

The Relative K -theory of an Algebraic Pair

by

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The Relative K -theory of an Algebraic Pair

Thesis directed by Prof. Markus Pflaum

In [18], Karoubi defined the relative K -theory of a Banach algebra which fit into a larger framework with various homology theories. The goal of this paper is to provide a definition for the relative groups $K_n^{\text{rel}}(A; I)$ which extends relative K -theory and fits in with the already existing algebraic, topological, and relative K -theory groups.

Dedication

This work is dedicated to the memory of my father, Joseph Martinez.

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Chapter 1

Introduction

K -theory began with Alexander Grothendieck's work in the 1950's on a generalized version of the Riemann-Roch theorem in algebraic geometry (See [14] for Grothendieck's original paper and [5] for the paper published by Borel and Serre). He first defined the group $K^0(X)$ of an algebraic variety in order to work with isomorphism classes of locally free sheaves on X with a group structure rather than the sheaves themselves. Grothendieck's K -groups are now a basic tool in algebraic geometry.

Atiyah and Hirzebruch applied this construction to topological spaces in order to formulate an alternate proof of the Atiyah-Singer index theorem [2]. Since then, the theory has split into algebraic and topological branches, with both theories proving to have wide applications to other fields of mathematics. In number theory, Vandiver's conjecture that a prime p will not divide the class number of a real subfield of $\mathbb{Q}(\zeta_p)$ is equivalent to the algebraic K -theory groups, $K_{4n}(\mathbb{Z})$, being trivial [33]. In geometric topology, the Wall finiteness obstruction to a finitely dominated space X being homotopy equivalent to a finite CW-complex is contained in the group $K_0(\mathbb{Z}[\pi_1(X)])$. See [30] for a discussion of this application.

Since the splitting of K -theory into the algebraic and topological branches, a problem of importance has been to determine which topological algebras have isomorphic topological and algebraic K -theory groups. It was conjectured by Karoubi in [17] and proven by Suslin and Wodzicki

in [34] that this is the case for stable C^* -algebras.

Regardless of whether the groups are isomorphic, there is always a map $\tau_n : K_n^{\text{alg}}(A) \rightarrow K_n^{\text{top}}(A)$ for every $n \geq 0$. In [18], Karoubi defined the relative K -theory groups as a way of measuring the obstruction to the maps τ_n being isomorphisms.

The goal of this paper is to provide a definition for the relative groups $K_n^{\text{rel}}(A; I)$ which extends the theory and fits in with the already existing algebraic, topological, and relative K -theory groups.

In order to define the relative K -theory groups, much of the paper will be spent developing the machinery needed. Many of the results are well known but are included for completeness.

To begin, some preliminary definitions and properties will be recalled in chapter 2 that will be of use later in the paper. The results of Karoubi rely on the ability to define the groups of topological and algebraic K -theory by starting with the infinite general linear group, $\text{GL}(A)$.

When A has a unit, the definition of algebraic K -theory, due to Quillen, is

$$K_n^{\text{alg}}(A) := \pi_n(B(\text{GL}(A))^+)$$

for $n \geq 1$. A description of the classifying space $B(G)$ of a group G as well as of the plus-construction is included in chapter 3. A more thorough development can be found in [29] or [23]. The topological K -theory of a unital Banach algebra is defined as

$$K_n^{\text{top}}(A) := \pi_{n-1}(\text{GL}(A))$$

for $n \geq 1$. Here $\text{GL}(A)$ is viewed as a topological group with topology induced by the topological structure on A . In order for this to look more like the algebraic K -theory, it is noted that $\pi_n(\text{GL}(A)) = \pi_{n+1}(B\text{GL}(A))$, which leads to

$$K_n^{\text{top}}(A) := \pi_n(B\text{GL}(A))$$

Since the plus-construction leaves a space invariant when its fundamental group is abelian, and by Bott periodicity $\pi_1(B\text{GL}(A)) = \pi_0(\text{GL}(A)) = \pi_2(\text{GL}(A))$ is abelian, the desired definition of

topological K -theory follows. That is,

$$K_n^{\text{top}}(A) := \pi_n(BGL(A)^+)$$

Again, it is important to remember that at every step, A has been considered with the Banach algebra topology, whereas for algebraic K -theory, the constructions are all done viewing A as only a ring.

In [18], Karoubi formed the homotopy fiber sequence

$$F \rightarrow B(GL(A)^\delta)^+ \rightarrow BGL(A)^+ = BGL(A)$$

and defined the n **th relative K -theory groups of A** by $K_n^{\text{rel}}(A) = \pi_n(F)$. Here, the second map is induced by the change of topology map and $GL(A)^\delta$ denotes $GL(A)$ with the discrete topology. As outlined above, there is a long exact sequence for $n \geq 1$

$$\dots \rightarrow K_{n+1}^{\text{top}}(A) \rightarrow K_n^{\text{rel}}(A) \rightarrow K_n^{\text{alg}}(A) \rightarrow K_n^{\text{top}}(A) \rightarrow \dots$$

so that the relative K -groups give a measure of the obstruction to the maps $K_n^{\text{alg}}(A) \rightarrow K_n^{\text{top}}(A)$ being isomorphisms.

There is then, for a unital Banach algebra A with closed two-sided ideal I , a homotopy commutative square

$$\begin{array}{ccc} B(GL(A)^\delta)^+ & \xrightarrow{\tau_*^A} & BGL(A) \\ \downarrow p_*^\delta & & \downarrow p_* \\ B(GL(A/I)^\delta)^+ & \xrightarrow{\tau_*^{A/I}} & BGL(A/I) \end{array}$$

where the maps p_* and p_*^δ are induced by the projection map $p : A \rightarrow A/I$ and τ_*^A and $\tau_*^{A/I}$ are induced by the change of topology maps. Expanding to include the homotopy fibers yields

$$\begin{array}{ccccc} & & F_{p_*^\delta} & \xrightarrow{\tilde{\tau}_*} & F_{p_*} \\ & & \downarrow & & \downarrow \\ F_{\tau_*^A} & \longrightarrow & B(GL(A)^\delta)^+ & \xrightarrow{\tau_*^A} & BGL(A) \\ \downarrow \tilde{p}_* & & \downarrow p_*^\delta & & \downarrow p_* \\ F_{\tau_*^{A/I}} & \longrightarrow & B(GL(A/I)^\delta)^+ & \xrightarrow{\tau_*^{A/I}} & BGL(A/I) \end{array}$$

The above diagram commutes up to homotopy as a result of the universal property of homotopy fibers.

In this paper, relative K -theory is extended by defining the relative K -groups of an algebraic pair (A, I) where I is a closed two sided ideal of A and proving the following theorem

Theorem 1. *Let A be a unital Banach algebra with a two-sided ideal I . There is a commutative diagram*

$$\begin{array}{ccccccccc}
K_{n+2}^{\text{alg}}(A) & \longrightarrow & K_{n+2}^{\text{top}}(A) & \longrightarrow & K_{n+1}^{\text{rel}}(A) & \longrightarrow & K_{n+1}^{\text{alg}}(A) & \longrightarrow & K_{n+1}^{\text{top}}(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{n+2}^{\text{alg}}(A/I) & \longrightarrow & K_{n+2}^{\text{top}}(A/I) & \longrightarrow & K_{n+1}^{\text{rel}}(A/I) & \longrightarrow & K_{n+1}^{\text{alg}}(A/I) & \longrightarrow & K_{n+1}^{\text{top}}(A/I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{n+1}^{\text{alg}}(A; I) & \longrightarrow & K_{n+1}^{\text{top}}(I) & \longrightarrow & K_n^{\text{rel}}(A; I) & \longrightarrow & K_n^{\text{alg}}(A; I) & \longrightarrow & K_n^{\text{top}}(I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{n+1}^{\text{alg}}(A) & \longrightarrow & K_{n+1}^{\text{top}}(A) & \longrightarrow & K_n^{\text{rel}}(A) & \longrightarrow & K_n^{\text{alg}}(A) & \longrightarrow & K_n^{\text{top}}(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{n+1}^{\text{alg}}(A/I) & \longrightarrow & K_{n+1}^{\text{top}}(A/I) & \longrightarrow & K_n^{\text{rel}}(A/I) & \longrightarrow & K_n^{\text{alg}}(A/I) & \longrightarrow & K_n^{\text{top}}(A/I)
\end{array}$$

with exact rows and columns for all $n \geq 1$.

Many interesting examples of Banach algebras (and C^* -algebras in particular) do not have a unit element. One way of dealing with this is to form the unitalization, \tilde{A} , of an algebra viewed as a ring, form the groups $K_n(\tilde{A})$ and then define

$$K_n^{\text{alg}}(A) := \ker \left(K_n^{\text{alg}}(\tilde{A}) \rightarrow K_n^{\text{alg}}(\mathbb{Z}) \right)$$

$$K_n^{\text{top}}(A) := \ker \left(K_n^{\text{top}}(\tilde{A}) \rightarrow K_n^{\text{top}}(\mathbb{C}) \right)$$

Karoubi's definition of relative K -theory may be extended to Banach algebras without unit by using the extension of the definition of the general linear group to nonunital algebras. With these definitions, the above theorem holds for nonunital algebras.

It is always true that $K_n^{\text{top}}(I) = K_n^{\text{top}}(A; I)$ and Suslin and Wodzicki [34] have shown that $K_n^{\text{alg}}(I) = K_n^{\text{alg}}(A; I)$ when I is a C^* -algebra, so Karoubi's discussion and definition give the long

exact sequence

$$\dots \rightarrow K_{n+1}^{\text{top}}(I) \rightarrow K_n^{\text{rel}}(I) \rightarrow K_n^{\text{alg}}(I) \rightarrow K_n^{\text{top}}(I) \rightarrow \dots$$

which can be substituted in to the large diagram of the theorem. It then follows that, in the case of C^* -algebras, $K_n^{\text{rel}}(A; I) = K_n^{\text{rel}}(I)$.

Chapter 2

Preliminaries

2.1 Preliminaries

Definition 2. A *Banach algebra* is an associative algebra A , equipped with a submultiplicative norm $\|\cdot\|_A$ which induces a complete topology on A .

2.1.1 Algebras of Functions

For this paper, it will be assumed unless otherwise stated that all Banach algebras are over the field of complex numbers. For any topological spaces X and Y , the set of continuous functions from X to Y will be denoted by $C(X, Y)$. When $Y = \mathbb{C}$ this notation will be simplified to $\mathbb{C}(X)$. The

If A is a Banach algebra and X is a compact Hausdorff space, the set of continuous functions $f : X \rightarrow A$ will be denoted by $A(X)$. If X is locally compact, $A(X)$ will be defined to be the continuous functions on the one-point compactification X_+ which vanish at ∞ . That is, $A(X) = \ker \{A(X_+) \rightarrow A(\infty)\}$. $A(X)$ is itself an algebra with addition and multiplication defined pointwise. With the supremum norm $\|f\| = \sup \{\|f(x)\|_A | x \in X\}$, $A(X)$ is a Banach algebra.

If X is a compact space, then $A(X)$ is the algebra $C(X, A)$

Example 2.1.1. If A is a Banach algebra, the set $A[0, 1]$ is the set of all paths in A while $A(0, 1]$ is the set of all paths beginning at 0.

Definition 3. Let A be a Banach algebra. The *suspension of A* is defined to be the Banach

algebra $S(A) := A(0, 1)$. We can then define the n^{th} suspension of a Banach algebra recursively by setting $S^n A = S(S^{n-1} A)$.

Note that the unit element in an algebra of functions would be the constant function $t \mapsto 1$. This means that the suspension of a Banach algebra will not be unital in general. This makes the business of defining the K -theory of an algebra somewhat problematic as the classic constructions involve the general linear group of A . In order to have a theory which includes such basic algebras as algebras of continuous operators, it is necessary to have the notion of a unitalization of an algebra.

Definition 4. For a given algebra A over a unital ring R , we define the algebra \tilde{A}_R to be the set $A \oplus R$ with multiplication given by $(a, m)(b, n) = (ab + na + mb, mn)$. A quick check shows that the element $(0, 1)$ is the identity in this algebra.

An algebra A may be endowed with the structure of a topological algebra by giving it the discrete topology. In such a case the algebra will be denoted by A^δ . Since every subset of A^δ is open, the addition and multiplication maps are clearly continuous. Furthermore, the identity map $\tau : A^\delta \hookrightarrow A$ is continuous. This map will be called the **change of topology map**.

2.2 $\text{GL}(A)$

Given a unital algebra, A , the infinite general linear group $\text{GL}(A)$ is defined as the direct limit, $\varinjlim \text{GL}_n(A)$ taken over the inclusion maps $i_n : \text{GL}_n \rightarrow \text{GL}_{n+1}$, $M \mapsto \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}$. More concretely, this is the space $\bigcup_n \text{GL}_n(A) / \sim$ where \sim is the equivalence relation generated by setting $M \sim i_n(M)$ if $M \in \text{GL}_n(A)$.

Definition 5. A **topological group** is a group equipped with a topology such that multiplication and inversion are continuous maps. A topological group is naturally a pointed space with the identity element as the base point.

Example 2.2.1. For any group G , it is clear that G^δ (G equipped with the discrete topology) is topological group as any map between spaces with discrete topologies is continuous.

A Banach space structure of A induces the structure of a topological group on $\mathrm{GL}_n(A)$ for every $n \geq 1$. $\mathrm{GL}(A)$ will then carry the colimit topology and this will be the structure that is assumed. $\mathrm{GL}(A)$ may also be endowed with the discrete topology, that is the topology induced by A^δ . This will be denoted by $\mathrm{GL}(A)^\delta$. The change of topology map $\tau : A^\delta \rightarrow A$ then induces the change of topology map on the general linear group as well.

Definition 6. *Let A be a unital Banach algebra. The subspace $\mathrm{GL}(A)_0 \subset \mathrm{GL}(A)$ is the path component of the identity element. $E_n(A)$ is the subgroup of $\mathrm{GL}_n(A)$ generated by the elementary matrices. Taking colimit of these subgroups results in the subgroup $E(A) \subset \mathrm{GL}(A)$.*

Lemma 7. $E(A) \subset \mathrm{GL}(A)_0$.

Proof. Consider the map $h(t) := 1 + taE_{ij}$. Since $h(0) = 1$ and $h(1) = 1 + aE_{ij}$ the claim is proven. \square

2.2.1 $\mathrm{GL}(A)$ for Nonunital Algebras

It is useful to use the previously defined topological groups to define the K -theory groups of an algebra. The immediate problem arises that a large number of Banach algebras of interest, including as an important example C^* -algebras, do not generally have identity elements. This motivates the following definition:

Definition 8. *Let A be an algebra, possibly without unit. The group $\mathrm{GL}(A)$ is defined to be the kernel of the map*

$$\mathrm{GL}(\tilde{A}_{\mathbb{Z}}) \rightarrow \mathrm{GL}(\mathbb{Z}) \tag{2.1}$$

Since $A = \ker(\tilde{A}_{\mathbb{Z}} \rightarrow \mathbb{Z})$ the definition ensures that

$$\mathrm{GL}(\ker(\tilde{A}_{\mathbb{Z}} \rightarrow \mathbb{Z})) = \ker(\mathrm{GL}(\tilde{A}_{\mathbb{Z}}) \rightarrow \mathrm{GL}(\mathbb{Z}))$$

In fact, GL is a left exact functor (see [24] p.201). Hence if $I \triangleleft A$ is an ideal embedding, $\mathrm{GL}(I)$ is the kernel of the map $\mathrm{GL}(A) \rightarrow \mathrm{GL}(A/I)$ as well. In particular, if A is an algebra over a unital

ring R , the ideal embedding of $A \triangleleft \tilde{A}_R$ results in

$$\mathrm{GL}(A) = \ker \left(\mathrm{GL}(\tilde{A}_R) \rightarrow \mathrm{GL}(R) \right)$$

From [29]: When R is a unital algebra with a two-sided ideal $I \triangleleft R$, the relative general linear group is

$$\mathrm{GL}(R; I) = \ker (\mathrm{GL}(R) \rightarrow \mathrm{GL}(R/I))$$

Due to left exactness, $\mathrm{GL}(R; I) = \mathrm{GL}(I)$, so for all algebras $\mathrm{GL}(A) = \mathrm{GL}(\tilde{A}; A)$. When A is a complex Banach algebra, the definition of $\mathrm{GL}(A) = \mathrm{GL}(\tilde{A}_{\mathbb{C}}; A)$ will be assumed. The smallest normal subgroup of $E(R)$ which contains the set of elementary matrices ae_{ij} with $a \in I$ will be denoted $E(R; I)$. For any complex algebra, A , the notation $E(A) := E(\tilde{A}_{\mathbb{C}}; A)$ will be used.

Chapter 3

Homotopy Theory

3.1 Introduction

Taking the point of view that K -theory is the homotopy theory of an appropriate space, it will be necessary to recall some key constructions of homotopy theory. The focus will be on the homotopy theory of pointed spaces, but thanks to Quillen [28], one can speak of homotopy in any category that has a model category structure (for descriptions, see [12] and [10]). As noted in [18], there is a construction of the algebraic and topological K -theory groups as simplicial homotopy groups built from the simplicial algebra $A_\bullet = C(\Delta^\bullet) \hat{\otimes} A$. The benefit to this approach is that it provides an explicit definition of the relative K -theory groups, and that this definition holds for algebras which are Fréchet, not just Banach. It will thus be useful to make use of simplicial homotopy theory as well.

In order to construct the functors desired, it is useful to have diagrammatic definitions with which to work. The first such example of this will be the most basic as well.

The standard definition of two maps $f, g : X \rightarrow Y$ being homotopic is that $f \simeq g$ if there is a continuous map $H : X \times [0, 1] \rightarrow Y$ such that $H|_{X \times \{0\}} = f$ and $H|_{X \times \{1\}} = g$.

Using the exponential law, this is equivalent to the following diagram commuting:

$$\begin{array}{ccc} & C([0, 1], Y) & \\ \nearrow H & \downarrow (ev_0, ev_1) & \\ X' & \xrightarrow{(f, g)} & Y \times Y \end{array}$$

where $H'(x)(t) = H(x, t)$. One can speak of the notion of homotopic morphisms in more general categories where the objects have set structure as long as the direct product of two objects and an appropriately defined path object of an object are in the category as well. In such a case the definition is the same as above with the path space $C([0, 1], Y)$ being replaced by $P(Y)$, the path object of Y .

Example 3.1.1. In the category of rings, one can use the polynomial ring $B[t]$ as the path object to obtain a relation between two ring homomorphisms. **Polynomial homotopy** is then the equivalence relation generated setting $f \simeq_{[t]} g$ for $f, g : A \rightarrow B$ if there is a ring morphism H making the following diagram commute:

$$\begin{array}{ccc} & & B[t] \\ & \nearrow H & \downarrow (ev_1, ev_2) \\ A & \xrightarrow{(f, g)} & B \times B \end{array}$$

This means that $f \simeq g$ if there is a finite sequence of maps $f = f_0, f_1, \dots, f_n = g$ such that $f_i \simeq_{[t]} f_{i+1}$ for $i = 0, \dots, n-1$.

If one were to mimic the development of topological K -theory of a ring A and polynomial homotopy in the place of the usual homotopy of spaces, the resulting theory is the Karoubi-Villamayor K -theory, of which a further development can be found in [20]

The set of homotopy classes of maps between topological spaces X and Y will be denoted by $[X; Y]$. Let $U \subset X$ and $V \subset Y$. The set of relative homotopy classes $[X, U; Y, V]$ is the set of homotopy classes of maps $f : X \rightarrow Y$ such that $f(U) \subset V$. An important example of this is when $(X, *_X)$ and $(Y, *_Y)$ are pointed spaces, the set of homotopy classes of maps in $\text{Hom}_{\text{Top}_*}((X, *_X), (Y, *_Y))$ is $[X, *_X; Y, *_Y]$.

Definition 9. For a space X , two points a and b are path connected, written $a \simeq b$, if there is a map $h \in A[0, 1]$ such that $h(0) = a$ and $h(1) = b$. The map h is called a homotopy from a to b . The set of equivalence classes under this relation is $\pi_0(X)$.

If $* \simeq *'$ in X it follows that $[X, *; Y, *_Y] = [X, *'; Y, *_Y]$ and $[Y, *_Y; X, *] = [Y, *_Y, X, *']$ for all pointed spaces $(Y, *_Y)$. For many spaces, there is either a natural choice for base point or at least there is a canonical choice of which path component to choose the base point to be in. In such a case the homotopy classes of pointed maps will be denoted simply by $[X; Y]$.

If G is a topological group, the path component of the identity element, G_0 , is a normal subgroup since continuity of multiplication and inversion imply that for any $g \in G_0$, the set $\{gh^{-1} : h \in G_0\}$ is itself connected and G_0 contains the identity so $\{gh^{-1} : h \in G_0\} \subset G_0$ so G_0 is indeed a subgroup. That G_0 is normal in G follows since the continuity of the group operations ensures that $g^{-1}G_0g \subset G_0$ for all $g \in G$. As sets, $\pi_0(G) = G/G_0$ so that $\pi_0(G)$ may be endowed with the group structure of G/G_0 .

Example 3.1.2. Recall that lemma 7 states that if A is a unital Banach algebra, then $E(A) \subset \text{GL}(A)_0$. By Whitehead's lemma, $E(A)$ is equal to the commutator subgroup of $\text{GL}(A)$ so that $\pi_0(\text{GL}(A))$ is abelian.

For $n \geq 0$ the unit sphere of dimension n will be denoted by \mathbb{S}^n and we choose as basepoint $(1, 0, \dots, 0)$ so that \mathbb{S}^n is a pointed space.

Definition 10. *The fundamental group of a pointed space X is defined to be*

$$\pi_1(X) = [\mathbb{S}^1; X]$$

More generally, the n th homotopy groups are defined for $n \geq 0$ by

$$\pi_n(X) = [\mathbb{S}^n; X]$$

Definition 11. *The path space of a pointed space $(X, *_x)$ is the space $PX = C_0((0, 1], X)$ of continuous maps $[0, 1] \rightarrow X$ for which $0 \mapsto *_x$. The loop space of $(X, *_x)$ is the space $\Omega X = C_0((0, 1), X)$ of continuous maps $[0, 1] \rightarrow X$ for which $0, 1 \mapsto *_x$.*

Note that when A is a Banach algebra, $\Omega A = SA$

Dual to the loop space construction is the reduced suspension.

Definition 12. *The reduced suspension of a pointed space $(X, *_X)$ is the space*

$$\Sigma X = (X \times [0, 1]) / (X \times \{0, 1\}) \cup (*_X \times [0, 1])$$

The loop space construction gives a way to describe higher homotopy groups in terms of lower ones.

Proposition 13. *For a pointed space X and $n \geq 1$, $\pi_n(X) = \pi_{n-1}(\Omega X)$.*

Proof. See proposition 2.10.5 in [1] where a more general claim is proven; namely that $[\Sigma X; Y] = [X; \Omega Y]$ for any pointed spaces X and Y via the map ν which sends $g \in C(\Sigma X, Y)$ to $g' \in C(X, \Omega Y)$ defined by $g'(x)(t) = g(x, t)$. The result on homotopy groups is then obtained since $\Sigma \mathbb{S}^n \approx \mathbb{S}^{n+1}$ for $n \geq 0$. \square

3.2 Homotopy Fibers

Definition 14. *For pointed spaces X and Y , the mapping path space of a pointed map $\phi : X \rightarrow Y$ is the space*

$$E_\phi = \{(x, \alpha) \in X \times C([0, 1], Y) : \alpha(1) = \phi(x)\}$$

The homotopy fiber of ϕ is the space

$$\mathcal{F}_\phi = \{(x, \alpha) \in X \times PY : \alpha(1) = \phi(x)\}$$

The homotopy fiber \mathcal{F}_f is defined by the pullback diagram

$$\begin{array}{ccc} \mathcal{F}_f & \xrightarrow{q_1} & X \\ \downarrow & & \downarrow f \\ PY & \xrightarrow{ev_1} & Y \end{array} \tag{3.1}$$

and is the fiber of the map $E_f \rightarrow Y$ over $*_Y$ defined by $(x, \alpha) \mapsto \alpha(0)$.

Further discussions of homotopy fibers can be found in [1], [15], and [35].

The homotopy fiber and the loop space functors enable the construction of a long exact homotopy sequence of a map $f : X \rightarrow Y$ by constructing the sequence

$$\cdots \xrightarrow{q_3} \mathcal{F}_{q_1} \xrightarrow{q_2} \mathcal{F}_f \xrightarrow{q_1} X \xrightarrow{f} Y \tag{3.2}$$

of homotopy fibrations and projections $q_i : (a, \gamma) \mapsto a$.

In the situation of the sequence 3.2, there are homotopy equivalences $\Omega X \simeq \mathcal{F}_{q_2}$ and $\Omega Y \simeq \mathcal{F}_{q_1}$ (proposition 3.3.20 in [1]) so the sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{q} \mathcal{F}_f \xrightarrow{q_1} X \xrightarrow{f} Y \quad (3.3)$$

is homotopy exact. That is, the sequence

$$[W, \Omega X] \xrightarrow{(\Omega f)^*} [W, \Omega Y] \xrightarrow{q_*} [W, \mathcal{F}_f] \xrightarrow{(q_1)^*} [W, X] \xrightarrow{f_*} [W, Y] \quad (3.4)$$

is exact for any pointed space W . Choosing W to be \mathbb{S}^n for $n \geq 1$ results in the exact sequence

$$\pi_n(\Omega X) \xrightarrow{(\Omega f)^*} \pi_n(\Omega Y) \xrightarrow{q_*} \pi_n(\mathcal{F}_f) \xrightarrow{(q_1)^*} \pi_n(X) \xrightarrow{f_n} \pi_n(Y)$$

so if f_i denotes the map induced by f on the i th homotopy groups and $\partial = q_* \nu_*$ where $\nu_* : \pi_{n+1}(Y) \rightarrow \pi_n(\Omega Y)$ is induced by the map ν in proposition 13 there is an exact sequence

$$\pi_{n+1}(X) \xrightarrow{f_{n+1}} \pi_{n+1}(Y) \xrightarrow{\partial} \pi_n(\mathcal{F}_f) \xrightarrow{(q_1)^*} \pi_n(X) \xrightarrow{f_n} \pi_n(Y)$$

3.2.1 Fibers Upon Fibers

Consider a commutative square of topological spaces and continuous maps

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ h \downarrow & & \downarrow k \\ Y & \xrightarrow{g} & Z \end{array} \quad (3.5)$$

Applying the results above, the square can be extended to a diagram which includes the homotopy fibers:

$$\begin{array}{ccccc} \mathcal{F}_f & \xrightarrow{p_W} & W & \xrightarrow{f} & X \\ & & h \downarrow & & \downarrow k \\ \mathcal{F}_g & \xrightarrow{p_Y} & Y & \xrightarrow{g} & Z \end{array} \quad (3.6)$$

Recalling the definition of the homotopy fiber as pullback yields the incomplete cube

$$\begin{array}{ccccc}
 \mathcal{F}_f & \xrightarrow{p_W} & W & & \\
 \downarrow p_{X(0,1]} & & \downarrow p_Y & \searrow h & \\
 & \mathcal{F}_g & \xrightarrow{\quad} & Y & \\
 & \downarrow & \downarrow f & \downarrow g & \\
 X(0,1] & \xrightarrow{\quad} & X & \xrightarrow{k} & Z \\
 & \searrow k_* & \downarrow ev_1^X & & \\
 & & Z(0,1] & \xrightarrow{ev_1^Z} & Z
 \end{array} \tag{3.7}$$

in which front, back, right, and bottom faces are commutative squares. Therefore it follows that

$$\begin{aligned}
 ev_1^X(k_* p_{X(0,1]}) &= k ev_1^X p_{X(0,1]} \\
 &= k f p_W \\
 &= g(h p_W)
 \end{aligned}$$

and by the universal property of \mathcal{F}_g there must be a map $\mathcal{F}_f \xrightarrow{p} \mathcal{F}_g$ making the whole cube commute.

In particular, The commutative square has been extended to the commutative diagram

$$\begin{array}{ccccc}
 \mathcal{F}_f & \xrightarrow{p_W} & W & \xrightarrow{f} & X \\
 p \downarrow & & h \downarrow & & k \downarrow \\
 \mathcal{F}_g & \xrightarrow{p_Y} & Y & \xrightarrow{g} & Z
 \end{array} \tag{3.8}$$

Furthermore, there is an explicit definition of this map

$$p(w, \delta) = (h(w), k\delta)$$

Since if $(w, \delta) \in \mathcal{F}_f$ it follows that $(h(w), k\delta) \in \mathcal{F}_g$ because $gh(w) = kf(w) = k(\delta(1))$. Now, since $p_Y(h(w), k\delta) = h(w) = h(p_W(w, \delta))$ and $p_{Z(0,1]}(h(w), k\delta) = k\delta = k_*(p_{X(0,1]}(w, \delta))$, the map

does indeed make the diagram

$$\begin{array}{ccccc}
 \mathcal{F}_f & & & & \\
 & \searrow^{p} & \searrow^{hop_W} & & \\
 & & \mathcal{F}_g & \longrightarrow & Y \\
 & \searrow^{k_* \circ p_{X(0,1]}} & \downarrow & & \downarrow \\
 & & Z(0,1] & \longrightarrow & Z
 \end{array}$$

commute and is continuous.

Iterating the above construction gives rise to a commutative diagram:

$$\begin{array}{ccccccccc}
 \dots & \xrightarrow{q_4} & \mathcal{F}_{q_2} & \xrightarrow{q_3} & \mathcal{F}_{q_1} & \xrightarrow{q_2} & \mathcal{F}_f & \xrightarrow{q_1} & W & \xrightarrow{f} & X \\
 & & \downarrow p_3 & & \downarrow p_2 & & \downarrow p_1 & & \downarrow h & & \downarrow k \\
 \dots & \xrightarrow{q'_4} & \mathcal{F}'_{q'_2} & \xrightarrow{q'_3} & \mathcal{F}'_{q'_1} & \xrightarrow{q'_2} & \mathcal{F}'_g & \xrightarrow{q'_1} & Y & \xrightarrow{g} & Z
 \end{array} \tag{3.9}$$

This process can be repeated with the homotopy fibers of the vertical maps to yield a large commutative diagram. The following result ensures that the order in which the diagram is built does not matter.

Proposition 15. *For a diagram with homotopy fibers*

$$\begin{array}{ccccc}
 & & \mathcal{F}_h & \xrightarrow{\tau} & \mathcal{F}_k \\
 & & \downarrow & & \downarrow \\
 \mathcal{F}_f & \longrightarrow & W & \xrightarrow{f} & X \\
 \downarrow p & & \downarrow h & & \downarrow k \\
 \mathcal{F}_g & \longrightarrow & Y & \xrightarrow{g} & Z
 \end{array} \tag{3.10}$$

in which the square

$$\begin{array}{ccc}
 W & \xrightarrow{f} & X \\
 h \downarrow & & \downarrow k \\
 Y & \xrightarrow{g} & Z
 \end{array}$$

commutes, the homotopy fibers \mathcal{F}_p and \mathcal{F}_τ are homeomorphic.

Proof. The homotopy fibers are the spaces

$$\begin{aligned}
\mathcal{F}_\tau &= \{((w, \beta), \omega) \in \mathcal{F}_h \times P\mathcal{F}_k : \omega(1) = \tau(w, \beta)\} \\
&= \{((w, \beta), (\alpha, \gamma)) \in \mathcal{F}_h \times P\mathcal{F}_k : (\alpha(1), \gamma(1)) = \tau(w, \beta) = (f(w), g\beta)\} \\
&= \{(w, \beta, \alpha, \gamma) \in W \times PY \times PX \times P(PZ) : \beta(1) = h(w), \gamma(t)(1) = k(\alpha(t)), \alpha(1) = f(w), \\
&\quad \gamma(1)(t) = g(\beta(t)) \forall t \in (0, 1]\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_p &= \{((w, \alpha), \theta) \in \mathcal{F}_f \times P\mathcal{F}_g : \theta(1) = p(w, \alpha)\} \\
&= \{((w, \alpha), (\beta, \gamma)) \in \mathcal{F}_f \times P\mathcal{F}_g : (\beta(1), \gamma(1)) = p(w, \alpha) = (h(w), k\alpha)\} \\
&= \{(w, \alpha, \beta, \gamma) \in W \times PX \times PY \times P(PZ) : \beta(1) = h(w), \gamma(1)(t) = k(\alpha(t)), \alpha(1) = f(w), \\
&\quad \gamma(t)(1) = g(\beta(t)) \forall t \in (0, 1]\}
\end{aligned}$$

For $\gamma \in P(PZ)$, let γ' be defined by $\gamma'(s)(t) = \gamma(t)(s)$. Then the map $\mathcal{F}_\tau \rightarrow \mathcal{F}_p$ defined by $(w, \beta, \alpha, \gamma) \mapsto (w, \alpha, \beta, \gamma')$ is a homeomorphism. \square

Proposition 16. *Let $\phi : W \rightarrow \mathcal{P} = \{(x, y) \in X \times Y : k(x) = g(y)\}$ be the map from W to the pullback of the diagram*

$$\begin{array}{ccc}
& & X \\
& & \downarrow k \\
Y & \xrightarrow{g} & Z
\end{array}$$

given by the universal property of \mathcal{P} with homotopy fiber \mathcal{F}_ϕ . For spaces and maps as in diagram 3.10, there is a homotopy equivalence between \mathcal{F}_ϕ and \mathcal{F}_τ .

Proof. This is a specific case of theorem 7.6.2 in [32] starting with the commutative square

$$\begin{array}{ccc}
W & \xrightarrow{f} & X \\
h \downarrow & & \downarrow k \\
Y & \xrightarrow{g} & Z
\end{array} \tag{3.11}$$

where k and g can be assumed to be fibrations by replacing X and Y by the mapping path spaces of k and g respectively and the diagram can be written including the fibers $F_h = h^{-1}(*_Y)$

and $F_k = k^{-1}(*_Z)$

$$\begin{array}{ccc}
 F_h & \xrightarrow{\tau} & F_k \\
 i_W \downarrow & & \downarrow i_X \\
 W & \xrightarrow{f} & X \\
 h \downarrow & & \downarrow k \\
 Y & \xrightarrow{g} & Z
 \end{array}$$

The map ϕ can be taken to be a fibration by replacing W with the mapping path space of ϕ . Once the following lemma is proven it will follow that the fibers $F_\phi = \phi^{-1}(*_{\mathcal{P}})$ and $F_\tau = \tau^{-1}(*_{F_k})$ are homotopy equivalent and the claim will be proven since in replacing spaces by the mapping path spaces, the fibers are actually the homotopy fibers of the original maps and by symmetry of the diagram, it will follow that \mathcal{F}_ϕ is homotopy equivalent to \mathcal{F}_p as well. \square

Lemma 17. *The square*

$$\begin{array}{ccc}
 F_h & \xrightarrow{\tau} & F_k \\
 i_A \downarrow & & \downarrow \psi \\
 W & \xrightarrow{\phi} & \mathcal{P}
 \end{array} \tag{3.12}$$

is a pullback square where the map $\psi : F_k \rightarrow \mathcal{P}$ is defined by $x \mapsto (*_Y, x)$.

Proof. Combining the diagrams given previously, one has the following commutative diagram:

$$\begin{array}{ccccc}
 F_h & \xrightarrow{\tau} & F_k & & \\
 i_W \downarrow & & \downarrow i_X & & \\
 W & \xrightarrow{f} & X & & \\
 \searrow \phi & & \downarrow p_X & & \\
 & \mathcal{P} & \xrightarrow{\quad} & X & \\
 \searrow h & \downarrow p_Y & & \downarrow k & \\
 & Y & \xrightarrow{g} & Z &
 \end{array}$$

Note that i_W and i_X are inclusion maps and $\tau = f|_{F_h}$. That ϕ is well defined follows because $k(x) = *_Z = g(*_Y)$ when $x \in F_k = k^{-1}(*_Z)$ so $(*_Y, x) \in \mathcal{P}$. Diagram 3.12 commutes since if $w \in F_h$, then $\phi(i_W(w)) = (h(w), i_W \circ f(w)) = (*_Y, i_X \circ \tau) = (*_Y, \tau(x)) = \psi(\tau(w))$.

Now suppose there is a space V with maps γ and δ making

$$\begin{array}{ccc} V & \xrightarrow{\delta} & F_k \\ \gamma \downarrow & & \downarrow \psi \\ W & \xrightarrow{\phi} & \mathcal{P} \end{array}$$

commute. Then for $v \in V$, $\psi(\delta(v)) = (*_Y, \delta(v)) = \phi(\gamma(v))$ so that $h(\gamma(v)) = *_Y$ and thus $\gamma(v) \in F_h$.

A map $V \rightarrow F_h$ can therefore be defined by $v \rightarrow \gamma(v)$ which makes the diagram

$$\begin{array}{ccccc} V & & & & \\ & \searrow \delta & & \searrow \psi & \\ & & F_h & \xrightarrow{\quad} & F_k \\ & \searrow \gamma & \downarrow & & \downarrow \\ & & W & \xrightarrow{\phi} & \mathcal{P} \end{array}$$

commute. That this map is unique is clear and so F_α must be the pullback as claimed. \square

Repeated application of propositions 13 and 15 yields the following corollary:

Corollary 18. *For the commutative square (3.5) there commutative diagram*

$$\begin{array}{ccccccccc} \Omega^2 A & \longrightarrow & \Omega^2 B & \longrightarrow & \Omega \mathcal{F}_f & \longrightarrow & \Omega A & \longrightarrow & \Omega B \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^2 X & \longrightarrow & \Omega^2 Y & \longrightarrow & \Omega \mathcal{F}_g & \longrightarrow & \Omega X & \longrightarrow & \Omega Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega \mathcal{F}_\alpha & \longrightarrow & \Omega \mathcal{F}_\beta & \longrightarrow & \mathcal{F}_p & \longrightarrow & \mathcal{F}_\alpha & \longrightarrow & \mathcal{F}_\beta \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega A & \longrightarrow & \Omega B & \longrightarrow & \mathcal{F}_f & \longrightarrow & A & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega X & \longrightarrow & \Omega Y & \longrightarrow & \mathcal{F}_g & \longrightarrow & X & \longrightarrow & Y \end{array} \tag{3.13}$$

which commutes and has homotopy exact rows and columns.

3.3 The Plus-Construction

Proposition 19 ([29] p. 268). *Consider a connected CW-complex X with $\pi_1 = \pi_1(X)$ and $\pi = [\pi_1, \pi_1]$ the commutator subgroup of π_1 . If π is perfect, then there is a CW-complex X^+ obtained by attaching 2-cells and 3-cells to X which has the following properties:*

- (1) $\pi_1(X^+) = \pi_1(X)_{\text{ab}} = \pi_1/\pi$
- (2) The map $\pi_1(X) \rightarrow \pi_1(X^+)$ induced by the inclusion map is the quotient map π_1/π
- (3) If M is a π_1/π -module, then $H_n(X^+, X; M) = 0$ for all $n \geq 0$.
- (4) If Y is a CW-complex which contains X as a subcomplex and satisfies the above conditions, then there is a homotopy equivalence $X^+ \rightarrow Y$ which is homotopic to the identity on X .

Construction. The space X^+ is constructed as follows. Choose representatives $\{g_i\}$ for a generating set of π . The Hurewicz theorem guarantees that the homology class of each g_i is trivial. Form the space X' by attaching one 2-cell e_i^2 for each generator g_i by using g_i as the attaching map. Let \tilde{X} be the covering space of X and let \tilde{X}' be covering space of X' with covering group π_1/π , \tilde{X}' is the universal covering of X' and $\pi_1(\tilde{X}) = \pi$ so that $H_1(\tilde{X}; \mathbb{Z}) = 0$. The relative homology groups $H_n(X', X; \mathbb{Z})$ and $H_n(\tilde{X}', \tilde{X}; \mathbb{Z})$ are zero for all $n \geq 2$ and when $n = 2$ they are, respectively, the free abelian group on $\{[e_i^2]\}$ and the free $\mathbb{Z}(\pi_1/\pi)$ -module on $\{[e_i^2]\}$. Forming the long exact sequence of homology groups associated to the pair (\tilde{X}', \tilde{X}) , it follows that $H_2(\tilde{X}') \cong H_2(\tilde{X}) \oplus \mathbb{Z}(\pi_1/\pi)[e_i^2]$. Since \tilde{X}' is simply connected, each class $[e_i^2]$ is in the image of the Hurewicz map, $\phi_{X'}$ for X' . It is therefore possible to choose a map $h_i : S^2 \rightarrow X'$ in $\phi_{X'}([e_i^2])$ for each i . X^+ is then obtained from X' by attaching 3-cells using the h_i as attaching maps. \square

The plus construction given above is functorial up to homotopy by and $f^+ : X^+ \rightarrow Y^+$ will denote a choice of map induced by $f : X \rightarrow Y$.

3.4 Classifying Spaces

The notion of the classifying space of a group is a key piece of the construction of algebraic K -theory and while not essential to defining topological K -theory, gives a useful description for the purpose of defining relative K -theory.

The classifying space of a discrete group G is obtained by first constructing a contractible CW-complex, EG on which G acts freely and cellularly and then forming the classifying space as

$BG = EG/G$. That such a space can always be found was famously proven by Eilenberg and MacLane (see [11]). The classifying space of a topological group is defined similarly by Milnor in [26] as follows.

Definition 20. *The n -fold join of a topological group G is the set*

$$G^{*n} = \left\{ ((t_1, g_1), \dots, (t_n, g_n)) \in \prod_{i=1}^n ([0, \infty) \times G / \sim) : \sum_{i=1}^n t_i = 1 \right\}$$

Where the equivalence \sim on $[0, \infty) \times G$ is defined by $(0, g) \sim (0, h)$ for all $g, h \in G$.

the space EG is the colimit of the diagram $G^{*1} \hookrightarrow G^{*2} \hookrightarrow \dots$ where the maps are the inclusions $G^{*n} \hookrightarrow G^{*(n+1)}$, $((t_1, g_1), \dots, (t_n, g_n)) \mapsto ((t_1, g_1), \dots, (t_n, g_n), (0, 1))$.

The classifying space BG is then defined to be the quotient of EG by the action of G given by $((t_1, g_1), (t_2, g_2), (t_3, g_3), \dots) \cdot g = ((t_1, g_1g), (t_2, g_2g), (t_3, g_3g), \dots)$. A continuous group homomorphism $f : G \rightarrow H$ induces a map on the n -fold joins $G^{*n} \rightarrow H^{*n}$. This map will respect the group action so there is an induced map $f_n : G^{*n}/G \rightarrow H^{*n}/H$. The map on classifying spaces is then obtained by taking the colimit as $n \rightarrow \infty$ results in a map $f_* : BG \rightarrow BH$. This construction makes it clear that $(id_G)_* = id_{BG}$ and $(f \circ h)_* = f_* \circ h_*$ and is therefore functorial.

For any group, forming the long exact sequence of the fiber sequence

$$G \rightarrow EG \rightarrow EG/G = BG \tag{3.14}$$

yields exact sequences

$$0 = \pi_{n+1}(EG) \rightarrow \pi_{n+1}(BG) \rightarrow \pi_n(G) \rightarrow \pi_n(EG) = 0 \tag{3.15}$$

so $\pi_{n+1}(BG) = \pi_n(G)$.

3.5 Homotopy Invariance

Definition 21. *Let \mathcal{C} be a category in which the objects have an underlying topological space structure. A functor $F : \mathcal{C} \rightarrow \mathbf{Ab}$ is said to be **homotopy invariant** if $f \simeq g$ implies $F(f) = F(g)$.*

Proposition 22. *A functor $F : \mathbf{BAlg} \rightarrow \mathbf{Ab}$ is homotopy invariant if and only if for every Banach algebra A , $F(A)$ is isomorphic to $F(A[0, 1])$ under the map induced by the inclusion $i : A \hookrightarrow A[0, 1]$ defined by sending an element $a \in A$ to the constant function $c_a : t \mapsto a$.*

Proof. Let \mathbf{C} be an arbitrary category and $F : \mathbf{BAlg} \rightarrow \mathbf{C}$ be a homotopy invariant functor. If $f \in A[0, 1]$ is a map such that $f(0) = a$, then $f \simeq c_a$ via the homotopy $H : A[0, 1] \times [0, 1] \rightarrow A[0, 1]$ defined by $H(s, t) := f(st)$. Since f is continuous, H is as well and $H(s, 0) = f(0) = c_a(s)$ and $H(s, 1) = f(s)$ for every $s \in [0, 1]$.

$j : A[0, 1] \rightarrow A$ be defined by $j(f) = f(0)$ induces a map $F(A[0, 1]) \rightarrow F(A)$ which is inverse to i_* and hence i_* is an isomorphism.

Conversely, suppose that i_* is an isomorphism. If $f \simeq g$, then there is a continuous map $H : A \rightarrow B[0, 1]$ such that $f = e_0 H$ and $g = e_1 H$. Then $F(g) = F(e_1 H) = F(e_1 i j H) = F(e_1 i H_0) = F(e_1 H_0) = F(e_0 H_0) = F(e_0 H) = F(f)$ where H_t sends the element $a \in A$ to the function with constant value $H(a)(t)$. This proves that F is homotopy invariant. \square

Example 3.5.1. The classifying space is homotopy invariant as a functor from the category of topological groups to the homotopy category of topological spaces. Given two homotopic maps $f, g : G \rightarrow G'$ with homotopy H , let $H_t : G \rightarrow G'$ be defined by $H_t(g) = H(g, t)$ for all $t \in [0, 1]$. Then the map $H_* : BG \times [0, 1] \rightarrow BG'$ defined by $H_*(x, t) = (H_t)_*(x)$ is a homotopy between f_* and g_* and so $f_* = g_*$ in the homotopy category of topological spaces.

Chapter 4

K -theory

4.1 Topological K -theory

K_0 was originally defined by Grothendieck for algebraic varieties [14]. The definition of topological K -theory was then made for compact Hausdorff spaces by Atiyah and Hirzebruch [2]. The original definition for the group $K^0(X)$ was the the Grothendieck group of $\text{Vect}(X)$, the monoid of isomorphism classes of (complex) vector bundles of X . The Serre-Swan theorem gives a monoid isomorphism between $\text{Vect}(X)$ and $\text{Proj}(\mathbb{C}(X))$, the monoid of isomorphism classes of finitely projective $\mathbb{C}(X)$ -modules where $\mathbb{C}(X)$ is the ring of continuous functions from X to \mathbb{C} . The monoid $\text{Proj}(\mathbb{C}(X))$ is isomorphic to $\text{Idem}(\mathbb{C}(X))$, the similarity classes of idempotent matrices in $M_\infty(\mathbb{C}(X)) = \varinjlim M_n(\mathbb{C}(X))$. Here the colimit is taken over the inclusions $M_n(\mathbb{C}(X)) \rightarrow M_{n+1}(\mathbb{C}(X))$ $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$. This enables elements of $K^0(X)$ to be viewed as formal differences of similarity classes of idempotent matrices.

The development of K -theory continues with the defining of

$$K^{-1}(X) := \text{GL}(\mathbb{C}(X))/\text{GL}(\mathbb{C}(X))_0 \tag{4.1}$$

with $\text{GL}(\mathbb{C}(X))_0$ denoting the connected component of the identity in $\text{GL}(\mathbb{C}(X))$. It turned out that this was equivalent to setting $K^{-1}(X) = K^0(\Sigma X)$ where Σ is the suspension of the space X . This lead to defining $K^{-n}(X) := K^0(\Sigma^n X)$ and the construction of the long exact sequence. The final piece of the puzzle was Bott periodicity [6], which states that $\Omega^2 \text{GL}(X)$ and $\text{GL}(X)$ are

homotopy equivalent. Bott's result showed that all of the groups of the topological K -theory of a space are either $K^0(X)$ or $K^{-1}(X)$

Using these results as motivation, it is possible to define the topological K -theory of a unital Banach algebra by replacing $\mathbb{C}(X)$ in the above definitions with a general Banach algebra A . See [9], [19], and [23] for more details of these constructions.

Definition 23. *Let A be a unital Banach algebra. $K_0(A)$ is the Grothendieck group of the monoid of similarity classes idempotent matrices in $M_\infty(A)$ and*

$$K_1^{\text{top}}(A) := \text{GL}(A)/\text{GL}(A)_0$$

Since $\text{GL}(A)/\text{GL}(A)_0 = \pi_0(\text{GL}(A))$, higher topological K -theory is defined as follows

Definition 24. *Let A be a unital Banach algebra. For $n \geq 1$, The n th topological K -theory group is*

$$K_n^{\text{top}}(A) = \pi_{n-1}(\text{GL}(A))$$

The definition is extended to nonunital Banach algebras by setting $K_n^{\text{top}}(A) := \ker(K_n^{\text{top}}(\tilde{A}_{\mathbb{C}}) \rightarrow K_n^{\text{top}}(\mathbb{C}))$.

Since $\text{GL}_n(\mathbb{C})$ is path connected for every n , it follows that $K_1^{\text{top}}(A) = K_1^{\text{top}}(\tilde{A}_{\mathbb{C}})$ since $K_n^{\text{top}}(\mathbb{C}) = 0$. Furthermore, since $\text{GL}(A) \rightarrow \text{GL}(\mathbb{C})$ is a fibration,

$$K_1^{\text{top}}(A) = \pi_0(\text{GL}(A)) = \text{GL}(A)/\text{GL}(A)_0$$

for all Banach algebras

If I is a closed two-sided ideal in A , starting with the Barrett-Puppe sequence of the map $p : \text{GL}(A) \rightarrow \text{GL}(A/I)$ induced by the projection $A \rightarrow A/I$ results in the long exact sequence of topological K -theory,

$$\dots \rightarrow K_2^{\text{top}}(A) \rightarrow K_2^{\text{top}}(A/I) \rightarrow K_1^{\text{top}}(I) \rightarrow K_1^{\text{top}}(A) \rightarrow K_1^{\text{top}}(A/I)$$

Bott periodicity gives that $K_{2n}^{\text{top}}(A) = K_0(A)$ and $K_{2n+1}^{\text{top}}(A) = K_1^{\text{top}}(A)$ for all $n \geq 0$ so the sequence becomes

$$\begin{array}{ccccc} K_1^{\text{top}}(I) & \longrightarrow & K_1^{\text{top}}(A) & \longrightarrow & K_1^{\text{top}}(A/I) \\ \uparrow & & & & \downarrow \\ K_0(A/I) & \longleftarrow & K_0(A) & \longleftarrow & K_0(I) \end{array} \quad (4.2)$$

This is precisely the long exact homotopy sequence of this projection map. For more details on this, see [9].

4.2 Algebraic K -theory

Algebraic K -theory starts with the same group, K_0 , as topological K -theory. Since $\text{Idem}(A)$ depends only on the ring structure of A the algebraic K -theory is taken to be the same as topological K -theory in degree 0. The higher algebraic K -theory groups are defined as follows:

Definition 25. *For a unital algebra A , the group $K_1^{\text{alg}}(A)$ is defined to be $\text{GL}(A)/E(A)$.*

It was the insight of Quillen [28] that this theory could be extended by use of classifying spaces and his plus-construction.

Definition 26. *Let A be a unital Banach algebra. For $n \geq 0$, define the group*

$$K_n^{\text{alg}}(A) := \pi_n(B(\text{GL}(A)^\delta)^+ \times K_0(A))$$

where $K_0(A)$ carries the discrete topology.

The K_0 term in this definition is there to make the definition agree with $K_n^{\text{alg}}(A)$ when $n = 0$. Since this paper is concerned with the case of $n \geq 1$, this can be omitted from the definition so that $K_n^{\text{alg}}(A) = \pi_n(B\text{GL}(A)^\delta)^+$ for $n \geq 1$.

Definition 27. *Let A be a unital Banach algebra and $I \subset A$ a closed two-sided ideal. Just as with topological K -theory, consider the map $p_*^\delta : (B\text{GL}(A)^\delta)^+ \rightarrow (B\text{GL}(A/I)^\delta)^+$ induced by the projection with homotopy fiber $\mathcal{F}_{p_*^\delta}$. For $n \geq 1$, the relative algebraic K -groups of the pair $(A; I)$ are defined to be*

$$K_n^{\text{alg}}(A; I) := \pi_n(\mathcal{F}_{p_*^\delta})$$

Recall that $E(R; I)$ is the smallest normal subgroup of $E(R)$ which contains the set of elementary matrices ae_{ij} with $a \in I$.

Theorem 28 (Relative Whitehead Lemma). *Let R be a unital ring with two-sided ideal $I \triangleleft R$. Then*

- (1) $E(A; I) \triangleleft \mathrm{GL}(A; I)$
- (2) $E(A; I) \triangleleft \mathrm{GL}(A)$
- (3) $\mathrm{GL}(A; I)/E(A; I)$ is the center of $\mathrm{GL}(A)/E(A; I)$
- (4) $E(A; I)$ is equal to the commutator subgroups $[E(A), E(A; I)]$ and $[\mathrm{GL}(A), E(A; I)]$
- (5) $K_1^{\mathrm{alg}}(A; I) = \mathrm{GL}(A; I)/E(A; I)$

Proof. See [29] Thm 2.5.3 □

Again, the definition is extended to nonunital algebras

Definition 29. *Let A be a complex algebra, possibly without unit.*

$$K_n^{\mathrm{alg}}(A) := K_n^{\mathrm{alg}}(\tilde{A}_{\mathbb{Z}}; A)$$

With these definitions, there is a long exact sequence

$$\dots \rightarrow K_{n+1}^{\mathrm{alg}}(A/I) \rightarrow K_n^{\mathrm{alg}}(A, I) \rightarrow K_n^{\mathrm{alg}}(A) \rightarrow K_n^{\mathrm{alg}}(A/I) \rightarrow \dots$$

for $n \geq 1$

Unfortunately, algebraic K -theory is not as well behaved as topological K -theory. For a general algebra, $K_n^{\mathrm{alg}}(A, I)$ and $K_n^{\mathrm{alg}}(I)$ are not isomorphic. Suslin and Wodzicki classified the algebras for which $K_n^{\mathrm{alg}}(I) \cong \pi_n(B(\mathrm{GL}(I)^\delta)^+)$ in [34]. An important subclass of algebras found to satisfy excision is the class of C^* -algebras. So for any C^* -algebra with ideal $I \subset A$, $\pi_n(B(\mathrm{GL}(I)^\delta)^+) = \pi_n(\mathcal{F}_{p_*}^\delta)$ for all $n \geq 0$.

In addition to lacking excision, algebraic algebraic K -theory does not have periodicity in general. In [16], Karoubi investigates periodicity in algebraic K -theory.

Chapter 5

Relative K -theory

While algebraic and topological K -theory differ for most Banach algebras after K_0 , the ability to define both as homotopy groups of a space built from $GL(A)$ suggests that they should be comparable in some way that is not too complicated. Such a comparison would enable the study of the more difficult algebraic K -theory by comparing it with the better understood topological K -theory. Karoubi's work in [18] explored this comparison by defining relative K -theory. Later work by Connes, Karoubi [8] and Weibel [36] put this theory into a larger context involving cyclic and Hochschild homology of Fréchet algebras.

5.1 $K_n^{\text{rel}}(A)$

Let A be a Banach algebra. Following the presentation in [18] and [8], the first step is to use the weak equivalence of X and ΩBX to obtain the new definition of

$$K_n^{\text{top}}(A) = \pi_{n-1}(\text{GL}(A)) = \pi_{n-1}(\Omega B\text{GL}(A)) = \pi_n(B\text{GL}(A)) \quad (5.1)$$

for $\text{GL}(A)$ viewed as a topological group, which implies that $\pi_n(B\text{GL}(A))$ is abelian for all $n \geq 1$ since $\pi_1(G)$ is abelian for any topological group G and $\pi_0(\text{GL}(A))$ is abelian by example 3.1.2. This is true in the case of nonunital Banach algebras as well because $K_1^{\text{top}}(A) = K_1^{\text{top}}(\tilde{A}_{\mathbb{C}})$. It then follows that $B\text{GL}(A) = B\text{GL}(A)^+$ so that

$$K_n^{\text{top}}(A) = \pi_n(B\text{GL}(A)^+) \quad (5.2)$$

From this it is clear that for $n \geq 1$, $K_n^{\text{alg}}(A) = \pi_n((BGL(A)^\delta)^+)$ and $K_n^{\text{top}}(A)$ are both the result of the same construction applied to the same starting set. The key difference between the two theories is the topology that the set $GL(A)$ has to begin with. The comparison map is then the map induced by the change of topology $\tau^A : GL(A)^\delta \rightarrow GL(A)$.

Definition 30. *Let A be a unital Banach algebra. For $n \geq 1$, the relative K -theory of A is defined by*

$$K_n^{\text{rel}}(A) := \pi_n(\mathcal{F}_{\tau_*^A}) \quad (5.3)$$

Where $\mathcal{F}_{\tau_*^A}$ is the homotopy fiber of $\tau_*^A : B(GL(A)^\delta)^+ \rightarrow BGL(A)^+$, the map induced by τ^A .

Forming the long exact homotopy sequence of the homotopy fiber sequence

$$\mathcal{F}_{\tau_*^A} \rightarrow B(GL(A)^\delta)^+ \rightarrow BGL(A)^+ = BGL(A) \quad (5.4)$$

yields the long exact sequence

$$K_{n+1}^{\text{alg}}(A) \rightarrow K_{n+1}^{\text{top}}(A) \rightarrow K_n^{\text{rel}}(A) \rightarrow K_n^{\text{alg}}(A) \rightarrow K_n^{\text{top}}(A) \quad (5.5)$$

for $n \geq 1$. Relative K -theory is then a measure of the obstruction to the map τ^A inducing isomorphisms on K -theory since if $K_n^{\text{rel}}(A) = 0$ then $K_n^{\text{alg}}(A) \rightarrow K_n^{\text{top}}(A)$ is injective and $K_{n+1}^{\text{alg}}(A) \rightarrow K_{n+1}^{\text{top}}(A)$ is surjective.

5.2 The Relative K -theory of (A, I)

Let A be a Banach algebra with closed two sided ideal $I \triangleleft A$. With the canonical projection maps $p^\delta : GL(A)^\delta \rightarrow GL(A/I)^\delta$ and $p : GL(A) \rightarrow GL(A/I)$ and the change of topology maps $\tau^A : GL(A)^\delta \rightarrow GL(A)$ and $\tau^{A/I} : GL(A/I)^\delta \rightarrow GL(A/I)$ there is a commutative diagram

$$\begin{array}{ccc} GL(A)^\delta & \xrightarrow{\tau^A} & GL(A) \\ \downarrow p^\delta & & \downarrow p \\ GL(A/I)^\delta & \xrightarrow{\tau^{A/I}} & GL(A/I) \end{array} \quad (5.6)$$

Applying the classifying space functor followed by the plus construction yields a commutative diagram

$$\begin{array}{ccc}
 (BGL(A)^\delta)^+ & \xrightarrow{(\tau_*^A)^+} & BGL(A) \\
 \downarrow (p_*^\delta)^+ & & \downarrow (p_*)^+ \\
 (BGL(A/I)^\delta)^+ & \xrightarrow{(\tau_*^{A/I})^+} & BGL(A/I)
 \end{array} \tag{5.7}$$

This diagram can be extended to a commutative diagram involving homotopy fibers

$$\begin{array}{ccccc}
 F_{(\tau_*^A)^+} & \longrightarrow & (BGL(A)^\delta)^+ & \xrightarrow{(\tau_*^A)^+} & BGL(A) \\
 \downarrow \tilde{p}_* & & \downarrow (p_*^\delta)^+ & & \downarrow p_* \\
 F_{(\tau_*^{A/I})^+} & \longrightarrow & (BGL(A/I)^\delta)^+ & \xrightarrow{(\tau_*^{A/I})^+} & BGL(A/I)
 \end{array}$$

Definition 31. Let A be a unital complex Banach algebra with closed two-sided ideal $I \triangleleft A$. For $n \geq 1$, The n^{th} relative K -theory group of the Banach algebra pair (A, I) is defined to be

$$K_n^{\text{rel}}(A; I) := \pi_n(\mathcal{F}_{\tilde{p}_*}) \tag{5.8}$$

The following theorem, which is the main result of this paper, shows that the groups $K_n^{\text{rel}}(A; I)$ as defined serve to fill a hole in the theories of algebraic, topological, and relative K -theory of Banach algebras.

Theorem 32. With the relative groups of relative K -theory, $K_n^{\text{rel}}(A; I)$, defined as above, there is a commutative diagram

$$\begin{array}{ccccccccc}
 K_{n+2}^{\text{alg}}(A) & \longrightarrow & K_{n+2}^{\text{top}}(A) & \longrightarrow & K_{n+1}^{\text{rel}}(A) & \longrightarrow & K_{n+1}^{\text{alg}}(A) & \longrightarrow & K_{n+1}^{\text{top}}(A) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 K_{n+2}^{\text{alg}}(A/I) & \longrightarrow & K_{n+2}^{\text{top}}(A/I) & \longrightarrow & K_{n+1}^{\text{rel}}(A/I) & \longrightarrow & K_{n+1}^{\text{alg}}(A/I) & \longrightarrow & K_{n+1}^{\text{top}}(A/I) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 K_{n+1}^{\text{alg}}(A; I) & \longrightarrow & K_{n+1}^{\text{top}}(I) & \longrightarrow & K_n^{\text{rel}}(A; I) & \longrightarrow & K_n^{\text{alg}}(A; I) & \longrightarrow & K_n^{\text{top}}(I) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 K_{n+1}^{\text{alg}}(A) & \longrightarrow & K_{n+1}^{\text{top}}(A) & \longrightarrow & K_n^{\text{rel}}(A) & \longrightarrow & K_n^{\text{alg}}(A) & \longrightarrow & K_n^{\text{top}}(A) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 K_{n+1}^{\text{alg}}(A/I) & \longrightarrow & K_{n+1}^{\text{top}}(A/I) & \longrightarrow & K_n^{\text{rel}}(A/I) & \longrightarrow & K_n^{\text{alg}}(A/I) & \longrightarrow & K_n^{\text{top}}(A/I)
 \end{array}$$

with exact rows and columns for all $n \geq 1$.

The proof of theorem will employ the use of proposition 15. In order to do so, it is necessary that diagram 5.7 commutes strictly, not just up to homotopy.

Lemma 33. *Let A be a Banach algebra with closed two-sided ideal $I \triangleleft A$. The diagram*

$$\begin{array}{ccc} BGL(A)^\delta & \xrightarrow{\tau_*^A} & BGL(A) \\ \downarrow p_*^\delta & & \downarrow p_* \\ BGL(A/I)^\delta & \xrightarrow{\tau_*^{A/I}} & BGL(A/I) \end{array} \quad (5.9)$$

commutes in the category of topological spaces.

Proof. As sets, $GL(A)^\delta = GL(A)$ and $GL(A/I)^\delta = GL(A/I)$ so as set maps, τ^A and $\tau^{A/I}$ are identity maps and $p^\delta = p$. Milnor's construction of the classifying space relies only on the group structure of a topological group. The different topologies on $GL(A)$ will change the topology of the resulting classifying spaces but both spaces have the same underlying sets. Therefore, $BGL(A)^\delta = BGL(A)$ and $BGL(A/I)^\delta = BGL(A/I)$ as sets so as set maps, τ_*^A and $\tau_*^{A/I}$ are identity maps and $p^\delta = p$. Thus the diagram commutes. \square

Applying the plus construction to the commutative diagram 5.9 results in a homotopy commutative diagram

$$\begin{array}{ccc} (BGL(A)^\delta)^+ & \xrightarrow{(\tau_*^A)^+} & (BGL(A))^+ \\ \downarrow (p_*^\delta)^+ & & \downarrow (p_*)^+ \\ (BGL(A/I)^\delta)^+ & \xrightarrow{(\tau_*^{A/I})^+} & (BGL(A/I))^+ \end{array} \quad (5.10)$$

What is needed is for this diagram to commute, not just commute up to homotopy. The issue is that when performing the plus construction as outlined in proposition (19) on each space individually, many choices are made which cannot generally be made to be consistent among the spaces in the diagram. This problem may be fixed in a number of ways as outlined by Loday in [23]. For complex unital topological algebras, one can take advantage of the natural map $\mathbb{C} \rightarrow A$. By fixing the choices in construction of the space $BE(\mathbb{C})^+$ the space $BGL(A)^+$ may then be defined by the

pushout diagram

$$\begin{array}{ccc} BE(\mathbb{C}) & \longrightarrow & BE(\mathbb{C})^+ \\ \downarrow & & \downarrow \\ BGL(A) & \longrightarrow & BGL(A)^+ \end{array}$$

so that the plus construction is functorial. Segal gives another functorial definition in [31] by proving that there is a canonical homotopy equivalence

$$\Omega B \left(\coprod_{n \geq 0} BGL_n(A) \right) \simeq \mathbb{Z} \times BGL(A)^+$$

where $\coprod_{n \geq 0} BGL_n(A)$ carries a topological monoid structure with addition given by the direct sum of matrices.

It will therefore be assumed for the remainder of the paper that the plus construction is performed functorially so that diagram (5.10) commutes and the theorem can be proven.

Proof of theorem 32: Fixing a functorial model for the plus construction such that $BGL(A)$ is a model for $(BGL(A))^+$ when $GL(A)$ carries the topology induced by the Banach space structure of A and including homotopy fibers results in the diagram

$$\begin{array}{ccccc} & & \mathcal{F}_{(p_*^\delta)^+} & \xrightarrow{\tilde{\tau}_*} & \mathcal{F}_{(p_*)^+} \\ & & \downarrow & & \downarrow \\ \mathcal{F}_{(\tau_*^A)^+} & \longrightarrow & (BGL(A)^\delta)^+ & \xrightarrow{(\tau_*^A)^+} & BGL(A) \\ \downarrow \tilde{p}_* & & \downarrow (p_*^\delta)^+ & & \downarrow p_* \\ \mathcal{F}_{(\tau_*^{A/I})^+} & \longrightarrow & (BGL(A/I)^\delta)^+ & \xrightarrow{(\tau_*^{A/I})^+} & BGL(A/I) \end{array}$$

The maps $\tilde{\tau}_*$ and \tilde{p}_* are the maps obtained by forming the incomplete cube given by diagram (3.7) of the pullback diagrams associated to the homotopy fibers and using the universal property of homotopy fibers. This means that the squares in the diagram involving the homotopy fibers commute and the lower right square commutes so this diagram is now of the type to which proposition 15 may be applied and so $\mathcal{F}_{\tilde{\tau}_*}$ is homeomorphic to $\mathcal{F}_{\tilde{p}_*}$.

Additionally, as in proposition 16 one obtains the following pullback square:

$$\begin{array}{ccc} \mathcal{P}_{(A;I)} & \longrightarrow & E_{p_*} \\ \downarrow & & \downarrow \\ E_{(\tau_*^{A/I})^+} & \longrightarrow & BGL(A/I) \end{array}$$

where E_f denotes the mapping path the mapping path space of a map f . There is a unique map $\phi : B(GL(A)^\delta)^+ \rightarrow \mathcal{P}_{(A;I)}$. Replacing $B(GL(A)^\delta)^+$ by E_ϕ gives the following diagram

$$\begin{array}{ccccc} \mathcal{F}_{\tilde{p}_*} & & \mathcal{F}_\phi & & \\ \downarrow & & \downarrow & & \\ \mathcal{F}_{(\tau_*^A)^+} & \longrightarrow & E_\phi & \searrow & \\ \downarrow \tilde{p}_* & \nearrow & \downarrow & \searrow & \\ \mathcal{F}_{(\tau_*^{A/I})^+} & \longrightarrow & \mathcal{P}_{(A;I)} & \longrightarrow & E_{p_*} \\ & & \downarrow & & \downarrow \\ & & E_{(\tau_*^{A/I})^+} & \longrightarrow & BGL(A/I) \end{array}$$

By lemma 17,

$$\begin{array}{ccc} \mathcal{F}_{(\tau_*^A)^+} & \longrightarrow & E_\phi \\ \tilde{p}_* \downarrow & & \downarrow \\ \mathcal{F}_{(\tau_*^{A/I})^+} & \longrightarrow & \mathcal{P}_{(A;I)} \end{array}$$

is a pullback square, hence $\mathcal{F}_{\tilde{p}_*}$ is homotopy equivalent to \mathcal{F}_ϕ . By symmetry, \mathcal{F}_ϕ is homotopy equivalent to $\mathcal{F}_{\tilde{\tau}_*}$ as well. There is therefore a diagram

$$\begin{array}{ccccc} \mathcal{F}_{\tilde{p}_*} & \longrightarrow & \mathcal{F}_{(p_*^\delta)^+} & \xrightarrow{\tilde{\tau}_*} & \mathcal{F}_{(p_*)^+} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_{(\tau_*^A)^+} & \longrightarrow & (BGL(A)^\delta)^+ & \xrightarrow{(\tau_*^A)^+} & BGL(A) \\ \downarrow \tilde{p}_* & & \downarrow (p_*^\delta)^+ & & \downarrow p_* \\ \mathcal{F}_{(\tau_*^{A/I})^+} & \longrightarrow & (BGL(A/I)^\delta)^+ & \xrightarrow{(\tau_*^{A/I})^+} & BGL(A/I) \end{array}$$

in which all of the rows and columns are homotopy fiber sequences and which commutes.

Corollary 5.11 then yields the homotopy commutative diagram

$$\begin{array}{ccccccccc}
\Omega^2(BGL(A)^\delta)^+ & \longrightarrow & \Omega GL(A) & \longrightarrow & \Omega \mathcal{F}_{(\tau_*^A)^+} & \longrightarrow & \Omega(BGL(A)^\delta)^+ & \longrightarrow & GL(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^2(BGL(A/I)^\delta)^+ & \longrightarrow & \Omega GL(A/I) & \longrightarrow & \Omega \mathcal{F}_{(\tau_*^{A/I})^+} & \longrightarrow & \Omega(BGL(A/I)^\delta)^+ & \longrightarrow & GL(A/I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega \mathcal{F}_{(p_*^\delta)^+} & \longrightarrow & \Omega \mathcal{F}_{p_*} & \longrightarrow & \mathcal{F}_{\tilde{p}_*} & \longrightarrow & \mathcal{F}_{(p_*^\delta)^+} & \longrightarrow & \mathcal{F}_{p_*} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega(BGL(A)^\delta)^+ & \longrightarrow & GL(A) & \longrightarrow & \mathcal{F}_{(\tau_*^A)^+} & \longrightarrow & (BGL(A)^\delta)^+ & \longrightarrow & BGL(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega(BGL(A/I)^\delta)^+ & \longrightarrow & GL(A/I) & \longrightarrow & \mathcal{F}_{(\tau_*^{A/I})^+} & \longrightarrow & (BGL(A/I)^\delta)^+ & \longrightarrow & BGL(A/I)
\end{array} \tag{5.11}$$

with homotopy exact rows and columns. The K -theory groups have the following characterizations

- $K_n^{\text{alg}}(A) = \pi_n(B(GL(A)^\delta)^+)$
- $K_n^{\text{top}}(A) = \pi_n(BGL(A))$
- $K_n^{\text{alg}}(A; I) = \pi_n(\mathcal{F}_{p_*^\delta})$
- $K_n^{\text{top}}(I) = \pi_n(\mathcal{F}_{p_*})$
- $K_n^{\text{rel}}(A) = \pi_n(\mathcal{F}_{\tau_*^A})$
- $K_n^{\text{rel}}(B) = \pi_n(\mathcal{F}_{\tau_*^B})$
- $K_n^{\text{rel}}(A) = \pi_n(\mathcal{F}_\phi)$

Thus taking the n th homotopy group results in the desired diagram. \square

Relative K -theory fits in with the larger theory of algebraic topology and functional analysis through the following commutative diagram which has exact horizontal sequences [8]

$$\begin{array}{ccccccc}
K_n^{\text{rel}}(A) & \longrightarrow & K_n^{\text{alg}}(A) & \longrightarrow & K_n^{\text{top}}(A) & \longrightarrow & K_{n-1}^{\text{rel}}(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
HC_{n-1}(A) & \longrightarrow & HH_n(A) & \longrightarrow & HC_n(A) & \longrightarrow & HC_{n-2}(A)
\end{array} \tag{5.12}$$

Here the map $K_n^{\text{rel}}(A) \rightarrow HC_{n-1}(A)$ is the relative Chern character as in [18], $K_n^{\text{alg}}(A) \rightarrow HH_n(A)$ is the Dennis trace and $K_n^{\text{top}}(A) \rightarrow HC_n(A)$ is the Chern character. This gives rise to the commutative diagram

$$\begin{array}{ccccccc} K_n^{\text{rel}}(A) & \longrightarrow & K_n^{\text{alg}}(A) & \longrightarrow & K_n^{\text{top}}(A) & \longrightarrow & K_{n-1}^{\text{rel}}(A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ HC_{n-1}(A) & \longrightarrow & HC_n^-(A) & \longrightarrow & HC_n^{\text{per}}(A) & \longrightarrow & HC_{n-2}(A) \end{array} \quad (5.13)$$

which again has exact rows in which all of the vertical maps are given by different versions of the Chern character. Properties and definitions of the functors and sequence in the diagram can be found in [7], [8], [18], and [23].

The Chern character $K_n^{\text{alg}}(A) \rightarrow HC_n^-(A)$ to the negative cyclic homology as in the above diagram and which is described in [23] and the relative Chern character have relative versions $K_n^{\text{alg}}(A; I) \rightarrow HC_n^-(A; I)$ and $K_n^{\text{rel}}(A; I) \rightarrow HC_{n-1}(A; I)$.

When A is an algebra with a closed ideal I and I has trivial topological K -theory groups, the sequence

$$0 = K_{n+1}^{\text{top}}(I) \rightarrow K_n^{\text{rel}}(A; I) \rightarrow K_n^{\text{alg}}(A; I) \rightarrow K_n^{\text{top}}(I) = 0$$

gives an isomorphism $K_n^{\text{rel}}(A; I) \cong K_n^{\text{alg}}(A; I)$ and so the relative Chern character gives a map $K_n^{\text{alg}}(A; I) \rightarrow HC_{n-1}(A; I)$.

If A and B are Banach algebras with a map $A \rightarrow B$ and $I \triangleleft A$ a closed ideal which maps isomorphically to a closed ideal $J \triangleleft B$ then I can be identified with J . Using the notation

$$\mathcal{F}^{\text{rel}}(A; I) = \text{hfiber} \left(\mathcal{F}_{(\tau_*^A)^+} \rightarrow \mathcal{F}_{(\tau_*^{A/I})^+} \right)$$

$$\mathcal{F}^{\text{alg}}(A; I) = \text{hfiber} \left((BGL(A)^\delta)^+ \rightarrow (BGL(A/I)^\delta)^+ \right)$$

$$\mathcal{F}^{\text{top}}(A; I) = \text{hfiber} (BGL(A) \rightarrow BGL(A/I))$$

define $K_n^{\text{rel}}(A, B; I)$, $K_n^{\text{alg}}(A, B; I)$ and $K_n^{\text{top}}(A, B; I)$ as the n th homotopy groups of the homotopy fibers of the maps

$$\mathcal{F}^{\text{rel}}(A; I) \rightarrow \mathcal{F}^{\text{rel}}(B; J)$$

$$\mathcal{F}^{\text{alg}}(A; I) \rightarrow \mathcal{F}^{\text{alg}}(B; J)$$

$$\mathcal{F}^{\text{top}}(A; I) \rightarrow \mathcal{F}^{\text{top}}(B; J)$$

respectively. Since topological K -theory satisfies excision, the map on topological K -theory is an isomorphism so that there is a commutative diagram with exact columns given by

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K_{n+1}^{\text{top}}(A; I) & \longrightarrow & K_{n+1}^{\text{top}}(B; I) & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 K_n^{\text{rel}}(A, B; I) & \longrightarrow & K_n^{\text{rel}}(A; I) & \longrightarrow & K_n^{\text{rel}}(B; I) & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 K_n^{\text{alg}}(A, B; I) & \longrightarrow & K_n^{\text{alg}}(A; I) & \longrightarrow & K_n^{\text{alg}}(B; I) & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & K_n^{\text{top}}(A; I) & \longrightarrow & K_n^{\text{top}}(B; I) & &
 \end{array}$$

which implies that $K_n^{\text{rel}}(A, B; I) \cong K_n^{\text{alg}}(A, B; I)$ for $n \geq 1$. Thus using the connection between relative K -theory and cyclic homology to extend the relative Chern character via the diagram

$$\begin{array}{ccccccc}
 K_n^{\text{rel}}(A, B; I) & \longrightarrow & K_n^{\text{rel}}(A; I) & \longrightarrow & K_n^{\text{rel}}(B; I) & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 HC_n(A, B; I) & \longrightarrow & HC_n(A; I) & \longrightarrow & HC_n(B; I) & &
 \end{array}$$

gives a map $K_n^{\text{alg}}(A, B; I) \rightarrow HC_n(A, B; I)$.

These connections between relative and algebraic K -theory and cyclic homology is worth exploring. In [21] and [22] Lesch, Moscovici, and Pflaum present constructions of K -theory invariants which they use in the study of pseudodifferential operators. providing further connections between K -theory and other homology theories through the relative K -theory groups should enable more results in this area.

Chapter 6

Outlook

6.1 Simplicial Methods

Using simplicial methods as discussed in the appendix, the weak equivalence between $|S_\bullet(X)|$ and X enables a more explicit description of the relative K -theory groups. The following methods work for more general Fréchet algebras, but for this paper, the focus will remain on Banach algebras.

Beginning with a Banach algebra, A , one can form the simplicial Banach algebras A , which is simply the Banach algebra A in each degree, and the singular simplicial algebra, $S_\bullet A$. Applying the general linear functor to each of these results in simplicial groups $\mathrm{GL}(A)$ and $\mathrm{GL}(S_\bullet A)$.

Since $\Omega B\mathrm{GL}(A) = \mathrm{GL}(A)$ and $\Omega B\mathrm{GL}(A_\bullet) = \mathrm{GL}(A_\bullet)$, it follows that there is a homotopy fiber sequence

$$\mathrm{GL}(A) \rightarrow \mathrm{GL}(S_\bullet A) \rightarrow \mathcal{F}_\bullet \rightarrow B\mathrm{GL}(A) \rightarrow B\mathrm{GL}(S_\bullet A)$$

so that \mathcal{F}_\bullet is weakly equivalent to $\mathrm{GL}(S_\bullet A)/\mathrm{GL}(A)$. [3] and [4] contain nice descriptions of the interactions between the plus construction and homotopy fibers. In particular [4] characterizes homotopy fiber sequences which are still homotopy fiber sequences after the plus construction is applied. The condition that is necessary and sufficient for such a sequence $F \rightarrow E \rightarrow B$ is that the maximal perfect subgroup of $\pi_1(B)$ acts on F^+ by maps freely homotopic to the identity. Since $\pi_1(B\mathrm{GL}(S_\bullet A))$ is abelian, this condition is satisfied so that applying the plus-construction results in the homotopy fiber sequence

$$(\mathrm{GL}(S_\bullet A)/\mathrm{GL}(A))^+ \rightarrow B\mathrm{GL}(A)^+ \rightarrow B\mathrm{GL}(S_\bullet A) \tag{6.1}$$

. It then follows that $K_n^{\text{rel}}(A) = \pi_n((\text{GL}(S_\bullet A)/\text{GL}(A))^+)$ for $n \geq 1$.

For a Banach algebra pair (A, I) , the groups $K_n^{\text{rel}}(A; I)$ are therefore the homotopy groups of the homotopy fiber of the map $(\text{GL}(S_\bullet A)/\text{GL}(A))^+ \rightarrow (\text{GL}(S_\bullet(A/I))/\text{GL}(A/I))^+$

When (A, B) is a C^* -algebra pair, it follows that this homotopy fiber is $(\text{GL}(S_\bullet I)/\text{GL}(I))^+$. In fact, using the five-lemma, one can view the result of Suslin and Wodzicki [34] as characterizing the Banach algebra pairs for which $K_\bullet^{\text{rel}}(A; I) \cong K_\bullet^{\text{rel}}(I)$.

6.2 Fréchet Algebra Issues

Fréchet algebras are a generalization of Banach algebras in which the topology is given by a countable family of submultiplicative seminorms. This generalization makes defining K -theory more difficult. In [27], Phillips provides a definition for the topological K -theory of a locally multiplicatively convex Fréchet algebra

Using simplicial methods as above, Karoubi's definition of relative K -theory applies to such Fréchet algebras as well and sequence 6.1 is still a homotopy fiber sequence (for details, see [18]). In trying to define the groups $K_n^{\text{rel}}(A; I)$ for locally multiplicatively convex Fréchet algebras, a problem that arises is that the diagram

$$\begin{array}{ccc} (BGL(A)^\delta)^+ & \xrightarrow{(\tau_*^A)^+} & BGL(A) \\ \downarrow (p_*^\delta)^+ & & \downarrow (p_*)^+ \\ (BGL(A/I)^\delta)^+ & \xrightarrow{(\tau_*^{A/I})^+} & BGL(A/I) \end{array}$$

cannot necessarily be made to commute for Fréchet algebras as it can for Banach algebras so that theorem 32 does not hold in the case of Fréchet algebras. A possible approach is to take a step back and apply proposition 15 to the diagram

$$\begin{array}{ccc} BGL(A)^\delta & \xrightarrow{\tau_*^A} & BGL(A) \\ \downarrow p_*^\delta & & \downarrow (p_*)^+ \\ BGL(A/I)^\delta & \xrightarrow{\tau_*^{A/I}} & BGL(A/I) \end{array}$$

to obtain the commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}_{\tilde{p}_*} & \longrightarrow & \mathcal{F}_{p_*^\delta} & \xrightarrow{\tilde{\tau}_*} & \mathcal{F}_{p_*} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{F}_{\tau_*^A} & \longrightarrow & BGL(A)^\delta & \xrightarrow{\tau_*^A} & BGL(A) \\
\downarrow \tilde{p}_* & & \downarrow p_*^\delta & & \downarrow p_* \\
\mathcal{F}_{\tau_*^{A/I}} & \longrightarrow & BGL(A/I)^\delta & \xrightarrow{\tau_*^{A/I}} & BGL(A/I)
\end{array}$$

in which all rows and columns are homotopy fiber sequences. Since $\pi_n(\mathcal{F}_{p_*}) = K_n^{\text{top}}(I)$ is abelian, all three rows as well as the column on the right are plus-constructive by [4]. The other two column sequences are not necessarily so that applying the plus construction to this results in the diagram

$$\begin{array}{ccccccccc}
\Omega^2(BGL(A)^\delta)^+ & \longrightarrow & \Omega GL(A) & \longrightarrow & \Omega \mathcal{F}_{\tau_*^A}^+ & \longrightarrow & \Omega(BGL(A)^\delta)^+ & \longrightarrow & GL(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^2(BGL(A/I)^\delta)^+ & \longrightarrow & \Omega GL(A/I) & \longrightarrow & \Omega \mathcal{F}_{\tau_*^{A/I}}^+ & \longrightarrow & \Omega(BGL(A/I)^\delta)^+ & \longrightarrow & GL(A/I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega \mathcal{F}_{p_*^\delta}^+ & \longrightarrow & \Omega \mathcal{F}_{p_*}^+ & \longrightarrow & \mathcal{F}_{\tilde{p}_*}^+ & \longrightarrow & \mathcal{F}_{p_*^\delta}^+ & \longrightarrow & \mathcal{F}_{p_*}^+ \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega(BGL(A)^\delta)^+ & \longrightarrow & GL(A) & \longrightarrow & \mathcal{F}_{\tau_*^A}^+ & \longrightarrow & (BGL(A)^\delta)^+ & \longrightarrow & BGL(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega(BGL(A/I)^\delta)^+ & \longrightarrow & GL(A/I) & \longrightarrow & \mathcal{F}_{\tau_*^{A/I}}^+ & \longrightarrow & (BGL(A/I)^\delta)^+ & \longrightarrow & BGL(A/I)
\end{array}$$

which is homotopy commutative and for which the rows are homotopy exact, but the only columns that are homotopy exact are the second and fifth columns. This means that taking the homotopy groups of this diagram does not give a diagram involving $K_n^{\text{alg}}(A; I)$ at all in general so it is unclear how to proceed in defining $K_n^{\text{rel}}(A; I)$ in this situation. In light of this, it would be of interest to characterize Fréchet algebras to which the definition of $K_n^{\text{rel}}(A; I)$ can be extended in such a way that a diagram is obtained as in the case of Banach algebras.

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Appendix A

Simplicial Methods

The following is a brief treatment of simplicial objects following closely a combination the lecture notes of Pflaum and the development in May's Book [25] where all of these definitions and results can be found. A modern treatment can be found in [13] as well.

A.1 Simplicial Objects

The category **Simp** (often denoted Δ^\bullet) has objects $\langle n \rangle = \{0, 1, \dots, n\}$ for $n = 1, 2, 3, \dots$. The morphisms are nondecreasing maps. The following maps, called the **face** and **degeneracy** maps,

$$\delta_{n,i} : \langle n-1 \rangle \rightarrow \langle n \rangle, \quad l \mapsto \begin{cases} l & 0 \leq l < i \\ l+1 & i \leq l < n \end{cases}$$
$$\sigma_{n,i} : \langle n+1 \rangle \rightarrow \langle n \rangle, \quad l \mapsto \begin{cases} l & 0 \leq l \leq i \\ l-1 & i < l \leq n+1 \end{cases}$$

are nondecreasing and hence morphisms in **Simp**. We will be particularly interested in these maps.

In the category **Simp**, the only isomorphisms are the identity morphisms and the face and degeneracy maps satisfy the following commutativity relations:

- $\delta_{n+1,j} \delta_{n,i} = \delta_{n+1,i} \delta_{n,j-1}$ for $i < j$
- $\sigma_{n-1,j} \sigma_{n,i} = \sigma_{n-1,i} \sigma_{n,j+1}$ for $i < j$

$$\bullet \sigma_{n,j} \delta_{n+1,i} = \begin{cases} \delta_{n,i} \sigma_{n-1,j-1} & i < j \\ id_{\langle n \rangle} & i = j, j+1 \\ \delta_{n,i-\sigma_{n-1,j}} & j+1 < i \end{cases}$$

In most cases the domain and range of the face and degeneracy maps is clear and so they will be written as δ_i and σ_j respectively. The face and degeneracy maps can be seen as generating the morphisms in \mathbf{Simp} because every morphism $f : \langle n \rangle \rightarrow \langle m \rangle$ can be written uniquely as

$$f = \delta_{i_r} \circ \dots \circ \delta_{i_1} \circ \sigma_{j_1} \circ \dots \circ \sigma_{j_s}$$

with $i_1 < \dots < i_r$ and $j_1 < \dots < j_s$.

Definition 34. Let \mathbf{C} be a category. A simplicial object in \mathbf{C} is a contravariant functor

$$X_\bullet : \mathbf{Simp} \rightarrow \mathbf{C}$$

If the functor is covariant, then we call it a cosimplicial object.

Because every morphism in \mathbf{Simp} can be factored into face and degeneracy maps as described above, a simplicial object in \mathbf{C} is characterized by the image of the objects $\langle n \rangle$ and the face and degeneracy maps in \mathbf{C} . Often a simplicial object will be described by this information rather than explicitly as a functor.

Example A.1.1. The geometric n -simplex, $\Delta_n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | t_i \in [0, 1], \sum t_i = 1\}$ with $\delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$ and $\sigma_j(t_0, \dots, t_{n+1}) = (t_0, \dots, t_j t_{j+1}, \dots, t_{n+1})$ is a cosimplicial object.

The subscript (n) will be used to signify that a point $t_{(n)} = (t_0, \dots, t_n)$ is in \mathbb{R}^{n+1} .

Definition 35. A point $t_{(n)} \in [0, 1]^n$ is called an interior point if $n = 0$ or if $t_{(n)} \in ([0, 1]^{n+1})^\circ$.

Note that any $t_{(n)}$ can be uniquely expressed as $\delta_{i_k} \circ \dots \circ \delta_{i_1} \tilde{t}_{(n-k)}$ where $\tilde{t}_{(n-k)} \in \Delta_{n-k}$ is an interior point and $0 \leq i_1 < \dots < i_k \leq n$.

Definition 36. Given a simplicial set X_\bullet , a point $x_{(n)}$ is called **nondegenerate** if there is no $y_{(n-1)}$ and i for which $x_{(n)} = s_i y_{(n-1)}$. A point which is not nondegenerate is called **degenerate**.

A.2 Fibrant Complexes

Definition 37. A **fibrant** complex is a simplicial complex with the property that if for every collection of $n + 1$ n -simplices $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}$ such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$ and $i, j \neq k$, there exists an $(n + 1)$ -simplex x such that $\partial_i x = x_i$ for $i \neq k$. This will be referred to as the **Kan extension condition**. Fibrant complexes are also referred to as Kan complexes.

An alternative definition of fibrant is that the map from X_\bullet to the terminal object Δ^0 is a fibration as defined below for simplicial sets.

Definition 38. A map $p : X_\bullet \rightarrow Y$ of simplicial sets is called a **fibration** if it satisfies the following extension condition: If $(x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ is an n -tuple of simplices in X such that $\partial_i x_j = \partial_{j-1} x_i$ for $i < j$ and $i, j \neq k$, and there is an n -simplex y of Y for which $\partial_i y = p(x_i)$ for all $i \neq k$, then there is an n -simplex x in X such that $\partial_i x = x_i$ for all $i \neq k$ and $p(x) = y$.

Example A.2.1. Let X be a topological space. The **singular complex** $S_\bullet(X)$, where $S_n(X)$ is the set of singular n -simplices $f : \Delta_n \rightarrow X$, is a fibrant complex with face and degeneracy maps given by

$$(\partial_i f)(t_0, \dots, t_{n-1}) = f(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

and

$$(s_i f)(t_0, \dots, t_{n+1}) = f(t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+1}, \dots, t_{n+1}).$$

This is because the union of a collection of $n + 1$ faces of Δ_{n+1} is a retract of Δ_{n+1} so that any continuous function on their union can be extended to all of Δ_{n+1} .

Example A.2.2. The underlying set of any simplicial group is fibrant.

A.3 Simplicial Homotopy

Definition 39. Two n -simplices, x and x' in a complex K are said to be **homotopic** if

$$(1) \quad \partial_i x = \partial_i x'.$$

(2) There is a simplex $y \in K_{n+1}$ such that

$$\partial_i y = \begin{cases} x & i = n \\ x' & i = n + 1 \\ s_{n-1} \partial_i x & 0 \leq i < n \end{cases}$$

In this case, the simplex y is called a **homotopy** from x to x' and we write $x \sim x'$.

Proposition 40. *If K is fibrant, then homotopy is an equivalence relation on the n -simplices of K for $n \geq 0$.*

Proof. To show that \sim is reflexive, suppose that $x = x'$ is an n -simplex. The element $s_n x$ is a homotopy from x to x since

$$(1) \quad \partial_n s_n x = \partial_{n+1} s_n x \text{ and}$$

$$(2) \quad \partial_i s_n x = s_{n-1} \partial_i x \text{ for } 0 \leq i < n.$$

To see that \sim is symmetric and reflexive, let $x, x', x'' \in K_n$ such that y' and y'' are homotopies from x to x' and from x to x'' respectively. Then since the $(n+1)$ -simplices

$$\partial_0 s_n s_n x', \dots, \partial_{n-1} s_n s_n x', y', y''$$

satisfy the compatibility condition for a fibrant complex, there must be an $(n+2)$ -simplex, z , for which $\partial_i z = \partial_i s_n s_n x'$ for $0 \leq i < n$, $\partial_n z = y'$ and $\partial_{n+1} z = y''$. We therefore have that

$$\partial_i \partial_{n+2} z = \begin{cases} x' & i = n \\ x'' & i = n + 1 \\ s_{n-1} \partial_i x' & 0 \leq i < n \end{cases}$$

so that $x' \sim x''$.

□

Definition 41. Let K be a complex with subcomplex L . Two n -simplices, x and x' are said to be **homotopic relative to L** if

$$(1) \partial_i x = \partial_i x'.$$

$$(2) \partial_0 x \sim \partial_0 x' \text{ in } L.$$

(3) there is a homotopy y from $\partial_0 x$ to $\partial_0 x'$ and a simplex $w \in K_{n+1}$ such that

$$\partial_i w = \begin{cases} y & i = 0 \\ x & i = n \\ x' & i = n + 1 \\ s_{n-1} \partial_i x & 1 \leq i < n \end{cases}$$

In this case, the simplex w is called a **relative homotopy** from x to x' and we write $x \sim_L x'$.

The proof that \sim_L is an equivalence relation is analogous to the proof above.

Definition 42. Let K be a complex with subcomplex L and $* \in L_0 \subset K_0$. We can form a subcomplex $*$ of L (and therefore of K) defined by $*_n = \{s_{n-1} \dots s_0 * \}$ for all n . We will denote this complex and every one of its simplices by $*$ when there is no possibility of confusion.

When K is fibrant, $(K, *)$ with the obvious maps is a **Kan pair**. If, additionally, L is a fibrant subcomplex, $(K, L, *)$ with the obvious maps is called a **Kan triple**.

Definition 43 (Simplicial homotopy). Let (K, ϕ) be a Kan pair and for $n \geq 0$ set

$$\tilde{K}_n := \{x \in K_n : \partial_i x = *, 0 \leq i \leq n\}.$$

We define

$$\pi(K, *) := \tilde{K}_n / \sim.$$

Let $(K, L, *)$ be a Kan triple and for $n > 0$ set

$$\tilde{K}(L)_n := \{x \in K_n : \partial_0 x \in L_{n-1}; \partial_i x = *, 1 \leq i \leq n\}.$$

We define

$$\pi(K, *) := \tilde{K}(L)_n / \sim_L .$$

The equivalence class $[x]$ of a simplex x is called the **homotopy class** of x .

We define a map

$$\partial : \pi(K, L, *) \rightarrow \pi(L, *)$$

by $\partial[x] = [\partial_0 x]$.

It is important to note that $\pi(K, *, *) = \pi(K, *)$ so that for a Kan triple $(K, L, *)$ with inclusions $i : L \rightarrow K$ and $j : * \rightarrow L$, there is a long exact sequence

$$\dots \rightarrow^{j^*} \pi_{n+1}(K, L, *) \rightarrow^{\partial^*} \pi_n(L, *) \rightarrow^{i^*} \pi_n(K, *) \rightarrow^{j^*} \pi_n(K, L, *) \rightarrow^{\partial^*} \dots$$

Definition 44. Let $(K, *)$ be a Kan pair, let $n \geq 1$ and let $[x], [y] \in \pi_n(K, *)$. Since the $n+1$ n -simplices $*, \dots, *, x, y$ satisfy the Kan extension condition, we can find an $(n+1)$ -simplex z such that

$$\partial_i z = \begin{cases} * & 0 \leq i < n-1 \\ x & i = n-1 \\ y & i = n+1 \end{cases}$$

We can then define a product on $\pi_n(K, *)$ by $[x][y] = [\partial_n z]$.

Now let $(K, L, *)$ be a Kan triple and let $n > 1$. For $[x], [y] \in \pi_n(K, L, *)$. Since $[\partial_0 x] = \partial[x]$ and $[\partial_0 y] = \partial[y]$ are elements of $\pi_{n-1}(L, *)$. It is possible to choose $z \in L_n$ such that $[\partial_0 x][\partial_0 y] = [\partial_{n-1} z]$. Then the $n+1$ n -simplices $z, *, \dots, *, x, y$ satisfy the Kan extension condition so there is an $(n+1)$ -simplex w for which

$$\partial_i w = \begin{cases} z & i = 0 \\ * & 0 < i < n-1 \\ x & i = n-1 \\ y & i = n+1 \end{cases}$$

Hence there is a product operation on $\pi_n(K, L, *)$ defined by $[x][y] = [\partial_n w]$.

With the multiplication defined above, $\pi_n(K, *)$ and $\pi_{n+1}(K, L, *)$ are groups for $n \geq 1$. If $n \geq 2$, the groups are abelian. Furthermore, π_n is a functor from the category of Kan pairs to the category of groups for $n \geq 1$ and to the category of sets for $n = 0$ and π_n is a functor from the category of Kan triples to the category of groups for $n \geq 2$ and to the category of sets for $n = 1$. In light of this, ∂ is a natural transformation for $n \geq 1$.

This makes the sequence

$$\dots \rightarrow^{j_*} \pi_{n+1}(K, L, *) \rightarrow^{\partial_*} \pi_n(L, *) \rightarrow^{i_*} \pi_n(K, *) \rightarrow^{j_*} \pi_n(K, L, *) \rightarrow^{\partial_*} \dots$$

an exact sequence of groups for $n \geq 1$.

In [?], in order to prove that $\pi_n(K_\bullet)$ is abelian when $n \geq 2$, a simplicial loop space functor Ω is constructed with the property that $\pi_n(\Omega K_\bullet) \cong \pi_{n+1}(K_\bullet)$. As with topological spaces, ΩK_\bullet is defined to be the fiber of the map from PK_\bullet , a suitable path space of K_\bullet , to K_\bullet .

Definition 45. Let f and g be simplicial maps from K to L . We say that f is **homotopic** to g and write $f \simeq g$ if there exist functions $h_i : K_q \rightarrow L_{q+1}$ with $0 \leq i \leq q$, such that

$$(1) \quad \partial_0 h_0 = f, \quad \partial_{q+1} h_q = g$$

$$(2) \quad \partial_{j+1} h_{j+1} = h_j \partial_{j+1}$$

$$(3) \quad \partial_i h_j = h_{j-1} \partial_i \text{ if } i < j$$

$$(4) \quad \partial_i h_j = h_j \partial_{i-1} \text{ if } i \geq j$$

$$(5) \quad s_i h_j = h_{j+1} s_i \text{ if } i \leq j$$

$$(6) \quad s_i h_j = h_j s_{i-1} \text{ if } i > j.$$

often the subscript will be dropped and the family of maps $\{h_i\}$ will be denoted by h . h is called a **homotopy** from f to g . This is denoted by $h : f \simeq g$ or simply $f \simeq g$ when it is not important to specify the maps.

If, additionally, K' and L' are subcomplexes of K and L respectively with $f(K'), g(K'), h(K') \subset L'$ then h is a **relative homotopy** from f to g if $h|_{K'}$ is a homotopy from $f|_{K'}$ to $g|_{K'}$.

If the identity map of K is homotopic relative to K' to a map from K onto K' which extends the inclusion of K' into K then K' is called a **deformation retract** of K .

K and L said are to be of the same **homotopy type** if there are maps $f : K \rightarrow L$ and $g : L \rightarrow K$ with $gf = 1_K$ and $fg = 1_L$.

Definition 46. A simplicial map $\pi : E \rightarrow B$ is called a **Kan fibration** if for every collection of $n + 1$ n -simplices, $x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1} \in E$ which satisfy the Kan extension condition and for every simplex $y \in B_{n+1}$ such that $\partial_i y = \pi(x_i)$ for $i \neq k$, there exists an simplex $x \in E_{n+1}$ such that $\partial x = x_i$ for $i \neq k$, and $\pi(x) = y$.

In this case E and B are respectively called the **total complex** and **base complex** of the **fiber space** (E, π, B) .

For the subcomplex $*$ of B . The complex $F = \pi^{-1}(*) \subset E$ is called the **fiber** over $*$ $\in B$ and there is **fiber sequence** sequence.

$$(F, *') \hookrightarrow (E, *') \xrightarrow{\pi} (B, *)$$

Definition 47. Let $X : \text{Simp} \rightarrow \text{Top}$ and consider the topological space

$$X^{\sqcup} = \bigsqcup_{n \in \mathbb{N}} X_n \times \Delta_n$$

A relation can be defined on X^{\sqcup} which is generated by the relations

$$(d_i x_{(n)}, t_{(n-1)}) \sim (x_{(n)}, \delta_i t_{(n-1)})$$

and

$$(s_i x_{(n)}, t_{(n+1)}) \sim (x_{(n)}, \sigma_i t_{(n+1)})$$

to obtain the space X^{\sqcup} / \sim which is called the **geometric realization** and denote by $|X_{\bullet}|$.

The geometric realization is functorial. In particular, given a continuous map $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$, the induced map $|f_{\bullet}| : |X_{\bullet}| \rightarrow |Y_{\bullet}|$ which sends $|x_{(n)}, t_{(n)}|$ to $|f(x_{(n)}), t_{(n)}|$ is continuous.

From a homotopic point of view, the geometric realization does indeed represent the “geometry” of a simplicial object since the simplicial homotopy groups of a fibrant simplicial set X_{\bullet} agree

with the topological homotopy groups of the the geometric realization $|X_\bullet|$. A nice consequence of this is that for any fibrant simplicial set X_\bullet , $|\Omega X_\bullet|$ is weakly equivalent to $\Omega|X_\bullet|$.

In addition, the geometric realization as a functor from \mathbf{SSet} , the category of simplicial sets, to \mathbf{Top} is left adjoint to the singular complex functor. That is, there is an isomorphism

$$\mathrm{Hom}_{\mathbf{Top}}(|X_\bullet|, Y) \cong \mathrm{Hom}_{\mathbf{SSet}}(X_\bullet, S_\bullet Y) \quad (\text{A.1})$$

which is natural in \mathbf{Top} and \mathbf{SSet} . There is then a weak equivalence between a space X and $|S_\bullet X|$ when X is a CW-complex.

There is a simplicial version of the classifying space which will be needed to define K -theory simplicially.

Definition 48. *Let G be a group. The **nerve** of G , $B_\bullet G$ is defined by $B_k G := G^k$. If the face and degeneracy maps are defined by*

$$d_i(g_1, \dots, g_n) = \begin{cases} (g_2, g_3, \dots, g_n) & i = 0 \\ (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) & 1 \leq i < n \\ (g_1, \dots, g_{n-1}) & i = n \end{cases}$$

and

$$s_i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, 1, g_i, \dots, g_n)$$

then $(B_\bullet G, d_i, s_i)$ is a simplicial set.

Computing the homotopy groups results in

$$\tilde{B}G_n = \begin{cases} G & n = 1 \\ 0 & n > 1 \end{cases} \quad (\text{A.2})$$

So $\pi_1(B_\bullet G) = G$ and $\pi_n(B_\bullet G) = 0$ for $n > 1$ so it follows that the geometric realization of the nerve of G is the classifying space of G .