ON THE STRUCTURE OF POLYNOMIALLY BOUNDED DOL SYSTEMS +

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SUMMARY

Polynomially bounded DOL systems are investigated. Necessary and sufficient conditions on the set of productions of a DOL system for yielding a polynomially bounded growth function are given. The relation between the minimal degree of a polynomial bounding a DOL system and the structure of its set of productions is investigated.

I. INTRODUCTION

Developmental systems (also called Lindenmayer systems or L-systems) have recently gained considerable attention in both formal language theory and theoretical biology (see, e.g., [2], [4], [6] and their references).

From the biological point of view, developmental systems have provided a useful theoretical framework within which the nature of cellular behavior in development can be discussed, computed and compared. From the formal language theory point of view they have provided us with an alternative to the now standard Chomsky framework (see, e.g., [3]) for defining languages.

Among the developmental systems which are under active investigation now are the so called D0L systems (see, e.g., [1], [2], [7], [8], [9], [10]). A D0L system has the following components

(i) A finite set of symbols, V, <u>the alphabet</u>,

(ii) A starting string, ω, the <u>axiom</u>,

(iii) A finite set of <u>productions</u> which tell us by what strings over the alphabet V a symbol may be replaced. This set is such that for each symbol in V it provides exactly one string over V by which the symbol can be replaced.

A D0L system generates a sequence of strings as follows.

The first string in the sequence is the axiom ω. The second string is obtained by replacing each symbol of ω by a string in accordance with the productions. Similarly, the third string is found from the second string, and so on.

One of the particularly interesting aspects in the theory of D0L systems is the theory of growth functions (see, e.g., [5], [7], [11], [12], [13]). A growth function is a function which yields, for each integer n, the length of the nth string in the sequence of its associated D0L system. Hence when we consider growth functions we are concerned with the length of a word rather than with the patterns of letters occurring within a word.

In this paper we investigate D0L systems in which the growth function is bounded by a polynomial (the so called polynomially bounded D0L systems). We study the necessary and sufficient conditions which the set of productions of a given D0L system must satisfy in order to be polynomially bounded. The way in which the detailed local properties of the components of a developmental system, i.e. the production rules, affect the behavior of the system as a whole, e.g., the growth function, is important from both the biological and formal language points of view. For a discussion of this point see, e.g., [5], [7] and [13].

We assume the reader to be familiar with basic formal language theory, e.g. in the scope of [3]. Our notation and terminology is that of [3] with the following extensions:

(i) If x is a word then $|x|$ denotes its length, Min x denotes the set of letters occurring in x, and if k is a positive integer then x^k abbreviates x catenated with itself k times (by definition x^0 is the empty word).

(ii) The empty word is denoted by Λ, and $\emptyset$ denotes the empty set.

(iii) If A is a finite set, then $\#A$ denotes its cardinality.

II. DOL SYSTEMS

In this section we review main notions from the theory of DOL systems needed for this paper.

Definition 1. A DOL system is a triple

$$G = \langle V, P, \omega \rangle,$$

where V is a finite nonempty set (of letters), called the alphabet of G,

ω is a nonempty word over V, called the axiom of G,

P is a finite nonempty set (of productions), each of which is of the form $a \to \alpha$, where $a \in V$ and $\alpha \in V^*$. We require that for each a in V there exists exactly one α in V^* such that $a \to \alpha$ is in P.

(We shall often write $a \underset{P}{\to} \alpha$ for "$a \to \alpha$ is in P").

Definition 2. Let $G = \langle V, P, \omega \rangle$ be a DOL system, and let x be in V^+, y be in V^*. We say that x directly derives y in G, denoted as $x \underset{G}{\Rightarrow} y$, if $x = a_1 \dots a_n$ for some $a_1, \dots, a_n$ in V and $y = \alpha_1 \dots \alpha_n$ for $\alpha_1, \dots, \alpha_n$ in V^* such that $a_1 \to \alpha_1$, $a_2 \to \alpha_2, \dots, a_n \to \alpha_n$ are in P. Also, by definition, $\Lambda \underset{G}{\Rightarrow} \Lambda$. We say that x derives y in G, denoted as $x \overset{*}{\underset{G}{\Rightarrow}} y$, if either $x = y$, or there exists a sequence $x_0, \dots, x_m$ $(m \geq 1)$ of words in V^* such that $x = x_0$, $y = x_m$, and $x_i \underset{G}{\Rightarrow} x_{i+1}$ for $i \in \{0, \dots, m-1\}$. The language of G (or the language generated by G), denoted as $L(G)$, is defined by

$L(G) = \{x : \omega \overset{*}{\underset{G}{\Rightarrow}} x\}$.

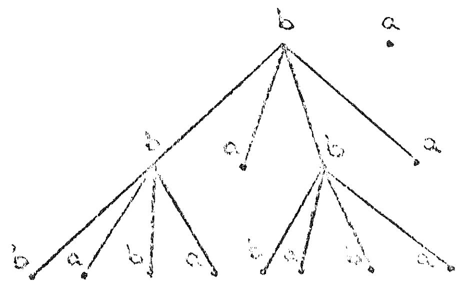
<u>Example 1.</u> $G = \langle\{a,b\}, \{a \to \Lambda, b \to (ba)^2\}, ba\rangle$ is a D0L system the language of which is $\{(ba)^{2^n} : n \geq 0\}$.

<u>Example 2.</u> $G = \langle\{a,b\}, \{a \to ab^2, b \to b\}, ab^2\rangle$ is a D0L system the language of which is $\{ab^{2n} : n \geq 1\}$.

<u>Definition 3.</u> Let $G = \langle V,P,\omega\rangle$ be a D0L system, and for $m \geq 1$ let $x_0, x_1, \ldots, x_m$ be a sequence of words in V^* such that x_0 is a nonempty word and $x_i \underset{G}{\Rightarrow} x_{i+1}$ for $i \in \{0,\ldots,m-1\}$. It is called a <u>derivation of x_m from x_0 in G.</u> (If $x_0 = \omega$ then we say that it is a <u>derivation of x_m in G</u>). For $0 \leq i \leq m$, $\delta_G(i,x_0)$ denotes the word x_i. When $x_0 = \omega$ then we simply write $\delta(G,i)$ instead of $\delta_G(i,\omega)$. If x is in V^+, then the <u>sequence for x in G</u>, is defined to be the infinite sequence $x, \delta_G(1,x), \delta_G(2,x), \ldots$. When $x = \omega$ then this sequence is called the <u>sequence of G</u>, or the sequence <u>generated by G</u>, and it is denoted by $E(G)$.

A particularly useful way of representing derivations in D0L systems are the so called <u>derivation graphs</u> (see, e.g. Herman & Rozenberg, 1973) which, for the purpose of this paper, are best explained by examples.

If G is the D0L system of Example 1, then the derivation ba, $(ba)^2$, $(ba)^4$ of $(ba)^4$ from ba is represented by the following derivation graph



Obviously in the same way, if $G = \langle V,P,\omega\rangle$ is a DOL system, and x is in V^+ then the sequence for x in G can be represented by a graph (which in general can be infinite). In particular if x is in V then such a graph is called the tree for x (in G). We say that the graph representing the sequence for x in G is of bounded width if the sequence contains a finite number of different words only.

Definition 4. A DOL system G is called polynomially bounded if there exists a polynomial[1)] F such that for every $m \geq 0$

$$|\delta(G,m)| \leq F(m).$$

(We also say that G is bounded by F).

Example 3. The DOL system of Example 1 is obviously not polynomially bounded, whereas the DOL system of Example 2 is polynomially bounded (e.g., by the polynomial $F(m) = 3m + 3$).

Definition 5. A DOL system G is called finite if $\Lambda \in L(G)$, otherwise it is called infinite.

Definition 6. Let $G = \langle V,P,\omega\rangle$ be an infinite D0L system, and let m,l be nonnegative integers such that $l \geq 2$. The *$\langle m,l\rangle$-decomposition of G* is the set $G(m)$, $G(m+1),\ldots,G(m+(l-1))$ of D0L systems (each of which is called a *component system* of this decomposition) such that for $j \in \{0,\ldots,l-1\}$, $G(m+j) = \langle V(m+j), P(m+j), \omega(m+j)\rangle$ where

$$V(m+j) = \bigcup_{s\geq 0} \text{Min } \delta(G,m+j+sl),$$

$$\omega(m+j) = \delta(G,m+j), \text{ and}$$

$a \to \alpha$ is in $P(m+j)$ if, and only if, $\delta_G(l,a) = \alpha$.

A set of D0L systems is called a *decomposition of G* if it is the $\langle m,l\rangle$-decomposition of G for some $m \geq 0$ and $l \geq 2$.

Example 4. For the D0L system $G = \langle\{a,b,c\}, \{a\to bc, b\to ac, c\to a\}, a\rangle$ the $\langle 0,2\rangle$-decomposition set consists of $G(0)=\langle\{a,b,c\}, \{a\to aca, b\to bca, c\to bc\}, a\rangle$ and $G(1)=\langle\{a,b,c\}, \{a\to aca, b\to bca, c\to bc\}, bc\rangle$.

It is instructive to observe how the graph representing $E(G)$ for a given D0L system G is sliced up when G is decomposed. In particular one should notice that the graph representing $E(G)$ is of bounded width if and only if each of the graphs representing the sequences of component systems in a given decomposition is of bounded width.

The following result explains how the language and the sequence of a given D0L system relate to the languages and the sequences of component systems in a given decomposition.

We give it here without proof, as it is easy, and besides (for a slightly different notion of a decomposition) it is given in [9].

Theorem 1. Let $G = \langle V,P,\omega \rangle$ be a D0L system, and $G(m), G(m+1), \ldots, G(m+(l-1))$ its (m,l)-decomposition. Then $L(G) = \{\delta(G,0), \ldots, \delta(G,m-1)\} \cup L(G(m)) \cup L(G(m+1)) \cup \cup \ldots \cup L(G(m+(l-1)))$,

and

$E(G)$ satisfies the following:

for $s \geq m$

$\delta(G,s) = \delta(G(m+r),q)$, where $s=m+ql+r$, with $0 \leq r \leq l$ and $q \geq 0$.

In this paper we investigate polynomially bounded D0L systems. Obviously, if a D0L system G is finite then it is bounded by a polynomial of degree zero. Furthermore the structure of a finite D0L system is not specially interesting, and it is decidable whether an arbitrary D0L system is finite. For these reasons in this paper we investigate infinite D0L systems only, and whenever in the sequel we write "a D0L system" we mean an infinite D0L system.

III. DOL SYSTEMS WITH RANK.

In this section the notion of the rank of a letter and of a DOL system is introduced and some elementary properties of this notion are proved. These are the central notions for this paper.

Definition 7 Let $G = \langle V,P,\omega \rangle$ be a DOL system. A letter a in V is called growing (in G) if for every positive integer n there exists a string α in V* such that $|\alpha|>n$ and $a \overset{*}{\underset{G}{\Rightarrow}} \alpha$. Otherwise a is called nongrowing (in G).

Note that the letter a is nongrowing in G if, and only if, the tree for a in G is of bounded width.

Example 5. If G is the DOL system from Example 2, then a is a growing letter in G, whereas b is a nongrowing letter in G.

Definition 8. Let $G = \langle V,P\ \omega \rangle$ be a DOL system. The reduction sequence of G, denoted as Red(G), is a sequence $G_0, G_1, \ldots, G_p$ of DOL systems defined recursively as follows

1) $G_0 = \langle V_0, P_0, \omega_0 \rangle = G$.

2) Let for $j \geq 0$, $U_j = \{a \in V_j\colon$ a is a nongrowing letter in $G_j\}$.

G_{j+1} is defined if, and only if, $U_j \neq \emptyset$ and $U_j \neq V_j$.

If G_{j+1} is defined, then $G_{j+1} = \langle V_{j+1}, P_{j+1}, \omega_{j+1} \rangle$ where

(i) $V_{j+1} = V_j - U_j$.

(ii) $\omega_{j+1} = a_1 \ldots a_n$ where $\omega_j = \alpha_0 a_1 \alpha_1 \ldots a_n \alpha_n$ for some $n \geq 1, \alpha_0, \ldots, \alpha_n$ in U_j^* and $a_1, \ldots, a_n \in V_{j+1}$.

(iii) For $a, a_1, \ldots, a_n$ in V_{j+1}, $a \to a_1 \ldots a_n$ is in P_{j+1} if, and only if, $a \underset{P_j}{\to} \alpha_0 a_1 \alpha_1 \ldots a_n \alpha_n$ for some $n \geq 1, \alpha_0, \ldots, \alpha_n$ in U_j^*.

Definition 9. Let $G = \langle V, P, \omega \rangle$ be a DOL system, where $\text{Red}(G) = G_0, G_1, \ldots, G_p$. A rank in G, denoted as ρ_G, is a (partial) function from (a subset of) V into the positive integers defined recursively as follows:

1) If a is a nongrowing letter in V then $\rho_G(a) = 1$

2) If a is a growing letter in V, then $\rho_G(a) = n+1$ if, and only if, $\rho_{G_1}(a) = n$.

If a is in V and $\rho_G(a)$ is defined then we say that a is a letter with rank.

Example 6. Let $G = \langle \{a,b,c\}, \{a \to ab, b \to bc, c \to c\}, abc \rangle$. The reduction sequence of G is the following sequence G_0, G_1, G_2 of DOL systems:

$G_0 = G, G_1 = \langle \{a,b\}, \{a \to ab, b \to b\}, ab \rangle$, $G_2 = \langle \{a\}, \{a \to a\}, a \rangle$

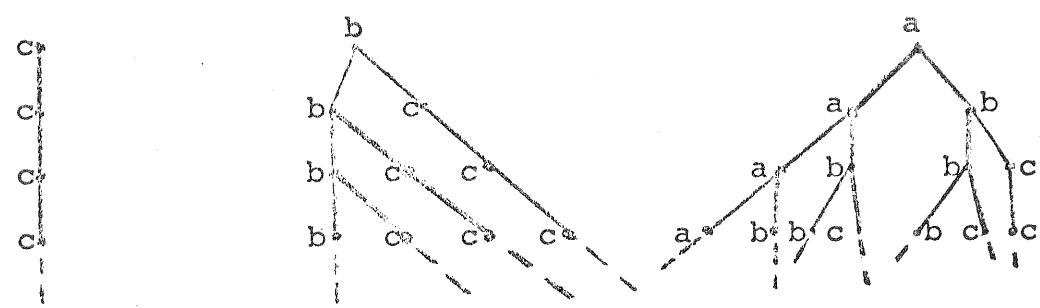
Ranks of letters are computed as follows:

$\rho_G(c)=1$,

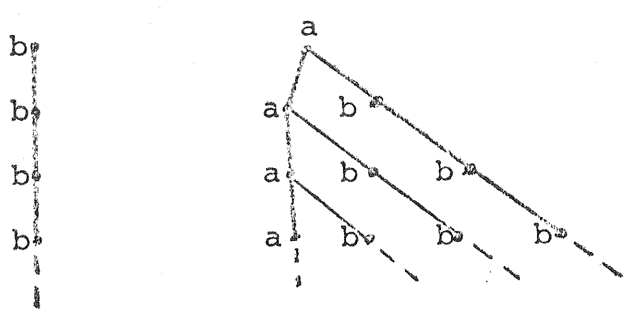
$\rho_G(b)=\rho_{G_0}(b)=\rho_{G_1}(b)+1=2$,

$\rho_G(a)=\rho_{G_0}(a)=\rho_{G_1}(a)+1=\rho_{G_2}(a)+2=3$

Trees for the letters in G look as follows:



Trees for the letters in G_1 are as follows:



The tree for a in G_2 is as follows:

Thus we see, that to obtain the tree for a letter, say a, in the (n+1)'th DOL system of the reduction sequence (providing that the rank of a is not smaller than n+2) one takes the tree for a in the n'th D0L system of the reduction sequence and deletes all the modes (and appropriate arcs) labeled by letters with rank (n+1).

The following result is a direct corollary of Definition 9, hence we give it without a proof.

Lemma 1. Let G = <V,P,ω> be a D0L system and let a be a letter with rank.

(i) If $G_0, \ldots, G_{n-1}$ are the first n D0L systems in the reduction sequence for G, then $\rho_G(a)=n$ if, and only if, $\rho_{G_{n-1}}(a)=1$.

(ii) If $a \overset{*}{\underset{G}{\Longrightarrow}} ubv$ for some u,v in V*, b in V, then b is a letter with rank and $\rho_G(b) \leq \rho_G(a)$.

The next three results are rather technical and they indicate the relation between the rank of a letter and the form of a production for this letter in a particular kind of D0L systems.

Lemma 2. Let G = <V,P,ω> be a D0L system such that if d is a growing letter in V and $d \underset{G}{\Longrightarrow} \alpha$ then $|\alpha| \geq 2$. Let $c \in V$ and $\rho_G(c)=n$. Then either $c \underset{P}{\rightarrow} \Lambda$ (in which case n=1) or there exists a letter a in V such that $\rho_G(a)=n$ and $c \underset{P}{\rightarrow} uav$

for some u,v in V*.

Proof. Let G be a D0L system satisfying the statement of the Lemma and c be a letter in V such that $\rho_G(c)=n$. We shall prove this Lemma by induction on the rank (n) of c.

(i) Let n=1. Then either $c \underset{P}{\to} \Lambda$ or according to Lemma 1 $c \to uav$ for some u,v in V*, a in V where $\rho_G(a)=1$. Hence the Lemma holds.

(ii) Let us assume that the Lemma holds for $n \leq l$ for some $l \geq 1$.

(iii) Let $n=l+1$. Now $n>1$ and by definition (of the rank) $\rho_{G_{n-1}}(c)=1$ where $G_0, G_1, \ldots$ is the reduction sequence of G. Thus from the inductive assumption it follows that $c \underset{P_{n-1}}{\longrightarrow} w_1 a w_2$ w_1, w_2 in V*, a in V where $\rho_G(a)=n$ and $c \underset{P}{\longrightarrow} uav$ for some u,v in V*. Thus Lemma 2 holds.

Lemma 3. Let G be a D0L system such that if d is a growing letter in G and $d \underset{G}{\Longrightarrow} \alpha$ then $|\alpha| \geq 2$. Then there exists a decomposition of G such that if H = <V,P,ω> is a component system in such a decomposition, c is in V and $\rho_H(c)=n>1$, then

either (i) There exists a letter b in V−{c} such that $\rho_H(b)=n-1$ and $c \underset{P}{\longrightarrow} ubv$ for some u,v in V* such that $uv \neq \Lambda$, or (ii) There exist letters a, b in V such that

$c \underset{P}{\rightarrow} u_1 a u_2 b u_3$ where u_1, u_2, u_3 are in V^* and

$\rho_H(a) = \rho_H(b) = n$.

Proof.

Let $G = \langle \Sigma, R, \sigma \rangle$ be a DOL system satisfying the statement of the Lemma. Let us decompose G in such a way that if $H = \langle V, P, \omega \rangle$ is a component DOL system in such a decomposition and $H_0 = H = \langle V_0, P_0, \omega_0 \rangle, H_1 = \langle V_1, P_1, \omega_1 \rangle, \ldots, H_{p-1} = \langle V_{p-1}, P_{p-1}, \omega_{p-1} \rangle$, for some $p > 1$, are the first p DOL systems in the reduction sequence of H, then for c in V_{p-1} either $c \overset{*}{\underset{H_{p-1}}{\Rightarrow}} \gamma$ implies $|\gamma| = 1$, or $c \underset{H_{p-1}}{\Rightarrow} \gamma$ implies $|\gamma| > 1$. (We leave to the reader the easy proof of the fact that such a decomposition is possible).

Let $H = \langle V, P, \omega \rangle$ be an arbitrary component DOL system of the above described decomposition of G. Let c be in V and $\rho_H(c) = n > 1$. Let the first n elements of the reduction sequence of H be $H_0 = H = \langle V_0, P_0, \omega_0 \rangle, H_1 = \langle V_1, P_1, \omega_1 \rangle, \ldots, H_{n-1} = \langle V_{n-1}, P_{n-1}, \omega_{n-1} \rangle$. Thus $\rho_{H_{n-1}}(c) = 1$, and we have two cases (exhausting all possibilities):

1) $c \underset{H_{n-1}}{\Rightarrow} \gamma$ for some γ such that $|\gamma| \geq 2$. Hence $c \underset{P_{n-1}}{\rightarrow} x_1 a x_2 b x_3$ for some x_1, x_2, x_3 in V^*_{n-1}, a,b in V_{n-1} where $\rho_{H_{n-1}}(a) = \rho_{H_{n-1}}(b) = 1$. Thus $c \underset{P}{\rightarrow} u_1 a u_2 b u_3$ for some u_1, u_2, u_3 in V^*, a,b in V where

$\rho_H(a)=\rho_H(b)=n$.

2) $c \underset{H_{n-1}}{\Rightarrow} c_1 \underset{H_{n-1}}{\Rightarrow} c_2 \Rightarrow \ldots \underset{H_{n-1}}{\Rightarrow} c_s \underset{H_{n-1}}{\Rightarrow} c_{s+1} \Rightarrow \ldots$ for some $s \geq 1$, $c_1, \ldots, c_s \varepsilon V_{n-1}, c_1=c_s$, where $\rho_{H_{n-1}}(c_i)=1$ for $1 \leq i \leq s$.

Thus $c_j \underset{H_{n-2}}{\Rightarrow} \gamma_1 c_{j+1} \gamma_2$ for some j in $\{1,\ldots,s\}, \gamma_1, \gamma_2$ in V^*_{n-2} for which $\gamma_1\gamma_2=\Lambda$; as otherwise $\rho_H(c)$ would be smaller than n; a contradiction. It is obvious that (because G has this property) for every growing letter d in V we have $|\alpha| \geq 2$ for every α such that $d \underset{P}{\rightarrow} \alpha$. But $c_j \underset{H_{n-2}}{\Rightarrow} \gamma_1 c_{j+1} \gamma_2$ implies that c is a growing letter in H (remember that $c \overset{*}{\underset{H}{\Rightarrow}} c_j$ and $n>1$ so that $n-2 \geq 0$) and as $\gamma_1\gamma_2$ is nonempty it contains at least one letter (say b) of rank $n-1$. Altogether $c \underset{H}{\Rightarrow} u_1 b u_2$ for some b in V, u_1, u_2 in V* such that $u_1 u_2 \neq \Lambda$ and $\rho_H(b)=n-1$ (hence $b \neq c$).

Thus the Lemma holds in both cases, which ends the proof of Lemma 3.

As a corollary from Lemma 3 we have the following result.

Corollary 1. Let G = <V,P,ω> be a D0L system such that if d is a growing letter in G and $d \underset{G}{\Rightarrow} \alpha$ then $|\alpha| \geq 2$. If c is in V and $\rho_G(c)=n>1$ then

$$c \underset{P}{\Rightarrow} u_1 a u_2 b u_3$$

for some u_1, u_2, u_3 in V^* and a,b in V, where $\rho_G(a)$ and $\rho_G(b)$ are not smaller than (n-1) and one of $\rho_G(a)$, $\rho_G(b)$ is equal to n.

Definition 10. A DOL system $G = \langle V,P,\omega\rangle$ is said to have a rank if every letter in V has a rank. If G has a rank, then the rank of G, denoted as $\rho(G)$, is defined by $\rho(G)=\max_{a\varepsilon V}\{\rho(a)\}$.

Example 7. If G is the DOL system of Example 1, then $\rho_G(a)=1$ but the letter b does not have a rank, hence G does not have a rank. If G is the DOL system of Example 2 then $\rho_G(b)=1$ and $\rho_G(a)=2$, hence G has a rank, and $\rho(G)=2$.

IV. D0L SYSTEMS WITH RANK VERSUS POLYNOMIALLY BOUNDED D0L SYSTEMS.

In this section the notions of a D0L system with rank and of a polynomially bounded D0L system are related to each other. In particular it is shown that a D0L system has a rank if, and only if, it is polynomially bounded.

__Lemma 4.__ Let $G = \langle V,P,\omega\rangle$ be a D0L system with a rank, where $\rho(G)=n+1$ for some $n\geq 0$. Then there exists a polynomial F of degree n such that for every $m\geq 0$

$$|\delta(G,m)|\leq F(m).$$

__Proof.__

Let G be a D0L system satisfying the statement of the Lemma. We shall prove this result by induction on n.

(i) Let $n=0$ so that $\rho(G)=1$.

In this case no letter in G is growing and so for every letter a in V there exists a constant C_a such that if $a \overset{*}{\underset{G}{\Rightarrow}} \alpha$ then $|\alpha|<C_a$. Let $C_G=\max\{C_a : a\in V\}$ and $K_G=C_G\cdot|\omega|$. Then, obviously, $|\delta(G,m)|<K_G$ for every $m\geq 0$ and so if we set F to be a polynomial such that $F(m)=K_G$ for every $m\geq 0$, the Lemma holds.

(ii) Let us assume that the Lemma is true for every $n\leq l$ for some $l\geq 1$.

(iii) Let $\rho(G)=l+1$. Let G, G_1 be the first two D0L systems in the reduction sequence of G. Then $\rho(G_1)=l$ and by inductive assumption there exists a polynomial F' such that F' is of degree $(l-1)$ such that $|\delta(G_1,m)|\leq F(m)$ for each $m\geq 0$.

Let C be a positive integer such that for every a in V for which $\rho_G(a)=1$ we have if $a \overset{*}{\underset{G}{\Longrightarrow}} \alpha$ then $|\alpha|<C$. Let D be the maximal length of the right-hand side of a production in the D0L system $\hat{G}$ obtained from G by erasing in all productions in P all letters of rank larger than 1. Finally, let $K=C\cdot D$.

The following equality obviously holds:

$$|\delta(G,m)|=U_1(G,m)+U_2(G,m)$$

where

$U_1(G,m)$ is the number of occurrences in $\delta(G,m)$ of letters of rank larger than one, and

$U_2(G,m)$ is the number of occurrences in $\delta(G,m)$ of letters of rank equal to one.

But $U_1(G,m)\leq F'(m)$ and so we have to estimate now $U_2(G,m)$.

Let us consider $\delta(G,m)$ for $m\geq 1$.

Every occurrence of every letter of rank 1 in $\delta(G,m)$ is either derived from an occurrence of a letter of rank 1 in the axiom (let us denote the number of such occurrences of letters in $\delta(G,m)$ by $U_{2,1}(G,m)$) or it is derived from an occurrence of a letter of rank larger than 1 in one of the strings $\delta(G,0),\delta(G,1),\ldots,\delta(G,m-1)$ (let us denote the number of such occurrences of letters in $\delta(G,m)$ by $U_{2,2}(G,m)$).

Hence $U_2(G,m)=U_{2,1}(G,m)+U_{2,2}(G,m)$.

Obviously $U_{2,1}(G,m)\leq C\cdot|\omega|$.

Let us consider now how many occurrences of letters of rank 1 in $\delta(G,m)$ were derived from occurrences of letters of rank larger than 1 on the level p where $0\leq p\leq m-1$.

Firstly, the number of all occurrences in $\delta(G,p)$ of all letters of rank larger than 1 is not larger than $F'(1)$.

Secondly, each occurrence in $\delta(G,p)$ of each letter of rank larger than one may derive at most D occurrences in $\delta(G,m)$ of letters of rank one.

Thirdly, each occurrence in $\delta(G,p)$ of a letter of rank 1 may derive at most C occurrences in $\delta(G,m)$ of letters of rank 1.

Thus altogether $\delta(G,m)$ contains no more than

$$C\cdot D\cdot F'(p)=K\cdot F'(p)$$

occurrences of letters of rank 1 which were derived from occurrences in $\delta(G,P)$ of letters of rank larger than 1. Consequently for $m \geq 1$

$$U_{2,2}(G,m) \leq \sum_{p=0}^{m-1} K \cdot F'(p)$$

and so for $m \geq 0$

$$|\delta(G,m)| \leq |\omega| + U_1(G,m) + U_2(G,m) = |\omega| + U_1(G,m) + U_{2,1}(G,m) + U_{2,2}(G,m) \leq$$
$$\leq F'(m) + (C+1) \cdot |\omega| + \sum_{p=0}^{m-1} K \cdot F'(p).$$

But $\sum_{p=0}^{m-1} K \cdot F'(p) = H(m)$ for some polynomial H of degree l (remember that the degree of F' is (l-1)).

Thus Lemma 4 holds.

The next result is an auxiliary result needed for the proof of the main theorem (Theorem 2) of this section.

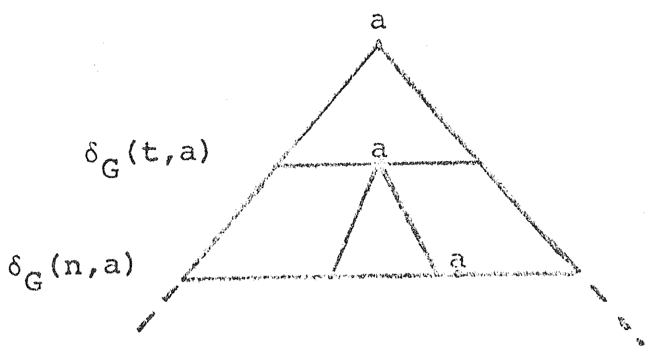
Lemma 5. If G = <V,P,ω> is a DOL system such that each letter in V is a growing letter (in G) then there exists a letter a in V such that $a \overset{*}{\underset{G}{\Rightarrow}} \alpha_1 a \alpha_2 a \alpha_3$ for some $\alpha_1, \alpha_2, \alpha_3$ in V*.

Proof.

Let G satisfy the conditions of the Lemma. As each letter in V is growing there exists a letter, say a, which occurs in more than one string in the derivation which starts with a.

If we consider the tree for a in G, then we have two possible cases.

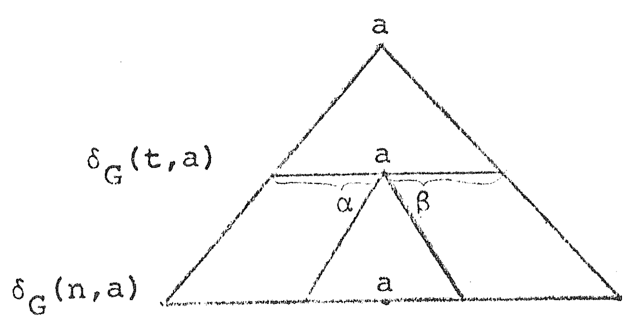
(i) The tree for a in G is of the form



for some positive integers t,n where $t<n$.

But then $\delta_G(nt,a)$ contains at least two occurrences of the letter a, and so the result holds.

(ii) The tree for a in G is of the form



for some positive integers t, n where $t<n$, and for some α,β in V^* such that $\alpha\beta\neq\Lambda$ (remember that each letter in G is growing) where no letter in $\alpha\beta$ derives a word containing an occurrence of the letter a.

Now we construct the D0L system $G^{(1)} = \langle V^{(1)}, P^{(1)}, \omega^{(1)}\rangle$ in such a way that $\omega^{(1)}=\alpha\beta$ and $V^{(1)}$, $P^{(1)}$ contain all letters from V and productions from P necessary to continue an arbitrarily long derivation (in G) starting with $\alpha\beta$.

Note that $G^{(1)}$ is such that each letter in $V^{(1)}$ is growing and so we can repeat the argument of this proof for $V^{(1)}$. In particular there exists a letter, say a_1, which occurs in more than one string in the derivation (in $G^{(1)}$) which starts with a_1.

If case (i) holds for the tree for a_1 in $G^{(1)}$, then we are done as then $a_1 \overset{*}{\underset{G^{(1)}}{\Rightarrow}} \alpha_1 a_1 \alpha_2 a_1 \alpha_3$ for some $\alpha_1, \alpha_2, \alpha_3$ in $(V^{(1)})^*$ and consequently $a \overset{*}{\underset{G}{\Rightarrow}} \beta a \beta_2 a \beta_3$ for some $\beta_1, \beta_2, \beta_3$ in V^* which proves the Lemma.

If case (ii) holds for the tree for a_1 in $G^{(1)}$, then we iterate the above procedure and construct the D0L system $G^{(2)}$.

Obviously, by iterating the above procedure either we find a letter a_i, for some $i \geq 1$, such that $a_i \overset{*}{\underset{G}{\Rightarrow}} \alpha_1 a_i \alpha_2 a_i \alpha_3$ (and hence the Lemma holds) or after at most $\#V$ steps we arrive at the D0L system $G^{(i)}$ containing one letter only, which is growing in $G^{(i)}$ (and hence also in G) in which case the Lemma also holds.

<u>Theorem 2</u>. Let $G = \langle V, P, \omega \rangle$ be a D0L system. The following three statements are equivalent:

I) There exists a polynomial F such that for every $m \geq 0$,

$$|\delta(G,m)| \leq F(m).$$

II) For every positive integer m and for every a in V, a occurs at most once in $\delta_G(m,a)$.

III) G is a DOL system with rank.

<u>Proof</u>:

(i) We shall prove first that I) implies II) by proving that if II) does not hold then I) does not hold either.

So, let us assume that II) does not hold. Then for some a in V and for some positive integer n_o we have

$$\delta_G(a,n_o) = \alpha_1 a \alpha_2 a \alpha_3$$

for some $\alpha_1, \alpha_2, \alpha_3$ in V*. But then for every positive integer s, $\delta_G(a,n_o s)$ will contain at least 2^s letters. Thus if l_o is a positive integer such that $\delta(G,l_o)$ contains an occurrence of the letter a, then for every positive integer s we have

$$|\delta(G,l_o+n_o s)| \geq 2^s$$

and consequently I) does not hold.

Thus (I) implies (II).

(ii) Next we shall prove that II) implies III) by proving that if III) does not hold then II) does not hold either.

So let us assume that III) does not hold. Thus if $G_o=\langle V,P,\omega\rangle$, $G_1=\langle V_1,P_1,\omega_1\rangle,\ldots,G_p=\langle V_p,P_p,\omega_p\rangle$ is the reduction sequence of G, then each letter in V_p is growing (in G_p). Hence from Lemma 5 it follows that there exists a letter a in V_p such that $a \overset{*}{\underset{G_p}{\Longrightarrow}} \alpha_1 a \alpha_2 a \alpha_3$ for some $\alpha_1,\alpha_2,\alpha_3$ in V_p^*) and consequently II) does not hold.

Thus II) implies III).

(iii)From Lemma 4 it follows directly that III) implies I). From (i), (ii) and (iii) the Theorem follows.

V. THE MINIMAL DEGREE OF A POLYNOMIAL BOUNDING A DOL SYSTEM.

In this section we prove (Theorem 3) that if G is a DOL system with rank then the minimal degree of a polynomial bounding G is $\rho(G)-1$.

The first two results show that if G is a DOL system with rank and it is a component system in a decomposition of G, then it is of the same rank as G is, and each letter in it has the same rank as it has in G.

Lemma 6. Let $G = \langle V,P,\omega\rangle$. be a DOL system with rank and $l \geq 2$. Let $G^l = \langle V_l,P_l,\omega_l\rangle$ be a DOL system such

that $\omega_l = \delta(G,l)$,

$a \underset{P_l}{\rightarrow} \beta$ if, and only if, $\delta_G(l,a) = \beta$,

and $V_l = \bigcup_{m\geq 1} \text{Min } \delta(G,ml)$.

Then

1) G^l is a DOL system with rank,

2) for every a in V_l, $\rho_G(a)=m$ if, and only if, $\rho_{G^l}(a)=m$,

3) $\rho(G) = \rho(G^l)$.

Proof.

Let G,l,G^l satisfy the conditions of Lemma 6.

1) G^l must have a rank as otherwise (see Theorem 2) G^l would not be polynomially bounded, whereas G is: a contradiction.

2) We shall prove this by induction on m. Let a be in V_1.

(i) For m=1 the result is obviously true.

(ii) Let us assume that the result is true for all positive integers up to and including m.

(iii) Let $G_0, G_1, \ldots G_m$ be the first (m+1) DOL systems in the reduction sequence for G and $G_0^1, G_1^1, \ldots, G_m^1$ be the first (m+1) DOL systems in the reduction sequence for G^1.

If $\rho_G(a)=m+1$, then the tree for a in G_m is of bounded width. But it was obtained from the tree for a in G by removing all nodes (and appropriate arcs) labeled by letters of rank (in G) not larger than m. But then, obviously, by the inductive assumption the tree for a in G_m^1 will also be of bounded width.

If $\rho_{G^1}(a)=m+1$, then by the inductive assumption $\rho_G(a) \geq m+1$. But if $\rho_G(a)>m+1$, then by repeating the previous argument we would have that the tree for a in G_m^1 is not of bounded width; a contradiction. Thus $\rho_G(a)=m+1$.

Hence $\rho_G(a)=\rho_{G^1}(a)$ and 2) holds.

3) This is a direct corollary from 2).

<u>Corollary 2</u>. If $G = \langle V,P,\omega\rangle$ is a DOL system with rank and $H = \langle V',P',\omega'\rangle$ is a component system in a decomposition of G, then

1) H is a DOL system with rank,

2) for every a in V', $\rho_G(a)=m$ if, and only if,

$\rho_H(a)=m$,

3) $\rho(G) = \rho(H)$.

The next result together with Lemma 4 will yield us the main result of this section (Theorem 3).

Lemma 7. If G is a DOL system with rank, then there exists a polynomial M of degree $\rho(G)-1$, such that for every $m \geq 0$

$$M(m) \leq |\delta(G,m)|.$$

Proof. Let $G = \langle V,P,\omega \rangle$ be a DOL system with rank. We shall prove this result by induction on $\rho(G)$.

(i) If $\rho(G)=1$ and we set M to be a polynomial such that $M(m)=0$ for all $m \geq 0$ then the result obviously holds.

(ii) Let us assume that if $\rho(G) \leq n$ then we can construct a polynomial M such that for every $m \geq 0$, $M(m) \leq |\delta(G,m)|$.

(iii) Let $\rho(G)=n+1$. From Lemma 1 it follows that $\omega = uav$ for some u,v in V^* and a in V, where $\rho_G(a)=n+1$. Let $G_0=G, G_1, \ldots, G_n$ be the reduction sequence of G. Hence $a \overset{*}{\underset{G_n}{\Rightarrow}} \alpha$ where $\alpha \neq \Lambda$ and α is such that if c

occurs in α and $c \overset{*}{\underset{G_n}{\Rightarrow}} \gamma$ then $|\gamma|=1$. In fact (due to Lemma 6 and to the fact that a linear transformation of a variable does not change the degree of polynomial) we can assume that G is already decomposed in such a way that each component system H is such that for d in the alphabet of H, if $d \underset{H_i}{\Rightarrow} \beta$ (where H_i is one of the D0L systems in the reduction sequence of H) and $|\beta|=1$ then $\beta=d$. Thus there is a letter b in V such that $\rho_G(b)=n+1$, $b \underset{G_n}{\Rightarrow} b$ and $a \overset{*}{\underset{G_n}{\Rightarrow}} \alpha_1, \alpha_2 \in V^*$.

Now, instead of looking directly for a polynomial M with the derived property we can look for a polynomial M' such that for all $m \geq 0$, $|\delta_G(m,a)| \geq M'(m)$, where the degree of M' is n. (It is obvious that M' then automatically satisfies the conditions required in the statement of this result for M). But rather than looking directly for M' we can look for a polynomial M" (of degree n) such that for all $m \geq 0$, $|\delta_G(m,b)| \geq M''(m)$. (Again, it is obvious, that M" satisfies the conditions required for M' and hence it also satisfies the conditions required for M).

Now from Corollary 1 and from the fact that $b \underset{G_n}{\rightarrow} b$ we have $b \underset{G}{\rightarrow} \alpha b \beta$ for some α, β in V^*, $\alpha\beta \neq \Lambda$ where $\alpha\beta$ contains an occurrence of a letter b_1 such that $\rho_G(b_1)=n$.

Let us first assume that β contains an occurrence of b_1. Then $\rho_G(m,b)=\gamma b\beta\delta_G(1,\beta)\delta_G(2,\beta)\ldots\delta_G(m-1,\beta)$ for some γ in V^*, where the highest rank of any letter occurring in β is n (remember that β contains b_1, where $\rho_G(b_1)=n$). Let us now construct a D0L system G_β in such a way that its axiom is β and G_β contains all the letters from V and all the productions from P which are needed to continue the derivation from β. As this new D0L system G_β is of rank n, from the inductive assumption we know that there exists a polynomial M_β of degree $(n-1)$ such that for every $m\geq 0$, $M_\beta(m)\leq|\delta_G(m,\beta)|$.

Now let M be a polynomial such that for every $m\geq 0$, $M(m)=\sum_{l=1}^{m-1} M_\beta(l)$.

But then, obviously, for every $m\geq 0$

$$M(m)\leq|\delta_G(m,b)|$$

and so

$$M(m)\leq|\delta_G(m,a)|$$

and so

$$M(m)\leq|\delta_G(m,\omega)|.$$

But M_β is a polynomial of degree $(n-1)$ and hence M is of degree n.

VI. THE EFFECTIVENESS OF THE RESULTS.

We end this paper with a discussion of effectiveness of the results proved in this paper. From Lemma 4, Lemma 7, Theorem 2, Theorem 3, their proofs and from the obvious fact that it is decidable whether an arbitrary D0L system has a rank and in the case it has one can effectively compute its rank, it follows that.

<u>Theorem 4.</u>

1) There exists an algorithm which given an arbitrary D0L system G decides whether or not G is polynomially bounded.

2) There exists an algorithm which given an arbitrary D0L system $G = \langle V,P,\omega \rangle$ with rank, constructs a polynomial M of degree $\rho(G)-1$ such that for every $m \geq 0$

$$M(m) \leq \delta(G,m).$$

3) There exists an algorithm which given an arbitrary D0L system G constructs a polynomial F bounding G, such that no polynomial with degree smaller than that of F bounds G.

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FOOTNOTES

1) By a polynomial in this paper we understand a polynomial of one variable with positive coefficient at the highest degree.

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