and-forward (DCF) protocol based on the classical compress-and-forward scheme of Cover and El Gamal was shown by Yuksel and Erkip. However, this scheme requires global channel state information (CSI) at the relay. In this work, the so-called quantize-map and forward (QMF) coding scheme is adopted as the achievability scheme with the added benefit that it achieves optimal tradeoff with only the knowledge of the (channel dependent) switching time at the relay node. Moreover, in special classes of the MIMO half-duplex relay channel, the optimal tradeoff is shown to be attainable even without this knowledge. Such a result was previously known only for the half-duplex relay channel with a single antenna at each node, also via the QMF scheme. More generally, the explicit characterization of the tradeoff curve in this work enables the in-depth comparisons herein of full-duplex versus half-duplex relaying as well as static versus dynamic relaying, both as a function of the numbers of antennas at the three nodes.

Dedication

To my parents for their support throughout the ups and downs.

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I am indebted to Prof. B. Sundar Rajan for introducing me to this wonderful world of wireless communication during my master's in IISc. I would like to thank my PhD thesis committee members Prof. B. C. Rider, F. Somenzi, P. Mathys, and Y. Liu. The subjects taught by Prof. Y. Liu and C. T. Mullis have greatly improved my understanding of Information theory and linear systems. Finally, I would like to acknowledge the contributions of my lab-mates and friends whose support in different levels gave me the energy to keep going. I wish to express my heartfelt thanks to - in alphabetical order - Angshuman, Arnab, Atashi, Chinmay, Deba, Kaniska, Palash, Prasanna and Rajesh.

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Chapter 1

Introduction to Multi-antenna, Multiuser Wireless communications

In this thesis we characterize the fundamental information theoretic performance limits[1] of two basic building blocks of multi-user wireless networks, namely the 2-user *interference channel* and the *relay channel*. The interference channel refers to a communication scenario where one or more transmitters try to send some information through a common channel to their corresponding receiver/s and in the process interfere with each other. As in a party, several conversations take place in the same room, where each conversation is affected by the noise produced by other conversations. As a practical example, consider the two adjacent cells of the cellular network depicted in figure 1.1, where user U_1 (U_2) has a message for BS_1 (BS_2) only. However, since both the communications

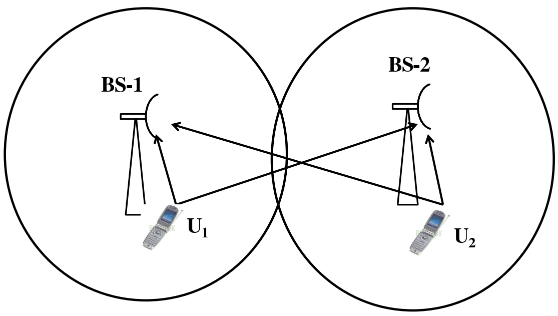


Figure 1.1: The 2-user interference channel in a cellular system.

take place through the same medium the transmitted signal from U_1, which is not desired at BS_2

[1] The various metrics of performance will be described shortly.

is also received by it, and interfere with its actual desired signal coming from U_2. The same is true for BS_1. The fundamental problem at hand for this channel is to characterize the best performance for both the U_1 to BS_1 and U_2 to BS_2 links in the face of interference as described above. The first appearance of this channel configuration and the problem can be traced back to a paper by C. E. Shannon in 1961 [1] and since then it had been an open research problem over more than 4 decades.

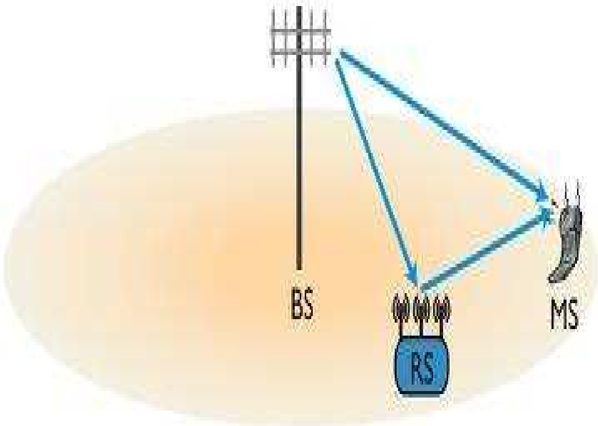


Figure 1.2: The 2-user interference channel in a cellular system.

The second communication environment that is analyzed in this thesis is the relay channel, which refers to a configuration where the communication between a source and destination is facilitated by a set of intermediate nodes - called the relay nodes or simply relays. A practical network where relay nodes are used for performance enhancement is depicted in figure 1.2. Note that in addition to the direct signal received by the mobile station (MS) from the base station (BS), it also receives an additional signal from the helping relay station (RS). The advantage of this set up can be two fold: 1) it can enhance the reliability of received signal at the mobile by providing an independent copy of the signal transmitted by the base station; and 2) it can increase the rate of information transfer from the BS to MS as well. The objective for this communication scenario is to characterize the maximum limit of such performance improvements.

The performance of these communication channels are measured in terms of the maximum amount of **information** that can be transferred *reliably* from the source/s to the receiver/s. Different measures to quantify the information and the degree of reliability can be defined and used to characterize the performance of a communication channel. For a systematic approach to such an analysis in what follows, we first describe the mathematical model which represents the physical communication scenario and then define the various metrics of performance formally.

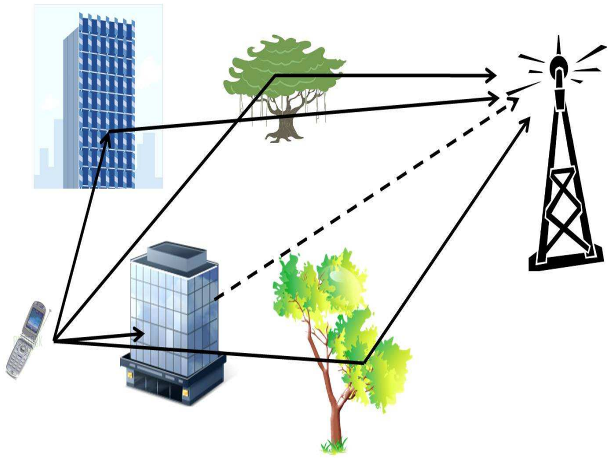


Figure 1.3: Signal propagation model for a point-to-point wireless communication channel.

Let us first consider a point-to-point wireless communication link such as the one from a mobile user to its base station, as shown in figure 1.3. The term wireless refers to the fact that signals on such a channel are transmitted from the source to the destination through a wireless or unguided medium [2]. Since the signal is not guided to follow any particular path, it can be reflected, scattered or refracted by different objects in between the source and destination node. As shown in figure 1.3, the signal can be reflected from or pass through (causing refraction) a building or get scattered from a tree. The receiver obtains several copies of the transmitted signal, each with a different phase and attenuation which are dependent on the path it follows. The accumulative effects of such multi-path propagation through a wireless channel on the received signal is known as the "fading" of the signal. An exact characterization of all these phase shifts and attenuation of the signals coming through different paths to the destination node is impossible. Instead, for the design and analysis of a practical communication systems a probabilistic model is used, where the received signal at sampling time m can be written as[2]

$$y[m] = g[m]x[m] + z[m], \ m \in \mathbb{N}. \tag{1.1}$$

$z[m]$ in the above equation represents the additive noise, mainly consists of the thermal noise of the receiving equipment, and assumed to be distributed independently across m and identically as a complex Gaussian random variable with unit variance and zero mean, i.e., $z[m] \sim \mathcal{CN}(0,1)$, i.i.d. across m. The fading nature of the wireless channel is captured by $g[m]$, which is also conventionally assumed to be distributed as $\mathcal{CN}(0,1)$. This assumption for the fading coefficient is popularly known as the Rayleigh fading assumption, because $|g|^2$ is distributed as a Rayleigh random variable. Both $g[m]$ and $z[m]$ are assumed to be independent of the input $x[m] \in \mathbb{C}$ for all m. Depending on how $g[m]$ changes with m the fading channel can be divided into different classes. When the fading coefficients changes with time index m we call it a time varying channel which will be discussed in more details in section 1.1, but in what follows we shall assume that $g[m] = g$, i.e. a time invariant channel. As mentioned earlier, the performance of a communication channel such as the one in (1.1) is characterized in terms of the amount of **information** that can be transferred from the source to the destination.

Information is a mathematical measure of the uncertainity of a random variable; the information content of a deterministic variable or constant is zero. For example, the outcome of an unbiased coin toss contains information, but the outcome of 2-headed coin toss does not give us any information. In general, any type of information can be represented as a set of indexes. For example, to transmit a voice signal, which is a continuous waveform, it is first sampled to obtain a discrete time signal and then each of these samples are quantized into say, k number of predetermined quantization levels. Then, it is is possible to send each of these samples by sending the quantization index to the receiver. Once these indexes are received, the receiver can reconstruct the samples and from them the continuous voice waveform, with some quantization error. The

[2] Note that equation (1.1) represents a sampled, baseband model of the actual communication channel. The destination node actually receives a continuous time signal, at a very high radio-frequency (RF). The received signal is then first brought down to baseband through frequency translation and then sampled to obtain the discrete-time model of equation (1.1).

quantization error can be decreased with increasing the number of quantization levels. These set of indexes can be thought as the set of possible values of a random variable, where probability of a sample point is equal to its probability of occurrence. The objective of a communication system is to recover the exact value of x sent by the transmitter. However, due to the presence of random noise the recovered signal at the destination can be different from what was actually sent, leading to a decoding error. For example, consider a random variable $\mathbf{x}$ which can take its value from $\{x_1, x_2, \cdots, x_k\}$ for some k, and the receiver adopts a decoding technique where on receiving $y[m]$ it decides x_i was sent if

$$i = \arg \min_{1 \leq l \leq k} \|y[m] - g[m]x_l[m]\|^2.$$

Then, there is a non-zero probability that the receiver will make the decision in favor of x_i when $x_{j \neq i}$ was actually sent (e.g., this happens when $z[m]$ is such that $\|z[m]\|^2 > \|z[m] + g[m]x_j[m] - g[m]x_i[m]\|^2$).

One of the most important measures of performance of a communication channel is the average information per use of the channel that can be transferred from the source to the destination node, with the probability of error going to zero. This rate is called an achievable rate on the channel and the supremum of all such rates is called the **Capacity** of the channel. Typically, this capacity is characterized by deriving upper and lower bounds to it and showing that they are identical. In the case of a single transmitter and single receiver or point-to-point communication channel, the **mutual information** between $\mathbf{x}$ and $\mathbf{y}$, which is defined as

$$I(\mathbf{x}; \mathbf{y}) = \int_{S_{xy}} P_{\mathbf{xy}}(x, y) \log \left(\frac{P_{\mathbf{xy}}(x, y)}{P_{\mathbf{x}}(x) P_{\mathbf{y}}(y)} \right) dxdy,$$

where $P_{\mathbf{xy}}(x, y)$ is the joint probability density function (pdf) of $\mathbf{x}, \mathbf{y}$ and S_{xy} is the support of the pdf, provides a lower bound to the capacity. That is, it can be proved that [3] any rate R that satisfy

$$R < \max_{P(\mathbf{x})} I(\mathbf{x}; \mathbf{y}),$$

can be achieved on the channel, where $P(\mathbf{x})$ represents the pdf of $\mathbf{x}$. On the other hand, an upper bound to the capacity is provided by the **Fano's inequality**, which essentially states that, if $\mathbf{x}^n$

and $\mathbf{y}^n$ are the inputs and outputs of n uses of a communication channel and R is the average rate of information per channel use with arbitrarily small probability of error, then

$$R \leq \frac{1}{n} I(\mathbf{x}^n; \mathbf{y}^n) + \epsilon_n, \tag{1.2}$$

where $\epsilon_n \to 0$ with $n \to \infty$. Now, using various properties[3] of the mutual information [3] it can be shown that the capacity of the channel is given as

$$C = \max_{P(\mathbf{x})} I(\mathbf{x}; g\mathbf{x} + \mathbf{z}). \tag{1.3}$$

In addition, if the channel input sequence is subjected to the following power constraint

$$\frac{1}{n} \sum_{i=1}^{n} |x[i]|^2 \leq \sigma_x^2,$$

then using the fact that Gaussian distribution maximizes the mutual information $I(\mathbf{x}; g\mathbf{x} + \mathbf{z})$ if the additive noise $\mathbf{z}$ is Gaussian [4], the right hand side of (1.3) can be evaluated as

$$C_{Gaussian} = \log(1 + |g|^2 \sigma_x^2).$$

As stated earlier, the capacity represents the maximum rate of information transfer in the sense that it is not possible to transfer information reliably at a larger rate than the capacity. However, in [5] it was proved that the capacity of this channel can be drastically improved by employing multiple antennas at both the transmitting and receiving node. The input output equation of the channel, with M antennas at the transmitter and N antennas at the receiver, can be written as

$$Y[m] = H[m]X[m] + Z[m], \text{ for } m \in \mathbb{N}, \tag{1.4}$$

where $X[m] \in \mathbb{C}^M$, $Z[m]$ is distributed i.i.d. as $\mathcal{CN}(0, I_N)$ for different m, the entries of $H[m] \in \mathbb{C}^{N \times M}$ are distributed i.i.d. as $\mathcal{CN}(0,1)$ and $H[m] = H$ for all m. Again just like the SISO case, an

[3] Let $\mathbf{x}^n = \{x[1], \cdots, x[n]\}$ and $\mathbf{y}^n = \{y[1], \cdots, y[n]\}$ are two n-length sequence of random variables, then

$$I(\mathbf{x}; \mathbf{y}) = h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y}), \text{ where } h(\mathbf{x}) \triangleq -\int_{S_x} P_{\mathbf{x}}(x) \log\left(P_{\mathbf{x}}(x)\right) dx;$$

$$h(\mathbf{x}|\mathbf{y}) \triangleq -\int_{S_{xy}} P_{\mathbf{xy}}(x,y) \log\left(P_{\mathbf{x}|\mathbf{y}}(x|y)\right) dxdy; \text{ and } h(\mathbf{x}^n) = \sum_{i=1}^{n} h(\mathbf{x}[i]|\mathbf{x}^{(i-1)}) \leq \sum_{i=1}^{n} h(\mathbf{x}[i]),$$

with equality in the last step if $\mathbf{x}[i]$'s are mutually independent. Also, $h(z) = \log(2\pi e \sigma^2)$, if $z \sim \mathcal{CN}(0, \sigma^2)$.

upper bound and a lower bound to the capacity of this channel can be computed using the Fano's inequality and optimizing the mutual information $I(X;Y)$, between the input vector $X \in \mathbb{C}^M$ and output vector $Y \in \mathbb{C}^N$, respectively. And using various properties of the mutual information it can be shown that the capacity of the channel is given by

$$C_{MIMO} = \max_{\{0 \preceq Q\}} \log\det\left(I_N + HQH^\dagger\right).$$

So far we have considered only point-to-point communication channels where there is a single transmitter and a single receiver only and is a simple communication channel in the sense that the received signal at the destination is corrupted only by the additive noise. In contrast to this, in a practical communication system there are multiple transmitters communicating to their respective receivers (multiple receivers) in the presence of interference from other undesired users. For example, consider the interference channel depicted in figure 1.1. If the input and output of the i-th cell are denoted by X_i and Y_i and the channel matrix from U_i to BS_j is denoted by H_{ij} then the input output relation for a single channel use can be written as

$$Y_1 = H_{11}X_1 + H_{21}X_2 + Z_1; \tag{1.5}$$

$$Y_2 = H_{12}X_1 + H_{22}X_2 + Z_2, \tag{1.6}$$

where Z_i represents the additive noise at BS_i. However, unlike the point-to-point channels described above due to the presence of undesired interfering signals from other user the characterization of the capacity of this channel is a relatively hard problem. In fact, despite more than 4 decades of research effort since its first mention in [1], the capacity region[4] of the 2-user interference channel (IC) had been an open research problem.

The capacity region is known for only a few special classes of interference channels characterized by some constraints on the channel matrices. Our contribution towards solving this problem is summarized next. In chapter 2 we characterize the capacity region of another class of interference

[4] If the users can communicate messages to their respective destinations at rates R_1 and R_2 with probability of error going to zero, then (R_1, R_2) is an achievable rate tuple. The set of all achievable rate tuples is called the achievable rate region and its closure is called the capacity region of the channel.

channel which we call the *strong in partial order* IC. In obtaining this result we prove a new outer bound to the difference of two specific mutual information terms. This novel outer bound in turn results in a new tight upper bound to the capacity region of the MIMO IC which for the class of *strong in partial order* ICs coincides with an achievable rate region and therefore, characterizes the capacity region. The application of the new outer bound need not be restricted to the 2-user MIMO IC and can be used wherever such a difference between two mutual information arises. For instance, such a difference term appears in the analysis of a 2-user MIMO broadcast channel (BC). Finding such tight upper bounds to the capacity region of the MIMO IC without any restriction on the channel matrices is hard.

In chapter 3 however, we adopt an alternative approach and solve the general problem although with respect to an approximate performance metric. In this alternative approach, we first derive a set of bounds that define a super set, $\mathcal{R}^u$, containing the capacity region and then derive an achievable rate region, $\mathcal{R}_a$ contained in the capacity region of the 2-user MIMO IC. It is then shown that, if $(R_1, R_2) \in \mathcal{R}^u$ then there exists a rate tuple $(\hat{R}_1, \hat{R}_2) \in \mathcal{R}_a$ such that $\hat{R}_i \geq (R_i - n_i)$ for $i = 1, 2$. Since the capacity region is contained within $\mathcal{R}^u$, this result proves that the achievable region is within n_i bit to the capacity region[5] . We also provide a simple and an explicit coding scheme which can achieve a rate region which is within a constant number of bits to the capacity region of the MIMO IC. It will be shown that to achieve any rate tuple (R_1, R_2) in the achievable region, each user needs to divide its message into 2 sub-messages called the **private** message, which is to be decoded only at its own receiver and the **public** message, which is to be decoded by both the receivers. We also specify the explicit rate splits among the private and public messages of each user. In particular, for the special case of SISO IC the result of this work complements the result reported in [6], where the explicit rate splits were not specified.

In Chapter 4, through high SNR approximations of the set of the aforementioned within-constant-gap upper and lower bounds to the capacity region of the MIMO IC, various insights

[5] Such a characterization of the capacity region of a communication channel within a constant number of bits is also widely referred to as the approximate capacity of the channel and the corresponding constant is known as the gap of approximation.

about the channel are revealed. For instance, it was found in [6] that treating interference from the undesired user as noise preserves the high SNR scaling of the optimal rates of a SISO IC when the cross-link gains of the channel are relatively small. However, this does not hold true on an IC with multiple antennas at different nodes.

While all of the aforementioned analysis is carried out assuming fixed or time-invariant channel matrices, in this thesis we also consider time varying channels. In contrast to the Capacity or approximate capacity metrics used for the previous analysis, on a time-varying channel we shall use a different performance metric known as the **diversity-multiplexing tradeoff** (DMT), which essentially characterizes the tradeoff that exist between the rate of information transfer and the reliability of reception on a communication channel. Using the DMT as performance metric we characterize the performance of the second class of communication channel known as the relay channel (e.g., see figure 1.2). In what follows, we shall give a brief overview of the DMT framework in the context of a SISO point-to-point channel.

1.1 Time-varying channel

Recall the input-output relation of the point-to-point channel given in equation (1.1), i.e.,

$$y[m] = g[m]x[m] + z[m], \text{ for } m \in \mathbb{N}, \tag{1.7}$$

where $g[m]$ represents the channel matrix at the sampling time m. Depending on how $g[m]$ varies with m, the fading channel can be divided into three different classes: 1) Fast fading channel, where $g[m]$ takes a new value for each m which are i.i.d. as $\mathcal{CN}(0,1)$; 2) Block fading or slow fading channel, where $g[m] = g_l$, for some $L > 1$, $(l-1)L \leq m \leq Ll$ and $l = 1, 2\cdots$; and 3) static channel, where $g[m] = g$ for all m.

On a slow or block fading channel, the capacity of the channel which is given by

$$C = \log(1 + |g_l|^2\sigma_x^2)$$

is a function of the instantaneous channel gain g_l, which changes i.i.d. at the beginning of each block, and therefore is a random quantity. For any given rate R, there is a probability that

$$\log(1 + |g_l|^2\sigma_x^2) < R,$$

i.e., the rate is not supportable on the channel. Such an event is widely known as the outage event. A more natural measure of performance on a slow fading channel is the outage probability, P_{out} defined as

$$P_{out} = \Pr\left(\log(1 + |g_l|^2\sigma_x^2) < R\right).$$

A high SNR analysis of this quantity yields the tradeoff between the rate and reliability of communication, i.e., the DMT of the channel. If the rate of information varies as $R = r\log(\rho)$ with ρ denoting the SNR, then the DMT analysis provides the the optimal diversity order $d(r)$, where the average probability of error decays as $\rho^{-d(r)}$.

Using DMT as a metric in Chapter 5 we characterize the performance of variation of the 2-user MIMO IC where only one of the receiver, say the 1-st receiver, is interfered and the other receiver is not affected by any interference coming from the other user (e.g., see figure 5.1). This particular variation of the 2-user MIMO IC is more popularly known as the Z-interference channel (ZIC). There are practical communication networks, such as the Femto cells, which can be modeled as a ZIC. The main technical barrier that was blocking the DMT characterization of the MIMO ZIC, before our work on this topic, was the lack of an explicit expression for the instantaneous capacity region of the channel. Note that in the 2-user IC of figure 2.1 if we substitute $H_{12} = 0$ we obtain a ZIC. Therefore, the required explicit expressions for the capacity region of the channel can be obtained from our earlier results on 2-user IC by substituting $H_{12} = 0$. Using this result however, only enables us to derive an explicit expression of the outage probability. Analyzing this outage probability further and characterizing its negative SNR exponent which is the DMT of the channel requires the joint eigen-value distribution of three mutually correlated random Wishart [7] matrices. Computing this joint distribution is another important step to solve this problem and a novel contribution of this thesis.

In Chapter 6, we characterize the DMT of the 3-node relay channel, which is the basic building block of a general class of wireless networks known as the cooperative networks. Among the most important steps in the characterization of DMT of a communication channel is to find an expression for the instantaneous capacity of the channel. However, like all the previous channel considered in this thesis the exact capacity of the relay channel is still an open problem. The way out of this situation is to find an approximate expression for the capacity which is still sufficient to characterize the DMT. We first find such an expression which is close to the instantaneous capacity for the relay channel and using it find the corresponding outage probability. Then, utilizing the joint eigeng-value distribution result mentioned earlier we characterize the negative SNR exponent of the outage probability of the channel which also represents the optimal diversity order of the channel.

1.2 Conclusion

4G wireless standards such as the WiMAX and 3GPP LTE recommend the usage of optimal interference management techniques and deployment of relay technology for achieving improved performance. The results and insights derived from this research is going to guide a system designer to efficiently design futuristic wireless networks. Further, the tools and techniques developed in this thesis can be used to characterize the performance of other multiuser network as well.

Chapter 2

Capacity region of a class of strong MIMO Interference Channels

2.1 Introduction

The 2-user interference channel is the simplest channel model in which multiple transmit-receive pairs communicate over a common noisy channel. This model was first mentioned in [1], and was studied in a series of works in [8–14] that considered certain special classes of the IC where the capacity regions of the so-called **very strong** IC, the **strong** IC and certain classes of **degraded** and **deterministic** ICs, respectively, were established. Different sets of inner and outer bounds considering the embedded multiple-access and broadcast and Z channels were derived in [12,15–19]. In spite of over 4 decades of research, the capacity region in the general case remained unsolved.

Recent results include the capacity regions of new and/or more general classes of channels than for which capacity was previously known, e.g., the sum capacity of the so-called *noisy* interference channels was found in [20–22]. The common feature of this line of work is that it focuses on a subset of channel parameters but seeks to solve the challenging problem of obtaining the exact capacity of the SISO channel.

Since most modern wireless communication systems feature multiple antennas at some or all terminals it is of interest to study the 2-user Gaussian MIMO IC. As compared to the result on the capacity of the strong SISO IC, the capacity of the MIMO IC is known [23] only for the so-called *aligned strong* interference regime, where the direct and cross link channel matrices satisfy a matrix equation. On an *aligned-strong* IC, the direct link's channel matrix is a matrix multiple of the cross link's channel, where the multiple satisfies some particular constraint. In general, the problem of characterizing the exact capacity of a MIMO IC even for small and special classes can be challenging; this point is also illustrated by [24] where the capacity region of a class of *very strong* SIMO ICs was characterized.

In this chapter, we characterize the capacity region of a class of 2-user MIMO Gaussian ICs which although have an overlap with the class of *aligned-strong* ICs characterized in [23], contains channels which are not *aligned-strong* and therefore enlarges the set of channels for which the exact capacity of 2-user MIMO IC can be characterized. To prove this result we first derive a new upper bound to the difference of two mutual information. Using this inequality we then prove that the capacity region of the *strong in partial order* ICs can be achieved by independent and Gaussian coding at each transmitter.

Notations 2.1 Let $\mathbb{C}$ and $\mathbb{R}^+$ represent the field of complex numbers and the set of non-negative real numbers, respectively. An $n \times m$ matrix with entries coming from $\mathbb{C}$ will be denoted by $A \in \mathbb{C}^{n \times m}$. We shall denote the conjugate transpose of the matrix A by $A^\dagger$ and determinant of it by $|A|$. I_n represents the $n \times n$ identity matrix, $\mathbf{0}_{n \times m}$ represents an all zero $n \times m$ matrix and $\mathbb{U}^{n \times n}$ represents the set of $n \times n$ unitary matrices. The fact that $(A - B)$ is a positive semi-definite (p.s.d.) (or positive definite (p.d.)) matrix is denoted by $A \succeq B$ (or $A \succ B$). $A \otimes B$ denotes the tensor product of the two matrices. If $x_t \in \mathbb{C}^{m \times 1}, \forall 1 \leq t \leq n$, then $x^n \triangleq [x_1^\dagger, \cdots, x_n^\dagger]^\dagger$. $I(x; y), I(x; y|z), h(x)$ and $h(x|y)$ represents the mutual information, conditional mutual information, differential entropy and conditional differential entropy of the arguments respectively.

2.2 Channel Model and Mathematical preliminaries

A 2-user IC as shown in Figure 2.1, where user i (Tx_i) has M_i antennas and receiver i (Rx_i) has N_i antennas, respectively for $i = 1, 2$, is considered. Such a MIMO IC will be referred to as a (M_1, N_1, M_2, N_2) MIMO IC, in the sequel. For characterizing the exact capacity region of a special class of these interference channels in this chapter we shall assume that $H_{ij} \in \mathbb{C}^{N_j \times M_i}$ models the channel matrix between Tx_i and Rx_j. For ease of analysis, we also assume that all the channel matrices are full rank with probability one however. For economy of notations hereafter these channel matrices will be denoted as $\mathcal{H}$, i.e., $\mathcal{H} = \{H_{11},\ H_{12},\ H_{21},\ H_{22}\}$. In this chapter, we shall consider a time-invariant or fixed channel where the channel matrices remain fixed for the entire duration of the communication. At time t, Tx_i chose a vector $X_{it} \in \mathbb{C}^{M_i \times 1}$ and sends $\sqrt{P_i} X_{it}$ into the channel, where for the input signals we assume the following power constraint:

$$\sum_{t=1}^{n} \frac{1}{n} \mathbb{E}\left(X_{it} X_{it}^{\dagger} \right) \preceq K_{x_i}, \forall\ i = 1, 2. \tag{2.1}$$

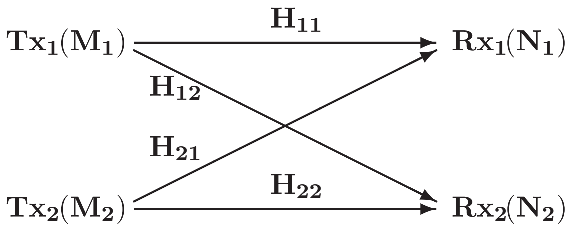


Figure 2.1: The (M_1, N_1, M_2, N_2) MIMO IC.

The received signals at time t can be written as

$$Y_{1t} = H_{11} X_{1t} + H_{21} X_{2t} + Z_{1t};\ Y_{1t} \in \mathbb{C}^{N_1 \times 1},$$

$$Y_{2t} = H_{22} X_{2t} + H_{12} X_{1t} + Z_{2t};\ Y_{2t} \in \mathbb{C}^{N_2 \times 1},$$

where $Z_{it} \in \mathbb{C}^{N_i \times 1}$ are i.i.d as $\mathcal{CN}(\mathbf{0}, I_{N_i})$ across i and t. In the sequel, an IC with channel matrices as in figure 2.1 will be denoted by $\mathcal{IC}\left([H_{11}, H_{12}, H_{21}, H_{22}]\right)$ or $\mathcal{IC}\left(\mathcal{H}\right)$.

Definition 2.1 (Strong IC in partial order) A 2-user (M_1, N_1, M_2, N_2) IC as shown in figure 2.1 is called a *strong in partial order* IC if the channel matrices satisfies the following constraints:

$$H_{ii}^{\dagger} H_{ii} \preceq H_{ij}^{\dagger} H_{ij},\ 1 \leq i \neq j \leq 2. \tag{2.2}$$

In the following section we shall first derive some information theoretic inequalities which will be used in the later part of the chapter to derive the main result, i.e., the capacity region of the 2-user *strong in partial order* IC.

2.3 An upper bound to the difference of two mutual informations

We derive a novel tight upper bound to the difference of two mutual information terms that appears in the process of upper bounding various rates of the 2-user MIMO IC. However, before delving into the derivation let us see how this term appears in the upper bounding process. For any achievable rate tuple (R_1, R_2) on the 2-user IC, using the Fano's inequality [3] we can find the following upper bound.

$$
\begin{aligned}
n(R_1+R_2) &\leq I(X_1^n;Y_1^n) + I(X_2^n;Y_2^n) + n\epsilon_n, \\
&\overset{(a)}{\leq} I(X_1^n;Y_1^n|X_2^n) + I(X_2^n;Y_2^n) + n\epsilon_n, \\
&= I(X_1^n;Y_1^n|X_2^n) + I(X_1^n,X_2^n;Y_2^n) - I(X_1^n;Y_2^n|X_2^n) + n\epsilon_n, \\
&= I(X_1^n,X_2^n;Y_2^n) + \underbrace{\{I(X_1^n;Y_1^n|X_2^n) - I(X_1^n;Y_2^n|X_2^n)\}}_{\mathcal{D}} + n\epsilon_n,
\end{aligned}
\tag{2.3}
$$

where $\epsilon_n \to 0$ as $n \to \infty$, step (a) follows from the fact that additional information at Rx_1 in the form of X_2^n does not reduce the mutual information and the subsequent equality follows from standard information theoretic inequalities. Note that equation (2.3) in the absence of the term $\mathcal{D}$ on the right hand side represents the sum-rate bound of a multiple access channel (MAC) formed by Tx_1, Tx_2 and Rx_2. Therefore, if $\mathcal{D} \leq 0$ it is possible to replace it by 0 and still obtain an upper bound which coincides with the MAC achievable rate region sum bound. This in turn implies that the rate region achievable by MAC optimal coding scheme is the capacity region of the channel. In what follows, through a set of lemmas and theorems we shall prove that on a *strong in partial order* IC, i.e., an IC in which the channel matrices satisfy equation (2.2), $\mathcal{D} \leq 0$ indeed. In [23], the capacity region of a class of aligned-strong IC was found using a similar method. Although the class of channels of this chapter has some overlap with the class of *aligned-strong* ICs, it contains channels which are not *aligned-strong* (e.g., see remark 2.3).

In Lemma 2.1 and Theorem 2.2 we derive the single letter version of the desired inequality for square invertible and rectangular channel matrices, respectively, which subsequently yields the desired upper bound to $\mathcal{D}$ and is stated in Corollary 2.1.

2.3.1 An upper bound to single-letter $\mathcal{D}$: The square and invertible case

Lemma 2.1 Let $G_1, G_2 \in \mathbb{C}^{M\times M}$ are two full-rank, complex matrices and $X_1 \in \mathbb{C}^{M\times 1}$ is a random vector with arbitrary distribution, where $\mathrm{Cov}(X_1) \preceq Q$ and $\tilde{Z}_1$ and $\tilde{Z}_2$ are i.i.d. as $\mathcal{CN}(0, I_M)$ which are also independent of X_1. If the matrices G_1 and G_2 satisfy the following condition

$$G_1^\dagger G_1 \preceq G_2^\dagger G_2. \tag{2.4}$$

then

$$I\left(X_1; G_1X_1 + \tilde{Z}_1\right) - I\left(X_1; G_2X_1 + \tilde{Z}_2\right) \leq 0.$$

Proof 2.1 (Proof of Lemma 2.1) The proof is based on an extremal inequality proved in [25] which is stated below for the reader's convenience:

Theorem 2.1 (Theorem 1 of [25]) Let Z_1 and Z_2 be two a Gaussian random n-vector with a positive definite covariance matrices K_1 and K_2 respectively and X is a random vector with arbitrary distribution, then the optimization problem, where μ is any real number,

$$\max_{\mathrm{Cov}(X)\preceq S} h(X + Z_1) - \mu h(X + Z_2)$$

is maximized by Gaussian input X.

Denoting the left hand side of the desired inequality by Γ we get the following set of inequalities

$$\begin{aligned}
\Gamma =& I\left(X_1; G_1X_1 + \tilde{Z}_1\right) - I\left(X_1; G_2X_1 + \tilde{Z}_2\right), && (2.5)\\
=& h\left(G_1X_1 + \tilde{Z}_1\right) - h\left(G_2X_1 + \tilde{Z}_2\right), && (2.6)\\
=& h\left(X_1 + G_1^{-1}\tilde{Z}_1\right) - h\left(X_1 + G_2^{-1}\tilde{Z}_2\right) + \tau, && (2.7)\\
\stackrel{(a)}{\leq}& \max_{\mathrm{Cov}(X_1)\preceq Q} h\left(X_1 + \hat{Z}_1\right) - h\left(X_1 + \hat{Z}_2\right) + \tau, && (2.8)
\end{aligned}$$

where $\tau = \log\left(G_1^\dagger G_1\right) - \log\left(G_2^\dagger G_2\right)$, in step (a) $\hat{Z}_i \sim \mathcal{CN}(0, (G_i^\dagger G_i)^{-1})$, for $i = 1, 2$. Evidently, the last equation is in a form addressed by Theorem 2.1 above and hence is maximized by Gaussian input. Let us further assume that the expression on the right hand side of equation (2.8) is maximized by an input with covariance matrix K_x^*. Under this assumption equation (2.8) takes the following form

$$\begin{aligned}
\Gamma \leq & \log\det\left(K_x^* + (G_1^\dagger G_1)^{-1}\right) - \log\det\left(K_x^* + (G_2^\dagger G_2)^{-1}\right) + \tau, && (2.9)\\
= & \log\det\left(G_1 K_x^* G_1^\dagger + I\right) - \log\det\left(G_2 K_x^* G_2^\dagger + I\right), && (2.10)\\
= & \log\det\left(G_1^\dagger G_1 K_x^* + I\right) - \log\det\left(G_2^\dagger G_2 K_x^* + I\right), && (2.11)\\
= & \log\det\left((K_x^*)^{\frac{1}{2}} G_1^\dagger G_1 (K_x^*)^{\frac{1}{2}} + I\right) - \log\det\left((K_x^*)^{\frac{1}{2}} G_2^\dagger G_2 (K_x^*)^{\frac{1}{2}} + I\right), && (2.12)\\
\leq & 0, && (2.13)
\end{aligned}$$

where the last step follows from the fact that $\log\det(.)$ is a monotonic function in the cone of positive semi-definite matrices and the fact that

$$(K_x^*)^{\frac{1}{2}} G_1^\dagger G_1 (K_x^*)^{\frac{1}{2}} \preceq (K_x^*)^{\frac{1}{2}} G_2^\dagger G_2 (K_x^*)^{\frac{1}{2}} \tag{2.14}$$

which in turn follows from equation (2.4).

2.3.2 An upper bound to single-letter $\mathcal{D}$: The non-square case

Theorem 2.2 Let $H_1 \in \mathbb{C}^{N_1 \times M}$ and $H_2 \in \mathbb{C}^{N_2 \times M}$ are full-rank matrices, where $N_1 \leq N_2$, $X_1 \in \mathbb{C}^{M \times 1}$ is a random vector with arbitrary distribution, $\text{Cov}(X_1) \preceq Q$ and $\tilde{Z}_1$ and $\tilde{Z}_2$ are i.i.d. as $\mathcal{CN}(0, I_{N_i})$ which are also independent of X_1. If the matrices H_1 and H_2 satisfy the following condition

$$H_1^\dagger H_1 \preceq H_2^\dagger H_2, \tag{2.15}$$

then

$$\Gamma = I\left(X_1; H_1 X_1 + \tilde{Z}_1\right) - I\left(X_1; H_2 X_1 + \tilde{Z}_2\right) \leq 0. \tag{2.16}$$

Proof 2.2 (Proof of Theorem 2.2) The proof is given in Appendix A.1.

Remark 2.1 Each of the terms, $H_iX + \tilde{Z}_i$ in equation (2.16) can be imagined as the output of an additive Gaussian channel. Further let us assume that the source is Gaussian and has CSIT. Then, in general if constraint (2.15) is not satisfied the source can direct its information in such a manner that no useful information reaches the output of the second channel, i.e., if $X \sim \mathcal{CN}(0, K_x)$, it is possible to choose $H_1,\ H_2$ such that $H_2K_x = 0$ but $H_1K_xH_1^\dagger \succeq 0$, then we get

$$\begin{aligned} I\left(X; H_1X + \tilde{Z}_1\right) - I\left(X; H_2X + \tilde{Z}_2\right) &= h\left(H_1X + \tilde{Z}_1\right) - h\left(H_2X + \tilde{Z}_2\right), \\ &= \log\det\left(H_1K_xH_1^\dagger + I_{N_1}\right) - \log\det\left(H_2K_xH_2^\dagger + I_{N_2}\right), \\ &= \log\det\left(H_1K_xH_1^\dagger + I_{N_1}\right) > 0,\ \text{if } [H_1K_xH_1^\dagger \succ 0]. \end{aligned}$$

Therefore, in general if equation (2.15) is not satisfied then it is possible for the left hand side of equation (2.16) to be greater than zero. However, the above lemma suggests that if equation (2.15) is satisfied, then it can not happen, i.e., the source can not direct its information in such a manner that no information reaches the output of the channel having channel matrix H_2 while there is some positive information at the output of the first channel.

In deriving the converse to the capacity region however, we shall need an upper bound to the difference of entropies of n-symbol extensions of the channel which is provided by the following Corollary.

Corollary 2.1 Let $H_1 \in \mathbb{C}^{N_1 \times M}$ and $H_2 \in \mathbb{C}^{N_2 \times M}$ are full-rank matrices, where $N_1 \leq N_2$, $\{X_t \in \mathbb{C}^{M \times 1}, 1 \leq t \leq n\}$ is a sequence of random vectors with arbitrary distribution satisfying (2.1) and $\tilde{Z}_{1t}$ and $\tilde{Z}_{2t}$ are mutually independent and i.i.d. as $\mathcal{CN}(0, I_{N_i})$ for all $1 \leq t \leq n$ which are also independent of X^n. If the matrices H_1 and H_2 satisfy equation (2.15), then

$$\Gamma_n = I\left(X^n; (I_n \otimes H_1)X^n + \tilde{Z}_1^n\right) - I\left(X^n; (I_n \otimes H_2)X^n + \tilde{Z}_2^n\right) \leq 0. \tag{2.17}$$

Proof 2.3 It follows from equation (2.15) and some elementary properties of tensor products of matrices that

$$(I_n \otimes H_1)^\dagger (I_n \otimes H_1) \preceq (I_n \otimes H_2)^\dagger (I_n \otimes H_2). \tag{2.18}$$

The proof then follows from Theorem 2.2.

In the following section we shall use the result in Corollary 2.1 to derive the capacity region of the 2-user MIMO IC and the 2-user ZIC.

2.4 Capacity of the 2-user strong MIMO IC

Theorem 2.3 The capacity region, $\mathcal{C}_{\mathrm{IC}}(\mathcal{H})$, of an (M, N, M, N) *strong in partial order* IC, where the channel matrices satisfy the following condition

$$H_{ii}^{\dagger} H_{ii} \preceq H_{ij}^{\dagger} H_{ij}, \ \forall i \neq j \in \{1, 2\}, \tag{2.19}$$

is given by the set of rate tuples satisfying the following constraints:

$$R_1 \leq \log \left| \left(I_{N_1} + H_{11} K_{x_1} H_{11}^{\dagger} \right) \right|; \tag{2.20}$$

$$R_2 \leq \log \left| \left(I_{N_2} + H_{22} K_{x_2} H_{22}^{\dagger} \right) \right|; \tag{2.21}$$

$$R_1 + R_2 \leq \log \left| \left(I_{N_2} + H_{12} K_{x_1} H_{12}^{\dagger} + H_{22} K_{x_2} H_{22}^{\dagger} \right) \right|; \tag{2.22}$$

$$R_1 + R_2 \leq \log \left| \left(I_{N_1} + H_{21} K_{x_2} H_{21}^{\dagger} + H_{11} K_{x_1} H_{11}^{\dagger} \right) \right|; \tag{2.23}$$

Proof 2.4 The proof is given in appendix A.2.

Remark 2.2 Without loss of generality one can assume that the channel coefficients in a SISO IC are given as $H_{ii} = 1$ and $H_{ij} = a_{ij}$ for $1 \leq i \neq j \leq 2$. Sato in [12] found the capacity region of the 2-user strong SISO IC under the constraint that $a_{ij}^2 \geq 1$, which is identical to (2.19). For a SIMO IC, where all the channel matrices are row vectors Viswanath and Jafar [24] characterized the capacity region of the strong channel under the condition

$$H_{ii}^{\dagger} H_{ii} = \|H_{ii}\|^2 \leq \|H_{ij}\|^2, \text{ for } 1 \leq i \neq j \leq 2,$$

which again coincides with the condition in (2.19). Therefore, Theorem 2.3 can reproduce the earlier results on strong ICs as special cases.

Recently, the capacity region of another class of strong MIMO IC, called the *aligned strong* ICs was characterized in [22]. This result however is not completely subsumed by our result. A detailed discussion is provided in the next remark.

Remark 2.3 (Difference to the *aligned-strong* channels) The class of ICs for which there exist matrices A_i and B_i such that $K_{x_i}B_i^\dagger = 0$, $A_i^\dagger A_i \preceq I$ and

$$H_{ii} = A_i H_{ij} + B_i, \text{ for } 1 \leq i \neq j \leq 2, \tag{2.24}$$

where K_{x_i} is the upper bound to the average covariance to the input at Rx_i, was named *aligned-strong* IC in [23] and their capacity region was characterized. Note that if K_{x_i} is full rank then $B_i = 0$. In this scenario, the existence of A_i implies

$$H_{ii}^\dagger H_{ii} = H_{ij}^\dagger A_i^\dagger A_i H_{ij} \preceq H_{ij}^\dagger H_{ij}, \tag{2.25}$$

i.e., condition (2.15) is satisfied. So the class of *aligned-strong* ICs in this scenario of full rank K_{x_i}'s forms a subset to the set of *strong-in-partial-order* ICs.

Moreover, the existence of such an A_i can be guaranteed only if H_{ij} is left invertible or has full column-rank. Because in that case A_i can be chosen as

$$A_i = H_{ii}\left(H_{ij}^\dagger H_{ij}\right)^{-1} H_{ij}^\dagger. \tag{2.26}$$

Therefore, whenever $H_{ij} \in \mathbb{C}^{N_j \times M}$ is full rank but $N_j < M_i$ there is no guarantee that such an A_i exists. However, such channels can still satisfy the *strong-in-partial order* condition, i.e., equation (2.15) and therefore the capacity of such channels can be characterized using the results of this chapter. Pictorially, the relations between the sets of channels which are *aligned-strong* and *strong-in-partial-order* can be depicted as shown in Figure 2.2

In Theorem 2.3 we have restricted the number of antennas to be same at both the transmitters and receivers. In what follows we shall lift this restriction in the receiver side and characterize the capacity region of a MIMO Z-IC.

2.4.1 Capacity region of a 2-user ZIC

In this subsection we derive the capacity region of a Z-interference channel (ZIC) with the first receiver being free of interference.

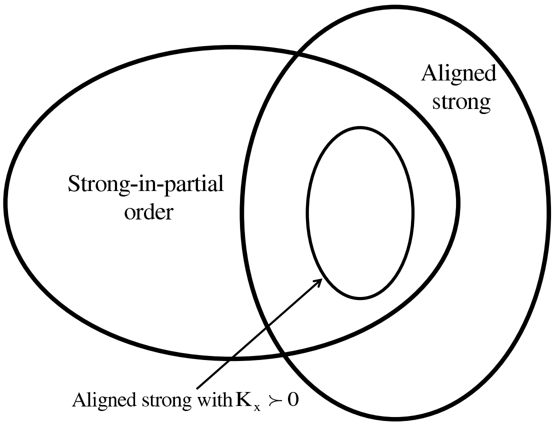


Figure 2.2: Comparison between the aligned-strong and strong-in-partial order ICs .

Theorem 2.4 The capacity region, $\mathcal{C}_{\text{ZIC}}(\mathcal{H})$, of an $(M, N_1, M, N_2 \geq N_1)$ *strong in partial order* ZIC, where the channel matrices satisfy the following condition

$$H_{11}^{\dagger}H_{11} \preceq H_{12}^{\dagger}H_{12}, \tag{2.27}$$

is given by the set of rate tuples satisfying the following constraints:

$$R_1 \leq \log\left|\left(I_{N_1} + H_{11}K_{x_1}H_{11}^{\dagger}\right)\right|; \tag{2.28}$$

$$R_2 \leq \log\left|\left(I_{N_2} + H_{22}K_{x_2}H_{22}^{\dagger}\right)\right|; \tag{2.29}$$

$$R_1 + R_2 \leq \log\left|\left(I_{N_2} + H_{12}K_{x_1}H_{12}^{\dagger} + H_{22}K_{x_2}H_{22}^{\dagger}\right)\right|; \tag{2.30}$$

Proof 2.5

- Achievability: The sum-rate bound in the expression for the capacity region represents the sum-bound present on a MAC formed formed by Tx_1, Tx_2 and Rx_2 and hence achievable through independent and Gaussian coding at both the transmitters.
- Converse: Can be proved similarly as Theorem 2.3 using Corollary 2.1.

2.5 Conclusion

As mentioned earlier, over the years the capacity region of the 2-user IC has been characterized only under specific conditions satisfied by the channel matrices $\mathcal{H}$; let us denote the set of such channel matrices by $\mathcal{H}_C$. In this chapter we have enlarged the size of this set $\mathcal{H}_C$ by characterizing the capacity region of the class of *strong in partial order* ICs. Even with this result, the capacity region characterization of the general MIMO IC is far from a being complete. Towards such a complete solution in the next chapter we adopt a different approach and characterize the capacity region of the 2-uer MIMO IC within a constant number of bits without any restriction on its channel coefficients.

Chapter 3

The capacity of the MIMO interference channel and its reciprocity to within a constant gap

3.1 Introduction

In chapter 2, we found tight upper and lower bounds to the capacity region of the channel which coincides with each other and hence characterizes the capacity region for the class of ICs called the *strong in partial order* IC, defined through a particular constraint satisfied by its channel matrices. Finding such tight upper and lower bounds for the IC without any constraint on the channel matrices is a hard problem which is why the characterization of the capacity region for this channel has been an open problem for so long. To go around this difficulty, we adopt a different approach and instead of finding upper and lower bounds which are identical we find bounds which are close to each other and therefore characterize the capacity region approximately within a small number of bits. This approach was used for the SISO channel by Etkin et al. [6], where the authors find an approximation of the capacity region of the two-user scalar Gaussian IC where the criterion of approximation is to specify the capacity region to within a constant gap independently of SNR and the direct and cross channel coefficients. Moreover, they obtain that result through a **simple** HK scheme, i.e., a single universal coding strategy for all channel parameters. The key feature of this simple HK scheme is that the the private message power is set so that it reaches the unintended receiver at the noise level. A 1 bit gap to capacity was proved in [6] using the simplified description of the HK rate region of [26]. Thus, the result of [6] characterizes the capacity region to within a constant gap that is independent of the SNR and all channel coefficients. Moreover, it identifies a simple HK scheme that has this property thereby also providing an explicit expression for the achievable rate region in terms of channel parameters. Since most modern wireless communication systems feature multiple antennas at some or all terminals, here we concentrate on the 2-user Gaussian MIMO IC.

In [27], Telatar and Tse consider an interesting class of two-user semi-deterministic discrete memoryless ICs which generalizes the class of deterministic ICs of [14] and is also applicable to the Gaussian MIMO IC and obtain outer bounds that are within a gap specified in terms of certain conditional mutual informations to the general HK achievable region [26]. The implication of this work to the 2-user MIMO IC is that the union of all the achievable rate sub-regions of the general HK scheme (one sub-region for each input distribution), is within a constant gap (of N_i bits, where N_i is the number of antennas at receiver i) to the outer bounds developed therein, and hence, to the capacity region. However, no specific achievable scheme is identified with the constant-gap-to-capacity property among the infinitely many possibilities that make up the the general HK scheme. In fact, it is unclear if there exists a **simple** HK scheme in general (corresponding to a single input distribution, as it does for the SISO case [6]) or even an **explicit** HK scheme (whose rate region is the union of rate regions achievable by a finite number of input distributions, with the constant-gap-to-capacity property. Moreover, the upper and lower bounds are not given as functions of the channel matrices in [27] which in turn bar them from being usable for further analysis (e.g., the generalized degrees of freedom (GDoF) analysis) as mentioned in [28].

In this chapter, we consider the 2-user Gaussian MIMO IC with an arbitrary number of antennas at each node. Without restricting the channel matrices in any way, we obtain constant-gap-to-capacity characterizations through a simple and an explicit HK scheme, neither of which involve time-sharing. The approach we adopt is as follows: using the general genie-aided approach of [27], we obtain a set of explicit upper bounds to the capacity region of the 2-user MIMO IC. Inspired by an interpretation of these bounds we propose a simple HK coding scheme which involves independent Gaussian superposition coding with certain explicit channel dependent covariance matrix assignments for the private and public messages of each user and show that this input distribution achieves a rate region that is within a constant gap to the capacity region. Since in the HK coding scheme the public message of an user gets decoded at the receiver of the other user, it is important to choose the "sub-rates" of the private and public messages of each user carefully because an arbitrary rate for the public message might not be supported if the corresponding cross link is **weak**. We also specify explicitly the set of these sub-rates from which if the users chose the rates for their private and public messages such a scenario never arise. In fact, a two-dimensional projection of this later set is actually yields the achievable rate region of the simple HK coding scheme. The gap to capacity (of this simple HK coding scheme) is further improved with an explicit HK scheme where the transmitters are allowed to use one of three simple superposition coding schemes depending on the operating rate pair. Interestingly, for a large class of MIMO ICs, this latter gap is smaller than the gap of N_i bits of [27]. This class includes, for example, SIMO ICs (with single-antenna transmitters and multiple antenna receivers) for which the gap is 1 bit, instead of N_i bits.

Using the explicit expressions for both the achievable rate region and the set of upper bounds to the capacity region of the MIMO IC, we then derive a *reciprocity* result which is that the capacity of a 2-user MIMO IC is within a constant gap to that of the channel obtained by interchanging the roles of the transmitters and receivers.

The rest of the chapter is organized as follows. Following a description of the notations used in this chapter in Section 3.2, we specify the system model. In Section 3.3, we derive a set of upper bounds to the capacity region and two different achievable rate regions achievable by one simple and one explicit HK coding scheme. Comparing the set of upper and lower bounds, the capacity region of the MIMO IC is characterized within a constant number of bits. As a byproduct of this analysis, we also prove the *reciprocity* of the capacity region of the MIMO IC in the approximate capacity sense in Section 3.3.4. Finally, Section 4.6 concludes the chapter. To keep the flow of the main ideas all but the simple proofs are given in the Appendices.

Notations 3.1 Let $\mathbb{C}$ and $\mathbb{R}^+$ represent the field of complex numbers and the set of non-negative real numbers, respectively. An $n \times m$ matrix with entries in $\mathbb{C}$ will be denoted as $A \in \mathbb{C}^{n\times m}$. The conjugate transpose of the matrix A is denoted as $A^{\dagger}$ and its determinant as $|A|$. Let $\|z\|^2$ represents the square of the absolute value of the complex number, i.e., if $z = (x + iy)$ then $\|z\|^2 = x^2 + y^2$. The trace of the matrix $A \in \mathbb{C}^{n\times n}$ is denoted as $\mathrm{Tr}(A)$, i.e., $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$. I_n represents the $n \times n$ identity matrix, $0_{m\times n}$ represents an all zero $m \times n$ matrix and $\mathbb{U}^{n\times n}$ represents the set of $n \times n$ unitary matrices. The k^{th} column of the matrix A will be denoted by $A^{[k]}$ whereas $A^{[k_1:k_2]}$ represents a matrix whose columns are same as the k_1^{th} to k_2^{th} columns of matrix A. $|\mathcal{A}|$ denotes the cardinality of the set $\mathcal{A}$. The fact that $(A - B)$ is a positive semi-definite (p.s.d.) (or positive definite (p.d.)) matrix is denoted by $A \succeq B$ (or $A \succ B$). $A \otimes B$ denotes the tensor or Kronecker product of the two matrices. If $x_t \in \mathbb{C}^{m\times 1}, \forall\ 1 \leq t \leq n$, then $x^n \triangleq [x_1^{\dagger}, \cdots, x_n^{\dagger}]^{\dagger}$. $\{A, B, C, D\}$ will represent an ordered set of matrices. Moreover, $I(X;Y), I(X;Y|Z), h(X)$ and $h(X|Y)$ represents the mutual information, conditional mutual information, differential entropy and conditional differential entropy of the random variable arguments, respectively. The quantities $x \wedge y$, $x \vee y$ and x^+ denote the minimum and maximum between x and y and the $\max\{x, 0\}$, respectively. All the logarithms in this chapter are with base 2. The distribution of a complex circularly symmetric Gaussian random vector with zero mean and covariance matrix Q is denoted as $\mathcal{CN}(0, Q)$.

3.2 Channel Model and Mathematical preliminaries

We consider the 2-user MIMO IC as was depicted in figure 2.1 in Chapter 2. However, here in addition we also incorporate a real-valued attenuation factor, denoted as η_{ij}, for the signal transmitted from Tx_i to receiver Rx_j and for the input we assume the following average power constraint rather than the covariance constraint in (2.1):

$$\frac{1}{n}\sum_{t=1}^{n} \mathrm{Tr}(Q_{it}) \ \leq\ 1, \tag{3.1}$$

for $i \in \{1, 2\}$, where $Q_{it} = \mathbb{E}(X_{it}X_{it}^{\dagger})$. Note that in the above power constraint Q_{it}'s can depend on the channel matrices. The received signals at time t can be written as

$$Y_{1t} = \sqrt{\rho_{11}}H_{11}X_{1t} + \sqrt{\rho_{21}}H_{21}X_{2t} + Z_{1t}, \tag{3.2}$$

$$Y_{2t} = \sqrt{\rho_{22}}H_{22}X_{2t} + \sqrt{\rho_{12}}H_{12}X_{1t} + Z_{2t}, \tag{3.3}$$

where $Z_{it} \in \mathbb{C}^{N_i \times 1}$ are i.i.d $\mathcal{CN}(\mathbf{0}, I_{N_i})$ across i and t, $\rho_{ii} = \eta_{ii}\sqrt{P_i}$ represents the signal-to-noise ratio (SNR) at receiver i and $\rho_{ij} = \eta_{ij}\sqrt{P_i}$ represents the interference-to-noise ratio (INR) at receiver j for $i \neq j \in \{1, 2\}$. In what follows, the MIMO IC with channel matrices, SNRs and INRs as described above will be denoted by $\mathcal{IC}(\mathcal{H}, \bar{\rho})$, where $\mathcal{H} = \{H_{11}, H_{12}, H_{21}, H_{22}\}$ and $\bar{\rho} = [\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}]$. The capacity region of $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ will be denoted by $\mathcal{C}(\mathcal{H}, \bar{\rho})$ and is defined as follows.

Let us assume that user i transmits information at a rate of R_i to Rx_i using the codebook $\mathcal{C}_{i,n}$ of n-length codewords with $|\mathcal{C}_{i,n}| = 2^{nR_i}$. Given a message $m_i \in \{1, \cdots, 2^{nR_i}\}$, the corresponding codeword $X_i^n(m_i) \in \mathcal{C}_{i,n}$ must satisfy the power constraint given in equation (5.1). From the received signal Y_i^n, the receiver obtains an estimate $\hat{m}_i$ of the transmitted message m_i using a decoding function $f_{i,n}$, i.e., $f_{i,n}(Y_i^n) = \hat{m}_i$. Let the average probability of error be denoted by $e_{i,n} = \mathbb{E}\left(\Pr\left(\hat{m}_i \neq m_i\right)\right)$.

A rate pair (R_1, R_2) is achievable if there exists a family of codebooks $\{\mathcal{C}_{i,n}, 1 \leq i \leq 2\}_n$ and decoding functions $\{f_{i,n}(.), 1 \leq i \leq 2\}_n$ such that $\max_i\{e_{i,n}\}$ goes to zero as the blocklength n goes to infinity. The capacity region $\mathcal{C}(\mathcal{H}, \bar{\rho})$ of $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ is defined as the closure of the set of achievable rate pairs.

Definition 3.1 An achievable rate region is said to be within n_i bits of the capacity region if for any given rate tuple $(R_1, R_2) \in \mathcal{C}(\mathcal{H}, \bar{\rho})$ the rate pair $((R_1 - n_1)^+, (R_2 - n_2)^+)$ lies in the achievable region.

3.3 Capacity to within a Constant Gap

In this section, we shall characterize the capacity region of the 2-user MIMO IC to within a constant number of bits where the constant is independent of SNRs, INRs and the channel matrices. Such a characterization involves establishing a rate region and showing that no rate tuple in the capacity region can be further from all the points in the achievable region by more than this constant. Such a characterization of the capacity region will some time be referred as the approximate capacity of the channel and the constant as the *gap* of approximation. A coding scheme which can achieve a rate region that is within a constant number of bits will be called an approximate capacity optimal (or constant-gap-to-capacity optimal) coding scheme.

In what follows, we shall first obtain a set of explicit upper bounds to the capacity region in terms of the channel matrices. These bounds enable us to give it an operational interpretation which in turn helps us identify a particular input distribution and superposition scheme (by specifying the covariance matrices for the private and public message of each user) leading to a simple HK coding scheme. The achievable rate region of this coding scheme and the corresponding gap to approximate capacity is computed in Section 3.3.2. Comparing these set of upper and lower bounds we prove that the two bounds are within n_i bits of each other, proving the constant gap capacity result, where

$$n_i = \max\left\{\left(m_{ii}\log(M_i) + m_{ij}\log(M_i+1)\right), \min\{N_i, M_s\}\log(M_x)\right\} + \hat{m}_{ji}, \text{ for } 1 \le i \ne j \le 2 \quad (3.4)$$

with $M_x = \max\{M_1, M_2\}$, $M_s = (M_1 + M_2)$, $m_{ij} = \min\{M_i, N_j\}$, and $\hat{m}_{ij} = m_{ij}\log\left(\frac{(M_i+1)}{M_i}\right)$.

In Section 3.3.3, an improvement is proposed by allowing the transmitters to select one of three carefully chosen superposition strategies depending on the rate pair to be achieved. It will be shown that the achievable region of this explicit HK coding scheme is within n_i^* bits to the capacity region, where

$$n_i^* = \min\{N_i, M_s\}\log(M_x) + \hat{m}_{ji}, \text{ for } 1 \le i \ne j \le 2. \quad (3.5)$$

Note that on a SIMO IC, $n_i^* = 1$. Finally, in Section 3.3.4, we prove the constant gap **reciprocity** of the MIMO IC, i.e., the capacity of the 2-user MIMO IC does not change by more than a constant number of bits if the roles of the transmitters and receivers are interchanged.

3.3.1 An upper bound to the capacity region

The set of upper bounds, to the capacity region for $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ derived in the following Lemma, will be denoted by $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$. For economy of notation, we define the matrices

$$K_i \triangleq \left(I_{M_i} + \rho_{ij} H_{ij}^{\dagger} H_{ij}\right)^{-1} \quad 1 \leq i \neq j \leq 2. \tag{3.6}$$

Lemma 3.1 (Upper Bound) For a given $\mathcal{H}$ and $\bar{\rho}$ the capacity region, $\mathcal{C}(\mathcal{H}, \bar{\rho})$ of a 2-user MIMO Gaussian IC, with input power constraint (5.1), is contained within the set of rate tuples $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$, i.e.,

$$\mathcal{C}(\mathcal{H}, \bar{\rho}) \subseteq \mathcal{R}^u(\mathcal{H}, \bar{\rho}),$$

where $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$ represents the set of rate pairs (R_1, R_2), satisfying the following constraints:

$$R_1 \leq \log\det\left(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger}\right); \tag{3.7}$$

$$R_2 \leq \log\det\left(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger}\right); \tag{3.8}$$

$$R_1 + R_2 \leq \log\det\left(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} H_{22}^{\dagger}\right) + \log\det\left(I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\right); \tag{3.9}$$

$$R_1 + R_2 \leq \log\det\left(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} H_{11}^{\dagger}\right) + \log\det\left(I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\right); \tag{3.10}$$

$$\begin{aligned} R_1 + R_2 \leq{} & \log\det\left(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\right) \\ & + \log\det\left(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\right); \end{aligned} \tag{3.11}$$

$$\begin{aligned} 2R_1 + R_2 \leq{} & \log\det\left(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} H_{11}^{\dagger}\right) + \log\det\left(I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\right) + \\ & \log\det\left(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\right); \end{aligned} \tag{3.12}$$

$$\begin{aligned} R_1 + 2R_2 \leq{} & \log\det\left(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} H_{22}^{\dagger}\right) + \log\det\left(I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\right) + \\ & \log\det\left(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\right). \end{aligned} \tag{3.13}$$

Proof 3.1 (Proof of Lemma 3.1) The proof is given in Appendix A.3.

Remark 3.1 It was mentioned in [27] that obtaining the approximate capacity results for the MIMO case starting from the result of [6] appears difficult. It seems that the difficulty lies in deriving a tight upper bound for the capacity region in the case when either $M_i > N_i$ or $M_i > N_{j \neq i}$. For example, consider the first sum rate upper bound in [6], where the second user's codeword is given to the receiver of the first user. Using Fano's inequality for the MIMO channel we have

$$\begin{aligned} n(R_1 + R_2) \leq & I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + n\epsilon_n, \text{ [here, } \epsilon_n \to 0 \text{ as } n \to \infty] \\ \leq & I(X_1^n; Y_1^n, X_2^n) + I(X_2^n; Y_2^n) + n\epsilon_n, \\ = & h(Y_1^n | X_2^n) - h(Z_1^n) + h(Y_2^n) - h(Y_2^n | X_2^n) + n\epsilon_n, \end{aligned}$$

which simplifies to

$$n(R_1 + R_2) \leq h(\sqrt{\rho_{11}}(I_n \otimes H_{11})X_1^n + Z_1^n) - h(\sqrt{\rho_{12}}(I_n \otimes H_{12})X_1^n + Z_2^n) + h(Y_2^n) - h(Z_1^n) + n\epsilon_n.$$

Now $h(Y_2^n)$ in the above equation can be easily upper bounded by Gaussian inputs and upper bounding the difference between the first two differential entropies is relatively easy when both H_{11} and H_{12} are square. A similar concept can be extended to the case when both of these matrices have a larger row dimension than the column dimension, using SVD of these matrices (e.g., this approach was used in [29], to extend the SISO result to the class of symmetric (M, N, M, N) MIMO ICs with $N \geq M$). However, this approach can not be applied when $M_1 > N_1$ or $M_1 > N_2$ because it is not clear how to upper bound the above difference in such a way that it is tight enough to yield an approximate capacity. In the genie-based model of [27] used in the proof in Appendix A.3, this problem does not arise.

Remark 3.2 Two sets of explicit upper bounds to the capacity region of the MIMO IC, denoted respectively as $\mathcal{R}_0(H, G)$ and $\mathcal{R}_{00}(H, G)$, were derived in [30] from the result in [27]. It can be easily verified that the first four bounds in Lemma 3.1 are identical to those in [30] and the fifth bound (on $R_1 + R_2$) can be shown to be equivalent to the 7^{th} bound of $\mathcal{R}_0(H, G)$ and $\mathcal{R}_{00}(H, G)$. However, the bounds on $(2R_1 + R_2)$ and $(R_1 + 2R_2)$ in [30] are incorrect. Figure 3.1 illustrates this fact by showing that on the SISO IC specified in Example 3.1, the bound on $(2R_1 + R_2)$ in $\mathcal{R}_{00}(H, G)$ contradicts with the achievability of some of the rate tuples.

Let us consider the bound on $(2R_1 + R_2)$ in $\mathcal{R}_{00}(H, G)$, which for the SISO channel can be written as

$$(2R_1 + R_2) \leq \log(1 + P\|H_{11}\|^2 + P\|H_{21}\|^2) + \log\left(1 + \frac{P\|H_{11}\|^2}{1 + P\|H_{21}\|^2}\right) + \log\left(1 + P\|H_{21}\|^2 + \frac{P\|H_{22}\|^2}{1 + P\|H_{12}\|^2}\right). \quad (3.14)$$

By the notation of [6], $\mathrm{SNR}_i = P_i\|H_{ii}\|^2$ and $\mathrm{INR}_j = P_i\|H_{ij}\|^2$ for $i \neq j \in \{1, 2\}$. Using this notation in the above equation we get

$$(2R_1 + R_2) \leq \log(1 + \mathrm{SNR}_1 + \mathrm{INR}_1) + \log\left(1 + \frac{\mathrm{SNR}_1}{1 + \mathrm{INR}_2}\right) + \log\left(1 + \mathrm{INR}_1 + \frac{\mathrm{SNR}_2}{1 + \mathrm{INR}_2}\right) \triangleq B_{UB}^{AOC}. \quad (3.15)$$

The corresponding bound in the achievable region for the **weak** SISO IC ($\mathrm{INR}_1 < \mathrm{SNR}_2$ and $\mathrm{INR}_2 < \mathrm{SNR}_1$) derived in [6] is given as (e.g., see Theorem 5, equation (61) in [6])

$$(2R_1 + R_2) \leq \log(1 + \mathrm{SNR}_1 + \mathrm{INR}_1) + \log\left(2 + \frac{\mathrm{SNR}_1}{\mathrm{INR}_2}\right) + \log\left(1 + \mathrm{INR}_2 + \frac{\mathrm{SNR}_2}{\mathrm{INR}_1}\right) - 3 \triangleq B_{LB}^{ETW}. \quad (3.16)$$

Comparing the two bounds in equation (3.15) and (3.16) we see

$$B_{UB}^{AOC} < B_{LB}^{ETW} + \log(1 + \mathrm{INR}_1) - \log(1 + \mathrm{INR}_2) + 3,$$

which implies that if INR_1 is sufficiently smaller than INR_2 i.e., $\log(1 + \mathrm{INR}_1) \leq \log(1 + \mathrm{INR}_2)$ then the upper bound in (3.15) can be strictly smaller than the lower bound (3.16). Suppose there exists an IC on which in addition to the fact that $B_{UB}^{AOC} < B_{LB}^{ETW}$ the bound (3.16) is active. This would imply that there exists achievable rate tuples (R_1, R_2) which satisfy equation (3.16) but violates (3.15) since $B_{UB}^{AOC} < B_{LB}^{ETW}$. Now, (R_1, R_2) is an achievable rate tuple and can not violate an upper bound unless it is incorrect. The following example proves the existence of such a channel.

Example 3.1 Consider a real SISO IC with $H_{11} = 45$, $H_{12} = 25$, $H_{21} = 3$, $H_{22} = 30$ and $P_1 = P_2 = P = 1$. For this channel we have $\mathrm{SNR}_1 = 2025 > \mathrm{INR}_2 = 625$ and $\mathrm{SNR}_2 = 900 > \mathrm{INR}_2 = 9$. Clearly, this is a **weak** interference channel (by the definition of [6]), for which an achievable rate region can be easily computed using equation (61) in Theorem 5 of [6] and is depicted in Fig. 3.1 as the polygon ABCDEF. On the other hand, putting the values of SNR_i's and INR_j's

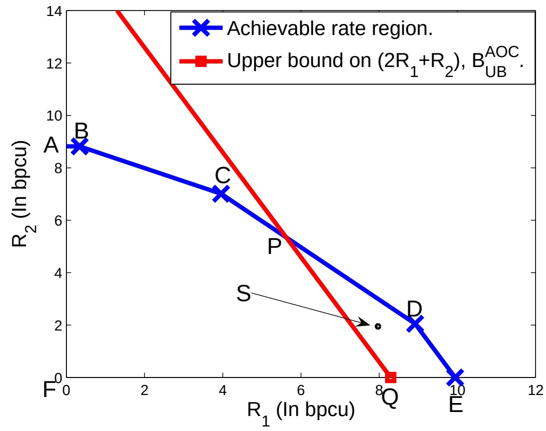


Figure 3.1: A comparison of an achievable rate region and the upper bound on $(2R_1 + R_2)$..

in equation (3.15) we get the upper bound on $(2R_1 + R_2)$ from [30], which is also depicted in the figure. According to this bound any point (e.g., point S in the figure), above the line segment PQ is not achievable, which evidently is not true.

3.3.2 A simple achievable scheme

Before describing the simple coding scheme, in what follows we shall briefly go through the original HK coding scheme [10] and some recent developments [31], [26] of this coding scheme for the discrete memoryless interference channel (DMIC). Later, we shall be applying these results for the Gaussian IC. On a DMIC with transition probability $P(Y_1, Y_2|X_1, X_2)$, for any set $\mathcal{P}^*$ of probability distributions P^* which factors as

$$P^*(Q, U_1, U_2, W_1, W_2, X_1, X_2) = P(Q)P(U_1|Q)P(U_2|Q)P(W_1|Q)P(W_2|Q)$$
$$P(X_1|U_1, W_1, Q)P(X_2|U_2, W_2, Q), \qquad (3.17)$$

where $P(X_1|U_1, W_1, Q)$ and $P(X_2|U_2, W_2, Q)$ equals either 0 or 1, let $\mathcal{R}^o_{\mathrm{HK}}(P^*) \triangleq \mathcal{R}^{(o,1)}_{\mathrm{HK}}(P^*) \cap \mathcal{R}^{(o,2)}_{\mathrm{HK}}(P^*)$ represents a set of real 4-tuples, where

$$\mathcal{R}^{(o,i)}_{\mathrm{HK}}(P^*) = \Big\{(S_1, T_1, S_2, T_2) : S_i \leq I(U_i; Y_i|W_i, W_j, Q) \triangleq I_{a_i}; \qquad (3.18a)$$
$$T_i \leq I(W_i; Y_i|U_i, W_j, Q) \triangleq I_{b_i}; \qquad (3.18b)$$
$$T_j \leq I(W_j; Y_i|U_i, W_i, Q) \triangleq I_{c_i}; \qquad (3.18c)$$
$$(S_i + T_i) \leq I(U_i, W_i; Y_i|W_j, Q) \triangleq I_{d_i}; \qquad (3.18d)$$
$$(S_i + T_j) \leq I(U_i, W_j; Y_i|W_i, Q) \triangleq I_{e_i}; \qquad (3.18e)$$
$$(T_i + T_j) \leq I(W_i, W_j; Y_i|U_i, Q) \triangleq I_{f_i}; \qquad (3.18f)$$
$$(S_i + T_i + T_j) \leq I(U_i, W_i, W_j; Y_i|Q) \triangleq I_{g_i}; \Big\} \qquad (3.18g)$$

for $i \neq j \in \{1, 2\}$. Further, for a set, $\mathcal{S}$ of 4-tuples (S_1, T_1, S_2, T_2), let $\Pi(\mathcal{S}) = \{(R_1, R_2) : 0 \leq R_i \leq (S_i + T_i), 1 \leq i \leq 2,$ for some $(S_1, T_1, S_2, T_2) \in \mathcal{S}\}$, i.e., it is a 2-dimensional projection of $\mathcal{S}$. Then from [10] we know that

Theorem 3.1 the set

$$\mathcal{R}^o_{\mathrm{HK}} = \Pi\left(\bigcup_{P^* \in \mathcal{P}^*} \mathcal{R}^{(o,1)}_{\mathrm{HK}}(P^*) \cap \mathcal{R}^{(o,2)}_{\mathrm{HK}}(P^*)\right) \qquad (3.19)$$

is an achievable region for the discrete memoryless IC.

Remark 3.3 Let U_1 (U_2) and W_1 (W_2) represent the private and common parts of the message to be transmitted by Tx_1 (Tx_2), which hereafter will be referred to as the private and common message of Tx_1 (Tx_2), respectively. Also, let S_i and T_i represents the rate of information carried by U_i and W_i, respectively, for $i \in \{1, 2\}$ and X_i is constructed from U_i and W_i in such a manner that the joint distribution $P^*(Q, U_1, W_1, U_2, W_2, X_1, X_2) \in \mathcal{P}^*$. Then, Theorem 3.1 essentially states that, for any $(S_1, T_1, S_2, T_2) \in \mathcal{R}^o_{\text{HK}}(P^*)$ the rate tuple $(S_1 + T_1, S_2 + T_2)$ is achievable on the DMIC.

Thus, for any given P^* Theorem 3.1 not only provides a set of achievable rate tuples of the channel in the form of $\Pi\left(\mathcal{R}^o_{\text{HK}}(P^*)\right)$, but also provides the set of 4-tuples from which each user can pick the rates of their private and public messages. However, to compute the achievable rate region of the channel, it is necessary to compute the auxiliary sets $\mathcal{R}^{(o,1)}_{\text{HK}}(P^*)$ and $\mathcal{R}^{(o,2)}_{\text{HK}}(P^*)$ first. This indirect way of computation can be avoided by using the equivalent description of $\Pi\left(\mathcal{R}^o_{\text{HK}}(P^*)\right)$ provided in Lemma 1 of [26], stated below for easy reference.

Lemma 3.2 (Lemma 1 in [26]) For a fixed $P^* \in \mathcal{P}^*$, let $\mathcal{R}^e_{\text{HK}}(P^*)$ be the set of (R_1, R_2) tuples satisfying:

$$R_1 \leq I(X_1; Y_1|W_2, Q); \tag{3.20a}$$

$$R_1 \leq I(X_1, Y_1|W_1, W_2, Q) + I(W_1; Y_2|X_2, Q); \tag{3.20b}$$

$$R_2 \leq I(X_2; Y_2|W_1, Q); \tag{3.20c}$$

$$R_2 \leq I(X_2; Y_2|W_1, W_2, Q) + I(W_2; Y_1|X_1, Q); \tag{3.20d}$$

$$R_1 + R_2 \leq I(X_2, W_1; Y_2|Q) + I(X_1; Y_1|W_1, W_2, Q); \tag{3.20e}$$

$$R_1 + R_2 \leq I(X_1, W_2; Y_1|Q) + I(X_2; Y_2|W_1, W_2, Q); \tag{3.20f}$$

$$R_1 + R_2 \leq I(X_1, W_2; Y_1|W_1, Q) + I(X_2, W_1; Y_2|W_2, Q); \tag{3.20g}$$

$$2R_1 + R_2 \leq I(X_1, W_2; Y_1|Q) + I(X_1; Y_1|W_1, W_2, Q) + I(X_2, W_1; Y_2|W_2, Q); \tag{3.20h}$$

$$R_1 + 2R_2 \leq I(X_2, W_1; Y_2|Q) + I(X_2; Y_2|W_1, W_2, Q) + I(X_1, W_2; Y_1|W_1, Q), \tag{3.20i}$$

then the Han-Kobayashi achievable region is given by $\mathcal{R}^e_{\text{HK}} = \cup_{P^* \in \mathcal{P}^*} \mathcal{R}^e_{\text{HK}}(P^*)$[1] .

It should be noted that the set of achievable rate region $\mathcal{R}^e_{\text{HK}}(P^*)$ in Lemma 3.2 is now specified directly as a set of (R_1, R_2) tuples, defined through constraints (3.20a)-(3.20i).

[1] We use the superscript "o" to refer to the *original* HK coding scheme [10] and and "e" to emphasize that $\mathcal{R}^e_{\text{HK}}(P^*)$ is an *equivalent* description of $\Pi\left(\mathcal{R}^o_{\text{HK}}(P^*)\right)$.

Remark 3.4 The Lemma was proved by showing that for any given $P^* \in \mathcal{P}^*$,

$$\mathcal{R}^e_{\mathrm{HK}}(P^*) = \Pi\left(\mathcal{R}^o_{\mathrm{HK}}(P^*)\right) = \Pi\left(\mathcal{R}^{(o,1)}_{\mathrm{HK}}(P^*) \cap \mathcal{R}^{(o,2)}_{\mathrm{HK}}(P^*)\right). \tag{3.21}$$

An equivalent description, for $\Pi\left(\mathcal{R}^o_{\mathrm{HK}}(P^*)\right)$ was first derived in [31], using Fourier-Motzkin elimination method on the set constraints given in equation (3.18), which have two additional constraints on $(2R_1 + R_2)$ and $(R_1 + 2R_2)$ besides those in Lemma 3.2. Later, in [26] these bounds were shown to be redundant, resulting in Lemma 3.2.

Using the standard technique of [32] (e.g., see Chapter 7) these DMC results can be applied for the Gaussian channel with discrete time but continuous input. To distinguish them from each other, the corresponding rate regions of $\mathcal{R}^o_{\mathrm{HK}}(P^*)$, $\mathcal{R}^{(o,i)}_{\mathrm{HK}}(P^*)$ and $\mathcal{R}^e_{\mathrm{HK}}(P^*)$, for the Gaussian channel will be denoted by $\mathcal{R}^{G_o}_{\mathrm{HK}}(P^*)$, $\mathcal{R}^{(G_o,1)}_{\mathrm{HK}}(P^*)$ and $\mathcal{R}^{G_e}_{\mathrm{HK}}(P^*)$, respectively.

Evidently, both the original (Theorem 3.1) and the alternative description (Lemma 3.2) of the HK coding scheme is a union of an infinite number of sub-regions, each corresponding to a particular input distribution and time sharing strategy. Since a complete characterization of this region is prohibitively complicated, we seek in some sense a single **good** input distribution and time sharing strategy. Indeed, we provide a novel and important operational interpretation of the bounds of Lemma 3.1 through which such a good choice of input distribution becomes apparent, leading to a simple HK coding scheme. Moreover, this simple HK coding scheme has a property of being universally good in that it achieves a rate region that is within a constant number of bits to the set of upper bounds of Lemma 3.1 independently of SNR and the channel parameters.

An interpretation of the bounds of Lemma 3.1: The first two bounds in $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$ come from the rate bound on a point-to-point channel. The first term of the third bound given in (5.11) represents the sum rate upper bound of a 2-user MAC having channel matrices H_{i2}, for $i = 1, 2$ and Gaussian input with zero mean and scaled identity matrix as covariance. The second term represents the mutual information on a point-to-point channel whose input covariance matrix is K_1 (see (3.6) for the definition of K_1). These terms can be given the following operational interpretation. The entire message of Tx_2 has to be decoded at the second receiver and some part of Tx_1 might be decoded at Rx_2. Let us call this the public message of the first user, denoted as W_1 having rate R_{1w}. Subsequently, let us denote the remaining part of the first user's message by U_1 having rate R_{1u} which will be referred to as the private message of the first user. Thus we have $R_1 = R_{1w} + R_{1u}$. Now, with respect to W_1 and X_2, Rx_2 acts as a MAC and thus has the following upper bound

$$R_{1w} + R_2 \leq \log\det\left(I_{N_2} + \rho_{12}H_{12}H_{12}^{\dagger} + \rho_{22}H_{22}H_{22}^{\dagger}\right).$$

On the other hand, since U_1 has to be decoded at Rx_1, it has the following point-to-point channel upper bound

$$R_{1u} \leq \log\det\left(I_{N_1} + \rho_{11}H_{11}K_1H_{11}^{\dagger}\right),$$

where K_1 is the covariance matrix of U_1. These two bounds together imply the third bound in Lemma 3.1. The 4^{th} bound can also be interpreted similarly just by interchanging the role of transmitters. The first term of the fifth bound can be thought as a bound on the private message of Tx_1 and the public message of Tx_2 which are to be decoded at Rx_1, i.e.,

$$R_{1u} + R_{2w} \leq \log\det\left(I_{N_1} + \rho_{21}H_{21}H_{21}^{\dagger} + \rho_{11}H_{11}K_1H_{11}^{\dagger}\right),$$

where the private message has a covariance matrix same as before. Similarly, the second term in the 5^{th} bound can be interpreted as an upper bound on $(R_{1w} + R_{2u})$, and together, they imply the fifth bound. The other terms of the remaining bounds can be similarly interpreted. This interpretation motivates a simple HK scheme, where Q is a deterministic number (no time-sharing), the i^{th} user's message is divided into a private and a public message and the private message has an input covariance matrix proportional to K_i.

Definition 3.2 (Input distribution for the simple coding scheme) Let the private and public messages of the users be encoded using mutually independent random Gaussian codewords and the overall codeword is a superposition of the two, i.e., the transmit signals for any particular channel use can be written as

$$X_1^g = U_1^g + W_1^g;$$
$$X_2^g = U_2^g + W_2^g, \tag{3.22}$$

where $U_i^g \sim \mathcal{CN}(\mathbf{0}, K_{iu})$ and $W_i^g \sim \mathcal{CN}(\mathbf{0}, K_{iw})$, represent symbols of the codewords of the private and public messages of user i, respectively and

$$K_{iu}(\mathcal{H}) \triangleq \mathbb{E}(U_i^g U_i^{g\dagger}) = \frac{K_i}{M_i} = \frac{1}{M_i}\left(I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij}\right)^{-1}; \tag{3.23}$$

$$K_{iw}(\mathcal{H}) \triangleq \mathbb{E}(W_i^g W_i^{g\dagger}) = \frac{1}{M_i}\left(I_{M_i} - K_i\right). \tag{3.24}$$

The scaling by $\frac{1}{M_i}$ is required to satisfy the power constraint (5.1). In the sequel, we shall refer to such a superposition coding scheme where the covariance matrices of the private and public messages of user i is given by K_{iu} and K_{iw} will be referred to as the $\mathcal{HK}\left(\{K_{iu}, K_{iw}, K_{iu}, K_{iw}\}\right)$ scheme. In particular, when K_{iu} and K_{iw} are as in equation (3.23) and (3.24), respectively the coding scheme will be denoted as $\mathcal{HK}^{(s)}$. Let us denote the distribution of the random variables as defined above by $P_s(U_1^g, W_1^g, X_1, U_2^g, W_2^g, X_2)$. Clearly, $P_s(U_1^g, W_1^g, X_1^g, U_2^g, W_2^g, X_2^g) \in \mathcal{P}^*$.

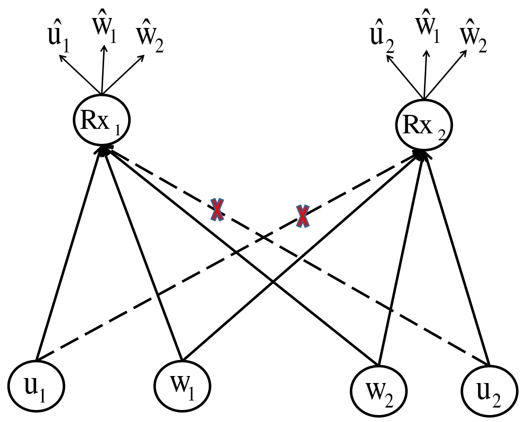


Figure 3.2: The equivalent virtual channel for the simple HK coding scheme.

Remark 3.5 The above choice ensures that the private message of user i, the covariance of whose contribution at Rx_j ($\sqrt{\rho_{ij}} H_{ij} U_i^g$) is given by

$$\rho_{ij} H_{ij} K_{iu} H_{ij}^\dagger = \frac{\rho_{ij}}{M_i} H_{ij}\left(I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij}\right)^{-1} H_{ij}^\dagger \preceq I_{M_i},$$

reaches the unintended receiver below the noise floor. The $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ coding scheme thus effectively divides each user into two virtual users as shown in Fig. 3.2. Note that the interference links from the first virtual user to Rx_2 and the fourth virtual user to Rx_1 are made very weak so that any signal along those links always reaches the receivers below noise floor. As shown in the figure, the channel can be thought as two interfering multiple access channels (MAC) where Rx_i jointly decodes U_i^g, W_i^g and $W_{j\neq i}^g$, treating U_j^g as noise for $1 \leq i \neq j \leq 2$.

Applying Theorem 3.1 and Lemma 3.2 for the Gaussian IC and evaluating it for the distribution, $P_s(.)$ of Definition 3.2 we get the following achievable region for the 2-user MIMO Gaussian IC.

Lemma 3.3 On a 2-user Gaussian MIMO IC, the simple $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ coding scheme can achieve the rate region, $\mathcal{R}_{\text{HK}}^{G_e}(P_s)$, which is a set of rate tuples, (R_1, R_2) where R_i's satisfy the following constraints

$$R_1 \leq I(X_1^g; Y_1^g | W_2^g); \tag{3.25a}$$

$$R_1 \leq I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g); \tag{3.25b}$$

$$R_2 \leq I(X_2^g; Y_2^g | W_1^g); \tag{3.25c}$$

$$R_2 \leq I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | X_1^g); \tag{3.25d}$$

$$R_1 + R_2 \leq I(X_2^g, W_1^g; Y_2^g) + I(X_1^g; Y_1^g | W_1^g, W_2^g); \tag{3.25e}$$

$$R_1 + R_2 \leq I(X_1^g, W_2^g; Y_1^g) + I(X_2^g; Y_2^g | W_1^g, W_2^g); \tag{3.25f}$$

$$R_1 + R_2 \leq I(X_1^g, W_2^g; Y_1^g | W_1^g) + I(X_2^g, W_1^g; Y_2^g | W_2^g); \tag{3.25g}$$

$$2R_1 + R_2 \leq I(X_1^g, W_2^g; Y_1^g) + I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(X_2^g, W_1^g; Y_2^g | W_2^g); \tag{3.25h}$$

$$R_1 + 2R_2 \leq I(X_2^g, W_1^g; Y_2^g) + I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(X_1^g, W_2^g; Y_1^g | W_1^g), \tag{3.25i}$$

where Y_i^g's are the outputs of the 2-user MIMO IC when its inputs are Gaussian as stated in Definition 3.2. Further,

$$\mathcal{R}_{\text{HK}}^{G_e}(P_s) = \Pi\left(\mathcal{R}_{\text{HK}}^{G_o}(P_s)\right) = \Pi\left(\mathcal{R}_{\text{HK}}^{(G_o,1)}(P_s) \cap \mathcal{R}_{\text{HK}}^{(G_o,2)}(P_s)\right),$$

where

$$\mathcal{R}_{\text{HK}}^{(G_o,i)}(P^*) = \Big\{(S_1,T_1,S_2,T_2) : S_i \le I(X_i^g;Y_i^g|W_i^g,W_j^g); \tag{3.26a}$$

$$T_i \le I(W_i^g;Y_i^g|U_i^g,W_j^g); \tag{3.26b}$$

$$T_j \le I(W_j^g;Y_i^g|W_i^g,U_i^g); \tag{3.26c}$$

$$(S_i+T_i) \le I(X_i^g;Y_i^g|W_j^g); \tag{3.26d}$$

$$(S_i+T_j) \le I(U_i^g,W_j^g;Y_i^g|W_i^g); \tag{3.26e}$$

$$(T_i+T_j) \le I(W_i^g,W_j^g;Y_i^g|U_i^g); \tag{3.26f}$$

$$(S_i+T_i+T_j) \le I(U_i^g,W_i^g,W_j^g;Y_i^g); \Big\} \tag{3.26g}$$

for $i \neq j \in \{1,2\}$ and

$$I(X_1^g;Y_1^g|W_1^g,W_2^g) = \log\det\Big(I_{N_1} + \rho_{11}H_{11}K_{1u}H_{11}^\dagger + \rho_{21}H_{21}K_{2u}H_{21}^\dagger\Big) - \tau_{21}; \tag{3.27}$$

$$I(W_1^g,;Y_1^g|W_2^g,U_1^g) = \log\det\Big(I_{N_1} + \rho_{11}H_{11}K_{1w}H_{11}^\dagger + \rho_{21}H_{21}K_{2u}H_{21}^\dagger\Big) - \tau_{21}; \tag{3.28}$$

$$I(W_2^g;Y_1^g|X_1^g) = \log\det\left(I_{N_1} + \frac{\rho_{21}}{M_2}H_{21}H_{21}^\dagger\right) - \tau_{21}; \tag{3.29}$$

$$I(X_1^g;Y_1^g|W_2^g) = \log\det\left(I_{N_1} + \frac{\rho_{11}}{M_1}H_{11}H_{11}^\dagger + \rho_{21}H_{21}K_{2u}H_{21}^\dagger\right) - \tau_{21}; \tag{3.30}$$

$$I(X_1^g,W_2^g;Y_1^g|W_1^g) = \log\det\left(I_{N_1} + \frac{\rho_{21}}{M_2}H_{21}H_{21}^\dagger + \rho_{11}H_{11}K_{1u}H_{11}^\dagger\right) - \tau_{21}; \tag{3.31}$$

$$I(W_1^g,W_2^g;Y_1^g|U_1^g) = \log\det\left(I_{N_1} + \frac{\rho_{21}}{M_2}H_{21}H_{21}^\dagger + \rho_{11}H_{11}K_{1w}H_{11}^\dagger\right) - \tau_{21}; \tag{3.32}$$

$$I(X_1^g,W_2^g;Y_1^g) = \log\det\left(I_{N_1} + \frac{\rho_{21}}{M_2}H_{21}H_{21}^\dagger + \frac{\rho_{11}}{M_1}H_{11}H_{11}^\dagger\right) - \tau_{21}, \tag{3.33}$$

$\tau_{ij} = \log\det(I_{N_j} + \rho_{ij}H_{ij}K_{iu}H_{ij}^\dagger)$ for $i \neq j\{1,2\}$ and $I(X_2^g;Y_2^g|W_2^g,W_1^g)$ through $I(X_2^g,W_1^g;Y_2^g)$ are obtained by swapping the indexes 1 and 2 in the above set of equations, where K_{iu} and K_{iw} are given by equation (3.23) and (3.24), respectively for $1 \le i \le 2$.

Proof 3.2 Equations (3.25) and (3.26) are simple application of the DMIC result of Lemma 3.2 and Theorem 3.1 to the Gaussian channel. Whereas equations (3.27)-(3.33) are obtained evaluating the different mutual information terms in equations (3.18) for the given distribution of U_1^g, U_2^g, W_1^g, W_2^g, X_1^g and X_2^g in Definition 3.2.

Note that in the simple HK coding scheme, each user has 2 messages: a private message (U_i^g), which is to be decoded at its own receiver and a common message (W_i^g), which is to be decoded at both the receivers. Therefore, to achieve a rate tuple $(R_1, R_2) \in \mathcal{R}_{\text{HK}}^{G_e}(P_s)$ the rates of these messages has to be chosen in such a way that, each of these messages can be decoded at their respective receivers with arbitrarily small probability of error and

$$S_i + T_i = R_i, \ \forall \ i \ \in \{1, 2\}, \tag{3.34}$$

where S_i and T_i are the rates of the private and public messages of user i, respectively. The second part of Lemma 3.3 provides such a set ($\mathcal{R}_{\text{HK}}^{G_o}(P_s)$) from which these "sub-rates" can be chosen. Since, $\mathcal{R}_{\text{HK}}^{G_e}(P_s) = \Pi\left(\mathcal{R}_{\text{HK}}^{G_o}(P_s)\right)$, for every $(R_1, R_2) \in \mathcal{R}_{\text{HK}}^{G_e}(P_s)$ by definition of $\Pi(.)$ there exists at least one 4-tuple $(S_1, T_1, S_2, T_2) \in \mathcal{R}_{\text{HK}}^{G_o}(P_s)$ such that $(S_i + T_i) = R_i$ for both $i = 1, 2$. On the other hand, by Theorem 3.1 for any $(S_1, T_1, S_2, T_2) \in \mathcal{R}_{\text{HK}}^{G_o}(P_s)$ if S_i and T_i represents the rates of information carried by U_i^g and W_i^g, respectively, then the simple HK scheme can achieve the rate tuple $(S_1 + T_1, S_2 + T_2)$ (recall Remark 3.3), i.e., U_i^g, W_i^g and W_j^g can be decoded at Rx_i with arbitrarily small probability of error, for $i \neq j \in \{1, 2\}$. So, the rate splitting strategy of the $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ scheme can be summarized as follows.

Rate splitting strategy for the simple HK coding scheme: For any $(R_1, R_2) \in \mathcal{R}_{\text{HK}}^{G_e}(P_s)$, the simple HK coding scheme of Definition 3.2 chose an (S_1, T_1, S_2, T_2) tuple from $\mathcal{R}_{\text{HK}}^{G_o}(P_s)$ in such a way that $(S_i + T_i) = R_i$[2] , assigns rate S_i to the private message U_i^g and T_i to the public message W_i^g of user i and transmit the signals using the superposition coding scheme specified earlier. On the decoding side, Rx_i can jointly decode U_i^g, W_i^g and W_j^g treating U_j^g as noise for $i \neq j \in \{1, 2\}$, with vanishing probability of error (e.g., see Theorem 3.1).

Example 3.2 Consider a 2-user Gaussian $(2, 3, 2, 2)$ IC with $\bar{\rho} = [20,\ 8,\ 12,\ 20]$ dB, where the

[2] The existence of such a quadruple is now guaranteed by Lemma 3.3.

channel matrices are given as follows

$$H_{11} = \begin{bmatrix} 1.1975 - 0.4385i & -0.0902 + 0.1895i \\ 0.3234 - 1.3614i & 0.1330 - 0.2564i \\ 0.7546 - 1.0080i & -0.3205 - 0.6958i \end{bmatrix} \quad H_{21} = \begin{bmatrix} 0.3816 - 0.8508i & 0.4450 - 0.4386i \\ -0.4892 - 0.2179i & -0.5346 - 0.1519i \\ 0.7665 - 1.0875i & 0.1689 + 0.7651i \end{bmatrix}$$

$$H_{12} = \begin{bmatrix} 0.9652 - 0.8085i & -0.3033 + 0.0055i \\ 0.6130 + 1.4479i & 0.6872 + 0.5280i \end{bmatrix} \quad H_{22} = \begin{bmatrix} -0.1209 - 0.4575i & -0.0040 + 0.0921i \\ -0.5730 + 1.1118i & -0.8223 - 0.5687i \end{bmatrix}$$

In Fig. 3.3 the dotted line represents the rate region achievable by the simple HK scheme and the solid line represents the superset $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$ which contains the capacity region of the channel.

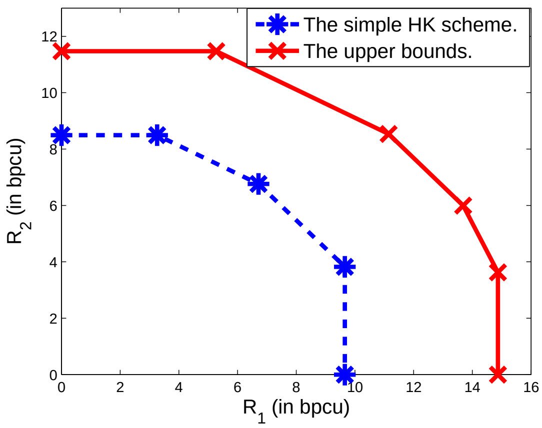


Figure 3.3: An achievable rate region of the simple HK scheme.

It is not unreasonable to think that this gap between the boundaries of the achievable rate region and the set $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$ can vary arbitrarily depending on the channel matrices. However, in what follows we shall show that this gap actually is bounded and can not be larger than a constant which is independent of the SNR, INR or the channel coefficients. This fact will be proved by showing that $\mathcal{R}_{\text{HK}}^{G_e}(P_s)$ contains a subset which is within a constant number of bits to the set of upper bounds. The following lemma specifies this subset.

Lemma 3.4 The achievable rate region of the simple $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ coding scheme employed on $\mathcal{IC}(\mathcal{H}, \bar{\rho})$, contains the region $\mathcal{R}_a(\mathcal{H}, \bar{\rho})$, which is a set of non-negative rate tuples satisfying the following constraints:

$$R_1 \leq \log\det\Big(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger}\Big) - n_1; \tag{3.35}$$

$$R_2 \leq \log\det\Big(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger}\Big) - n_2; \tag{3.36}$$

$$R_1 + R_2 \leq \log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} H_{22}^{\dagger}\Big) \tag{3.37}$$

$$+ \log\det\Big(I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\Big) - (n_1 + n_2); \tag{3.38}$$

$$R_1 + R_2 \leq \log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} H_{11}^{\dagger}\Big) \tag{3.39}$$

$$+ \log\det\Big(I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\Big) - (n_1 + n_2); \tag{3.40}$$

$$R_1 + R_2 \leq \log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\Big) \tag{3.41}$$

$$+ \log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\Big) - (n_1 + n_2); \tag{3.42}$$

$$2R_1 + R_2 \leq \log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} H_{11}^{\dagger}\Big) + \log\det\Big(I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\Big) + \tag{3.43}$$

$$+ \log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\Big) - (2n_1 + n_2); \tag{3.44}$$

$$R_1 + 2R_2 \leq \log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} H_{22}^{\dagger}\Big) + \log\det\Big(I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\Big) \tag{3.45}$$

$$+ \log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\Big) - (n_1 + 2n_2), \tag{3.46}$$

where K_i's are as specified before (see equation (3.6)) and n_i's are given by equation (3.4) for $1 \leq i \leq 2$.

Proof 3.3 (Proof of Lemma 3.4(Outline)) Note that the rate region $\mathcal{R}_{\mathrm{HK}}^{G_e}(P_s)$ is a polygon or an area in the first quadrant of (R_1, R_2)-plane restricted by different straight line segments. Two sides of this polygon are the R_1 and R_2 axes and each of the other sides of it is part of a straight line obtained by replacing the inequality by equality in one of hte constraints in equations (3.25a)-(3.25i).

If the right hand side term of any of these bounds is replaced by a term smaller in magnitude, the corresponding side in the polygon moves parallely towards the center. If this is done to all the constraints, all the sides move towards the center resulting in a smaller polygon completely inside the previous one. In Appendix A.5 we shall show that the bounds in (3.35)-(3.46) are obtained by replacing the right hand sides of (3.25a)-(3.25i) by terms smaller in magnitude. So, $\mathcal{R}_a(\mathcal{H},\bar{\rho})$ describes a polygon which is completely inside $\mathcal{R}_{\text{HK}}^{G_e}(P_s)$ and hence achievable by $\mathcal{HK}^{(s)}$.

Note that each bound of Lemma 3.4 differs from the corresponding bound in Lemma 3.1 only by a constant, from which we get the following constant gap to capacity result.

Theorem 3.2 The rate region $\mathcal{R}_a(\mathcal{H},\bar{\rho})$ of Lemma 3.4, which is achievable by the simple HK scheme $\mathcal{HK}(\{K_{1u},K_{1w},K_{2u},K_{2w}\})$, is within n_i bits to the capacity region of the Gaussian MIMO IC, where n_i is given by equation (3.4).

Proof 3.4 We need to prove that for any given $(R_1,R_2)\in\mathcal{C}(\mathcal{H},\bar{\rho})$, there exists a rate pair $(\hat{R}_1,\hat{R}_2)\in\mathcal{R}_a(\mathcal{H},\bar{\rho})$ such that $\hat{R}_i\geq R_i-n_i$ for $1\leq i\leq 2$, or equivalently, $((R_1-n_1)^+,(R_2-n_2)^+)\in\mathcal{R}_a(\mathcal{H},\bar{\rho})$. This can be proved using Lemma 3.1 and 3.4 as follows. The proof is by contradiction. Using Lemma 3.1, we have

$$(R_1,R_2)\in\mathcal{C}(\mathcal{H},\bar{\rho}) \;\Rightarrow (R_1,R_2)\in\mathcal{R}^u(\mathcal{H},\bar{\rho}).$$

Now, denoting $\hat{R}_i=(R_i-n_i)^+$ for $i=1,2$, let us assume that $(\hat{R}_1,\hat{R}_2)\notin\mathcal{R}_a(\mathcal{H},\bar{\rho})$. This implies that one or more of the bounds of Lemma 3.4 are not satisfied by the rate pair $(\hat{R}_1,\hat{R}_2)$. Without loss of generality, we assume that $R_i\geq n_i,\ \forall\, i$ because the other case follows trivially and the 3^{rd} bound is not satisfied, i.e.,

$$\begin{aligned}
(\hat{R}_1+\hat{R}_2) =&(R_1+R_2-(n_1+n_2)),\\
&>\log\det\left(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger}+\rho_{22}H_{22}H_{22}^{\dagger}\right)\\
&\quad+\log\det\left(I_{N_1}+\rho_{11}H_{11}K_1H_{11}^{\dagger}\right)-(n_1+n_2);\\
\Rightarrow\quad (R_1+R_2) &>\log\det\left(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger}+\rho_{22}H_{22}H_{22}^{\dagger}\right)\\
&\quad+\log\det\left(I_{N_1}+\rho_{11}H_{11}K_1H_{11}^{\dagger}\right).
\end{aligned}$$

However, this suggest that $(R_1,R_2)\notin\mathcal{R}^u(\mathcal{H},\bar{\rho})$, which clearly is a contradiction.

3.3.3 An explicit coding scheme

In this subsection we shall show that if the powers (covariance matrices for the corresponding codewords) for the private and public messages of each user are chosen carefully, then the 2-nd and 4-th constraints of Lemma 3.3 can be made redundant. To prove this in what follows, we shall first find the scenarios under which these bounds (bounds of equation (3.25b) or (3.25d)) can be tighter than the corresponding bounds of equations (3.25a) or (3.25c). Subsequently, it will be shown that by carefully choosing the covariance matrices for the private and public messages of each user, it is possible to ensure that such scenarios never arise. Whence we get a rate region, in which the rate tuples are constrained by only equations (3.25a), (3.25c) and (3.25e)-(3.25i), is achievable (e.g., see Lemma 3.5). It will be also shown that this new rate region contains a subset which is within n_i^* number of bits to the capacity region of the channel.

Consider a rate tuple (R_1, R_2), where R_i's satisfy all the constraints of equation (3.25) but (3.25b), i.e.,

$$I(X_1^g, Y_1^g|W_1^g, W_2^g) + I(W_1^g; Y_2^g|X_2^g) \; < R_1 \leq I(X_1^g, Y_1^g|W_2^g). \tag{3.47}$$

The maximum value of R_1 in such a rate tuple is restricted only by the bound in equation (3.25b). However, comparing the two sides of equation (3.47) and (3.25b)) we see that the first term on the left hand side of equation (3.47) differs from that on the right hand side only due to the extra W_1^g in the conditioning. If all of the power is allocated to the private message only (i.e., $X_1^g = U_1^g$ and $W_1^g = \phi$) then the first term on the left hand side alone is equal to the right hand side and equation (3.47) can not be true. So, some fraction of the total power available at Tx_1 is being used to send W_1^g which decreases the term $I(X_1^g, Y_1^g|W_1^g, W_2^g)$. However, this decrease is much more that the corresponding increase in the second term on the left hand side (which also imply that the cross link from Tx_1 to Rx_2 is relatively *weaker* than the direct link).

The main flaw of the encoding technique in the above scenario is that a significant fraction of energy is spent to send some common information (W_1^g) through a **weak** channel to a receiver (Rx_2) where the message is not even desirable. Intuitively it seems, instead of wasting a lot of energy on a **weak** channel it will be better if Tx_1 chose $K_{1w} = \mathbf{0}$ and assigns all its energy to the private message. As mentioned earlier, if we put $X_1^g = U_1^g$ and $W_1^g = 0$ in equation (3.47), the strict inequality becomes equality, i.e.,

$$I(X_1^g, Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g) = I(U_1^g, Y_1^g | W_2^g) = I(X_1^g; Y_1^g | W_2^g), \tag{3.48}$$

and the two bounds in equation (3.25a) and (3.25b) becomes identical. With such a power split it might turn out that the rate pair (R_1, R_2) is actually achievable.

Example 3.3 (A case with no common message) Consider the 2-user Gaussian $(2, 3, 2, 2)$ IC of Example 3.2. Computing the right hand sides of the bounds in equations (3.25a) and (3.25b) for this channel we get

$$I(X_1^g, Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g) = 9.6572 \ < I(X_1^g, Y_1^g | W_2^g) = 11.8524.$$

In Fig. 3.4 the dotted line represents the rate region achievable by the simple HK scheme of the previous section and the solid line represents the rate region achievable by the simple HK coding scheme when Tx_1 uses all its power to send the private message only, i.e., $K_{1w} = \mathbf{0}$. This figure illustrates that, it is indeed possible to achieve a rate tuple outside the rate region $\mathcal{R}_{\mathrm{HK}}^{G_e}(P_s)$. For example, on the particular channel of Fig. 3.4 the point A, is achievable by the coding scheme $\mathcal{HK}(\{\frac{1}{M_1} I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\})$, which is not achievable by the simple $\mathcal{HK}^{(s)}$ scheme.

Remark 3.6 Fig. 3.4 points out another salient but important point regarding the usage of full power for the private message only. Note that the achievable rate region of the $\mathcal{HK}(\{\frac{1}{M_1} I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\})$ (the region marked by the solid line, in the figure) is not strictly larger than $\mathcal{R}_{\mathrm{HK}}^{G_e}(P_s)$. For instance point B, in Fig. 3.4 can not be achieved by $\mathcal{HK}(\{\frac{1}{M_1} I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\})$, while it is achievable by the simple HK scheme of the previous subsection. Thus, it is not helpful to set $W_1^g = \mathbf{0}$, whenever (3.25b) is tighter than (3.25a).

The above discussion motivates the following channel dependent covariance splitting[3] strategy for the private and public messages of each user.

[3] In what follows, we shall use "power splitting strategy" and "covariance splitting strategy" synonymously.

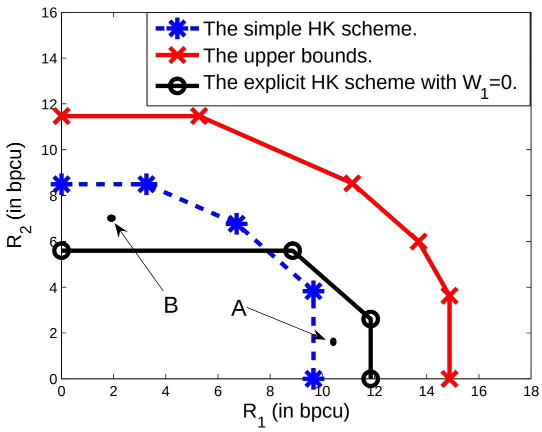


Figure 3.4: Comparison of the achievable rate regions of the simple HK scheme and the HK scheme with no public message for the first user.

Definition 3.3 (A coding scheme with rate dependent power split) Let us consider the set of rate tuples (R_1, R_2), where the R_i's satisfy constraints (3.25a), (3.25c) and (3.25e)-(3.25i) and denote it by $\mathcal{R}_2$. Now, consider a slightly modified version of the simple HK coding scheme of definition 3.2, where each of the transmitters choose the power of their private and public messages depending on the rate tuple to be achieved in the following manner:

(1) **For** $(R_1, R_2) \in \mathcal{R}_2$ **such that** R_i**'s violate constraint** (3.25b): Tx_1 assigns all its available energy to its private message only, i.e., the coding scheme $\mathcal{HK}(\{\frac{1}{M_1}I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\}) \triangleq \mathcal{HK}^{(s_1)}$ is used.

(2) **For** $(R_1, R_2) \in \mathcal{R}_2$ **such that** R_i**'s violate constraint** (3.25d): Tx_2 assigns all its available energy to its private message only, i.e., the coding scheme $\mathcal{HK}(\{K_{1u}, K_{1w}, \frac{1}{M_2}I_{M_2}, \mathbf{0}\}) \triangleq \mathcal{HK}^{(s_2)}$ is used.

(3) **For** $(R_1, R_2) \in \mathcal{R}^{G_e}_{\mathbf{HK}}(P_s)$**, i.e.,** R_i**'s violate neither** (3.25b) **nor** (3.25d)[4] : the simple HK coding scheme $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ is used, where K_{iu} and K_{iw} are chosen according equation (3.23) and (3.24) for both $i = 1, 2$.

An HK coding scheme, which use mutually independent Gaussian codewords to encode the private (U_i) and public (W_i) messages of each user, where the covariance matrices for the different messages are chosen as described above, will be referred to as the explicit HK coding scheme and will be denoted by $\widetilde{\mathcal{HK}}$.

Remark 3.7 Note that, the use of $\mathcal{HK}^{(s_1)}$ or $\mathcal{HK}^{(s_2)}$ in $\widetilde{\mathcal{HK}}$ scheme depends on the channel matrices and is not necessary always. For example, when the channel matrices are such that equation (3.25a) and (3.25c) are tighter than (3.25b) and (3.25d), respectively then the simple HK scheme alone is sufficient to achieve $\mathcal{R}_2$.

[4] It will be shown in the proof of Lemma 3.5 that there does not exist any rate tuple $(R_1, R_2) \in \mathcal{R}_2$ which violates both (3.25b) and (3.25d) simultaneously.

Lemma 3.5 The explicit HK coding scheme $\widetilde{\mathcal{HK}}$ employed on $\mathcal{IC}(\mathcal{H}, \bar{\rho})$, contain the rate region $\mathcal{R}_2$, where $\mathcal{R}_2$ is a set of rate tuples (R_1, R_2), which satisfy the following constraints

$$R_1 \leq I(X_1^g; Y_1^g | W_2^g); \tag{3.49a}$$

$$R_2 \leq I(X_2^g; Y_2^g | W_1^g); \tag{3.49b}$$

$$R_1 + R_2 \leq I(X_2^g, W_1^g; Y_2^g) + I(X_1^g; Y_1^g | W_1^g, W_2^g); \tag{3.49c}$$

$$R_1 + R_2 \leq I(X_1^g, W_2^g; Y_1^g) + I(X_2^g; Y_2^g | W_1^g, W_2^g); \tag{3.49d}$$

$$R_1 + R_2 \leq I(X_1^g, W_2^g; Y_1^g | W_1^g) + I(X_2^g, W_1^g; Y_2^g | W_2^g); \tag{3.49e}$$

$$2R_1 + R_2 \leq I(X_1^g, W_2^g; Y_1^g) + I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(X_2^g, W_1^g; Y_2^g | W_2^g); \tag{3.49f}$$

$$R_1 + 2R_2 \leq I(X_2^g, W_1^g; Y_2^g) + I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(X_1^g, W_2^g; Y_1^g | W_1^g), \tag{3.49g}$$

where $I(.; .|.)$'s are given by equation (3.27)-(3.33).

Remark 3.8 Note that the set of bounds in the above Lemma are the same as in Lemma 3.3 after removing bounds (3.25b) and (3.25d). This rate region can be achieved by the explicit HK scheme where the common message of each user is always decoded at the unintended receiver. In other words, using the simple rate dependent power splitting strategy the entire rate region $\mathcal{R}_2$ can be achieved without considering the so called **don't care conditions**. However, it should be emphasized at this point that even considering the **don't care conditions**[5] the equivalent rate region is not same as $\mathcal{R}_2$. This region for the DMIC was recently computed in [31] (e.g., see Theorem D), and was shown that the equivalent rate region contains two extra **non-redundant** bounds in addition to those in equation (3.49).

Proof 3.5 (Proof of Lemma 3.5 (Outline)) We know that every point in $\mathcal{R}_{\text{HK}}^{G_e}(P_s)$ is achievable by the $\mathcal{HK}^{(s)}$ scheme. Using this result we prove the Lemma by showing that every rate tuple that lie in $\mathcal{R}_2$ bun not in $\mathcal{R}_{\text{HK}}^{G_e}(P_s)$ can be achieved by the modified or the *explicit* coding scheme $\widetilde{\mathcal{HK}}$ of definition 3.3.

[5] That is, removing the constraints (3.26b),(3.26c) and (3.26f) from the expressions of $\mathcal{R}_{\text{HK}}^{(G_o,i)}$, for $i = 1, 2$.

It can be easily seen that, when $(R_1, R_2) \in \mathcal{R}_2$ but $(R_1, R_2) \notin \mathcal{R}_{\mathrm{HK}}^{G_e}(P_s)$ both of (3.25a) and (3.25d) can not be violated simultaneously, because that would imply that $(R_1 + R_2)$ violates equation (3.49e) contradicting the assumption that $(R_1, R_2) \in \mathcal{R}_2$. On the other hand, in Appendix 3.5 it will be shown that when (3.25b) is violated for a given $(R_1, R_2) \in \mathcal{R}_2$, the $\mathcal{HK}(\{\frac{1}{M_1} I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\})$ scheme, which assigns all the power of user one to its private message, can achieve the rate tuple and when (3.25d) is violated, the HK scheme of definition 3.2 with $K_{2w} = \mathbf{0}$ and $K_{2u} = \frac{1}{M_2} I_{M_2}$ can achieve the rate tuple. This however, is exactly the coding technique used by the coding scheme of Definition 3.3. Therefore, the $\widetilde{\mathcal{HK}}$ scheme can achieve any rate point in $\mathcal{R}_2$.

It is clear that, depending on the rate tuple to be achieved, the explicit HK scheme use one of the three simple HK coding schemes specified in definition 3.3. The corresponding input distribution when Tx_i spends all its power to send the private message only, was denoted by P_{s_i} (e.g., see Appendix A.6) in the proof of Lemma 3.5, where $P_{s_i}(.) \in \mathcal{P}^*$, for $i = 1, 2$. The achievable rate region of the simple HK scheme with input distribution P_{s_i} is given by $\mathcal{R}_{\mathrm{HK}}^{G_e}(P_{s_i})$, where $\mathcal{R}_{\mathrm{HK}}^{G_e}(P_{s_i})$ can be computed as in Lemma 3.3. As was argued earlier, to achieve a point in this rate region, it is important to chose the sub-rates carefully. In particular, to achieve any rate tuple $(R_1, R_2) \in \mathcal{R}_{\mathrm{HK}}^{G_e}(P_{s_i})$, the corresponding sub-rate for the private and public messages can be chosen from $\mathcal{R}_{\mathrm{HK}}^{G_o}(P_{s_i})$ since $\mathcal{R}_{\mathrm{HK}}^{G_e}(P_{s_i}) = \Pi\left(\mathcal{R}_{\mathrm{HK}}^{G_o}(P_{s_i})\right)$ by Lemma 3.3. This suggest the following rate splitting strategy for the **explicit** HK scheme, $\widetilde{\mathcal{HK}}$.

Rate splitting strategy for the explicit HK coding scheme: Depending on the rate tuple to be achieved when the input distribution of the coding is P, where $P \in \{P_s, P_{s_1}, P_{s_2}\}$, the sub-rates for the different private and public messages are chosen from $\mathcal{R}_{\mathrm{HK}}^{G_o}(P)$, where $\mathcal{R}_{\mathrm{HK}}^{G_o}(P)$ can be computed from equation (3.26) by using distribution P in place of P_s.

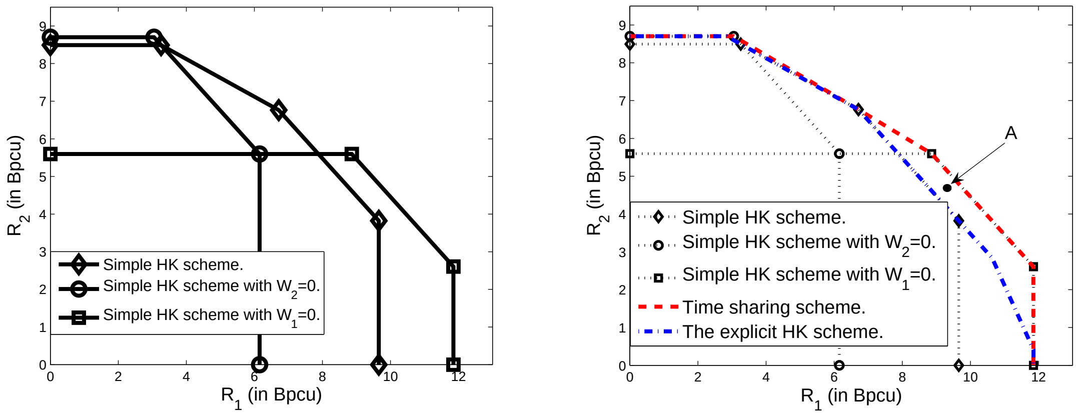
Remark 3.9 Since the sub-rates are chosen from $\mathcal{R}_{\mathrm{HK}}^{G_o}(P)$, when Tx_i spends all its power to send the private message only, i.e., $W_i^g = \phi$, it is expected that $\mathcal{R}_{\mathrm{HK}}^{G_o}(P)$ should not allow any positive rate for the common message. Putting $W_i^g = \phi$ in equation (3.26b) it can be easily seen that it is indeed the case, i.e., $T_i \leq 0$.

Remark 3.10 It is worth pointing out some of the differences between the explicit coding scheme of this chapter and that in [6], where a similar coding scheme was used to characterize the capacity region of the SISO IC within one bit. The authors in [6] use a superposition coding scheme where each users private and public messages are encoded using independent Gaussian random codewords with covariances P_{iu} and $(P_i - P_{iu})$, respectively for $i = 1, 2$. Here, P_i is the total average power of Tx_i and P_{iu} depends on the cross channel coefficients as follows (see equation (57) and (58) of [6])

$$P_{iu} = \min\{P_i, \frac{1}{\|H_{ij}\|^2}\}, \; i \neq j \in \{1, 2\}. \tag{3.50}$$

In the notation of the present chapter this coding scheme is identical to $\mathcal{HK}(\{\frac{P_{1u}}{P_1}, \left(1 - \frac{P_{1u}}{P_1}\right), \frac{P_{2u}}{P_2}, \left(1 - \frac{P_{1u}}{P_1}\right)\})$, when $P_i \geq \frac{1}{\|H_{ij}\|^2}\}$. On the other hand, it is identical to $\mathcal{HK}(\{1, 0, \frac{P_{2u}}{P_2}, \left(1 - \frac{P_{1u}}{P_1}\right)\})$, when only $P_1 < \frac{1}{\|H_{12}\|^2}\}$, $\mathcal{HK}(\{\frac{P_{1u}}{P_1}, \left(1 - \frac{P_{1u}}{P_1}\right), 1, 0\})$, when only $P_2 < \frac{1}{\|H_{21}\|^2}\}$ and it is identical to $\mathcal{HK}(\{1, 0, 1, 0\})$, when $P_i \leq \frac{1}{\|H_{ij}\|^2}\}$ for both $i = 1, 2$. So, depending on the channel coefficients the coding scheme is equivalent to one of the four schemes just described. However, for a given channel the coding scheme and power allocation of [6] is fixed and does not changes with the rate tuple to be achieved whereas the explicit coding scheme of this chapter utilizes one of the three different power splitting schemes depending on the rate tuple to be achieved. On the other hand, in contrast to [6], the $\widetilde{\mathcal{HK}}$ scheme explicitly specify the sub-rates of the different messages for each rate tuple to be achieved in $\mathcal{R}_2$.

Remark 3.11 Note the subtle difference between the $\widetilde{\mathcal{HK}}$ scheme and a coding scheme which time shares between the three simple HK schemes of definition 3.3. In general, the latter can achieve a larger rate region. Such is the case because the $\widetilde{\mathcal{HK}}$ scheme does not use $\mathcal{HK}(\{\frac{1}{M_1}I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\})$ or $\mathcal{HK}(\{K_{1u}, K_{1w}, \frac{1}{M_2}I_{M_2}, \mathbf{0}\})$ to achieve any point that is not inside $\mathcal{R}_2$. Whereas, both of $\mathcal{HK}(\{\frac{1}{M_1}I_{M_1}, \mathbf{0}, \; K_{2u}, K_{2w}\})$ and $\mathcal{HK}(\{K_{1u}, K_{1w}, \frac{1}{M_2}I_{M_2}, \mathbf{0}\})$ may achieve points which does not lie in $\mathcal{R}_2$ but clearly achievable by time sharing scheme. Figure 3.5 illustrates this point by an example, where Fig. 3.5(a) depicts the achievable rate regions of the three simple HK schemes for the channel of Example 3.2 and in Fig. 3.5(b), the rate region bounded by the dotted and dashed line represents the achievable region of the explicit HK scheme, $\mathcal{R}_2$ and the dashed line represents the rate region $\mathcal{R}_{TS}$, achievable by time sharing. Point A in the latter represents a rate tuple which lies in the achievable region of $\mathcal{HK}(\{\frac{1}{M_1}I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\})$ and hence also lie in $\mathcal{R}_{TS}$ but it is clearly outside $\mathcal{R}_2$ and hence not achievable by $\widetilde{\mathcal{HK}}$.



(a) Achievable rate regions of the component schemes.

(b) Rate regions of the explicit scheme and time sharing.

Figure 3.5: Comparison of the achievable rate regions of the explicit scheme and the region achievable by time sharing among the component schemes, on the channel of Ex. 3.2.

From the fact that, the rate region $\mathcal{R}_2$ in general is larger than $\mathcal{R}_{\text{HK}}^{G_e}(P_s)$, it might be expected that $\mathcal{R}_2$ contains a subset, whose boundary is also at most a constant number of bits away from the set of upper bounds in Lemma 3.1. However, this constant now might be smaller than n_i. Indeed, the following Lemma provides such a subset of rate tuples.

Lemma 3.6 Let $\mathcal{R}_a^*(\mathcal{H}, \bar{\rho})$ be a set of non-negative rate tuples (R_1, R_2) which satisfy the following constraints:

$$\begin{aligned}
R_1 &\le \log\det\Big(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger}\Big) - n_1^*;\\
R_2 &\le \log\det\Big(I_{N_2} + \rho_{22} H_{22} H_{22}^{\dagger}\Big) - n_2^*;\\
R_1 + R_2 &\le \log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} H_{22}^{\dagger}\Big)\\
&\qquad + \log\det\Big(I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\Big) - (n_1^* + n_2^*);\\
R_1 + R_2 &\le \log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} H_{11}^{\dagger}\Big)\\
&\qquad + \log\det\Big(I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\Big) - (n_1^* + n_2^*);\\
R_1 + R_2 &\le \log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\Big)\\
&\quad + \log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\Big) - (n_1^* + n_2^*);\\
2R_1 + R_2 &\le \log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} H_{11}^{\dagger}\Big) + \log\det\Big(I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\Big)\\
&\quad + \log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\Big) - (2n_1^* + n_2^*);\\
R_1 + 2R_2 &\le \log\det\Big(I_{N_2} + \rho_{12} H_{12} H_{12}^{\dagger} + \rho_{22} H_{22} H_{22}^{\dagger}\Big) + \log\det\Big(I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^{\dagger}\Big)\\
&\quad + \log\det\Big(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger} + \rho_{11} H_{11} K_1 H_{11}^{\dagger}\Big) - (n_1^* + 2n_2^*),
\end{aligned}$$

where n_i^* given by equation (3.5). Then, $\mathcal{R}_a^*(\mathcal{H}, \bar{\rho})$ is an achievable rate region on $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ and is achievable by the explicit HK coding scheme $\widetilde{\mathcal{HK}}$, i.e.,

$$\mathcal{R}_a^*(\mathcal{H}, \bar{\rho}) \subseteq \mathcal{R}_2.$$

Proof 3.6 The proof follows trivially from the proof of Lemma 3.2, in particular, from equation (A.27).

Theorem 3.3 The achievable rate region $\mathcal{R}_a^*(\mathcal{H}, \bar{\rho})$, given by Lemma 3.6, is within n_i^* bits to the capacity region of the Gaussian MIMO IC, where n_i^* is given by equation (3.5).

Proof 3.7 The proof is identical to that of Theorem 3.2.

Corollary 3.1 The achievable rate region $\mathcal{R}_a^*(\mathcal{H}, \bar{\rho})$ is within n_i^* bits to $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$.

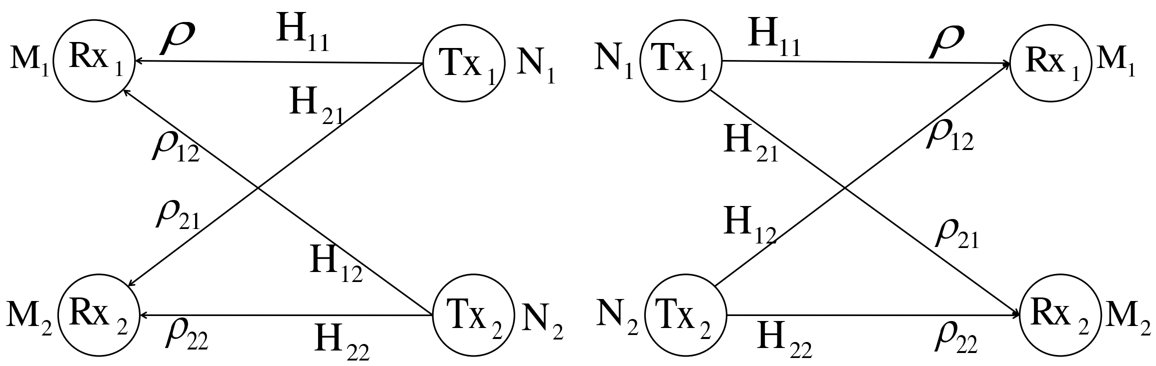
Proof 3.8 Follows from the proof of Theorem 3.2, with $\mathcal{R}_a(\mathcal{H}, \bar{\rho})$ replace by $\mathcal{R}_a^*(\mathcal{H}, \bar{\rho})$ and n_i by n_i^*.

Example 3.4 Capacity of the SIMO IC within 1 bit: On a $(1, N_1, 1, N_2)$ IC, $n_1^* = n_2^* = 1$, thus the explicit $\mathcal{HK}^{(3)}$ scheme can achieve a rate region which is within 1 bit of the capacity region for any SNRs, INRs and the channel vectors. This result is different from that reported in [24] where the exact sum capacity of the **strong** SIMO IC with $\|H_{ii}\|^2 \leq \|H_{ij}\|^2$ for $1 \leq i \neq j \leq 2$, was characterized. While [24] provides the exact sum capacity for the strong SIMO IC, our 1 bit approximation is valid for all channel coefficients. Further, this approximation is tighter than that reported in [27] and [28], where the capacity approximation within N_i bits was proved.

Remark 3.12 Although the above approximate characterization is not always better than that reported in [27], it provides a closer approximation for a large class of interference channels. In particular, for all the interference channels on which $n_i^* < N_i$ the explicit $\mathcal{HK}^{(3)}$ scheme provides a tighter approximation. Among the other interesting aspects of the approximate characterization of this section are (a) we have a set of explicit expressions for the achievable region and upper bounds to the capacity region, which for instance, can be used for a further analysis such as the evaluation of the generalized degrees of freedom (which will be carried out in chapter 4) and the diversity-multiplexing tradeoff (DMT) analysis obtained by the authors in [33] and [34]; and (b) an explicit coding scheme, involving just three superposition strategies was shown to be approximate capacity optimal in contrast to the result of [27] where no light is shed on what simple or explicit scheme, if any, out of all possible input distributions and all possible time sharing schemes, would be approximate capacity optimal.

3.3.4 Reciprocity of the approximate capacity region

For a communication channel with an unequal number of antennas at the source and destination nodes, how does the capacity (or any other performance metric) change if the information flows in the opposite direction (i.e., the roles of the transmitters and the receivers are interchanged)? The property of maintaining the same performance even if the direction of flow of information is reversed is widely known as the *reciprocity* of the channel. For instance, the following reciprocity of the point-to-point MIMO channel was proved in [5]: the capacity of a MIMO point-to-point channel is unchanged when the roles of the transmitters and receivers are interchanged provided the power constraint is appropriately scaled. In [35] the degrees of freedom (DoF) region of a (M_1, N_1, M_2, N_2) MIMO IC was shown to be the same as that of a (N_1, M_1, N_2, M_2) IC. In this section, we prove a reciprocity result for the (M_1, N_1, M_2, N_2) MIMO IC by showing that reciprocity actually holds in the much stronger constant-gap-to-capacity sense.



(a) Information flowing in the reverse direction in the (M_1, N_1, M_2, N_2) IC.

(b) Equivalent channel, information flows in the forward direction.

Figure 3.6: Information flowing in the reverse direction on an 2-user MIMO IC and its corresponding forward information flow model.

Fig. 3.6(a) illustrates an (M_1, N_1, M_2, N_2) MIMO IC with channel parameters H and $\bar{\rho}$ with roles of the transmitters and receivers interchanged so that information flows in the reverse direction. Fig. 3.6(b) shows its equivalent model where the information flows in the forward direction. Clearly, the capacity of the reverse channel is the same as that of $\mathcal{IC}\left(\mathcal{H}^r, \bar{\rho}^r\right)$ where $\mathcal{H}^r = \{H_{11}^T, H_{21}^T, H_{12}^T, H_{22}^T\}$ and $\bar{\rho}^r = [\rho, \rho_{21}, \rho_{12}, \rho_{22}]$. The capacity region of the reverse channel is denoted as $\mathcal{C}\left(\mathcal{H}^r, \bar{\rho}^r\right)$.

Let us define the counterparts in the reverse channel of the capacity gap parameters of the forward channel in (3.5) as

$$m_i^* \triangleq \min\{M_i, N_s\} \log(N_x) + \tilde{m}_{ij}, \qquad 1 \leq i \neq j \leq 2, \tag{3.51}$$

where $N_s = (N_1 + N_2)$, $N_x = \max\{N_1, N_2\}$ and $\tilde{m}_{ij} = m_{ij} \log\left(\frac{(N_j+1)}{N_j}\right)$ for $1 \leq i \neq j \leq 2$.

To prove the reciprocity in the constant gap to capacity sense, the capacity regions of $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ and $\mathcal{IC}(\mathcal{H}^r, \bar{\rho}^r)$ must be shown to be within a constant number of bits to each other. We start with a result on the outer bounds.

Lemma 3.7 The outer bound $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$ from Lemma 3.1 of the forward channel $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ and the outer bound $\mathcal{R}^u(\mathcal{H}^r, \bar{\rho}^r)$ (obtained in the same way as in Lemma 3.1 but for the reverse channel $\mathcal{IC}(\mathcal{H}^r, \bar{\rho}^r)$) define the same set of rate tuples, i.e.,

$$\mathcal{R}^u(\mathcal{H}, \bar{\rho}) = \mathcal{R}^u(\mathcal{H}^r, \bar{\rho}^r). \tag{3.52}$$

Proof 3.9 The proof is given in Appendix A.7.

Corollary 3.1 proves that the explicit HK scheme, $\mathcal{HK}^{(3)}$, achieves a rate region on $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ which is within n_i^* bits to a set of rate tuple $\mathcal{R}^u(\mathcal{H}, \bar{\rho})$ which contains its capacity region. Clearly, the counterpart of this explicit HK coding scheme for the reverse channel (with suitable changes in the channel matrices, INRs and the number of antennas) can achieve a rate region on $\mathcal{IC}(\mathcal{H}^r, \bar{\rho}^r)$ which is within m_i^* bits to $\mathcal{R}^u(\mathcal{H}^r, \bar{\rho}^r)$, where m_i^* is given by equation (3.51). However, from Lemma 3.7 we know that

$$\mathcal{R}^u(\mathcal{H}, \bar{\rho}) = \mathcal{R}^u(\mathcal{H}^r, \bar{\rho}^r).$$

Thus the capacity regions of the two interference channels can not differ by more than $\max\{m_i^*, n_i^*\}$ bits proving the following theorem.

Theorem 3.4 The capacity regions of $\mathcal{IC}(\mathcal{H}, \bar{\rho})$ and $\mathcal{IC}(H^r, \bar{\rho}^r)$ are within $\max\{m_i^*, n_i^*\}$ bits to each other, i.e., if $(R_1, R_2) \in \mathcal{C}(\mathcal{H}, \bar{\rho})$, then there exists a rate tuple $(R_1^r, R_2^r) \in \mathcal{C}(\mathcal{H}^r, \bar{\rho}^r)$, the capacity region of the reverse channel, such that

$$|(R_i - R_i^r)| \leq \max\{m_i^*, n_i^*\}, \ \forall\ 1 \leq i \leq 2.$$

Proof 3.10 Let $(R_1, R_2) \in \mathcal{C}(\mathcal{H}, \bar{\rho})$. From Corollary 3.1, there exist a rate tuple $(\hat{R}_1, \hat{R}_2) \in \mathcal{R}^u(\mathcal{H}, \bar{\rho})$ such that

$$0 \leq (\hat{R}_i - R_i) \leq n_i^*, \ \forall\ 1 \leq i \leq 2. \tag{3.53}$$

Further, from Lemma 3.7 we have $(\hat{R}_1, \hat{R}_2) \in \mathcal{R}^u(\mathcal{H}^r, \bar{\rho}^r)$. Next, applying Corollary 3.1 for $\mathcal{IC}(\mathcal{H}^r, \bar{\rho}^r)$, we have that there exists a rate tuple $(R_1^r, R_2^r) \in \mathcal{C}(\mathcal{H}^r, \bar{\rho}^r)$ such that

$$0 \leq (\hat{R}_i - R_i^r) \leq m_i^*, \ \forall\ 1 \leq i \leq 2. \tag{3.54}$$

Note that equations (3.53) and (3.54) provide ranges of R_i and R_i^r and the magnitude of the difference between them is maximum when one takes its largest value and the other its smallest, i.e.,

$$|(R_i - R_i^r)|_{max} = m_i^* \text{ or } n_i^*,$$

which proves the theorem.

Remark 3.13 Note that reciprocity holds for the 2-user MIMO IC without power scaling and this may seem counter-intuitive given the point-to-point MIMO channel result of [5]. The reason is that reciprocity was shown here in the approximate capacity sense. The difference due to not scaling power gets absorbed in the gap that already exists between the exact capacity and the achievable region.

3.4 Conclusion

An approximate capacity region of the 2-user MIMO IC with an arbitrary number of antennas at each node is characterized. It is shown that a simple (or universal) and an explicit HK coding schemes which inherently perform a form of joint interference alignment in the signal space and in the signal level can achieve the capacity region within a constant gap. For a class of ICs, this gap is the tightest approximation to the capacity region of the MIMO IC found to date and this includes the SIMO ICs for which the gap is 1 bit independently of the number of antennas at the receivers.

The explicit upper and lower bounds to the capacity region are used to prove the *reciprocity* of the MIMO IC in the constant-gap-to-capacity sense.

Chapter 4

The Generalized Degrees of Freedom Region of the MIMO Interference Channel

4.1 Introduction

In this chapter we characterize the generalized degrees of freedom (GDoF) region of the 2-user MIMO IC. The so called degrees of freedom (DoF) of a communication channel characterizes the scaling factors of the rates supported by the channel at asymptotic values of the SNR and INR of the channel and therefore is a coarser metric than the capacity. However, as shall demonstrate in this chapter, the DoF metric is very useful for characterizing the performance of a channel for a large range of parameters such as the various SNRs and INRs of the channel. Since the performance of a 2-user IC serves as the upper and lower bounds to various other multi-user interference network whose capacity or approximate capacity is not available, the GDoF of the 2-user IC can serve as a performance bound to those networks. Moreover, the tools and techniques derived in this analysis can also be used to characterize the performance of other multiuser networks as well which is demonstrated by computing the GDoF region of the 2-user MIMO multiple access cannel (MAC) in subsection 4.5.4.

The GDoF region metric, as its name suggests, generalizes the notion of the conventional degrees of freedom (DoF) region metric by additionally emphasizing the signal level as a signaling dimension. It therefore characterizes the simultaneously accessible fractions of spatial and signal-level dimensions (per channel use) by the two users in the limit of high signal-to-noise ratio (SNR) while the ratios of the SNRs and INRs relative to a reference SNR, each expressed in the dB scale, are held constant, with each constant taken, in the most general case, to be arbitrary. The GDoF region was obtained for the SISO IC in [6] based on the constant gap to capacity result found therein. The symmetric GDoF, $d_{\text{sym}}(\alpha)$, which is the maximum common GDoF achievable by each of the two users, for the symmetric SISO IC with equal SNRs and equal INRs for the two users, i.e, with $\text{INR} = \text{SNR}^{\alpha}$ was evaluated in [6] to be the well-known "W" curve. The W-curve clearly delineates the very weak, weak, moderate, strong and very strong interference regimes, depending on the value of α, pointing to the optimal (upto GDoF accuracy) interference management techniques as a function of the severity or mildness of the interference.

There have been several other recent works on characterizing the GDoF of various channels. For example, in [29], the symmetric GDoF of a class of symmetric MIMO ICs – for which the SNRs at each receiver are the same and the INRs at each receiver are also the same, with $\text{INR} = \text{SNR}^{\alpha}$– and where both transmitters have M antennas and both receivers have N antennas, with the restriction $N \geq M$, was obtained and found to be a "W" curve also. In [36], the symmetric GDoF in the perfectly symmetric (with all direct links having identical gains and all cross links having identical gains) scalar K-user interference network was found (see also [37]). In [38], the symmetric GDoF was obtained for the $(N + 1)$-user symmetric SIMO IC with N antennas at each receiver and with equal direct link SNRs and equal cross link INRs. The symmetric GDoF of a symmetric model of the scalar X-channel with real-valued channel coefficients was found in [39].

In this work, we obtain the GDoF region of the general MIMO IC with an arbitrary number of antennas at each node and in the most general case where the signal-to-noise ratios (SNR) and interference-to-noise ratios (INRs) vary with arbitrary exponents to a nominal SNR. This result is made possible by the recent constant gap to capacity characterization for the general MIMO IC in [27, 40]. The GDoF result of this chapter thus generalizes the GDoF region of the SISO IC found in [6] to the MIMO IC. It also recovers the symmetric GDoF result of [29] for the class of symmetric MIMO ICs considered therein. Moreover, the single and unified constant-gap-to-capacity achievability scheme of [40] considered here, unlike that in [29], does not require the restriction on the numbers of antennas at the different nodes or on the values of SNR exponents and is GDoF optimal in the most general case. The main result of this work also recovers the conventional DoF region result obtained in [35] for the MIMO IC by setting all SNR exponents to unity. In addition to providing several insights that include whatever is common between certain symmetric (in numbers of antennas) MIMO ICs and SISO ICs and what is not, the GDoF result of this chapter gives rise to new insights into optimal signaling strategies that make jointly optimal use of the available spatial and signal level dimensions.

The single, unified achievable scheme studied in depth here that is GDoF optimal (and indeed constant-gap-to-capacity optimal [40]) is a simple Han-Kobayashi coding scheme with mutually independent Gaussian input for the private and public messages of each user without time-sharing. The private and public message can be thought of consisting of several information streams. The private information streams are either directed along the null space of the corresponding cross-link channel matrix or transmitted at power levels that ensure that they reach the unintended receiver below the noise floor. Such a scheme therefore jointly and optimally employs both **signal-level interference alignment** [41] as well as **transmit beamforming or signal-space interference alignment** [42] techniques. For a given DoF tuple in the GDoF region of the channel, we also explicitly specify the DoFs carried by the private and public messages of each user.

The rest of the chapter is organized as follows. In Section 4.2 we describe the channel model and the GDoF optimal coding scheme. Section 4.3 contains the main result of this chapter, namely, the GDoF region of the general MIMO IC. Specializations of this result to the SISO IC and to the DoF region of the MIMO IC are also given in Section 4.3 which recover the results of [6] and [35], respectively. Explicit specifications of the DoF-splitting between private and public sub-messages are obtained. The reciprocity property of the GDoF region (which denotes the invariability of GDoF with respect to direction of information flow) is described in 4.4 as are specializations of the GDoF region to obtain the symmetric GDoF of the symmetric (M, N, M, N) MIMO IC, thereby recovering as a special case the result of [29] for $M \leq N$. In Section 4.5 several novel insights revealed by the GDoF analysis are given. Section 4.6 concludes the chapter.

Notations 4.1 Let $\mathbb{C}$ and $\mathbb{R}^+$ represent the field of complex numbers and the set of non-negative real numbers, respectively. An $n \times m$ matrix with entries coming from $\mathbb{C}$ will be denoted by $A \in \mathbb{C}^{n\times m}$ and its entry in the i^{th} row and j^{th} column will be denoted by $[A]_{ij}$. We shall denote the transpose and the conjugate transpose of the matrix A by A^T and $A^\dagger$ respectively. I_n represents the $n\times n$ identity matrix, $0_{m\times n}$ represents an all zero $m\times n$ matrix and $\mathbb{U}^{n\times n}$ represents the set of $n\times n$ unitary matrices. The k^{th} column (row) of the matrix A will be denoted by $A^{[k]}$ ($A^{(k)}$) whereas $A^{[k_1:k_2]}$ ($A^{(k_1:k_2)}$) will represent a matrix whose columns (rows) are same as the k_1^{th} to k_2^{th} columns (rows) of matrix A. If $x^{(k)} \in \mathbb{C}, \forall\ 1 \leq k \leq n$, then $\mathbf{x} \triangleq [x^{(1)}, x^{(2)}, \cdots, x^{(n)}]^T$. $\{A, B, C, D\}$ will represent an ordered set of matrices. $I(x; y)$ and $I(x; y|z)$ will represent the mutual information and conditional mutual information of the arguments, respectively. $(x \wedge y)$, $(x \vee y)$ and $(x)^+$ represents the minimum and maximum between x and y and maximum between x and 0, respectively. We also use Landau notations for error terms in approximations. $o(1)$ denotes a term which goes to zero asymptotically and $\mathcal{O}(1)$ denotes a term which is bounded above by some constant. We say x is of the order of y if $\lim_{y\to\infty} \frac{x}{y} = 0$. All the logarithms in this chapter are with base 2. We denote the distribution of a complex circularly symmetric Gaussian random vector with zero mean and covariance matrix Q, by $\mathcal{CN}(0, Q)$. Finally, the indicator function 1(S) is defined as follows

$$1(\mathrm{S}) = \begin{cases} 1, & \text{if S is true;} \\ 0, & \text{if S is false.} \end{cases}$$

4.2 Channel Model and preliminaries

In this section, we specify the particulars of the channel matrices of the two-user MIMO IC, a high SNR interpretation of the achievable scheme of [40] and then give asymptotic (high SNR) approximations upto $O(1)$ of key quantities that arise in the bounds on capacity region. These approximations are used later to derive the main result of this chapter on the GDoF region of the MIMO IC.

For the high SNR analysis of this chapter we assume that the entries of the channel matrices (e.g., see figure 2.1), H_{ij} are drawn i.i.d. from a continuous and unitarily invariant [7] distribution, i.e., $UH_{ij}V$ is identically distributed to H_{ij} for any $U \in \mathbb{U}^{N_j \times N_j}$ and $V \in \mathbb{U}^{M_i \times M_i}$ which ensures that the channel matrices are full rank with probability one (w.p.1). This class of distributions will be denoted by $\mathcal{P}$ in the rest of the chapter.

The performance on the MIMO IC should depend on the strength of the interference relative to the desired signal level on the dB scale with, for example, a better DoF performance expected when interference strength is much less or much higher than the signal strength as in the SISO IC [6]. This variation of performance due to relative difference in strengths of SNRs and INRs can not be captured (at high SNR) by a DoF analysis alone, i.e., if they differ say by only a constant. To characterize the DoF region under such a scenario we thus let the SNRs and INRs vary exponentially with respect to a nominal SNR, ρ, with different scaling factors as follows:

$$\lim_{\log(\rho)\to\infty} \frac{\log(\rho_{ij})}{\log(\rho)} = \alpha_{ij}, \text{ where } \alpha_{ij} \in \mathbb{R}^+ \text{ and } i, j \in \{1, 2\}. \tag{4.1}$$

We assume $\rho_{11} = \rho$ or $\alpha_{11} = 1$, without loss of generality and in the rest of the chapter $\bar{\rho}$ and $\bar{\alpha}$ will be used interchangeably to indicate the power levels of different links of the channel, i.e., the capacity of the channel will also be denoted by $\mathcal{IC}(\mathcal{H}, \bar{\alpha})$, where $\bar{\alpha} = [\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}]$. As mentioned earlier, this technique of varying different SNRs and INRs was first introduced in [6] to characterize the DoF region of the SISO 2-user IC and the corresponding DoF region was called the **generalized** DoF (GDoF) region. In the following subsection we shall provide a formal definition of the GDoF region of the channel.

4.2.1 Generalized Degrees of Freedom Region

Definition 4.1 The GDoF region, $\mathcal{D}_o(\bar{M}, \bar{\alpha})$, of $\mathcal{IC}(\mathcal{H}, \bar{\alpha})$ is defined as

$$\mathcal{D}_o(\bar{M}, \bar{\alpha}) = \left\{(d_1, d_2) : \; d_i = \lim_{\rho_{ii}\to\infty} \frac{R_i}{\log(\rho_{ii})}, i \in \{1, 2\} \text{ such that } (R_1, R_2) \in \mathcal{C}(\mathcal{H}, \bar{\alpha})\right\}. \tag{4.2}$$

Since the capacity region of a MIMO IC is not known and a constant number of bits is insignificant in the GDoF analysis, to derive the GDoF region we shall use the constant-gap-to-capacity result found by the authors in [40]. In particular, since a constant number of bits is insignificant in the GDoF analysis, the $\mathcal{C}(\mathcal{H}, \bar{\alpha})$ in the definition of the GDoF region can be replaced by either $\mathcal{R}^u(\mathcal{H}, \bar{\alpha})$ or $\mathcal{R}_a(\mathcal{H}, \bar{\alpha})$ to compute the GDoF region of the MIMO IC. We state this fact as a lemma for easy further reference.

Lemma 4.1 The GDoF region of the MIMO IC is given as

$$\mathcal{D}_o(\bar{M}, \bar{\alpha}) = \left\{(d_1, d_2) : \; d_i = \lim_{\rho_{ii} \to \infty} \frac{R_i}{\log(\rho_{ii})}, i \in \{1, 2\} \text{ and } (R_1, R_2) \in \mathcal{R}^u(\mathcal{H}, \bar{\alpha})\right\}, \tag{4.3}$$

where $\mathcal{R}^u(\mathcal{H}, \bar{\alpha}) = \mathcal{R}^u(\mathcal{H}, \bar{\rho})$ is given by Lemma 3.1.

Proof 4.1 From Lemma 3.1 and 3.4 we have the following

$$\mathcal{R}_a(\mathcal{H}, \bar{\alpha}) \quad \subseteq \mathcal{C}(\mathcal{H}, \bar{\alpha}) \quad \subseteq \mathcal{R}^u(\mathcal{H}, \bar{\alpha}). \tag{4.4}$$

To obtain the desired result we use in the definition of the GDoF region of (4.2), the above set inclusions along with the fact that n_i's in Lemma 3.4 are independent of ρ and $\mathcal{H}$.

4.2.2 Asymptotic Approximations

In the derivation of the GDoF region of the 2-user MIMO IC quantities like the sum rate upper bound on 2- and 3-user MIMO multiple-access channels (MACs) will appear frequently. Thus, in the following two lemmas, we provide asymptotic approximations up to $O(1)$ of such quantities for different $\bar{\alpha}$ and number of antennas.

Lemma 4.2 Let $H_1 \in \mathbb{C}^{u \times u_1}$ and $H_2 \in \mathbb{C}^{u \times u_2}$ are two full rank (w.p.1) channel matrices such that $H = [H_1 \; H_2]$ is also full rank w.p.1. Then for asymptotic ρ

$$\log\det\left(I_u + \rho^a H_1 H_1^\dagger + \rho^b H_2 H_2^\dagger\right) = f\left(u, (a, u_1), (b, u_2)\right) \log(\rho) + \mathcal{O}(1), \tag{4.5}$$

where for any $u \in \mathbb{R}^+$ and $(a_i, u_i) \in \mathbb{R}^2$ for $i \in \{1, 2\}$,

$$f\left(u, (a_1, u_1), (a_2, u_2)\right) \triangleq \begin{cases} \min\{u, u_1\} a_1^+ + \min\{(u - u_1)^+, u_2\} a_2^+, \; \text{if } a_1 \geq a_2; \\ \min\{u, u_2\} a_2^+ + \min\{(u - u_2)^+, u_1\} a_1^+, \; \text{if } a_1 < a_2. \end{cases} \tag{4.6}$$

Proof 4.2 This result was proved in [29] when $(u_1 + u_2) \geq u$. The proof for the case when $(u_1 + u_2) < u$ is given in Appendix A.8.

Remark 4.1 If $H_1 \in \mathbb{C}^{u \times u_1}$ and $H_2 \in \mathbb{C}^{u \times u_2}$ are mutually independent and $H_1, H_2 \in \mathcal{P}$ then $[H_1 H_2] \in \mathcal{P}$ and therefore is a full rank matrix w.p.1. That is, if H_1 and H_2 represent the two incoming channel matrices at any of the receivers in the MIMO IC then Lemma 4.2 holds.

Lemma 4.3 Let $H_i \in \mathbb{C}^{u \times u_i}$ for $i = 1, 2, 3$ be three channel matrices with statistics described at the beginning of this section, then for asymptotic ρ

$$\log\det\left(I_u + \sum_{i=1}^{3} \rho^{a_i} H_i H_i^{\dagger}\right) = g\left(u, (a_1, u_1), (a_2, u_2), (a_3, u_3)\right)\log(\rho) + \mathcal{O}(1), \tag{4.7}$$

where for any $u \in \mathbb{R}^{+}$ and $(a_i, u_i) \in \mathbb{R}^2$ for $i \in \{1, 2, 3\}$,

$$\begin{aligned} g\left(u, (a_1, u_1), (a_2, u_2), (a_3, u_3)\right) \triangleq \min\{u, u_{i_1}\}a_{i_1}^{+} + \min\{(u - u_{i_1})^{+}, u_{i_2}\}a_{i_2}^{+} + \\ \min\{(u - u_{i_1} - u_{i_2})^{+}, u_{i_3}\}a_{i_3}^{+}, \end{aligned} \tag{4.8}$$

for $i_1, i_2, i_3 \in \{1, 2, 3\}$ such that $a_{i_1} \geq a_{i_2} \geq a_{i_3}$.

Proof 4.3 The proof is given in Appendix A.9.

Remark 4.2 Suppose $a_1 \geq \max\{a_2, a_3\}$ in Lemma 4.3, then $g(.)$ can also be written as

$$g\left(u, (a_1, u_1), (a_2, u_2), (a_3, u_3)\right) = \min\{u, u_1\}a_1^{+} + f\left((u - u_1)^{+}, (a_2, u_2), (a_3, u_3)\right).$$

For example, $g\left(10, (.5, 3), (1, 4), (1.2, 2)\right) = 2(1.2) + f\left(8, (.5, 3), (1, 4)\right) = 2.4 + 4(1) + 3(.5) = 7.9$.

Remark 4.3 g(.) in Lemma 4.3 represents the sum DoFs achievable on a 3-user MIMO multiple-access channel (MAC) with u antennas at the receiver, u_i antennas at the i^{th} transmitter, where the SNR of the i^{th} user is ρ^{a_i} for $i \in \{1, 2, 3\}$. Similarly, $f(.)$ in Lemma 4.2 can be interpreted as the sum GDoF achievable on a 2-user MIMO MAC.

In section 4.3 we shall use these asymptotic approximation Lemmas to characterize the GDoF region of the MIMO IC. However, before that let us have another look at the approximate capacity optimal coding scheme, which from the discussion of subsection 4.2.1 is also the GDoF optimal coding scheme.

4.2.3 Another look at the simple HK coding scheme

Recall that in the simple HK coding scheme $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ (e.g., see Definition 3.2) K_{iu} and K_{iw} represents the convenience matrices of the private and public messages at transmitter i, respectively, for $i \in \{1, 2\}$. Let the singular value decomposition (SVD) of the channel matrix H_{ij} be given by $H_{ij} = V_{ij}\Sigma_{ij}U_{ij}^{\dagger}$, where $V_{ij} \in \mathbb{U}^{N_j \times N_j}$ and $U_{ij} \in \mathbb{U}^{M_i \times M_i}$ are unitary matrices and $\Sigma_{ij} \in \mathbb{C}^{N_j \times M_i}$ is a rectangular matrix containing the singular values along its diagonal. Using the SVD of the matrix H_{ij}, the covariance matrices for U_i and W_i of equation (3.23) and (3.24) can alternatively be written as

$$K_{iu} = U_{ij} \begin{bmatrix} \frac{1}{M_i}\left(I_{\min\{M_i,N_j\}} + \rho^{\alpha_{ij}}\Lambda_{ij}\right)^{-1} & \mathbf{0} \\ \mathbf{0} & \frac{1}{M_i} I_{(M_i-N_j)^+} \end{bmatrix} U_{ij}^{\dagger} = U_{ij} D_{ij} U_{ij}^{\dagger}, \tag{4.9}$$

where Λ_{ij} is a diagonal matrix containing the non-zero eigenvalues of $H_{ij}^{\dagger}H_{ij}$ and denoting the quantity $M_i(1 + \rho^{\alpha_{ij}}\lambda_{ij}^{(k)}) = r_{ik}$ where $\lambda_{ij}^{(k)}$ is the k^{th} non-zero eigenvalues of $H_{ij}^{\dagger}H_{ij}$ for $1 \le k \le m_{ij} = \min\{M_i, N_j\}$ we have

$$[D_{ij}]_{kk} = \begin{cases} r_{ik}^{-1}, & \text{for } 1 \le k \le m_{ij}; \\ \frac{1}{M_i}, & \text{for } m_{ij} + 1 \le k \le M_i. \end{cases} \tag{4.10}$$

Similarly, we have

$$K_{iw} = U_{ij}\left(\frac{1}{M_i} I_{M_i} - D_{ij}\right) U_{ij}^{\dagger} = U_{ij}^{[1:m_{ij}]} \widetilde{D}_{ij} (U_{ij}^{[1:m_{ij}]})^{\dagger}, \tag{4.11}$$

where $[\widetilde{D}_{ij}]_{kk} = (\frac{1}{M_i} - \frac{1}{r_{ik}})$ for $1 \le k \le m_{ij}$. Now, it is well known that a Gaussian vector V with covariance matrix K can be expressed as $V = Ax$, where x is a Gaussian vector with identity as covariance matrix if $AA^{\dagger} = K$. Using this result along with equations (4.9) and (4.11) we can write

$$\begin{aligned} U_i =& U_{ij}\sqrt{D_{ij}}\mathbf{x}_{ip} = \sum_{l=1}^{M_i} \sqrt{[D_{ij}]_{ll}} x_{ip}^{(l)} U_{ij}^{[l]}; \\ W_i =& U_{ij}\sqrt{\widetilde{D}_{ij}}\mathbf{x}_{ic} = \sum_{k=1}^{m_{ij}} \sqrt{[\widetilde{D}_{ij}]_{kk}} x_{ic}^{(k)} U_{ij}^{[k]}, \end{aligned} \tag{4.12}$$

where $\mathbf{x}_{ic} = [x_{ic}^{(1)}, \cdots, x_{ic}^{(m_{ij})}]^T \sim \mathcal{CN}(0, I_{m_{ij}})$ and $\mathbf{x}_{ip} = [x_{ip}^{(1)}, \cdots, x_{ip}^{(M_i)}]^T \sim \mathcal{CN}(0, I_{M_i})$ are mutually independent normal Gaussian vectors. Substituting this in the expression for X_i we see that, the transmit signal at Tx_i can be written as

$$X_i = \sum_{k=1}^{m_{ij}} \sqrt{[\widetilde{D}_{ij}]_{kk}} x_{ic}^{(k)} U_{ij}^{[k]} + \sum_{l=1}^{m_{ij}} \sqrt{[D_{ij}]_{ll}} x_{ip}^{(l)} U_{ij}^{[l]} + \sum_{m=1+m_{ij}}^{M_i} \frac{1}{\sqrt{M_i}} x_{ip}^{(m)} U_{ij}^{[m]}. \tag{4.13}$$

In the above equation, $x_{ic}^{(l)}$ for $1 \leq l \leq m_{ij}$ and $x_{ip}^{(k)}$ for $1 \leq k \leq M_i$ represent the l^{th} and k^{th} stream of the public and private information along directions $U_{ij}^{[l]}$ and $U_{ij}^{[k]}$, respectively, for user i.

Remark 4.4 Note that each of the terms in the second sum of the right hand side of (4.13) have power proportional to ρ_{ij}^{-1}. Hence, all the streams encoded through $x_{ip}^{(k)}$ for $1 \leq k \leq m_{ij}$ after passing through the cross channel with strength ρ_{ij} reach Rx_j at the noise floor. This technique can be considered as a form of **interference alignment at the signal level**.

On the other hand, in the SVD of the matrix H_{ij}, the last $(M_i - N_j)^+$ columns of Σ_{ij} are all zeros and hence $H_{ij} U_{ij}^{[k]} = 0$ for $N_j < k \leq M_i$. In other words, each of the $U_{ij}^{[k]}$'s for $m_{ij} < k \leq M_i$ lie in the null space of the matrix H_{ij}. Therefore, any stream sent along one of these directions reaches Rx_j in a subspace which is perpendicular to the subspace in which the useful signals of Rx_2 lie. That is, each user can be said to align the interference to the undesired user in a particular subspace, which is a simple form of **signal space interference alignment**. This explains why we call the streams carried by $x_{ip}^{(k)}$, $1 \leq k \leq M_i$, *private* streams.

Thus the specific choice of the covariance matrices K_{iu} in $\mathcal{HK}\left(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\}\right)$ for $i = 1, 2$ amounts to employing a technique to jointly utilize both types of interference alignments described above.

4.3 The GDoF region of the MIMO IC

Using the explicit expression for the upper bounds to the capacity region of the MIMO IC from Lemma 3.1 and using it in Lemma 4.1 we get the main result of this chapter.

Theorem 4.1 The GDoF region of $\mathcal{IC}(\bar{M}, \bar{\alpha})$ is the set of DoF tuples (d_1, d_2), denoted by $\mathcal{D}_o(\bar{M}, \bar{\alpha})$, where $d_i \in \mathbb{R}^+$ for $i = 1, 2$ satisfy the following conditions:

$$
\begin{aligned}
d_1 \leq & \min\{M_1, N_1\}; \\
d_2 \leq & \min\{M_2, N_2\}; \\
(d_1 + \alpha_{22} d_2) \leq & f\left(N_2, (\alpha_{12}, M_1), (\alpha_{22}, M_2)\right) + f\left(N_1, (\beta_{12}, m_{12}), (\alpha_{11}, (M_1 - N_2)^+)\right); \\
(d_1 + \alpha_{22} d_2) \leq & f\left(N_1, (\alpha_{21}, M_2), (\alpha_{11}, M_1)\right) + f\left(N_2, (\beta_{21}, m_{21}), (\alpha_{22}, (M_2 - N_1)^+)\right); \\
(d_1 + \alpha_{22} d_2) \leq & g\left(N_1, (\alpha_{21}, M_2), (\beta_{12}, m_{12}), (1, (M_1 - N_2)^+)\right) + \\
& g\left(N_2, (\alpha_{12}, M_1), (\beta_{21}, m_{21}), (\alpha_{22}, (M_2 - N_1)^+)\right); \\
(2d_1 + \alpha_{22} d_2) \leq & f\left(N_1, (\alpha_{21}, M_2), (\alpha_{11}, M_1)\right) + f\left(N_1, (\beta_{12}, m_{12}), (\alpha_{11}, (M_1 - N_2)^+)\right) + \\
& g\left(N_2, (\alpha_{12}, M_1), (\beta_{21}, m_{21}), (\alpha_{22}, (M_2 - N_1)^+)\right); \\
(d_1 + 2\alpha_{22} d_2) \leq & f\left(M_2, (\alpha_{21}, N_1), (\alpha_{22}, N_2)\right) + f\left(N_2, (\beta_{21}, m_{21}), (\alpha_{22}, (M_2 - N_1)^+)\right) + \\
& g\left(N_1, (\alpha_{21}, M_2), (\beta_{12}, m_{12}), (1, (M_1 - N_2)^+)\right),
\end{aligned}
$$

where $\beta_{ij} = (\alpha_{ii} - \alpha_{ij})^+$, functions $f(., ., .)$ and g(.,.,.,.) are as defined in equation (4.6) and (4.8), respectively, for $i \neq j \in \{1, 2\}$ and and $m_{ij} \triangleq \min\{M_i, N_j\}$ as defined before.

Proof 4.4 (Proof of Theorem 4.1(Outline)) From Lemma 4.1 we see that the GDoF region of the 2-user MIMO IC is simply the scaled version of the rate region $\mathcal{R}^u(\mathcal{H}, \bar{\alpha})$. Thus to evaluate the GDoF region we simply need to scale all the terms in each of the equations of $\mathcal{R}^u(\mathcal{H}, \bar{\alpha})$. For example, consider the third bound in equation (5.11)

$$R_1 + R_2 \leq I_{b3}.$$

Dividing both sides by $\log(\rho_{11})$ and taking the limit we get

$$
\begin{aligned}
& \lim_{\rho_{11} \to \infty} \frac{R_1 + R_2}{\log(\rho_{11})} \leq \lim_{\rho_{11} \to \infty} \frac{I_{b3}}{\log(\rho_{11})}; \\
\Rightarrow \quad & \lim_{\rho \to \infty} \left\{ \frac{R_1}{\log(\rho)} + \frac{\alpha_{22} R_2}{\alpha_{22} \log(\rho)} \right\} \leq \lim_{\rho \to \infty} \frac{I_{b3}}{\log(\rho)}; \\
\Rightarrow \quad & \lim_{\rho \to \infty} \frac{R_1}{\log(\rho)} + \lim_{\rho_{22} \to \infty} \frac{\alpha_{22} R_2}{\log(\rho_{22})} \leq \lim_{\rho \to \infty} \frac{I_{b3}}{\log(\rho)}; \\
\Rightarrow \quad & d_1 + \alpha_{22} d_2 \leq \lim_{\rho \to \infty} \frac{I_{b3}}{\log(\rho)}.
\end{aligned}
$$

Following the same steps for other bounds we get

$$\mathcal{D}_o(\bar{M},\bar{\alpha}) = \Big\{(d_1,d_2): \ d_1 \leq \lim_{\rho\to\infty} \frac{I_{b1}}{\log(\rho)}; \tag{4.14}$$

$$d_2 \leq \lim_{\rho\to\infty} \frac{I_{b2}}{\log(\rho)}; \tag{4.15}$$

$$d_1 + \alpha_{22} d_2 \leq \lim_{\rho\to\infty} \frac{I_{b3}}{\log(\rho)}; \tag{4.16}$$

$$d_1 + \alpha_{22} d_2 \leq \lim_{\rho\to\infty} \frac{I_{b4}}{\log(\rho)}; \tag{4.17}$$

$$d_1 + \alpha_{22} d_2 \leq \lim_{\rho\to\infty} \frac{I_{b5}}{\log(\rho)}; \tag{4.18}$$

$$2d_1 + \alpha_{22} d_2 \leq \lim_{\rho\to\infty} \frac{I_{b6}}{\log(\rho)}; \tag{4.19}$$

$$d_1 + 2\alpha_{22} d_2 \leq \lim_{\rho\to\infty} \frac{I_{b7}}{\log(\rho)}; \Big\}, \tag{4.20}$$

To prove the theorem we have to evaluate the right hand side limits, which can be done by finding the asymptotic approximations of the different I_{bi}s of Lemma 3.1 using Lemmas 4.2 and 4.3. The detailed proof is given in Appendix A.10.

The above theorem is specialized next to the SISO IC (by putting $M_1 = M_2 = N_1 = N_2 = 1$) in the following corollary, yielding its GDoF region.

Corollary 4.1 The GDoF region of the SISO IC is given as

$$\mathcal{D}_{\text{SISO}} = \Big\{(d_1,d_2): d_1 \leq 1; \tag{4.21a}$$

$$d_2 \leq 1; \tag{4.21b}$$

$$(d_1 + \alpha_{22} d_2) \leq \max\{\alpha_{22}, \alpha_{12}\} + (1-\alpha_{12})^+; \tag{4.21c}$$

$$(d_1 + \alpha_{22} d_2) \leq \max\{1, \alpha_{21}\} + (\alpha_{22}-\alpha_{21})^+; \tag{4.21d}$$

$$(d_1 + \alpha_{22} d_2) \leq \max\{\alpha_{21}, (1-\alpha_{12})^+\} + \max\{\alpha_{12}, (\alpha_{22}-\alpha_{21})^+\}; \tag{4.21e}$$

$$(2d_1 + \alpha_{22} d_2) \leq \max\{1, \alpha_{21}\} + (1-\alpha_{12})^+ \max\{\alpha_{12}, (\alpha_{22}-\alpha_{21})^+\}; \tag{4.21f}$$

$$(d_1 + 2\alpha_{22} d_2) \leq \max\{\alpha_{22}, \alpha_{12}\} + \max\{\alpha_{21}, (1-\alpha_{12})^+\} + (\alpha_{22}-\alpha_{21})^+ \Big\} \tag{4.21g}$$

Remark 4.5 The region of Corollary 4.1 provides a single unified formula for the GDoF region of the SISO IC for all interference regimes. It can be specialized to obtain GDoF regions for different interference regimes given in Section V of [6] such as weak interference, mixed interference, strong interference, etc., separate formulas for each of which are given therein. For instance, in the **weak** interference regime defined by $\alpha_{12} \leq 1$ and $\alpha_{21} \leq \alpha_{22}$, we have

$$\max\{\alpha_{22}, \alpha_{12}\} + (1 - \alpha_{12})^+ = \max\{\alpha_{22}, \alpha_{12}\} + (1 - \alpha_{12}) = 1 + (\alpha_{22} - \alpha_{12})^+;$$
$$\max\{1, \alpha_{21}\} + (\alpha_{22} - \alpha_{21})^+ = \max\{1, \alpha_{21}\} + (\alpha_{22} - \alpha_{21}) = \alpha_{22} + (1 - \alpha_{21})^+.$$

Substituting these identities in equation (4.21) we recover equation (78) of [6] which represents the GDoF region of the SISO IC in the weak interference regime (note that α_{22}, α_{21} and α_{12} are denoted as α_1, α_2 and α_3 in [6]). Similarly for instance, equations (82) and (84) of [6] can be recovered for the mixed and strong interference channels, respectively, by simplifying the result of Corollary 4.1 according to the defining conditions on the α's for those regimes.

Remark 4.6 The conventional DoF region of the MIMO IC obtained in [35] can also be recovered from Theorem 4.1 by putting $\alpha_{ij} = 1$, for $1 \leq i, j \leq 2$ in Theorem 4.1 and simplifying the different bounds. Consequently, we get

$$\begin{aligned}
\mathcal{D}_{\text{DoF}} = \Big\{(d_1, d_2) : d_1 &\leq \min\{M_1, N_1\}; && \text{(4.22a)}\\
d_2 &\leq \min\{M_1, N_1\}; && \text{(4.22b)}\\
(d_1 + d_2) &\leq (N_2 \wedge (M_1 + M_2)) + N_1 \wedge (M_1 - N_2)^+; && \text{(4.22c)}\\
(d_1 + d_2) &\leq (N_1 \wedge (M_1 + M_2)) + N_2 \wedge (M_2 - N_1)^+; && \text{(4.22d)}\\
(d_1 + d_2) &\leq (N_1 \wedge M_2) + ((N_1 - M_2)^+ \wedge (M_1 - N_2)^+) + \\
&\quad (N_2 \wedge M_1) + ((M_1 - N_2)^+ \wedge (M_2 - N_1)^+); && \text{(4.22e)}\\
(2d_1 + d_2) &\leq (N_1 \wedge (M_1 + M_2)) + N_1 \wedge (M_1 - N_2)^+ + \\
&\quad (N_2 \wedge M_1) + ((M_1 - N_2)^+ \wedge (M_2 - N_1)^+); && \text{(4.22f)}\\
(d_1 + 2d_2) &\leq (N_2 \wedge (M_1 + M_2)) + N_2 \wedge (M_2 - N_1)^+ + \\
&\quad (N_1 \wedge M_2) + ((N_1 - M_2)^+ \wedge (M_1 - N_2)^+)\Big\} && \text{(4.22g)}
\end{aligned}$$

Corollary 4.2 (The main result of [35]) The DoF region of the 2-user MIMO IC is given as

$$\begin{aligned}\mathcal{D}_{\text{DoF}} = \Big\{(d_1, d_2) : d_1 \leq & \min\{M_1, N_1\}; \\ d_2 \leq & \min\{M_2, N_2\}; \\ (d_1 + d_2) \leq & \min\{(M_1 + M_2), (N_1 + N_2), \max(M_1, N_2), \max(M_1, N_2)\}\Big\}.\end{aligned}$$

Proof 4.5 We obtain this result starting from equation (4.22). Let us consider the sum bound of equation (4.22c),

$$\begin{aligned}(d_1 + d_2) \leq & (N_2 \wedge (M_1 + M_2)) + N_1 \wedge (M_1 - N_2)^+, \\ = & \{N_2 \wedge (M_1 + M_2)\}1(N_2 \geq M_1) + \{N_2 + (N_1 \wedge (M_1 - N_2))\}1(N_2 < M_1), \\ = & \{N_2 \wedge (M_1 + M_2)\}1(N_2 \geq M_1) + \{(N_2 + N_1) \wedge M_1\}1(N_2 < M_1), \\ = & \min\{(M_1 + M_2), (N_1 + N_2), \max(N_2, M_1)\}.\end{aligned}$$

Similarly, simplifying the bound in equation (4.22d) it can be shown that

$$(d_1 + d_2) \leq \min\{(M_1 + M_2), (N_1 + N_2), \max(N_1, M_2)\}.$$

Moreover, in Appendix A.11 it will be shown that the bounds in equations (4.22e)-(4.22g) are looser than those in equation (4.22a)-(4.22d). Finally, combining the simplified forms of the 3-rd and 4-th bounds above, the claim is proved.

Remark 4.7 The GDoF region of the 2-user MIMO multiple-access channel (MAC) with an arbitrary number of antennas at the three terminals can also be found as a by-product of the analysis of the MIMO IC. This is detailed in Section 4.5.4.

The compact yet complicated form of the various bounds of Theorem 4.1 - although very general as demonstrated by the above two specializations - may hinder a complete understanding of the intuitive structure of the GDoF region. To bring out this feature of the theorem, in the following remark we provide an operational interpretation of its various bounds.

Remark 4.8 (Operational interpretation of the different bounds) We know that the GDoF optimal coding scheme divides each user's message into two sub-messages. Let the DoFs of the private and the public messages of user i be denoted by d_{ip} and d_{ic}, respectively. Note that H_{12} has a $(M_1 - N_2)^+$-dimensional null space along which Tx_1 can send private information to its desired receiver at an SNR of $\rho^{\alpha_{11}}$. Along the remaining m_{12} dimensions Tx_1 can send private information only at a power level of $\rho^{-\alpha_{12}}$ which reaches Rx_1 at a power level of $\rho^{(\alpha_{11}-\alpha_{12})^+}$. Thus with respect to the private information of Tx_1, Rx_1 is a MAC with 2 virtual transmitters having SNRs $\rho^{\alpha_{11}}$ and $\rho^{(\alpha_{11}-\alpha_{12})^+}$ and $(M_1 - N_2)^+$ and m_{12} transmit antennas, respectively. Hence, from Lemma 4.2, we have

$$d_{1p} \leq f\left(N_1, ((\alpha_{11} - \alpha_{12})^+, m_{12}), (\alpha_{11}, (M_1 - N_2)^+)\right).$$

On the other hand, since d_{1c} is decoded at Rx_2, Rx_2 is a MAC receiver with respect to W_1 (having an SNR of $\rho^{\alpha_{12}}$ and M_1 transmit antennas) and X_2 (having an SNR of $\rho^{\alpha_{22}}$) and from Lemma 4.2 (recall Remark 4.3) we have

$$(d_{1c} + \alpha_{22}d_2) \leq f\left(N_2, (\alpha_{12}, M_1), (\alpha_{22}, M_2)\right).$$

Combining the above two equations we get the 3^{rd} bound of the GDoF region. The 4^{th} bound can be similarly interpreted just by interchanging the roles of Rx_1 and Rx_2. As explained above, the two parts of the private message of Tx_1 can be thought of as two virtual users to the MAC receiver Rx_1; in addition to them, Tx_2 can send a maximum of $m_{21}\alpha_{21}$ public DoFs to Rx_1 through W_2, which can be interpreted as the 3^{rd} virtual user (with SNR $\rho^{\alpha_{21}}$ and m_{21} transmit antennas) to the MAC receiver at Rx_1, and therefore, Lemma 4.3 provides the following sum DoF uppper bound

$$(d_{1p} + \alpha_{22}d_{2c}) \leq g\left(N_1, (\alpha_{21}, M_2), (\beta_{12}, m_{12}), (1, (M_1 - N_2)^+)\right).$$

A similar consideration regarding the DoFs decodable at Rx_2 gives

$$(d_{1c} + \alpha_{22}d_{2p}) \leq g\left(N_2, (\alpha_{12}, M_1), (\beta_{21}, m_{21}), (\alpha_{22}, (M_2 - N_1)^+)\right).$$

Combining the last two equations we get the 5^{th} bound of Theorem 4.1. The other two bounds of the theorem can be similarly interpreted.

4.3.1 DoF-Splitting strategy

In order to completely specify the GDoF optimal coding scheme it is also necessary to specify the DoFs carried by the private and public messages, which are denoted by d_{ip} and d_{ic}, at the i-th transmitter besides the distributions and power levels of the codewords for $i = 1, 2$. Moreover, unlike in DoF optimal coding scheme the DoF carried by an information stream is dependent on the crosslink channel gains and can be a fraction. For instance, consider a receive dimension is effected by interference coming at ρ^{α}. Then, in that particular dimension signals only having DoF less than $(1-\alpha)$ can be received. Therefore, given a DoF pair $(d_1, d_2) \in \mathcal{D}_o(\bar{M}, \bar{\alpha})$ it is not straight forward how much of each d_i is carried by the private and public messages at Tx_i. The lemma below addresses this issue and completes the specification of the GDoF optimal coding scheme by providing a set of 4-tuples, $\mathcal{G}(\bar{M}, \bar{\alpha}) = \{(d_{1c}, d_{1p}, d_{2c}, d_{2p})\}$, which is achievable on the 2-user MIMO IC by the GDoF optimal coding scheme of Section 4.2.3. The $\mathcal{G}(\bar{M}, \bar{\alpha})$ region has the property that, for any $(d_1, d_2) \in \mathcal{D}_o(\bar{M}, \bar{\alpha})$ (which is specified in Theorem 4.1), there exists an $(d_{1c}, d_{1p}, d_{2c}, d_{2p}) \in \mathcal{G}(\bar{M}, \bar{\alpha})$ such that $(d_{ip} + d_{ic}) = d_i$ for $i = 1, 2$.

Lemma 4.4 The DoF pair $(d_1, d_2) \in \mathcal{D}_o(\bar{M}, \bar{\alpha})$ only if there exists a 4-tuple $(d_{1c}, d_{1p}, d_{2c}, d_{2p}) \in \mathcal{G}(\bar{M}, \bar{\alpha})$ such that $d_i = (d_{ic} + d_{ip})$ for $i = 1, 2$, where $\mathcal{G}(\bar{M}, \bar{\alpha}) = \mathcal{G}_1(\bar{M}, \bar{\alpha}) \cap \mathcal{G}_2(\bar{M}, \bar{\alpha})$ with $\mathcal{G}_1(\bar{M}, \bar{\alpha})$ defined below (and with $\mathcal{G}_2(\bar{M}, \bar{\alpha})$ obtained by interchanging the indexes 1 and 2 in the expression for $\mathcal{G}_1(\bar{M}, \bar{\alpha})$),

$$\mathcal{G}_1(\bar{M}, \bar{\alpha}) = \Big\{(d_{1p}, d_{1c}, d_{2c}) : \ \alpha_{11} d_{1p} \leq f\left(N_1, (\beta_{12}, m_{12}), (\alpha_{11}, (M_1 - N_2)^+)\right)\alpha_{11}; \tag{4.23a}$$

$$\alpha_{11} d_{1c} \leq \min\{N_1, M_1, N_2\}\alpha_{11}; \tag{4.23b}$$

$$\alpha_{22} d_{2c} \leq \min\{N_1, M_2\}\alpha_{21}; \tag{4.23c}$$

$$\alpha_{11}(d_{1p} + d_{1c}) \leq \min\{M_1, N_1\}\alpha_{11}; \tag{4.23d}$$

$$(\alpha_{11} d_{1p} + \alpha_{22} d_{2c}) \leq g\left(N_1, (\alpha_{21}, M_2), (\beta_{12}, m_{12}), (1, (M_1 - N_2)^+)\right); \tag{4.23e}$$

$$(\alpha_{11} d_{1c} + \alpha_{22} d_{2c}) \leq f\left(N_1, (\alpha_{21}, M_2), (\alpha_{11}, m_{12})\right); \tag{4.23f}$$

$$(\alpha_{11} d_{1p} + \alpha_{11} d_{1c} + \alpha_{22} d_{2c}) \leq f\left(N_1, (\alpha_{21}, M_2), (\alpha_{11}, M_1)\right); \Big\} \tag{4.23g}$$

with $\beta_{12} = (\alpha_{11} - \alpha_{12})^+$ and functions $f(.,.,.)$ and g(.,.,.,.) are as defined in equation (4.6) and (4.8), respectively.

Proof 4.6 The proof is given in Appendix A.12.

As explained in Subsection 4.2.3, in the GDoF optimal coding scheme the private and public messages of each user are essentially a weighted sum of several independent streams of information, each stream directed along a beam which is dependent on the channel matrix of the cross link emerging from the corresponding transmitter. The direction of these beams and their weights are chosen in such a manner (e.g., see equations (4.9)-(4.13)) that the effective covariance matrix of the overall codeword corresponding to each of the message is as given by equation (3.23) and (3.24)[1] . As for decoding, it is clear that with respect to U_i, W_i and W_j, Rx_i sees a MAC channel for $i \neq j \in \{1, 2\}$ and for any $(d_{1p}, d_{1c}, d_{2p}, d_{2c})$-tuple belonging to the achievable region (see Lemma 4.4 in Appendix A.12) Rx_i can decode U_i, W_i and W_j with probability of error going to zero. Therefore, any decoding scheme which is capacity optimal on a MAC will be GDoF optimal for the MIMO IC if each receiver tries to decode the 2 public messages and its own private message while treating the other private message as noise.

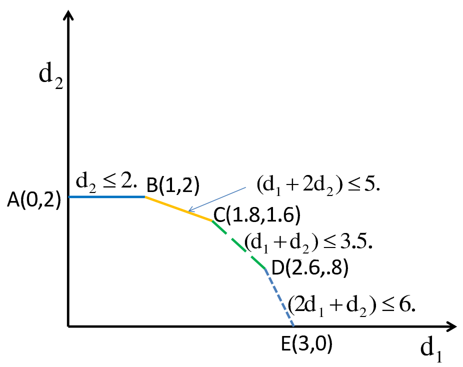


Figure 4.1: GDoF region of the (3, 3, 2, 2) IC with $\bar{\alpha} = (1, \frac{3}{5}, \frac{3}{5}, 1)$.

[1] Instead of sending independent streams of information (which is without loss of GDoF optimality), if coding is also done across different streams, it is possible to achieve a larger error exponents.

Example 4.1 (A MIMO IC with weak interference) Figure 4.1 depicts the GDoF region of a $(3,3,2,2)$ MIMO IC with $\bar{\alpha} = [1, \frac{3}{5}, \frac{3}{5}, 1]$. Clearly, it is sufficient to illustrate the achievability of the vertices of the GDoF region since any point on the line joining any two vertices can be achieved via time-sharing. The time sharing argument, however, is just a matter of convenience, and is not necessary[2] to achieve a point in the GDoF region. Note that points A or E can be achieved simply by turning off Tx_1 or Tx_2, respectively. To analyze the achievability of the other corner points we need to know the DoFs carried by the private and public messages of each user. For the $(3,3,2,2)$ IC with $\bar{\alpha} = [1, \frac{3}{5}, \frac{3}{5}, 1]$, Lemma 4.4 gives

$$\begin{array}{rlcrl} d_{1p} \leq & 1.8; & & d_{2p} \leq & .8; \\ d_{1c} \leq & 2; & & d_{2c} \leq & 2; \\ d_{2c} \leq & 1.2; & & d_{1c} \leq & 1.2; \\ (d_{1p}+d_{1c}) \leq & 3; & \text{and} & (d_{2p}+d_{2c}) \leq & 2; \\ (d_{1p}+d_{2c}) \leq & 2.2; & & (d_{2p}+d_{1c}) \leq & 1.2; \\ (d_{1c}+d_{2c}) \leq & 2.6; & & (d_{2c}+d_{1c}) \leq & 2; \\ (d_{1p}+d_{1c}+d_{2c}) \leq & 3; & & (d_{2p}+d_{2c}+d_{1c}) \leq & 2; \end{array} \tag{4.24}$$

Achievability of point B: From the set of bounds in equation (4.24) we see the only choice for the different DoFs for the public and private messages of the two users are given as $d_{1p} = 1$, $d_{1c} = 0$, $d_{2p} = .8$ and $d_{2c} = 1.2$. Since the first user needs to send only private information having DoF 1, it is best to send it in the direction of the null space of H_{12}, i.e.,

$$X_1 = \frac{1}{\sqrt{3}} x_{1p}^{(3)} U_{12}^{[3]}. \tag{4.25}$$

On the other hand, the structure of the codeword for the second user is also clear from equation (4.13),

$$X_2 = \sum_{k=1}^{2} \frac{\sqrt{\rho_{21}\lambda_{21}^{(k)}}}{\sqrt{2(1+\rho_{21}\lambda_{21}^{(k)})}} x_{2c}^{(k)} U_{21}^{[k]} + \sum_{l=1}^{2} \frac{1}{\sqrt{2(1+\rho_{21}\lambda_{21}^{(l)})}} x_{2p}^{(l)} U_{21}^{[l]}, \tag{4.26}$$

where $x_{2c}^{(k)}$ and $x_{2p}^{(k)}$ carries .6 and .4 DoFs, respectively for both $k = 1, 2$.

[2] For an example illustrating this point the reader is referred to [43].

Decoding: Rx_1 first projects the received signal on the 2 dimensional space which is perpendicular to $H_{11}U_{12}^{[3]}$ to remove the effect of $x_{1p}^{(3)}$ by zero forcing. In the resulting 2 dimensional signal space, only contribution from W_2 is present, carrying a DoF of 1.2. This can be decoded because the link from Tx_2 to Rx_1 is a 2×2 point-to-point MIMO channel with effective SNR of $\rho^{.6}$. Once decoded, Rx_1 removes its effect from the original received signal (the received signal before zero-forcing) and then it gets a interference-free channel from Tx_1 to itself. It can hence decode U_1. On the other hand, Rx_2 does not face any interference[3] from Tx_1 so that it can decode W_2 while treating U_2 as noise. This is possible because treating U_2 as noise only raises the noise floor to $\rho^{.4}$ while the received signal power of W_2 is at ρ which implies it can decode .6 DoFs from each receive dimension. Next, subtracting the contribution of W_2 from the received signal, Rx_2 can decode U_2.

Achievability of point C: Since Rx_2 can support only 2 DoFs at point C, we have $d_{1c} \leq .4$. Combining this with equation (4.24) we get $d_{1c} = .4$, $d_{1p} = 1.4$, $d_{2p} = .8$ and $d_{2c} = .8$. For this choice of the different rates, the transmit signals at Tx_1 and Tx_2 are given by

$$X_1 = \sum_{k=1}^{2} \frac{\sqrt{\rho_{12}\lambda_{12}^{(k)}}}{\sqrt{3(1+\rho_{12}\lambda_{12}^{(k)})}} x_{1c}^{(k)} U_{12}^{[k]} + \sum_{l=1}^{2} \frac{1}{\sqrt{3(1+\rho_{12}\lambda_{12}^{(l)})}} x_{1p}^{(l)} U_{12}^{[l]} + \frac{1}{\sqrt{3}} x_{1p}^{(3)} U_{12}^{[3]}, \tag{4.27}$$

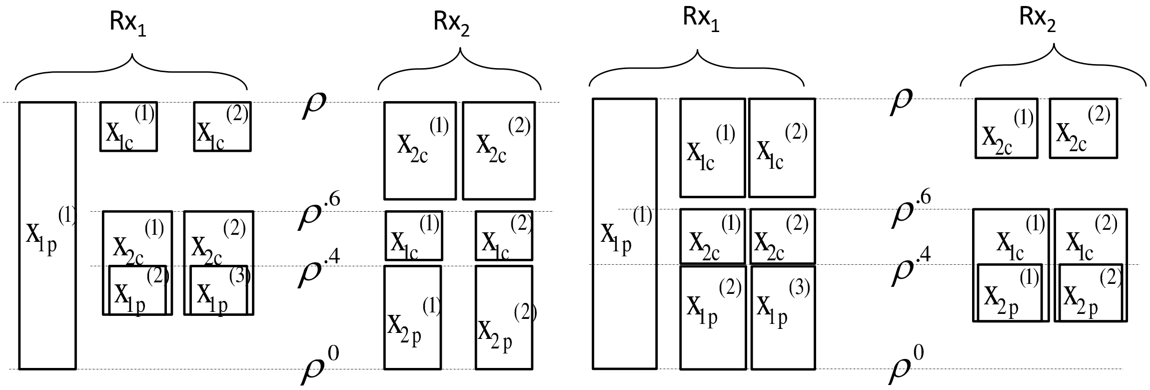
where $x_{1c}^{(k)}$ and $x_{1p}^{(k)}$ carries .2 DoFs, for both $k = 1, 2$ and $x_{1p}^{(3)}$ carries 1 DoF and

$$X_2 = \sum_{k=1}^{2} \frac{\sqrt{\rho_{21}\lambda_{21}^{(k)}}}{\sqrt{2(1+\rho_{21}\lambda_{21}^{(k)})}} x_{2c}^{(k)} U_{21}^{[k]} + \sum_{l=1}^{2} \frac{1}{\sqrt{2(1+\rho_{21}\lambda_{21}^{(l)})}} x_{2p}^{(l)} U_{21}^{[l]}, \tag{4.28}$$

where $x_{2c}^{(k)}$ and $x_{2p}^{(k)}$ carries .4 DoFs each, for both $k = 1, 2$. The different signals at both the receivers are depicted in Fig. 4.2(a), where each stream is represented by a box the top level of which marks its signal strength and the vertical height is proportional to the DoFs carried by it. Note that, $x_{1p}^{(1)}$ though transmitted at a power level of 1, does not appear at Rx_2 since it is transmitted along the null space of the channel from Tx_1 to Rx_2.

Decoding: The decoding procedure at Rx_1 is exactly the same as in the previous case. Rx_2 on the other hand, can decode W_2, W_1 and U_2, respectively, in that order through successive interference cancellation, i.e., it first decodes W_2, treating W_1 and U_2 (both of which are received

[3] The interference that reach below noise floor is irrelevant in the GDoF computation.



(a) Receive signal spaces at DoF pair (1.8, 1.6).

(b) Receive signal spaces at DoF pair (2.6, .8).

Figure 4.2: GDoF region of a (3, 3, 2, 2) MIMO IC and its explicit achievable scheme.

below $\rho^{\cdot 6}$) as noise. Subtracting the contribution of W_2, it next decodes W_1 treating U_2 as noise. Finally, subtracting the contribution of W_1 it decodes U_2. It should be noted that during the decoding of each of these messages the noise floor is actually at the power level of the messages being treated as noise.

Achievability of point D: Again from (4.24) we get $d_{1p} = 1.8$, $d_{1c} = .8$, $d_{2p} = .4$ and $d_{2c} = .4$, for which X_i for $1 \leq i \leq 2$ can be written as

$$X_1 = \sum_{k=1}^{2} \frac{\sqrt{\rho_{12}\lambda_{12}^{(k)}}}{\sqrt{3(1+\rho_{12}\lambda_{12}^{(k)})}} x_{1c}^{(k)} U_{12}^{[k]} + \sum_{l=1}^{2} \frac{1}{\sqrt{3(1+\rho_{12}\lambda_{12}^{(l)})}} x_{1p}^{(l)} U_{12}^{[l]} + \frac{1}{\sqrt{3}} x_{1p}^{(3)} U_{12}^{[3]}, \tag{4.29}$$

where $x_{1c}^{(k)}$ and $x_{1p}^{(k)}$ carries .4 DoFs each, for both $k = 1, 2$ and $x_{1p}^{(3)}$ carries 1 DoF and

$$X_2 = \sum_{k=1}^{2} \frac{\sqrt{\rho_{21}\lambda_{21}^{(k)}}}{\sqrt{2(1+\rho_{21}\lambda_{21}^{(k)})}} x_{2c}^{(k)} U_{21}^{[k]} + \sum_{l=1}^{2} \frac{1}{\sqrt{2(1+\rho_{21}\lambda_{21}^{(l)})}} x_{2p}^{(l)} U_{21}^{[l]}, \tag{4.30}$$

where $x_{2c}^{(k)}$ and $x_{2p}^{(k)}$ carries .2 DoFs each, for both $k = 1, 2$. The different received signals at both the receivers are depicted in Fig. 4.2(b). It is clear from Fig. 4.2(b) that a MAC receiver can decode all the messages.

4.4 The Symmetric GDoF region of the (M, N, M, N) MIMO IC

Suppose the roles of the transmitters and receivers of the MIMO IC $\mathcal{IC}(\mathcal{H}, \bar{\alpha})$ are interchanged. In the notations defined in Section 4.2, this resulting IC (hereafter referred to as the "reciprocal" channel) can be denoted by $\mathcal{IC}(\mathcal{H}^r, \bar{\alpha}^r)$, where $\mathcal{H}^r = \{H_{11}^T, H_{21}^T, H_{12}^T, H_{22}^T\}$ and $\bar{\alpha}^r = [\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22}]$. Clearly, $\mathcal{D}_o(\bar{M}^r, \bar{\alpha}^r)$ denotes the GDoF region of the reciprocal channel where $\bar{M}^r = (N_1, M_1, N_2, M_2)$.

Corollary 4.3 (Reciprocity of the GDoF region) The GDoF region of the MIMO IC is same as that of its *reciprocal* channel i.e.,

$$\mathcal{D}_o(\bar{M}, \bar{\alpha}) = \mathcal{D}_o(\bar{M}^r, \bar{\alpha}^r).$$

Proof 4.7 It was proved in Theorem 3.4 of Chapter 3 that the capacity region of a 2-user MIMO IC and its *reciprocal* channel are within a constant (independent of ρ) number of bits to each other. The corollary is easily proved by using this result in the definition of the GDoF region of the IC in equation (4.2), which states that the GDoF regions of two channels with capacity regions differing by only a constant number of bits are the same.

In other words, the GDoF region of the channel does not change if the roles of the transmitters and the receivers are interchanged. Note that this is a more general result than the reciprocity of the conventional DoF region proved in [35]. In what follows, we define the symmetric GDoF metric.

Definition 4.2 (Symmetric GDoF) Let $\mathcal{C}_s(\alpha) = \sup(R_1 + R_2)$ with $(R_1, R_2) \in \mathcal{C}(\mathcal{H}, \bar{\alpha})$ where $\bar{\alpha} = [1, \alpha, \alpha, 1]$ and $\sup \mathcal{A}$ represents the supremum of the set of elements in $\mathcal{A}$. Then the **symmetric GDoF** of the channel, denoted by d_s, is defined as

$$d_s \triangleq \lim_{\rho \to \infty} \frac{\mathcal{C}_s(\alpha)}{2 \log(\rho)}.$$

It is clear from Definition 4.1 and the above equation that

$$d_s = \frac{\sup_{\mathcal{D}_o(\bar{M}, \bar{\alpha})}(d_1 + d_2)}{2}. \tag{4.31}$$

The symmetric GDoF region of the 2-user MIMO IC with (M, N, M, N) IC with $M \leq N$ was computed in [29][4] which can be recovered by putting the specific values of the antennas and $\bar{\alpha} = [1, \alpha, \alpha, 1]$ in Theorem 4.1 and is given as

$$d_s \leq \min\{M, \hat{D}(\alpha)\} \tag{4.32}$$

where

$$\hat{D}(\alpha) = \begin{cases} M - (2M - N)\alpha, & 0 \leq \alpha < \frac{1}{2}; \\ (N - M) + (2M - N)\alpha, & \frac{1}{2} \leq \alpha \leq \frac{2}{3}; \\ M - \frac{\alpha}{2}(2M - N), & \frac{2}{3} \leq \alpha \leq 1; \\ \frac{N}{2} + \frac{M}{2}(\alpha - 1), & 1 \leq \alpha. \end{cases} \tag{4.33}$$

[4] For the rest of this section we shall assume $M \leq N$.

However, the techniques developed in [29] was not sufficient to derive the GDoF of the channel with more input antennas than the number of output antennas. Using the results of this chapter, the GDoF of such a channel can be computed in two different ways:

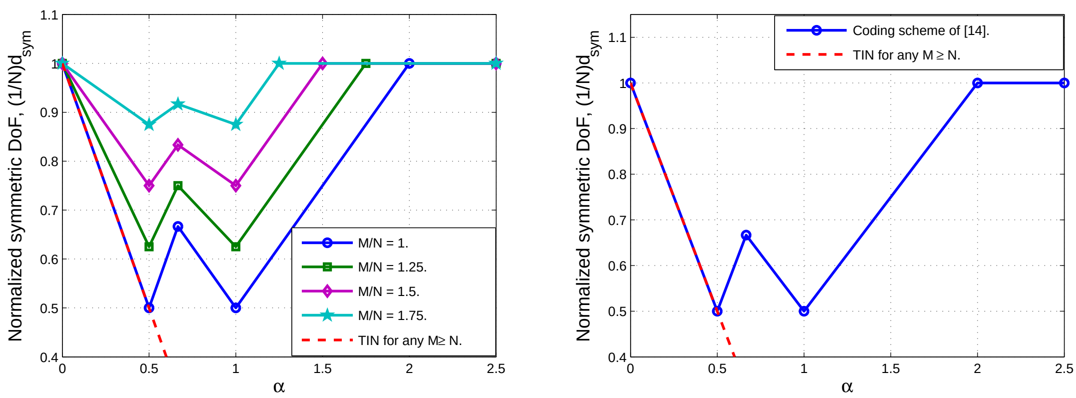
(1) *From the reciprocity result:* From Corollary 4.3 we know that the GDoF region of the (N, M, N, M) IC is given by equation (4.32), since this is the reciprocal channel of the (M, N, M, N) IC.

(2) *From Theorem 4.1:* The same result can be obtained by substituting $M_1 = M_2 = N$, $M = N_1 = N_2$ and $\bar{\alpha} = [1, \alpha, \alpha, 1]$ in Theorem 4.1.

Note that in both of the above cases, the number of transmit antennas is greater than or equal to the number of antennas at the receivers.

Remark 4.9 It must be noted that the achievable schemes on the two channels are entirely different. While for (M, N, M, N) IC the coding scheme need not depend on the channel matrices at the transmitters (see the achievability scheme of [29]), for (N, M, N, M) IC the covariance matrices are necessarily functions of the channel matrices. Hence, a naive extension of the scheme of [29] to the case of (N, M, N, M) IC is not GDoF optimal. In fact, such a scheme wouldn't even be DoF optimal because while on a MIMO IC with less number of antennas at the receiving end of each cross links, receive zero-forcing is sufficient to achieve the DoF region, for the opposite case, knowledge of channel state information at the transmitters (CSIT) is necessary to achieve DoF-optimal performance [44–46] such as through transmit beamforming [35].

Remark 4.10 (A scheme that ignores CSIT:) The GDoF optimal coding scheme of [29] does not utilize any CSIT. The approach in [29] was to divide the range of α into the five regimes delineated in the SISO IC case in [6] and employ the main idea of the achievable schemes that are known to be GDoF-optimal in the SISO case (e.g., treat interference as noise in the very weak interference regime; set the power level of private messages so they arrive at the noise level at the unintended receiver in the moderate and weak interference regimes following the prescription of [6], send only common messages in the strong and very strong interference regimes). Hence, this coding scheme effectively only employs signal level interference alignment without any form of transmit beamforming. While on an (M, N, M, N) IC beamforming is not necessary because neither of the cross-links have a null space, it is so on an (N, M, N, M) IC. Therefore, the coding scheme of [29] when applied naively to the (N, M, N, M) IC cannot achieve the fundamental GDoF region of the channel. Fig. 4.3(a) and Fig. 4.3(b) show the GDoF achievable by the GDoF optimal coding scheme of this chapter in comparison with the coding scheme used in [29]. Comparing the GDoF curves of the two schemes, it is clear that the No-CSIT coding scheme of [29] fails to achieve the fundamental GDoF of the (N, M, N, M) IC.



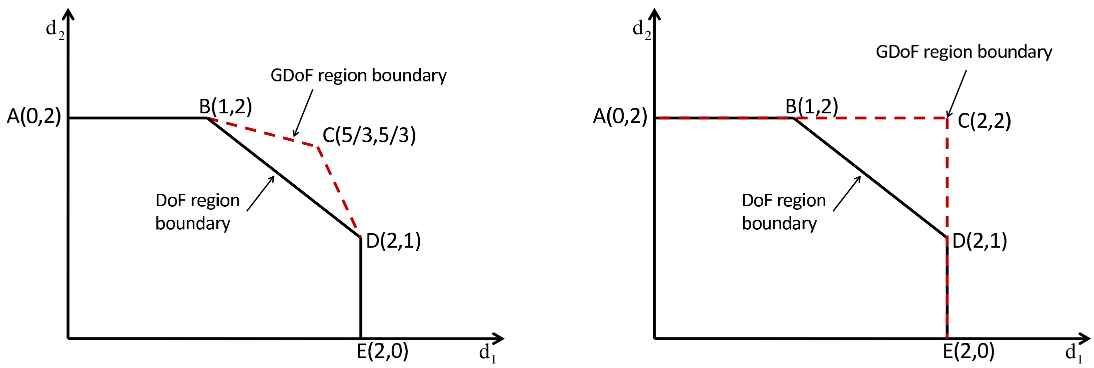
(a) The fundamental GDoF of the channel. (b) GDoF achievable by the coding scheme of [29].

Figure 4.3: Symmetric GDoF of the (N, M, N, M) IC.

4.5 Further insights and results

4.5.1 Only Tx/Rx ZF Beam-forming is not GDoF optimal

The fundamental GDoF gives a finer high SNR approximation than the DoF approximation and therefore reveals insights that are not revealed by the DoF analysis. Figure 4.4(a) illustrates this point by comparing the DoF and GDoF region of the $(3, 2, 3, 2)$ IC with $\bar{\alpha} = [1, \frac{2}{3}, \frac{2}{3}, 1]$. It is known from [35] that only transmit/receive zero-forcing beam-forming is sufficient to achieve any

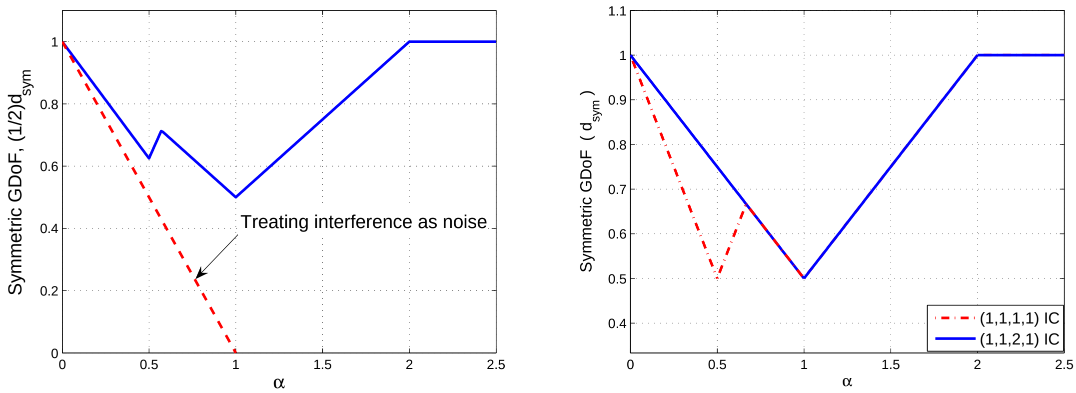


(a) $\alpha = \frac{2}{3}$

(b) $\alpha = \frac{3}{2}$

Figure 4.4: GDoF region of the $(3, 2, 3, 2)$ MIMO IC with $\alpha_{11} = \alpha_{22} = 1$ and $\alpha_{12} = \alpha_{21} = \alpha$.

point in the DoF region of the channel. The DoF region achievable using this scheme is shown in Fig. 4.4 as against the fundamental GDoF region. It is easily seen that forgoing the opportunity to align signals in the signal-level dimension leads to a strictly GDoF suboptimal performance. In particular, this technique can not achieve any point in the triangular region BCD. However, the coding scheme of Section 4.2 which in addition to beamforming, also employs signal-level interference alignment, can achieve all the points in the region BCD.



(a) Symmetric GDoF region of the $(3, 2, 2, 3)$ IC.

(b) Symmetric GDoF region of a $(1, 1, 2, 1)$ MIMO IC

Figure 4.5: Sub-optimality of TIN and deviation of the GDoF boundary from the well known "W" shape.

4.5.2 Sub-optimality of treating interference as noise

Another fundamental difference of the MIMO IC from the SISO IC revealed by the GDoF analysis is this: in general, treating interference as noise (TIN) is **not** GDoF optimal on a MIMO IC even in the *very weak interference* regime, i.e., when $\alpha \leq \frac{1}{2}$. This is seen in Fig. 4.3(a) where the dotted line, which represents the symmetric GDoF achievable by TIN, is strictly sub-optimal with respect to the fundamental GDoF of the channel for $\alpha \leq \frac{1}{2}$ whenever $M/N > 1$. See also Fig. 4.5(a) which illustrates this point for the $(3, 2, 2, 3)$ MIMO IC.

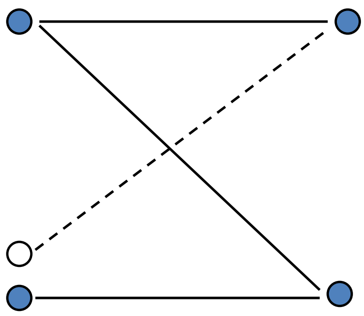


Figure 4.6: Diagonalization of the cross links using ZF and BF.

4.5.3 Deviation from the "W" shape

Unlike in the SISO IC, the symmetric GDoF region of a MIMO IC in general need not maintain the "W" shape. The deviation in general is due to asymmetry in the numbers of antennas. For example, consider the $(1,1,2,1)$ IC with $\alpha_{ii} = 1$ and $\alpha_{ij} = \alpha$, for $i \neq j \in \{1,2\}$. The best achievable symmetric DoF ($d_{sym} = d_1 = d_2$) on this channel denoted by d is

$$d = \begin{cases} 1 - \frac{\alpha}{2}, & 0 \leq \alpha \leq 1; \\ \frac{\alpha}{2}, & 1 \leq \alpha \leq 2; \\ 1, & 2 \leq \alpha. \end{cases}$$

which is depicted in Figure 4.5(b). Diagonalizing the cross-link from Tx_2 to Rx_1 and then turning off the subchannel which interferes with Rx_1 gives the GDoF equivalent channel of Figure 4.6 which is a SISO "Z" IC. The symmetric GDoF region of this channel is indeed "V" shaped as found in [6]. Although a little more involved, the distorted "W" of Figure 4.5(a) for the $(3,2,2,3)$ MIMO IC can be explained similarly.

4.5.4 GDoF region of the 2-User MIMO MAC

Both the set of lower and upper bounds to the capacity region of the 2-user MIMO IC contain terms that also appear in the capacity region of a 2-user MIMO MAC channel. Thus, as a by product we can obtain the GDoF region of the MIMO MAC channel.

Consider a MIMO MAC with two transmitters having M_1 and M_2 antennas, respectively, and with N receive antennas at the common receiver. The input-output relation for this channel can be written as

$$Y = \sqrt{\rho} H X_1 + \sqrt{\rho^\alpha} G X_2 + Z,$$

where $X_i \in \mathbb{C}^{M_i \times 1}$ is the transmitted signal from user i, where $Y \in \mathbb{C}^{N \times 1}$ is the received signal and $H \in \mathbb{C}^{N \times M_1}$ and $G \in \mathbb{C}^{N \times M_2}$ are the channel matrices from users 1 and 2 to the receiver, respectively, both of which are assumed to have full rank, and $Z \sim \mathcal{CN}(\mathbf{0}, I_N)$ is additive white Gaussian noise. Without loss of generality, we assume that the SNR of the second user is represented as ρ^α.

Let $\mathcal{C}_{MAC}(H,G)$ denote the capacity region of the 2-user MIMO MAC defined above. The GDoF region is defined as

$$\mathcal{D}_{MAC} = \left\{ (d_1, d_2) : \; d_i = \lim_{\rho \to \infty} \frac{R_i}{\log(\rho)}, i \in \{1,2\} \text{ and } (R_1, R_2) \in \mathcal{C}_{MAC}(H,G) \right\}.$$

The result below gives the GDoF region of the 2-user MIMO MAC.

Corollary 4.4 The GDoF region of the 2-user MIMO MAC defined above is given as

$$\begin{aligned} \Big\{ (d_1, d_2) : d_1 \leq & \min\{M_1, N\}; \\ d_2 \leq & \min\{M_2, N\}\alpha; \\ (d_1 + d_2) \leq & f\left(N, (\alpha, M_2), (1, M_1)\right) \Big\}, \end{aligned}$$

where $f(.,.,.)$ is given by equation (4.6).

Proof 4.8 Following the analysis of the MIMO IC in [40], it can be easily shown that an achievable rate region of the MIMO MAC is given as

$$\begin{aligned} \mathcal{R}_A = \Big\{ (R_1, R_2) : R_1 &\leq \log\det\left(I_N + \rho H H^\dagger\right) - N\log(M_1); \\ R_2 &\leq \log\det\left(I_N + \rho^\alpha G G^\dagger\right) - N\log(M_2); \\ R_1 + R_2 &\leq \log\det\left(I_N + \rho H H^\dagger + \rho^\alpha G G^\dagger\right) - N\log(\max\{M_1, M_2\}) \Big\}, \end{aligned}$$

and an upper bound is given as

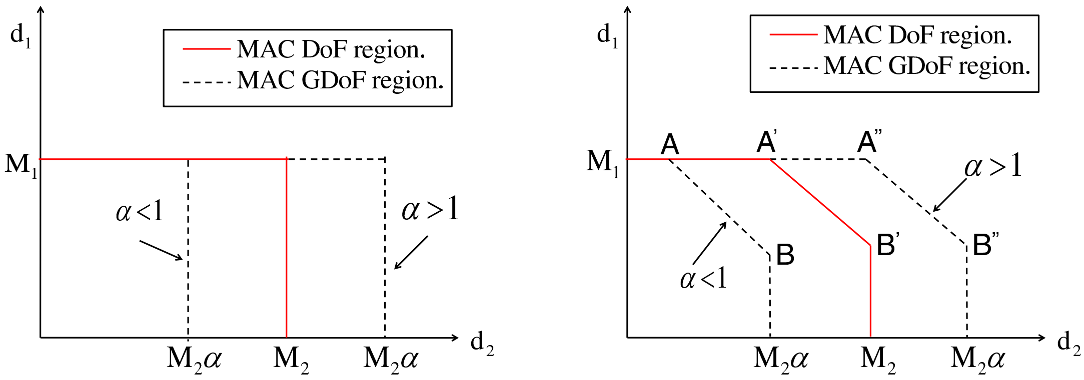
$$\mathcal{R}^U = \Big\{(R_1, R_2) : R_1 \leq \log\det\left(I_N + \rho HH^\dagger\right);$$
$$R_2 \leq \log\det\left(I_N + \rho^\alpha GG^\dagger\right);$$
$$R_1 + R_2 \leq \log\det\left(I_N + \rho HH^\dagger + \rho^\alpha GG^\dagger\right)\Big\}.$$

Note that the two regions differ only by constant (independent of SNR) number of bits. The desired result now follows by replacing $\mathcal{C}_{MAC}$ in the definition of the GDoF region by $\mathcal{R}^U$ or $\mathcal{R}_A$, since a constant number of bits are insignificant in the GDoF analysis.

Remark 4.11 The GDoF regions for the case when $N \geq (M_1 + M_2)$ is depicted in Fig. 4.7(a) and the case when $\max\{M_1, M_2\} < N < (M_1 + M_2)$ is depicted in Fig. 4.7(b), where $A = (M_1, (N - M_1)\alpha)$, $B = ((N - M_2)\alpha + M_1(1 - \alpha), M_2\alpha)$, $A^{'} = (M_1, (N - M_1))$, $B^{'} = ((N - M_2), M_2)$, $A^{''} = (M_1, (N - M_1) + M_2(\alpha - 1))$ and $B^{''} = ((N - M_2), M_2\alpha)$. Although, the GDoF analysis reveals the possibility of achieving a larger sum DoFs when one of the link's strength is exponentially larger that the other ($\alpha > 1$), it is not as interesting as the MIMO IC since the GDoF region of the MAC can be achieved using independent Gaussian codes with scaled identity input covariances at each transmitter and joint decoding just as in a MAC with $\alpha = 1$. In other words, this DoF-optimal scheme is also GDoF-optimal.

4.6 Conclusion

The GDoF analysis of this chapter, unifies and generalizes the earlier results on GDoF of SISO IC [6], the DoF region [35] of MIMO IC and the symmetric GDoF [29] of MIMO IC through a single achievable scheme for all. The coding schemes in [35] and [29] are strictly suboptimal in the GDoF sense on a general 2-user MIMO IC in one case or other. The analysis here reveals various insights about the MIMO IC including the fact that in general, partially decoding the unintended user's message is necessary to be GDoF optimal even in the so called very weak interference regime. The two types of signaling dimensions available on a MIMO IC – namely, signal space and signal level – are jointly and optimally exploited in the GDoF optimal scheme.



(a) $N \geq (M_1 + M_2)$.

(b) $\max\{M_1, M_2\} < N < (M_1 + M_2)$.

Figure 4.7: The GDoF region of the MIMO MAC.

Chapter 5

The diversity-multiplexing tradeoff of the MIMO Z interference channel

5.1 Introduction

The model used for the channel coefficients of a communication system is another important design parameter. All of the performance characterizations of the previous chapters assume time-invariant channel coefficients, whereas the channel coefficients of a practical wireless network vary with time and are said to undergo fading. Most of the previous results on interference channels which assume a fading channel model use a much coarser performance metric such as the degrees of freedom (DoF) [45] or the generalized DoF (GDoF) [47]. These metrics can only characterize the rate scaling factor/s with average SNR, of the corresponding channel and do not reveal any information about the reliability of communication. Diversity-multiplexing tradeoff introduced in [48] for the point-to-point channel captures this relationship between rate and reliability of communication. Encouraged by the importance of knowing the best achievable reliability on a channel while communicating at a particular rate, in this chapter we choose DMT as our performance metric. Further, as a first step towards understanding the general interference network, we choose the 2-user Z interference channel as our channel model in this chapter. In the 2-user Z interference channel (ZIC), 2 transmitters communicate to their corresponding receivers via the same signal space, while only one of the transmitters interfere with the other receiver.

Besides being one of the basic building blocks of the general interference network, ZICs also emerge as the natural information theoretic model for various practical wireless communication scenario such as femto-cells [49]. Also, the ZIC is a special case of the 2-user IC. Thus optimal (with respect to some metric) coding and decoding schemes on a ZIC may reveal useful insights for the 2-user IC as well. For instance, the optimal DMT of the ZIC is an upper bound for the 2-user MIMO IC both with and without CSIT. These facts make the analysis of the ZIC an important step towards a better understanding of the general multiuser wireless system. Motivated by the aforementioned facts in this chapter we analyze the DMT of the MIMO ZIC. However, unlike the DMT framework in a point-to-point (PTP) channel, [48] where there is a single communication link which can be characterized by a single SNR, in a multiuser setting such as the one at hand, it is only natural to allow the SNRs and INRs of different links to vary with different exponentials with respect to a nominal SNR, denoted as ρ. This technique was first used in [6] to analyze the DoF region which the authors referred to as the Generalized DoF (GDoF) region, of the 2-user SISO IC. Later, this technique was extended to the DMT scenario of the SISO IC in [50] and [51]. Following similar approach, we allow the different INR and SNRs at the receivers to vary exponentially with respect to ρ with different scaling factors. We refer to the corresponding DMT as the generalized DTM (GDMT) to distinguish it from the case when SNR=INR in all the links.

In this chapter, we first derive the DMT of the MIMO ZIC with CSIT and arbitrary number of antennas at each node. The achievability is based on a simple Han-Kobayashi [10] coding scheme, where the signal to be transmitted by the 2nd user depends only on the channel matrix to the first receiver, whereas the transmitted signal of the first user does not use any CSIT. The converse is proved by deriving a set of upper bounds to the achievable DMT, from a set of upper bounds to the capacity region of the ZIC. The set of upper bounds to the capacity region in turn is obtained assuming a genie aided interfered receiver. The computation of the DMT of the MIMO ZIC involves the asymptotic joint eigenvalue distribution of two specially correlated random Wishart matrices which was recently derived by the authors in [52] in a different context. Using this distribution result, the fundamental DMT of the MIMO ZIC channel with CSIT is established as the solution of a convex optimization problem. While it is argued that in general the optimization problem can be solved using numerical methods, closed-form solutions are computed for several special cases. Secondly, we characterize the achievable DMT of a transmission scheme which does not utilize CSIT. Comparing the achievable DMT of this scheme with the F-CSIT DMT of the ZIC, we identify two classes of MIMO ZICs on which the No-CSIT scheme can achieve the F-CSIT DMT. The first class of ZICs have equal number of antennas at all the nodes and a stronger INR than a certain threshold (e.g.,Theorem 5.6) and the second class of ZICs have a larger number of antennas at the interfered node than a certain threshold (e.g., Theorem 5.7). The above result thus effectively characterizes the DMT of these channels without CSIT because the F-CSIT DMT of the channel is an upper bound to the No-CSIT DMT. For other channel configurations, this achievable DMT represents a lower bound to the fundamental No-CSIT DMT of the channel.

An early work in this direction is [53], where the authors derive an achievable DMT on a SISO ZIC with No-CSIT. The DMT of the SISO ZIC with F-CSIT can be obtained from [54], where the DMT (F-CSIT) of the 2-user SISO IC was derived. In this work, we focus on the MIMO case. In [30], an upper bound to the DMT of a 2-user MIMO IC with CSIT was derived for the case in which all nodes have same number of antennas and the direct and cross links have the same SNRs and INRs, respectively. This result, if specialized for the MIMO ZIC, provides only an upper bound. Our result will prove that for the special case considered in [30], this upper bound is actually tight on a ZIC with F-CSIT. However, the result of this chapter on MIMO ZIC is much more general, in the sense that we consider arbitrary number of antennas at each node and arbitrary scaling parameters for the different SNRs and the INR of the system. Moreover, we also characterize the DMT of the channel with no CSI at the transmitters for some specific channel configurations.

The rest of the chapter is organized as follow. In section 5.2, we first provide a description of the channel model considered in this chapter and define the SNRs and INRs of the different links which is followed by the definition of the DMT framework on a MIMO ZIC in subsection 5.2.1. In subsection 5.2.2 we derive upper and lower bounds to the instantaneous capacity regions of the channel which is used in section 5.3 to establish the DMT of the channel as a solution of an convex optimization problem. In subsection 5.3.1 we derive analytic solutions to this general optimization problem and hence the DMT of various classes of ZICs with specific number of antennas and SNR and/or INR parameters. In section 5.4 we characterize the DMT of the channel with no-CSIT which is then followed by the conclusion in section 6.6. Some of the proofs are relegated to the appendix for a better flow of the main concepts of the chapter.

Notations: We denote the conjugate transpose of the matrix A as $A^{\dagger}$ and its determinant as $|A|$. $\mathbb{C}$ and $\mathbb{R}$ represent the field of complex and real numbers, respectively. The set of real numbers $\{x \in \mathbb{R} : a \leq x \leq b\}$ will be denoted by $[a, b]$. Furthermore, $(x \wedge y)$, $(x \vee y)$ and $(x)^{+}$ represent the minimum of x and y, the maximum of x and y, and the maximum of x and 0, respectively. All the logarithms in this chapter are with base 2. We denote the distribution of a complex circularly symmetric Gaussian random vector with zero mean and covariance matrix Q as $\mathcal{CN}(0, Q)$. Any two functions $f(\rho)$ and $g(\rho)$ of ρ, where ρ is the signal to noise ratio (SNR) defined later, are said to be exponentially equal and denoted as $f(\rho)\doteq g(\rho)$ if, $\lim_{\rho\to\infty} \frac{\log(f(\rho))}{\log(\rho)} = \lim_{\rho\to\infty} \frac{\log(g(\rho))}{\log(\rho)}$. The same is true for $\dot{\geq}$ and $\dot{\leq}$.

5.2 Channel Model and Preliminaries

We consider a MIMO ZIC as shown in Figure 5.1, where user 1 (Tx_1) and user 2 (Tx_2) have M_1 and M_2 antennas and receiver 1 (Rx_1) and 2 (Rx_2) have N_1 and N_2 antennas, respectively. This channel will be referred hereafter as a (M_1, N_1, M_2, N_2) ZIC. A slow fading Rayleigh distributed channel model is considered where $H_{ij} \in \mathbb{C}^{N_j \times M_i}$ represents the channel matrix between Tx_i and Rx_j. It is assumed that H_{11}, H_{21} and H_{22} are mutually independent and contain mutually independent and identically distributed (i.i.d.) $\mathcal{CN}(0, 1)$ entries. These channel matrices remain fixed for a particular fade duration of the channel and changes in an i.i.d. fashion, in the next. Perfect channel state information is assumed at both the receivers (CSIR) and both the transmitters (CSIT). At time t, Tx_i chooses a vector $X_{it} \in \mathbb{C}^{M_i \times 1}$ and sends $\sqrt{P_i}X_{it}$ over the channel, where for the input signals we assume the following short term average power constraint:

$$\frac{1}{N} \sum_{t=1+N(k-1)}^{N+N(k-1)} \mathrm{tr}(Q_{it}) \leq 1, \forall\ k \geq 1,\ i = 1, 2,\ \text{where } Q_{it} = \mathbb{E}\left(X_{it}X_{it}^{\dagger}\right), \tag{5.1}$$

where N represents the number of channel uses for which the channel matrices remain fixed, or in other words the fade duration.

Remark 5.1 Note that since the transmitters are not allowed to allocate power across different fades of the channel, the channel is still in the outage setting, i.e., the delay limited capacity of each of the links of the ZIC is zero [55]. Thus, the DMT characterization of this outage limited channel make sense.

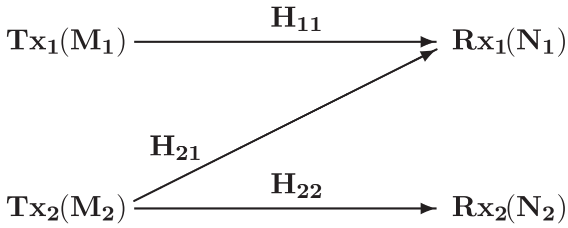


Figure 5.1: Channel model for the ZIC.

The received signals at time t can be written as

$$Y_{1t} = \eta_{11}\sqrt{P_1}H_{11}X_{1t} + \eta_{21}\sqrt{P_2}H_{21}X_{2t} + Z_{1t},$$
$$Y_{2t} = \eta_{22}\sqrt{P_2}H_{22}X_{2t} + Z_{2t},$$

where $Z_{it} \in \mathbb{C}^{N_i \times 1}$ are i.i.d as $\mathcal{CN}(\mathbf{0}, I_{N_i})$ across i and t and η_{ij} represents the signal attenuation factor [51] from Tx_i to Rx_j. The above equations can be equivalently written in the following form.

$$Y_{1t} = \sqrt{\mathrm{SNR}_{11}}H_{11}\hat{X}_{1t} + \sqrt{\mathrm{INR}_{21}}H_{21}\hat{X}_{2t} + Z_{1t}; \tag{5.2}$$
$$Y_{2t} = \sqrt{\mathrm{SNR}_{22}}H_{22}\hat{X}_{2t} + Z_{2t}, \tag{5.3}$$

where the normalized inputs $\hat{X}_i$s satisfy equation (5.1) with equality and SNR_{ii} and INR_{ji} are the signal-to-noise ratio and interference-to-noise ratio, respectively at receiver i, which from now onwards will be denoted by ρ_{ii} and ρ_{ji}, respectively. Further, the difference in performance due to relative difference in strengths of SNRs and INRs of the different links of the channel can not be captured through the DMT metric, if they differ by only a constant factor. To overcome this problem and characterize the DMT under a more general scenario of arbitrary SNR and INR strengths, we let the different SNRs and INRs to vary exponentially with respect to a nominal SNR, ρ with different scaling factors as follows:

$$\alpha_{11} = \frac{\log(\mathrm{SNR}_{11})}{\log(\rho)},\ \alpha_{22} = \frac{\log(\mathrm{SNR}_{22})}{\log(\rho)}, \tag{5.4}$$

$$\alpha_{21} = \frac{\log(\mathrm{INR}_{21})}{\log(\rho)}. \tag{5.5}$$

For brevity, in the sequel we shall use the following notations: $\mathcal{H} = \{H_{11}, H_{21}, H_{22}\}$, $\bar{\rho} = [\rho_{11}, \rho_{21}, \rho_{22}]$ and $\bar{\alpha} = [\alpha_{11}, \alpha_{21}, \alpha_{22}]$.

Diversity order of a point-to-point channel [48] is defined as the negative SNR exponent of the average probability of error at the receiver. Since in the present channel model there are 2 receivers and more than one SNR and INR parameters, in the next subsection, we provide the definitions of diversity order and the multiplexing gains appropriate to the ZIC.

5.2.1 Definition of DMT of a MIMO ZIC

Let us start by defining the DMT of the channel formally. Let Tx_i is transmitting information using a codebook $\mathcal{C}_i(\rho)$ having $2^{LR_i(\rho)}$ codewords, each of length L, at a rate $R_i(\rho)$, then the corresponding multiplexing gain is denoted by r_i where

$$r_i = \lim_{\rho\to\infty} \frac{R_i(\rho)}{\log(\rho)},\ \text{for}\ \ i = 1, 2. \tag{5.6}$$

Remark 5.2 Note that the maximum asymptotic rate supportable by the first direct link is $R_1^{\max} = \min\{M_1, N_1\} \log(\rho_{11})$, when the second user is silent. Putting this into equation (5.6) we have

$$r_i = \min\{M_1, N_1\}\alpha_{11} > \min\{M_1, N_1\},\ \text{if}\ \alpha_{11} > 1. \tag{5.7}$$

Apparently, it seems that the direct link can support a multiplexing-gain strictly larger than $\min\{M_1, N_1\}$. However, this is only a consequence of the fact that the multiplexing gain r_1 in (5.6) is defined with respect to (w.r.t.) the nominal SNR ρ and therefore, a by product of the more general mathematical model that we assume here in this chapter. With respect to the direct link's

SNR ρ_{11} the multiplexing-gain is still $\min\{M_1, N_1\}$, irrespective of the value of α_{11}, i.e.,

$$\hat{r}_i = \lim_{\rho_{11}\to\infty} \frac{R_1(\rho)}{\log(\rho_{11})} = \min\{M_1, N_1\}. \tag{5.8}$$

Alternatively, this apparent difference can also be removed by equating the nominal SNR ρ to the direct links SNR ρ_{11} which amounts to setting $\alpha_{11} = 1$ in equation (5.7).

Now, to define the diversity order, let $\mathcal{P}_{e,\mathcal{C}_i}(\bar{\rho}, r_1, r_2)$ represents the maximum of the average probability of errors at the receivers (averaged over the random channel, Gaussian additive noise at the receivers and different codewords of a codebook) at a multiplexing gain pair (r_1, r_2) and SNR of ρ, and $\mathcal{P}_e^*(\bar{\rho}, r_1, r_2)$ represents the minimum $\mathcal{P}_{e,\mathcal{C}_i}(\bar{\rho}, r_1, r_2)$ among all possible coding schemes, i.e.,

$$\mathcal{P}_e^*(\bar{\rho}, r_1, r_2) = \min_{\{\text{All possible coding scheme},\mathcal{C}_i(\rho)\}} \mathcal{P}_{e,\mathcal{C}_i}(\bar{\rho}, r_1, r_2), \tag{5.9}$$

then the corresponding diversity order [48] is defined as

$$d^*_{\text{ZIC}}(r_1, r_2) = \lim_{\rho\to\infty} \frac{-\log\left(\mathcal{P}_e^*(\bar{\rho}, r_1, r_2)\right)}{\log(\rho)}. \tag{5.10}$$

Note that the diversity order $d^*_{\text{ZIC}}(r_1, r_2)$ is a function of the relative scaling parameters of the different links $\bar{\alpha}$. However, for brevity we shall not mention them explicitly in it's notation.

The typical approach to characterize the DMT of a communication channel, whose exact instantaneous end-to-end mutual information (IMI) is not known, is to find an upper and a lower bound to it. Then, from this upper and lower bound to the IMI a lower and upper bound to the appropriately defined outage event is derived, respectively. If the later set of bounds have identical negative SNR exponents then that represents the DMT of the corresponding channel. In this chapter we adopt the same approach and therefore, need a subset and a superset to the IMI region of the channel, which we specify in the next subsection.

5.2.2 A Subset and a Superset to the instantaneous mutual information region

In this subsection, we shall first derive a set of upper bounds to the various end-to-end mutual information defining a superset to the IMI region of the MIMO ZIC. Next, we shall propose a simple superposition coding scheme, which can achieve a IMI region, with its various bounds within constant (independent of SNR and channel coefficients) number of bits to those of the superset. These bounds will then be used to derive the fundamental DMT of the channel.

Lemma 5.1 The IMI region of the 2-user MIMO ZIC with F-CSIT, for a given realization of channel matrices $\mathcal{H}$, denoted by $\mathcal{C}(\mathcal{H},\bar{\rho})$, is contained in the set of real-tuples $\mathcal{R}^u(\mathcal{H},\bar{\rho})$, where $\mathcal{R}^u(\mathcal{H},\bar{\rho})$ represents the set of rate pairs (R_1,R_2) such that $R_1,R_2 \geq 0$ and satisfy the following constraints:

$$\begin{aligned}
&R_i \leq \log\left|\left(I_{N_i}+\rho_{ii}H_{ii}H_{ii}^{\dagger}\right)\right| \triangleq I_{bi},\ i\in\{1,2\};\\
&R_1+R_2 \leq \log\left|\left(I_{N_1}+\rho_{21}H_{21}H_{21}^{\dagger}+\rho_{11}H_{11}H_{11}^{\dagger}\right)\right|+\\
&\quad \log\left|\left(I_{N_2}+\rho_{22}H_{22}\left(I_{M_2}+\rho_{21}H_{21}^{\dagger}H_{21}\right)^{-1}H_{22}^{\dagger}\right)\right| \triangleq I_{bs}.
\end{aligned}$$

Proof 5.1 (Proof of Lemma 5.1) The expression for the super set $\mathcal{R}^u(\mathcal{H},\bar{\rho})$ is obtained by substituting $H_{12}=0_{N_2\times M_1}$ in Lemma 1 of [40], which converts the 2-user IC into an ZIC.

In what follows, we find a subset to the IMI region of the channel. Consider a coding scheme where the first transmitter uses a random Gaussian code book and the second user uses a superposition code as follows

$$X_2 = U_2 + W_2, \tag{5.11}$$

where U_2 (hereafter mentioned as the private part of the message) and W_2 (public part of the message) are mutually independent complex Gaussian random vectors with covariance matrices as follows:

$$\begin{aligned}
\mathbb{E}(X_1X_1^{\dagger}) = I_{M_1},\ \ \mathbb{E}(W_2W_2^{\dagger}) = \frac{I_{M_2}}{2} \text{ and}\\
\mathbb{E}(U_2U_2^{\dagger}) = \frac{1}{2}\left(I_{M_2}+\rho^{\alpha_{21}}H_{21}^{\dagger}H_{21}\right)^{-1}.
\end{aligned}\tag{5.12}$$

Remark 5.3 Note that this covariance split satisfies the power constraint in equation (5.1).

Remark 5.4 The above described coding scheme is clearly a special case of the Han-Kobayashi coding scheme where the first transmitter's message does not have any private part. Also the DMT characterized in this chapter represents the best DMT achievable on a ZIC when both transmitters have full CSIT. It will be shown shortly that the above coding scheme which use the knowledge of only H_{21} at Tx_2 can achieve the F-CSIT DMT.

Lemma 5.2 For a given channel realization $\mathcal{H}$, the above described coding scheme can achieve an IMI region $\mathcal{R}^l(\mathcal{H},\bar{\rho})$, where $\mathcal{R}^l(\mathcal{H},\bar{\rho})$ represents the set of rate pairs (R_1,R_2) such that $R_1,R_2 \geq 0$ and satisfies the following constraints:

$$\begin{aligned} R_i \leq & \log\left|\left(I_{N_i} + \rho_{ii}H_{ii}H_{ii}^\dagger\right)\right| - n_i \triangleq I_{li}, \ i \in \{1,2\}; \\ R_1 + R_2 \leq & \log\left|\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger + \rho_{11}H_{11}H_{11}^\dagger\right)\right| + \\ & \log\left|\left(I_{N_2} + \rho_{22}H_{22}\left(I_{M_2} + \rho_{21}H_{21}^\dagger H_{21}\right)^{-1}H_{22}^\dagger\right)\right| - (n_1+n_2) \triangleq I_{ls}, \end{aligned}$$

where

$$n_i = \max\left\{(m_{ii}\log(M_i) + m_{ij}\log(M_i+1)), \min\{N_i, M_s\}\log(M_x)\right\} + \hat{m}_{ji}, \text{ for } 1 \leq i \neq j \leq 2 \quad (5.13)$$

with $M_x = \max\{M_1, M_2\}$, $M_s = (M_1+M_2)$, m_{ij} representing the rank of the matrix H_{ij}, and $\hat{m}_{ij} = m_{ij}\log\left(\frac{(M_i+1)}{M_i}\right)$. Note that $m_{ij} \leq \min\{M_i, N_j\}$.

Proof 5.2 (Proof of Lemma 5.2) The achievability of the rate region on the ZIC follows by substituting $H_{12} = 0_{N_2\times M_1}$ in Lemma 4 of [40], which converts the 2-user IC into an ZIC.

It has been explained earlier that the superset, $\mathcal{R}^u(\mathcal{H},\bar{\rho})$ given by Lemma 5.1 and the subset, $\mathcal{R}^l(\mathcal{H},\bar{\rho})$, given by Lemma 5.2 to $\mathcal{C}(\mathcal{H},\bar{\rho})$ can be used to derive a set of upper and lower bounds to DMT of the channel, respectively. In what follows, we prove this fact formally. Let Tx_i is operating at a rate $R_i(\rho)$ bits per channel use where the corresponding multiplexing-gain is $r_i = \lim_{\rho\to\infty}\frac{R_i(\rho)}{\log(\rho)}$ for $i=1,2$ and the outage event $\mathcal{O}$ is defined as follows

$$\mathcal{O} = \{\mathcal{H} : (R_1,R_2) \notin \mathcal{C}(\mathcal{H},\bar{\rho})\}. \quad (5.14)$$

Following similar method as in [48] it can be easily proved that

$$\mathcal{P}_e^*(\bar{\rho}, r_1, r_2) \doteq \Pr(\mathcal{O}), \quad (5.15)$$

where $\mathcal{P}_e^*(\bar{\rho}, r_1, r_2)$ represents the minimum average probability of error achievable on the ZIC, as defined in subsection 5.2.1. Now, from Lemma 5.1 and 5.2 for any realization of the channel matrices $\mathcal{H}$ we have,

$$\begin{aligned}
&\mathcal{R}^l(\mathcal{H},\bar{\rho}) \subseteq \mathcal{C}(\mathcal{H},\bar{\rho}) \subseteq \mathcal{R}^u(\mathcal{H},\bar{\rho});\\
&\text{Or, } \{\mathcal{H} : (R_1,R_2) \notin \mathcal{R}^u(\mathcal{H},\bar{\rho})\} \subseteq \mathcal{O} \subseteq \{\mathcal{H} : (R_1,R_2) \notin \mathcal{R}^l(\mathcal{H},\bar{\rho})\};\\
&\text{Or, } \Pr\{(R_1,R_2) \notin \mathcal{R}^u(\mathcal{H},\bar{\rho})\} \dot{\leq} \mathcal{P}_e^*(\bar{\rho}, r_1, r_2) \dot{\leq} \Pr\left\{(R_1,R_2) \notin \mathcal{R}^l(\mathcal{H},\bar{\rho})\right\};\\
&\text{Or, } \Pr\{(R_1,R_2) \notin \mathcal{R}^u(\mathcal{H},\bar{\rho})\} \dot{\leq} \rho^{-d^*_{\mathrm{ZIC}}(r_1,r_2)} \dot{\leq} \Pr\left\{(R_1,R_2) \notin \mathcal{R}^l(\mathcal{H},\bar{\rho})\right\};\\
&\text{Or, } \Pr\{\cup_i\{I_{bi} \leq R_i\}\} \dot{\leq} \rho^{-d^*_{\mathrm{ZIC}}(r_1,r_2)} \dot{\leq} \Pr\{\cup_i\{I_{li} \leq R_i\}\};\\
&\text{Or, } \max_{i\in\{1,2,s\}} \Pr\{I_{bi} \leq R_i\} \dot{\leq} \rho^{-d^*_{\mathrm{ZIC}}(r_1,r_2)} \dot{\leq} \sum_{i=1,2,s} \Pr\{I_{li} \leq R_i\}, && (5.16)\\
&\text{Or, } \max_{i\in\{1,2,s\}} \Pr\{I_{bi} \leq R_i\} \dot{\leq} \rho^{-d^*_{\mathrm{ZIC}}(r_1,r_2)} \dot{\leq} \max_{i\in\{1,2,s\}} \Pr\{I_{li} \leq R_i\}, && (5.17)
\end{aligned}$$

where $R_s = (R_1 + R_2)$ and I_{bi}'s and I_{li}'s are as defined in Lemma 5.1 and Lemma 5.2, respectively. Note that, $I_{bi} = I_{li} + n_i$, for $i = 1, 2$ and $I_{bs} = I_{ls} + (n_1 + n_2)$ where n_i's given by equation (5.13) are constants independent of ρ, for $i \in \{1, 2\}$, which becomes insignificant at asymptotic SNR. Therefore, at asymptotic values of ρ equation (5.17) is equivalent to

$$\rho^{-d^*_{\mathrm{ZIC}}(r_1,r_2)} \doteq \max_{i\in\{1,2,s\}} \Pr\{I_{bi} \leq R_i\},$$

which can be written as

$$d^*_{\mathrm{ZIC}}(r_1,r_2) = \min_{i\in\mathcal{I}} d_{O_i}(r_i), \text{ where} \tag{5.18}$$

$$d_{O_i}(r_i) = \lim_{\rho\to\infty} -\frac{\Pr\left(I_{bi} \leq r_i \log(\rho)\right)}{\log(\rho)}, \tag{5.19}$$

for all $i \in \mathcal{I} = \{1, 2, s\}$ and $r_s = (r_1 + r_2)$.

The only remaining step to characterize the DMT completely is to evaluate the probabilities in equation (5.19), which in turn requires the statistics of the mutual information terms I_{b_i}'s. It will be shown in the next section that this statistics and therefrom the DMT of the channel can be characterized, if only the joint distribution of the eigenvalues of 2 mutually correlated random Wishart matrices are known.

5.3 Explicit DMT of the ZIC

In this section, we shall evaluate the different SNR exponents, $d_{O_i}(r_i)$'s, of the various outage events given in equation (5.18), which would yield the explicit DMT expression for the ZIC. Substituting the right hand sides of the first and second bound's in Lemma 5.1 in equations (5.19) we have

$$d_{O_i}(r_i) = \lim_{\rho\to\infty} -\frac{\Pr\left(\sum_{k=1}^{\min\{M_i,N_i\}}(\alpha_{ii}-\upsilon_{i,k})^+ \leq r_i\right)}{\log(\rho)},\ i \in \{1,2\}, \tag{5.20}$$

where $\upsilon_{i,k}$'s are the negative SNR exponents of the ordered eigenvalues of the matrices $H_{ii}H_{ii}^\dagger$. The joint distribution of $\{\upsilon_{i,k}\}_{k=1}^{\min\{M_i,N_i\}}$ was specified in [48]. Using this distribution and a similar technique as in [48], it can be shown that

$$d_{O_i}(r_i) = \min \sum_{k=1}^{\min\{M_i,N_i\}} (M_i+N_i+1-2k)\upsilon_{i,k} \tag{5.21a}$$

$$\text{subject to:} \quad \sum_{k=1}^{\min\{M_i,N_i\}} (\alpha_{ii}-\upsilon_{i,k})^+ \leq r; \tag{5.21b}$$

$$0 \leq \upsilon_{i,1} \leq \cdots \leq \upsilon_{i,\min\{M_i,N_i\}}. \tag{5.21c}$$

Similar optimization problem will recur in the remaining part of the chapter. So we formally state the solution of the above problem in the following Lemma for convenience of reference later.

Lemma 5.3 If $d(r)$ represents the optimal solution of the optimization problem,

$$\min \sum_{i=1}^{m}(M+N+1-2i)\mu_i \tag{5.22a}$$

$$\text{subject to:} \quad \sum_{i=1}^{m}(\alpha-\mu_i)^+ \leq r; \tag{5.22b}$$

$$0 \leq \mu_1 \leq \cdots \leq \mu_m, \tag{5.22c}$$

then,

$$d(r) = \alpha d_{M,N}\left(\frac{r}{\alpha}\right),\ \text{for } 0 \leq r \leq m\alpha, \tag{5.23}$$

where $m = \min\{M,N\}$ and $d_{M,N}(r)$ represents the DMT of a $M \times N$ point-to-point channel and is a piecewise linear curve joining the points $(M-k)(N-k)$ for $k = 0, 1, \cdots m$.

Proof 5.3 (Proof) Putting $\mu_i^{'} = \frac{\mu_i}{\alpha}$ in the optimization problem (5.22) we get,

$$\frac{d(r)}{\alpha} = \min \sum_{i=1}^{m} (M + N + 1 - 2i)\mu_i^{'} \tag{5.24a}$$

$$\text{subject to: } \sum_{i=1}^{m} (1 - \mu_i^{'})^+ \leq \frac{r}{\alpha}; \tag{5.24b}$$

$$0 \leq \mu_1^{'} \leq \cdots \leq \mu_m^{'}. \tag{5.24c}$$

The solution of this modified optimization problem was derived in [48] and is given by

$$\frac{d(r)}{\alpha} = d_{M,N}\left(\frac{r}{\alpha}\right), \text{ for } 0 \leq \frac{r}{\alpha} \leq m,$$
$$\text{or, } d(r) = \alpha d_{M,N}\left(\frac{r}{\alpha}\right), \text{ for } 0 \leq r \leq m\alpha.$$

The solution of the optimization problem (5.21) is now evident from Lemma 5.3, and is given by

$$d_{O_i}(r_i) = \alpha_{ii} d_{M_i,N_i}\left(\frac{r_i}{\alpha_{ii}}\right), \ \forall\ r_i \in [0, \min\{M_i, N_i\}\alpha_{ii}] \text{ and } i \in \{1, 2\}, \tag{5.25}$$

where $d_{m,n}(r)$ is the optimal diversity order of a point-to-point MIMO channel with m transmit and n receive antennas, at integer values of r and is point wise linear between integer values of r. To evaluate $d_{O_s}(r_s)$, we write the bound I_{bs} of Lemma 5.1 in the following way

$$\begin{aligned} I_{bs} =& \log\left|\left(I_{M_1} + \rho_{11} H_{11}^{\dagger}\left(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger}\right)^{-1} H_{11}\right)\right| + \log\left|\left(I_{N_2} + \rho_{22} H_{22}\left(I_{M_2} + \rho_{21} H_{21}^{\dagger} H_{21}\right)^{-1} H_{22}^{\dagger}\right)\right| \\ &+ \log\left|\left(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger}\right)\right|, \\ \stackrel{(a)}{=}& \left\{\sum_{i=1}^{p}(1 + \rho^{\alpha_{21}}\lambda_i) + \sum_{j=1}^{q_1}(1 + \rho^{\alpha_{11}}\mu_j) + \sum_{k=1}^{q_2}(1 + \rho^{\alpha_{22}}\pi_k)\right\}, \end{aligned}$$

where in step (a), $p = \min\{M_2, N_1\}$, $q_1 = \min\{M_1, N_1\}$, $q_2 = \min\{M_2, N_2\}$ and we have denoted the ordered non-zero (with probability 1) eigenvalues of $W_1 = H_{11}^{\dagger}\left(I_{N_1} + \rho_{21} H_{21} H_{21}^{\dagger}\right)^{-1} H_{11}$, $W_2 = H_{22}\left(I_{M_2} + \rho_{21} H_{21}^{\dagger} H_{21}\right)^{-1} H_{22}^{\dagger}$ and $W_3 = H_{21} H_{21}^{\dagger}$ by $\mu_1 \geq \cdots \geq \mu_{q_1} > 0$, $\pi_1 \geq \cdots \geq \pi_{q_2} > 0$ and $\lambda_1 \geq \cdots \geq \lambda_p > 0$, respectively. Now, using the transformations $\lambda_i = \rho^{-\upsilon_i}$, for $1 \leq i \leq p$, $\mu_j = \rho^{-\beta_j}$, for $1 \leq j \leq q_1$ and $\pi_k = \rho^{-\gamma_k}$, $1 \leq k \leq q_2$ in the above equation and substituting that in turn in equation (5.19) we get

$$\rho^{-d_{O_s}(r_s)} \dot{=} \Pr\left(\left\{\sum_{i=1}^{p}(\alpha_{21}-\upsilon_i)^+ + \sum_{j=1}^{q_1}(\alpha_{11}-\beta_j)^+ + \sum_{k=1}^{q_2}(\alpha_{22}-\gamma_k)^+\right\} < r_s\right). \quad (5.26)$$

To evaluate this expression we need to derive the joint distribution of $\vec{\gamma}, \vec{\beta}$ and $\vec{\upsilon}$ where $\vec{\gamma} = \{\gamma_1, \cdots, \gamma_{q_2}\}$ and similarly $\vec{\upsilon} = \{\upsilon_1, \cdots, \upsilon_p\}$ and $\vec{\beta} = \{\beta_1, \cdots, \beta_{q_1}\}$. Note that W_1, W_2 and W_3 are not independent and hence neither are $\vec{\gamma}, \vec{\beta}$ and $\vec{\upsilon}$. However, this distribution can be computed using Theorems 1 and 2 of [52]. Using this joint distribution, equation (5.26) and a similar argument as in [48] $d_{O_s}(r_s)$ can be derived as the solution of an convex optimization problem as stated in the following Lemma.

Lemma 5.4 The negative SNR exponent of the outage event corresponding to the sum bound in Lemma 5.2, i.e., $d_{O_s}(r_s)$, is equal to the minimum of the following objective function:

$$\begin{aligned}\mathcal{F}^{\text{FCSIT}}_{(M_1,N_1,M_2,N_2)} =& \sum_{i=1}^{p}(M_2+N_1+M_1+N_2+1-2i)\upsilon_i + \sum_{j=1}^{q_1}(M_1+N_1+1-2j)\beta_j \\ &+ \sum_{k=1}^{q_2}(M_2+N_2+1-2k)\gamma_k - (M_1+N_2)p\alpha_{21} \\ &+ \sum_{k=1}^{q_2}\sum_{i=1}^{\min\{(M_2-k),N_2\}}(\alpha_{21}-\upsilon_i-\gamma_k)^+ + \sum_{j=1}^{q_1}\sum_{i=1}^{\min\{(N_1-j),M_1\}}(\alpha_{21}-\upsilon_i-\beta_j)^+ \end{aligned} \quad (5.27)$$

constrained by

$$\sum_{i=1}^{p}(\alpha_{21}-\upsilon_i)^+ + \sum_{j=1}^{q_1}(\alpha_{11}-\beta_j)^+ + \sum_{k=1}^{q_2}(\alpha_{22}-\gamma_k)^+ < r_s; \quad (5.28a)$$

$$0 \le \upsilon_1 \le \cdots \le \upsilon_p; \quad (5.28b)$$

$$0 \le \beta_1 \le \cdots \le \beta_{q_1}; \quad (5.28c)$$

$$0 \le \gamma_1 \le \cdots \le \gamma_{q_2}; \quad (5.28d)$$

$$(\upsilon_i + \beta_j) \ge \alpha_{21}, \ \forall (i+j) \ge (N_1+1); \quad (5.28e)$$

$$(\upsilon_i + \gamma_k) \ge \alpha_{21}, \ \forall (i+k) \ge (M_2+1). \quad (5.28f)$$

Proof 5.4 The proof is given in Appendix A.13.

Differentiating the objective function with respect to $\{\upsilon_i\}$, $\{\beta_j\}$ and $\{\gamma_k\}$, it can be easily verified that (5.27) is a convex function of these variables. The constraints on the other hand are all linear. Therefore, equations (5.27) and (5.28) represent an convex optimization problem (e.g., see subsection 4.2.1 in [56]) and hence can be solved efficiently using numerical methods. Since we have already found expressions for $d_{\mathcal{O}_i}$ for $i = 1, 2$ as in equation (5.25), Lemma 5.4 provides the last piece of the puzzle required to characterize the DMT of the MIMO ZIC, by evaluating $d_{\mathcal{O}_s}$. This is stated formally in the following Theorem.

Theorem 5.1 The DMT of the (M_1, N_1, M_2, N_2) ZIC, with FCSIT and short term power allocation scheme, is given as

$$d^{*,\text{FCSIT}}_{\text{ZIC}}(r_1, r_2) = \min_{i\in\{1,2,s\}} d_{\mathcal{O}_i}(r_i),$$

where $d_{\mathcal{O}_i}(r_i)$ for $i = 1, 2$ and $i = s$ are given by equation (5.25) and Lemma 5.4, respectively.

Although the computation of $d_{\mathcal{O}_s}(r_s)$ and hence characterization of the DMT of a general ZIC with arbitrary number of antennas at each node require application of numerical methods, in what follows, we shall provide closed form expressions for it for various special cases. Since the DMT with FCSIT acts as an upper bound to the DMT of the channel with no-CSIT or only CSIR, these expressions facilitates an easy characterization of the gap between perfect and no CSIT performances of the channel. The no-CSIT DMT of the channel will be characterized in section 5.4, for different range of values of α_{21} and no of antennas at different nodes.

The central idea in all of the following proof is the fact that the steepest descent method provides a global optimal value of a convex optimization problem: it is well known that the steepest descent method provides a local optimal value of the objective function. However, a local optimal solution is equal to a global one as well, for a convex function [56]. Combining the above two facts we conclude that the value obtained by the steepest descent method is actually the global minimum of the objective function. The first case considered is a class of channels where all the nodes have equal number of antennas.

Theorem 5.2 Consider the MIMO ZIC with $M_1 = M_2 = N_1 = N_2 = n$ and SNRs and INRs of different links are as described in Section 5.2 with $\alpha_{21} = \alpha$ and $\alpha_{22} = 1 = \alpha_{11}$. The best achievable diversity order on this channel, with F-CSIT and short term average power constraint (5.1), at multiplexing gain pair (r_1, r_2), is given by

$$d^{*,\text{FCSIT}}_{\text{ZIC},(n,n,n,n)}(r_1, r_2) = \min\left\{d_{n,n}(r_1), d_{n,n}(r_2), d^{\text{FCSIT}}_{s,(n,n,n,n)}(r_s)\right\} \tag{5.29}$$

where $d_{\mathcal{O}_s}(r_s)$ for the special channel configuration being considered is denoted by $d^{\text{FCSIT}}_{s,(n,n,n,n)}(r_s)$, with

$$d^{\text{FCSIT}}_{s,(n,n,n,n)}(r_s) = \begin{cases} \alpha d_{n,3n}(\frac{r_s}{\alpha}) + 2n^2(1-\alpha), \text{ for } 0 \leq r_s \leq n\alpha; \\ 2(1-\alpha)d_{n,n}(\frac{(r_s - n\alpha)}{2(1-\alpha)}), \text{ for } n\alpha \leq r_s \leq n(2-\alpha). \end{cases} \tag{5.30}$$

if $\alpha \leq 1$ and

$$d^{\text{FCSIT}}_{s,(n,n,n,n)}(r_s) = \begin{cases} d_{n,3n}(r_s) + n^2(\alpha - 1), \ 0 \leq r_s \leq n; \\ (\alpha - 1)d_{n,n}\left(\frac{r_s - n}{\alpha - 1}\right), \ n \leq r_s \leq n\alpha, \end{cases} \tag{5.31}$$

for $1 \leq \alpha$.

Proof 5.5 (Proof of Theorem 5.2) The proof is given in Appendix A.14.

Remark 5.5 Note that for $\alpha = 1$, the optimal DMT becomes $d^{\text{FCSIT}}_{s,(n,n,n,n)}(r_s) = \min\{d_{n,n}(r_1), d_{n,n}(r_2), d_{n,3n}(r_1$ - which is exactly the upper bound to the DMT of the 2-user MIMO IC with n antennas at each node, derived in [30]. Since in the ZIC, the second receiver is free of interference, the fundamental DMT of the ZIC serves as an upper bound to the DMT of the 2-user IC.

5.3.1 The DMT of a Femto-Cell

A practical communication channel following the ZIC signal model appears in the so called Femtocell environment. The Femtocell [49] concept is an outcome of the telecommunication industry's efforts to provide high-throughput, high quality services into the user's home. Consider the scenario depicted in figure 5.2, where the larger circle represents the macro cell serviced by the macro cell base station (MCBS). Within this macro cell is the smaller circle represents a small area where the signal from the MCBS either does not reach with enough strength or does not reach at all, hereafter referred to as the Femtocell. To provide coverage in this region a smaller user deployed base station connected to the backbone can be used, which is called the Femto cell BS (FCBS). This FCBS can provide mobile services to the users within the Femtocell just like a WiFi access point. The basic difference between the FCBS and WiFi access point is that the former operates in a licensed band.

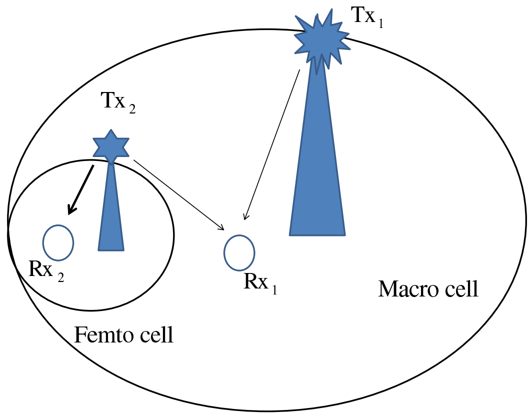


Figure 5.2: Femtocell channel model: down link.

Now, let us consider the downlink communication on such a channel with one mobile user in both the Femtocell and the macro-cell. Note that since the MCBS signal does not reach the mobile user within the Femtocell, the signal input-output follows the ZIC model. To model the larger SNR of the Femtocell direct link we can assume that $\alpha_{22} \geq 1$. In what follows, we shall derive the DMT of this channel.

Theorem 5.3 Consider a ZIC with $M_1 = M_2 = N_1 = N_2 = n$, $\alpha_{22} = \alpha \geq 1$ and $\alpha_{11} = \alpha_{21} = 1$, F-CSIT and short term average power constraint given by (5.1). The optimal diversity order of this channel at a multiplexing gain pair (r_1, r_2) is given by

$$d^{\text{Femto}}_{(n,n,n,n)}(r_1, r_2) = \min\left\{d_{n,n}(r_1), d_{n,n}(r_2), d^{\text{Femto}}_{s,(n,n,n,n)}(r_s)\right\}$$

where $d_{\mathcal{O}_s}(r_s)$ for the special channel configuration being considered, is denoted by $d^{\text{Femto}}_{s,(n,n,n,n)}(r_s)$, and

$$d^{\text{Femto}}_{s,(n,n,n,n)}(r_s) = \begin{cases} d_{n,3n}(r_s) + n^2(\alpha - 1), & \text{for } 0 \leq r_s \leq n; \\ (\alpha - 1)d_{n,n}(\frac{(r_s - n)}{(\alpha - 1)}), & \text{for } n \leq r_s \leq n\alpha. \end{cases}$$

Proof 5.6 The proof is given in Appendix A.15.

Remark 5.6 Note that the fundamental DMT of the ZIC with single antenna nodes and $\alpha_{22} = \alpha, \alpha_{11} = \alpha_{21} = 1$ was derived in [57]. This clearly is a special case of Theorem 5.3 and can be obtained by putting $n = 1$.

Typically, in a multiuser communication scenario one end – say the base station in a cellular network – can host more antennas than the other. Motivated by this fact in what follows we consider a case which addresses the DMT of such a practical communication network, i.e., where $M_1 = M_2 = M \leq \min\{N_1, N_2\}$.

Theorem 5.4 Consider the ZIC with $M_1 = M_2 = M \leq \min\{N_1, N_2\}$, $\alpha_{11} = \alpha_{22} = \alpha_{21} = 1$, F-CSIT and short term average power constraint given by (5.1). The optimal diversity order achievable on this channel at a multiplexing gain pair (r_1, r_2) is given by

$$d^{\text{FCSIT}}_{M,N_1,M,N_2}(r_1, r_2) = \min\left\{d_{M,N_1}(r_1), d_{M,N_2}(r_2), d^{\text{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s)\right\}$$

where $d_{\mathcal{O}_s}(r_s)$ for the special channel configuration being considered, is denoted by $d^{\text{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s)$, and

$$d^{\text{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s) = \begin{cases} d_{M,(M+N_1+N_2)}(r_s) + M(N_1 - M); & 0 \leq r_s \leq M; \\ d_{2M,N_1}(r_s); & M \leq r_s \leq \min\{N_1, 2M\}. \end{cases}$$

Proof 5.7 The proof is given in Appendix A.16.

Let us now quantify the loss due the use of sub-optimal coding schemes on ZICs, with respect to the fundamental DMT of the channel achievable by sophisticated coding schemes such as the superposition coding scheme described in subsection 5.2.2. In Figure 5.3, explicit F-CSIT DMT curves for a few antenna configurations are plotted and compared against the performance of

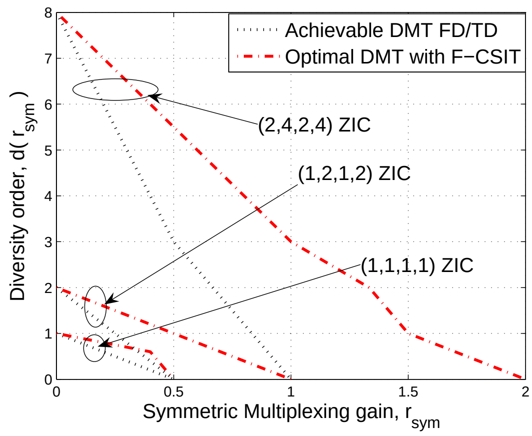


Figure 5.3: Optimal DMT of different ZICs with $\bar{\alpha} = [1, 1, 1]$.

orthogonal schemes such as frequency division (FD) or time division (TD) multiple-access which do not even utilize CSIT. It can be noticed from the figure that the gain due to CSIT, over the orthogonal access schemes can be significant, particularly in MIMO ZICs. While using better coding-decoding schemes this gap can be reduced, in general with CSIT a better performance can be achieved. However, to evaluate this gap in performance due to lack of CSIT exactly, it is necessary to know the best DMT achievable on the channel without any CSIT. Popular approaches of characterization of the No-CSIT DMT involves either evaluating the optimal instantaneous mutual information region of the channel without CSIT exactly or finding subsets and supersets of it which are within a constant number of bits to each other, as was done in the case of F-CSIT considered so far. For the no CSIT case either of the above two objectives are very hard to achieve. However, in the following section, we bypass this approach and characterize the No-CSIT DMT of the ZIC for some specific values of the different channel parameters such as the number of antennas at the different nodes and the INR parameter α.

5.4 DMT with No CSIT

To avoid the above mentioned difficulties in this section, we first note that the F-CSIT DMT derived in the previous sections can serve as an upper bound to the No-CSIT DMT of the channel.

Then, we derive the achievable DMT of a No-CSIT transmit-receive scheme, which for two special classes of ZICs meets the upper bound and therefore, represents the fundamental No-CSIT DMT of the corresponding ZICs. In what follows, we shall describe the No-CSIT transmit-receive scheme first.

Let both the users encode their messages using independent Gaussian signals. Moreover, we consider a decoder at Rx_1 which does joint maximum-likelihood (ML) decoding of both the messages. However, since Rx_1 is not interested in the signal transmitted by Tx_2, the event where only the second user's message is decoded incorrectly is not considered as an error event. Rx_2 uses an ML decoder to decode its own message. Hereafter, we will refer to this scheme as the *Individual ML (IML) decoder* and the encoding-decoding scheme as the Independent coding IML decoding scheme or the IIML scheme.

An achievable rate region of the individual ML decoder and independent Gaussian coding at both the users is given by the following set of rate tuples

$$\mathcal{R}_{\text{IML}} = \Big\{(R_1, R_2) : R_1 \leq \log\det\left(I_{N_1} + \frac{\rho}{M_1} H_{11} H_{11}^\dagger\right) \triangleq I_{c_1}; \tag{5.32a}$$

$$R_2 \leq \log\det\left(I_{N_2} + \frac{\rho}{M_2} H_{22} H_{22}^\dagger\right) \triangleq I_{c_2}; \tag{5.32b}$$

$$(R_1 + R_2) \leq \log\det\left(I_{N_1} + \frac{\rho}{M_1} H_{11} H_{11}^\dagger + \frac{\rho^\alpha}{M_2} H_{21} H_{21}^\dagger\right) \triangleq I_{c_s}; \Big\} \tag{5.32c}$$

Note in the above set of equations we do not have a constraint on R_2 due to the point-to-point link from Tx_2 to Rx_1 because of the IML decoding definition, i.e., Rx_2 does not consider it as an error event if only the message of Tx_2 is decoded erroneously. Using the above expression for the achievable rate region, the corresponding achievable DMT of this transmit-receive scheme, i.e., mutually independent Gaussian coding at each transmitters and IML decoder at the interfered receiver, can be easily computed using standard techniques. The result is specified in the following Lemma.

Lemma 5.5 If we denote the achievable diversity order of the IML scheme, at multiplexing gain pair (r_1, r_2), by $d^{\text{IML}}_{(M_1,N_1,M_2,N_2)}(r_1, r_2)$ then,

$$d^{\text{IML}}_{(M_1,N_1,M_2,N_2)}(r_1, r_2) \;\geq\; \min_{i\in\{1,2,s\}} \{d^{\text{IML}}_{i,(M_1,N_1,M_2,N_2)}(r_i)\}, \tag{5.33}$$

where $r_s = (r_1 + r_2)$ and

$$d^{\text{IML}}_{i,(M_1,N_1,M_2,N_2)}(r_i) = \lim_{\rho\to\infty} -\frac{\log\left(\Pr\left(I_{ci} \leq r_i\right)\right)}{\log(\rho)},\; \forall\, i \in \{1, 2, s\}. \tag{5.34}$$

Proof 5.8 The proof is given in Appendix A.17.

Note that I_{c_1} and I_{c_2} represents the mutual information of a point-to-point channel with channel matrices H_{11} and H_{22}, respectively. Therefore, by the results of [48] we have

$$d^{\text{IML}}_{i,(M_1,N_1,M_2,N_2)}(r_i) = d_{M_i,N_i}(r_i),\; 0 \leq r_i \leq \min\{M_i, N_i\}, \tag{5.35}$$

and $i \in \{1, 2\}$. To analyze the outage event due to the third bound of the achievable rate region, we first approximate I_{c_s} by another term which does not differ from it by more than a constant. Note that

$$\begin{aligned}
&\log\det\left(I_{N_1} + \rho H_{11}H_{11}^{\dagger} + \rho^{\alpha} H_{21}H_{21}^{\dagger}\right) - N_1 \log(\max\{M_1, M_2\}),\\
&\leq I_{c_s} \leq \log\det\left(I_{N_1} + \rho H_{11}H_{11}^{\dagger} + \rho^{\alpha} H_{21}H_{21}^{\dagger}\right) \triangleq I'_{c_s}.
\end{aligned} \tag{5.36}$$

Since a constant independent of ρ, does not matter in the high SNR analysis, to compute d^{IML}_{s} we can use I'_{c_s} in place of I_{c_s}. Next, we write I'_{c_s} in the following manner:

$$I'_{c_s} = \log\det\left(I_{N_1} + \rho H_{11}H_{11}^{\dagger} + \rho^{\alpha} H_{21}H_{21}^{\dagger}\right), \tag{5.37}$$

$$= \log\det\left(I_{M_1} + \rho \widetilde{H}_{11}^{\dagger}\widetilde{H}_{11}\right) + \log\det\left(I_{N_1} + \rho^{\alpha} H_{21}H_{21}^{\dagger}\right), \tag{5.38}$$

$$\stackrel{(a)}{=} \left\{\sum_{j=1}^{q_1}(1 + \rho^{\alpha_{11}}\mu_j) + \sum_{i=1}^{p}(1 + \rho^{\alpha_{21}}\lambda_i)\right\}, \tag{5.39}$$

where $\widetilde{H}_{11} = \left(I_{N_1} + \rho^{\alpha} H_{21}H_{21}^{\dagger}\right)^{-\frac{1}{2}} H_{11}$ and in step (a), $p = \min\{M_2, N_1\}$, $q_1 = \min\{M_1,\; N_1\}$. Also, we have denoted the ordered non-zero (with probability 1) eigenvalues of $W_1 = H_{11}^{\dagger}\left(I_{N_1} + \rho_{21} H_{21}H_{21}^{\dagger}\right)^{-1}$ and $W_2 = H_{21}H_{21}^{\dagger}$ by $\mu_1 \geq \cdots \geq \mu_{q_1} > 0$ and $\lambda_1 \geq \cdots \geq \lambda_p > 0$, respectively. Now, using the transformations $\lambda_i = \rho^{-\upsilon_i}$, for $1 \leq i \leq p$, $\mu_j = \rho^{-\beta_j}$, for $1 \leq j \leq q_1$ in the above equation and substituting that in turn in equation (5.34) we get

$$\rho^{-d^{\text{IML}}_{s,(M_1,N_1,M_2,N_2)}(r_s)} \dot{=} \Pr\left(\left\{\sum_{i=1}^{p}(\alpha_{21}-\upsilon_i)^{+} + \sum_{j=1}^{q_1}(\alpha_{11}-\beta_j)^{+}\right\} < r_s\right). \tag{5.40}$$

To evaluate this expression we need to derive the joint distribution of $\vec{\beta}$ and $\vec{\upsilon}$. Note that, since W_1 and W_2 are mutually correlated and so are $\vec{\beta}$ and $\vec{\upsilon}$. As already stated earlier, in general characterizing the joint distribution of the eigenvalues of such mutually correlated random matrices is a hard problem. However, in what follows, we show that this distribution can be computed using Theorems 1 and 2 of [52], which in turn facilitates the characterization of $d_s^{\text{IML}}(r_s)$.

Lemma 5.6 The negative SNR exponent of the outage event corresponding to the sum bound in (5.32), i.e., $d^{\text{IML}}_{s,(M_1,N_1,M_2,N_2)}(r_s)$, is equal to the minimum of the following objective function:

$$\begin{aligned} d^{\text{IML}}_{s,(M_1,N_1,M_2,N_2)}(r_s) = \min_{(\vec{\upsilon},\vec{\beta})} &\sum_{i=1}^{p}(M_2+N_1+M_1+1-2i)\upsilon_i + \sum_{j=1}^{q_1}(M_1+N_1+1-2j)\beta_j \\ &- M_1 p\alpha_{21} + \sum_{j=1}^{q_1}\sum_{i=1}^{\min\{(N_1-j),M_1\}}(\alpha_{21}-\upsilon_i-\beta_j)^{+}; \end{aligned} \tag{5.41a}$$

$$\text{constrained by: } \sum_{i=1}^{p}(\alpha_{21}-\upsilon_i)^{+} + \sum_{j=1}^{q_1}(\alpha_{11}-\beta_j)^{+} \leq r_s; \tag{5.41b}$$

$$0 \leq \upsilon_1 \leq \cdots \leq \upsilon_p; \tag{5.41c}$$

$$0 \leq \beta_1 \leq \cdots \leq \beta_{q_1}; \tag{5.41d}$$

$$(\upsilon_i + \beta_j) \geq \alpha_{21},\ \forall (i+j) \geq (N_1+1). \tag{5.41e}$$

Proof 5.9 The proof is given in Appendix A.18.

Theorem 5.5 (A lower bound to the No-CSIT DMT of the ZIC) (1) The optimal diversity order achievable by the IIML scheme described above, on a (M_1, N_1, M_2, N_2) ZIC without CSIT is given as

$$d^{\text{No-CSIT}}_{\text{LB, ZIC}}(r_1, r_2) = \min_{i\in\{1,2,s\}} d^{\text{IML}}_{i,(M_1,N_1,M_2,N_2)}(r_i),$$

where $d^{\mathrm{IML}}_{i,(M_1,N_1,M_2,N_2)}(r_i)$ for $i = 1, 2$ and $i = s$ are given by equation (5.35) and Lemma 5.6, respectively.

(2) This DMT also represents a lower bound to the No-CSIT DMT of the (M_1, N_1, M_2, N_2) ZIC.

Proof 5.10 The first part of the Theorem follows from Lemma 5.5, and the second part of the Theorem follows from the fact that the IIML scheme is only one of the numerous transmit-receive schemes that are possible on the ZIC.

Although the computation of $d^{\mathrm{IML}}_{s,(M_1,N_1,M_2,N_2)}(r_s)$ and hence characterization of the lower bound to the No-CSIT DMT of a general ZIC with arbitrary number of antennas at each node require application of numerical methods, in what follows, we shall provide closed form expressions for it for various special cases. We shall see that for two special classes of ZICs this lower bound meets the upper bound, i.e., the FCSIT DMT of the channel. We start with the case where all the nodes have equal number of antennas.

Lemma 5.7 On a MIMO ZIC with n antennas at all the nodes, $\alpha_{11} = \alpha_{22} = 1$ and $\alpha_{21} = \alpha$, the IML decoder can achieve the following DMT

$$d^{\mathrm{IML}}_{(n,n,n,n)}(r_1, r_2) = \min\left\{d_{n,n}(r_1), d_{n,n}(r_2), d^{\mathrm{IML}}_{s,(n,n,n,n)}(r_s)\right\}$$

where $d^{\mathrm{IML}}_{s,(n,n,n,n)}(r_s)$ is given as

$$d^{\mathrm{IML}}_{s,(n,n,n,n)}(r_s) = \begin{cases} d_{n,2n}(r_s) + n^2(\alpha - 1), \ 0 \le r_s \le n; \\ (\alpha - 1)d_{n,n}(\frac{r_s - n}{\alpha - 1}), \ n \le r_s \le n\alpha. \end{cases} \tag{5.42}$$

Proof 5.11 The desired result is obtained by following the same steps as in the 2nd part of Theorem 5.2.

Figure 5.4 illustrates that the IML decoder can achieve the DMT (with F-CSIT) of the MIMO ZIC on a region of low multiplexing gains.

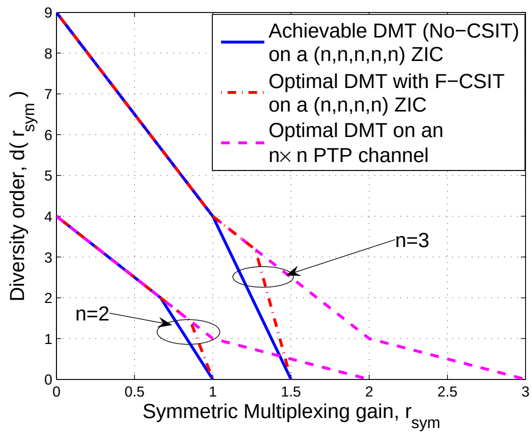


Figure 5.4: Comparison of the DMT on a ZIC with $\bar{\alpha} = [1, 1, 1]$ to PTP performance.

On the other hand, comparing equations (5.31) to (5.42) we see that the IML decoder can achieve the F-CSIT DMT for high multiplexing gain values as well, when $\alpha \geq 1$. This fact raises the natural question: is it possible for the IML decoder to achieve the F-CSIT DMT for all multiplexing gains and if it is, under what circumstances? It turns out that if the interference is strong enough then the IML decoder can achieve the F-CSIT DMT for all symmetric multiplexing gains. For example, Figure 5.5 illustrates this effect on a ZIC with 2 antennas at all the nodes.

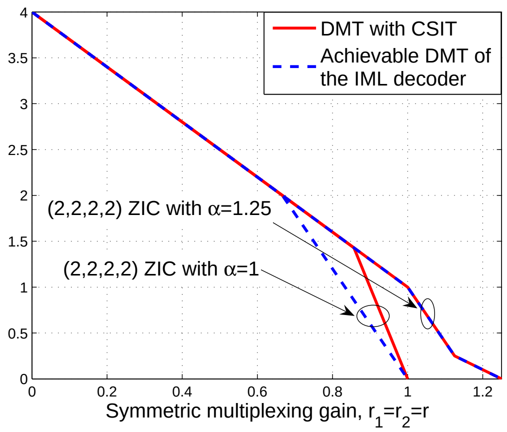


Figure 5.5: The DMT on the $(2, 2, 2, 2)$ ZIC with different $\bar{\alpha}$.

This characteristics of the DMT on MIMO ZICs for general n is captured by the following Lemma.

Theorem 5.6 The DMT specified in Lemma 5.7 represents the fundamental DMT of the channel with No-CSIT and $r_1 = r_2 = r$, if

$$\alpha \geq 1 + \frac{d_{n,n}\left(\frac{n}{2}\right)}{n^2}. \tag{5.43}$$

Proof 5.12 Detailed proof will be provided in Appendix A.19.

Lemma 5.8 Consider the MIMO ZIC as in Theorem 5.4, but no CSI at the transmitters. The achievable diversity order of the IIML scheme on this channel, at a multiplexing gain pair (r_1, r_2), is given by

$$d^{\text{IML}}_{(M,N_1,M,N_2)}(r_1, r_2) = \min\left\{d_{M,N_1}(r_1), d_{M,N_2}(r_2), d_{2M,N_1}(r_s)\right\}.$$

Proof 5.13 From Lemma 5.5 and equation (5.35) it is clear that to prove the Lemma it is sufficient to derive an expression for $d^{\text{IML}}_{s,(M,N_1,M,N_2)}(r_s)$. Towards that, for convenience we use I'_{c_s} instead of I_{c_s} to evaluate the corresponding outage event since the two are within a constant number which does not matter at asymptotic SNR as was shown in equation (5.36). Using this in equation (5.34) along with the facts that $M_1 = M_2 = M$ and $\alpha = 1$ we get

$$\begin{aligned}\rho^{-d^{\text{IML}}_{s,(M,N_1,M,N_2)}(r_s)} &\doteq \Pr\left\{\log\det\left(I_{N_1} + \rho H_{11}H_{11}^{\dagger} + \rho H_{21}H_{21}^{\dagger}\right) \leq r_s\log(\rho)\right\}, \\ &= \Pr\left\{\log\det\left(I_{N_1} + \rho H_e H_e^{\dagger}\right) \leq r_s\log(\rho)\right\},\end{aligned} \tag{5.44}$$

where $H_e = [H_{11}\ H_{21}] \in \mathbb{C}^{N_1\times(2M)}$ is identically distributed as the other channel matrices, since H_{11} and H_{21} are mutually independent. However the right hand side of the last equation represents the outage probability of an $N_1 \times 2M$ point-to-point MIMO channel whose negative SNR exponent was computed in [48] and is given by $d_{N_1,2M}(r_s)$. Using this in equation (5.44) we get,

$$\begin{aligned}\rho^{-d^{\text{IML}}_{s,(M,N_1,M,N_2)}(r_s)} &\doteq \rho^{-d_{N_1,2M}(r_s)}, \\ \text{or, } d^{\text{IML}}_{s,(M,N_1,M,N_2)}(r_s) &= d_{N_1,2M}(r_s).\end{aligned}$$

Substituting this and equation (5.35) into equation (5.33) we obtain the desired result.

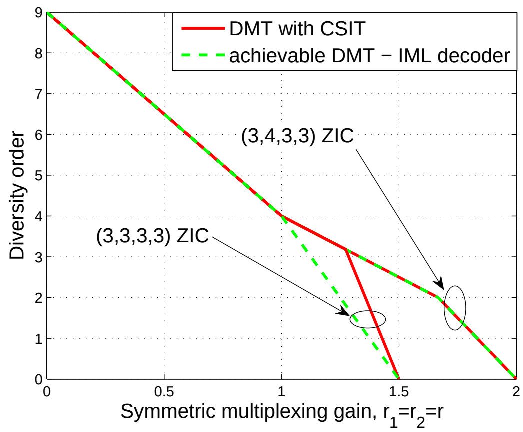


Figure 5.6: Effect of a larger number of antennas at the interfered node on ZICs with $\bar{\alpha} = [1, 1, 1]$.

Figure 5.6 depicts the comparison of the achievable DMT of the IML decoder with that of the fundamental F-CSIT DMT of the channel on two different MIMO ZICs. Comparing the performance improvement of the IML decoder on the $(3, 4, 3, 3)$ ZIC with respect to that on the $(3, 3, 3, 3)$ ZIC, we realize that a larger number of antennas at the interfered receiver can completely compensate for the lack of CSIT. Again by the argument that the FCSIT DMT represents an upper bound to the No-CSIT DMT of the channel, the observation from Fig. 5.6 imply that the DMT of Lemma 5.8 represents the fundamental DMT of the $(3, 4, 3, 3)$ ZIC with only CSIR. It turns out that, this channel is only a member of a large class of ZICs for which the No-CSIT DMT can be characterized. This class of channels is specified in the next Theorem.

Theorem 5.7 The DMT specified in Lemma 5.8 represents the fundamental DMT of the channel with No-CSIT and $r_1 = r_2 = r$, if

$$N_1 \geq M + \frac{d_{M,\min\{N_1,N_2\}}\left(\frac{M}{2}\right)}{M}.$$

Proof 5.14 Comparing Theorem 5.4 and Lemma 5.8, the desired result can be obtained following the similar steps as in the proof of Theorem 5.6.

5.5 Conclusion

The DMT of the MIMO ZIC with CSIT is characterized. It is shown that the knowledge of H_{21} at the 2nd transmitter only is sufficient to achieve the F-CSIT DMT of the channel. The No-CSIT DMT of two special class of ZICs have been characterized revealing useful insights about the system such as: a stronger interference or a larger number of antennas at the interfered receiver can completely compensate for the lack of CSIT on a ZIC. Characterizing the DMT of the general ZIC with no CSIT is an interesting problem for future research.

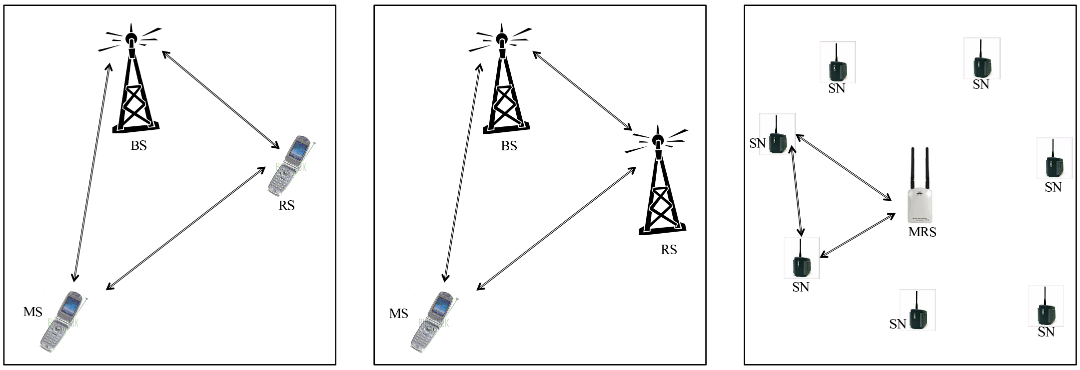
Chapter 6

The Diversity Multiplexing Tradeoff of the MIMO Half-Duplex Relay Channel

6.1 Introduction

Cooperative communication techniques can advantageously utilize the fading environment of a wireless network to provide better reliability and/or rate [58, 59]. The simplest theoretical abstraction of a cooperative communication network is the 3-node relay channel (RC), where the relay node helps the communication between the source and destination nodes by forwarding an appropriately processed version of the source message received at the relay node to the destination. Moreover, multiple antennas at the three nodes can markedly boost rate and reliability performance by allowing for the exploitation of the inherent combined MIMO and cooperative communication gains.

MIMO relay channel communications can be considered for various applications. For instance, Fig. 6.1 depicts three different cooperative communication scenarios to which the theory of this work applies. Fig. 6.1(a) depicts a cellular network, denoted CN_1, wherein a mobile user (or mobile set (MS)) uses another mobile user as the relay station (RS) to communicate its message to and from the base station (BS). This cooperative model was proposed in [58]. Fig. 6.1(b) depicts a scenario where, in a cellular network (denoted CN_2), a particular cell area is divided into more than one sub-cell and each sub-cell is served by an additional dedicated node (a smaller BS) to provide better quality of service. Thus each user in these sub-cells can use this dedicated node to relay their messages to and from the BS. The CN_2 network is different from CN_1 in the sense that in it the relay station can host a larger number of antennas. It is under consideration to be implemented in



(a) CN_1: A mobile set acts as a relay

(b) CN_2: A smaller base station acts as a relay

(c) CN_3: A sensory network with a mobile relay station (MRS)

Figure 6.1: Three Examples of Cooperative Networks.

LTE-advanced and WiMAX technologies [60] and being standardized for broadband wireless access by the IEEE 802.16's relay task group [61] for expanded throughput and coverage with deployment of relay stations of complexity and cost lower than that of legacy base stations but higher than that of mobile stations [61]. A third example of a cooperative network (denoted CN_3) is the sensor network of Fig. 6.1(c) (cf. [62]), where a more capable mobile relay station (MRS) (i.e., with more antennas) helps several less capable sensor nodes (SN) to communicate with each other. It is possible to give other examples, see for instance, the application of cooperative communication in ad-hoc networks in [63]. Note that the numbers of antennas at the different nodes vary across the applications and also depend on whether the communication is uplink or downlink (such as in the CN_1 and CN_2 networks), which points to the importance of studying MIMO relay channels with an arbitrary number of antennas at each node.

The relay has two phases of operation: the listen phase, in which it receives the signal from the source, and the transmit phase, in which it transmits some version of the received signal to the transmitter. If the relay can simultaneously operate in both phases it is called a full-duplex (FD) relay and the corresponding channel is called a full duplex relay channel (FD-RC). Otherwise, if the relay can only operate in one phase at a time it is called a half-duplex (HD) relay and the corresponding channel a half duplex relay channel (HD-RC). Due to the large difference between the power levels of the transmitted and received signals however, it is difficult, if not impossible, to design FD relays cost- and space-efficiently. The focus of this chapter is hence on MIMO HD-RCs.

Cooperative protocols proposed and analyzed for the HD-RC can be divided into different classes. If a protocol uses the CSI at the relay to opportunistically decide the switching time – the time at which it switches between the listen and transmit phases – it is called a **dynamic** protocol. Dynamic protocols considered in the literature include the dynamic decode-and-forward (DDF) protocol of [64] and the dynamic compress-and-forward (DCF) protocol of [65]. Otherwise, if the relay is restricted to switch between the listen and transmit phases at a pre-determined, channel independent time, it is called a **static** protocol. An important example is the static compress-and-forward (SCF) protocol of [65,66]. An HD-RC on which protocols are restricted to be static is called a static HD-RC, and one on which they are not, is called a dynamic HD-RC (or simply HD-RC) since any static protocol can be thought as a special case of dynamic protocols. On the other hand, the transmit-receive phases on an HD-RC can be thought of as states and additional information can be conveyed to the receiver through the sequence of these states. A cooperative protocol that uses these states to send additional information is called a **random** protocol, otherwise it is a **fixed** protocol (see [65, 67]).

In this chapter we focus on the general three-node MIMO HD-RC, i.e., in which there are an arbitrary number of antennas at each node and in which there are no constraints on the relay operation so that it can operate via the static or dynamic and random or fixed mode. In order to avoid repeated use of a complete descriptor of a channel we will use simplified ones when the meaning is unambiguous from the context. For example, we may refer to the dynamic MIMO HD-RC sometimes simply as the relay channel because this channel is the central focus of this chapter. Similarly, we may refer to the MIMO FD-RC or the static MIMO HD-RC as the FD-RC or the static RC, respectively, when the meaning is clear.

In spite of its apparent simplicity, neither the capacity nor the diversity-multiplexing tradeoff [48] of the 3-node MIMO HD-RC is known till date. However, in a recent paper [68] the capacity of this channel was characterized within a constant number of bits. It was proved that the so called quantize-map and forward (QMF) scheme can achieve a rate which is within a constant number of bits to the cut-set upper bound of the channel.[1] On a slow fading HD-RC however, the instantaneous end-to-end mutual information, and therefore the cut-set bound of the channel, is a random quantity. A meaningful measure of performance on this channel is hence the **outage probability** which is a measure of reliability as a function of the communication rate in that it represents the (minimum) probability with which a particular rate cannot be supported on the channel. The result of [68] provides upper and lower bounds on this outage probability, both in terms of the instantaneous cut-set upper bound of the channel, denoted as $\bar{C}(\mathcal{H})$ for a channel

[1] More recently, it was proved [69] that the classical CF protocol [70] can also achieve the same rate as the QMF protocol on the 3-node relay channel.

realization of $\mathcal{H}$, as

$$\Pr\{\bar{C}(\mathcal{H}) < R\} \leq \mathcal{P}_{out} \leq \Pr\{\bar{C}(\mathcal{H}) < R - \kappa\},$$

where R is the operating rate and κ is a positive constant independent of the channel parameters and the signal-to-noise ratio (SNR) of the channel (e.g., see Theorem 8.5 in [68]). The exact evaluation of the outage probability requires both these bounds to be tight which in turn requires the exact capacity of the channel. Instead, in this chapter we focus on the asymptotic (in SNR) behavior of the tradeoff between rate and reliability as captured by the DMT metric, first introduced in the context of the point-to-point MIMO channel by Zheng and Tse in [48]. Since it was proved that a random protocol can increase the capacity by at most one bit in [65], there is no distinction between random and fixed protocols in the DMT framework. Thus, from the DMT perspective, characterizing the DMT of the HD-RC by allowing for dynamic operation of the relay but restricting it to the fixed mode still amounts to characterizing the fundamental DMT of the HD-RC. It is noted that the DMT of the static MIMO HD-RC for the symmetric configuration, where the destination and source have equal number of antennas, was recently obtained by Leveque et al in [66]. It is shown here that in general a restriction that relay protocols be static fundamentally limits DMT performance over the MIMO HD-RC.

Since its first introduction and pioneering work by Van der Meulen [71] and the subsequent significant progress made by Cover and El Gamal in [70], the relay channel and its more general versions have been analyzed from both the capacity perspective in [63, 68, 72–75] and from the diversity, or more generally, the DMT perspectives for the 3-node relay network in quasi-static fading channels in [52, 64–66, 76–81]. The earliest works demonstrating the improved reliability of the relay channel in terms of the diversity gain compared to the corresponding point-to-point (PTP) channel were reported in [76–79], where a number of simple cooperative protocols were proposed and their DMT performance was analyzed. Later in [64, 79, 80], more efficient protocols were introduced. Notable among these were the dynamic decode-and-forward (DDF) protocol which is DMT optimal on a single antenna relay channel for a range of low multiplexing gains and the so called enhanced dynamic decode-and-forward (EDDF) of [80]. All of the above protocols were analyzed for the relay channel with single antenna nodes ([79] considers multiple antennas at the destination).

Multiple antenna relay channels were studied by Yuksel and Erkip in [65], where the DMTs of a number of protocols were evaluated and the DMT optimality of the compress-and-forward (CF) coding scheme of [70] for the MIMO FD- and HD-RCs was proved. In the DCF protocol of [65], the relay node utilizes all the instantaneous channel realizations, i.e., **global CSI**, to compute the quantized signal and the optimal switching time of the relay node. However, global CSI at the relay is not necessary to achieve DMT optimal performance as we discuss next.

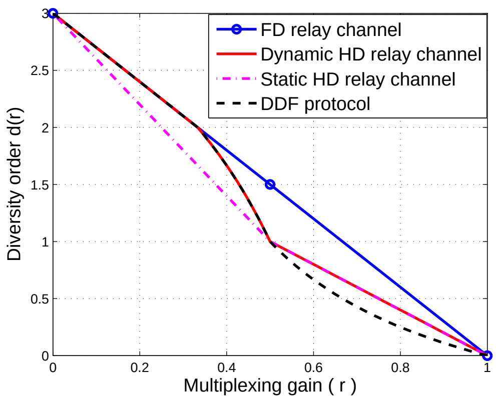
The static QMF protocol of [68] can achieve the cut-set bound of the HD-RC to within a constant gap that is independent of CSI and SNR for a **fixed** scheduling of the relay, i.e., for an a priori fixed time t_d at which the relay switches from listen to the transmit mode (e.g., see Theorem 8.3 in [68]), and do so without knowledge of CSI at the relay. However, on a slow fading dynamic HD-RC the cut-set bound, denoted by $\bar{C}(\mathcal{H}, t_d)$, is a function of global CSI including t_d and hence the optimal switching time t_d^* that maximizes the cut-set bound can be a function of the instantaneous channel matrices. If (just) this switching time information is hence available at the relay node it then follows that the QMF protocol can achieve a rate that is within a constant gap to $\bar{C}(\mathcal{H}, t_d^*)$ without requiring global CSI at the relay. Henceforth, the QMF protocol that operates with a **dynamic** (channel-dependent) switching time of the relay node will be referred to as the **dynamic QMF** protocol. Since a constant gap is irrelevant in the DMT metric, the dynamic QMF protocol in which the relay switches from listen to transmit modes at t_d^* achieves the fundamental DMT of the MIMO HD-RC with only knowledge of t_d^* at the relay, as opposed to global CSI $\mathcal{H}$ required by the DCF protocol of [65]. The above discussion shows that the DMT of the static MIMO HD-RC found in [66] for the static CF protocol requiring global CSI at the relay also applies to the static QMF protocol and hence to the static MIMO HD-RC without any CSI at the relay.

In this chapter, we are interested in establishing the DMT of the dynamic MIMO HD-RC.

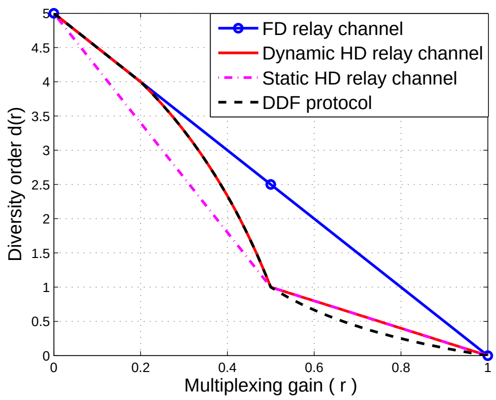
While the optimality in the DMT metric of the DCF was shown in [65] and that of the dynamic QMF protocol [68] is evident from the discussion above, the characterization of this optimal performance, i.e., the fundamental DMT of the MIMO HD-RC, is not yet known and is the subject of this chapter. The key mathematical tool that prevented its computation thus far is the joint eigenvalue distribution of three mutually correlated random Wishart matrices. Here we obtain this distribution as a stepping stone to characterizing the DMT of the MIMO HD-RC. Not only is this distribution result interesting in its own right as a problem in random matrices, it is also arises in establishing the DMT of the MIMO interference channel as was done by the authors in [33, 34].

The explicit DMT of the MIMO HD-RC evidently would serve as a theoretical benchmark against which the performances of the various cooperative protocols proposed and analyzed in the literature can be compared. Further, cooperative protocols which are suboptimal but cost-efficient provide the system designer with an option to trade performance and complexity if their performance loss can be quantified relative to optimal performance. Moreover, the answers of a number of interesting and open questions hinge on the explicit characterization of the DMT of the MIMO HD-RC. For instance, while the DMT of the MIMO FD-RC is an upper bound to that of the MIMO HD-RC, it is not known whether the latter is strictly worse than that of the former. The question is especially intriguing in light of the result by Pawar **et at** in [82] where it was shown that the DMT of the single-antenna (or single-input, single-output (SISO)) HD-RC is identical to that of the FD-RC. Comparing with the DMT of the MIMO FD-RC which was found in [65], this question can be resolved if the explicit DMT of the MIMO HD-RC can be characterized. There are also open questions regarding the comparative performances of the static and dynamic MIMO HD-RCs. Although intuitively it seems that the dynamic HD-RC should have a better DMT than the static HD-RC, there is no theoretical proof of this thesis to date. For instance, in the SISO case, there is no difference in the DMTs of the static and dynamic HD-RCs as shown in [82] because the DMT of the static QMF protocol coincides with that of the SISO FD-RC. The question here is whether this result continues to hold in the more general static and dynamic MIMO HD-RCs. This question can be answered if the DMT of the (dynamic) MIMO HD-RC were to be found, since in this case, one could simply compare with the DMT of the static MIMO HD-RC of [66].

This chapter answers the two questions raised above in the negative. In particular, the results of this chapter, examples from which are shown in Fig. 6.2(a) and Fig. 6.2(b) depicting the DMTs of the HD-RC with single-antenna source and destinations but with two and four antennas at the relay (the $(1, 2, 1)$ and[2] $(1, 4, 1)$ RCs (applicable for example, to CN_3 of Fig. 6.1(c)), respectively, show that in general neither is the DMT of the static MIMO HD-RC always equal to that of the corresponding dynamic MIMO HD-RC, nor is the DMT of the MIMO FD-RC always identical to that of the corresponding MIMO HD-RC.



(a) $(1, 2, 1)$ relay channel



(b) $(1, 4, 1)$ relay channel

Figure 6.2: Comparison of DDF and SCF protocol with the fundamental DMT of the (1,k,1) relay channel.

Besides resolving the above discussed problems, the explicit DMT computed in this chapter provides sharper answers about the MIMO HD-RC. They include, but are not limited to the following:

- While in general the DMT performance of the MIMO HD-RC is inferior to that of the corresponding MIMO FD-RC, it is found that for two classes of channels, namely (a) the (m, k, n) RCs with $m > n \geq k$ and (b) the $(n, 1, n)$ RCs, the DMTs of the HD- and FD-RCs are identical (see Remark 6.11). While the observation of case (a) is based on empirical

[2] A relay channel with m, k and n antennas at the source, relay and destination, respectively will be denoted by (m, k, n)-RC, hereafter.

results (e.g., see Remark 6.3), case (b) is proved analytically. Therefore, for these classes of RCs, an FD relay does not improve the DMT performance over that of the corresponding HD-RC.

- In general, for a set of high multiplexing gain values, the optimal DMT of the MIMO HD-RC can be achieved without CSI at the relay node. Again empirical results show that, on other RCs besides the above described two classes, as the number of antennas at the relay node increases, the size of this set increases (e.g., see Fig. 6.6(b)).

- It is well known from [65] that the fundamental DMT of (m, k, n) FD-RC is given by

$$\min\left\{d_{(m+k),n}(r), d_{m,(n+k)}(r)\right\}, \; 0 \leq r \leq \min\{m, n\}, \tag{6.1}$$

  where $d_{p,q}(r)$ represents the DMT of the $p \times q$ MIMO point-to-point channel [48]. From this it is clear that an additional antenna at the relay node strictly improves the DMT performance of an FD-RC at all multiplexing-gains. However, this is not true for the MIMO HD-RC. When k is large enough, an extra antenna at the relay node does not further improve the DMT of the HD-RC for high multiplexing gains (see Remark 6.5).

- Finally, it is proved that the DMT of the $(1, k, 1)$ and the $(n, 1, n)$ HD-RC can be achieved by the QMF and the DDF protocol without CSI at the relay node, i.e., neither global CSI nor even the switching time information is necessary at the relay node. Earlier, in [83] DMT of the multi-hop MIMO relay channel, where there is no direct link between the source and destination, was characterized without any CSI at the relay node. In contrast, for a relay channel where there is a direct path to the destination as well, this chapter provides the first result regarding the achievability of the DMT of a non-SISO HD-RC without CSI at the relay node.

The rest of the chapter is organized as follows. In Section 6.2, we describe the system model and provide some preliminaries including the asymptotic joint distribution of the eigenvalues of three specially correlated random matrices which will be used later to derive the fundamental DMT of the MIMO HD-RC. In Section 6.3, we characterize the fundamental DMT of the MIMO HD-RC as the solution of a simple optimization problem in three steps: 1) in Subsection 6.3.1 we obtain an upper bound on the instantaneous capacity; 2) in Subsection 6.3.2, we obtain a lower bound on the instantaneous capacity as the achievable rate of the dynamic QMF protocol, which is within a constant gap from the upper bound, and finally, in Subsection 6.3.3, we characterize the DMT as a solution to an optimization problem, which we subsequently simplify to a 2-variable optimization problem. In Section 6.4, we provide closed-form expressions for the DMT of different classes of MIMO HD-RCs including the class of symmetric (n, k, n) RCs and then prove in Section 6.5 that the DMT of $(1, k, 1)$ RC and $(n, 1, n)$ HD-RC can be achieved without CSI at the relay node. Section 6.6 concludes the chapter.

Proof 6.1 (Notations) $(x)^+$, $x \wedge y$, $|\mathcal{S}|$, $\det(X)$ and $(X)^\dagger$ represent the number $\max\{x, 0\}$, the minimum of x and y, the size of the set $\mathcal{S}$, the determinant and the conjugate transpose of the matrix, X, respectively. The symbol diag(.) represents a square diagonal matrix of corresponding size with the elements in its argument on the diagonal. I_n represents an $n \times n$ identity matrix. We denote the field of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. The set of real numbers between $r_1 \in \mathbb{R}$ and $r_2(\geq r_1) \in \mathbb{R}$ will be denoted by $[r_1, r_2]$. The set of all $n \times m$ matrices with complex entries is denoted as $\mathbb{C}^{n\times m}$. The distribution of a complex Gaussian random vector with zero mean and covariance matrix Σ is denoted as $\mathcal{CN}(0, \Sigma)$. The trace of a square matrix A, is denoted as $\mathrm{Tr}(A)$. $A \succeq B$ (or $A \succ B$) would mean that $(A - B)$ is a positive-semidefinite (psd) matrix (or positive-definite (pd) matrix), respectively. $\Pr(\mathcal{E})$ represents the probability of the event $\mathcal{E}$. All the logarithms in this text are to the base 2. Finally, any two functions $f(\rho)$ and $g(\rho)$ of ρ, where ρ is the signal-to-noise ratio (SNR) defined later, are said to be exponentially equal and denoted as $f(\rho)\doteq g(\rho)$ if,

$$\lim_{\rho\to\infty} \frac{\log(f(\rho))}{\log(\rho)} = \lim_{\rho\to\infty} \frac{\log(g(\rho))}{\log(\rho)}, \tag{6.2}$$

and $\dot{\leq}$ and $\dot{\geq}$ signs are defined similarly.

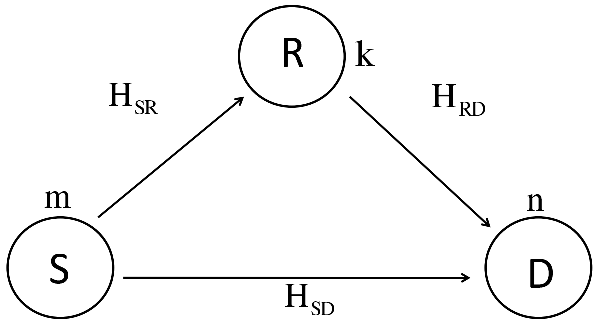


Figure 6.3: System model of the MIMO 3-node relay channel.

6.2 System model and preliminaries

We consider a quasi-static, Rayleigh faded MIMO HD-RC, with a single relay node as shown in Fig. 6.3. It is assumed that the source, destination and relay nodes have m, n and k antennas, respectively. Let $H_{SR} \in \mathbb{C}^{k\times m}$, $H_{SD} \in \mathbb{C}^{n\times m}$ and $H_{RD} \in \mathbb{C}^{n\times k}$ model the fading channel matrices between the source and relay, source and destination and relay and destination nodes, respectively. For economy of notation, the set of these channel matrices is denoted as $\mathcal{H}$. It is assumed that these matrices are mutually independent and their elements are independent and identically distributed (i.i.d.) as $\mathcal{CN}(0, 1)$.

The channel matrices remain constant within a block of L_b channel uses, where L_b is the block length of the source codeword. Suppose that during the first $t_d L_b$ symbol times the relay node only listens to the source transmission and during the remaining $(1 - t_d)L_b$ symbol times it transmits its own codeword $X_r \in \mathbb{C}^{k\times(1-t_d)L_b}$, where $t_d \in (0, 1)$. In what follows, the listening phase and the transmitting phases of the relay node will be denoted by p_1 and p_2, respectively, and the fraction t_d is called the **switching time**. Since the relay node operates dynamically, this switching time should be chosen to maximize the end-to-end instantaneous mutual information and can thus depend on all of $\mathcal{H}$.

We assume that the destination and relay nodes have global CSI $\mathcal{H}$, but the source node does not have any CSI. The relay node in this channel model is more capable than that of a relay channel with only receive CSI (CSIR) at all the nodes.[3] We assume short term power constraints at the source and relay, i.e., these nodes cannot allocate power across different fades of the channel as a function of $\mathcal{H}$, see equation (6.6).

Further, we also assume that the source and relay nodes transmit information at fixed rates; in particular, the relay node does not use the available transmit CSI (CSIT) to transmit information at a variable, channel dependent rate. Note that an information outage can be avoided on a communication link if CSIT is used to allocate power across different fading blocks (cf. [55, 84, 85]) under the so-called long-term power constraint and/or transmit information at a variable rate as a function of the instantaneous channel realizations. It was shown in [85] that the DMT of a point-to-point MIMO channel can be improved by either of these two techniques.

Denoting the signals transmitted by the source at time t in phases one and two as $X_{S_1}[t]$ and $X_{S_2}[t]$, respectively, and the signal transmitted by the relay at time t as $X_R[t]$ (in phase two), the received signals at the destination and relay nodes in phase one are given as

$$\begin{aligned} Y_{D_1}[t] &= H_{SD}X_{S_1}[t] + Z_{D_1}[t], && (6.3) \\ Y_R[t] &= H_{SR}X_{S_1}[t] + Z_R[t], && (6.4) \end{aligned}$$

and the received signal at the destination node in phase two is given as

$$Y_{D_2}[t] = H_{SD}X_{S_2}[t] + H_{RD}X_R[t] + Z_{D_2}[t],$$

where $Z_{D_1}[t], Z_{D_2}[t] \in \mathbb{C}^{n\times 1}$ and $Z_R[t] \in \mathbb{C}^{k\times 1}$ represent mutually independent additive noise random vectors at the destination and relay nodes, respectively. All the entries of these random vectors are assumed to be independent and identically distributed (i.i.d.) as $\mathcal{CN}(0,1)$. The power

[3] Note that the DMT of the relay channel with only CSIR at all the nodes can not be better than the DMT of the relay channel considered in this chapter, since the relay node can choose not to use any CSI except H_{SR}. In fact, intuitively it seems that the latter may use the additional information (H_{SD} and H_{RD}) at the relay for instance to optimize the time to switch from its listening mode to the transmit mode and achieve a better DMT than the former, on which the switching time can only be a function of H_{SR}. Interestingly, in this chapter we shall prove that depending on the number of antennas at different nodes the DMT of the above two relay channels (with and without global CSI at the relay and destination node) can be identical, cf. Section 6.5.

constraints at the relay and the source nodes are[4]

$$\frac{1}{L_b}\left\{\sum_{t=1}^{\lceil t_d L_b \rceil} \mathrm{Tr}\left(\mathbb{E}\left(X_{S_1}[t]X_{S_1}[t]^{\dagger}\right)\right) + \sum_{t=(\lceil t_d L_b \rceil+1)}^{L_b} \mathrm{Tr}\left(\mathbb{E}\left(X_{S_2}[t]X_{S_2}[t]^{\dagger}\right)\right)\right\} \leq \rho; \tag{6.5}$$

$$\frac{1}{(L_b - \lceil t_d L_b \rceil)} \sum_{t=(\lceil t_d L_b \rceil+1)}^{L_b} \mathrm{Tr}\left(\mathbb{E}\left(X_R[t]X_R[t]^{\dagger}\right)\right) \leq \rho. \tag{6.6}$$

Let $\{\mathcal{C}(\rho)\}$ be a sequence of codebooks, where for each ρ the corresponding codebook $\mathcal{C}(\rho)$ consists of $2^{L_b R(\rho)}$ codewords, each of which is a $m \times L_b$ matrix satisfying equation (6.5). The sequence of codebooks is said to have a multiplexing gain of r if

$$\lim_{\rho\to\infty} \frac{R(\rho)}{\log(\rho)} = r.$$

Further, suppose for such a coding scheme $\mathcal{C}(\rho)$, $P_e(\mathcal{C}(\rho), \rho)$ represents the average probability of decoding error at the destination node (averaged over the Gaussian noise, channel realizations and the different codewords of the codebook) at an SNR of ρ, then the optimal diversity order of the channel at a multiplexing gain r is defined as

$$d^*(r) \triangleq \lim_{\rho\to\infty} \frac{-\log\left(P_e^*(\rho)\right)}{\log(\rho)}, \tag{6.7}$$

where $P_e^*(\rho)$ represents the minimum average probability of error achievable on a relay channel minimized over the collection of all possible coding schemes, $\mathscr{C}(\rho)$, i.e.,

$$P_e^*(\rho) \triangleq \min_{\{\mathcal{C}(\rho)\in\mathscr{C}\}} P_e(\mathcal{C}(\rho), \rho). \tag{6.8}$$

In Subsection 6.3.3, we shall show that the optimal diversity order at a multiplexing gain of r can be written as

$$d^*(r) = \lim_{\rho\to\infty} -\frac{\log\left(\Pr\left\{r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \leq r\right\}\right)}{\log(\rho)}, \tag{6.9}$$

where $r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta})$ is given by (6.50) in Subsection 6.3.3 and $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\delta}$ are vectors containing the negative SNR exponents of the eigenvalues (see equations (6.12)-(6.14) in the following section) of

[4] Allowing distinct powers at the source and relay nodes of ρ and $c\rho$, respectively, where c is a constant independent of ρ, does not alter the diversity-multiplexing tradeoff. We assume $c = 1$ for ease of disposition.

$H_{SD}H_{SD}^{\dagger}$, $H_{SR}(I_m + \rho H_{SD}^{\dagger}H_{SD})^{-1}H_{SR}^{\dagger}$ and $H_{RD}^{\dagger}(I_n + \rho H_{SD}H_{SD}^{\dagger})^{-1}H_{RD}$, respectively. Evidently, to evaluate the DMT, we need the joint probability density function (pdf) of the eigenvalues of these matrices, which we obtain next.

6.2.1 Joint eigenvalue distribution of three mutually correlated Wishart matrices

Let us denote the matrices $H_{SD}H_{SD}^{\dagger}$, $H_{SR}(I_m+\rho H_{SD}^{\dagger}H_{SD})^{-1}H_{SR}^{\dagger}$ and $H_{RD}^{\dagger}(I_n+\rho H_{SD}H_{SD}^{\dagger})^{-1}H_{RD}$ as W_1, W_2 and W_3, respectively. It is evident from their structure that these matrices are not mutually independent. In general, finding the joint pdf of 2 or more mutually correlated random matrices is a difficult problem in the theory of random matrices. However, in this section we show that by exploiting the specific structure of these matrices and the distribution of the constituent matrices, we can compute a closed form expression for the joint pdf of their eigenvalues.

Let $0 < \lambda_u \leq \cdots \leq \lambda_1$, $0 < \mu_p \leq \cdots \leq \mu_1$ and $0 < \gamma_q \leq \cdots \leq \gamma_1$ be the ordered non-zero eigenvalues with probability 1 (w.p. 1) of W_1, W_2 and W_3, respectively. Define $\bar{\lambda} = [\lambda_1, \cdots \lambda_u]$, $\bar{\mu} = [\mu_1, \cdots \mu_p]$ and $\bar{\gamma} = [\gamma_1, \cdots \gamma_q]$ with $u = \min\{m, n\}$, $p = \min\{m, k\}$ and $q = \min\{n, k\}$. It is convenient to denote the joint pdf of the three sets of eigenvalues as $F_{W_1W_2W_3}(\bar{\lambda}, \bar{\mu}, \bar{\gamma})$ and similarly their marginal and conditional pdfs, i.e., the marginal pdf of $\bar{\lambda}$ is denoted as $F_{W_1}(\bar{\lambda})$, the conditional pdf of $\bar{\mu}$ conditioned on $\bar{\lambda}$ is denoted as $F_{W_2|W_1}(\bar{\mu}|\bar{\lambda})$, etc. Consider the following lemma which provides the first step towards simplifying the problem at hand.

Lemma 6.1 The eigenvalues of W_2 are independent of the eigenvalues of W_3 given the eigenvalues of W_1, i.e.,

$$F_{W_2W_3|W_1}(\bar{\mu}, \bar{\gamma}|\bar{\lambda}) = F_{W_2|W_1}(\bar{\mu}|\bar{\lambda})F_{W_3|W_1}(\bar{\gamma}|\bar{\lambda}). \tag{6.10}$$

Proof 6.2 (Proof) The proof is provided in Appendix B.1.

Using the above lemma, the joint pdf of the eigenvalues of the three matrices can be expressed as

$$F_{W_1W_2W_3}(\bar{\lambda}, \bar{\mu}, \bar{\gamma}) = F_{W_2|W_1}(\bar{\mu}|\bar{\lambda})F_{W_3|W_1}(\bar{\gamma}|\bar{\lambda})F_{W_1}(\bar{\lambda}). \tag{6.11}$$

The joint pdf of the eigenvalues of W_1, which is a central Wishart matrix, can be found for example in [86] whereas the conditional pdfs $F_{W_2|W_1}(\bar{\mu}|\bar{\lambda})$ and $F_{W_3|W_1}(\bar{\gamma}|\bar{\lambda})$ involve complicated functions such as determinants whose components are hypergeometric functions of the eigenvalues (e.g., see the proof of Theorem 1 in [52]). However, since we are interested only in a high SNR analysis, it is sufficient to obtain $F_{W_2|W_1}(\bar{\mu}|\bar{\lambda})$ and $F_{W_3|W_1}(\bar{\gamma}|\bar{\lambda})$ exactly just up to their SNR exponents, i.e., approximate expressions which have the same SNR exponents as the exact joint pdf. For this purpose, we use the following theorem from [52].

Theorem 6.1 (Theorem 1 in [52]) Let $H_1 \in \mathbb{C}^{N_2 \times N_1}$ and $H_2 \in \mathbb{C}^{N_2 \times N_3}$ be two mutually independent random matrices with independent, identically distributed (i.i.d.) $\mathcal{CN}(0,1)$ entries. Suppose that $\xi_1 \geq \xi_2 \geq \cdots \xi_v > 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_u > 0$ are the ordered non-zero eigenvalues w.p. 1 of $V_1 \triangleq H_1^\dagger (I_{N_2} + \rho H_2 H_2^\dagger)^{-1} H_1$ and $V_2 \triangleq H_2 H_2^\dagger$, respectively, with $u = \min\{N_2, N_3\}$ and $v = \min\{N_1, N_2\}$, and where all the eigenvalues are assumed to vary exponentially with SNR in the following sense:[5]

$$\lambda_i = \rho^{-\alpha_i}, \ 1 \leq i \leq u; \tag{6.12}$$

$$\mu_j = \rho^{-\beta_j}, \ 1 \leq j \leq p; \tag{6.13}$$

$$\gamma_l = \rho^{-\delta_l}, \ 1 \leq l \leq q. \tag{6.14}$$

Then, the conditional asymptotic pdf of the eigenvalues $\bar{\xi}$ given $\bar{\lambda}$ is given as

$$\mathbf{f_1}(\bar{\xi}|\bar{\lambda}) \doteq \prod_{j=1}^{v} (\xi_j^{(N_1+N_2-2j)} e^{-\xi_j}) \prod_{\substack{(n_1=1,n_2=1) \\ ((n_1+n_2)=(N_2+1))}}^{(u,v)} \left(e^{-\rho \xi_{n_2} \lambda_{n_1}} \right) \prod_{i=1}^{u} (1+\rho\lambda_i)^{N_1} \prod_{j=1}^{v} \prod_{i=1}^{(N_2-j)\wedge N_3} \left(\frac{1 - e^{-\rho \xi_j \lambda_i}}{\rho \xi_j \lambda_i} \right).$$

Note that the above theorem gives the conditional pdf of the joint eigenvalues of V_1 given the eigenvalues of V_2 exactly up to its exponential order. This asymptotic distribution is simpler to obtain than its exact counterpart and is also sufficient for the DMT analysis. It can be easily verified [7] that the first product term corresponds to the joint distribution of the eigenvalues of

[5] This assumption greatly simplifies an otherwise very complicated expression of the pdf. Further, in the context of the problem being analyzed in this chapter, the usefulness of this assumption will be evident shortly.

$H_1^\dagger H_1$. The additional three product terms appear because V_1 is a Wishart matrix with a non-identity covariance matrix. To see this, note that V_1 converges to $H_1^\dagger H_1$ if each of the eigenvalues of $\rho H_2 H_2^\dagger$ tends to zero. Indeed, putting $\rho\lambda_i \to 0$, $\forall i$ in the above expression, it is easily shown that the last three terms converge to 1 giving the joint distribution of $H_1^\dagger H_1$.

Clearly, Theorem 6.1 can be used to derive the asymptotic conditional joint pdf of the eigenvalues of W_3 given the eigenvalues of W_1. Consequently, we have

$$F_{W_3|W_1}(\bar{\gamma}|\bar{\lambda}) \doteq \prod_{l=1}^{q}(\gamma_l^{(k+n-2l)} e^{-\gamma_l}) \prod_{\substack{(i=1,l=1)\\((i+l)=(n+1))}}^{(u,q)} \left(e^{-\rho\gamma_l\lambda_i}\right) \prod_{i=1}^{u}(1+\rho\lambda_i)^k \prod_{l=1}^{q} \prod_{i=1}^{(n-l)\wedge m} \left(\frac{1-e^{-\rho\gamma_l\lambda_i}}{\rho\gamma_l\lambda_i}\right) \quad (6.15)$$

Next, since for each realization of H_{SD} the eigenvalues of $H_{SD}H_{SD}^\dagger$ and $H_{SD}^\dagger H_{SD}$ are the same, the conditional joint pdf of the eigenvalues of W_2 given the eigenvalues of W_1 can also be derived from Theorem 6.1 and is hence given as

$$F_{W_2|W_1}(\bar{\mu}|\bar{\lambda}) \doteq \prod_{j=1}^{p}(\mu_j^{(k+m-2j)} e^{-\mu_j}) \prod_{\substack{(i=1,j=1)\\((i+j)=(m+1))}}^{(u,p)} \left(e^{-\rho\mu_j\lambda_i}\right) \prod_{i=1}^{u}(1+\rho\lambda_i)^k \prod_{j=1}^{p} \prod_{i=1}^{(m-j)\wedge n} \left(\frac{1-e^{-\rho\mu_j\lambda_i}}{\rho\mu_j\lambda_i}\right) \quad (6.16)$$

Substituting equations (6.15) and (6.16) in (6.11) and importing the expression for $F_{W_1}(\bar{\lambda})$ from [86] the joint pdf of $(\bar{\lambda}, \bar{\mu}, \bar{\gamma})$ up to its exponential order can be obtained.

Recall next that for the DMT analysis we need the joint pdfs of the negative SNR exponents of these eigenvalues, i.e., those of the transformed variables $(\bar{\alpha}, \bar{\beta}, \bar{\delta})$ defined via equations (6.12)-(6.14). Using these change of variables in equation (6.11) we get the joint pdf of $(\bar{\alpha}, \bar{\beta}, \bar{\delta})$ (where each vector in this triple is simply the vector of the corresponding random variables), denoted as $f_{W_1W_2W_3}(\bar{\alpha}, \bar{\beta}, \bar{\delta})$, given as

$$f_{W_1W_2W_3}(\bar{\alpha}, \bar{\beta}, \bar{\delta}) = \left(\Pi_{j=1}^{p}|J(\mu_j)|\right) g_{W_2|W_1}(\bar{\beta}|\bar{\alpha}) \left(\Pi_{l=1}^{q}|J(\gamma_l)|\right) g_{W_3|W_1}(\bar{\delta}|\bar{\alpha}) \left(\Pi_{i=1}^{u}|J(\lambda_i)|\right) g_{W_1}(\bar{\alpha}) \quad (6.17)$$

where $g_{W_2|W_1}(\bar{\beta}|\bar{\alpha})$, $g_{W_3|W_1}(\bar{\delta}|\bar{\alpha})$ and $g_{W_1}(\bar{\alpha})$ are obtained by replacing the three sets of arguments in $(\bar{\lambda}, \bar{\mu}, \bar{\gamma})$ using the transformations (6.12)-(6.14) in $F_{W_2|W_1}(\bar{\mu}|\bar{\lambda})$, $F_{W_3|W_1}(\bar{\gamma}|\bar{\lambda})$ and $F_{W_1}(\bar{\lambda})$, respectively. The quantities $J(\lambda_i) = -\rho^{-\alpha_i}\ln\rho$, $J(\mu_j) = -\rho^{-\beta_j}\ln\rho$ and $J(\gamma_l) = -\rho^{-\delta_l}\ln\rho$ represent the Jacobians of the transformations in equations (6.12)-(6.14), respectively.

We next evaluate the three sets of products of Jacobians and the associated g functions in the overall product expression in the right hand side of equation (6.17) up to exponential order. We begin with $\left(\Pi_{j=1}^{p}|J(\mu_j)|\right) g_{W_2|W_1}(\bar{\beta}|\bar{\alpha})$ first. Using the transformations (6.12) and (6.13) in equation (6.16) we get

$$g_{W_2|W_1}(\bar{\beta}|\bar{\alpha}) \doteq \prod_{j=1}^{p}\left(\rho^{-(k+m-2j)\beta_j} e^{-\rho^{-\beta_j}}\right) \prod_{\substack{(i=1,j=1)\\(s.t.\ i+j=m+1)}}^{(u,p)} \left(e^{-\rho^{(1-\beta_j-\alpha_i)}}\right) \prod_{i=1}^{u}(1+\rho^{-\alpha_i})^k \prod_{j=1}^{p}\prod_{i=1}^{(m-j)\wedge n}\left(\frac{1-e^{-\rho^{(1-\beta_j-\alpha_i)}}}{\rho^{(1-\beta_j-\alpha_i)}}\right). \tag{6.18}$$

For asymptotic SNR ($\rho \to \infty$) we have

$$\lim_{\rho\to\infty} e^{-\rho^{-\beta_j}} = 0, \text{ if } \beta_j < 0 \text{ for any } 1 \le j \le p; \tag{6.19}$$

$$\lim_{\rho\to\infty} e^{-\rho^{(1-\beta_j-\alpha_i)}} = 0, \text{ if } (\alpha_i+\beta_j) < 1 \text{ for any } (i+j) \ge (m+1); \tag{6.20}$$

$$\lim_{\rho\to\infty}\left(\frac{1-e^{-\rho^{(1-\beta_j-\alpha_i)}}}{\rho^{(1-\beta_j-\alpha_i)}}\right) = \begin{cases} \rho^{-(1-\beta_j-\alpha_i)} & \text{if } (\beta_j+\alpha_i) \le 1; \\ 1 & \text{otherwise;} \end{cases} \quad \left[\because \lim_{x\to 0}\frac{1-e^{-x}}{x} = 1\right]. \tag{6.21}$$

Substituting the above asymptotic approximations and the fact that the limiting value of a product of convergent sequences is equal to the product of the individual limiting values in equation (6.18), we get

$$\left(\Pi_{j=1}^{p}|J(\mu_j)|\right) g_{W_2|W_1}(\bar{\beta}|\bar{\alpha}) \doteq \begin{cases} \rho^{-E_2(\bar{\alpha},\bar{\beta})}, & \text{if } (\bar{\alpha},\bar{\beta}) \in \mathcal{S}_2; \\ 0, & \text{otherwise,} \end{cases} \tag{6.22}$$

where $\mathcal{S}_2 = \{(\bar{\alpha},\bar{\beta}) : 0 \le \alpha_1 \le \cdots \le \alpha_u; 0 \le \beta_1 \le \cdots \le \beta_p; (\beta_j+\alpha_i) \ge 1,\ \forall (i+j) \ge (m+1)\}$ and

$$E_2(\bar{\alpha},\bar{\beta}) = \left[\sum_{j=1}^{p}(m+k-2j+1)\beta_j - k\sum_{i=1}^{u}(1-\alpha_i)^+ + \sum_{\substack{i,j=1\\j+i\le m}}^{u,p}(1-\alpha_i-\beta_j)^+\right]. \tag{6.23}$$

Similarly, it can be shown that

$$\left(\Pi_{l=1}^{q}|J(\gamma_l)|\right) g_{W_3|W_1}(\bar{\delta}|\bar{\alpha}) \doteq \begin{cases} \rho^{-E_3(\bar{\alpha},\bar{\delta})}, & \text{if } (\bar{\alpha},\bar{\delta}) \in \mathcal{S}_3; \\ 0, & \text{otherwise,} \end{cases} \tag{6.24}$$

where $\mathcal{S}_3 = \{(\bar{\alpha},\bar{\beta}) : 0 \le \alpha_1 \le \cdots \le \alpha_u; 0 \le \delta_1 \le \cdots \le \delta_q; (\delta_l+\alpha_i) \ge 1,\ \forall (i+l) \ge (n+1)\}$ and

$$E_3(\bar{\alpha},\bar{\delta}) = \left[\sum_{l=1}^{q}(n+k-2l+1)\delta_l - k\sum_{i=1}^{u}(1-\alpha_i)^+ + \sum_{\substack{i,l=1\\l+i\le n}}^{u,q}(1-\alpha_i-\delta_l)^+\right]. \tag{6.25}$$

Finally, using the expression for the pdf of $\bar{\alpha}$ given in [48] we have

$$\left(\Pi_{i=1}^{u}|J(\lambda_i)|\right) g_{W_1}(\bar{\alpha}) \doteq \begin{cases} \rho^{-\sum_{i=1}^{u}(m+n-2i+1)\alpha_i}, & \text{if } 0 \leq \alpha_1 \leq \cdots \leq \alpha_u; \\ 0, & \text{otherwise.} \end{cases} \tag{6.26}$$

Finally, substituting equations (6.22), (6.24) and (6.26) into equation (6.17) we get the main result of this section, namely, the joint distribution of $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ up to exponential order, which we state in the following theorem.

Theorem 6.2 If $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$ are the vectors containing the negative SNR exponents of the ordered eigenvalues of the matrices W_1, W_2 and W_3, respectively, as defined in the transformations (6.12)-(6.14), then the joint distribution of $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ is given up to exponential order as

$$f_{W_1W_2W_3}(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \doteq \begin{cases} \rho^{-E(\bar{\alpha},\bar{\beta},\bar{\delta})}, \text{ if } (\bar{\alpha}, \bar{\beta}, \bar{\delta}) \in \mathcal{S}; \\ 0, \text{ if } (\bar{\alpha}, \bar{\beta}, \bar{\delta}) \notin \mathcal{S}, \end{cases} \tag{6.27}$$

where

$$\mathcal{S} = \mathcal{S}_2 \cap \mathcal{S}_3 = \left\{ (\bar{\alpha}, \bar{\beta}, \bar{\delta}) : \begin{array}{c} (\alpha_i + \beta_j) \geq 1, \ \forall (i+j) \geq (m+1); \\ (\alpha_i + \delta_l) \geq 1, \ \forall (i+l) \geq (n+1); \\ 0 \leq \alpha_1 \leq \cdots \leq \alpha_u, \\ 0 \leq \beta_1 \leq \cdots \leq \beta_p, \\ 0 \leq \delta_1 \leq \cdots \leq \delta_q, \end{array} \right\}, \tag{6.28}$$

and

$$\begin{aligned} E(\bar{\alpha}, \bar{\beta}, \bar{\delta}) = & \sum_{i=1}^{u}(n+m-2i+1)\alpha_i + \sum_{j=1}^{p}(k+m-2j+1)\beta_j + \sum_{l=1}^{q}(k+n-2l+1)\delta_j \\ & -2k\sum_{i=1}^{u}(1-\alpha_i)^+ + \sum_{\substack{i,j=1 \\ j+i\leq m}}^{u,p}(1-\alpha_i-\beta_j)^+ + \sum_{\substack{i,l=1 \\ l+i\leq n}}^{u,q}(1-\alpha_i-\delta_l)^+. \end{aligned} \tag{6.29}$$

6.3 DMT of the MIMO HD-RC

Assuming global CSI $\mathcal{H}$ at the relay node, it was proved in [65] that the DCF protocol based on the CF scheme of [70] can achieve the DMT of the MIMO HD-RC. The actual DMT was however not obtained therein. Here, using the asymptotic eigenvalue distribution result of Theorem 6.2 of the previous section, the DMT of the MIMO HD-RC is first characterized as the solution of a convex optimization problem (see Theorem 6.3) and then simplified to a two-variable optimization problem (see Theorem 6.4). Moreover, since it is shown that the dynamic QMF protocol achieves this fundamental DMT, only knowledge of the optimal switching time is required at the relay to achieve the DMT of the MIMO HD-RC. This is in contrast to the DCF protocol of [65] which requires global CSI at the relay. Later in Section 6.5, it is proved that even the switching time information, while sufficient, is not necessary under certain conditions. In particular, it is shown that the DMT of the $(n, 1, n)$ and $(1, k, 1)$ HD-RCs can be achieved without CSI at the relay. This is also the case for more general classes of MIMO HD-RCs but only for sufficiently high multiplexing gains.

To characterize the DMT, we first prove that $P_e^*(\rho)$ (see (6.8)), the minimum average probability of decoding error achievable on the channel at an SNR of ρ, is exponentially equal (recall definition in (6.2)) to the probability of an appropriately defined **outage** event. In Subsections 6.3.1 and 6.3.2 we derive an upper bound and a lower bound for the outage probability, respectively, which are in turn exponentially equal. The lower bound to the outage probability is based on an upper bound on the instantaneous cut-set bound of the channel. The upper bound on the outage probability is derived from an achievable rate expression of the QMF protocol operating dynamically on the relay channel. Finally, analyzing these bounds in Section 6.3.3, we derive the DMT of the channel by computing the negative SNR exponent of the outage probability.

It is well known that on a slow fading point-to-point channel the maximum rate at which information can be reliably transferred to the receiver depends on the channel realization, and is hence a random quantity. In what follows, this rate will be referred to as the instantaneous capacity of the channel. For a particular channel realization, if the rate of transmission is larger than the instantaneous capacity of a point-to-point channel, we say the channel is in outage. The same is true for a relay channel, where in addition a relay node helps the end-to-end transmission between the source and the destination nodes. Further, on a dynamic HD-RC, the instantaneous capacity of the channel also depends on the switching time of the relay node and should be chosen optimally to maximize it. Let $\hat{t}_d(\mathcal{H})$ represent the optimal switching time and let the instantaneous capacity be denoted as $C_o\left(\mathcal{H}, \hat{t}_d(\mathcal{H})\right)$. Using this notation we next define the outage event.

Definition 6.1 (Outage event) The HD-RC is said to be in outage if for a particular channel realization, $\mathcal{H}$ and SNR ρ, and the rate of transmission, $R = r\log(\rho)$ (in bits per channel use (Bpcu)) is larger than its instantaneous capacity. The corresponding outage event is denoted as $\mathcal{O}$, so that

$$\mathcal{O} \triangleq \left\{\mathcal{H} : C_o\left(\mathcal{H}, \hat{t}_d(\mathcal{H})\right) < r\log(\rho)\right\}. \tag{6.30}$$

Let $\Pr(\mathcal{O})$ denote the outage probability and let $d_O(r)$ denote its diversity order, i.e., $d_O(r) \triangleq \lim_{\rho\to\infty} -\frac{\log(\Pr(\mathcal{O}))}{\log(\rho)}$. We have the following lemma.

Lemma 6.2 The minimum probability of decoding error achievable on the MIMO HD-RC, $P_e^*(\rho)$ (see (6.8)) is exponentially equal to the outage probability. Hence the corresponding diversity orders are also equal, so that

$$P_e^*(\rho) \doteq \Pr(\mathcal{O}) \quad \Longrightarrow \quad d^*(r) = d_O(r), \tag{6.31}$$

where $d^*(r)$ is defined in (6.7).

Proof 6.3 (Proof of Lemma 6.2) The proof is identical to that in [48].

In the next section, an upper bound on the DMT of the MIMO HD-RC is obtained.

6.3.1 An upper bound on instantaneous capacity (and DMT)

From the discussion in Section 6.1 we have that an upper bound on the DMT for the family of fixed and dynamic protocols is also an upper bound on the achievable DMT of any (cooperative) communication scheme on the MIMO HD-RC. Thus we restrict attention, without loss of generality, to the family of fixed and dynamic protocols. Assuming that the relay node switches from the listening mode to the transmit mode at time t_d, we have that any achievable rate R on the relay channel for which the error probability can be made arbitrarily small is upper bounded using the cut-set bounds for the HD-RC [87, 88] so that

$$R \leq \max_{\{t_d, P(X_S,X_R)\}} \min\left\{I_{C_S}(t_d), I_{C_D}(t_d)\right\} = \max_{t_d} \bar{C}(\mathcal{H}, t_d), \tag{6.32}$$

where $\bar{C}(\mathcal{H}, t_d)$ denotes the cut-set bound of the channel for a given t_d and

$$I_{C_S}(t_d) = t_d I\left(X_S; Y_D, Y_R | p_1\right) + (1-t_d) I\left(X_S; Y_D | X_R, p_2\right); \tag{6.33}$$

$$I_{C_D}(t_d) = t_d I\left(X_S; Y_D | p_1\right) + (1-t_d) I\left(X_S, X_R; Y_D | p_2\right), \tag{6.34}$$

represent the maximum mutual information that can flow across the cuts around the source and destination, respectively.

The following two-part lemma provides (i) upper bounds to both I_{C_S} and I_{C_D} and (ii) a lower bound to $\bar{C}(\mathcal{H}, t_d)$.

Lemma 6.3 i. The cut-set mutual informations $I_{C_S}(t_d)$ and $I_{C_D}(t_d)$ in equations (6.33) and (6.34), are upper bounded as

$$\max_{\{P_{X_S,X_R}\}} I_{C_S}(t_d) \leq I'_{C_S}(t_d) \triangleq t_d \log\left(L_{S,RD}\right) + (1-t_d)\log\left(L_{SD}\right), \tag{6.35}$$

$$\max_{\{P_{X_S,X_R}\}} I_{C_D}(t_d) \leq I'_{C_D}(t_d) \triangleq t_d \log\left(L_{SD}\right) + (1-t_d)\log\left(L_{SR,D}\right), \tag{6.36}$$

where $H_{S,RD} \triangleq \begin{bmatrix} H_{SR} \\ H_{SD} \end{bmatrix}$, $H_{SR,D} \triangleq [H_{SD}\ H_{RD}]$ and

$$L_{SD} \triangleq \det\left(H_{SD}H_{SD}^{\dagger}\rho + I_n\right), \tag{6.37}$$

$$L_{SR,D} \triangleq \det\left(\rho H_{SR,D}H_{SR,D}^{\dagger} + I_n\right); \tag{6.38}$$

$$L_{S,RD} \triangleq \det\left(\rho H_{S,RD}H_{S,RD}^{\dagger} + I_{n+k}\right). \tag{6.39}$$

ii. Moreover, the cut-set bound $\bar{C}(\mathcal{H}, t_d)$ is lower bounded as follows:

$$\bar{C}(\mathcal{H}, t_d) \geq \min\{I'_{C_S}(t_d), I'_{C_D}(t_d)\} - (m+k).$$

Proof 6.4 See Appendix B.2.

Now, continuing from equation (6.32) we have

$$
\begin{aligned}
R \leq & \max_{\{t_d, P(X_S, X_R)\}} \min \left\{ I_{C_S}(t_d), I_{C_D}(t_d) \right\}, \\
\leq & \max_{\{t_d\}} \min \left\{ \max_{\{P(X_S,X_R)\}} I_{C_S}(t_d), \max_{\{P(X_S,X_R)\}} I_{C_D}(t_d) \right\}, \\
\overset{(a)}{\leq} & \max_{\{t_d\}} \min \left\{ I'_{C_S}(t_d), I'_{C_D}(t_d) \right\} = \max_{t_d} R_U(t_d),
\end{aligned} \tag{6.40}
$$

where in step (a) we used the set of upper bounds from the first part of Lemma 6.3 and the definition $R_U(t_d) = \min\left\{ I'_{C_S}(t_d), I'_{C_D}(t_d)\right\}$. Note that t_d in equation (6.40) can be a function of the channel matrices since we are considering a dynamic HD-RC. Since the right hand side of equation (6.40) is maximized when $I'_{C_S}(t_d) = I'_{C_D}(t_d)$, equating equations (6.35) and (6.36) we get the optimal value for the switching time as

$$
t_d^* = \frac{\log\left(\frac{L_{SR,D}}{L_{SD}}\right)}{\log\left(\frac{L_{SR,D}}{L_{SD}}\right) + \log\left(\frac{L_{S,RD}}{L_{SD}}\right)}. \tag{6.41}
$$

Putting this value of t_d in equation (6.40) we get

$$
R \leq \frac{\log\left(\frac{L_{SR,D}}{L_{SD}}\right)\log\left(\frac{L_{S,RD}}{L_{SD}}\right)}{\log\left(\frac{L_{SR,D}}{L_{SD}}\right) + \log\left(\frac{L_{S,RD}}{L_{SD}}\right)} + \log\left(L_{SD}\right) \triangleq R_U^*. \tag{6.42}
$$

Since any rate up to the instantaneous capacity $C_o\left(\mathcal{H}, \hat{t}_d(\mathcal{H})\right)$ is achievable, we have $C_o\left(\mathcal{H}, \hat{t}_d(\mathcal{H})\right) \leq R_U^*$. This inequality when used along with the definition of the outage probability in (6.30) yields

$$
\mathcal{O}(r) = \left\{\mathcal{H} : C_o\left(\mathcal{H}, \hat{t}_d(\mathcal{H})\right) < r\log(\rho)\right\} \supseteq \left\{\mathcal{H} : R_U^* < r\log(\rho)\right\} \triangleq \mathcal{O}_U(r), \tag{6.43}
$$

from which we have a lower bound on the outage probability, $\Pr\{\mathcal{O}(r)\} \geq \Pr\{\mathcal{O}_U(r)\}$. Using (6.31), we then obtain an exponential lower bound on the minimum achievable probability of decoding error, and hence an upper bound on the DMT as

$$
P_e^*(\rho) \dot{\geq} \Pr\{\mathcal{O}_U(r)\} \quad \Longrightarrow \quad d^*(r) \leq d_U(r) \tag{6.44}
$$

where $d_U(r)$ is the diversity order of $\Pr\{\mathcal{O}_U(r)\}$, i.e., $d_U(r) \triangleq \lim_{\rho\to\infty} -\frac{\log(\Pr(\mathcal{O}_U(r)))}{\log(\rho)}$.

6.3.2 A lower bound on instantaneous capacity (and the DMT) via the QMF scheme

Since the instantaneous capacity of a slow fading channel is the supremum of the achievable rates of all possible coding schemes, the achievable rate of a particular coding scheme yields a lower bound to it. We first derive such a lower bound for the HD-RC by computing the achievable rate of the QMF protocol [68] which when substituted in the definition of the outage event results in an upper bound to the outage probability yielding in turn the desired lower bound to the DMT.

Recall that the cut-set upper bound to the instantaneous capacity of the channel for a given listen-transmit scheduling of the relay (i.e., fixed t_d) node was denoted by $\bar{C}(\mathcal{H}, t_d)$ [87,88]. In [68] it was proved that for a given t_d, the QMF protocol can achieve a rate $R_q(\mathcal{H}, t_d)$ on a relay channel with channel matrices $\mathcal{H}$, where

$$R_q(\mathcal{H}, t_d) \geq \bar{C}(\mathcal{H}, t_d) - \tau, \tag{6.45}$$

and τ is independent of both the channel matrices and ρ. The above rate satisfying equation (6.45) can be achieved by the QMF protocol for any given t_d as long as it is known to the relay node. In particular, putting $t_d = t_d^*$ (given by equation (6.41)) in equation (6.45) we get

$$R_q(\mathcal{H}, t_d^*) \geq \bar{C}(\mathcal{H}, t_d^*) - \tau. \tag{6.46}$$

In other words, a rate which is within constant number of bits to $\bar{C}(\mathcal{H}, t_d^*)$ can be achieved by the QMF protocol. Note that, t_d^* is a function of the instantaneous channel realizations, $\mathcal{H}$ (e.g., see equation (6.41)) and can be computed by the relay node since we assume global CSI at the relay node.

From the second part of Lemma 6.3 we have

$$\bar{C}(\mathcal{H}, t_d^*) \geq \min\{I'_{C_S}(t_d^*), I'_{C_D}(t_d^*)\} - (m+k) \geq R_U^* - (m+k),$$

where the last step follows from the fact that $I'_{C_S}(t_d^*) = I'_{C_D}(t_d^*) = R_U^*$ (see equation (6.41)). Now, substituting the last lower bound to $\bar{C}(\mathcal{H}, t_d^*)$ in equation (6.46) we get

$$R_q(\mathcal{H}, t_d^*) \geq R_U^* - \underbrace{(m+k+\tau)}_{R_0} = R_U^* - R_0 \triangleq R_L^*, \tag{6.47}$$

where $R_0 = (m+k+\tau)$.

Clearly, the instantaneous capacity $C_o\left(\mathcal{H}, \hat{t}_d(\mathcal{H})\right)$ is larger than or equal to any achievable rate on the channel, i.e., $C_o\left(\mathcal{H}, \hat{t}_d(\mathcal{H})\right) \geq R_L^*$. This inequality along with the definition of outage probability in (6.30) yields

$$\mathcal{O}(r) = \{\mathcal{H} : C_o\left(\mathcal{H}, \hat{t}_d(\mathcal{H})\right) < r\log(\rho)\} \subseteq \{\mathcal{H} : R_L^* < r\log(\rho)\} \triangleq \mathcal{O}_L(r),$$

which in turn implies that $P_e^*(\rho) \doteq \Pr\{\mathcal{O}(r)\} \leq \Pr\{\mathcal{O}_L(r)\}$, where the exponential equality is from (6.31). Now, since $R_L^* = R_U^* - R_0$ and R_0 is independent of the SNR (ρ) and $\mathcal{H}$, we have at asymptotically high SNR that

$$\begin{aligned}\Pr\{\mathcal{O}_L(r)\} =& \Pr\{R_L^* < r\log(\rho)\} = \Pr\{R_U^* - R_0 < r\log(\rho)\} \\ \doteq& \Pr\{R_U^* < r\log(\rho)\} = \Pr\{\mathcal{O}_U(r)\}.\end{aligned}$$

Hence, $P_e^*(\rho) \dot{\leq} \Pr\{\mathcal{O}_U(r)\}$ and combining with (6.44) we have that $\Pr\{\mathcal{O}_U(r)\}$ characterizes $P_e^*(\rho)$ exactly up to exponential order, i.e.,

$$P_e^*(\rho) \doteq \Pr\{\mathcal{O}_U(r)\}$$

so that the DMT of the MIMO HD-RC can be expressed as

$$d^*(r) = d_U(r) = \lim_{\rho\to\infty} -\frac{\log\left(\Pr\{\mathcal{O}_U(r)\}\right)}{\log(\rho)}. \tag{6.48}$$

with $\mathcal{O}_U(r)$ defined in (6.43) in terms of R_U^* which in turn is defined in (6.42). In the next section, we evaluate this DMT.

Remark 6.1 The QMF protocol can achieve the DMT of the MIMO HD-RC with knowledge of only t_d^*, the switching time that maximizes the cut-set bound (in lieu of the true optimal switching time $\hat{t}_d$)., i.e., it does not require the explicit knowledge of H_{SR}, H_{SD} and H_{RD}. However, although in the QMF protocol the relay node does not require global CSI, the destination node requires global CSI. In particular, the channel realization H_{SR} has to be forwarded by the relay node to the destination which can not directly measure the channel between the source and relay node. The other two channel matrices, i.e., H_{SD} and H_{RD} can be estimated by the destination node itself.

6.3.3 The DMT as a solution to an optimization problem

Evidently, to obtain $d^*(r)$ the probability distribution of R_U^*, which is a function of the three channel matrices, is needed. However, by simplifying the expression for R_U^*, it is shown that just the joint eigenvalue distribution of the three composite channel matrices W_i, for $1 \leq i \leq 3$, defined in Section 6.2.1, suffices. Further simplification shows that only the joint distribution of the SNR exponents of these eigenvalues is sufficient to obtain $d^*(r)$.

Lemma 6.4 The optimal diversity order of the MIMO HD-RC can be written as

$$d^*(r) = \lim_{\rho \to \infty} -\frac{\log\left(\Pr\left\{r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \leq r\right\}\right)}{\log(\rho)}, \tag{6.49}$$

where $r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta})$ is given as

$$r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \triangleq \left[\frac{\sum_{l=1}^{q}(1-\delta_l)^+ \sum_{j=1}^{p}(1-\beta_j)^+}{\sum_{l=1}^{q}(1-\delta_l)^+ + \sum_{j=1}^{p}(1-\beta_j)^+}\right] + \left(\sum_{i=1}^{u}(1-\alpha_i)^+\right); \tag{6.50}$$

and α_i's, β_j's and δ_l's are the negative SNR exponents of the eigenvalues of $H_{SD}H_{SD}^\dagger$, $H_{SR}(I_m + \rho H_{SD}^\dagger H_{SD})^{-1}H_{SR}^\dagger$ and $H_{RD}^\dagger(I_n + \rho H_{SD}H_{SD}^\dagger)^{-1}H_{RD}$, respectively.

Proof 6.5 (Proof) The proof is given in Appendix B.3.

Using the joint pdf of $\{\bar{\alpha}, \bar{\beta}, \bar{\delta}\}$ given by equation (6.27), $\Pr\left\{r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \leq r\right\}$ is evaluated and using equation (6.49), the optimal diversity order $d^*(r)$ is obtained, leading to the following theorem.

Theorem 6.3 The solution of the following optimization problem yields the fundamental DMT of the MIMO HD-RC:

$$\min_{(\bar{\alpha}, \bar{\beta}, \bar{\delta})} F\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right) \tag{6.51}$$

subject to the following constraints

$$\sum_{i=1}^{u}(1-\alpha_i)+\frac{\sum_{j=1}^{p}(1-\beta_j)\sum_{l=1}^{q}(1-\delta_l)}{\sum_{j=1}^{p}(1-\beta_j)+\sum_{l=1}^{q}(1-\delta_l)} \leq r, \tag{6.52}$$

$$\alpha_{m-j+1}+\beta_j \geq 1 \; \forall 1 \leq j \leq p, \tag{6.53}$$

$$\alpha_{n-l+1}+\delta_l \geq 1 \; \forall 1 \leq l \leq q, \tag{6.54}$$

$$0 \leq \alpha_1 \leq \cdots \leq \alpha_u \leq 1, \tag{6.55}$$

$$0 \leq \beta_1 \leq \cdots \leq \beta_p \leq 1, \tag{6.56}$$

$$0 \leq \delta_1 \leq \cdots \leq \delta_q \leq 1; \tag{6.57}$$

where

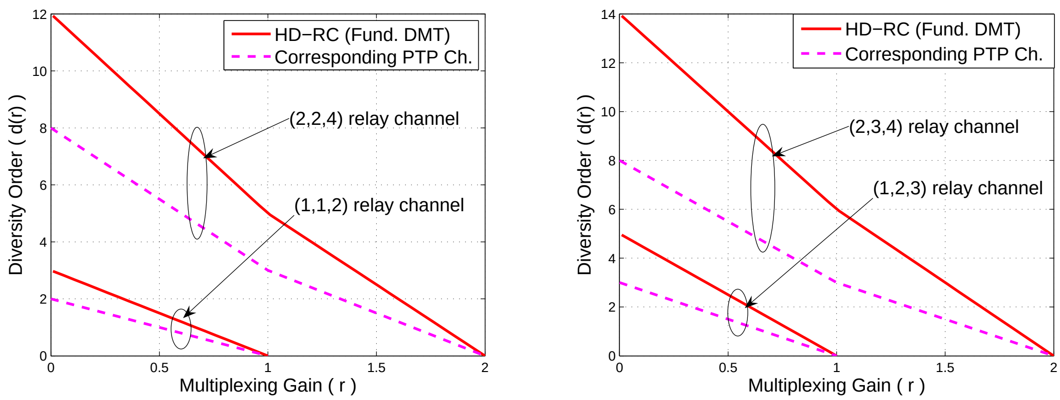
$$F\left(\bar{\alpha},\bar{\beta},\bar{\delta}\right)=\sum_{i=1}^{u}(n+m+2k-2i+1)\alpha_i+\sum_{j=1}^{p}(k+m-2j+1)\beta_j+\sum_{l=1}^{q}(k+n-2l+1)\delta_j-2ku$$
$$+\sum_{\substack{i,j=1\\ j+i\leq m}}^{u,p}(1-\alpha_i-\beta_j)^{+}+\sum_{\substack{i,l=1\\ l+i\leq n}}^{u,q}(1-\alpha_i-\delta_l)^{+}. \tag{6.58}$$

Proof 6.6 (Outline of proof) It is clear from equation (6.49) that the optimal diversity order is equal to the negative SNR exponent of $\Pr\left\{r^*(\bar{\alpha},\bar{\beta},\bar{\delta}) \leq r\right\}$. Using the joint pdf of $(\bar{\alpha},\bar{\beta},\bar{\delta})$ obtained in Subsection 6.2.1, this probability can be written as an integral of the pdf over the subset of the sample space of $(\bar{\alpha},\bar{\beta},\bar{\delta})$ where $r^*(\bar{\alpha},\bar{\beta},\bar{\delta}) \leq r$ (call it $\mathcal{D}$). From Laplace's method it follows that this integral is dominated by a term having the minimum negative SNR exponent over $\mathcal{D}$. The details are provided in Appendix B.4.

Remark 6.2 It is well known that the fundamental DMTs of the (m,n) and the (n,m) point-to-point MIMO channels are identical. From (6.1) it is also clear that the DMTs of the (m,k,n) and the (n,k,m) MIMO FD-RCs are identical. The above theorem proves that this **reciprocity** property of DMT extends to the MIMO HD-RC as well as can be seen from the symmetry in m and n of the optimization problem of (6.51). In other words, the fundamental DMTs on the (m,k,n) and (n,k,m) MIMO HD-RCs are identical. Henceforth, we let $m \geq n$ without loss of generality.

Note that $\sum_{i=1}^{u}\alpha_i$, $\sum_{j=1}^{p}\beta_j$ and $\sum_{l=1}^{q}\delta_l$ are affine functions of the α_i's, β_j's and δ_l's, respectively. Furthermore, by computing its Hessian, it can be easily proved that the function $\frac{(p-x)(q-y)}{(p-x)+(q-y)}$

is not convex with respect to x and y. Thus, it is evident that $\frac{(p-\sum_{j=1}^{p}\beta_j)(q-\sum_{l=1}^{q}\delta_l)}{(p-\sum_{j=1}^{p}\beta_j)+(q-\sum_{l=1}^{q}\delta_l)}$ is a not a convex function. Hence the left hand side of the inequality constraint (6.52) is not a convex function either. Therefore, the optimization problem in Theorem 6.3 is not a convex optimization problem and hence is not amenable to the convex programming methods [56]. Moreover, the number of variables in the optimization also grow with m, k and n linearly. To overcome these problems, in what follows we shall find an equivalent optimization problem with only two variables (independent of m, k and n), which can then be solved by exhaustive search in the closed domain of the problem.



(a) Relay channels of CN_1 (b) Relay channels of CN_2

Figure 6.4: The fundamental DMTs of the MIMO HD-RC and the corresponding point-to-point MIMO channel.

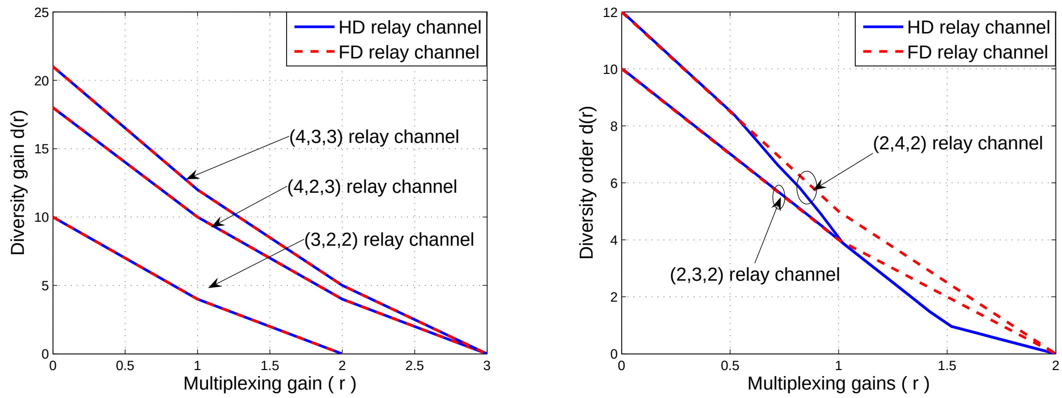
Theorem 6.4 The fundamental diversity-muliplexing tradeoff of the (m, k, n) HD-RC is given as

$$d^*(r) = \min_{\{a\in\mathcal{R},\ b\in\mathcal{B}\}} F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta\left(\frac{b(r-a)}{(b-r+a)}\right)\right), \tag{6.59}$$

where the interval $\mathcal{R}$ is specified in equation (B.28) in Appendix B.5, $\mathcal{B} = \left[\frac{s_m(r-a)}{(s_m-r+a)}, b_m\right]$, $b_m = \min\{p, (m-a)\}$, $s_m = \min\{q, (n-a)\}$ and ϕ_i's are as defined in equations (B.18)-(B.20) in Appendix B.5.

Proof 6.7 (Proof) The proof is given in Appendix B.5.

Example 6.1 We illustrate the advantage of relaying relative to point-to-point communication by considering the networks CN_1 and CN_2 of Fig. 6.1(a) and Fig. 6.1(b). Fig. 6.4(a) applies to the uplink of CN_1 (in which $m = k < n$) and depicts the DMT performance of the relay channel with respect to that achievable on the corresponding point-to-point channel. Similarly, Fig. 6.4(b) applies to the uplink of CN_2 (in which $m < k < n$). These figures clearly demonstrate the superior performance of cooperative MIMO over point-to-point MIMO communication.



(a) Cases where DMTs of FD- and HD-RCs are identical

(b) Cases where DMTs of FD- and HD-RCs are not identical

Figure 6.5: DMTs Comparisons for MIMO HD- vs. FD-RCs.

Remark 6.3 The explicit numerical computation of the fundamental DMT reveals several interesting characteristics of the MIMO HD-RC. For example, for the class of (m, k, n) HD-RCs where $m > n \geq k$ we found that the DMT is identical to that of the corresponding MIMO FD-RC. It appears to be difficult however to show this analytically. Note that this scenario applies to the downlink of the two networks CN_1 (with $m > n = k$) and CN_2 (with $m > n > k$) of Fig. 6.1(a) and Fig. 6.1(b), respectively. Fig. 6.5(a) illustrates this fact for a few specific examples of MIMO RCs. Thus, for the class of MIMO HD-RCs for which $m > n \geq k$, the half-duplex constraint does not appear to be restrictive in terms of DMT performance. In general however, the MIMO HD-RC has different DMT characteristics than the MIMO FD-RC. For instance, see Fig. 6.5(b) (which is relevant for the sensor network CN_3 of Fig. 6.1(c)). This will be further discussed in the next section.

Conjecture 6.1 For the class of (m, k, n) HD-RCs for which $m > n \geq k$ the DMT is equal to that of the corresponding MIMO FD-RC.

6.4 Closed form expressions for the DMTs of a few classes of relay channels

A closed form expression of the DMT would provide more insights about the system than a numerical solution. Motivated by this fact, we next provide closed-form solutions for the fundamental DMT of special classes of MIMO HD-RCs specified by the relationship between the numbers of antennas at the three nodes, including the (n, k, n) (henceforth, called symmetric since $m = n$) HD-RC.

Theorem 6.5 The optimal diversity order $d^*(r)$, at a multiplexing gain of r, of the (n, k, n) HD-RC is upper bounded by $d_U^s(r)$ (defined below) as

$$d^*(r) \leq d_U^s(r) \triangleq \begin{cases} \min_{\{1 \leq i \leq (3+p)\}} d_{U_i}(r), & k \leq n, \\ \min_{\{1 \leq i \leq (3+2p)\}} d_{U_i}(r), & k \geq n, \end{cases} \tag{6.60}$$

where recall that $p = \min\{m, k\} = \min\{k, n\}$ (since $m = n$) and for $N = 1, \cdots, p$, and recalling that $d_{(n_t, n_r)}(\cdot)$ represents the fundamental DMT of a MIMO point-to-point channel with n_t transmit and n_r receive antennas, we define

$$d_{U_1}(r) = d_{(n,n+k)}(r), \quad \text{for} \quad 0 \leq r \leq n, \tag{6.61}$$

$$d_{U_2}(r) = d_{(2n,2n)}(2r), \quad \text{for} \quad n - \frac{p}{2} \leq r \leq n, \tag{6.62}$$

$$d_{U_3}(r) = n^2 + \sum_{l=1}^{p}(n + k - 2l + 1)\left(1 - \left(\frac{pr}{(p-r)} - l + 1\right)^+\right)^+, \quad \text{for} \quad 0 \leq r \leq \frac{p}{2}, \tag{6.63}$$

$$d_{U_{(3+N)}}(r) = N^2 + d_{(n-N),(n+2k-N)}\left(r - \frac{N}{2}\right), \quad \text{for} \quad \frac{N}{2} \leq r \leq \min\left\{n - \frac{N}{2}, n - \frac{N^2}{(2p-N)}\right\} \tag{6.64}$$

$$d_{U_{(3+p+N)}}(r) = \sum_{i=1}^{(n-N)} (2n + k - N - 2i + 1)\left(1 - (a_N - i + 1)^+\right)^+ + N^2, \text{ for } \frac{Nn}{(N+n)} \leq r \leq n - \frac{N}{2} \tag{6.65}$$

and a_N is given by equation (B.43) in Appendix B.6.

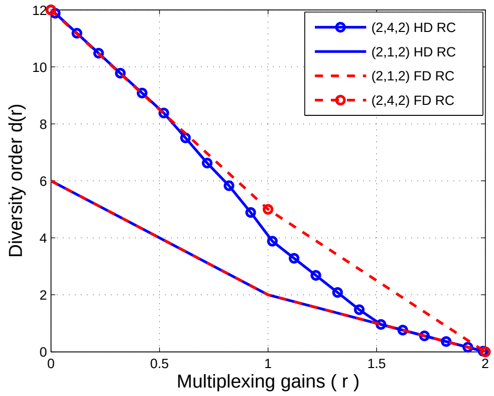
Proof 6.8 (Proof) The proof is given in Appendix B.6.

Evaluating the exact DMT of the (n, k, n) relay channel using Theorem 6.4 for several values of n and k we found that $d^*(r) = d_U^s(r)$ which leads us to make the following conjecture that the upper bound $d_U^s(r)$ is in fact tight.

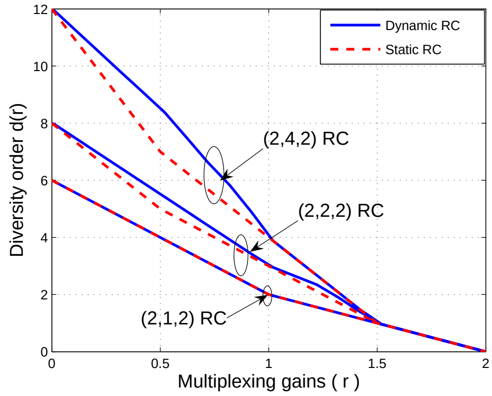
Conjecture 6.2 On a symmetric (n, k, n) relay channel $d^*(r) = d_U^s(r)$, where $d_U^s(r)$ is given by equation (6.60).

Remark 6.4 From the expression of $d_{U_2}(r)$ in equation (6.62) we see that this particular upper bound does not depend on k for $k \geq n$. Thus, when this bound is active, adding an extra antenna at the relay node does not improve the DMT performance of the channel. This is an interesting difference between the HD- and FD-RCs since for FD-RCs every additional antenna at the relay improves the diversity order for all values of multiplexing gains (recall (6.1)). Empirical results show that $d_{U_2}(r)$ is a tight bound for the DMT on the (n, k, n) HD-RC for $r \geq \frac{n}{2}$ and $k \geq \lceil \frac{3n}{2} \rceil$. For example, Fig. 6.5(b) illustrates this fact by showing that while adding an extra antenna on the $(2, 3, 2)$ FD-RC uniformly increases the achievable diversity orders at all multiplexing gains, the achievable diversity order on the corresponding HD-RC does not change for $r \geq 1$.

Remark 6.5 Another interesting fact revealed by Theorem 6.5 is that, as the number of antennas increases at the relay node, the difference in the DMT performance between the FD-RC and the HD-RC increases. From the expression of the upper bound $d_{U_3}(r)$ in equation (6.63) we see that at a multiplexing gain of $r = \frac{p}{2}$ the diversity order achievable on the HD-RC is upper bounded by n^2 but on an FD-RC this is clearly not the case where the diversity order increases with k. Fig. 6.6(a) demonstrates this phenomenon on a $(2, k, 2)$ relay channel which is applicable to the CN_3 scenario of Fig. 6.1(c). Intuitively, the above phenomenon occurs because as the number of antennas at the relay increases the signal forwarded by the relay node can significantly contribute to enhancing the diversity of the received signal at the destination node and hence the half-duplex constraint becomes increasingly more restrictive (relative to the FD relay) because the relay node can not transmit in the listening phase.



(a) MIMO HD-RC Vs. MIMO FD-RC.



(b) Static Vs. Dynamic HD-RCs.

Figure 6.6: A comparison of DMTs on Dynamic vs. Static and HD- vs. FD-RCs.

Remark 6.6 A similar argument holds for the comparison between the DMT performance of the static and dynamic HD-RCs. In contrast to a static channel, since on a dynamic channel the switching time varies depending on the instantaneous channel matrices, it is expected that a larger number of antennas at the relay node make a bigger difference on the DMT performance.

The following result gives an explicit formula for the DMT of the $(1, k, 1)$ HD-RC.

Theorem 6.6 The optimal DMT of a half-duplex $(1, k, 1)$ HD-RC is given as

$$d^*_{(1,k,1)}(r) = \begin{cases} (k+1)(1-r), \ 0 \le r \le \frac{1}{(k+1)}; \\ 1 + k\left(\frac{1-2r}{1-r}\right), \ \frac{1}{(k+1)} \le r \le \frac{1}{2}; \\ 2(1-r), \ \frac{1}{2} \le r \le 1. \end{cases} \tag{6.66}$$

Proof 6.9 (Proof) The proof is given in Appendix B.7.

Remark 6.7 A comparison with the DMT of the class of static $(1, k, 1)$ HD-RCs (derived in [66]), numerical examples of which are given in Fig. 6.2(a) and Fig. 6.2(b), reveals that the DMT of such static HD-RCs are strictly smaller than that for their dynamic counterparts for $r \le \frac{1}{2}$.

The next result gives an explicit DMT formula for the class of symmetric HD-RCs with single-antenna relays.

Theorem 6.7 The DMT of the $(n, 1, n)$ HD-RC is given by a piece-wise linear curve whose corner points at integer values of r are given as

$$\begin{aligned} d^*_{(n,1,n)}(r) &= d_{(n+1,n)}(r) \\ &= (n-r)(n+1-r), \quad 0 \le r \le n. \end{aligned} \tag{6.67}$$

Proof 6.10 (Proof) The proof involves computation of an upper and a lower bound to $d^*_{(n,1,n)}(r)$ and showing that they are identical. The upper bound, $d^u_{(n,1,n)}(r)$ is based on the DMT of the FD relay channel and the DMT of the static $(n, 1, n)$ RC, denoted as $d^{stat}_{(n,1,n)}(r)$ serves as a lower bound (since dynamic protocols include static protocols as a special case).

6.4.1 An expression for $d^{u}_{(n,1,n)}(r)$

Clearly, the the performance of the FD relay channel can not be exceeded by a HD relay channel. The DMT of the FD $(n,1,n)$-RC can easily be computed from [65] and is given by

$$d_{n,(n+1)}(r),\ 0 \leq r \leq n.$$

This in turn by the above argument imply that

$$d^{*}_{(n,1,n)}(r) \leq d^{u}_{(n,1,n)}(r) = d_{n,(n+1)}(r),\ 0 \leq r \leq n. \tag{6.68}$$

6.4.2 Lower bound

Clearly, the DMT of the static $(n,1,n)$ HD-RC, denoted as $d^{stat}_{(n,1,n)}(r)$ serves as a lower bound to the fundamental DMT of the HD relay channel which in general is not restricted to operate in a static manner. In the following we shall derive an expression for the DMT of the static $(n,1,n)$ HD-RC.

The DMT of the symmetric (n,k,n) static HD-RC was established as the solution of a convex optimization problem in [66] and an analytic expression for only an upper bound[6] to the DMT was provided therein. Here we obtain an exact closed form solution to that optimization problem analytically, for the case of $k = 1$. Our starting point is thus equation (13) in [66] which is restated here for convenience,

$$d^{stat}_{(n,1,n)}(r) = \min_{\{(\bar{\alpha},\beta_1)\in\mathcal{T}\}} \sum_{i=1}^{n}(2n-2i+1)\alpha_i + n\beta_1 - \sum_{i=1}^{n}(1-\alpha_i)^{+} + \sum_{i=1}^{n-1}(1-\beta_1-\alpha_i)^{+} \tag{6.69}$$

where

$$\mathcal{T} = \left\{(\bar{\alpha},\beta_1) : \sum_{i=1}^{n}(1-\alpha_i)^{+} + \frac{1}{2}(1-\beta_1)^{+} \leq r; 0 \leq \alpha_1 \leq \alpha_2 \leq, \cdots \leq \alpha_n; 0 \leq \beta_1; (\beta_1+\alpha_n) \geq 1\right\} \tag{6.70}$$

Using an argument similar to that in the proof of Theorem 6.3 in Appendix B.4, it can be shown that a further restriction to $\alpha_n, \beta_1 \in [0,1]$ can be made without changing the solution of (6.69) but

[6] However, in [66] based on numerical simulations it was claimed that this upper bound is tight and represents the DMT of the channel.

it greatly simplifies the problem as

$$d^{stat}_{(n,1,n)}(r) = \min_{\{(\bar{\alpha},\beta_1)\in\hat{\mathcal{T}}\}} \sum_{i=1}^{n}(2n-2i+2)\alpha_i + n\beta_1 - n + \sum_{i=1}^{n-1}(1-\beta_1-\alpha_i)^+. \tag{6.71}$$

where

$$\hat{\mathcal{T}} = \left\{(\bar{\alpha},\beta_1) : \sum_{i=1}^{n}(1-\alpha_i) + \frac{1}{2}(1-\beta_1) \leq r; 0 \leq \alpha_1 \leq \alpha_2 \leq, \cdots \leq \alpha_n \leq 1; 0 \leq \beta_1 \leq 1; (\beta_1+\alpha_n) \geq 1\right\} \tag{6.72}$$

The proof that the solution of (6.71) is indeed identical to $d_{n,(n+1)}(r)$, follows from induction and the details are relegated to Appendix B.8.

The Theorem then follows from the fact that the lower bound and upper bounds derived above coincides.

Before proceeding to the next section, we summarize the findings of the previous two sections. The explicit computation of the DMT of the MIMO HD-RC enabled the proof of the existence of classes of HD-RCs whose DMT performances are (a) strictly inferior to that of the corresponding FD-RCs (with certain m, k, n) and (b) equal to that of the corresponding FD-RCs (when $m > n \geq k$, see Remark 6.3, which was proved empirically for a large number of values of m, k and n satisfying the constraint). Furthermore, closed-form solutions for the two-variable optimization problem are obtained for two classes of symmetric HD-RCs, namely, the $(n, 1, n)$ and the $(1, k, 1)$ HD-RCs (see Theorems 6.6 and 6.7). More generally, the explicit DMT is upper bounded for the symmetric (n, k, n) HD-RCs and is conjectured to be tight (see Theorem 6.5 and Conjecture 6.2). These solutions reveal certain special characteristics of the HD-RC which are different from that of the FD-RC and the static HD-RC. For example, while an extra antenna at the relay node uniformly improves the DMT for an FD-RC this is not the case for the HD-RC. Moreover, the greater the number of antennas at the relay, the greater is the difference between the DMT performances of the FD- and HD-RCs. It is also observed that within the class of HD-RCs, static operation of the relay in general limits performance relative to unconstrained (or dynamic) operation of the relay with the difference in DMT performance becoming more pronounced with an increasing number of antennas at the relay.

6.5 Achievability of the DMT without switching time at the relay node

In the previous section we established the fundamental DMT for the MIMO HD-RC. It hence represents the DMT achievable by the best cooperative protocol among all admissible ones with global CSI at both the relay and the destination. The QMF scheme of Section 6.3.2 however achieves this fundamental tradeoff with just the relay having knowledge of the switching time t_d^* defined in (6.41) (with the destination node having global CSI). In this section, we explore the question of whether there are situations in which even the switching time knowledge at the relay is not necessary. Note that the DCF protocol was shown to be optimal from the DMT perspective in [65] so that the DMT of the previous section is achievable by this protocol too but it requires global CSI at the relay. In the case of the SISO HD-RC, the QMF scheme of [68] was shown in [82] to achieve, with switching time of 1/2 (and hence without knowledge of switching time at relay), the fundamental DMT of the SISO FD-RC which in turn is an upper bound for the SISO HD-RC, so that dynamic operation of the relay in this case does not help from the DMT perspective. Does this result generalize to MIMO HD-RCs? It turns out that it does in some cases. In particular, we show that for the $(n, 1, n)$ MIMO HD-RCs the DMT (given by Theorem 6.7) can indeed be achieved by the QMF protocol with a channel independent switching time. In other words, on this class of RCs even global CSI at the relay node does not help in terms of DMT performance, generalizing the SISO HD-RC result of [82]. Moreover, for the class of $(1, k, 1)$ HD-RCs we show that for all multiplexing gains $r \in [0, 1/2]$ the optimal tradeoff curve can be achieved by the dynamic decode-and-forward (DDF) protocol analyzed for the general MIMO HD-RC by the authors in [52] and for all multiplexing gains $r \in [1/2, 1]$ it is achieved by the static QMF protocol and neither of these protocols requires knowledge of H_{SD} and H_{RD} at the relay node.

We shall show that while on the two specific classes of RCs as specified earlier the optimal diversity order at all multiplexing gains can be achieved without any CSI at the relay node, the same is also true in general but only for higher multiplexing gain values (e.g., see Fig. 6.6(b)).

Remark 6.8 The DMT of the static MIMO HD-RC was obtained in [66] with the static CF protocol as the achievability scheme which requires global CSI at the relay. From Theorem 8.5 in [68] however, we have that on the static HD-RC, the QMF protocol, which doesn't require any CSI at the relay node (cf. Section VIII-A of [68]), can achieve the instantaneous capacity within a constant gap. Since a constant number of bits is insignificant in DMT metric, the QMF protocol can hence achieve the DMT of the static HD-RC.

Remark 6.9 If the DMT of the static and dynamic HD-RCs are identical over some range of multiplexing gains then the optimal diversity orders at those values of the multiplexing gains can be achieved without any CSI at the relay node.

Example 6.2 From Fig. 6.6(b) it is clear that the optimal diversity order achievable on the $(2, 2, 2)$ and $(2, 4, 2)$ HD-RCs can be achieved by the QMF protocol without any CSI at the relay for $r \in [1.5,\ 2]$ and $r \in [1,\ 2]$, respectively. For all other values of r, the relay node requires the optimal switching time (or t_d^*) to achieve the maximum diversity orders achievable by a dynamic protocol.

We turn our attention now to the MIMO FD-RC. The optimality of the CF protocol in the DMT metric was proved in [65] (and the DMT itself was found to be given by (6.1)) but this protocol requires global CSI as discussed earlier.

Remark 6.10 It was shown in [68] that the (full-duplex version of the) QMF protocol can achieve the instantaneous capacity of the FD-RC to within a constant number of bits. Hence, the DMT of the MIMO FD-RC can be achieved by the QMF protocol without any CSI at the relay node and global CSI at the destination node.

As an immediate application of Remark 6.8 and Theorem 6.6 we get the following result.

Corollary 6.1 The fundamental DMT of the $(1, k, 1)$ relay channel can be achieved by the DDF protocol, for multiplexing gains in $\left[0, \frac{1}{2}\right]$ and by the static QMF protocol for multiplexing gains in the interval $\left[\frac{1}{2}, 1\right]$. While the DDF protocol requires only CSIR, the QMF protocol does not require any CSI at the relay node, i.e., neither the DDF nor the QMF protocol requires global CSI at the relay node.

Proof 6.11 (Proof) Theorem 6.6 provides the optimal DMT on a $(1, k, 1)$ relay channel. Comparing it with the DMT of the DDF protocol on this channel derived in [52] which is restated here for convenience, namely,

$$d^{\text{DDF}}_{(1,k,1)}(r) = \begin{cases} (k+1)(1-r),\ 0 \leq r \leq \frac{1}{(k+1)}; \\ 1 + k\left(\frac{1-2r}{1-r}\right),\ \frac{1}{(k+1)} \leq r \leq \frac{1}{2}; \\ \left(\frac{(1-r)}{r}\right),\ \frac{1}{2} \leq r \leq 1, \end{cases}$$

it is evident that the fundamental DMT of the $(1, k, 1)$ HD-RC can be achieved by the DDF protocol for $0 \leq r \leq \frac{1}{2}$. Further this DMT can be achieved by the DDF protocol with only the knowledge of H_{SR} at the relay node.

On the other hand, it was proved in [66] that the DMT of the static $(1, k, 1)$ HD-RC is $2(1-r)$ for $\frac{1}{2} \leq r \leq 1$. For $\frac{1}{2} \leq r \leq 1$, this DMT can be achieved by the QMF protocol without any CSI at the relay by Remark 6.9.

The key enabling result for Corollary 6.1 beyond the DMTs of the DDF and the static HD-RC is the explicit DMT of the $(1, k, 1)$ HD-RC of Theorem 6.6. Moreover, to the best of our knowledge, Corollary 6.1 is the first result on the achievability of the DMT of a non-SISO HD-RC without global CSI at the relay node. This result however requires two different protocols for the two ranges of multiplexing gains. In this sense, the above result doesn't truly generalize the result of [82] in which it is shown that the QMF protocol achieves the DMT of the SISO HD-RC.

Example 6.3 Comparing the DMT of the DDF protocol with the fundamental DMT of the $(1, 2, 1)$ relay channel depicted in Fig. 6.2(b), we see that the DDF protocol is DMT optimal on this channel for a multiplexing gain in the range $[0, \frac{1}{2}]$. Moreover, since the static DMT is strictly smaller than that of the corresponding dynamic channel in this range, the CF and QMF protocols require global CSI and the optimal switching time information at the relay node, respectively, to achieve optimal

DMT performance. However, the DDF protocol needs only source-to-relay CSI at the relay node. Clearly, the cooperative protocol of choice in this case (i.e., with $r \in [0, \frac{1}{2}]$) is the DDF protocol.

In what follows, we identify a class of non-SISO RCs, namely the $(n, 1, n)$ HD-RCs, on which the DMT of the channel can be achieved by a single protocol, namely the static QMF protocol with no CSI at the relay node, thereby generalizing the result of [82]. This result is shown by proving that the DMTs of the static and dynamic $(n, 1, n)$ HD-RCs are identical. In other words, for this class of HD-RCs dynamic operation of the relay does not help from the DMT perspective. We start by first finding in closed form the DMT of the static $(n, 1, n)$ HD-RC.

Theorem 6.8 The DMT of the static and dynamic $(n, 1, n)$ HD-RCs are identical, i.e., is given by

$$d^{stat}_{(n,1,n)}(r) = d^{*}_{(n,1,n)}(r) = d_{(n+1),n}(r), \quad 0 \leq r \leq n. \tag{6.73}$$

Hence, the DMT of the $(n, 1, n)$ HD-RC can be achieved by the static QMF protocol with no CSI at the relay node (i.e., without the knowledge of even the optimal switching time).

Proof 6.12 (Proof) The second equality in (6.73) was proved in Theorem 6.7 and it was shown that the DMT is identical to that achievable on a static HD-RC. In other words, the static QMF protocol (which, by Remark 6.8, achieves the DMT of the static HD-RC without any CSI at the relay node) achieves DMT of the dynamic $(n, 1, n)$ HD-RC without any CSI at the relay node.

Remark 6.11 Note that the DMT of the FD $(n, 1, n)$ RC is also given be $d_{(n+1),n}(r)$ [65]. Therefore, on the $(n, 1, n)$ HD-RC, the DMT of the $(n, 1, n)$ FD-RC can be achieved by an HD relay without any CSI at the relay node.

Remark 6.12 Although, for a large number of $n \in \mathbb{N}$ the DMT of the static $(n, 1, n)$ RC was computed in [66] and was observed to be equal to $d_{(n+1),n}(r)$, the analysis of [66] only proves that $d_{(n+1),n}(r)$ represents an upper bound to the DMT of the static $(n, 1, n)$ RC. Therefore, the conclusion of Theorem 6.8 cannot be obtained from the result of [66].

6.6 Conclusion

The fundamental DMT of the three-terminal (m, k, n) HD-RC is characterized. This allows an in-depth comparison of half-duplex and full duplex relaying as well as dynamic and static operation of the relay as a function of the numbers of antennas at the three nodes. Unlike in the single-antenna relay channel, half-duplex relaying in general results in a penalty relative to a full-duplex relaying and an improved performance relative to static half-duplex relaying at high SNR performance as measured by the DMT metric. The achievability of the fundamental DMT is shown via the dynamic QMF protocol [68] which requires only the knowledge of the optimal switching time at the relay. Classes of HD-RCs for which dynamic operation of the relay doesn't improve performance over that of static relaying are identified. For such RCs the knowledge of switching time is not needed either. The problem of characterizing the DMT of the relay channel with multiple relays is one for future research as is the problem of finding finite block-length coding schemes that are DMT optimal.

Appendix A

MIMO IC

A.1 Proof of Theorem 2.2

Let us denote the two random vectors involved in equation (2.16) by $\tilde{W}_1$ and $\tilde{W}_2$ respectively, i.e.,

$$\tilde{W}_1 = H_1 X_1 + \tilde{Z}_1;\ \tilde{W}_2 = H_2 X_1 + \tilde{Z}_2,$$

and the svd of the matrices H_i as follows

$$H_1 = V_1 \Sigma_1 U_1^\dagger;\ H_2 = V_2 \Sigma_2 U_2^\dagger,$$

where $U_i \in \mathbb{U}^{N_i \times N_i}$, $V_i \in \mathbb{U}^{M \times M}$ and Σ_i contains the singular values of H_i in the decreasing order along its diagonal. Since a unitary transformation does not change the differential entropy of a random vector we have

$$h\left(\tilde{W}_i\right) = h\left(W_i = \Sigma_i U_i^\dagger X_1 + Z_i\right),\ 1 \leq i \leq 2,$$

where $Z_i = V_i^\dagger \tilde{Z}_i \sim \tilde{Z}_i$. Now, depending on the relative values of N_i and M we modify the vector W_i in two different ways:

- Case ($N_i \geq M$): In this case, the last $(N_i - M)$ rows of Σ_i have only zeros and W_i can be written as

$$\begin{aligned} W_i =& \Sigma_i U_i^\dagger X_1 + Z_i = \begin{bmatrix} \hat{\Sigma}_i \\ 0 \end{bmatrix} U_i^\dagger X_1 + Z_i, \\ =& \begin{bmatrix} \hat{W}_i \\ Z_{ie} \end{bmatrix}, \text{ where } \hat{W}_i = \hat{\Sigma}_i U_i^\dagger X_1 + \hat{Z}_i, \end{aligned}$$

$\hat{Z}_i \sim \mathcal{CN}(0, I_M)$ and $Z_{ie} \sim \mathcal{CN}(0, I_{(N_i-M)^+})$ are mutually independent and $\hat{\Sigma}_i$ contains the non-zero singular values of H_i. Since Z_{ie} is independent of $\hat{Z}_i$ and X_1 we can write

$$\begin{aligned} h(W_i) =& h(\hat{W}_i) + h(Z_{ie}|\hat{W}_i) = h(\hat{W}_i) + h(Z_{ie}) \\ =& h(\hat{W}_i) + (N_i - M)\log(2\pi e). \end{aligned} \tag{A.1}$$

- Case ($N_i < M$): In this case the last $(M - N_i)$ columns of Σ_i are zeros, i.e.,

$$\Sigma_i = \begin{bmatrix} \tilde{\Sigma}_i & 0_{N_i \times (M-N_i)^+} \end{bmatrix},$$

where $\tilde{\Sigma}_i$ contains along its diagonal the non-zero singular values of H_i. Defining $\hat{\Sigma}_i$ as

$$\hat{\Sigma}_i = \begin{bmatrix} \tilde{\Sigma}_i + \delta I_{N_i} & 0_{N_i \times (M-N_i)^+} \\ 0_{(M-N_i)^+ \times N_i} & \delta I_{(M-N_i)^+ \times (M-N_i)^+} \end{bmatrix},$$

the differential entropy of W_i can be written as

$$\begin{aligned} h(W_i) =& h(\Sigma_i U_i^\dagger X_1 + Z_i), \\ =& \lim_{\delta \to 0} \Big[h\left(\hat{\Sigma}_i U_i^\dagger X_1 + \hat{Z}_i\right) \\ & \quad - h\left(L U_i^\dagger X_1 + Z_{ie}|W_i\right)\Big], \\ \overset{(a)}{=}& \lim_{\delta \to 0} \left[h\left(\hat{W}_i\right)\right] - h\left(Z_{ie}|W_i\right), \\ \overset{(b)}{=}& \lim_{\delta \to 0} \left[h\left(\hat{W}_i\right)\right] - (M - N_i)\log(2\pi e), \end{aligned} \tag{A.2}$$

where $\hat{Z}_i = [Z_i^T \ Z_{ie}^T]^T \sim \mathcal{CN}(0, I_M)$ and $Z_{ie} \sim \mathcal{CN}(0, I_{(M-N_i)})$ is independent of Z_i and X_1 and $L = \left[0_{(M-N_i)^+ \times N_i} \ \delta I_{(M-N_i)^+ \times (M-N_i)^+}\right]$. In step (a) we applied the limit in the second term and used the notation $\hat{W}_i = \hat{\Sigma}_i U_i^\dagger X_1 + \hat{Z}_i$. Whereas, step (b) follows because Z_{ie} is independent of W_i.

Denoting $\hat{\Sigma}_i U_i^\dagger$ by $\hat{H}_i$ it can be proved that

$$\hat{H}_1^\dagger \hat{H}_1 \preceq \hat{H}_i^\dagger \hat{H}_2 \tag{A.3}$$

For $N_i \geq M$, this fact follows directly from the assumption of the Lemma. For $N_i < M$, it can be proved as follows. From the assumption (2.15) we have

$$\begin{aligned}
U_1\Sigma_1^\dagger \Sigma_1 U_1^\dagger \preceq& U_2\Sigma_2^\dagger \Sigma_2 U_2^\dagger,\\
\text{or, } U_1\Sigma_1^\dagger \Sigma_1 U_1^\dagger + \delta^2 U_1 I_M U_1^\dagger \preceq& U_2\Sigma_2^\dagger \Sigma_2 U_2^\dagger + \delta^2 U_2 I_M U_2^\dagger,\\
\text{or, } \lim_{\delta\to 0}\ U_1\hat{\Sigma}_1^\dagger \hat{\Sigma}_1 U_1^\dagger \preceq& U_2\hat{\Sigma}_2^\dagger \hat{\Sigma}_2 U_2^\dagger,\\
\text{or, } \lim_{\delta\to 0}\ \hat{H}_1^\dagger \hat{H}_1 \preceq& \hat{H}_2^\dagger \hat{H}_2.
\end{aligned}$$

Using equation (A.1) and (A.2) in the desired expression of the Lemma we get

$$\begin{aligned}
\Gamma \triangleq& h\left(H_1 X_1 + \tilde{Z}_1\right) - h\left(H_2 X_1 + \tilde{Z}_2\right) - h(\tilde{Z}_1) + h(\tilde{Z}_2),\\
=& \lim_{\delta\to 0} h\left(\hat{H}_1 X_1 + \tilde{Z}_1\right) - h\left(\hat{H}_2 X_1 + \tilde{Z}_2\right).
\end{aligned}$$

Note that although the above equation is valid for all values of M and N_i allowed in the Lemma, for $N_1, N_2 < M$ it is independent of δ. Moreover, since $\hat{H}_i$ for $i = 1, 2$ are full rank and satisfies equation (2.15), we can use Lemma 2.1 which results in the following

$$\Gamma \leq 0. \tag{A.4}$$

A.2 Proof of Theorem 2.3

The proof is divided into two parts: first we prove the converse, where it is shown that any rate pair in the capacity region satisfies the set of constraints specified in (5.11)-(3.10). Then we prove that any rate pair in the capacity region can also be achieved by Gaussian and independent coding at both the receivers, which is also an optimal coding scheme for the compound MAC channel present within the 2-user IC.

Before going into the details of the proof we define some notations. Let $S_{ij,t} = H_{ij}X_{it} + Z_{jt}$ represents the signl/interference from user i plus the additive noise at receiver j at time t. Then $S_{ij}^n = [S_{ij,1}^\dagger \; S_{ij,2}^\dagger \; \cdots S_{ij,n}^\dagger]^\dagger$ can be written as

$$\begin{aligned} S_{ij}^n = & \begin{bmatrix} H_{ij} & 0 & \cdots & 0 \\ 0 & H_{ij} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{ij} \end{bmatrix} \begin{bmatrix} X_{i1} \\ X_{i2} \\ \vdots \\ X_{in} \end{bmatrix} + \begin{bmatrix} Z_{j1} \\ Z_{j2} \\ \vdots \\ Z_{jn} \end{bmatrix} \\ = & (I_n \otimes H_{ij})\, X_i^n + Z_j^n. \end{aligned}$$

If we denote the signal received at Rx_i over n channel uses by Y_i^n. Then using Fano's inequality and the fact that conditioning does not increase the average entropy the following inequalities can be easily proved.

$$\begin{aligned} h(Y_i^n) \leq & \, n \log\det\left(I_{N_i} + H_{ji}K_{x_j}H_{ji}^\dagger + H_{ii}K_{x_i}H_{ii}^\dagger\right) \\ & + nN_i \log(2\pi e), \;\; i \neq j \in \{1,2\}, \end{aligned} \tag{A.5}$$

$$\begin{aligned} h((I_n \otimes H_{ij})X_i^n + Z_j^n) \leq & \, n \log\det\left(I_{N_j} + H_{ij}K_{x_i}H_{ij}^\dagger\right) \\ & + nN_j \log(2\pi e), \;\; i, j \in \{1,2\}. \end{aligned} \tag{A.6}$$

A.2.1 An upper bound to the capacity region

To derive the upper bounds we will assume that a genie provides some additional information about the signal of interest to one or both of the receivers.

(1) From Fano's lemma we have

$$\begin{aligned} nR_1 \leq & I(X_1^n; Y_1^n) + n\epsilon_n \overset{(a)}{\leq} I(X_1^n; Y_1^n, X_2^n) + n\epsilon_n, \\ \overset{(b)}{=} & I(X_1^n; X_2^n) + I(X_1^n; Y_1^n | X_2^n) + n\epsilon_n, \\ = & h((I_n \otimes H_{11})X_1^n + Z_1^n) - h(Z_1^n) + n\epsilon_n \\ \overset{(c)}{\leq} & n \log\det\left(I_{N_1} + H_{11}K_{x_1}H_{11}^\dagger\right) + n\epsilon_n, \end{aligned}$$

where step (a) follows from the fact that some extra information (X_2^n) at the receiver does not reduce the average mutual information, step (b) follows from the fact that $I(X_1^n; X_2^n) = 0$, since X_{1t} and X_{2t} are independent and step (c) follows from equation (A.6). Now dividing both sides by n in the limit when $n \to \infty$, we get

$$R_1 \leq \log\det\left(I_{N_1} + H_{11}K_{x_1}H_{11}^{\dagger}\right),$$

(2) Derivation of the second bound is similar.

(3) To derive the third upper bound we assume that a genie provides the side information (X_2^n) to Rx_1. Starting with the Fano's lemma we get

$$\begin{aligned}
n(R_1 + R_2) &\leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) + n\epsilon_n \\
&\overset{(a)}{\leq} I(X_1^n; Y_1^n, X_2^n) + I(X_2^n; Y_2^n) + n\epsilon_n, \\
&= I(X_1^n; X_2^n) + I(X_1^n; Y_1^n|X_2^n) + I(X_2^n; Y_2^n) + n\epsilon_n,
\end{aligned}$$

where the inequality in step (a) follows due to the extra information provided by the genie. Since X_1^n and X_2^n are independent we have $I(X_1^n; X_2^n) = 0$ using this in the last equation we get

$$\begin{aligned}
& n(R_1 + R_2) \\
&\leq h(Y_1^n|X_2^n) - h(Z_1^n) + h(Y_2^n) - h(Y_2^n|X_2^n) + n\epsilon_n, \\
&= h((I_n \otimes H_{11})X_1^n + Z_1^n) - h(Z_1^n) \\
&\qquad + h(Y_2^n) - h((I_n \otimes H_{12})X_1^n + Z_2^n) + n\epsilon_n, \\
&\leq h((I_n \otimes H_{11})X_1^n + Z_1^n) - h((I_n \otimes H_{12})X_1^n + Z_2^n) \\
&\qquad + h(Y_2^n) - h(Z_1^n) + n\epsilon_n, \\
&\overset{(b)}{\leq} h(Y_2^n) - h(Z_2^n) + n\epsilon_n, \\
&\overset{(c)}{\leq} n\log\det\left(I_{N_2} + H_{12}K_{x_1}H_{12}^{\dagger} + H_{22}K_{x_2}H_{22}^{\dagger}\right) + n\epsilon_n
\end{aligned}$$

where step (b) follows from Corollary 2.1 and step (c) follows from equation (A.5). Now, dividing both sides by n and taking limit with respect to n, we get the third bound since $\epsilon_n \to 0$ as $n \to \infty$.

(4) The fourth bound can be derived similarly assuming the genie provides the side information (X_1^n) to Rx_2.

A.2.2 Achievability

As an achievable coding scheme we allow both of the transmit signals to be decoded by both the receivers. Essentially, the IC becomes a compound multiple-access channel (CMAC). The achievable rate region of the discrete memoryless CMAC is given as

$$\mathcal{R}_{\text{CMAC}} = \mathcal{R}_{\text{MAC}-R_1} \cap \mathcal{R}_{\text{MAC}-R_2}, \tag{A.7}$$

where $\mathcal{R}_{\text{MAC}-R_i}$ denotes the MAC formed by the transmitters and receiver i, for $i = 1, 2$,

$$\begin{aligned} \mathcal{R}_{\text{MAC}-R_1} = \Big\{ (R_1, R_2) : R_1 \leq &\ \ I(X_1; Y_1 | X_2); \\ R_2 \leq &\ \ I(X_2; Y_1 | X_1); \\ R_1 + R_2 \leq &\ \ I(X_1, X_2; Y_1) \Big\} \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} \mathcal{R}_{\text{MAC}-R_2} = \Big\{ (R_1, R_2) : R_1 \leq &\ \ I(X_1; Y_2 | X_2); \\ R_2 \leq &\ \ I(X_2; Y_2 | X_2); \\ R_1 + R_2 \leq &\ \ I(X_1, X_2; Y_2) \Big\} \end{aligned} \tag{A.9}$$

Moreover, we assume that each transmitter uses a random i.i.d. Gaussian code book which can be written as

$$X_{it} \sim \mathcal{CN}(0, K_{x_i}),\ 1 \leq i \leq 2,\ \forall\ t. \tag{A.10}$$

Note that this distribution satisfies the input power constraint (2.1). Substituting this into equation (A.8) and (A.9) we obtain the achievable rate regions for the two MACs as follows.

$$\begin{aligned} \mathcal{R}^g_{\text{MAC}-R_1} = \Big\{ (R_1, R_2) : R_1 \leq &\ \ \log\det\left(I_{N_1} + H_{11} K_{x_1} H_{11}^\dagger\right); \\ R_2 \leq &\ \ \log\det\left(I_{N_1} + H_{21} K_{x_2} H_{21}^\dagger\right); \\ R_1 + R_2 \leq &\ \ \log\det\left(I_{N_1} + H_{21} K_{x_2} H_{21}^\dagger + H_{11} K_{x_1} H_{11}^\dagger\right) \Big\} \end{aligned} \tag{A.11}$$

and

$$\begin{aligned}\mathcal{R}^g_{\mathrm{MAC}-R_2} = \Big\{(R_1, R_2) : R_1 \leq \;& \log\det\left(I_{N_2} + H_{12}K_{x_1}H_{12}^\dagger\right);\\ R_2 \leq \;& \log\det\left(I_{N_2} + H_{22}K_{x_2}H_{22}^\dagger\right);\\ R_1 + R_2 \leq \;& \log\det\left(I_{N_2} + H_{12}K_{x_1}H_{12}^\dagger + H_{22}K_{x_2}H_{22}^\dagger\right)\Big\}\end{aligned} \tag{A.12}$$

On the other hand, from the constraint among the channel matrices that defines the strong in partial order IC we get

$$\log\det\left(I_{N_1} + H_{11}K_{x_1}H_{11}^\dagger\right) \leq \log\det\left(I_{N_2} + H_{12}K_{x_1}H_{12}^\dagger\right); \tag{A.13}$$

$$\log\det\left(I_{N_2} + H_{22}K_{x_2}H_{22}^\dagger\right) \leq \log\det\left(I_{N_1} + H_{21}K_{x_2}H_{21}^\dagger\right). \tag{A.14}$$

Substituting the above capacity expressions for $\mathcal{R}^g_{\mathrm{MAC}-R_i}$ in equation (A.7) and using the last two inequalities we get the following achievable rate region for the CMAC which as explained earlier is also achievable on the 2-user MIMO IC.

$$\begin{aligned}\mathcal{R}_{\mathrm{GIC}} = \Big\{R_1 \leq \;& \log\det\left(I_{N_1} + H_{11}K_{x_1}H_{11}^\dagger\right);\\ R_2 \leq \;& \log\det\left(I_{N_2} + H_{22}K_{x_2}H_{22}^\dagger\right);\\ R_1 + R_2 \leq \;& \log\det\left(I_{N_2} + H_{12}K_{x_1}H_{12}^\dagger + H_{22}K_{x_2}H_{22}^\dagger\right);\\ R_1 + R_2 \leq \;& \log\det\left(I_{N_1} + H_{21}K_{x_2}H_{21}^\dagger + H_{11}K_{x_1}H_{11}^\dagger\right)\Big\}.\end{aligned}$$

A.3 Proof of Lemma 3.1

The set of upper bounds to the achievable rate region will be derived in two steps. In the first step, the different mutual information terms in Fano's inequality are expanded in terms of the corresponding differential entropies. The genie-aided signaling strategies of [27] are employed that provide side information to the receivers so that no negative entropy term involving inputs appear in the upper bound, thereby allowing for a single-letterization of the resulting bounds. The positive differential entropies are then upper bounded using Lemma A.1 and its corollaries in Appendix A.4.

(1) From Fano's lemma we have

$$
\begin{aligned}
nR_1 \leq & I(X_1^n; Y_1^n) + n\epsilon_n \\
\overset{(a)}{\leq} & I(X_1^n; Y_1^n, X_2^n) + n\epsilon_n, \\
= & I(X_1^n; X_2^n) + I(X_1^n; Y_1^n | X_2^n) + n\epsilon_n, \;\; [I(X_1^n; X_2^n) = 0, \text{ since } X_{1t} \text{ and } X_{2t} \text{ are independent}] \\
= & h(\sqrt{\rho_{11}}(I_n \otimes H_{11})X_1^n + Z_1^n) - h(Z_1^n) + n\epsilon_n \\
\overset{(b)}{\leq} & n \log\det\left(I_{N_i} + \rho_{11} H_{11} H_{11}^{\dagger}\right) - \log\det(I_{N_1}) + n\epsilon_n, \\
= & n \log\det\left(I_{N_i} + \rho_{11} H_{11} H_{11}^{\dagger}\right) + n\epsilon_n,
\end{aligned}
$$

where step (a) follows from the fact that the extra information (X_2^n) at the receiver does not reduce mutual information and step (b) follows from Corollary A.3 in Appendix A.4. Now dividing both sides by n and taking the limit as $n \to \infty$, we have

$$
R_1 \leq \log\det\left(I_{N_1} + \rho_{11} H_{11} H_{11}^{\dagger}\right),
$$

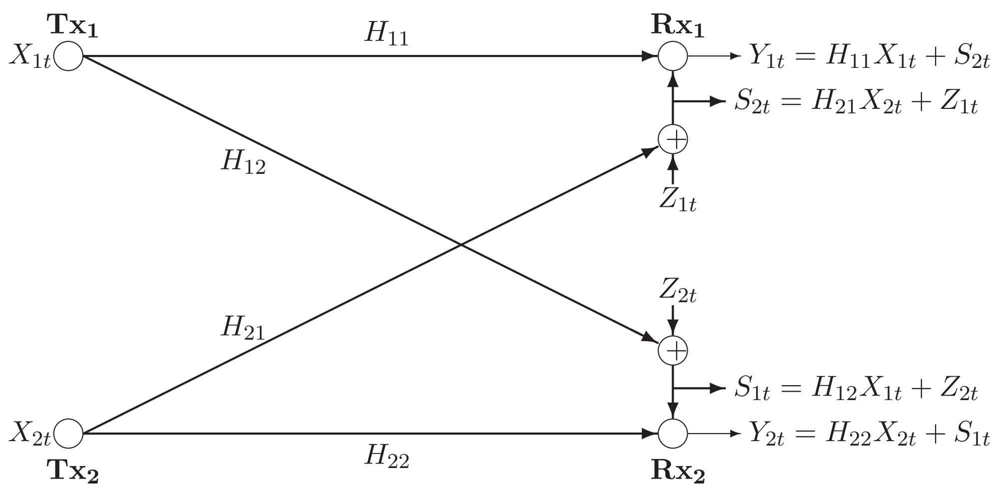


Figure A.1: 2-user MIMO IC with genie aided receivers.

(2) The second bound can be obtained similarly.

(3) The third upper bound is derived with the genie providing the side information (S_1^n, X_2^n) to Rx_1 where S_{it} is the interference plus noise at Rx_j $j \neq i$ (as shown in Fig. A.1). Since

additional information does not reduce mutual information we have from Fano's lemma

$$
\begin{aligned}
n(R_1+R_2) \leq & I(X_1^n;Y_1^n) + I(X_2^n;Y_2^n) + n\epsilon_n \\
\overset{(a)}{\leq} & I(X_1^n;Y_1^n,S_1^n,X_2^n) + I(X_2^n;Y_2^n) + n\epsilon_n, \\
= & I(X_1^n;X_2^n) + I(X_1^n;S_1^n|X_2^n) + I(X_1^n;Y_1^n|S_1^n,X_2^n) + I(X_2^n;Y_2^n) + n\epsilon_n.
\end{aligned}
$$

Again $I(X_1^n;X_2^n) = 0$, so that

$$
\begin{aligned}
n(R_1+R_2) \leq & h(S_1^n|X_2^n) - h(Z_2^n) + h(Y_1^n|S_1^n,X_2^n) - h(Z_1^n) + h(Y_2^n) - h(Y_2^n|X_2^n) + n\epsilon_n, \\
= & h(S_1^n) - h(Z_2^n) + h(Y_1^n|S_1^n,X_2^n) - h(Z_1^n) + h(Y_2^n) - h(S_1^n) + n\epsilon_n, \\
= & h(\sqrt{\rho_{11}}(I_n \otimes H_{11})X_1^n + Z_1^n|S_1^n) + h(Y_2^n) - n(N_1+N_2)\log(2\pi e) + n\epsilon_n, \\
\overset{(b)}{\leq} & n \log\det\left(I_{N_1} + \rho_{11}H_{11}\left(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12}\right)^{-1} H_{11}^\dagger\right) + n\epsilon_n + \\
& n\log\det\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger + \rho_{22}H_{22}H_{22}^\dagger\right),
\end{aligned}
$$

where (b) follows from Corollaries A.1 and A.2 in Appendix A.4. Now, dividing both sides by n and taking the limit as $n \to \infty$, we get the third bound since $\epsilon_n \to 0$ as $n \to \infty$.

(4) The fourth bound can be obtained similarly assuming the genie provides the side information (S_2^n, X_1^n) to Rx_2.

(5) To derive the fifth upper bound we assume that the genie gives side information S_i^n to Rx_i.

Using Fano's lemma we have

$$\begin{aligned}
n(R_1+R_2) \le& I(X_1^n;Y_1^n)+I(X_2^n;Y_2^n)+n\epsilon_n,\\
\le& I(X_1^n;Y_1^n,S_1^n)+I(X_2^n;Y_2^n,S_2^n)+n\epsilon_n,\\
=& I(X_1^n;S_1^n)+I(X_1^n;Y_1^n|S_1^n)+I(X_2^n;S_2^n)+I(X_2^n;Y_2^n|S_2^n)+n\epsilon_n,\\
=& h(S_1^n)+h(Y_1^n|S_1^n)-h(Y_1^n|S_1^n,X_1^n)+h(S_2^n)+h(Y_2^n|S_2^n)\\
&\quad -h(Z_2^n)-h(Z_1^n)-h(Y_2^n|S_2^n,X_2^n)+n\epsilon_n,\\
=& h(S_1^n)+h(Y_1^n|S_1^n)-h(S_2^n|S_1^n)+h(S_2^n)+h(Y_2^n|S_2^n)-h(S_1^n|S_2^n)\\
&\quad -h(Z_2^n)-h(Z_1^n)+n\epsilon_n,\\
=& h(Y_1^n|S_1^n)+h(Y_2^n|S_2^n)-n(N_1+N_2)\log(2\pi e)\\
&\quad +n\epsilon_n,\ [\because S_i^n \text{is independent of } S_j^n,\ \text{for } i\neq j]\\
\overset{(d)}{\le}& n\log\det\left(I_{N_1}+\rho_{21}H_{21}H_{21}^\dagger+\rho_{11}H_{11}\left(I_{M_1}+\rho_{12}H_{12}^\dagger H_{12}\right)^{-1}H_{11}^\dagger\right)+\\
&\quad n\log\det\left(I_{N_2}+\rho_{12}H_{12}H_{12}^\dagger+\rho_{22}H_{22}\left(I_{M_2}+\rho_{21}H_{21}^\dagger H_{21}\right)^{-1}H_{22}^\dagger\right)+n\epsilon_n
\end{aligned}$$

where in step (d) we used Lemma A.1 of Appendix A.4, twice.

(6) Again from Fano's lemma we have

$$\begin{aligned}
n(2R_1+R_2) \le& I(X_1^n;Y_1^n)+I(X_1^n;Y_1^n,)+I(X_2^n;Y_2^n)+n\epsilon_n,\\
\le& I(X_1^n;Y_1^n)+I(X_1^n;Y_1^n,S_1^n,X_2^n)+I(X_2^n;Y_2^n,S_2^n)+n\epsilon_n,
\end{aligned}$$

where the last step again follows from the fact that extra information does not reduce mutual information. Next, using the chain rule of mutual information we get

$$
\begin{aligned}
n(2R_1+R_2) \leq & I(X_1^n;Y_1^n)+I(X_1^n;S_1^n|X_2^n)+I(X_1^n;Y_1^n|S_1^n,X_2^n)+I(X_2^n;S_2^n)+I(X_2^n;Y_2^n|S_2^n)+n\epsilon_n, \\
= & h(Y_1^n)-h(Y_1^n|X_1^n)+h(S_1^n|X_2^n)-h(S_1^n|X_2^n,X_1^n)+h(Y_1^n|S_1^n,X_2^n)- \\
& \quad h(Y_1^n|S_1^n,X_2^n,X_1^n)+h(S_2^n)-h(S_2^n|X_2^n)+h(Y_2^n|S_2^n)-h(Y_2^n|S_2^n,X_2^n)+n\epsilon_n, \\
= & h(Y_1^n)-h(S_2^n)+h(S_1^n)-h(Z_2^n)+h((I_n\otimes H_{11})X_1^n+Z_1^n|S_1^n)-h(Z_1^n)+ \\
& \quad h(S_2^n)-h(Z_1^n)+h(Y_2^n|S_2^n)-h(S_1^n|S_2^n)+n\epsilon_n, \\
= & h(Y_1^n)+h(\sqrt{\rho_{11}}(I_n\otimes H_{11})X_1^n+Z_1^n|S_1^n)+h(Y_2^n|S_2^n)-n(2N_1+N_2)\log(2\pi e)+n\epsilon_n, \\
\overset{(e)}{\leq} & n\log\det\left(I_{N_1}+\rho_{21}H_{21}H_{21}^{\dagger}+\rho_{11}H_{11}H_{11}^{\dagger}\right)+ \\
& \quad n\log\det\left(I_{N_1}+\rho_{11}H_{11}\left(I_{M_1}+\rho_{12}H_{12}^{\dagger}H_{12}\right)^{-1}H_{11}^{\dagger}\right)+ \\
& \quad n\log\det\left(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger}+\rho_{22}H_{22}\left(I_{M_2}+\rho_{21}H_{21}^{\dagger}H_{21}\right)^{-1}H_{22}^{\dagger}\right)+n\epsilon_n,
\end{aligned}
$$

where in step (e) we used Corollary A.2, Corollary A.1 and Lemma A.1 of Appendix A.4. Finally, dividing both sides by n and taking the limit with respect to n, we get equation (3.12).

(7) The 7^{th} bound can be similarly derived as the last one.

A.4 Proof of Lemma A.1

From equation (5.2) we know that $S_{it}=\sqrt{\rho_{ij}}H_{ij}X_{it}+Z_j$ represents the interference from user i plus the additive noise at receiver j at time t and $S_i^n=[S_{i1}^{\dagger}\ S_{i2}^{\dagger}\ \cdots S_{in}^{\dagger}]^{\dagger}$ can be written as

$$
S_i^n=\sqrt{\rho_{ij}}\begin{bmatrix} H_{ij} & 0 & \cdots & 0 \\ 0 & H_{ij} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_{ij} \end{bmatrix}\begin{bmatrix} X_{i1} \\ X_{i2} \\ \vdots \\ X_{in} \end{bmatrix}+\begin{bmatrix} Z_{i1} \\ Z_{i2} \\ \vdots \\ Z_{in} \end{bmatrix}=\sqrt{\rho_{ij}}\left(I_n\otimes H_{ij}\right)X_i^n+Z_i^n.
$$

Similarly, the output at Rx_i over n channel uses can be written as

$$
Y_i^n=\sqrt{\rho_{ii}}(I_n\otimes H_{ii})X_i^n+\sqrt{\rho_{ji}}(I_n\otimes H_{ji})X_j^n+Z_i^n,\ \text{for } i\neq j\in\{1,2\},
$$

where $X_{it} \in \mathbb{C}^{M_i \times 1}, \ \forall\, t \leq n$ satisfies the power constraints of equation (5.1).

Lemma A.1 For the n-length vector sequences Y_i^n and S_i^n as described above

$$h(Y_i^n|S_i^n) \leq n \log\det\left(I_{N_i} + \rho_{ji}H_{ji}H_{ji}^{\dagger} + \rho_{ii}H_{ii}\left(I_{M_i} + \rho_{ij}H_{ij}^{\dagger}H_{ij}\right)^{-1}H_{ii}^{\dagger}\right) + nN_i\log(2\pi e),$$

for $i \neq j \in \{1,2\}$.

Proof A.1 We shall prove the Lemma for $i = 1$ with $i = 2$ case being identical. Denoting the covariance matrix of a zero-mean random vector V by $Cov(V)$, i.e., $Cov(V) = \mathbb{E}(VV^{\dagger})$ and the composite vector V_t by

$$V_t = \begin{bmatrix} S_{1t} \\ Y_{1t} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_{12}}H_{12}X_{1t} + Z_{2t} \\ \sqrt{\rho_{11}}H_{11}X_{1t} + \sqrt{\rho_{21}}H_{21}X_{2t} + Z_{1t} \end{bmatrix}, \ \forall\, t \leq n, \tag{A.15}$$

it can be easily verified that

$$K_{J_t} \triangleq \mathbb{E}\left(V_tV_t^{\dagger}\right) = \begin{bmatrix} \rho_{12}H_{12}Q_{1t}H_{12}^{\dagger} + I_{N_2} & \sqrt{\rho_{12}}\sqrt{\rho_{11}}H_{12}Q_{1t}H_{11}^{\dagger} \\ \sqrt{\rho_{11}}\sqrt{\rho_{12}}H_{11}Q_{1t}H_{12}^{\dagger} & \rho_{11}H_{11}Q_{1t}H_{11}^{\dagger} + \rho_{21}H_{21}Q_{2t}H_{21}^{\dagger} + I_{N_1} \end{bmatrix}, \ \forall\, t \leq n. \tag{A.16}$$

Note that in the above computation we assumed that input distribution has zero mean, which is standard since a non-zero mean only contributes to power inefficiency. Let us define

$$\begin{aligned} \widehat{S}_1^* &= \sqrt{\rho_{12}}H_{12}X_1^G + Z_2, \\ \widehat{Y}_1^* &= \sqrt{\rho_{11}}H_{11}X_1^G + \sqrt{\rho_{21}}H_{21}X_2^G + Z_1, \end{aligned}$$

where $X_i^G \sim \mathcal{CN}(0, \frac{1}{n}\sum_{i=1}^{n} Q_{it})$ for $1 \leq i \neq j \leq 2$ and X_1^G and X_2^G are mutually independent. It can be easily verified that $\widehat{S}_1^*$ and $\begin{bmatrix} \widehat{S}_1^* \\ \widehat{Y}_1^* \end{bmatrix}$ are Gaussian vectors with covariance matrices $\bar{K}$ and $\bar{K}_J$, respectively, where $\bar{K} = \rho_{12}H_{12}\bar{Q}_1H_{12}^{\dagger} + I_{N_2}$, $\bar{Q}_i = \frac{1}{n}\sum_{t=1}^{n} Q_{it}$ and

$$\bar{K}_J = \begin{bmatrix} \rho_{12}H_{12}\bar{Q}_1H_{12}^{\dagger} + I_{N_2} & \sqrt{\rho_{12}}\sqrt{\rho_{11}}H_{12}\bar{Q}_1H_{11}^{\dagger} \\ \sqrt{\rho_{11}}\sqrt{\rho_{12}}H_{11}\bar{Q}_1H_{12}^{\dagger} & \rho_{11}H_{11}\bar{Q}_1H_{11}^{\dagger} + \rho_{21}H_{21}\bar{Q}_2H_{21}^{\dagger} + I_{N_1} \end{bmatrix} \tag{A.17}$$

which follows from the fact that any linear transformation of a Gaussian vector is also Gaussian (Proposition 5.2, [89]) and the sum of several mutually independent Gaussian vectors is also Gaussian. In other words,

$$Cov\begin{bmatrix}\widehat{S}_1^* \\ \widehat{Y}_1^*\end{bmatrix} = \frac{1}{n}\sum_{t=1}^{n} Cov\begin{bmatrix}S_{1t} \\ Y_{1t}\end{bmatrix}.$$

Under the constraints satisfied on $\widehat{S}_1^*$, $\widehat{Y}_1^*$ and $V_t,\ 1 \le t \le n$, it was proved in Lemma 2 of [23] that

$$\begin{aligned}
h(Y_1^n|S_1^n) \le & n h(\widehat{Y}_1^*|\widehat{S}_1^*) \\
= & n\left(h\left(\begin{bmatrix}\widehat{S}_1^* \\ \widehat{Y}_1^*\end{bmatrix}\right) - h(\widehat{S}_1^*)\right), \\
= & n\left(\log\det(\bar{K}_J) - \log\det(\bar{K})\right).
\end{aligned}$$

Putting the values of $\bar{K}_J$ and $\bar{K}$ in the above equation and after some simplification, we get

$$\begin{aligned}
h(Y_1^n|S_1^n) \le & n \log\det\Big(I_{N_1} + \rho_{11}H_{11}\bar{Q}_1H_{11}^\dagger + \rho_{21}H_{21}\bar{Q}_2H_{21}^\dagger - \\
& \rho_{11}\rho_{12}H_{11}\bar{Q}_1H_{12}^\dagger\left(I_{N_2} + \rho_{12}H_{12}\bar{Q}_1H_{12}^\dagger\right)^{-1}H_{12}\bar{Q}_1H_{11}^\dagger\Big) + nN_1\log(2\pi e), \\
\overset{(c)}{=} & n\log\det\left(I_{N_1} + \rho_{21}H_{21}\bar{Q}_2H_{21}^\dagger + \rho_{11}H_{11}\bar{Q}_1^{\frac{1}{2}}\left(I_{M_2} + \rho_{12}\bar{Q}_1^{\frac{1}{2}}H_{12}^\dagger H_{12}\bar{Q}_1^{\frac{1}{2}}\right)^{-1}\bar{Q}_1^{\frac{1}{2}}H_{11}^\dagger\right) \\
& + nN_1\log(2\pi e), \\
\overset{(d)}{\le} & n\log\det\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger + \rho_{11}H_{11}\left(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12}\right)^{-1}H_{11}^\dagger\right) \\
& + nN_1\log(2\pi e),\ [\because \bar{Q}_i \preceq I_{M_i}]
\end{aligned}$$

where step (c) follows from the Woodbury identity and the last step follows from the fact that $\log\det(.)$ is a monotonically increasing function on the cone of p.d. matrices and Lemma A.2 (below) with $G_1 = \bar{Q}_1^{\frac{1}{2}}$, $A = H_{12}^\dagger H_{12}$ and $G_2 = I_{M_1}$.

Lemma A.2 Let $0 \preceq G_1 \preceq G_2$ and $0 \preceq A$ are p.s.d. matrices of size n, then for any given $\pi \in \mathbb{R}^+$

$$G_1\left(I + \pi G_1AG_1\right)^{-1}G_1 \preceq G_2\left(I + \pi G_2AG_2\right)^{-1}G_2.$$

Proof A.2 Let $\epsilon \in \mathbb{R}^+$, $G_{1\epsilon} = (G_1 + \epsilon I)$ and $G_{2\epsilon} = (G_2 + \epsilon I)$. For any such ϵ, we have

$$
\begin{aligned}
& G_{2\epsilon} \succeq G_{1\epsilon} \succ 0, \quad \text{or} \\
& G_{1\epsilon}^{-2} \succeq G_{2\epsilon}^{-2} \succ 0, \quad \text{or} \\
& \left(G_{1\epsilon}^{-2} + \pi A\right) \succeq \left(G_{2\epsilon}^{-2} + \pi A\right) \succ 0, \quad \text{or} \\
& \left(G_{1\epsilon}^{-2} + \pi A\right)^{-1} \preceq \left(G_{2\epsilon}^{-2} + \pi A\right)^{-1}, \quad \text{or} \\
& G_{1\epsilon} \left(I + \pi G_{1\epsilon} A G_{1\epsilon}\right)^{-1} G_{1\epsilon} \preceq G_{2\epsilon} \left(I + \pi G_{2\epsilon} A G_{2\epsilon}\right)^{-1} G_{2\epsilon}.
\end{aligned}
$$

From the definition of partial order between p.s.d. matrices we get

$$
\begin{aligned}
& x \left(G_{1\epsilon} \left(I + \pi G_{1\epsilon} A G_{1\epsilon}\right)^{-1} G_{1\epsilon}\right) x^\dagger \leq \; x \left(G_{2\epsilon} \left(I + \pi G_{2\epsilon} A G_{2\epsilon}\right)^{-1} G_{2\epsilon}\right) x^\dagger, \; \forall \, x \in \mathbb{C}^{1\times n}; \\
& \Rightarrow \lim_{\epsilon \to 0} x \left(G_{1\epsilon} \left(I + \pi G_{1\epsilon} A G_{1\epsilon}\right)^{-1} G_{1\epsilon}\right) x^\dagger \leq \lim_{\epsilon \to 0} \; x \left(G_{2\epsilon} \left(I + \pi G_{2\epsilon} A G_{2\epsilon}\right)^{-1} G_{2\epsilon}\right) x^\dagger, \; \forall \, x \in \mathbb{C}^{1\times n}; \\
& \Rightarrow \; x \left(G_1 \left(I + \pi G_1 A G_1\right)^{-1} G_1\right) x^\dagger \leq \; x \left(G_2 \left(I + \pi G_2 A G_2\right)^{-1} G_2\right) x^\dagger, \; \forall \, x \in \mathbb{C}^{1\times n}.
\end{aligned}
$$

Invoking the definition of partial ordering once again, the lemma is proved.

The following corollaries can be proved from Lemma A.1. In particular, Corollary A.1 by setting $H_{ji} = 0$, Corollary A.2 by setting $H_{ij} = 0$ and Corollary A.3 by putting $H_{ji} = 0$ and $H_{ij} = 0$.

Corollary A.1

$$
\begin{aligned}
h(\sqrt{\rho_{ii}}(I_n \otimes H_{ii})X_i^n + Z_i^n | S_i^n) \leq & n \log\det \left(I_{N_i} + \rho_{ii} H_{ii} \left(I_{M_i} + \rho_{ij} H_{ij}^\dagger H_{ij}\right)^{-1} H_{ii}^\dagger\right), \\
& + nN_i \log(2\pi e), \; i \neq j \in \{1, 2\}.
\end{aligned}
$$

Corollary A.2

$$
h(Y_i^n) \leq n \log\det \left(I_{N_i} + \rho_{ji} H_{ji} H_{ji}^\dagger + \rho_{ii} H_{ii} H_{ii}^\dagger\right) + nN_i \log(2\pi e), \; i \neq j \in \{1, 2\}.
$$

Corollary A.3

$$
h(\sqrt{\rho_{ii}}(I_n \otimes H_{ii})X_i^n + Z_i^n) \leq n \log\det \left(I_{N_i} + \rho_{ii} H_{ii} H_{ii}^\dagger\right) + nN_i \log(2\pi e), \; i \neq j \in \{1, 2\}.
$$

A.5 Proof of Lemma 3.4

As stated in the Outline, it is only required to show that the right hand sides of the different bounds in the Lemma are actually lower bounds to the corresponding terms in (3.25a)-(3.25i). However, first we shall derive some common inequalities which will be used throughout the proof. From the definition of p.s.d. matrices [90] we get

$$\left(\rho_{ij}H_{ij}K_{iu}H_{ij}^{\dagger}\right) = \left(\frac{\rho_{ij}}{M_i}H_{ij}\left(I_{M_i}+\rho_{ij}H_{ij}^{\dagger}H_{ij}\right)^{-1}H_{ij}^{\dagger}\right) \succeq \mathbf{0}, \quad \left[\because \ \left(I_{M_i}+\rho_{ij}H_{ij}^{\dagger}H_{ij}\right)^{-1} \succ \mathbf{0}\right] \tag{A.18}$$

On the other hand, for any given $\mathbf{0} \neq x \in \mathbb{C}^{1\times N_j}$ we have

$$\begin{aligned} x\left(\frac{\rho_{ij}}{M_i}H_{ij}\left(I_{M_i}+\rho_{ij}H_{ij}^{\dagger}H_{ij}\right)^{-1}H_{ij}^{\dagger}\right)x^{\dagger} =& \frac{\rho_{ij}}{M_i}\left(xU_{ij}\right)\Sigma_{ij}\left(I_{M_i}+\rho_{ij}\Sigma_{ij}^{\dagger}\Sigma_{ij}\right)^{-1}\Sigma_{ij}^{\dagger}\left(xU_{ij}\right)^{\dagger}, \\ \overset{(a)}{\leq}& \frac{1}{M_i}(xU_{ij})(xU_{ij})^{\dagger} \leq \frac{xx^{\dagger}}{M_i}, \end{aligned}$$

where $\Sigma_{ij} \in \mathbb{C}^{N_j\times M_i}$ is the singular value matrix of H_{ij}, i.e., $H_{ij} = U_{ij}\Sigma_{ij}V_{ij}^{\dagger}$ and step (a) follows from the fact that $\rho_{ij}\Sigma_{ij}\left(I_{M_i}+\rho_{ij}\Sigma_{ij}^{\dagger}\Sigma_{ij}\right)^{-1}\Sigma_{ij}^{\dagger} \preceq I_{N_j}$. However, the last inequality along with the definition of p.s.d. matrices imply

$$\rho_{ij}H_{ij}K_{ij}H_{ij}^{\dagger} = \left(\frac{\rho_{ij}}{M_i}H_{ij}\left(I_{M_i}+\rho_{ij}H_{ij}^{\dagger}H_{ij}\right)^{-1}H_{ij}^{\dagger}\right) \preceq \frac{1}{M_i}I_{N_j}. \tag{A.19}$$

Each of the eigenvalues of the matrix on the left hand side of equation (A.19) is smaller than or equal to 1. However, it has only $\min\{M_i, N_j\} = m_{ij}$ non-zero eigenvalues since the rank of H_{ij} is m_{ij}, which in turn implies that

$$\log\det\left(I_{N_j}+\rho_{ij}H_{ij}K_{iu}H_{ij}^{\dagger}\right) \leq m_{ij}\log\left(\frac{(1+M_i)}{M_i}\right) = \hat{m}_{ij}, \tag{A.20}$$

for all $1 \leq i \neq j \leq 2$.

As a first step towards deriving the lower bounds, in what follows, we shall first derive lower bounds for the different mutual information terms of equation (3.27)-(3.33). From equation (3.27) we obtain

$$\begin{aligned}
I(X_1^g;Y_1^g|W_1^g,W_2^g) =& \log\det\left(\frac{\rho_{11}}{M_1}H_{11}K_1H_{11}^\dagger + \frac{\rho_{21}}{M_2}H_{21}K_2H_{21}^\dagger + I_{N_1}\right) \\
&- \log\det\left(\rho_{21}H_{21}K_{2u}H_{21}^\dagger + I_{N_1}\right), \\
\overset{(a)}{\geq}& \log\det\left(\frac{\rho_{11}}{M_1}H_{11}K_1H_{11}^\dagger + \frac{1}{M_1}I_{N_1}\right) - \hat{m}_{21}, \\
=& \log\det\left(\rho_{11}H_{11}K_1H_{11}^\dagger + I_{N_1}\right) - (m_{11}\log(M_1) + \hat{m}_{21}), \qquad \text{(A.21)}
\end{aligned}$$

where step (a) follows from the fact that $\log\det(.)$ is a monotonically increasing function over the cone of positive-definite matrices with respect to the partial ordering and equations (A.18) and (A.20). The last equality follows from the fact that H_{11} is a rank m_{11} matrix. Similarly,

$$I(X_2^g;Y_2^g|W_2^g,W_1^g) \geq \log\det\left(\rho_{22}H_{22}K_2H_{22}^\dagger + I_{N_2}\right) - (m_{22}\log(M_2) + \hat{m}_{12}). \qquad \text{(A.22)}$$

From equation (3.29) we obtain

$$\begin{aligned}
I(W_2^g;Y_1^g|X_1^g) =& \log\det\left(\frac{\rho_{21}}{M_2}H_{21}H_{21}^\dagger + I_{N_1}\right) - \log\det\left(\rho_{21}H_{21}K_{2u}H_{21}^\dagger + I_{N_1}\right), \\
\geq& \log\det\left(\frac{\rho_{21}}{M_2}H_{21}H_{21}^\dagger + I_{N_1}\right) - \hat{m}_{21}, \ [\because \ \text{(A.20)}] \\
\geq& \log\det\left(\frac{\rho_{21}}{M_2}H_{21}H_{21}^\dagger + \frac{1}{M_2}I_{N_1}\right) - \hat{m}_{21}, \\
=& \log\det\left(\rho_{21}H_{21}H_{21}^\dagger + I_{N_1}\right) - (m_{21}\log(M_2) + \hat{m}_{21}), \qquad \text{(A.23)}
\end{aligned}$$

where the last step follows from the fact that the rank of H_{21} is m_{21}. Similarly, we have

$$I(W_1^g;Y_2^g|X_2^g) \geq \log\det\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger\right) - (m_{12}\log(M_1) + \hat{m}_{12}). \qquad \text{(A.24)}$$

From equation (3.30)

$$\begin{aligned}
I(X_1^g;Y_1^g|W_2^g) =& \log\det\left(\frac{\rho_{11}}{M_1}H_{11}H_{11}^\dagger + \rho_{21}H_{21}K_{2u}H_{21}^\dagger + I_{N_1}\right) - \log\det\left(\rho_{21}H_{21}K_{2u}H_{21}^\dagger + I_{N_1}\right), \\
\overset{(b)}{\geq}& \log\det\left(\frac{\rho_{11}}{M_1}H_{11}H_{11}^\dagger + I_{N_1}\right) - \hat{m}_{21} \geq \log\det\left(\frac{\rho_{11}}{M_1}H_{11}H_{11}^\dagger + \frac{1}{M_1}I_{N_1}\right) - \hat{m}_{21}, \\
=& \log\det\left(\rho_{11}H_{11}H_{11}^\dagger + I_{N_1}\right) - (m_{11}\log(M_1) + \hat{m}_{21}), \qquad \text{(A.25)}
\end{aligned}$$

where steps (b) follows from equations (A.18) and (A.20) and the fact that $\log\det(.)$ is a monotonically increasing function over the cone of positive-definite matrices. The last equality follows from the fact that H_{11} is a rank m_{11} matrix. Similarly, we have

$$I(X_2^g;Y_2^g|W_1^g) \;\geq\; \log\det\left(\rho_{22}H_{22}H_{22}^{\dagger}+I_{N_2}\right)-(m_{22}\log(M_2)+\hat{m}_{12}). \tag{A.26}$$

Next, we focus on the term $I(X_2^g, W_1^g; Y_2^g)$.

$$\begin{aligned}
I(X_2^g,W_1^g;Y_2^g) =&\log\det\left(\frac{\rho_{12}}{M_1}H_{12}H_{12}^{\dagger}+\frac{\rho_{22}}{M_2}H_{22}H_{22}^{\dagger}+I_{N_2}\right)-\log\det\left(\rho_{12}H_{12}K_{1u}H_{12}^{\dagger}+I_{N_2}\right),\\
\geq&\log\det\left(\frac{\rho_{12}}{M_x}H_{12}H_{12}^{\dagger}+\frac{\rho_{22}}{M_x}H_{22}H_{22}^{\dagger}+\frac{1}{M_x}I_{N_2}\right)-\hat{m}_{12},\ [\because\ (\text{A.20})]\\
=&\log\det\left(\rho_{12}H_{12}H_{12}^{\dagger}+\rho_{22}H_{22}H_{22}^{\dagger}+I_{N_2}\right)-(\min\{N_2,M_s\}\log(M_x)+\hat{m}_{12}),
\end{aligned}$$

where the last step follows from the fact that the matrix $\left(\frac{\rho_{12}}{M_x}H_{12}H_{12}^{\dagger}+\frac{\rho_{22}}{M_x}H_{22}H_{22}^{\dagger}\right)$ has rank $\min\{N_2, M_s\}$, M_s and M_x are as defined in Section 3.3. Similarly, we have

$$I(X_1^g,W_2^g;Y_1^g) \geq\log\det\left(\rho_{21}H_{21}H_{21}^{\dagger}+\rho_{11}H_{11}H_{11}^{\dagger}+I_{N_1}\right)-(\min\{N_1,M_s\}\log(M_x)+\hat{m}_{21}).$$

From equation (3.31)

$$\begin{aligned}
I(X_1^g,W_2^g;Y_1^g|W_1^g) =&\log\det\left(\rho_{11}H_{11}K_{1u}H_{11}^{\dagger}+\frac{\rho_{21}}{M_2}H_{21}H_{21}^{\dagger}+I_{N_1}\right)\\
&-\log\det\left(\rho_{21}H_{21}K_{2u}H_{21}^{\dagger}+I_{N_1}\right),\\
\geq&\log\det\left(\frac{\rho_{11}}{M_1}H_{11}K_1H_{11}^{\dagger}+\frac{\rho_{21}}{M_2}H_{21}H_{21}^{\dagger}+I_{N_1}\right)-\hat{m}_{21},\ [\because\ (\text{A.20})]\\
\geq&\log\det\left(\frac{\rho_{11}}{M_x}H_{11}K_1H_{11}^{\dagger}+\frac{\rho_{21}}{M_x}H_{21}H_{21}^{\dagger}+\frac{1}{M_x}I_{N_1}\right)-\hat{m}_{21},\\
\overset{(a)}{=}&\log\det\left(\rho_{11}H_{11}K_1H_{11}^{\dagger}+\rho_{21}H_{21}H_{21}^{\dagger}+I_{N_1}\right)\\
&-(\min\{N_1,M_s\}\log(M_x)+\hat{m}_{21})\triangleq I_{e_1}^L,
\end{aligned}$$

where step (a) follows from the fact that the sum of the first two matrices inside $\log\det(.)$ has a rank of $\min\{N_1, M_s\}$ and similarly,

$$\begin{aligned}
I(X_2^g,W_1^g;Y_2^g|W_2^g) \geq&\log\det\left(I_{N_2}+\rho_{12}H_{12}H_{12}^{\dagger}+\rho_{22}H_{22}K_2H_{22}^{\dagger}\right)\\
&-(\min\{N_2,M_s\}\log(M_x)+\hat{m}_{12}).
\end{aligned}$$

Note that each of the lower bounds are difference of a channel dependent term and a constant. To denote the constants by a common notation we define $n_i^* = \min\{N_i, M_s\}\log(M_x) + \hat{m}_{ji}$, for

$i \neq j \in \{1, 2\}$. Using this notations and equations (A.21)-(A.27) we get the following bounds:

$$R_1 \leq \log\det\left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger\right) - n_1^*; \tag{A.27a}$$

$$R_1 \leq \log\det\left(I_{M_1} + \rho_{11} H_{11}^\dagger H_{11} + K_1^{-1}\right) - (m_{11}\log(M_1) + m_{12}\log(M_1+1)) - \hat{m}_{21}; \tag{A.27b}$$

$$R_2 \leq \log\det\left(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger\right) - n_2^*; \tag{A.27c}$$

$$R_2 \leq \log\det\left(I_{M_2} + \rho_{22} H_{22}^\dagger H_{22} + K_2^{-1}\right) - (m_{22}\log(M_2) + m_{21}\log(M_2+1)) - \hat{m}_{12}; \tag{A.27d}$$

$$\begin{aligned} R_1 + R_2 \leq\ & \log\det\left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger\right) \\ & + \log\det\left(I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^\dagger\right) - (n_1^* + n_2^*); \end{aligned} \tag{A.27e}$$

$$\begin{aligned} R_1 + R_2 \leq\ & \log\det\left(I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} H_{11}^\dagger\right) \\ & + \log\det\left(I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^\dagger\right) - (n_1^* + n_2^*); \end{aligned} \tag{A.27f}$$

$$\begin{aligned} R_1 + R_2 \leq\ & \log\det\left(I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_1 H_{11}^\dagger\right) \\ & + \log\det\left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} K_2 H_{22}^\dagger\right) - (n_1^* + n_2^*); \end{aligned} \tag{A.27g}$$

$$\begin{aligned} 2R_1 + R_2 \leq\ & \log\det\left(I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} H_{11}^\dagger\right) + \log\det\left(I_{N_1} + \rho_{11} H_{11} K_1 H_{11}^\dagger\right) \\ & + \log\det\left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} K_2 H_{22}^\dagger\right) - (2n_1^* + n_2^*); \end{aligned} \tag{A.27h}$$

$$\begin{aligned} R_1 + 2R_2 \leq\ & \log\det\left(I_{N_2} + \rho_{12} H_{12} H_{12}^\dagger + \rho_{22} H_{22} H_{22}^\dagger\right) + \log\det\left(I_{N_2} + \rho_{22} H_{22} K_2 H_{22}^\dagger\right) \\ & + \log\det\left(I_{N_1} + \rho_{21} H_{21} H_{21}^\dagger + \rho_{11} H_{11} K_1 H_{11}^\dagger\right) - (n_1^* + 2n_2^*), \end{aligned} \tag{A.27i}$$

Clearly, the right hand sides of each of the bounds in equation (A.27) is smaller than those in the corresponding bound in equation (3.25). However, in equation (A.27) there are two bounds on each R_i. To combine them into one we define

$$n_i = \max\{(m_{ii}\log(M_i) + m_{ij}\log(M_i+1)) + \hat{m}_{ji}, n_i^*\},\ \forall\ i \neq j \in \{1, 2\}.$$

Using this notation the first four bounds on R_1 and R_2 in equation (A.27) can be written as

$$R_1 \leq \log\det\left(I_{N_1} + \rho_{11} H_{11} H_{11}^\dagger\right) - n_1;$$

$$R_2 \leq \log\det\left(I_{N_2} + \rho_{22} H_{22} H_{22}^\dagger\right) - n_2,$$

where we have used the fact that K_i^{-1} is a p.d. matrix and $\log\det(.)$ is a monotonically increasing function in the cone of p.s.d. matrices. The remaining bounds in Lemma 3.4 follows from equation (A.27) and the fact that $n_i \geq n_i^*$, for all $i \in \{1,2\}$.

A.6 Proof of Lemma 3.5

As stated in the outline, first we show that when $(R_1, R_2) \in \mathcal{R}_2$, equations (3.25b) and (3.25d) both can not be violated simultaneously. Suppose, it is not true, i.e., $(R_1, R_2) \in \mathcal{R}_2$ but

$$\begin{aligned} R_1 >& I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g); \\ R_2 >& I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | X_1^g). \end{aligned}$$

Adding the above two equations we get

$$\begin{aligned} R_1 + R_2 >& I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g) + I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | X_1^g), \\ \geq& I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | W_2^g) + I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | W_1^g), \\ =& I(X_1^g, W_2^g; Y_1^g | W_1^g) + I(X_2^g, W_1^g; Y_2^g | W_2^g), \end{aligned}$$

which violates equation (3.49e) and contradicts the assumption that $(R_1, R_2) \in \mathcal{R}_2$. Therefore, whenever $(R_1, R_2) \in \mathcal{R}_2$ but $(R_1, R_2) \notin \mathcal{R}_{\text{HK}}^{G_e}(P_s)$ only one among (3.25b) and (3.25d) is violated and not both.

Next we shall show that such a rate tuple is always achievable by the explicit HK coding scheme. Suppose $(R_1, R_2) \in \mathcal{R}_2$ but $(R_1, R_2) \notin \mathcal{R}_{\text{HK}}^{G_e}(P_s)$ because it violates equation (3.25b), i.e.,

$$R_1 > I(X_1^g; Y_1^g | W_1^g, W_2^g) + I(W_1^g; Y_2^g | X_2^g). \tag{A.28}$$

From definition 3.3 we know that in this case, Tx_1 transmit only private message, i.e., the coding scheme $\mathcal{HK}(\{\frac{1}{M_1} I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\})$ is used. To distinguish the codewords of the first user in this case to those used in the simple HK scheme we use the following notations:

$$\tilde{X}_1^g = \tilde{U}_1^g \sim \mathcal{CN}(\mathbf{0}, \frac{1}{M_1} I_{M_1}), \tilde{W}_1^g = \phi,\ \tilde{W}_2^g = W_2^g \text{ and } \tilde{U}_2^g = U_2^g, \tag{A.29}$$

are mutually independent and $\tilde{X}_2^g = \tilde{W}_2^g + \tilde{U}_2^g$, where $\tilde{Y}_i^g$'s are the outputs of the channel (e.g., see equations (5.2)) when $\tilde{X}_i^g$'s are the inputs. Evidently, if the joint distribution of these random vectors be denoted by $P_{s_1}(\tilde{X}_1^g, \tilde{U}_1^g, \tilde{W}_1^g, \tilde{X}_2^g, \tilde{U}_2^g, \tilde{W}_2^g)$, then $P_{s_1}(.) \in \mathcal{P}^*$. Putting $P^* = P_{s_1}$ in the expression for $\mathcal{R}_{\text{HK}}^e(P^*)$ we get the achievable region of the coding scheme $\mathcal{HK}(\{\frac{1}{M_1}I_{M_1}, \mathbf{0}, K_{2u}, K_{2w}\})$ as

$$\begin{aligned}
\mathcal{R}_{\text{HK}}^{G_e}(P_{s_1}) = \Big\{(R_1, R_2):\ \ R_1 \leq& I(\tilde{X}_1^g; \tilde{Y}_1^g|\tilde{W}_2^g);\\
R_2 \leq& I(\tilde{X}_2^g; \tilde{Y}_2^g);\\
R_2 \leq& I(\tilde{W}_2^g; \tilde{Y}_1^g|\tilde{X}_1^g) + I(\tilde{X}_2^g; \tilde{Y}_2^g|\tilde{W}_2^g);\\
R_1 + R_2 \leq& I(\tilde{X}_1^g, \tilde{W}_2^g; \tilde{Y}_1^g) + I(\tilde{X}_2^g; \tilde{Y}_2^g|\tilde{W}_2^g)\Big\}
\end{aligned}$$

which can easily be shown to be equivalent to

$$\begin{aligned}
\mathcal{R}_{\text{HK}}^{G_e}(P_{s_1}) = \Big\{(R_1, R_2):\ \ R_1 \leq& I(X_1^g; Y_1^g|W_2^g); && \text{(A.30)}\\
R_2 \leq& I(X_2^g; Y_2^g); && \text{(A.31)}\\
R_2 \leq& I(W_2^g; Y_1^g|X_1^g) + I(X_2^g; Y_2^g|W_2^g); && \text{(A.32)}\\
R_1 + R_2 \leq& I(X_1^g, W_2^g; Y_1^g) + I(X_2^g; Y_2^g|W_2^g)\Big\} && \text{(A.33)}
\end{aligned}$$

using the relations between $\tilde{V}_i^g$'s and V_i^g's defined earlier, where $V \in \{U, W, X, Y\}$. In what follows, we shall shown that $(R_1, R_2) \in \mathcal{R}_{\text{HK}}^{G_e}(P_{s_1})$ and hence achievable by $\widetilde{\mathcal{HK}}$.

From (3.49a) we know that

$$R_1 \leq I(X_1^g; Y_1^g|W_2^g). \tag{A.34}$$

From (A.28) and (3.49d) we obtain

$$R_2 \leq I(X_2^g; Y_2^g). \tag{A.35}$$

From (A.28) and (3.49e) we obtain

$$\begin{aligned}
R_2 <& I(W_2^g; Y_1^g|W_1^g) + I(X_2^g; Y_2^g|W_2^g); && \text{(A.36)}\\
\leq& I(W_2^g; Y_1^g|X_1^g) + I(X_2^g; Y_2^g|W_2^g); && \text{(A.37)}
\end{aligned}$$

and from (A.28) and (3.49f)

$$R_1 + R_2 \leq I(X_1^g, W_2^g; Y_1^g) + I(X_2^g; Y_2^g | W_2^g). \tag{A.38}$$

This proves that $(R_1, R_2) \in \mathcal{R}_{\text{HK}}^{G_e}(P_{s_1})$ and hence achievable by $\widetilde{\mathcal{HK}}$.

The other case when $(R_1, R_2) \in \mathcal{R}_2$ but equation (3.25d) is violated, i.e.,

$$R_2 > I(X_2^g; Y_2^g | W_1^g, W_2^g) + I(W_2^g; Y_1^g | X_1^g), \tag{A.39}$$

can be similarly proved. Again from definition 3.3 we know that in this case the coding scheme $\mathcal{HK}$ $(\{K_{1u}, K_{1w}, \frac{1}{M_2} I_{M_2}, \mathbf{0}\})$ is used. Defining

$$\hat{X}_2^g = \hat{U}_2^g \sim \mathcal{CN}(\mathbf{0}, \frac{1}{M_2} I_{M_2}), \hat{W}_2^g = \phi,\ \hat{W}_1^g = W_1^g,\ \hat{U}_1^g = U_1^g, \tag{A.40}$$

and $\hat{X}_1^g = \hat{W}_1^g + \hat{U}_1^g$, where $\hat{Y}_i^g$'s are the outputs of the channel (e.g., see equations (5.2)) when $\hat{X}_i^g$'s are the inputs, it is clear that the joint distributions of these variables, $P_{s_2}(\hat{X}_1^g, \hat{U}_1^g, \hat{W}_1^g, \hat{X}_2^g, \hat{U}_2^g, \hat{W}_2^g) \in \mathcal{P}^*$. Finally, putting $P^* = P_{s_2}$ in the expression for $\mathcal{R}_{\text{HK}}^{e}(P^*)$ the achievable region of the coding scheme $\mathcal{HK}(\{K_{1u}, K_{1w}, \frac{1}{M_2} I_{M_2}, \mathbf{0}\})$ can be computed. Now, combining equation (A.39) and (3.49), in the same way as the previous case, it can be shown that $(R_1, R_2) \in \mathcal{R}_{\text{HK}}^{G_e}(P_{s_2})$ and hence achievable by $\widetilde{\mathcal{HK}}$.

Finally, if $(R_1, R_2) \in \mathcal{R}_{\text{HK}}^{G_e}(P_s)$, then by Lemma 3.3 we know that $\mathcal{HK}(\{K_{1u}, K_{1w}, K_{2u}, K_{2w}\})$ or the simple HK scheme can achieve the rate tuple.

A.7 Proof of Lemma 3.7

We shall prove this lemma in two steps. In step one, we shall prove

$$\mathcal{R}^u(\mathcal{H}, \bar{\rho}) = \mathcal{R}^u\left(\tilde{\mathcal{H}}, \bar{\rho}^r\right),$$

where $\tilde{\mathcal{H}} = \{H_{11}^\dagger, H_{21}^\dagger, H_{12}^\dagger, H_{22}^\dagger\}$ and in the second step we shall prove that

$$\mathcal{R}^u\left(\tilde{\mathcal{H}}, \bar{\rho}^r\right) = \mathcal{R}^u(\mathcal{H}^r, \bar{\rho}^r).$$

Clearly, the above two equalities prove the lemma.

Step1: Let us consider the interference channel $\mathcal{IC}\left(\tilde{\mathcal{H}}, \bar{\rho}^r\right)$. Following a similar method as in Lemma 3.1 we can derive an upper bound to the capacity region of this IC. Let the corresponding bounds of $\mathcal{R}^u\left(\tilde{\mathcal{H}}, \bar{\rho}^r\right)$ be denoted by I_k^r, $1 \leq k \leq 7$. In what follows, we shall first prove that $I_{b3} = I_4^r$, $I_{b4} = I_3^r$ and $I_{bk} = I_k^r$ for $k \in \{1, 2, 5, 6, 7\}$.

Towards proving the first equality, from equation (5.11) we get

$$\begin{aligned}
I_{b3} = &\log\det\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger + \rho_{22}H_{22}H_{22}^\dagger\right) + \log\det\left(I_{N_1} + \rho_{11}H_{11}\left(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12}\right)^{-1} H_{11}^\dagger\right), \\
= &\log\det\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger + \rho_{22}H_{22}H_{22}^\dagger\right) + \log\det\left(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12} + \rho_{11}H_{11}^\dagger H_{11}\right) \\
&- \log\det\left(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12}\right), \\
= &\log\det\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger + \rho_{22}H_{22}H_{22}^\dagger\right) + \log\det\left(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12} + \rho_{11}H_{11}^\dagger H_{11}\right) \\
&- \log\det\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger\right), \quad [\because \; \log\det(I + AB) = \log\det(I + BA)] \\
= &\log\det\left(I_{M_2} + \rho_{22}H_{22}^\dagger\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger\right)^{-1} H_{22}\right) + \log\det\left(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12} + \rho_{11}H_{11}^\dagger H_{11}\right), \\
= &I_4^r.
\end{aligned}$$

Similarly, it can be proved that $I_{b4} = I_3^r$. The equality of the first two bounds follow trivially from the identity $\log\det(I + AB) = \log\det(I + BA)$. Now, towards proving the fifth bound we see

$$\begin{aligned}
I_{b5}(1) &= \log\det\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger + \rho_{11}H_{11}K_1H_{11}^\dagger\right) \\
&= \log\det\left(I_{N_1} + \rho_{11}\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right)^{-1} H_{11}K_1H_{11}^\dagger\right) + \log\det\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right) \\
&= \log\det\left(I_{M_1} + \rho_{11}H_{11}^\dagger\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right)^{-1} H_{11}K_1\right) + \log\det\left(I_{M_2} + \rho_{21}H_{21}^\dagger H_{21}\right) \\
&= \log\det\left(K_1^{-1} + \rho_{11}H_{11}^\dagger\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right)^{-1} H_{11}\right) + \log\det(K_1) + \log\det\left(K_2^{-1}\right) \\
&= \log\det\left(I_{M_1} + \rho_{12}H_{12}^\dagger H_{12} + \rho_{11}H_{11}^\dagger\left(I_{N_1} + \rho_{21}H_{21}H_{21}^\dagger\right)^{-1} H_{11}\right) \\
&\qquad + \log\det(K_1) - \log\det\left(K_2\right).
\end{aligned}$$

Similarly, it can be easily proved that

$$\begin{aligned}
I_{b5}(2) &= \log\det\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger + \rho_{22}H_{22}K_2H_{22}^\dagger\right) \\
&= \log\det\left(I_{M_2} + \rho_{21}H_{21}^\dagger H_{21} + \rho_{22}H_{22}^\dagger\left(I_{N_2} + \rho_{12}H_{12}H_{12}^\dagger\right)^{-1} H_{22}\right) \\
&\qquad + \log\det(K_2) - \log\det\left(K_1\right).
\end{aligned}$$

Combining the last two equations we get

$$\begin{aligned}
I_{b5} =& I_{b5}(1) + I_{b5}(2), \\
=& \log\det\left(I_{M_1} + \rho_{12}H_{12}^{\dagger}H_{12} + \rho_{11}H_{11}^{\dagger}\left(I_{N_1} + \rho_{21}H_{21}H_{21}^{\dagger}\right)^{-1}H_{11}\right) \\
& + \log\det\left(I_{M_2} + \rho_{21}H_{21}^{\dagger}H_{21} + \rho_{22}H_{22}^{\dagger}\left(I_{N_2} + \rho_{12}H_{12}H_{12}^{\dagger}\right)^{-1}H_{22}\right), \\
=& I_5^r.
\end{aligned}$$

Proving the equality of the other two bounds is similar. Hence, the set upper bounds for the capacity region of $\mathcal{IC}\left(\tilde{\mathcal{H}}, \bar{\rho}^r\right)$ defines the same set of rate tuples as $\mathcal{R}^u\left(\mathcal{H}, \bar{\rho}\right)$.

Step2: Suppose S is a p.s.d. matrix and S^* represents its complex conjugate, i.e., the matrix obtained by replacing all its entries by the corresponding complex conjugates. Then, using the fact that its eigen-values are real, it can be easily be proved that

$$\log\det(I + S) = \log\det(I + S^*).$$

However, note that all the terms in the different bounds of Lemma 3.1 are of the form just described. This in turn proves that if we replace all the channel matrices of a 2-user MIMO IC by their complex conjugates the set of upper bounds remain the same. From this fact, it easily follows that

$$\mathcal{R}^u\left(\tilde{\mathcal{H}}, \bar{\rho}^r\right) = \mathcal{R}^u\left(\mathcal{H}^r, \bar{\rho}^r\right).$$

A.8 Proof of Lemma 4.2

We shall consider two different cases: 1) $(u_1 + u_2) \geq u$; and 2) $(u_1 + u_2) < u$. The first case was proved in Lemma 1 of [29] which gives

$$\log\det\left(I_u + \rho^a H_1H_1^{\dagger} + \rho^b H_2H_2^{\dagger}\right) = \min\{u, u_1\}a\log(\rho) + (u - u_1)^+ b\log(\rho) + \mathcal{O}(1). \tag{A.41}$$

However when $(u_1+u_2)<u$ we have the following

$$
\begin{aligned}
\log\det\left(I_u+\rho^a H_1H_1^\dagger+\rho^b H_2H_2^\dagger\right) =& \log\det\left(I_u+[H_1\ H_2]\begin{bmatrix}\rho^a I_{u_1} & 0\\ 0 & \rho^a I_{u_2}\end{bmatrix}[H_1\ H_2]^\dagger\right)\\
=& \log\det\left(I_{(u_1+u_2)}+H^\dagger H\begin{bmatrix}\rho^a I_{u_1} & 0\\ 0 & \rho^a I_{u_2}\end{bmatrix}\right),\ [\because H=[H_1\ H_2]]\\
\overset{(a)}{=}& \log\det\left(H^\dagger H\begin{bmatrix}\rho^a I_{u_1} & 0\\ 0 & \rho^a I_{u_2}\end{bmatrix}\right)+o(1)\\
=& (u_1a+u_2b)\log(\rho)+\mathcal{O}(1), \qquad \text{(A.42)}
\end{aligned}
$$

where step (a) follows from the fact that $H^\dagger H$ is full rank, since H is full rank by assumption. Finally combining equations (A.41) and (A.42) we have the desired result.

A.9 Proof of Lemma 4.3

Without loss of generality, let us assume that $a_1\geq\max\{a_2,a_3\}$. In the proof we shall use the following notations:

$$
L_t=\log\det\left(I_u+\sum_{i=1}^{3}\rho^{a_i}H_iH_i^\dagger\right),\ \Lambda=\begin{bmatrix}\rho^{a_2}I_{u_2} & 0\\ 0 & \rho^{a_3}I_{u_3}\end{bmatrix},\ \text{and } H_{23}=[H_2\ H_3],
$$

where the entries of H_{23} are iid that come form a continuous distribution and hence H_{23} is is full rank w.p.1. Using the identity $\log\det(I+AB)=\log\det(I+BA)$ we get

$$
\begin{aligned}
L_t =& \log\det\left(I_u+\rho^{a_1}H_1H_1^\dagger+H_{23}\Lambda H_{23}^\dagger\right),\\
=& \log\det\left(I_u+\rho^{a_1}H_1H_1^\dagger\right)+\log\det\left(I_{u_2+u_3}+\Lambda H_{23}^\dagger\left(I_u+\rho^{a_1}H_1H_1^\dagger\right)^{-1}H_{23}\right),\\
=& \min\{u,u_1\}a_1\log(\rho)+\log\det\left(I_{u_2+u_3}+\Lambda H_{23}^\dagger\left(I_u+\rho^{a_1}H_1H_1^\dagger\right)^{-1}H_{23}\right)+\mathcal{O}(1), \quad \text{(A.43)}
\end{aligned}
$$

Next, we approximate the second term on the right hand side of the last equation as

$$
\begin{aligned}
L_{t2} =& \log\det\left(I_{u_2+u_3}+\Lambda H_{23}^\dagger\left(I_u+\rho^{a_1}H_1H_1^\dagger\right)^{-1}H_{23}\right)\\
=& \log\det\left(I_{u_2+u_3}+\Lambda H_{23}^\dagger U\begin{bmatrix}\left(I_{\min\{u,u_1\}}+\rho^{a_1}\Lambda_1\right)^{-1} & 0\\ 0 & I_{(u-u_1)^+}\end{bmatrix}U^\dagger H_{23}\right), \quad \text{(A.44)}
\end{aligned}
$$

where in the last step we have used the eigen-decomposition of the matrix $H_1 H_1^\dagger$, where Λ_1 is a diagonal matrix containing the positive eigenvalues only and U is the unitary matrix containing all the eigen-vectors. Since both $H_2, H_3 \in \mathcal{P}$, $H \triangleq U^\dagger H_{23}$ is identically distributed as H_{23} and thus $H \in \mathcal{P}$. Suppose the rows of the matrix H are divided into two sub sets: $G_1^\dagger = H^{(1:\min\{u,u_1\})}$ and $G_2^\dagger = H^{(\min\{u,u_1\}+1:u)}$, then both $G_1^\dagger, G_2^\dagger \in \mathcal{P}$ and from the last equation we get

$$\begin{aligned}
L_{t2} =& \log\det\left(I_{u_2+u_3} + \Lambda G_1 \left(I_{u_1} + \rho^{a_1}\Lambda_1\right)^{-1} G_1^\dagger + \Lambda G_2 G_2^\dagger\right), \\
\overset{(c)}{=}& \log\det\left(I_{u_2+u_3} + \Lambda G_2 G_2^\dagger\right) + \mathcal{O}(1), \\
=& \log\det\left(I_{(u-u_1)^+} + G_2^\dagger \Lambda G_2\right) + \mathcal{O}(1), \qquad \text{(A.45)}
\end{aligned}$$

where step (c) follows from the fact that $a_1 \geq \max\{a_2, a_3\}$. Since $G_2^\dagger \in \mathcal{P}$, it is full rank and so are $(G_2^\dagger)^{[1:u_2]} = G_{21}$ and $(G_2^\dagger)^{[u_2+1:u_2+u_3]} = G_{22}$. Putting these into equation (A.45) we get

$$\begin{aligned}
L_{t2} =& \log\det\left(I_{u-u_1} + \rho^{a_2} G_{21} G_{21}^\dagger + \rho^{a_3} G_{22} G_{22}^\dagger\right) + \mathcal{O}(1), \\
=& \left(\min\{(u-u_1)^+, u_2\} a_2 + \min\{(u-u_1-u_2)^+, u_3\} a_3\right) \log(\rho) + \mathcal{O}(1),
\end{aligned}$$

where the last step follows from Lemma 4.2. Finally, substituting this into equation (A.43), we get the desired result.

A.10 Proof of Theorem 4.1

1^{st} *and* 2^{nd} *bound*: We start with the first two constraints in equation (5.11) and (3.8),

$$\begin{aligned}
I_1 \triangleq& \log\det\left(I_{N_1} + \rho H_{11} H_{11}^\dagger\right) = \min\{M_1, N_1\} \log(\rho) + \mathcal{O}(1); \;\; [\because \text{ Lemma 4.2 with } a = 1 \text{ and } b = 0] \\
I_2 \triangleq& \log\det\left(I_{N_1} + \rho^{\alpha_{22}} H_{22} H_{22}^\dagger\right) = \min\{M_2, N_2\} \alpha_{22} \log(\rho) + \mathcal{O}(1). \;\; [\because \text{ Lemma 4.2 with } a = \alpha_{22} \text{ and } b = 0]
\end{aligned}$$

Putting these into equations (4.14) and (4.15) we get

$$d_1 \leq \lim_{\rho\to\infty} \frac{I_1}{\log(\rho)} = \min\{M_1, N_1\}; \qquad \text{(A.46)}$$

$$d_2 \leq \lim_{\rho\to\infty} \frac{I_2}{(\alpha_{22}\log(\rho))} = \min\{M_2, N_2\}. \qquad \text{(A.47)}$$

3^{rd} *and* 4^{th} *bound*: Using Lemma 4.2 we get

$$\begin{aligned}
I_{31} \triangleq & \log\det\left(I_{N_2} + \rho^{\alpha_{12}} H_{12} H_{12}^\dagger + \rho^{\alpha_{22}} H_{22} H_{22}^\dagger\right) \\
= & f\left(N_2, (\alpha_{12}, M_1), (\alpha_{22}, M_2)\right) \log(\rho) + \mathcal{O}(1), \qquad \text{(A.48)}
\end{aligned}$$

where $f(.,.,.)$ is as defined in equation (4.6). The second term in equation (5.11) can again be approximated by using Lemma 4.2 twice (recall Remark 4.1) as follows.

$$\begin{aligned}
I_{32} = & \log\det\left(I_{N_1} + \rho_{11} H_{11} \left(I_{M_1} + \rho_{12} H_{12}^\dagger H_{12}\right)^{-1} H_{11}^\dagger\right), \\
= & \log\det\left(I_{M_1} + \rho_{12} H_{12}^\dagger H_{12} + \rho_{11} H_{11}^\dagger H_{11}\right) - \log\det\left(I_{M_1} + \rho_{12} H_{12}^\dagger H_{12}\right), \\
= & f\left(M_1, (\alpha_{12}, N_2), (\alpha_{11}, N_1)\right) \log(\rho) - \min\{M_1, N_2\}\alpha_{12} \log(\rho) + \mathcal{O}(1), \qquad \text{(A.49)} \\
= & f\left(N_1, ((1-\alpha_{12})^+, m_{12}), (1, (M_1 - N_2)^+)\right) \log(\rho) + \mathcal{O}(1), \qquad \text{(A.50)}
\end{aligned}$$

where the last step follows from the definition of $f(.)$, i.e., equation (4.6). Next, putting equations (A.48), and (A.49) in equation (4.16) we get

$$\begin{aligned}
d_1 + \alpha_{22} d_2 \leq & \lim_{\rho\to\infty} \frac{I_3}{\log(\rho)} = \lim_{\rho\to\infty} \frac{I_{31} + I_{32}}{\log(\rho)}; \\
= & f\left(N_2, (\alpha_{12}, M_1), (\alpha_{22}, M_2)\right) + f\left(N_1, ((1-\alpha_{12})^+, m_{12}), (1, (M_1 - N_2)^+)\right).
\end{aligned}$$

Similarly, approximating the terms in equation (3.10) by Lemma 4.2 and putting it in equation (4.17) we get

$$d_1 + \alpha_{22} d_2 \leq f\left(N_1, (\alpha_{21}, M_2), (\alpha_{11}, M_1)\right) + f\left(N_2, ((\alpha_{22} - \alpha_{21})^+, m_{21}), (\alpha_{22}, (M_2 - N_1)^+)\right).$$

5^{th} *bound*: Note that neither of the terms in equation (3.11) are in a form on which we can apply Lemma 4.2 or 4.3. However, as we shall see next, these terms can be expressed in an alternative format on which Lemma 4.3 can be used. Let the eigenvalue decomposition of the matrix $H_{12}^\dagger H_{12}$ is given as

$$H_{12}^\dagger H_{12} = U \Lambda U^\dagger, \text{ where } \Lambda = \begin{bmatrix} \Lambda^+ & 0 \\ 0 & 0_{(M_1 - N_2)^+} \end{bmatrix}$$

and Λ^{+} is a diagonal matrix containing only the positive eigenvalues. Using this decomposition we get

$$\begin{aligned}
I_{51} \triangleq& \log\det\left(I_{N_1} + \rho^{\alpha_{21}} H_{21}H_{21}^{\dagger} + \rho H_{11}\left(I_{M_1} + \rho_{12}H_{12}^{\dagger}H_{12}\right)^{-1} H_{11}^{\dagger}\right) \\
=& \log\det\left(I_{N_1} + \rho^{\alpha_{21}} H_{21}H_{21}^{\dagger} + \rho H_{11}U\begin{bmatrix} (I_{m_{12}} + \rho^{\alpha_{12}}\Lambda^{+})^{-1} & 0 \\ 0 & I_{(M_1-N_2)^{+}} \end{bmatrix} U^{\dagger}H_{11}^{\dagger}\right), \\
=& \log\det\left(I_{N_1} + \rho^{\alpha_{21}} H_{21}H_{21}^{\dagger} + \rho\widetilde{H}\begin{bmatrix} (I_{m_{12}} + \rho^{\alpha_{12}}\Lambda^{+})^{-1} & 0 \\ 0 & I_{(M_1-N_2)^{+}} \end{bmatrix} \widetilde{H}^{\dagger}\right), \quad \left[\because\ \widetilde{H} = H_{11}U\right]
\end{aligned}$$

Note that $\widetilde{H}$ is identically distributed to H_{11}. Therefore, both $G_1 = \widetilde{H}^{[1:m_{12}]}$ and $G_2 = \widetilde{H}^{[(m_{12}+1):M_1]}$ have the same distribution as specified in section 4.2, having all the properties of a typical channel matrix of the 2-user MIMO IC. Substituting this in the last equation we get

$$\begin{aligned}
I_{51} =& \log\det\left(I_{N_1} + \rho^{\alpha_{21}} H_{21}H_{21}^{\dagger} + \rho G_1\left(I_{m_{12}} + \rho^{\alpha_{12}}\Lambda^{+}\right)^{-1} G_1^{\dagger} + \rho G_2 G_2^{\dagger}\right) \\
=& \log\det\left(I_{N_1} + \rho^{\alpha_{21}} H_{21}H_{21}^{\dagger} + \rho^{(1-\alpha_{12})} G_1 G_1^{\dagger} + \rho G_2 G_2^{\dagger}\right) + o(1).
\end{aligned}$$

Clearly, we can now apply Lemma 4.3 on equation (A.51),

$$I_{51} = g\left(N_1, (\alpha_{21}, M_2), ((1-\alpha_{12})^{+}, m_{12}), (1, (M_1 - N_2)^{+})\right)\log(\rho) + \mathcal{O}(1). \tag{A.51}$$

Applying similar technique for the other term in equation (3.11) we get

$$I_{52} = g\left(N_2, (\alpha_{12}, M_1), ((\alpha_{22}-\alpha_{21})^{+}, m_{21}), (\alpha_{22}, (M_2 - N_1)^{+})\right)\log(\rho) + \mathcal{O}(1). \tag{A.52}$$

Finally, using this expressions for I_{51} and I_{52} in equation (4.18) we get the 5^{th} bound for the GDoF region.

6^{th} and 7^{th} bound: Note that equations (3.12) and (3.13) involves terms whose approximations are already computed. Using those approximations we get the remaining 2 bounds of the GDoF region completing the proof.

A.11 Proof of Corollary 4.2

We have to prove that the bounds in equations (4.22e)-(4.22g) are looser than the others. To analyze the 5-th bound, we start from equation (4.22e),

$$
\begin{aligned}
(d_1+d_2) \leq & (N_1 \wedge M_2) + ((N_1-M_2)^+ \wedge (M_1-N_2)^+) + (N_2 \wedge M_1) + ((M_1-N_2)^+ \wedge (M_2-N_1)^+);\\
= & \begin{cases} M_2+N_2+\{(N_1-M_2)\wedge(M_1-N_2)\}, & \text{if } N_1 > M_2 \text{ and } M_1 > N_2;\\ N_1+N_2, & \text{if } N_1 \leq M_2 \text{ and } M_1 > N_2;\\ M_1+M_2, & \text{if } N_1 > M_2 \text{ and } M_1 \leq N_2;\\ M_1+N_1+\{(N_2-M_1)\wedge(M_2-N_1)\}, & \text{if } N_1 \leq M_2 \text{ and } M_1 \leq N_2; \end{cases}\\
= & \begin{cases} \min\{(M_1+M_2),(N_1+N_2)\}, & \text{if } N_1 > M_2 \text{ and } M_1 > N_2;\\ N_1+N_2, & \text{if } N_1 \leq M_2 \text{ and } M_1 > N_2;\\ M_1+M_2, & \text{if } N_1 > M_2 \text{ and } M_1 \leq N_2;\\ \min\{(M_1+M_2),(N_1+N_2)\}, & \text{if } N_1 \leq M_2 \text{ and } M_1 \leq N_2; \end{cases}
\end{aligned}
$$

Clearly, this is looser than both the 3^{rd} and the 4^{th} bound. Consider next the 6^{th} bound which, from equation (4.22f), is given as

$$
\begin{aligned}
(2d_1+d_2) \leq & (N_1 \wedge (M_1+M_2)) + N_1 \wedge (M_1-N_2)^+ + (N_2 \wedge M_1) + ((M_1-N_2)^+ \wedge (M_2-N_1)^+);\\
= & \begin{cases} (N_1 \wedge (M_1+M_2)) + (N_1 \wedge (M_1-N_2)) + N_2, & \text{if } M_1 \geq N_2;\\ (N_1 \wedge (M_1+M_2)) + M_1 + ((N_2-M_1) \wedge (M_2-N_1)^+), & \text{if } M_1 < N_2; \end{cases}\\
= & \begin{cases} (N_1 \wedge (M_1+M_2)) + (N_1 \wedge (M_1-N_2)) + N_2, & \text{if } M_1 \geq N_2;\\ (N_1 \wedge (M_1+M_2)) + \min\{N_2, (M_1+M_2-N_1)\}, & \text{if } M_1 < N_2 \text{ and } M_2 > N_1;\\ (N_1 \wedge (M_1+M_2)) + M_1, & \text{if } M_1 < N_2 \text{ and } M_2 \leq N_1; \end{cases}\\
= & \begin{cases} (N_1 \wedge (M_1+M_2)) + \min\{(N_1+N_2), M_1\}, & \text{if } M_1 \geq N_2;\\ \min\{(M_1+M_2), (N_1+N_2)\}, & \text{if } M_1 < N_2 \text{ and } M_2 > N_1;\\ M_1 + (N_1 \wedge (M_1+M_2)), & \text{if } M_1 < N_2 \text{ and } M_2 \leq N_1; \end{cases}\\
= & \begin{cases} (N_1 \wedge (M_1+M_2)) + \min\{(N_1+N_2), \max(N_2, M_1)\}, & \text{if } M_1 \geq N_2;\\ \min\{M_1 + \max(M_2, N_1), N_1 + \max(N_2, M_1)\}, & \text{if } M_1 < N_2 \text{ and } M_2 > N_1;\\ M_1 + \min\{\max(N_1, M_2), (M_1+M_2)\}, & \text{if } M_1 < N_2 \text{ and } M_2 \leq N_1; \end{cases}
\end{aligned}
$$

It is clear that this bound is looser than the sum of the 1st and 3rd or the sum of the 1st and the 4th bounds. The proof of the fact that the same is true for the 7th bound is identical and is hence skipped.

A.12 Proof of Equivalent GDoF region

In the HK coding scheme [10], each user's message is divided into two parts called the private (U_i) and public (W_i) messages with rate S_i and T_i, respectively. It was proved in [10] that for any given probability distribution P(.) which factors as

$$
\begin{aligned}
P(Q, W_1, U_1, W_2, U_2, X_1, X_2) = & P(Q)P(U_1|Q)P(W_1|Q)P(U_2|Q)P(W_2|Q)\\
& P(X_1|U_1, U_2, Q)P(X_2|U_2, W_2, Q), \qquad \text{(A.53)}
\end{aligned}
$$

the rate region $\mathcal{R}^{o}_{\text{HK}}(P) = \mathcal{R}^{(o,1)}_{\text{HK}}(P) \cap \mathcal{R}^{(o,2)}_{\text{HK}}(P)$ is achievable where

$$\begin{aligned}
\mathcal{R}^{(o,i)}_{\text{HK}}(P) = \Big\{(S_1, T_1, S_2, T_2) :\ S_i &\leq I(U_i; Y_i|W_i, W_j, Q);\\
T_i &\leq I(W_i; Y_i|U_i, W_j, Q);\\
T_j &\leq I(W_j; Y_i|W_i, U_i, Q);\\
(S_i + T_i) &\leq I(U_iW_i; Y_i|W_j, Q);\\
(S_i + T_j) &\leq I(U_iW_j; Y_i|W_i, Q);\\
(T_i + T_j) &\leq I(W_iW_j; Y_i|U_i, Q);\\
(S_i + T_i + T_j) &\leq I(U_iW_iW_j; Y_i|Q); \Big\}
\end{aligned}$$

for $i \neq j \in \{1, 2\}$. Let the 2-dimensional projection of the set $\mathcal{R}^{o}_{\text{HK}}(P)$ be denoted by $\Pi\,(\mathcal{R}^{o}_{\text{HK}})$, which is defined as follows

$$\Pi\,(\mathcal{R}^{o}_{\text{HK}}(P)) = \{(0 \leq R_1 \leq (S_1 + T_1), 0 \leq R_2 \leq (S_2 + T_2)) : (S_1, T_1, S_2, T_2) \in \mathcal{R}^{o}_{\text{HK}}(P)\}.$$

Clearly, if $(R_1, R_2) \in \Pi\,(\mathcal{R}^{o}_{\text{HK}}(P))$ then there exists a 4-tuple $(S_1, T_1, S_2, T_2) \in \mathcal{R}^{o}_{\text{HK}}(P)$ such that $(S_i + T_i) = R_i$ for $i = 1, 2$, and vice versa. This is true for any distribution satisfying (A.53). Using the Fourier-Motzkin elimination method, a compact formula for the rate region $\Pi\,(\mathcal{R}^{o}_{\text{HK}}(P))$ was recently derived in Lemma 1 of [26], which when evaluated for the input distributions specified in Section 3.2, results in an achievable rate region containing the rate region given in Lemma 3.4. Let us denote the rate region $\mathcal{R}^{o}_{\text{HK}}(P)$ by $\mathcal{R}^{G}_{\text{HK}}$, when P is same as the distributions specified in Section 3.2. Using the technique in the proof of Lemma 3.4 it then follows that $\mathcal{R}^{G}_{\text{HK}} = \mathcal{R}^{G1}_{\text{HK}} \cap \mathcal{R}^{G2}_{\text{HK}}$, where

$$\begin{aligned}
\mathcal{R}_{\text{HK}}^{Gi} = \Big\{(S_i, T_i, T_j):\ S_i \leq & \log\det\left(I_{N_i} + \rho_{ii} H_{ii} K_{iu} H_{ii}^{\dagger} + \rho_{ji} H_{ji} K_{ju} H_{ji}^{\dagger}\right) - \tau_{ji};\\
T_i \leq & \log\det\left(I_{N_i} + \rho_{ii} H_{ii} K_{iw} H_{ii}^{\dagger} + \rho_{ji} H_{ji} K_{ju} H_{ji}^{\dagger}\right) - \tau_{ji};\\
T_j \leq & \log\det\left(I_{N_i} + \frac{\rho_{ji}}{M_j} H_{ji} H_{ji}^{\dagger}\right) - \tau_{ji};\\
(S_i + T_i) \leq & \log\det\left(I_{N_i} + \frac{\rho_{ii}}{M_i} H_{ii} H_{ii}^{\dagger} + \rho_{ji} H_{ji} K_{ju} H_{ji}^{\dagger}\right) - \tau_{ji};\\
(S_i + T_j) \leq & \log\det\left(I_{N_i} + \frac{\rho_{ji}}{M_j} H_{ji} H_{ji}^{\dagger} + \rho_{ii} H_{ii} K_{iu} H_{ii}^{\dagger}\right) - \tau_{ji}\\
(T_i + T_j) \leq & \log\det\left(I_{N_i} + \frac{\rho_{ji}}{M_j} H_{ji} H_{ji}^{\dagger} + \rho_{ii} H_{ii} K_{iw} H_{ii}^{\dagger}\right) - \tau_{ji};\\
(S_i + T_i + T_j) \leq & \log\det\left(I_{N_i} + \frac{\rho_{ji}}{M_j} H_{ji} H_{ji}^{\dagger} + \frac{\rho_{ii}}{M_i} H_{ii} H_{ii}^{\dagger}\right) - \tau_{ji}; \Big\}
\end{aligned}$$

for $i \neq j \in \{1, 2\}$ and τ_{ji}'s for $1 \leq i \neq j \leq 2$ are constants independent of ρ or channel matrices. The GDoF region corresponding to the above achievable rate region can be defined as follows

$$\mathcal{G}_i(\bar{M}, \bar{\alpha}) = \Big\{(d_{1p}, d_{1c}, d_{2p}, d_{2c}) : d_{ip} = \lim_{\rho_{ii}\to\infty} \frac{S_i}{\log(\rho_{ii})}, d_{ic} = \lim_{\rho_{ii}\to\infty} \frac{T_i}{\log(\rho_{ii})} \text{ for } i = 1, 2,$$
$$\text{and } (S_1, T_1, S_2, T_2) \in \mathcal{R}_{\text{HK}}^{Gi}\Big\}.$$

Using this definition, and following a similar approach as in Theorem 4.1, we get equation (4.23).

From the above analysis on one hand, we have the achievable rate region $\mathcal{R}_{\text{HK}}^{G}$ for the Gaussian IC, which is $\mathcal{R}_{\text{HK}}^{o}(P)$ evaluated for the distribution of Subsection 4.2.3. on the other hand, we have $\mathcal{R}_a(\mathcal{H}, \bar{\alpha})$, which is a *subset* of the rate region obtained when $\Pi\left(\mathcal{R}_{\text{HK}}^{o}(P)\right)$ is evaluated at the distribution in Subsection 4.2.3. This two facts together imply that

$$\mathcal{R}_a(\mathcal{H}, \bar{\alpha}) \subseteq \Pi\left(\mathcal{R}_{\text{HK}}^{G}\right), \tag{A.54}$$

i.e., for any rate pair $(R_1, R_2) \in \mathcal{R}_a(\mathcal{H}, \bar{\alpha})$ there exists a 4-tuple $(S_1, T_1, S_2, T_2) \in \mathcal{R}_{\text{HK}}^{G}$ such that $(S_i + T_i) = R_i$ for $i = 1, 2$. In other words, $\mathcal{R}_a(\mathcal{H}, \bar{\alpha})$ is a subset of the 2-dimensional projection of the set $\mathcal{R}_{\text{HK}}^{G}$. Since $\mathcal{G}(\bar{M}, \bar{\alpha})$ and $\mathcal{D}_o(\bar{M}, \bar{\alpha})$ are the high SNR scaled versions of the rate regions $\mathcal{R}_{\text{HK}}^{G}$ and $\mathcal{R}_a(\mathcal{H}, \bar{\alpha})$, respectively the same is true for them. That is $\mathcal{D}_o(\bar{M}, \bar{\alpha})$ is a subset of the 2-dimensional projection of the set $\mathcal{G}(\bar{M}, \bar{\alpha})$ or for every $(d_1, d_2) \in \mathcal{D}_o(\bar{M}, \bar{\alpha})$, there exists a 4-tuple $(d_{1p}, d_{1c}, d_{2p}, d_{2c}) \in \mathcal{G}(\bar{M}, \bar{\alpha})$ such that $(d_{ip} + d_{ic}) = d_i$ for $i = 1, 2$.

A.13 Proof of Lemma 5.4

Recall that, the negative SNR exponents of the eigenvalues of W_1, W_2 and W_3 are denoted by $\{\beta_j\}_{j=1}^{q_1}$, $\{\gamma_k\}_{k=1}^{q_2}$ and $\{\upsilon_i\}_{i=1}^{p}$, respectively. Interestingly, these matrices have exactly the same structure specified in Theorem 1 and 2 of [52], except from the $\rho^{\alpha_{21}}$ which can be easily taken care of. Therefore, using these Theorems we obtain the following conditional distributions:

$$f_{W_1|W_3}(\bar{\beta}|\bar{\upsilon}) \doteq \begin{cases} \rho^{-E_1(\bar{\beta},\bar{\upsilon})} & \text{if } \ (\bar{\beta},\bar{\upsilon}) \in \mathcal{B}_1; \\ 0 & \text{if } \ (\bar{\beta},\bar{\upsilon}) \notin \mathcal{B}_1 \end{cases} \tag{A.55}$$

where

$$E_1(\bar{\beta},\bar{\upsilon}) = \left\{ \sum_{j=1}^{q_1} \left((M_1+N_1+1-2j)\beta_j + \sum_{i=1}^{\min\{(N_1-j),M_1\}} (\alpha_{21}-\upsilon_i-\beta_j)^+ \right) - M_1 \sum_{i=1}^{p} (\alpha_{21}-\upsilon_i)^+ \right\} \tag{A.56}$$

$$\mathcal{B}_1 = \left\{ (\bar{\beta},\bar{\upsilon}) : \ 0 \le \upsilon_1 \le \cdots \le \upsilon_p; \ 0 \le \beta_1 \le \cdots \le \beta_{q_1}; \ (\upsilon_i+\beta_j) \ge \alpha_{21}, \ \forall (i+j) \ge (N_1+1) \right\} \tag{A.57}$$

and

$$f_{W_2|W_3}(\bar{\gamma}|\bar{\upsilon}) \doteq \begin{cases} \rho^{-E_2(\bar{\gamma},\bar{\upsilon})} & \text{if } \ (\bar{\gamma},\bar{\upsilon}) \in \mathcal{B}_2; \\ 0 & \text{if } \ (\bar{\gamma},\bar{\upsilon}) \notin \mathcal{B}_2 \end{cases} \tag{A.58}$$

where

$$E_2(\bar{\gamma},\bar{\upsilon}) = \left\{ \sum_{k=1}^{q_2} \left((M_2+N_2+1-2k)\gamma_k + \sum_{i=1}^{\min\{(M_2-k),N_2\}} (\alpha_{21}-\upsilon_i-\gamma_k)^+ \right) - N_2 \sum_{i=1}^{p} (\alpha_{21}-\upsilon_i)^+ \right\} \tag{A.59}$$

$$\mathcal{B}_2 = \left\{ (\bar{\beta},\bar{\upsilon}) : \ 0 \le \upsilon_1 \le \cdots \le \upsilon_p; \ 0 \le \gamma_1 \le \cdots \le \gamma_{q_2}; \ (\upsilon_i+\gamma_k) \ge \alpha_{21}, \ \forall (i+k) \ge (M_2+1) \right\} \tag{A.60}$$

These pdfs are easily obtained from equation (10) of [52] by changing it properly due to the presence of the $\rho^{\alpha_{21}}$ instead of ρ in [52].

It was proved in [91] that given the non-zero eigenvalues of W_3, W_1 and W_2 are conditionally independent random matrices. Since the non-zero eigenvalues of $H_{21}H_{21}^{\dagger}$ and $H_{21}^{\dagger}H_{21}$ are exactly the same for each realization of the matrix H_{21}, given $\bar{\upsilon}$, $\bar{\beta}$ and $\bar{\gamma}$ are conditionally independent of each other. Intuitively, it is a well known fact in the literature of random matrix theory that the eigenvalues of W_1 are dependent on the matrix $W3$ only through it's eigenvalues. The same is true for W_2. Therefore, given the eigenvalues of W_3, the eigenvalues of W_1 and W_2 are conditionally independent[1] . Using this fact we have the following:

$$f_{W_1,W_2,W_3}(\bar{\beta},\bar{\gamma},\bar{\upsilon}) = f_{W_1,W_2|W_3}(\bar{\beta},\bar{\gamma}|\bar{\upsilon}) f_{W_3}(\bar{\upsilon}), \tag{A.61}$$

$$\overset{(a)}{=} f_{W_1|W_3}(\bar{\beta}|\bar{\upsilon}) f_{W_2|W_3}(\bar{\gamma}|\bar{\upsilon}) f_{W_3}(\bar{\upsilon}). \tag{A.62}$$

Now, substituting equations (A.55), (A.58) into the above equation we obtain the joint distribution $f_{W_1,W_2,W_3}(\bar{\beta},\bar{\gamma},\bar{\upsilon})$, where the marginal distributions of υ_i's are given by

$$f_{W_3}(\bar{\upsilon}) \dot{=} \begin{cases} \rho^{-\sum_{i=1}^{p}(M_2+N_1+1-2i)\upsilon_i} & \text{if } \ 0 \le \upsilon_1 \le \cdots \le \upsilon_p; \\ 0 & \text{otherwise,} \end{cases} \tag{A.63}$$

which was derived in [48]. Using this joint distribution $f_{W_1,W_2,W_3}(\bar{\beta},\bar{\gamma},\bar{\upsilon})$ and the Laplace's method and following the similar approach as in [48] equation (5.26) can be evaluated to obtain the following:

$$\begin{aligned} d_{\mathcal{O}_s}(r_s) = \min & \sum_{i=1}^{p}(M_2+N_1+1-2i)\upsilon_i + \sum_{j=1}^{q_1}(M_1+N_1+1-2j)\beta_j + \sum_{k=1}^{q_2}(M_2+N_2+1-2k)\gamma_k \\ & - (M_1+N_2)\sum_{i=1}^{p}(\alpha_{21}-\upsilon_i)^+ + \sum_{k=1}^{q_2}\sum_{i=1}^{\min\{(M_2-k),N_2\}}(\alpha_{21}-\upsilon_i-\gamma_k)^+ \\ & + \sum_{j=1}^{q_1}\sum_{i=1}^{\min\{(N_1-j),M_1\}}(\alpha_{21}-\upsilon_i-\beta_j)^+; \end{aligned} \tag{A.64a}$$

$$\text{constrained by: } \sum_{i=1}^{p}(\alpha_{21}-\upsilon_i)^+ + \sum_{j=1}^{q_1}(\alpha_{11}-\beta_j)^+ + \sum_{k=1}^{q_2}(\alpha_{22}-\gamma_k)^+ < r_s; \tag{A.64b}$$

$$0 \le \upsilon_1 \le \cdots \le \upsilon_p; \tag{A.64c}$$

$$0 \le \beta_1 \le \cdots \le \beta_{q_1}; \tag{A.64d}$$

$$0 \le \gamma_1 \le \cdots \le \gamma_{q_2}; \tag{A.64e}$$

$$(\upsilon_i+\beta_j) \ge \alpha_{21}, \ \forall (i+j) \ge (N_1+1); \tag{A.64f}$$

$$(\upsilon_i+\gamma_k) \ge \alpha_{21}, \ \forall (i+k) \ge (M_2+1). \tag{A.64g}$$

[1] For a detailed proof of this fact the reader is referred to Lemma 1 of [91].

Finally, the desired result follows from the fact that by restricting $\alpha_i \leq \alpha_{21}$ for all $i \leq p$ does not change the optimal solution of the above problem. This can be proved as follows: suppose the optimal solution have $\alpha_i > \alpha_{21}$ for some i. Now, since the objective function is monotonically decreasing function of α_i for all i, substituting $\alpha_i = \alpha_{21}$ does not violate any of the constraints but reduces the objective function. But, that means the earlier solution was not really the optimal solution. Therefore, in the optimal solution we must have $\alpha_i \leq \alpha_{21}$ for all $i \leq p$. However, with this constraint we have

$$(\alpha_{21} - \upsilon_i)^+ = (\alpha_{21} - \upsilon_i), \ \forall\ i \leq p.$$

Substituting this in equation (A.64a) we get the desired result.

A.14 Proof of Theorem 5.2

The main steps of the proof can be described as follows. First, we simplify the optimization problem in equations (5.27) and (5.28) by putting specific values of the different parameters, such as M_i, N_i and α_{ij} as stated in the statement of the Theorem, in equations (5.27) and (5.28). Then we calculate a local minimum of this simplified optimization problem for each value of r, using the steepest descent method, which by the previous argument then represents a global minimum. The later part of the problem, i.e., the computation of the local minimum will be carried out in two steps: in step one, we consider the case when $\alpha \leq 1$ and then in the second step we consider the remaining case. Let us start by deriving the simplified optimization problem.

Substituting $M_1 = M_2 = N_1 = N_2 = n$, $\alpha_{11} = \alpha_{22} = 1$ and $\alpha_{21} = \alpha$ in the optimization problem of Lemma 5.4 we obtain the following:

$$\min \mathcal{O}_s \triangleq \min_{(\vec{v},\vec{\gamma},\vec{\beta},\bar{\alpha})} \sum_{i=1}^{n}(4n+1-2i)\upsilon_i + \sum_{j=1}^{n}(2n+1-2j)\beta_j + \sum_{k=1}^{n}(2n+1-2k)\gamma_k$$
$$-2n^2\alpha + \sum_{k=1}^{n}\sum_{i=1}^{(n-k)}(\alpha-\upsilon_i-\gamma_k)^+ + \sum_{j=1}^{n}\sum_{i=1}^{(n-j)}(\alpha-\upsilon_i-\beta_j)^+; \tag{A.65a}$$

$$\text{constrained by: } \sum_{i=1}^{n}(\alpha-\upsilon_i)^+ + \sum_{j=1}^{n}(1-\beta_j)^+ + \sum_{k=1}^{n}(1-\gamma_k)^+ < r_s; \tag{A.65b}$$

$$0 \le \upsilon_1 \le \cdots \le \upsilon_n; \tag{A.65c}$$

$$0 \le \beta_1 \le \cdots \le \beta_n; \tag{A.65d}$$

$$0 \le \gamma_1 \le \cdots \le \gamma_n; \tag{A.65e}$$

$$(\upsilon_i + \beta_j) \ge \alpha,\ \forall (i+j) \ge (n+1); \tag{A.65f}$$

$$(\upsilon_i + \gamma_k) \ge \alpha,\ \forall (i+k) \ge (n+1). \tag{A.65g}$$

As stated earlier, in what follows, we solve this optimization problem in two steps; in the first step, we assume $\alpha \le 1$.

A.14.1 Step 1: $(\alpha \le 1)$

To apply the steepest descent method we first compute the rate of change of the objective function with respect to the various parameters. Differentiating the objective function in equation (A.65a) we obtain the following:

$$\left.\frac{\partial \mathcal{O}_s}{\partial \alpha_i}\right|_{\alpha_1=0,\cdots,\alpha_{i-1}=0,\alpha_{i+1}=1,\cdots,\alpha_n=1,\beta_1=1,\gamma_1=1} = (4n+1-2i),\ 1 \le i \le n; \tag{A.66}$$
$$\left.\frac{\partial \mathcal{O}_s}{\partial \beta_1}\right|_{\alpha_1=1,\gamma_1=1} = (2n-1) \le (4n+1-2i),\ \forall\ 1 \le i \le n; \tag{A.67}$$
$$\left.\frac{\partial \mathcal{O}_s}{\partial \gamma_1}\right|_{\alpha_1=1,\beta_1=1} = (2n-1) \le (4n+1-2i),\ \forall\ 1 \le i \le n. \tag{A.68}$$

Note that it is sufficient, to consider the decay of the function with respect to (w.r.t.) β_1 and γ_1 only, because of the decreasing slope of the objective function with increasing index of β and

γ and equation (A.65d) and (A.65e). Therefore, it is clear from the slopes of the function given above that for $(i-1)\alpha \leq r_s \leq i\alpha$, the steepest descent of the function is along decreasing values of α_i, while $\beta_1 = \gamma_1 = 1$. Note that for these values of β_1 and γ_1 the last two terms of equation (A.65a) vanishes and equations (A.65d)-(A.65g) becomes redundant and as a result the optimization problem of (A.65) simplifies to the following:

$$\min \sum_{i=1}^{n} (4n+1-2i)\upsilon_i + 2n^2(1-\alpha); \tag{A.69a}$$

$$\text{constrained by: } \sum_{i=1}^{n} (\alpha - \upsilon_i)^+ \leq r_s; \tag{A.69b}$$

$$0 \leq \upsilon_1 \leq \cdots \leq \upsilon_n. \tag{A.69c}$$

The solution of this optimization problem follows from Lemma 5.3 and is given by,

$$d_s(r_s) = \alpha d_{n,3n}\left(\frac{r_s}{\alpha}\right) + 2n^2(1-\alpha),\ 0 \leq r_s \leq n\alpha. \tag{A.70}$$

The above solution also imply that for $\alpha \geq n\alpha$, the optimal solution have $\alpha_i = 0,\ \forall i$, which when substituted in equation (A.65) we obtain the following problem.

$$\min \sum_{j=1}^{n} (2n+1-2j)\beta_j + \sum_{k=1}^{n} (2n+1-2k)\gamma_k - 2n^2, \tag{A.71a}$$

$$\text{constrained by: } \sum_{j=1}^{n} (1-\beta_j)^+ + \sum_{k=1}^{n} (1-\gamma_k)^+ < (r_s - n\alpha); \tag{A.71b}$$

$$\alpha \leq \beta_1 \leq \cdots \leq \beta_n; \tag{A.71c}$$

$$\alpha \leq \gamma_1 \leq \cdots \leq \gamma_n. \tag{A.71d}$$

Note that the last two summands in the objective function (A.65) is zero because of equations (A.71c) and (A.71d). Now, from the symmetry of the optimization problem (A.71) w.r.t. β_i and γ_i, we can assume without loss of generality that the optimal solution will have $\beta_i = \gamma_i,\ \forall i$. Substituting this and $\delta_i = \beta_i - \alpha$ in equation (A.71) we get the following equivalent optimization problem:

$$\min 2\sum_{j=1}^{n}(2n+1-2j)\delta_j, \tag{A.72a}$$

$$\text{constrained by: } \sum_{j=1}^{n}(1-\alpha-\delta_j)^+ \le \left(\frac{r_s-n\alpha}{2}\right); \tag{A.72b}$$

$$0 \le \delta_1 \le \cdots \le \delta_n. \tag{A.72c}$$

The solution of this optimization problem follows again from Lemma 5.3 and is given by

$$\begin{aligned} d_s(r_s) =& 2(1-\alpha)d_{n,n}\left(\frac{r_s-n\alpha}{2(1-\alpha)}\right), \ 0 \le \frac{r_s-n\alpha}{2} \le n(1-\alpha) \\ =& 2(1-\alpha)d_{n,n}\left(\frac{r_s-n\alpha}{2(1-\alpha)}\right), \ n\alpha \le r_s \le n(2-\alpha). \end{aligned} \tag{A.73}$$

Combining equations (A.70) and (A.73) we obtain equation (5.30) of Theorem 5.2 and we have completed the first step of this proof. In what follows we consider the remaining case, when $\alpha \ge 1$.

A.14.2 Step 2: ($\alpha \ge 1$)

Differentiating the objective function in equation (A.65a) we obtain the following:

$$\left.\frac{\partial \mathcal{O}_s}{\partial \alpha_1}\right|_{\alpha_i=\alpha\ \forall\ i\ge 2,\beta_1=1,\gamma_1=1} = \begin{cases} (4n-1), & \text{for } \alpha_1 \ge (\alpha-1); \\ (2n-1), & \text{for } 0 \le \alpha_1 \le (\alpha-1); \end{cases} \tag{A.74}$$

$$\left.\frac{\partial \mathcal{O}_s}{\partial \beta_1}\right|_{\alpha_i=\alpha\ \forall\ i\ge 2,\gamma_1=1} = \left.\frac{\partial \mathcal{O}_s}{\partial \gamma_1}\right|_{\alpha_i=\alpha\ \forall\ i\ge 2,\beta_1=1} = \begin{cases} (2n-1), & \text{for } \alpha_1 \ge (\alpha-1); \\ (2n-2), & \text{for } 0 \le \alpha_1 \le (\alpha-1); \end{cases} \tag{A.75}$$

$$\left.\frac{\partial \mathcal{O}_s}{\partial \alpha_1}\right|_{\alpha_1=(\alpha-1),\alpha_i=\alpha\ \forall\ i\ge 3,\beta_1=1,\gamma_1=1} = (4n-3), \text{for } \alpha_2 \ge (\alpha-1). \tag{A.76}$$

Comparing equations (A.74) and (A.75), we realize that for $0 \le r_s \le 1$, the steepest descent is along the direction of decreasing α_1, while $\beta_1 = \gamma_1 = 1$. On the other hand, comparing equations (A.74), (A.75) and (A.76) it is clear that beyond $r_s = 1$, decreasing α_2 has the steepest descent than β_1, γ_1 and even α_1. In the same way it can be proved that for $(i-1) \le r_s \le i$, the steepest descent of the function is along decreasing values of α_i, while $\beta_1 = \gamma_1 = 1$. Note that for these values of β_1 and γ_1 the last two terms of equation (A.65a) vanishes and equations (A.65d)-(A.65g) becomes redundant and as a result the optimization problem of (A.65) simplifies to the following:

$$\min \sum_{i=1}^{n} (4n+1-2i)\upsilon_i - 2n^2(\alpha-1); \tag{A.77a}$$

$$\text{constrained by: } \sum_{i=1}^{n} (\alpha-\upsilon_i)^+ \leq r_s; \tag{A.77b}$$

$$(\alpha-1) \leq \upsilon_1 \leq \cdots \leq \upsilon_n. \tag{A.77c}$$

By the aforementioned argument, each of the υ_i's can decrease with the constraint $\upsilon_i \geq (\alpha-1)$ with increasing r_s. Therefore, substituting $\upsilon_i' = \upsilon_i - (\alpha-1)$ for all i in the above set of equations we obtain the following equivalent optimization problem:

$$\min \sum_{i=1}^{n} (4n+1-2i)\upsilon_i' + n^2(\alpha-1); \tag{A.78a}$$

$$\text{constrained by: } \sum_{i=1}^{n} (1-\upsilon_i')^+ \leq r_s; \tag{A.78b}$$

$$0 \leq \upsilon_1' \leq \cdots \leq \upsilon_n', \tag{A.78c}$$

which in turn by Lemma 5.3 have the following optimal value

$$d_s(r_s) = \alpha d_{n,3n}(r_s) + n^2(\alpha-1), \; 0 \leq r_s \leq n. \tag{A.79}$$

It is clear from this solution that for $r_s \geq n$, $\alpha_i \leq (\alpha-1)$ for all i, and the optimization problem (A.65) reduces to the following:

$$\min \mathcal{O}_s \triangleq \min_{(\vec{\upsilon},\vec{\gamma},\vec{\beta},\bar{\alpha})} \sum_{i=1}^{n}(4n+1-2i)\upsilon_i + \sum_{j=1}^{n}(2n+1-2j)\beta_j + \sum_{k=1}^{n}(2n+1-2k)\gamma_k$$
$$-2n^2\alpha + \sum_{k=1}^{n}\sum_{i=1}^{(n-k)}(\alpha-\upsilon_i-\gamma_k)^+ + \sum_{j=1}^{n}\sum_{i=1}^{(n-j)}(\alpha-\upsilon_i-\beta_j)^+; \tag{A.80a}$$

$$\text{constrained by: } \sum_{i=1}^{n}(\alpha-1-\upsilon_i) + \sum_{j=1}^{n}(1-\beta_j)^+ + \sum_{k=1}^{n}(1-\gamma_k)^+ \le (r_s-n); \tag{A.80b}$$

$$0 \le \upsilon_1 \le \cdots \le \upsilon_n \le (\alpha-1); \tag{A.80c}$$

$$0 \le \beta_1 \le \cdots \le \beta_n; \tag{A.80d}$$

$$0 \le \gamma_1 \le \cdots \le \gamma_n; \tag{A.80e}$$

$$(\upsilon_i+\beta_j) \ge \alpha,\ \forall (i+j) \ge (n+1); \tag{A.80f}$$

$$(\upsilon_i+\gamma_k) \ge \alpha,\ \forall (i+k) \ge (n+1). \tag{A.80g}$$

Note the upper bound in equation (A.80c), which is different from equation (A.65c). Also, since we are seeking for the minimum value of the objective function which decreases with every β_i and γ_j, in the optimal solution these parameters must take their minimum value, which from equations (A.80f) and (A.80g) is given by

$$\beta_j|_{\min} = \gamma_j|_{\min} = \alpha - \upsilon_{n-j+1}, \tag{A.81}$$

which along with the ordering among the υ_i's, β_j's and γ_j's imply that all the terms in the last two summands of the objective function are non-negative. Substituting these minimum values and thereby eliminating β_j's and γ_j's from the optimization problem we obtain the following equivalent optimization problem:

$$\min \sum_{i=1}^{n} (2n+1-2i)\upsilon_i, \tag{A.82a}$$

$$\text{constrained by:} \ \sum_{i=1}^{n} (\alpha - 1 - \upsilon_i) \leq (r_s - n); \tag{A.82b}$$

$$0 \leq \upsilon_1 \leq \cdots \leq \upsilon_n \leq (\alpha - 1), \tag{A.82c}$$

where the simplification of the objective function involves the following algebraic computations

$$\begin{aligned}
\mathcal{O}_s =& \sum_{i=1}^{n}(4n+1-2i)\upsilon_i + 2\sum_{j=1}^{n}(2n+1-2j)(\alpha - \upsilon_{n-j+1}) - 2n^2\alpha \\
&+ \sum_{k=1}^{n}\sum_{i=1}^{(n-k)}(\alpha - \upsilon_i - (\alpha - \upsilon_{n-k+1})) + \sum_{j=1}^{n}\sum_{i=1}^{(n-j)}(\alpha - \upsilon_i - (\alpha - \upsilon_{n-j+1})); \\
=& \sum_{i=1}^{n}(4n+1-2i)\upsilon_i + 2\sum_{j=1}^{n}(2n+1-2j)(\alpha - \upsilon_{n-j+1}) - 2n^2\alpha \\
&+ n(n-1)\alpha - \sum_{i=1}^{n} 2(n-i)\upsilon_i - \sum_{k=1}^{n}(\alpha - \upsilon_{n-k+1}) - \sum_{j=1}^{n}(n-j)(\alpha - \upsilon_{n-j+1}); \\
=& \sum_{i=1}^{n}(2n+1)\upsilon_i + 2\sum_{j=1}^{n}(n+1-j)(\alpha - \upsilon_{n-j+1}) - n(n+1)\alpha \\
=& \sum_{i=1}^{n}(2n+1-2i)\upsilon_i + 2\sum_{j=1}^{n}(n+1-j)\alpha - n(n+1)\alpha \\
=& \sum_{i=1}^{n}(2n+1-2i)\upsilon_i.
\end{aligned}$$

The solution of this optimization problem follows again from Lemma 5.3 and is given by

$$\begin{aligned}
d_s(r_s) =& (\alpha - 1) d_{n,n}\left(\frac{r_s - n}{(\alpha - 1)}\right), \ 0 \leq (r_s - n) \leq n(\alpha - 1) \\
=& (\alpha - 1) d_{n,n}\left(\frac{r_s - n}{(\alpha - 1)}\right), \ n \leq r_s \leq n\alpha.
\end{aligned} \tag{A.83}$$

Combining equations (A.79) and (A.83) we obtain equation (5.31) of Theorem 5.2.

A.15 Proof of Theorem 5.3

When we substitute the specific values of the different parameters in equation (5.28), it reduces to the following

$$d^{\text{Femto}}_{s,(n,n,n,n)}(r_s) = \min \sum_{i=1}^{n}(4n+1-2i)\upsilon_i + \sum_{j=1}^{n}(2n+1-2j)\beta_j + \sum_{k=1}^{n}(2n+1-2k)\gamma_k$$
$$- 2n^2 + \sum_{k=1}^{n}\sum_{i=1}^{(n-k)}(1-\upsilon_i-\gamma_k)^+ + \sum_{j=1}^{n}\sum_{i=1}^{(n-j)}(1-\upsilon_i-\beta_j)^+; \tag{A.84a}$$
$$\text{constrained by: } \sum_{i=1}^{n}(1-\upsilon_i)^+ + \sum_{j=1}^{n}(1-\beta_j)^+ + \sum_{k=1}^{n}(\alpha-\gamma_k)^+ < r_s; \tag{A.84b}$$
$$0 \le \upsilon_1 \le \cdots \le \upsilon_n; \tag{A.84c}$$
$$0 \le \beta_1 \le \cdots \le \beta_n; \tag{A.84d}$$
$$0 \le \gamma_1 \le \cdots \le \gamma_n; \tag{A.84e}$$
$$(\upsilon_i + \beta_j) \ge 1,\ \forall (i+j) \ge (n+1); \tag{A.84f}$$
$$(\upsilon_i + \gamma_k) \ge 1,\ \forall (i+k) \ge (n+1). \tag{A.84g}$$

Differentiating the objective function with respect to υ_i, $\forall\ i, \beta_1$ and γ_1 we find that

$$\left.\frac{\partial \mathcal{F}_{\text{Femto}}}{\partial \upsilon_i}\right|_{\beta_1=1,\gamma_1=\alpha} = (4n+1-2i) \ge \left.\frac{\partial \mathcal{F}_{\text{Femto}}}{\partial \beta_1}\right|_{\gamma_1=\alpha,\upsilon_k=0,\ \forall k<i,\upsilon_k=1,\forall k>i} = (2n-i), \tag{A.85}$$
$$= \left.\frac{\partial \mathcal{F}_{\text{Femto}}}{\partial \gamma_1}\right|_{\beta_1=1,\upsilon_k=0,\ \forall k<i,\upsilon_k=1,\forall k>i},\ \forall\ i \le n, \tag{A.86}$$

where we have denoted the objective function by $\mathcal{F}_{\text{Femto}}$. It is clear form these values that, for $(i-1) \le r_s \le i$, the steepest descent is along decreasing υ_i with $\beta_1 = 1$ and $\gamma_1 = \alpha$. Putting this in equation (A.84) we get

$$d^{\text{Femto}}_{s,(n,n,n,n)}(r_s) = \min \sum_{i=1}^{n}(4n+1-2i)\upsilon_i + n^2(\alpha-1); \tag{A.87a}$$
$$\text{constrained by: } \sum_{i=1}^{n}(1-\upsilon_i)^+ \le r_s; \tag{A.87b}$$
$$0 \le \upsilon_1 \le \cdots \le \upsilon_n. \tag{A.87c}$$

Now, using Lemma 5.3 we obtain the minimum value of the the above optimization problem, which can be written as

$$d^{\text{Femto}}_{s,(n,n,n,n)}(r_s) = d_{n,3n}(r_s) + n^2(\alpha - 1),\ 0 \le r_s \le n. \tag{A.88}$$

Next, we obtain the optimal value of the objective function for values of $r_s \ge n$. Note that for any $r_s \ge n$, $v_i = 0\ \forall\ i$, which along with equations (A.84f) and (A.84g) imply that $\beta_j \ge 1$, and $\gamma_j \ge 1\ \forall j$. Putting this in equation (A.84) we get

$$d^{\text{Femto}}_{s,(n,n,n,n)}(r_s) = \min \sum_{i=1}^{n} (2n + 1 - 2i)\gamma_i - n^2, \tag{A.89a}$$

$$\text{constrained by:} \sum_{i=1}^{n} (\alpha - \gamma_i)^+ \le (r_s - n); \tag{A.89b}$$

$$1 \le \gamma_1 \le \cdots \le \gamma_n. \tag{A.89c}$$

To bring the above problem into a form amenable to Lemma 5.3 we use the following variable transform in the above set of equations: $\gamma'_i = \gamma_i - 1$. This results in the following equivalent optimization problem.

$$d^{\text{Femto}}_{s,(n,n,n,n)}(r_s) = \min \sum_{i=1}^{n} (2n + 1 - 2i)\gamma'_i, \tag{A.90a}$$

$$\text{constrained by:} \sum_{i=1}^{n} (\alpha - 1 - \gamma'_i)^+ \le (r_s - n); \tag{A.90b}$$

$$0 \le \gamma'_1 \le \cdots \le \gamma'_n, \tag{A.90c}$$

which in turn by Lemma 5.3 attains the following optimal value:

$$d^{\text{Femto}}_{s,(n,n,n,n)}(r_s) = (\alpha - 1) d_{n,n}\left(\frac{(r_s - n)}{(\alpha - 1)}\right),\ 0 \le \frac{(r_s - n)}{(\alpha - 1)} \le n \tag{A.91}$$

$$= (\alpha - 1) d_{n,n}\left(\frac{(r_s - n)}{(\alpha - 1)}\right),\ n \le r_s \le n\alpha. \tag{A.92}$$

Finally, combining equations (A.88) and (A.91) we obtain the desired result.

A.16 Proof of Theorem 5.4

Substituting $M_1 = M_2 = M$ and $\alpha_{ij} = 1$ in equation (5.28) we obtain

$$\begin{aligned} d^{\text{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s) =& \sum_{i=1}^{M}(2M + N_1 + N_2 + 1 - 2i)\upsilon_i + \sum_{j=1}^{M}(M + N_1 + 1 - 2j)\beta_j + \sum_{k=1}^{M}(M + N_2 + 1 - 2k)\gamma_k \\ & - (M + N_2)M + \sum_{k=1}^{M}\sum_{i=1}^{(M-k)}(1 - \upsilon_i - \gamma_k)^+ + \sum_{j=1}^{M}\sum_{i=1}^{\min\{(N_1-j),M\}}(1 - \upsilon_i - \beta_j)^+; \end{aligned} \tag{A.93a}$$

$$\text{constrained by: } \sum_{i=1}^{p}(1 - \upsilon_i)^+ + \sum_{j=1}^{q_1}(1 - \beta_j)^+ + \sum_{k=1}^{q_2}(1 - \gamma_k)^+ < r_s; \tag{A.93b}$$

$$0 \le \upsilon_1 \le \cdots \le \upsilon_p; \tag{A.93c}$$

$$0 \le \beta_1 \le \cdots \le \beta_{q_1}; \tag{A.93d}$$

$$0 \le \gamma_1 \le \cdots \le \gamma_{q_2}; \tag{A.93e}$$

$$(\upsilon_i + \beta_j) \ge 1,\ \forall (i + j) \ge (N_1 + 1); \tag{A.93f}$$

$$(\upsilon_i + \gamma_k) \ge 1,\ \forall (i + k) \ge (M + 1). \tag{A.93g}$$

Differentiating the objective function with respect to $\upsilon_i,\ \forall\ i, \beta_1$ and γ_1 we find that

$$\left.\frac{\partial \mathcal{F}_{\text{a-FCSIT}}}{\partial \upsilon_i}\right|_{\beta_1=1,\gamma_1=1} \ge \begin{cases} \left.\frac{\partial \mathcal{F}_{\text{a-FCSIT}}}{\partial \beta_1}\right|_{\gamma_1=1,\alpha_k=0,\ \forall k<i,\alpha_k=1,\forall k>i}, \\ \left.\frac{\partial \mathcal{F}_{\text{a-FCSIT}}}{\partial \gamma_1}\right|_{\beta_1=1,\alpha_k=0,\ \forall k<i,\alpha_k=1,\forall k>i}, \end{cases} \quad \forall\ i \le n, \tag{A.94}$$

where we have denoted the objective function by $\mathcal{F}_{\text{a-FCSIT}}$. It is clear form these values that, for $(i-1) \le r_s \le i$, the steepest descent is along decreasing υ_i with $\beta_1 = \gamma_1 = 1$ and $i \le n$. Putting this in equation (A.93) we get

$$d^{\text{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s) = \min \sum_{i=1}^{n}(2M + N_1 + N_2 + 1 - 2i)\upsilon_i + M(N_1 - M); \tag{A.95a}$$

$$\text{constrained by: } \sum_{i=1}^{M}(1 - \upsilon_i)^+ \le r_s; \tag{A.95b}$$

$$0 \le \upsilon_1 \le \cdots \le \upsilon_n. \tag{A.95c}$$

Now, using Lemma 5.3 we obtain the minimum value of the the above optimization problem, which can be written as

$$d^{\text{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s) = d_{M,M+N_1+N_2}(r_s) + M(N_1 - M),\ 0 \le r_s \le M. \tag{A.96}$$

Next, we evaluate the optimal value of the objective function for values of $r_s \geq M$. Note that for any $r_s \geq M$, $\upsilon_i = 0\ \forall\ i$, which along with equations (A.93f) and (A.93g) imply that $\beta_j \geq 1$ for $j \geq (N_1 + 1 - M)$ and $\gamma_k \geq 1\ \forall k$. Clearly, the objective function is minimized for $\beta_j = 1$ for $j \geq (N_1 + 1 - M)$ and $\gamma_k = 1\ \forall k$. Putting this in equation (A.93) we get

$$d^{\text{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s) = \min \sum_{j=1}^{M} (M + N_1 + 1 - 2j)\beta_j - M^2 + \sum_{j=1}^{M} \min\{(N_1 - j), M\}(1 - \beta_j)^+, \tag{A.97a}$$

$$\text{constrained by:} \sum_{i=1}^{M} (1 - \beta_j) \le (r_s - M); \tag{A.97b}$$

$$0 \le \beta_1 \le \cdots \le \beta_M, \tag{A.97c}$$

where $\beta_u = 1$ for $u \geq \min\{(N_1 - M), M\}$. Since $\beta_u = 1$ only for $u \geq \min\{(N_1 - M), M\}$ the last term in equation (A.97a) reduces to

$$\sum_{j=1}^{M} \min\{(N_1 - j), M\}(1 - \beta_j)^+ = \sum_{j=1}^{\tau} M(1 - \beta_j),$$

where we denote $\min\{(N_1 - M), M\} = \tau$. When we substitute this, the last optimization problem further reduces to the following:

$$\begin{aligned} d^{\text{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s) &= \min \sum_{j=1}^{\tau} (N_1 + 1 - 2j)\beta_j + (M - \tau)(N_1 - \tau) - M^2 + \tau M, \qquad \text{(A.98a)} \\ &= \min \sum_{j=1}^{\tau} (N_1 + 1 - 2j)\beta_j + (M - \tau)(N_1 - M - \tau), \\ &= \min \sum_{j=1}^{\tau} (\max\{N_1 - M, M\} + \tau + 1 - 2j)\beta_j, \end{aligned}$$

$$\text{constrained by:} \sum_{i=1}^{\tau} (1 - \beta_j) \le (r_s - M); \tag{A.98b}$$

$$0 \le \beta_1 \le \cdots \le \beta_\tau. \tag{A.98c}$$

Using Lemma 5.3 in the above optimization then it yields the following solution:

$$\begin{aligned} d^{\mathrm{FCSIT}}_{s,(M,N_1,M,N_2)}(r_s) =& d_{\tau,\max\{N_1-M,M\}}(r_s-M),\ 0\leq r_s-M\leq\tau; \\ =& d_{2M,N_1}(r_s),\ M\leq r_s\leq\min\{N_1,2M\}. \end{aligned} \tag{A.99}$$

Finally, combining equations (A.96) and (A.99) we obtain the desired result.

A.17 Proof of Lemma 5.5

Let us denote the event that a target rate tuple $(r_1\log(\rho), r_2\log(\rho))$ does not belong to $\mathcal{R}_{\mathrm{IML}}$ by $\mathcal{O}_{\mathrm{IML}}$, i.e.,

$$\mathcal{O}_{\mathrm{IML}} = \Big\{\mathcal{H}:\ (r_1\log(\rho), r_2\log(\rho))\notin\mathcal{R}_{\mathrm{IML}}\Big\}.$$

Now, let us denote the maximum among the average probability of errors at both the receivers be denoted by $\mathcal{P}_e$, then using Bayes' rule we get

$$\mathcal{P}_e = \mathcal{P}_{e|\mathcal{O}_{\mathrm{IML}}}\Pr\{\mathcal{O}_{\mathrm{IML}}\} + \mathcal{P}_{e|\mathcal{O}^c_{\mathrm{IML}}}\Pr\{\mathcal{O}^c_{\mathrm{IML}}\}, \tag{A.100}$$

$$\leq \Pr\{\mathcal{O}_{\mathrm{IML}}\} + \mathcal{P}_{e|\mathcal{O}^c_{\mathrm{IML}}}, \tag{A.101}$$

where $\mathcal{P}_{e|\mathcal{E}}$ denote the conditional average probability of error given the event $\mathcal{E}$. Note that, the above equation holds for any SNR. When the target rate tuple belongs to $\mathcal{R}_{\mathrm{IML}}$, letting the block length be sufficiently large the probability of error given $\mathcal{O}^c_{\mathrm{IML}}$ can be be made arbitrarily close to zero. Therefore, letting the block length of the code goes to infinity at both side of the above equation we obtain

$$\mathcal{P}_e \ \leq \Pr\{\mathcal{O}_{\mathrm{IML}}\} = \Pr\left\{\{I_{c_1}\leq r_1\log(\rho)\}\cup\{I_{c_2}\leq r_2\log(\rho)\}\cup\{I_{c_s}\leq r_s\log(\rho)\}\right\}, \tag{A.102}$$

$$\leq \sum_{i=1,2,s}\Pr\left\{I_{c_i}\leq r_i\log(\rho)\right\}, \tag{A.103}$$

$$\overset{(a)}{\doteq} \max_{i=1,2,s}\Pr\left\{I_{c_i}\leq r_i\log(\rho)\right\}, \tag{A.104}$$

$$\doteq \max_{i=1,2,s}\rho^{-d^{\mathrm{IML}}_{i,(M_1,N_1,M_2,N_2)}(r_i)} = \rho^{-\min_{i=1,2,s}\{d^{\mathrm{IML}}_{i,(M_1,N_1,M_2,N_2)}(r_i)\}}, \tag{A.105}$$

where step (a) follows from the fact that in the asymptotic SNR the largest term dominates and the last step follows from equation (5.34). Finally, the desired result follows from the fact that $\mathcal{P}_e \dot{=} \rho^{-d^{\text{IML}}_{(M,N_1,M,N_2)}(r_1,r_2)}$.

A.18 Proof of Lemma 5.6

The joint distribution of $(\bar{\beta}, \bar{v})$ can be obtained from equation (A.61) substituting $f_{W_2|W_3}(.) = 1$. The rest of the proof follows the same steps as the proof of Lemma 5.4 and hence skipped to avoid repeating.

A.19 Proof of Theorem 5.6

Since the FCSIT DMT is an upper bound to the No-CSIT DMT, it is sufficient to prove that when the condition of equation (5.43) is satisfied the expressions of Theorem 5.2 and Lemma 5.7 are identical.

It is clear from the comparison of equations (5.31) and (5.42) that for $r_s = 2r \geq n$ they are identical for all values of $\alpha \geq 1$ and hence when α satisfies equation (5.43). Now, it is only necessary to find a condition when the expressions in equations (5.31) and (5.42) are identical even when $r_s < n$, which is what we derive next and turns out to be identical to the condition of equation (5.43).

It is clear from equation (5.42) and (5.31) that,

$$d^{\text{IML}}_{s,(n,n,n,n)}(r_s) < d^{\text{FCSIT}}_{s,(n,n,n,n)}(r_s), \text{ for } r_s \leq n.$$

Therefore, the DMTs given by Theorem 5.2 and Lemma 5.7 are identical only if for $r_s \leq n$, the single user performance is dominating, i.e.,

$$\begin{aligned} d_{n,n}(r) \leq & d^{\text{IML}}_{s,(n,n,n,n)}(r_s = 2r); \\ d_{n,n}(r) \leq & d_{n,2n}(2r) + n^2(\alpha - 1), \end{aligned} \tag{A.106}$$

where in the last step we have substituted the value of $d^{\text{IML}}_{s,(n,n,n,n)}(2r)$ from equation (5.42). Since $d^{\text{IML}}_{s,(n,n,n,n)}(2r)$ decays much faster than $d_{n,n}(r)$ with increasing r and both are continuous functions of r, equation (A.106) will be valid for all $r \leq \frac{n}{2}$ if it is true for $r = \frac{n}{2}$. Substituting this in equation (A.106) we get

$$\begin{aligned} d_{n,n}\binom{n}{2} \leq & n^2(\alpha - 1), \\ \text{or, } \alpha \geq & 1 + \frac{d_{n,n}\binom{n}{2}}{n^2}. \end{aligned}$$

Appendix B

Relay channel

B.1 Proof of Lemma 6.1

Let the singular value decomposition of H_{SD} be given as $U\Lambda_0 V^\dagger$, where $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ are mutually independent unitary random matrices distributed uniformly over the set of square unitary matrices of corresponding dimensions (e.g., see equation (3.9) in [92]). Using this fact we can write

$$H_{SD}H_{SD}^\dagger = U\Lambda U^\dagger, \text{ and } H_{SD}^\dagger H_{SD} = V\Lambda_2 V^\dagger, \tag{B.1}$$

where the sets of non-zero elements of Λ_2 and Λ are identical. In particular, given one, the other is fixed. Putting this in the expressions for W_2 and W_3 we get

$$W_2 = H_{SR}V\left(I_m + \Lambda_2\right)^{-1} V^\dagger H_{SR}^\dagger = \widetilde{H}_{SR}\left(I_m + \Lambda_2\right)^{-1}\widetilde{H}_{SR}^\dagger, \tag{B.2}$$

$$W_3 = H_{RD}^\dagger U\left(I_n + \Lambda\right)^{-1} U^\dagger H_{RD} = \widetilde{H}_{RD}^\dagger\left(I_n + \Lambda\right)^{-1}\widetilde{H}_{RD}, \tag{B.3}$$

where $\widetilde{H}_{SR} = H_{SR}V$ and $\widetilde{H}_{RD} = U^\dagger H_{RD}$ are mutually independent random matrices that have the same distributions as H_{SR} and H_{RD}, respectively, since H_{SR} and H_{RD} are unitarily invariant (cf. [7]). Letting $A = \widetilde{H}_{SR}\left(I_m + \Lambda_2\right)^{-\frac{1}{2}}$ and $B = \widetilde{H}_{RD}^\dagger\left(I_n + \Lambda\right)^{-\frac{1}{2}}$ we realize that both A and B still have mutually independent Gaussian entries conditioned on Λ. Computing the conditional correlation between the two we get

$$\begin{aligned}
\mathbb{E}\left(B^\dagger A|\Lambda_2,\Lambda\right) =& \mathbb{E}\left(\left(I_n+\Lambda\right)^{-\frac{1}{2}}\widetilde{H}_{RD}\widetilde{H}_{SR}\left(I_m+\Lambda_2\right)^{-\frac{1}{2}}|\Lambda_2,\Lambda\right),\\
=& \mathbb{E}\left(\left(I_n+\Lambda\right)^{-\frac{1}{2}}U^\dagger H_{RD}H_{SR}V\left(I_m+\Lambda_2\right)^{-\frac{1}{2}}|\Lambda_2,\Lambda\right) = 0_{k\times k},
\end{aligned}$$

where the last step follows from the fact that components of H_{SR} and H_{RD} are zero mean and mutually independent. This, along with the fact that A and B are Gaussian [89] proves that conditioned on Λ_2 or Λ, they are independent. This in turn implies that $W_2 = AA^\dagger$ and $W_3 = BB^\dagger$ are independent given Λ. Consequently, the eigenvalues of W_2 are independent of the eigenvalues of W_3 given Λ.

B.2 Proof of Lemma 6.3

B.2.1 Proof of Part i

The proof consists of upper bounding in two steps the mutual information terms of the form $I(X; X + Z)$, subject to a sum power constraint on X. First, we use the fact Gaussian input is optimal, and then in the second step, we use the monotonically increasing property of the $\log\det(.)$ function in the cone of positive semi-definite matrices.

Suppose $Y = HX + Z$, where $Z \sim \mathcal{CN}(0, I)$, $H \in \mathbb{C}^{N\times M}$ and $\mathrm{Cov}(X) \preceq Q$, then it is well known [87] that

$$\max_{\{\mathrm{Cov}(X)\preceq Q\}} I(HX + Z; X) = I(HX_G + Z; X_G) = \log\det\left(I + HQH^\dagger\right), \tag{B.4}$$

where $X_G \sim \mathcal{CN}(0, Q)$. Similarly, for a sum power constraint we have

$$\begin{aligned}\max_{\{\mathrm{Tr}(\mathrm{Cov}(X))\preceq\rho\}} I(HX + Z; X) &= \max_{\mathrm{Tr}(Q)\le\rho} \max_{\{\mathrm{Cov}(X)\preceq Q\}} I(HX + Z; X)\\ &\overset{(a)}{=} \max_{\mathrm{Tr}(Q)\le\rho} \log\det\left(I + HQH^\dagger\right), &\text{(B.5)}\\ &\le \log\det\left(I + \rho HH^\dagger\right), &\text{(B.6)}\end{aligned}$$

where step (a) follows from (B.4) and the last step follows from the fact that $Q \preceq \rho I$ and $\log\det(.)$ is a monotonically increasing function in the cone of semi-definite matrices.

Using equation (B.6) we have

$$\begin{aligned}\max_{P(X_S,X_R)} I(X_S, X_R; Y_D|p_2) &= \max_{P(X_S,X_R)} I\left(H_{SR,D}\begin{bmatrix} X_S \\ X_R\end{bmatrix} + Z_D; \begin{bmatrix} X_S \\ X_R\end{bmatrix}\right)\\ &\le \log\det\left(I_n + \rho H_{SR,D}H_{SR,D}^\dagger\right) = \log(L_{SR,D}),\end{aligned}$$

where $H_{SR,D} = [H_{SD}\ H_{RD}]$. Using a similar method we obtain

$$\max_{P(X_S,X_R)} I(X_S;Y_D|X_R,p_2) \leq \log\det\left(I_n + \rho H_{SD}H_{SD}^{\dagger}\right) = \log(L_{SD});$$
$$\max_{P(X_S,X_R)} I(X_S;Y_D|p_1) \leq \log\det\left(I_n + \rho H_{SD}H_{SD}^{\dagger}\right) = \log(L_{SD});$$
$$\max_{P(X_S,X_R)} I(X_S;Y_R,Y_D|p_1) \leq \log\det\left(I_n + \rho H_{S,RD}H_{S,RD}^{\dagger}\right) = \log(L_{S,RD}).$$

Finally, substituting the above set of upper bounds in equation (6.33) and (6.34) we get

$$\max_{P(X_S,X_R)} I_{C_S}(t_d) \leq t_d \log\left(L_{S,RD}\right) + (1-t_d)\log\left(L_{SD}\right) \triangleq I_{C_S}^{'}(t_d),$$
$$\max_{P(X_S,X_R)} I_{C_D}(t_d) \leq t_d \log\left(L_{SD}\right) + (1-t_d)\log\left(L_{SR,D}\right) \triangleq I_{C_D}^{'}(t_d).$$

This proves the first part of the lemma.

B.2.2 Proof of Part ii

Let P^* represent the distribution where $X_S \sim \mathcal{CN}(0, \frac{\rho}{m}I_m)$ and $X_R \sim \mathcal{CN}(0, \frac{\rho}{k}I_k)$ and X_S and X_R are mutually independent. Note that P^* satisfies the input power constraints at the source and relay given in (6.5) and (6.6). Then denoting the mutual information $I(X_S, X_R; Y_D|p_2)$ evaluated at P^* by $I(X_S, X_R; Y_D|p_2)\Big|_{P^*}$ we see

$$\begin{aligned}
I(X_S, X_R; Y_D|p_2)\Big|_{P^*} &= I\left(H_{SR,D}\begin{bmatrix} X_S \\ X_R \end{bmatrix} + Z_D; \begin{bmatrix} X_S \\ X_R \end{bmatrix}\right)\Bigg|_{P^*}, \\
&= \log\det\left(I_n + \frac{\rho}{m}H_{SD}H_{SD}^{\dagger} + \frac{\rho}{k}H_{RD}H_{RD}^{\dagger}\right), \\
&\overset{(a)}{\geq} \log\det\left(\frac{I_n}{m+k} + \frac{\rho}{m+k}H_{SD}H_{SD}^{\dagger} + \frac{\rho}{m+k}H_{RD}H_{RD}^{\dagger}\right), \\
&= \log\det\left(I_n + \rho H_{SR,D}H_{SR,D}^{\dagger}\right) - (m+k), \\
&= \log(L_{SR,D}) - m
\end{aligned} \tag{B.7}$$

where step (a) follows from the fact that $\log\det(.)$ is a monotonically increasing function in the cone of positive semi-definite matrices. Using a similar method we get

$$I(X_S;Y_D|X_R,p_2)\Big|_{P^*} \geq \log\det\left(I_n+\rho H_{SD}H_{SD}^{\dagger}\right)-m=\log(L_{SD})-m; \tag{B.8}$$

$$I(X_S;Y_D|p_1)\Big|_{P^*} \geq \log\det\left(I_n+\rho H_{SD}H_{SD}^{\dagger}\right)-m=\log(L_{SD})-m; \tag{B.9}$$

$$I(X_S;Y_R,Y_D|p_1)\Big|_{P^*} \geq \log\det\left(I_n+\rho H_{S,RD}H_{S,RD}^{\dagger}\right)-m=\log(L_{S,RD})-m. \tag{B.10}$$

Now, from the definition of $\bar{C}(\mathcal{H},t_d)$ in equation (6.32) we get

$$\begin{aligned}
\bar{C}(\mathcal{H},t_d) =& \max_{\{P(X_S,X_R)\}} \min\{I_{C_S}(t_d), I_{C_D}(t_d)\},\\
&\overset{(a)}{\geq} \max_{\{P(X_S,X_R)=P^*\}} \min\left\{I_{C_S}(t_d), I_{C_D}(t_d)\right\},\\
&= \min\left\{I_{C_S}(t_d)\Big|_{P^*}, I_{C_D}(t_d)\Big|_{P^*}\right\},\\
&\overset{(b)}{\geq} \min\{I'_{C_S}(t_d)-m, I'_{C_D}(t_d)-t_dm-(1-t_d)(m+k)\},\\
&\geq \min\{I'_{C_S}(t_d), I'_{C_D}(t_d)\}-(m+k),
\end{aligned}$$

where step (a) follows from the fact that instead of maximizing over all possible input distributions, we are evaluating the right hand side of the equation at a particular distribution P^* and in step (b) we substituted the set of lower bounds from equations (B.7)-(B.10) in the expressions for $I_{C_S}(t_d)$ and $I_{C_D}(t_d)$.

B.3 Proof of Lemma 6.4

We shall prove that

$$R_U^* = r^*(\bar{\alpha},\bar{\beta},\bar{\delta})\log(\rho),$$

which, when substituted in equation (6.48), proves the lemma. From the expression of $L_{S,RD}$ in equation (6.39) and using elementary properties of determinants, we have

$$\begin{aligned}
\log(L_{S,RD}) =& \log\det\left(I_{n+k} + \rho H_{S,RD} H_{S,RD}^{\dagger}\right), \\
=& \log\det\left(I_m + \rho H_{S,RD}^{\dagger} H_{S,RD}\right) \overset{(a)}{=} \log\det\left(I_m + \rho H_{SR}^{\dagger} H_{SR} + \rho H_{SD}^{\dagger} H_{SD}\right), \\
=& \log\det\left(I_m + \rho H_{SR}^{\dagger} H_{SR}\left(I_m + \rho H_{SD}^{\dagger} H_{SD}\right)^{-1}\right) + \log\det\left(I_m + \rho H_{SD}^{\dagger} H_{SD}\right), \\
=& \log\det\left(I_k + \rho H_{SR}\left(I_m + \rho H_{SD}^{\dagger} H_{SD}\right)^{-1} H_{SR}^{\dagger}\right) + \log(L_{SD}).
\end{aligned}$$

where in step (a) we have used the fact that $H_{S,RD}^{\dagger} = [H_{SR}^{\dagger}\ H_{SD}^{\dagger}]$. Hence, we have

$$\log\left(\frac{L_{S,RD}}{L_{SD}}\right) = \log\det\left(I_k + \rho H_{SR}\left(I_m + \rho H_{SD}^{\dagger} H_{SD}\right)^{-1} H_{SR}^{\dagger}\right).$$

Similarly, from the expression for $L_{SR,D}$ in equation (6.38) we get

$$\log\left(\frac{L_{SR,D}}{L_{SD}}\right) = \log\det\left(I_k + \rho H_{RD}^{\dagger}\left(I_n + \rho H_{SD} H_{SD}^{\dagger}\right)^{-1} H_{SR}\right). \tag{B.11}$$

Now, assuming $0 < \lambda_u \leq \cdots \leq \lambda_1$, $0 < \mu_p \leq \cdots \leq \mu_1$ and $0 < \gamma_q \leq \cdots \leq \gamma_1$ represent the ordered non-zero (w.p. 1) eigenvalues of the matrices $W_1 \triangleq H_{SD} H_{SD}^{\dagger}$, $W_2 \triangleq H_{SR}(I_n + \rho H_{SD}^{\dagger} H_{SD})^{-1} H_{SR}^{\dagger}$ and $W_3 \triangleq H_{RD}^{\dagger}(I_n + \rho H_{SD} H_{SD}^{\dagger})^{-1} H_{RD}$ and substituting these in equation (6.42) we get

$$R_U^* = \frac{\log\left(\prod_{j=1}^{p}(1+\rho\mu_j)\right)\log\left(\prod_{l=1}^{q}(1+\rho\gamma_l)\right)}{\log\left(\prod_{j=1}^{p}(1+\rho\mu_j)\right) + \log\left(\prod_{l=1}^{q}(1+\rho\gamma_l)\right)} + \log\left(\prod_{i=1}^{u}(1+\rho\alpha_i)\right), \tag{B.12}$$

To further simplify the expression on the right hand side of the above equation we use the following transformations: $\lambda_i = \rho^{-\alpha_i},\ 1 \leq i \leq u$, $\mu_j = \rho^{-\beta_j},\ 1 \leq j \leq p$ and $\gamma_l = \rho^{-\delta_l},\ 1 \leq l \leq q$ and get

$$R_U^* = \log(\rho)\left[\frac{\sum_{l=1}^{q}(1-\delta_l)^{+}\sum_{j=1}^{p}(1-\beta_j)^{+}}{\sum_{l=1}^{q}(1-\delta_l)^{+} + \sum_{j=1}^{p}(1-\beta_j)^{+}}\right] + \log(\rho)\left(\sum_{i=1}^{u}(1-\alpha_i)^{+}\right) \triangleq r^*(\bar{\alpha},\bar{\beta},\bar{\delta})\log(\rho). \tag{B.13}$$

B.4 Proof of theorem 6.3

From equation (6.48) $d^*(r)$ is equal to the negative SNR exponent of $\Pr\{\mathcal{O}_U(r)\}$. However, from Lemma 6.4, $\Pr\{\mathcal{O}_U(r)\}$ is exponentially equal to $\Pr\{r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta}) < r\}$. Hence,

$$\Pr\{\mathcal{O}_U(r)\} \doteq \rho^{-d^*(r)} \tag{B.14}$$

$$\doteq \Pr\{r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta}) < r\} \tag{B.15}$$

$$= \int_{\{(\bar{\alpha},\bar{\beta},\bar{\delta}) \in \mathcal{O}_U(r)\}} f\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right) \, d\bar{\alpha} \, d\bar{\beta} \, d\bar{\delta},$$

$$= \int_{\{(\bar{\alpha},\bar{\beta},\bar{\delta}) \in \mathcal{O}_U(r) \cap \mathcal{S}\}} \rho^{-E\left(\bar{\alpha},\bar{\beta},\bar{\gamma}\right)} \, d\bar{\alpha} \, d\bar{\beta} \, d\bar{\delta}, \tag{B.16}$$

where $\mathcal{O}_U(r) = \{(\bar{\alpha}, \bar{\beta}, \bar{\delta}) : \ r^*(\bar{\alpha}, \bar{\beta}, \bar{\delta}) < r\}$ and $f(.)$ is the joint pdf of $(\bar{\alpha}, \bar{\beta}, \bar{\delta})$.

Roughly, the above integral is a sum of an infinite number of terms of the form $\rho^{-E(\bar{\alpha},\bar{\beta},\bar{\delta})}$, one for each $(\bar{\alpha}, \bar{\beta}, \bar{\delta})$-tuple in $\mathcal{O}_U(r)$. Laplace's method suggest that at the asymptotic SNR only the term having minimum negative exponent dominates, i.e.,

$$d^*(r) = \min_{\{(\bar{\alpha},\bar{\beta},\bar{\delta}) \in \mathcal{S} \cap \mathcal{O}_U(r)\}} E\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right), \tag{B.17}$$

where $\mathcal{S}$ represents the support set of the joint pdf of $(\bar{\alpha}, \bar{\beta}, \bar{\delta})$ given in equation (6.28) and the expression of $E(.)$ is given in equation (6.29). Suppose at a given r, the objective function attains the minimum value for an $\bar{\alpha} \in \mathcal{S}$ where $\alpha_i > 1$ for one or more i's. Let $\tilde{\bar{\alpha}} = \min\{[1, 1, \cdots, 1], \bar{\alpha}\}$, where the minimization is component-wise. Clearly, $\tilde{\bar{\alpha}} \in \mathcal{S}$ but at this point $E(.)$ has a strictly smaller value. This proves that in the optimal solution, $\alpha_i \in [0, 1]$ for all i. The same is true for $\bar{\beta}$ and $\bar{\gamma}$. This, however, simplifies both the objective function and the constraint set giving the optimization problem in equations (6.51)-(6.57) and (6.58)in the statement of Theorem 6.3.

B.5 Proof of Theorem 6.4

The proof essentially contains two simplifying steps which consecutively simplify the optimization problem of Theorem 6.3. In the first step a transformation of variables yields an equivalent problem having three variables independent of the values of m, k and n. Analyzing the domain of definition of the equivalent problem we find that in the optimal solution one of the variables is a function of the other two resulting in an optimization problem having only two variables. We start with the first step.

Step 1: The objective function in (6.58) decreases strictly monotonically as α_i is decreased for any i and the rate of decrease with α_i is smaller for a larger value of i. The same is true for $\bar{\beta}$ and $\bar{\delta}$. Thus, following a similar method as in [48], it can be shown that if $\sum_{i=1}^{u}(1-\alpha_i) = a$, $\sum_{j=1}^{p}(1-\beta_j) = b$, $\sum_{l=1}^{q}(1-\delta_l) = s$ and $\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right)$ satisfy equations (6.55)-(6.57), then the optimal choice of $\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right)$ that minimizes $F(.)$ is given by $(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s))$, where

$$\phi_\alpha(a) = [\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_u]^T \;:\; \hat{\alpha}_i = \left(1-(a-i+1)^+\right)^+,\; 1 \le i \le u, \tag{B.18}$$

$$\phi_\beta(b) = [\hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_p]^T \;:\; \hat{\beta}_j = \left(1-(b-j+1)^+\right)^+,\; 1 \le j \le p, \tag{B.19}$$

$$\phi_\delta(y) = [\hat{\delta}_1, \hat{\delta}_2, \cdots, \hat{\delta}_t]^T \;:\; \hat{\delta}_l = \left(1-(s-l+1)^+\right)^+,\; 1 \le l \le q. \tag{B.20}$$

Denoting by $\mathcal{T}(a,b,s)$ the following set

$$\left\{(\bar{\alpha}, \bar{\beta}, \bar{\delta}) : \sum_{i=1}^{u}(1-\alpha_i) = a, \sum_{j=1}^{p}(1-\beta_j) = b, \sum_{l=1}^{q}(1-\delta_l) = s,\; \left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right) \text{ satisfy equations (6.55)-(6.57)}\right\},$$

we have from the above argument that

$$\min_{\{\mathcal{T}(a,b,s)\}} F\left((\bar{\alpha}, \bar{\beta}, \bar{\delta})\right) = F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s)\right). \tag{B.21}$$

Let us now define the following set of new variables

$$\mathcal{O}_1 = \left\{(a,b,s) : a + \frac{bs}{b+s} \le r,\; (a+b) \le m,\; (a+s) \le n,\; 0 \le a \le u,\; 0 \le b \le p,\; 0 \le s \le q\right\}. \tag{B.22}$$

It is clear from the definition of $\mathcal{T}(a,b,s)$ that,

$$\hat{\mathcal{O}}_1 \triangleq \bigcup_{\{(a,b,s) \in \mathcal{O}_1\}} \mathcal{T}(a,b,s) \supset \hat{\mathcal{O}(r)}. \tag{B.23}$$

Since the minimum of a function over a set is not larger than the minimum of that function over a subset of it, the above relation along with equation (B.21) imply that

$$\min_{\{(a,b,s) \in \mathcal{O}_1\}} F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s)\right) = \min_{\{(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \in \hat{\mathcal{O}}_1\}} F\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right) \tag{B.24}$$

$$\le \min_{\{(\bar{\alpha}, \bar{\beta}, \bar{\delta}) \in \hat{\mathcal{O}(r)}\}} F\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right). \tag{B.25}$$

Before proceeding further we take note of a few properties of the newly defined variables a, b and s. From the definition of ϕ_i's it is clear that if $(a,b,s) \in \mathcal{O}_1$, then $(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s))$ satisfy equations (6.52) and (6.55)-(6.57). Suppose for some $(i+j) = (m+1)$, $(\hat{\alpha}_i + \hat{\beta}_j) < 1$, then it can be shown that $\sum_{i=1}^{u}(1-\hat{\alpha}_i) + \sum_{j=1}^{p}(1-\hat{\beta}_j) > m$, which is impossible. Thus $(\hat{\alpha}_i + \hat{\beta}_j) \geq 1$ for all $(i+j) \geq (m+1)$. Similarly, it can be shown that $(\hat{\alpha}_i + \hat{\delta}_l) \geq 1$ for all $(i+l) \geq (n+1)$, which in turn imply that the $(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s))$ tuple also satisfies equations (6.53) and (6.54). That is $(a,b,s) \in \mathcal{O}_1 \;\Rightarrow\; (\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s)) \in \hat{\mathcal{O}(r)}$ which implies that

$$\min_{\{(\bar{\alpha},\bar{\beta},\bar{\delta}) \in \hat{\mathcal{O}(r)}\}} F\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right) \leq \min_{\{(a,b,s) \in \mathcal{O}_1\}} F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s)\right).$$

Combining this with equation (B.24), we get

$$\min_{\{(\bar{\alpha},\bar{\beta},\bar{\delta}) \in \hat{\mathcal{O}(r)}\}} F\left(\bar{\alpha}, \bar{\beta}, \bar{\delta}\right) \;=\; \min_{\{(a,b,s) \in \mathcal{O}_1\}} F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s)\right). \tag{B.26}$$

Therefore, we have an equivalent optimization problem to that presented in Theorem 6.3, but with fewer variables, i.e., $d^*(r)$ can be equivalently written as

$$d^*(r) = \min_{\{(a,b,s) \in \mathcal{O}_1\}} F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(s)\right). \tag{B.27}$$

When $(a+b) = m$ or $(a+s) = n$, the objective function has a property which we state now that will be helpful to solve the minimization problem in the next section.

Claim B.1 The fundctions $F\left(\phi_\alpha(a), \phi_\beta(m-a), \phi_\delta(s)\right)$ and $F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(n-a)\right)$ are monotonically decreasing with a for a given s and b, respectively, whereas $F\left(\phi_\alpha(a), \phi_\beta(m-a), \phi_\delta(n-a)\right)$ is monotonically decreasing with a.

Proof B.1 It can be shown using equations (B.18)-(B.20) that when $(a+b) = m$ we have

$$(\hat{\alpha}_i + \hat{\beta}_j) = 1,\ \forall (i+j) = (m+1) \text{ and } (\hat{\alpha}_i + \hat{\beta}_j) \leq 1,\ \forall (i+j) \leq m.$$

Using these relations in the expression for $F\left(\bar{\bar{\alpha}}, \bar{\bar{\beta}}, \bar{\bar{\delta}}\right)$, after some algebra we get

$$F\left(\bar{\bar{\alpha}}, \bar{\bar{\beta}}, \bar{\bar{\delta}}\right) = \sum_{i=1}^{p}(m+n+k+1-2i)\hat{\alpha}_i + \sum_{l=1}^{q}(n+k+1-2l)\hat{\delta}_l - ku + \sum_{\substack{i,l=1\\ l+i \leq n}}^{u,q} (1-\hat{\alpha}_i-\hat{\delta}_l)^+.$$

The above function is a strictly monotonically increasing function of $\hat{\alpha}_i$ for each $1 \leq i \leq u$. Each of the $\hat{\alpha}_i$'s in turn is a monotonically decreasing function of a which makes the above function a monotonically decreasing function of a.

Similarly, it can be shown that $F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta(n-a)\right)$ is a monotonically decreasing function of a. However, when both $(a+b)=m$ and $(a+s)=n$, then we have

$$(\hat{\alpha}_i + \hat{\beta}_j) = 1, \forall\ (i+j) = (m+1) \text{ and } (\hat{\alpha}_i + \hat{\beta}_j) \leq 1,\ \forall\ (i+j) \leq m;$$

$$(\hat{\alpha}_i + \hat{\delta}_l) = 1, \forall\ (i+l) = (n+1) \text{ and } (\hat{\alpha}_i + \hat{\delta}_l) \leq 1,\ \forall\ (i+l) \leq n.$$

Using this in the expression for $F\left(\bar{\hat{\alpha}}, \bar{\hat{\beta}}, \bar{\hat{\delta}}\right)$ we get

$$F\left(\phi_\alpha(a), \phi_\beta(m-a), \phi_\delta(n-a)\right) = F\left(\bar{\hat{\alpha}}, \bar{\hat{\beta}}, \bar{\hat{\delta}}\right) = \sum_{i=1}^{p} (m+n+1-2i)\hat{\alpha}_i,$$

which by a similar argument as above is a monotonically decreasing function of a.

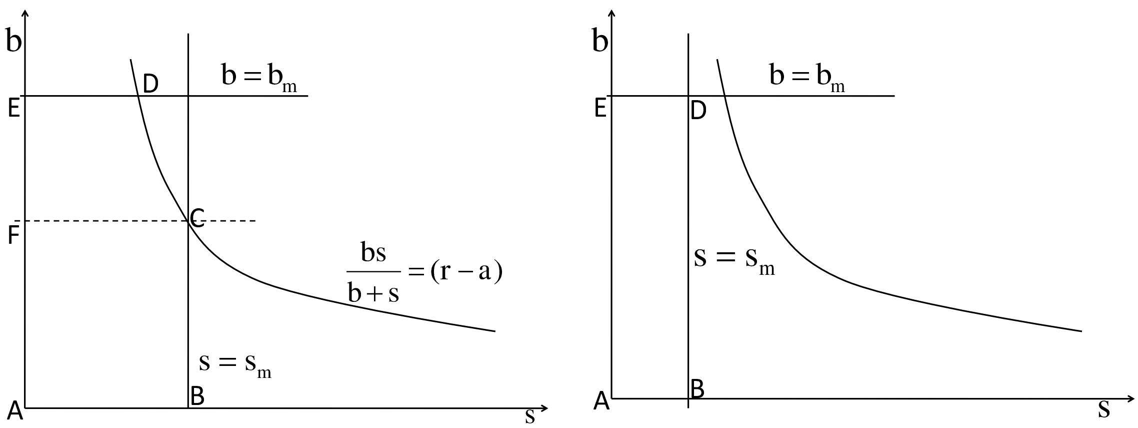
Step 2: In this final step, we determine the minimum of $F(.)$ on $\mathcal{O}_1$ and establish the theorem. Note that if $a, b, s \in \mathcal{O}_1$, then $b \leq \min\{(m-a), p\}$ and $s \leq \min\{(n-a), q\}$. Let us denote these maximum values of b and s by b_m and s_m, respectively. Depending on the value of a the set of feasible (b, s) pairs takes on different shapes as shown in the following figures. For example, when $a \in \mathcal{R}_1 = \{a : \frac{b_m(r-a)}{(b_m-r+a)} \leq s_m\}$ the feasible set of (b, s) pairs is the region ABCDE shown in Fig. B.1(a).

For any given value of a the following observations will help us solve the problem:

- The optimal (b, s) pair always lies on the boundary BCDE or BDE, because the objective function is monotonically decreasing with both b and s.

- By the same argument the optimal point on the line segment BC and ED are C and D, respectively.

Now, we argue that the optimal solution does not lie in $\mathcal{O}_1 \cap \mathcal{R}^c$. Note that when $a \in \mathcal{R}^c$ the optimal solution for the (b, s) tuple is point D where $(b, s) = (b_m, s_m)$. However, when $b = b_m$ we have either $b = p$ or $b = (n-a)$. In both of these cases the objective function is monotonically



(a) $\mathcal{R} = \left\{ a : \frac{b_m(r-a)}{(b_m - r + a)} \leq s_m \right\}$ (b) $\mathcal{R}^c = \left\{ a : s_m < \frac{b_m(r-a)}{(b_m - r + a)} \right\}$

Figure B.1: Sets of feasible (b, s) tuples for different values of a.

decreasing with a (e.g., see Claim B.1). The same is true for s. Therefore, it is clear from the definition of $\mathcal{O}_1$ that when $(b,s) = (b_m, s_m)$, a should be (also can be) increased until the constraint $a + \frac{b_m s_m}{b_m + s_m} \le r$ becomes active. In that case however, we have $a \in \mathcal{R}$ because

$$\frac{b_m s_m}{b_m + s_m} = (r-a) \quad \Longrightarrow \quad \frac{b_m(r-a)}{(b_m - r + a)} = s_m.$$

So, the objective function does not attain its minimum value when $a \in \mathcal{R}^c$ and we need to optimize the objective function only over the set $\mathcal{O}_1 \cap \mathcal{R}$. In the definition of $\mathcal{R}$ the condition in terms of s_m and b_m can be converted to constraints on a as

$$\mathcal{R} = \left[\max\{r - \frac{pq}{(p+q)}, r - \sqrt{(m-r)(n-r)}, a_n^*, a_m^*\}, r\right], \tag{B.28}$$

where

$$\begin{aligned} a_n^* &= \left(\frac{n+r}{2}\right) - \sqrt{\left(\frac{n-r}{2}\right)^2 + q(n-r)}; \\ a_m^* &= \left(\frac{m+r}{2}\right) - \sqrt{\left(\frac{m-r}{2}\right)^2 + p(m-r)}. \end{aligned} \tag{B.29}$$

Also by the previous argument the optimal (b,s) tuple lies on the arc CD and satisfies $\frac{bs}{(b+s)} = (r-a)$. Further on the arc CD b can take any value between point, E where $b = b_m$ and F, where $b = \frac{s_m(r-a)}{(s_m - r + a)}$ and thus lies in the range $\mathcal{B} = \left[\frac{s_m(r-a)}{(s_m-r+a)}, b_m\right]$. Using these facts, we see that the optimal solution is given by

$$d^*(r) = \min_{\{a \in \mathcal{R},\ b \in \mathcal{B}\}} F\left(\phi_\alpha(a), \phi_\beta(b), \phi_\delta\left(\frac{b(r-a)}{(b-r+a)}\right)\right). \tag{B.30}$$

B.6 Proof of Theorem 6.5

To prove the theorem we evaluate the minimum value of the objective function in the optimization problem of Theorem 6.4 over different carefully chosen subsets of the feasible set. The choice of these subsets also helps us to obtain a closed form expression for the optimal solution in each subset. The union of these sets might not be equal to the feasible set. The minimum of these different optimal solutions represent the minimum value of the objective function over a subset of the feasible set and hence yields only an an upper bound to the actual minimum. The proof is divided into different cases and each case considers a particular subset of the feasible set.

Case 1 ($\mathcal{O}_1 \cap \{a = r\}$): We know the optimal (a, b, s) tuple satisfies $a + \frac{bs}{(b+s)} = r$. So, $a = r$ implies either $b = 0$ or $s = 0$. Since we are considering the symmetric case ($m = n$), without loss of generality we assume $s = 0$. From the definition of ϕ_δ we get $\hat{\delta}_l = 1$ for $1 \leq l \leq q$. Since the objective function is monotonically decreasing with b for a given a and s to minimize the objective function the maximum possible value of b should be chosen, i.e., $b = b_m = \min\{(n - r), p\}$. We need to consider two different cases: 1) $(n - r) \leq p$ and 2) $(n - r) \geq p$. In the first case, $b = (n - r)$ and $(a + b) = n$ which along with Claim B.1 implies that

$$\begin{aligned} d_{11}(r) \triangleq \min F =& \sum_{i=1}^{n}(2n + k - 2i + 1)\hat{\alpha}_i + \sum_{j=1}^{p}(k + n - 2j + 1)\hat{\delta}_j - kn + \sum_{\substack{i,j=1 \\ j+i \leq n}}^{n,q}(1 - \hat{\alpha}_i - \hat{\delta}_j), \\ \overset{(a)}{=}& \sum_{i=1}^{n}(2n + k - 2i + 1)\hat{\alpha}_i = d_{n,(n+k)}(r),\ (n - p) \leq r \leq n, \end{aligned} \tag{B.31}$$

where step (a) is obtained by putting $\hat{\delta}_l = 1,\ \forall l$ and the last step follows from the definition of $\phi_\alpha(a)$. Next we consider the case when $(n-a) \geq p$ and $b = p$, which in turn imply $\hat{\beta}_j = 0,\ 1 \leq j \leq p$. Putting this in the objective function, we get

$$\begin{aligned} d_{12}(r) \triangleq \min F =& \sum_{i=1}^{n}(2n + 2k - 2i + 1)\hat{\alpha}_i + -kn + \sum_{\substack{i,j=1 \\ j+i \leq n}}^{n,p}(1 - \hat{\alpha}_i), \\ =& \sum_{i=1}^{n}(2n + k - 2i + 1)\alpha_i = d_{n,(n+k)}(r),\ 0 \leq r \leq (n - p). \end{aligned} \tag{B.32}$$

Combining equations (B.31) and (B.32) we get the minimum value of the objective function over the chosen subset

$$d_{U_1} \triangleq \min\{d_{11}(r), d_{12}(r)\} = d_{n,(n+k)}(r),\ 0 \leq r \leq n. \tag{B.33}$$

Case 2 ($\mathcal{O}_1 \cap \{b = s = (n - a)\}$): Putting $b = s = (n - a)$ in the constraint $a + \frac{bs}{(b+s)} = r$ which is always active we get $a = (2r - n)$. Now, $a \in \mathcal{R}$ if and only if

$$\max\left\{r - \frac{p}{2}, a_n^*, 2r - n\right\} \leq (2r - n) \quad \Longrightarrow \quad (n - \frac{p}{2}) \leq r. \tag{B.34}$$

Since $(a+b) = n = (a+s)$, we know from Claim B.1 that the objective function gets simplified to

$$d_{U_2}(r) \triangleq \min F = \sum_{i=1}^{n}(2n-2i+1)\alpha_i \tag{B.35}$$

$$= d_{n,n}(2r-n) \tag{B.36}$$

$$= d_{2n,2n}(2r),\ (n-\frac{p}{2}) \leq r \leq n\ [\because a = (2r-n)]. \tag{B.37}$$

Case 3 ($\mathcal{O}_1 \cap \{a=0, b=p\}$): We know from Theorem 6.4 that $a \in \mathcal{R}$ if and only if

$$\max\left\{r-\frac{p}{2}, a_n^*, 2r-n\right\} \leq 0 \quad \Longrightarrow \quad (n-\frac{p}{2}) \leq r. \tag{B.38}$$

Further, from the definition of ϕ_α and ϕ_β we get $\hat{\beta}_j = 0,\ \forall j \leq p$ and $\hat{\alpha}_i = 1,\ \forall i \leq n$. Putting this in the objective function we have

$$d_{U_3}(r) \triangleq \min F = n(n+2k) + \sum_{l=1}^{p}(k+n-2l+1)\hat{\delta}_j - 2kn,$$

$$= n^2 + \sum_{l=1}^{p}(n+k-2l+1)\left(1-\left(\frac{pr}{(p-r)}-l+1\right)^+\right)^+,\ 0 \leq r \leq \frac{p}{2},. \tag{B.39}$$

where the last step follows from the fact that the optimizing (a,b,s) tuple satisfies[1] $a + \frac{bs}{(b+s)} = r$.

Case 4 ($\mathcal{O}_1 \cap \{b=s=N, 1 \leq N \leq p\}$): From the constraint $a + \frac{bs}{(b+s)} = r$ we get $a = r - \frac{N}{2}$. Now, $a \in \mathcal{R}$ if and only if

$$\max\left\{r-\frac{p}{2}, (2r-n), a_n^*\right\} \leq r-\frac{N}{2}, \implies \frac{N}{2} \leq r \leq \min\left\{n-\frac{N}{2}, n-\frac{N^2}{(2p-N)}\right\}. \tag{B.40}$$

Since $b = s = N$, from the definition of ϕ_i's, we have $\hat{\delta}_j, \hat{\beta}_j = 1,\ \forall j \geq (N+1)$ and $\hat{\delta}_j, \hat{\beta}_j = 0,\ \forall j \leq N$. Putting this in the objective function we have

$$d_{U_{(3+N)}}(r) \triangleq \min F = \sum_{i=1}^{n}(2n+2k-2i+1)\hat{\alpha}_i + \sum_{j=(N+1)}^{p} 2(k+n-2j+1) + -2kn + 2\sum_{j=1}^{N}\sum_{i=1}^{(n-j)}(1-\hat{\alpha}_i),$$

$$= \sum_{i=1}^{(n-N)}(2n+2k-2N-2i+1)\hat{\alpha}_i + N^2,$$

$$= N^2 + d_{(n-N),(n+2k-N)}\left(r-\frac{N}{2}\right),\ \frac{N}{2} \leq r \leq \min\left\{n-\frac{N}{2}, n-\frac{N^2}{(2p-N)}\right\}. \tag{B.41}$$

[1] Recall the optimal solution lies on the arc CD in Fig. B.1(a).

Case 5 $(\mathcal{O}_1 \cap \{b = (n-a), s = N, 1 \le N \le p\})$: We further assume $k \ge n$, i.e., $p = q = n$. Using the fact that the optimal solution always lies on the arc CD in Fig. B.1(a) we have

$$a + \frac{N(n-a)}{(N+n-a)} = r. \tag{B.42}$$

Solving the above equation for a we get

$$a_N = \frac{(n+r)}{2} - \sqrt{\left(\frac{(n-r)}{2}\right)^2 + N(n-r)}, \quad 1 \le N \le p. \tag{B.43}$$

Now, $a \in \mathcal{R}$ if and only if

$$\max\left\{a_n^*, (2r-n), r - \frac{n}{2}\right\} \le \frac{(n+r)}{2} - \sqrt{\left(\frac{(n-r)}{2}\right)^2 + N(n-r)}, \tag{B.44}$$

which implies

$$\max\left\{\frac{Nn}{(N+n)}, n-p\right\} \le r \le n - \frac{N}{2}. \tag{B.45}$$

From Claim B.1 we get

$$d_{U_{(3+p+N)}}(r) \triangleq \min F = \sum_{i=1}^{n}(2n+k-2i+1)\hat{\alpha}_i + \sum_{j=1}^{n}(k+n-2j+1)\hat{\delta}_j - kn + \sum_{i=1}^{n}\sum_{l=1}^{(n-i)}(1-\hat{\alpha}_i-\hat{\delta}_l).$$

Since $s = N$, from the definition of ϕ_δ, we have $\hat{\delta}_j = 1,\ \forall j \ge (N+1)$ and $\hat{\delta}_j = 0,\ \forall j \le N$. Putting this in the above equation we get

$$\begin{aligned}
d_{U_{(3+p+N)}}(r) &= \sum_{i=1}^{n}(2n+k-2i+1)\hat{\alpha}_i - N(n+k-N) + \sum_{i=1}^{n}\sum_{l=1}^{(n-i)\wedge N}(1-\hat{\alpha}_i), \\
&\overset{(a)}{=} \sum_{i=1}^{n-N}(2n+k-N-2i+1)\hat{\alpha}_i - \frac{N(2k-N+1)}{2} + \sum_{i=(n-N+1)}^{n}(n+k-i+1), \\
&= \sum_{i=1}^{(n-N)}(2n+k-N-2i+1)\left(1-(a_N-i+1)^+\right)^+ + N^2, \quad \frac{Nn}{(N+n)} \le r \le n - \frac{N}{2},
\end{aligned}$$

where step (a) follows from the fact that $\sum_{i=1}^{n}(n-i)\wedge N = \frac{N(2n-N-1)}{2}$.

B.7 Proof of Theorem 6.6

The optimization problem of Theorem 6.4 is solved analytically for $m = n = 1$. Denoting the optimal solution for this case by $d_{(1,k,1)}(r)$ and specializing Theorem 6.4, we get

$$d_{(1,k,1)}(r) = \min_{\{a\in\mathcal{R}, b\in\mathcal{B}\}} (2k+1)(1-a) + k(1-b) + k(1-s) - 2k, \tag{B.46}$$

where $s = \frac{b(r-a)}{(b-r+a)}$ and $\mathcal{R}$ and $\mathcal{B}$ can be computed from equation (B.28) and Theorem 6.4 by setting $m = n = 1$. Since the objective function is symmetric with respect to b and s, without loss of generality, we can assume that $b \geq s = \frac{b(r-a)}{(b-r+a)}$, which in turn implies $b \geq 2(r-a)$. Differentiating with respect to b, we see that the function in (B.46) is a monotonically decreasing function of b for any given a when $b \geq 2(r-a)$. It is thus minimized when $b = \max \mathcal{B} = \max \left[\frac{(1-a)(r-a)}{(1-r)} \; (1-a) \right] = (1-a)$. Putting this in the above equation we get

$$
\begin{aligned}
d_{(1,k,1)}(r) &= \min_{\{a\}} \; (2k+1)(1-a) + ka + k \left(1 - \frac{(1-a)(r-a)}{(1-r)} \right) - 2k \\
&= \min_{\{a\}} \; (k+1)(1-r) + \underbrace{(r-a) \left(1 - \frac{k(r-a)}{(1-r)} \right)}_{T(r-a)}, \qquad \text{(B.47)}
\end{aligned}
$$

where $0 \leq (r-a) \leq (r \wedge (1-r))$. Note that $T(r-a)$ in the above equation is a concave function of $(r-a)$ of the form depicted in Fig. B.2 (in this figure, $k = 3$ and $r = .35$) which intersects the x-axis at $(r-a) = R_1 = \frac{(1-r)}{k}$. Thus the objective function is minimized for the following values of $(r-a)$

$$
(r-a) = \begin{cases} 0, & \text{if } (r-a) \leq \frac{(1-r)}{k}; \\ (r \wedge (1-r)), & \text{if } (r-a) > \frac{(1-r)}{k} \end{cases} \implies (r-a) = \begin{cases} 0, & \text{if } \; 0 \leq r \leq \frac{1}{(1+k)}; \\ r, & \text{if } \; \frac{1}{(1+k)} \leq r \leq \frac{1}{2}; \\ (1-r), & \text{if } \; \frac{1}{2} \leq r \leq 1. \end{cases}
$$

Putting these values for optimal $(r-a)$ in equation (B.47), we get the DMT of the $(1,k,1)$ HD-RC as in (6.66), thus proving the theorem.

B.8 Proof of Theorem 6.7 (Contd.)

Here we prove the following identity by mathematical induction

$$
d^{stat}_{(n,1,n)}(r) = d_{(n+1),n}(r), \; 0 \leq r \leq n. \qquad \text{(B.48)}
$$

For $n = 1$, the result is given in [82]. Assuming that (B.48) is true for $n = (N-1)$, we prove that it is also true for $n = N$. Now, from the objective function in equation (6.71) and the constraint (6.72)

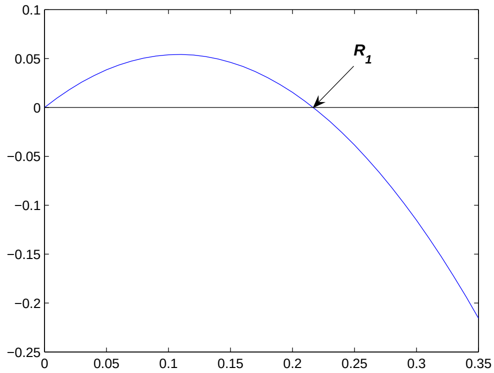


Figure B.2: Plot of $T(r - a)$ vs. $(r - a)$.

it is clear that for $0 \leq r \leq 1$, the objective function decays at the fastest rate if α_1 is decreased with increasing r.[2] Therefore, for $0 \leq r \leq 1$, the optimal solution is $\alpha_1 = (1-r)^+$, $\alpha_i = 1$ for $2 \leq i \leq N$ and $\beta_1 = 1$. Putting this in equation (6.71) we get

$$\begin{aligned} d^{stat}_{(N,1,N)}(r) =& 2N(1-r) + d_{(N-1),N}(0),\ 0 \leq r \leq 1; \\ =& d_{(N+1),N}(r),\ 0 \leq r \leq 1. \end{aligned} \tag{B.49}$$

On the other hand, since $\alpha_1 = (1-r)^+$ for $r \geq 1$, substituting $\alpha_1 = 0$ in equation (6.71) we see that the optimization problem can be written as

$$\begin{aligned} d^{stat}_{(N,1,N)}(r) &= \min_{\{(\bar{\alpha},\beta_1)\}} \sum_{i=2}^{N} (2N-2i+2)\alpha_i + (N-1)\beta_1 - (N-1) + \sum_{i=2}^{N-1} (1-\beta_1-\alpha_i)^+, \\ &= \min_{\{(\bar{\hat{\alpha}},\beta_1)\}} \sum_{i=1}^{(N-1)} (2(N-1)-2i+2)\hat{\alpha}_i + (N-1)\beta_1 - (N-1) + \sum_{i=1}^{N-1} (1-\beta_1-\hat{\alpha}_i)^+, \end{aligned}$$

subject to the following constraints

$$\sum_{i=1}^{(N-1)} (1-\hat{\alpha}_i) + \frac{1}{2}(1-\beta_1) \leq (r-1);\ 0 \leq \hat{\alpha}_1 \leq \hat{\alpha}_2 \leq, \cdots \leq \hat{\alpha}_{(N-1)} \leq 1;\ 0 \leq \beta_1 \leq 1 \text{ and } (\beta_1 + \hat{\alpha}_{(N-1)}) \geq 1.$$

Evidently, the solution of (B.50) at a given r is the DMT of the static $(N-1,1,N-1)$ HD-RC evaluated at $r-1$, which by the induction assumption is

$$\begin{aligned} d^{stat}_{(N,1,N)}(r) &= d^{stat}_{(N-1,1,N-1)}(r-1),\ 0 \leq (r-1) \leq (N-1) \\ &= d_{(N-1),N}(r-1),\ 0 \leq (r-1) \leq (N-1) \\ &= d_{(N+1),N}(r),\ 1 \leq r \leq N. \end{aligned}$$

Combining this with equation (B.49) we get $d^{stat}_{(N,1,N)}(r) = d_{(N+1),N}(r),\ 0 \leq r \leq N$ Hence, by induction we have for all $n \in \mathbb{N}$,

$$d^{stat}_{(n,1,n)}(r) = d_{(n+1),n}(r),\ 0 \leq r \leq n.$$

[2] For $0 \leq r \leq \frac{1}{2}$, the objective function decays at the same rate if β_1 is decreased, but then for $\frac{1}{2} \leq r \leq 1$, the objective function decreases at a strictly smaller rate than if α_1 was decreased from the beginning.

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