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DOUBLE VARIATIONAL EVALUATION
OF LINEAR ANTENNA CURRENT

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Double Variational Evaluation of Linear Antenna Current

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A network approach for determining the current and input admittance for N elements, thin wire dipole arrays are examined. A double variational procedure is used to approximate this current and the antenna end effects to obtain quick numerical results. Numerical results for two and three element parallel dipoles of length $2h$ and for a two element Yagi type antenna are included. Comparison with other approaches found in the literature is quite good.

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CHAPTER I

INTRODUCTION

For many years electrical engineers have attempted to mathematically simulate the currents and/or impedances of arrays of finite length antennas in order to avoid building purely by trial-and-error. For any number of dipole antennas of varied length (such as shown in figure 1.1) there have been many methods and approximations used to find the reflections caused by the terminations of the wires, to find the effects of each of the wires on each of the other wires, and to find the effects of various wire diameter sizes.

A common starting point for these modeling techniques is the integral equation for the current on one wire of radius a . If we choose very thin ($a \ll \lambda_0$), perfectly conducting wires, the electric field in the z direction (the tangential direction) is the only important component. The resulting integral equation is

$$E_z(z) = \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{\substack{\text{surface} \\ \text{of wire}}} I(z') K(z, z') dz' \quad (1.1)$$

for z on the surface of the wire where $E_z(z)$ is the tangential electric field, $I(z')$ is the current on the wire, $K(z, z')$ is the appropriate Green's function, and k_0 is the free space wave number. For thin wires the kernel $K(z, z')$ is given in the cylindrical coordinate system by

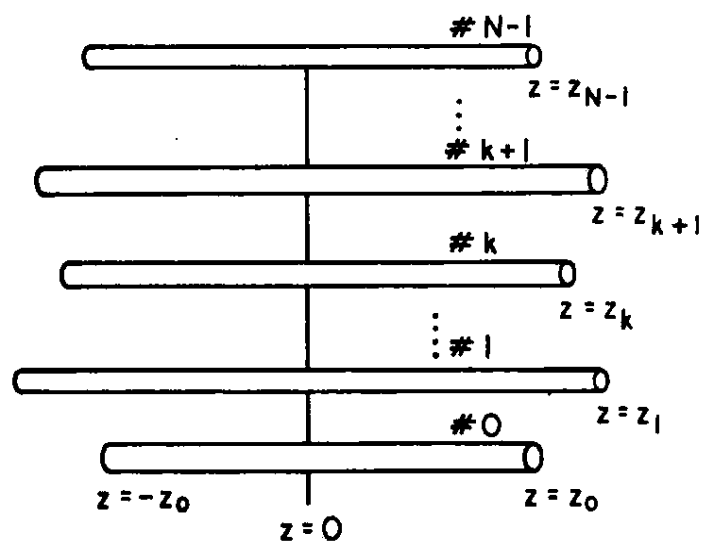


Figure 1.1 Array of N dipole antennas of varied length

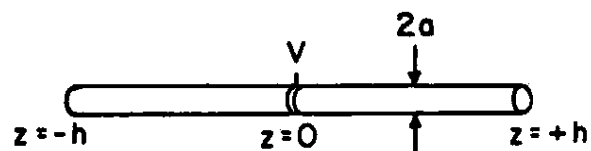


Figure 1.2 Center-fed dipole antenna of length $2h$

$$K(z, z') = -i\omega\mu \exp(-ik_0 R) / 4\pi k_0^2 R \quad (1.2)$$

where $R = ((z-z')^2 + a^2)^{1/2}$ and μ is the permeability. This kernel is approximate, assuming the current is at the center rather than the surface of the wire (which is good for thin wires). On a perfectly conducting wire the boundary condition requires the tangential component of the electric field to vanish, so the incident electric field along the wire equals the source field for a voltage source at $z = 0$

$$E_z(z) = -V_0 \delta(z) \quad (1.3)$$

where $\delta(z)$ is the Dirac delta function and V_0 is a constant.

Difficulty arises in solving the integral equation since the Green's function $K(z, z')$ is not easily integrated analytically. Since $I(z')$ is unknown and may be a mathematically involved expression, approximation techniques are utilized to derive $I(z')$.

Probably the most commonly used technique to approximate $I(z')$ is the moment method. For the center-fed dipole antenna of length $2h$ (see figure 1.2), the integral equation (1.1) becomes

$$E_z(z) = \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-h}^h I(z') K(z, z') dz' \quad (1.4)$$

for $-h \leq z \leq h$. This method assumes the current can be approximated by a finite series:

$$I(z') = \sum_{n=1}^N I_n F_n(z')$$

where the I_n are complex coefficients and the $F_n(z')$ are

expansion functions. In one version of this method, equation (1.1) becomes a system of simultaneous equations:

$$E_z(z_m) \approx \left(\frac{\partial z}{\partial z^2} + k_0^2 \right) \int_{-h}^h \sum_{n=1}^N I_n F_n(z') K(z_m, z') dz' \quad (1.6)$$

where z_m is a point on the m^{th} segment on the antenna. By selecting an appropriate function for $F_n(z')$, the equation can be solved. If $F_n(z')$ is a step function, the equation is quickly solved. However, often this $F_n(z')$ does not match the physical problem closely enough so a more complex function for $F_n(z')$ must be chosen. The resulting integral can then become very difficult to solve, even numerically. There are also N of these integrals so that a numerical solution would be very cumbersome, especially for long antennas needing many increments.¹

Another method used to approximate the center-fed dipole antenna integral equation (1.4) is that of King and Middleton. Working with the vector potential component $A_z(z)$ version of the integral equation, the King-Middleton method uses the approximation that the ratio of the vector potential component $A_z(z)$ just outside the antenna to the current $I_z(z)$ on the antenna is for all practical purposes a constant value (except at the ends of the antenna where the current vanishes but the vector potential component is merely small). By approximating the current $I_z(z)$ by an appropriate function $f(z)$, then $I_z(z')$ may be written as

$$I_z(z') \approx I_z(z) f(z') / f(z) = I_z(z) g(z, z') \quad (1.7)$$

This will lead to the integral expression

$$\begin{aligned}
4\pi A_z(z)/\mu I_z(z) &= \int_{-h}^h g(z,z')K(z,z')dz \\
&= \psi + \gamma(z)
\end{aligned} \tag{1.8}$$

where ψ is a constant and $\gamma(z)$ is the hopefully small difference between ψ and the true ratio of $A_z(z)$ and $I_z(z)$. Equation (1.8) can then be solved by iteration, and with a good initial approximation for the current, few iterations of $g(z,z')$ in expression (1.8) would be necessary. This method does use the thin wire approximation $k_0 a \ll 1$ and the approximation $h \gg a$.²

Approximations based on the Weiner-Hopf integral for the current reflection coefficient have been done in many papers including Chang, Lee, and Rispin;³ Rispin and Chang;⁴ and Shen, Wu, and King.⁵ This method takes a higher level of mathematical understanding than methods such as the moment method and is not applicable if the lengths of the antennas involved are not identical. A useful result found using the Weiner-Hopf technique was found by Shen. Shen demonstrates that the current of the center-fed dipole (as shown in figure 1.2) can be represented as the sum of the incident current (the infinitely long antenna current I_∞) and the currents reflected at the ends $z = \pm h$, which are proportional to the incident current $I_\infty(h \mp z)$ so that the total current $I_z(z)$ on the dipole is

$$I_z(z) = I_\infty(z) + C(h)I_\infty(h-z) + C(-h)I_\infty(z+h) . \tag{1.9}$$

Once the reflection coefficient is known, the coefficients $C(h)$ and $C(-h)$ are determined using the end conditions on the dipole

antenna--that the currents into the ends of the antenna ($z = \pm h$) must equal the current out of the end of the antenna, so that $I_z(\pm h) = 0$.⁵ This means then that the current on a dipole antenna or on an array of dipole antennas can be determined simply by finding the current on an infinitely long set of wires of identical characteristics to the dipoles and the reflection coefficient.

L. A. Vainshtein pursued a double variational technique to describe the current on an infinitely long thin wire antenna. We look at his development of this technique in Chapter II for one and two element arrays of infinitely long, parallel, thin wire antennas. We develop a system mode approach to expand this technique to N wires in Chapter III. Also in Chapter III we modify his functional expression to yield a different, but still double variational, functional expression for the reflected current of an array of truncated thin wire antennas.

With these results and Shen's expression for total current in equation (1.9) we also obtain expressions for the total current, the reflection coefficients, and the admittances of parallel, thin wire, dipole antennas in Chapter III. One and two element dipole antennas and a three wire circular array of dipoles demonstrate this method.

Finally in Chapter IV, we extend the double variational technique to evaluate the current on a simple two element Yagi antenna; something not possible with the current Weiner-Hopf analysis.

We will see that the double variational approach to determine the current, the reflection coefficients and the input admittances can be advantageous since it takes less computer time and space than the moment method; it can be extended to physical problems not possible with the Weiner-Hopf technique, and it works well for longer wires where the King-Middleton approach becomes inaccurate. It also has good agreement with results from other methods.

CHAPTER II

THE DOUBLE VARIATIONAL FORMULATION

Vainshtein developed a numerical technique based on double variational principles to describe the current on a thin cylindrical conductor of infinite length. The results of the infinitely long, thin wire analysis are necessary for our later evaluation of dipole antenna current and admittance expressions, so we will first review Vainshtein's work for the infinitely long, thin wire.

2.1 Double Variational Principle

In general, the double variational technique Vainshtein developed can be used on any integral or integral-differential equation of the form

$$LI + \mathcal{E}_{\text{source}} = 0 \quad (2.1)$$

where L is a linear integral or integral-differential operator operating on I , I is the unknown current function on the surface of the object being studied, and $\mathcal{E}_{\text{source}}$ is the known electric field source also on the surface. To develop the double variational expression for this equation, Vainshtein uses two source functions \mathcal{E}_{α} and \mathcal{E}_{β} for the same surface S , which correspond to currents I_{α} and I_{β} respectively so that equation (2.1) becomes

$$\begin{aligned}
 LI_{\alpha} + e_{\alpha} &= 0 \quad \text{and} \\
 LI_{\beta} + e_{\beta} &= 0.
 \end{aligned}
 \tag{2.2}$$

Vainshtein stated that it is necessary (as we will see later) for this formulation that the vector products

$$\langle e_{\alpha}, I_{\beta} \rangle = \langle I_{\beta}, e_{\alpha} \rangle. \tag{2.3}$$

(The vector notation for $\langle e_{\alpha}, I_{\beta} \rangle$ for electromagnetics is given by $\langle e_{\alpha}, I_{\beta} \rangle = \int_S (e_{\alpha}, I_{\beta}) dS$ where (e_{α}, I_{β}) is a scalar product of the vectors \bar{e}_{α} and \bar{I}_{β} at a given point on the surface S .) It is also necessary that L be chosen so that there is symmetry so that when the operator L operates on I_{α} and I_{β}

$$\langle I_{\alpha}, LI_{\beta} \rangle = \langle I_{\beta}, LI_{\alpha} \rangle \tag{2.4}$$

for whatever functions I_{α} and I_{β} are chosen.

If $e_{\alpha} = -\delta(z-z_{\alpha})$ and similarly for e_{β} (the situation encountered in antenna problems) using the expressions of equations (2.3) and (2.4), a functional

$$\begin{aligned}
 F_{\alpha\beta} &= \langle I_{\alpha}, LI_{\beta} \rangle / \{ \langle e_{\alpha}, I_{\beta} \rangle \langle e_{\beta}, I_{\alpha} \rangle \} \\
 &= -1/I_{\alpha}(z_{\beta}) = -1/I_{\beta}(z_{\alpha})
 \end{aligned}
 \tag{2.5}$$

is defined, where I_{α} and I_{β} are trial functions while $I_{\alpha}(z_{\beta})$ and $I_{\beta}(z_{\alpha})$ are the currents being approximated. This functional $F_{\alpha\beta}$ takes on its exact value if either trial function I_{α} or I_{β} is exact (i.e. solves equation (2.2)). Moreover, a small first-order error δI_{α} or δI_{β} in both trial functions was shown by

Vainshtein⁶ to result only in a second-order error in $F_{\alpha\beta}$, proportional roughly to $(\delta I_\alpha)(\delta I_\beta)$. When $F_{\alpha\beta}$ has these properties, we say it is double variational.

2.2 Double Variational Principle for an Infinitely Long, Thin Wire Conductor

To write an integro-differential equation of (1.1) in the form of (2.1) for an infinitely long, thin wire we must define the operator L :

$$\begin{aligned} LI_\alpha &= \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-\infty}^{\infty} dz' I_\alpha(z') K(z, z') \\ LI_\beta &= \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-\infty}^{\infty} dz' I_\beta(z') K(z, z') \end{aligned} \quad (2.6)$$

and the source functions:

$$\begin{aligned} \mathcal{E}_\alpha &= -V_\alpha \delta(z - z_\alpha) \quad \text{and} \\ \mathcal{E}_\beta &= -V_\beta \delta(z - z_\beta) \end{aligned} \quad (2.7)$$

(we will choose $V_\alpha = V_\beta = 1$ for this section since there is only one wire) where $K(z, z')$ is the kernel as defined in equation (1.2). As stated in Chapter 1, for the thin wire ($a \ll \lambda$) only the longitudinal current is significant. For the thin, perfectly conducting cylindrical wire, good trial functions for the current are

$$\begin{aligned} (I_\alpha(z))_0 &= -\exp(-ik_0|z|) \quad \text{and} \\ (I_\beta(z))_0 &= -\exp(-ik_0|z - z_\beta|) \end{aligned} \quad (2.8)$$

(assuming without loss of generality that $z_\alpha = 0$). Equation (2.3) is true for this e_α and I_β since

$$\langle e_\alpha, I_\beta \rangle = - \int_S \delta(z) \exp(-k_0 |z - z_\beta|) dS = \langle I_\beta, e_\alpha \rangle .$$

Equation (2.4) for this wire is shown to be true in Appendix A.

The functional in equation (2.5) becomes

$$F_{\alpha\beta} = \left\{ \int_{-\infty}^{\infty} dz \exp(-ik_0 |z|) \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-\infty}^{\infty} dz' \exp(-ik_0 |z' - z_\beta|) K(z, z') \right\} / \left\{ \left| \int_{-\infty}^{\infty} dz \delta(z) \exp(-ik_0 |z - z_\beta|) \right| \left| \int_{-\infty}^{\infty} dz \delta(z + z_\beta) \exp(-ik_0 |z|) \right| \right\} .$$

The double integral in the numerator is evaluated in Appendix A. Using the shifting property of the delta function, the integrals in the denominator are easily integrated so that

$$F_{\alpha\beta} = -\zeta_0 \exp(ik_0 z) |E_1(ik_0 r^-) + \exp(2ik_0 z) E_1(ikr^+)| / 2\pi \quad (2.10)$$

and from (2.5)

$$I_\infty^\alpha(z) = 2\pi \exp(-ik_0 z) / \zeta_0 f(z) \quad (2.11)$$

where $f(z) = E_1(ikr^-) + \exp(2ik_0 z) E_1(ikr^+)$ (z_β has been set to z), $r^\pm = (z^2 + a^2)^{1/2} \pm z$, and ζ_0 is the intrinsic impedance ($\zeta_0 = (\mu_0/\epsilon_0)^{1/2}$). Since $V = 1$, the input conductance is simply $Y_\infty(0) = I_\infty^\alpha(0) = (\pi/\zeta_0)/E_1(ik_0 a)$. Numerical calculations for $I_\infty^\alpha(z)$ when the source is at $z = 0$ are compared to those of Rispin and Chang for $k_0 a = 0.1$ in figure 2.1a. As can be seen the results are very good for $k_0 z > 1.5$. Some variation is seen for $k_0 z < 1.5$ for the real part of the current. The input conductance is also

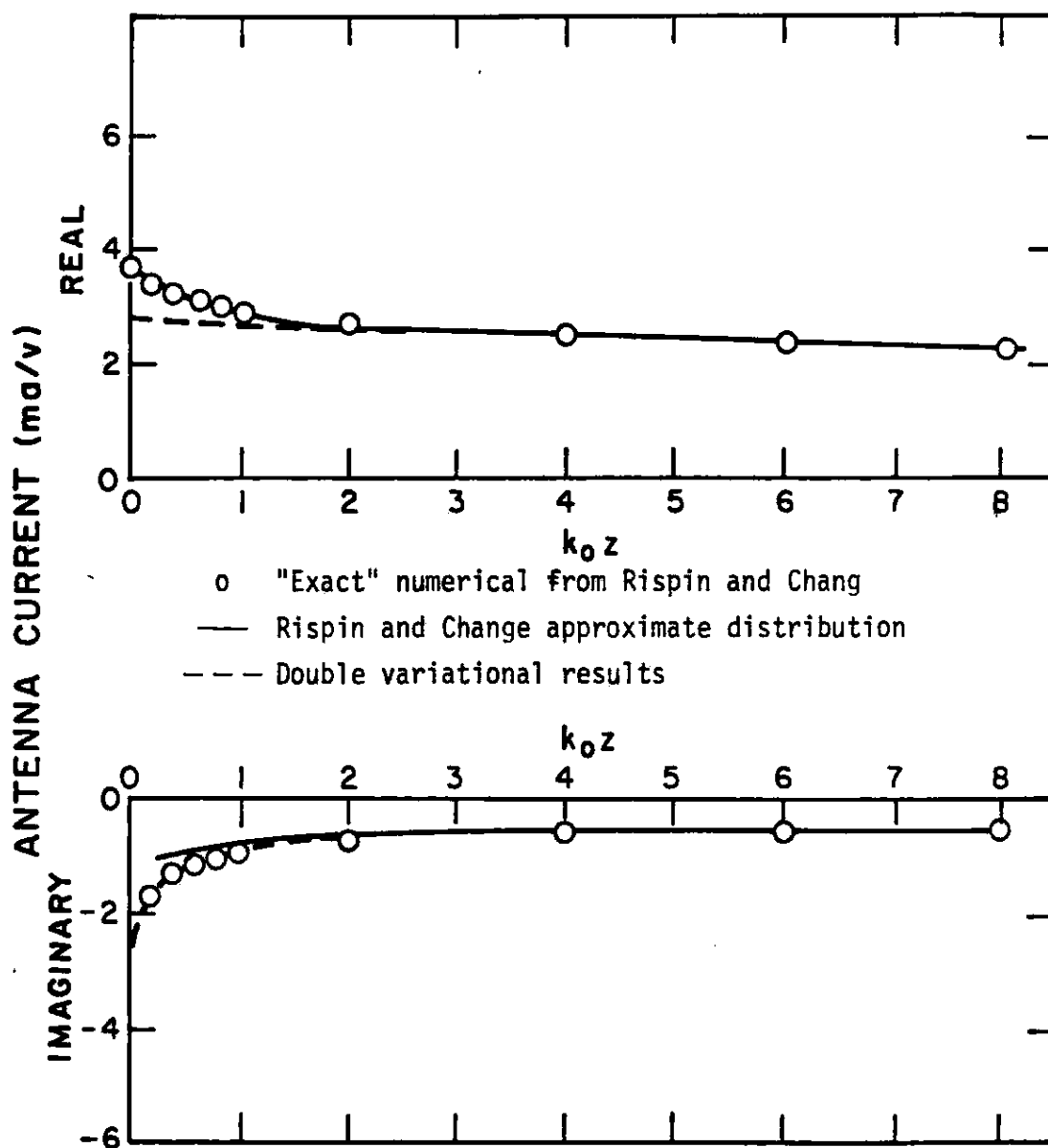


Figure 2.1a Current distribution on an infinitely long, thin wire antenna with delta function voltage source at $z = z_0 = 0$ for $k_0 a = 0.1$

compared with that of Rispin and Chang in figure 2.1b with good comparison for $k_0 a < 10^{-2}$.

2.3 Two Infinitely Long, Thin Wire Antennas Double

Variational Formulation

For two infinitely long, thin wire ($a \ll \lambda_0$), parallel antennas (figure 2.2) the double variational development is very similar to the development for the single wire case. The difference, of course, is that now there can be currents and voltages on the wires either in phase for both wires (antenna mode) or 180° out of phase (transmission line mode).

For the two wires, separated by a distance d and each wire with diameter a , voltage equations can be written as

$$\begin{aligned} V_0 &= Z_{00}I_0 + Z_{01}I_1 \\ V_1 &= Z_{10}I_0 + Z_{11}I_1 \end{aligned} \tag{2.12}$$

where V_0 and V_1 are the voltages applied to conductors #0 and #1, respectively; I_0 and I_1 are the currents on conductor #0 and #1; and Z_{00} , Z_{01} , Z_{10} , and Z_{11} are the various mutual and self conductor impedances. By setting

$$V_0 = V_a + V_t \tag{2.13a}$$

$$V_1 = V_a - V_t \tag{2.13b}$$

$$I_0 = I_a + I_t \tag{2.13c}$$

and

$$I_1 = I_a - I_t. \tag{2.13d}$$

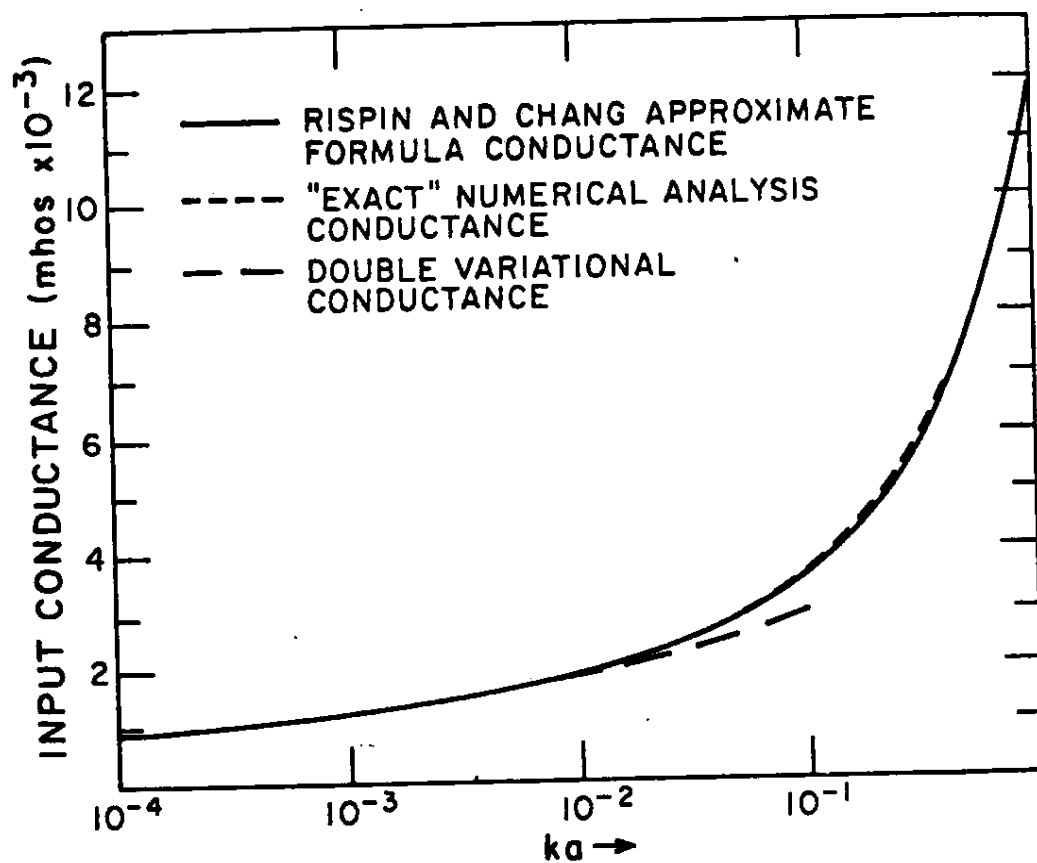


Figure 2.1b Input conductance of an infinitely long, thin wire antenna

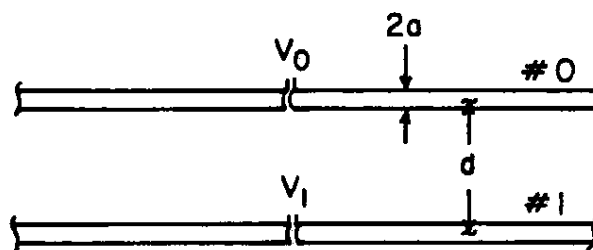


Figure 2.2 Two infinitely long, thin wire antenna

where V_a , V_t , I_a , and I_t are the antenna mode voltage, transmission line voltage, antenna mode current, and transmission line mode current, respectively, we can write the voltage equations as

$$V_a = Z_a I_a \quad (2.14)$$

$$V_t = Z_t I_t$$

where Z_a and Z_t are independent of each other for two identical wires ($Z_{00} = Z_{11}$ and $Z_{01} = Z_{10}$).

Appropriate kernels for these mode representations are necessary. Since the antenna mode is produced when in phase voltages are used, an appropriate K is

$$K_a(z, z') = -i\zeta_0 \{ \exp(-ik_0 R_0)/k_0 R_0 + \exp(-ik_0 R_1)/k_0 R_1 \} / 4\pi. \quad (2.15a)$$

The transmission line mode is the result of voltages input 180° out of phase, so that

$$K_t(z, z') = -i\zeta_0 \{ \exp(-ik_0 R_0)/k_0 R_0 - \exp(-ik_0 R_1)/k_0 R_1 \} / 4\pi \quad (2.15b)$$

is appropriate. The radii in (2.15) are defined by

$$R_0 = ((z-z')^2 + a^2)^{1/2} \quad \text{and}$$

$$R_1 = ((z-z')^2 + d^2)^{1/2}.$$

The next step in the analysis is to select appropriate trial functions. Because we are still working with thin perfectly conducting wires, we choose

$$(I_{\alpha}(z))_0 = -\exp(-ik_0|z|) \quad \text{and}$$

$$(I_{\beta}(z))_0 = -\exp(-ik_0|z-z_{\beta}|)$$

as we did in section 2.2 for the single wire case, where now I_{α} and I_{β} represent either I_a or I_t as appropriate. We again have the functional expression given in equation (2.9) where $K(z, z')$ is either $K_a(z, z')$ or $K_t(z, z')$ as stated in equations (2.15). Upon integrating the integrals in equation (2.9) for the two wire case (as shown in Appendix A), we find antenna mode and transmission line mode currents:

$$I_{\alpha}^{\infty}(z) = 2\pi \exp(-ik_0 z') / \epsilon_0 f_{\alpha}(z) \quad (2.9)$$

$$\text{where } f_{\alpha}(z) = E_1(ik_0 r_a^{\pm}) \pm E_1(ik_0 r_d^{\pm}) +$$

$$\exp(2ik_0 z) [E_1(ik_0 r_a^{\pm}) \pm E_1(ik_0 r_d^{\pm})]$$

$$\text{and } r_a^{\pm} = (z^2 + a^2)^{1/2} \pm z \quad \text{and} \quad r_d^{\pm} = (z^2 + d^2)^{1/2} \pm z .$$

For thin wires and for z in the far field, the transmission line mode current simplifies to (see Appendix A for details)

$$I_t^{\infty}(z) = \exp(-ik_0 z) / [\epsilon_0 \ln(d/a) / \pi] = V \exp(-ik_0 z) / Z_c$$

The characteristic impedance can be seen to be

$$Z_c = \epsilon_0 \ln(d/a) / \pi$$

which agrees with the characteristic impedance for a transmission line where $z^2 \gg a^2, d^2$.

CHAPTER III

SYSTEM MODE EVALUATION OF N THIN WIRE ANTENNA

For the one and two wire antennas it was easy to use a physically simple, conductor to conductor approach to solving the integral equation for the current, as was done in Chapter II. To analyze finite length antennas with N elements, where N is greater than two, the problems are easier to solve if we use a set of voltage sequences and current sequences with "black box" terminations as appears in figure 3.1.

3.1 General System Mode Definitions

Defining such a set of sequences as a system, the system current modes J can be written as

$$\overline{J} = \{J_m; m = 0, 1, \dots, N-1\}$$

$$\text{where } J_m(z) = \sum_{\mu=0}^{N-1} I_{\mu}(z) \exp(-i2\mu m\pi/N)/N \quad (3.1)$$

where $I_{\mu}(z)$ is the μ^{th} conductor current so that

$$I_{\mu}(z) = \sum_{m=0}^{N-1} J_m(z) \exp(i2\mu m\pi/N) . \quad (3.2)$$

Similarly the system voltage modes can be defined as

$$\overline{E} = \{E_m, m = 0, 1, \dots, N-1\} \quad (3.3)$$

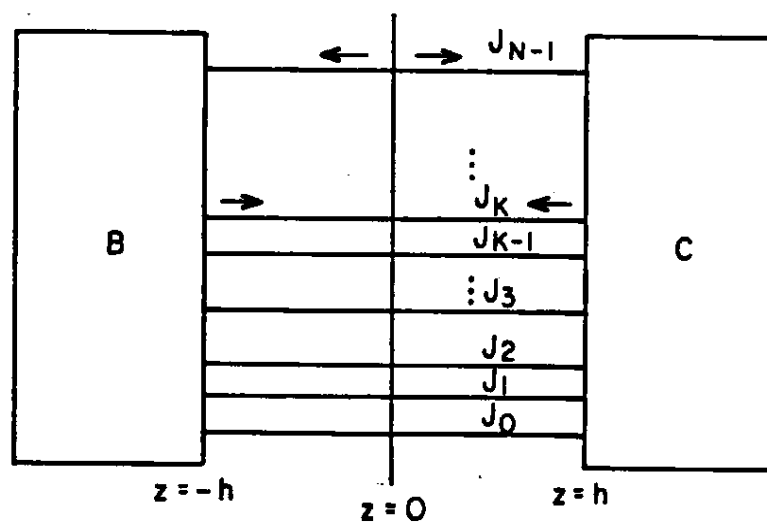


Figure 3.1 System mode analysis model

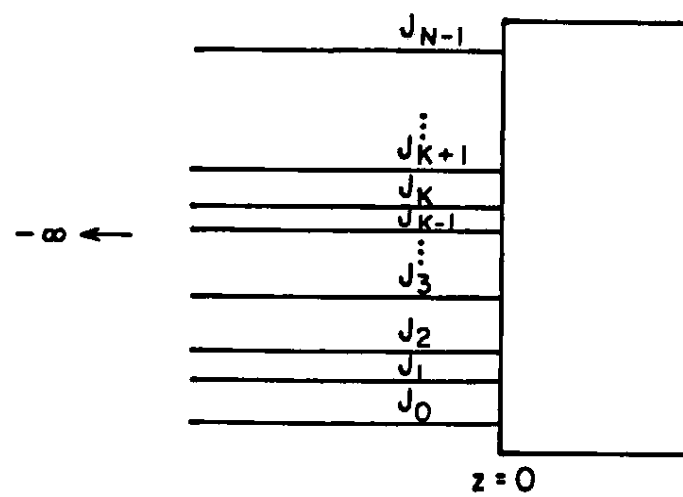


Figure 3.2 Semi-infinite "black box" termination

where the individual voltage applied to each wire is

$$\bar{V} = \{V_n, n = 0, 1, \dots, N-1\} \quad (3.4)$$

for conductor field sources $\phi(z) = -V_p \delta(z-z_p)$ so that

$$\bar{E} = (\bar{S}^*)^{-1} \bar{V} \quad (3.5)$$

where \bar{S}^* is the conjugate of \bar{S} and

$$\bar{S} = S_{mn}; m, n = 0, 1, \dots, N-1$$

$$S_{mn} = \exp(2\pi i m n / N) .$$

As was mentioned in Chapter I, Shen found that the total current can be found by summing the current incident at z (the infinitely long wire current, $I_\infty(z)$), the current reflected from the $z = h$ end of the wire, and the current reflected from the $z = -h$ end of the wire (these reflected currents were found to be proportional to $I_\infty(h+z)$ and $I_\infty(h-z)$) so that the total current on the γ^{th} conductor is

$$I_\gamma^{\text{total}}(z) = I_{\infty\gamma}(|z|) + B_\gamma I_{\infty}(h+z) + C_\gamma I_{\infty}(h-z) \quad (3.6)$$

where the coefficients B_γ and C_γ can be determined by using the end conditions of the antennas once the reflection coefficients are known. This similarly can be done for the system mode analysis so that the total system mode current can be evaluated as

$$\bar{J}(z) = \bar{Y}_\infty(|z|)\bar{E}(0) + \bar{Y}_\infty(h-z)\bar{E}(h) + \bar{Y}_\infty(h+z)\bar{E}(-h) \quad (3.7)$$

for the model in figure 3.1. In this expression $\bar{Y}_\infty(z)$ is the

transfer admittance matrix resulting from the m^{th} system mode current at z due to the n^{th} system mode voltage at $z = 0$; and $\bar{E}(0)$, $\bar{E}(h)$ and $\bar{E}(-h)$ are system mode voltages at 0 , h , and $-h$, respectively. The $\bar{J}(z)$, then, is the total system mode current at z (from all N wires); $\bar{Y}_{\infty}(|z|)\bar{E}(0)$ is the current at z from the input voltage; $\bar{Y}_{\infty}(h-z)\bar{E}(h)$ is the current at z due to the voltage resulting from the "black box" termination at h ; and $\bar{Y}_{\infty}(h+z)\bar{E}(-h)$ is the current at z due to the voltage resulting from the "black box" termination at $-h$.

We can also define two matrices \bar{Q}_B and \bar{Q}_C as the junction impedance matrices for the "black box" terminations at $-h$ and h , respectively, which yield the n^{th} reflected system mode voltage amplitude at the junctions due to the m^{th} system mode incident current. These Q matrices are defined by the reflected current equation for the semi-infinitely long wires, so that $\bar{J}^r(z) = \bar{Y}_{\infty}(|z|)\bar{Q}\bar{J}^{\text{inc}}(0)$ for figure 3.2. $\bar{E}(h)$ and $\bar{E}(-h)$ can be determined by multiplying the currents incident at h and $-h$, respectively, by the respective \bar{Q}_C or \bar{Q}_B :

$$\bar{E}(h) = \bar{Q}_C \{ \bar{Y}_{\infty}(h)\bar{E}(0) + \bar{Y}_{\infty}(2h)\bar{E}(-h) \} \quad (3.8)$$

$$\bar{E}(-h) = \bar{Q}_B \{ \bar{Y}_{\infty}(h)\bar{E}(0) + \bar{Y}_{\infty}(2h)\bar{E}(h) \} \quad (3.9)$$

From equations (3.7), (3.8), and (3.9) we can also determine the input admittance matrix \bar{Y} for the finite antenna case where $\bar{J}(0) = \bar{Y}\bar{E}(0)$. First, however, it is necessary to determine \bar{Y} for the infinitely long wire system and \bar{Q} for reflection from an end.

3.2 Infinitely Long N Wire System Mode Analysis

In addition to the discrete Fourier transform variables given in equations (3.1) and (3.2) and the transform variables of equations (3.3) to (3.5), it is advantageous to define a system kernel variable:

$$\bar{G} = \{G_{mn}; m, n = 0, 1, \dots, N-1\} \quad (3.10)$$

so that $G_{mn} = \sum_{pq=0}^{N-1} K_{pq}(z-z') \exp(i2\pi[mq+np]/N)/N$ where K_{pq} is defined by

$$K_{pq}(z, z') = -i\omega\mu \exp(-ik_0 R_{pq}) / 4\pi k_0^2 R_{pq}$$

$$\text{where } R_{pq} = ((z-z')^2 + a_{pq}^2)^{1/2}, \quad a_{pq} = \begin{cases} a, & \text{if } p = q \\ d_{pq}, & \text{if } p \neq q, \end{cases}$$

and d_{pq} is the distance between conductor p and conductor q . The phasing factor $\exp(i2\pi(mq+np)/N)$ allows excitation of different modes.

Using the system mode variables given in (3.1), (3.2), and (3.10), we can write Vainshtein's functional expression for the double variational technique as

$$F_{\alpha\beta} = -1/J_{S_\beta}^\alpha(z_\beta) = \sum_{mn=0}^{N-1} \langle J_m^\alpha, M_{mn} J_n^\beta \rangle / \{J_{S_\beta}^\alpha(z_\beta) J_{S_\alpha}^\beta(z_\alpha)\} \quad (3.11)$$

where $J_{S_\beta}^\alpha(z_\beta)$ is the β^{th} mode current (resulting from an excitation mode α) at z_β and M_{mn} is the operator

$M_{mn} J_n^\beta = \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-\infty}^{\infty} G_{mn}(z, z') J_n^\beta(z') dz'$. The α state of excitation is defined by

$$D \sum_{q=0}^{N-1} \int_{-\infty}^{\infty} K_{pq}(z, z') I_q(z') dz' = -\delta(z - z_\alpha) \exp(-i2\pi s_\alpha / N) / N \quad (3.12)$$

where s_α is an integer associated with α that sets the phase differences between conductors, and $D = (\partial^2 / \partial z^2 + k_0^2)$.

If the trial functions

$$J_m^\alpha(z) = A_m^\alpha \exp(-ik_0|z|) ; \quad m = 0, 1, \dots, N-1 \quad (3.13a)$$

and $J_n^\beta(z) = A_n^\beta \exp(-ik_0|z - z_\beta|) ; \quad n = 0, 1, \dots, N-1 \quad (3.13b)$

where A_m and A_n are complex amplitudes constant with respect to z , are used in equation (3.11); the functional becomes

$$\begin{aligned} F_{\alpha\beta} = & \sum_{pq=0}^{N-1} \sum_{mn=0}^{N-1} A_n^\beta A_m^\alpha \exp(i2\pi(mq + np)/N) \\ & \cdot \int_{-\infty}^{\infty} dz \exp(-ik_0|z - z_\beta|) D \int_{-\infty}^{\infty} dz' \exp(-ik_0|z'|) K_{pq}(z, z') / \\ & \{A_{s_\beta}^\alpha A_{s_\alpha}^\beta \exp(-ik_0|z_\beta|) \exp(-ik_0|-z_\beta|)\} \end{aligned} \quad (3.14)$$

The double integral in (3.14) is identical in form to that evaluated in Appendix A. Using this result (3.14) becomes

$$F_{\alpha\beta} = -(\zeta_0 / 2\pi N) \sum_{mn=0}^{N-1} A_n^\beta A_m^\alpha P_{mn}(z_\beta) / (A_{s_\beta}^\alpha A_{s_\alpha}^\beta) \quad (3.15)$$

where

$$\begin{aligned} P_{mn}(z_\beta) = & -(\zeta_0 / 2\pi N) \sum_{pq=0}^{N-1} \exp(2\pi i(mq + np)/N) \exp(-ik_0 z_\beta) \\ & \cdot \{E_1(ikr^-) + \exp(2ik_0 z_\beta) E_1(ikr^+)\} \end{aligned} \quad (3.16)$$

$$\text{where } r^\pm = ((z_\beta^2 + a_{pq}^2)^{1/2} \pm z), \quad a_{pq} = \begin{cases} a, & \text{if } p = q \\ d_{pq}, & \text{if } p \neq q \end{cases}$$

and d_{pq} is the distance between conductor p and conductor q .

The incident current now is

$$J_{S_\beta}^\alpha(z) \approx -A_{S_\beta}^\alpha A_{S_\alpha}^\beta / \sum_{mn=0}^{N-1} A_n^\beta A_m^{\alpha p}{}_{mn}(z) \quad (3.17)$$

As stated above $\bar{Y}_\infty(|z|)E(0) = \bar{J}_{S_\beta}^\alpha(z)$, so $\bar{Y}_\infty(|z|)$ can be determined from (3.17).

The problem remaining is to determine the coefficients A_m and A_n . These coefficients must be determined using the physical layout of the wires and the phasing of the input voltage. Two of these coefficients are usually set to one; the remaining coefficients fall out using the double variational property. To demonstrate the determination of these coefficients, we look at the simple two wire, infinitely long transmission line.

3.3 The Infinitely Long, Two Wire Line Current

To demonstrate how to use equation (3.17) we again evaluate the two wire, infinitely long case using the system mode approach. For the infinitely long, parallel two thin wire line with the diameter of the wire a and separated by a distance d (as shown in figure 2.2), the system mode currents are $J_0 = (I_0 + I_1)/2$ and $J_1 = (I_0 - I_1)/2$ so that J_0 is the antenna mode current and J_1 is the transmission line mode current. We can use equations (3.16) and (3.17) to find that

$$J_{S_\beta}^\alpha(z) = -A_{S_\beta}^\alpha A_{S_\alpha}^\beta / \sum_{mn=0}^1 A_m^\alpha A_n^{\beta p}{}_{mn}(z) \quad (3.18)$$

where

$$P_{mn}(z) = -(\zeta_0/4\pi) \exp(-ik_0 z) \sum_{pq=0}^1 \exp(\pi i(mp+nq)) \\ \cdot \{E_1(ik_0 r^-) + \exp(2ik_0 z) E_1(ikr^+)\}.$$

The individual P_{mn} terms are readily determined to be

$$P_{00}(z) = -(\zeta_0/2\pi) \{E_1(ik_0 r_a^-) + E_1(ik_0 r_d^-) + \exp(2ik_0 z) [E_1(ik_0 r_a^+) \\ + E_1(ikr_d^+)]\} \exp(ik_0 z) \quad (3.19a)$$

$$P_{01} = P_{10}(z) = 0 \quad (3.19b)$$

and

$$P_{11}(z) = -(\zeta_0/2\pi) \{E_1(ik_0 r_a^-) - E_1(ik_0 r_d^-) + \exp(2ik_0 z) [E_1(ikr_a^+) \\ - E_1(ikr_d^+)]\} \exp(ik_0 z) \quad (3.19c)$$

where $r_a^\pm = (z^2 + a^2)^{1/2} \pm z$ and $r_d^\pm = (z^2 + d^2)^{1/2} \pm z$.

This reduces (3.15) and (3.17) to

$$F_{\alpha\beta} = -1/J_{s_\beta}^\alpha(z) = (A_{00}^{\alpha\beta} P_{00} + A_{11}^{\alpha\beta} P_{11}) / (A_{s_\alpha}^\beta A_{s_\beta}^\alpha) \quad (3.20)$$

Now we must determine the coefficients. The first case is

$s_\alpha = s_\beta = 0$. Assuming $A_0^\alpha = A_0^\beta = 1$ (since α and β are even), then

$$F_{00} = [P_{00} + A_{11}^{\alpha\beta} P_{11}].$$

Using the double variational property so that $\partial F_{\alpha\beta} / \partial A_1^\alpha = 0$,

we see that

$$\partial F_{00} / \partial A_1^\alpha = 0 = A_{11}^{\beta} P_{11}.$$

This implies that $A_1^\beta = 0$, since P_{11} is not identically zero.

The second case is $s_\alpha = 1$ and $s_\beta = 0$, so that

$$J_0^\alpha(z) = J_1^\beta(z) = -A_1^\beta A_0^\alpha / [A_0^\alpha A_0^\beta P_{00} + A_1^\alpha A_1^\beta P_{11}]$$

setting $A_0^\beta = A_1^\alpha$, then

$$J_1^\beta(z) = J_0^\alpha(z) = -A_1^\beta A_0^\alpha / [A_0^\alpha P_{00} + A_1^\beta P_{11}]$$

so that

$$\partial J_1^\beta / \partial A_0^\alpha = 0 = -[A_1^\beta (A_0^\alpha P_{00} + A_1^\beta P_{11}) - A_1^\beta A_0^\alpha P_{00}] / [A_0^\alpha P_{00} + A_1^\beta P_{11}]^2$$

so that $A_1^\beta = 0$. Therefore $J_1^\beta(z) = J_0^\alpha(z) = 0$. Similarly

$J_0^\beta(z) = J_1^\alpha(z) = 0$ for $s_\alpha = 0$ and $s_\beta = 1$.

For the remaining case $s_\alpha = s_\beta = 1$,

$$F_{11} = (A_0^\alpha A_0^\beta P_{00} + A_1^\alpha A_1^\beta P_{11}) / A_1^\beta A_1^\alpha$$

so that for $A_1^\alpha = A_1^\beta = 1$, the double variational property

$\partial F_{11} / \partial A_0^\alpha = 0$ gives

$$\partial F_{11} / \partial A_0^\alpha = A_0^\beta P_{00} = 0$$

implying that $F_{11} = P_{11}$ and $J_1(z) = J_1(z) = -1/P_{11}(z)$. The

resulting transfer admittance matrix is

$$\bar{Y}(z) = \begin{bmatrix} -1/P_{00} & 0 \\ 0 & -1/P_{11} \end{bmatrix} \quad (3.21)$$

where $P_{00}(z)$ and $P_{11}(z)$ are given in equation (3.19) for the infinitely long two wire line. This is identical to the result in section 2.3.

3.4 The Reflected Current Functional

In order to determine the junction impedance matrices $\bar{\bar{Q}}_B$ and $\bar{\bar{Q}}_C$, we would like a functional pertaining to the reflected current rather than to the total current of Vainshtein's functional. We will develop such a functional using the conductor currents, I_p , of Chapter II. Since the system mode currents, J_m , are simply the sum of I_p multiplied by constants, the system mode currents will have a similar functional that is double variational.

By defining on a given conductor

$$I^\alpha = I^{\alpha,i} + I^{\alpha,r} \quad (3.22)$$

where I^α is the total current, $I^{\alpha,i}$ is the incident current, and $I^{\alpha,r}$ is the reflected current, we can write

$$\begin{aligned} LI^{\alpha,i} &= (L_+ + L_-)I^{\alpha,i} = D \int_{-\infty}^{\infty} I^{\alpha,i}(z')K(z,z')dz' \\ &= \mathcal{E}_\alpha(z) \end{aligned} \quad (3.23)$$

where $L_+ I^{\alpha,i} = D \int_0^{\infty} I^{\alpha,i}(z')K(z,z')dz$ and $L_- I^{\alpha,i} = D \int_{-\infty}^0 I^{\alpha,i}(z')K(z,z')dz$. Because of the reciprocity of $K(z,z')$ $= K(z',z)$ and because K is a function of $(z-z')^2$ only, it can be seen that $\partial K(z,z')/\partial z = \partial K(z',z)/\partial z$, so that (details in Appendix B)

$$\langle I^{\beta,i}, L_- I^\alpha \rangle = \langle I^\alpha, L I^{\beta,i} \rangle_- \quad (3.24)$$

(where $\langle A, B \rangle_- = \int_{-\infty}^0 AB dz$) and that

$$\begin{aligned}
 VI^{\beta,r}(-z_\alpha) &= -\langle i^\beta, L_+ I^{\alpha,i} \rangle + \langle i^\beta, L_- I^{\alpha,r} \rangle \\
 &= \langle i^\beta, L i^\alpha \rangle
 \end{aligned} \tag{3.25}$$

where the scattered currents are defined by $i^\beta = \begin{cases} I^{\beta,r}, & z < 0 \\ -I^{\beta,i}, & z > 0 \end{cases}$.

Defining $I^{\beta,r} = I_0^{\beta,r} + \partial I^{\beta,r}$ and $I^{\alpha,r} = I_0^{\alpha,r} + \partial I^{\alpha,r}$ (where $\partial I^{\beta,r}$ and $\partial I^{\alpha,r}$ are small changes in the $I^{\beta,r}$ and $I^{\alpha,r}$, respectively), while the Green's function, K , remains unchanged; $I^{\beta,r}(-z_\alpha)$ can be shown to be stationary (see Appendix B for details). Since $I^{\beta,r}(-z_\alpha)$ is stationary and because $LI^\beta = e_\beta$, $I^{\beta,r}(-z_\alpha)$ is double variational as written in equation (3.25).

3.5 The Semi-infinitely Long N Wire Reflected Current Analysis

For the N wire semi-infinitely long array of figure 3.3, with the wires all ending at $z = 0$, with each wire of diameter a , and with the i^{th} and j^{th} wires separated by a distance d_{ij} , the system mode expression for the reflected current from equation (3.25) is

$$j_{s_\alpha}^{\beta,r}(z_\beta) = \sum_{mn=0}^{N-1} \langle j_m^\alpha(z), M_{mn}(z, z') j_n^\beta(z') \rangle \tag{3.26}$$

where $j_m(z) = \begin{cases} j_m^r(z), & z < 0 \\ -j_m^i(z), & z > 0 \end{cases}$, $j_m^r(z)$ is the reflected current,

$j_m^i(z)$ is the incident current, and M_{mn} is described in equation (3.11). As was done for the infinitely long case, we must select trial functions for j_m^α and j_n^β . Choosing

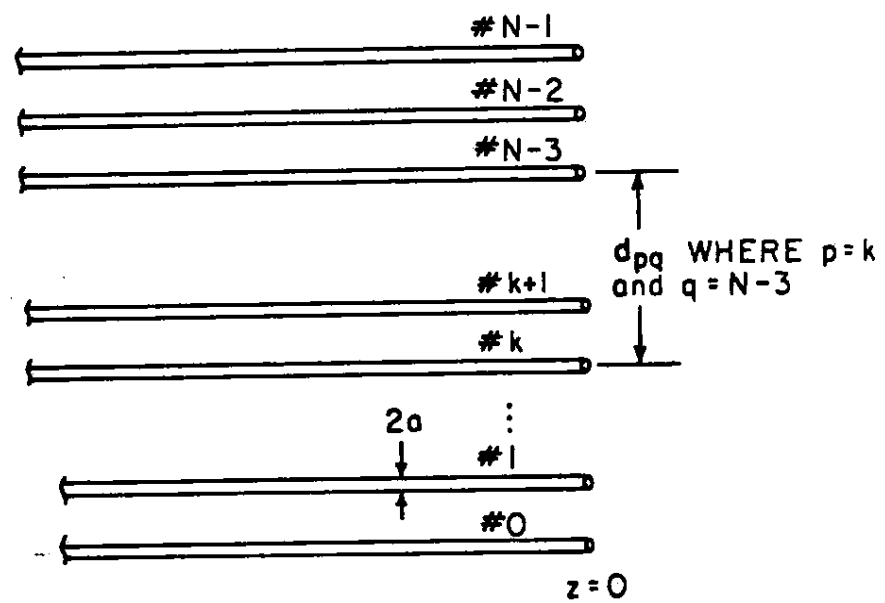


Figure 3.3 N semi-infinitely long wires

$$j_m^\alpha(z) = -J_m^\alpha(z_\alpha) \exp(-ik_0|z|) \quad (3.28a)$$

$$j_n^\beta(z) = -J_n^\beta(z_\beta) \exp(-ik_0|z|) \quad (3.28b)$$

where $J_m^\alpha(z_\alpha)$ and $J_n^\beta(z_\beta)$ are incident currents as described by equation (3.17)

Using the trial functions in (3.26) and integrating as in Appendix A, the reflected current becomes

$$J_{S_\alpha}^{\beta,r}(-z_\beta) = \sum_{mn=0}^{N-1} J_m^\alpha(z_\alpha) J_n^\beta(z_\beta) Q_{mn} \quad (3.29)$$

$$\text{where } Q_{mn} = -(\epsilon_0/\pi N) \sum_{pq=0}^{N-1} \exp(2\pi i [mp + nq]/N) \left[E_1(ik_0 a_{pq}) \right]$$

where $a_{pq} = \begin{cases} a & \text{if } p = q \\ d_{pq} & \text{if } p \neq q \end{cases}$. This Q_{mn} is the mn^{th} term of the junction impedance matrix.

As an example of equation (3.29), we examine two, semi-infinitely long wires truncated at $z = 0$ of figure 3.4. The incident currents $J_0^\alpha(z)$ and $J_1^\alpha(z)$ are given by equation (3.21) when multiplied by $\bar{E}(0)$ so that $J_0^\alpha(z) = -1/P_{00}(z)$ and $J_1^\alpha(z) = -1/P_{11}(z)$ where P_{00} and P_{11} are given in equations (3.19).

From equation (3.29), we find that $Q_{01} = Q_{10} = 0$, showing that there is no mode conversion in reflection. This means then that a reflection coefficient matrix with the only nonzero terms being diagonal terms (R_{00} and R_{11}) can be found. Since these diagonal terms are independent of each other, we can write expressions for these in terms of scalar equations. Rewriting (3.29) we have

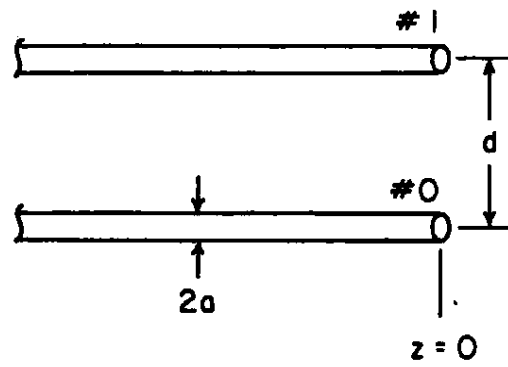


Figure 3.4 Two wire truncated antenna

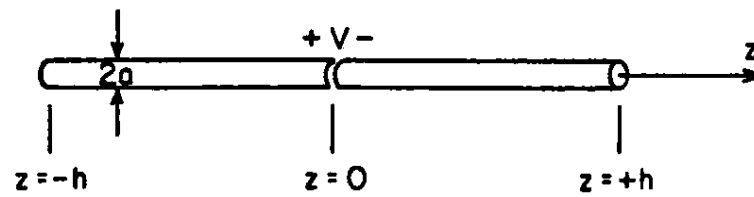


Figure 3.5 Single wire, finite length antenna

$$J_{s_\alpha}^{\beta,r}(z_\beta) = \sum_{m=0}^1 J_m^\alpha(z_\alpha) J_m^\beta(z_\beta) Q_{mm} \quad (3.30)$$

since $Q_{mn} = 0$ if $m \neq n$. In order to find a reflection coefficient, it is necessary to examine the reflected current at a z far from $z = 0$ --the discontinuity from truncation. Looking at the reflected current in terms of the incident current at the end ($J_m^\alpha(z_\alpha)$), the reflection coefficient R_{mm} , and an impedance transfer function $\exp(-ik_0 z_\beta)$; the transmission line mode reflected current is (since $J_0^\alpha(z_\alpha) = 0$)

$$\begin{aligned} J_1^{\beta,r}(-z_\beta) &= J_1^\alpha(z_\alpha) J_1^\beta(z_\beta) Q_{11} \\ &= J_1^\alpha(z_\alpha) R_{11} \exp(-ik_0 z_\beta) \end{aligned}$$

so that the reflection coefficient is found from

$$\begin{aligned} R_{11} &= \lim_{z_\beta \rightarrow \infty} J_1^{\beta,r}(-z_\beta) \exp(ik_0 z_\beta) / J_1^\alpha(z_\alpha) \\ &= \lim_{z_\beta \rightarrow \infty} J_1^\beta(z_\beta) Q_{11} \exp(ik_0 z_\beta) \\ &= \lim_{z_\beta \rightarrow \infty} \left[Y_\infty(z_\beta) \right]_{11} \exp(ik_0 z_\beta) Q_{11} . \end{aligned} \quad (3.31)$$

(details in Appendix B), the transmission line mode reflection coefficient for the truncated two lines is

$$R_{11} = -\exp(-ik_0 d / \ln(d/a)) \cdot \exp\{-(k_0 d)^2 (1 - 2/\ln(d/a)) / (4 \ln(d/a))\} \quad (3.32)$$

In her 1970 paper, N.I. Shameyeva determined a transmission line mode current reflection coefficient for the thin, truncated, two wires of figure 3.4. For a time convention of $\exp(-i\omega t)$, her

reflection coefficient was

$$R = -|R| \exp(zik\alpha)$$

where $|R| = \exp\{-(k_0 d)^2 [1 - 2 \ln(d/a)] / [4 \ln(d/a)]\}$ and the open end correction α was $\alpha = d / [2 \ln(d/a)]$.⁷ Since our time convention is $\exp(+i\omega t)$, it is easily seen that we are in agreement.

3.6 Single Dipole Antenna System Mode Analysis

As a simple example of the system mode analysis for a finite length antenna, we first look at a simple dipole antenna of length $2h$, with wire diameter a , and being center-fed as shown in figure 3.5. The system mode current of equation (3.1) is simply $\bar{J} = J_0 = I_0$; and the system voltage is $\bar{E} = E_0 = 1$. Equation (3.4) becomes a linear equation:

$$J(z) = Y_{\infty 0}(|z|)E_0(0) + Y_{\infty 0}(h-z)E_0(h) + Y_{\infty 0}(h+z)E_0(-h)$$

$$\text{where } E_0(\pm h) = Q_{CB} \left[Y_{\infty 0}(h)E_0(0) + Y_{\infty 0}(2h)E_0(\mp h) \right].$$

The incident current (and equivalently the transfer admittance $Y_{\infty 0}(z)$) from equation (3.17) is seen to be

$$J_0^\alpha(z) = Y_{\infty 0}(z) = -A_0^\alpha A_0^\beta / (A_0^\beta A_0^\alpha P_{00}(z)) = -1/P_{00}(z)$$

$$\text{where } P_{00}(z) = -(\zeta_0/2\pi) \exp(-ik_0 z) \left[E_1(ik_0 r^-) + \exp(2ik_0 z) E_1(ik_0 r^+) \right]$$

and $r^\pm = (z^2 + a^2) \pm z$ from equation (3.16). From equation (3.29),

the junction impedance matrix can be found so that $Q_{00} =$

$$-(\zeta_0/\pi) E_1(ik_0 a). \text{ Due to the symmetry of the problem } Q_{B00} = Q_{C00},$$

so the system mode voltages $E(h)$ and $E(-h)$ are quickly determined to be

$$E(h) = E(-h) = Q_{00}Y_{\infty}(h)/[1 - Q_{00}Y_{\infty}(2h)] \quad (3.34)$$

This results in a total current expression of

$$J(z) = Y_{\infty}(|z|) + [Y_{\infty}(h-z) + Y_{\infty}(h+z)]Q_{00}Y_{\infty}(h)/[1 - Q_{00}Y_{\infty}(2h)] \quad (3.35)$$

so that the input conductance is simply

$$Y(0) = Y_{\infty}(0) + 2[Y_{\infty}(h)]^2 Q_{00}/[1 - Q_{00}Y_{\infty}(2h)] . \quad (3.36)$$

The results for equation (3.36) for the single dipole antenna are compared with the results of the King Middleton method² for various values of k_0h in figure 3.6 with good agreement.

3.7 Two Dipole Antennas System Mode Analysis

Since we determined the system mode transfer admittance matrix (for the infinitely long wires) in section 3.3 and the junction impedance matrix in section 3.5, we need only determine the system mode voltages $E(h)$ and $E(-h)$ to determine the two dipole antenna current for the antenna of figure 3.7. Since there is no coupling between the antenna mode and transmission line mode ($Q_{01} = Q_{10} = 0$ as shown in section 3.5), equation (3.6), the linear equivalent of equation (3.7) can be used:

$$J_0(z) = (J_{\infty}(|z|))_0 + C_0(J_{\infty}(h-z))_0 + B_0(J_{\infty}(h+z))_0 \quad (3.37)$$

1
1
1
1
1

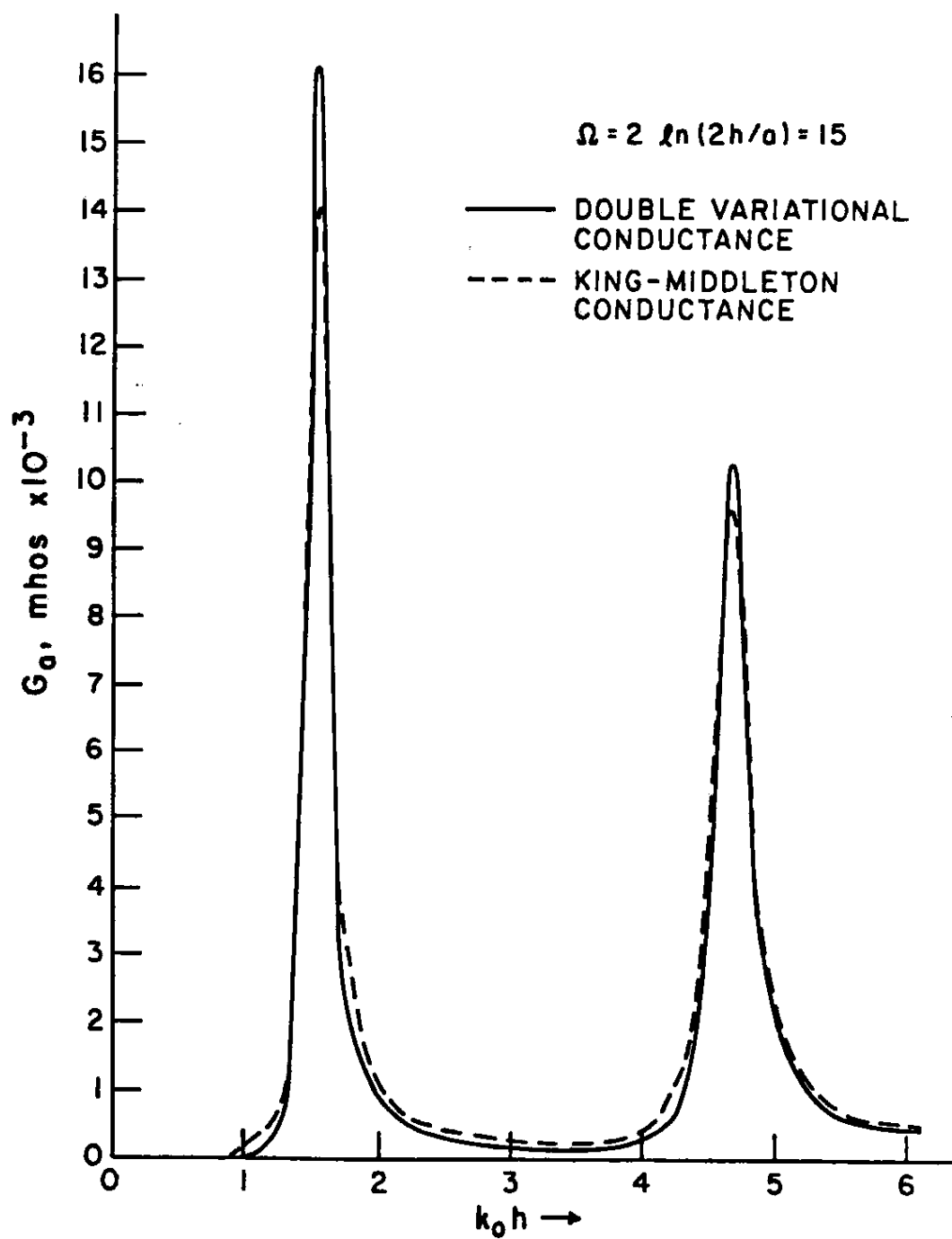


Figure 3.6a Single wire antenna input conductance vs $k_0 h$

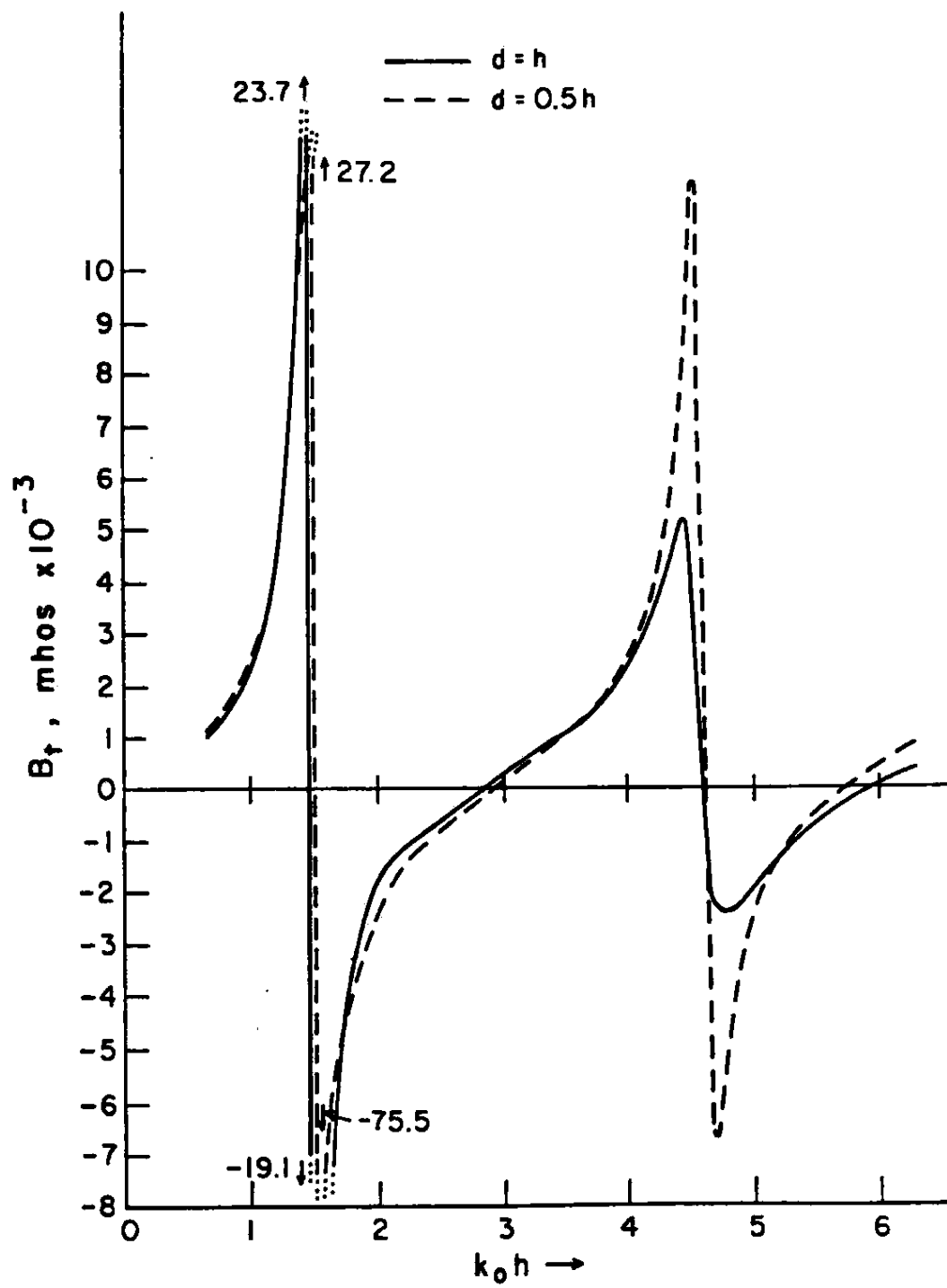


Figure 3.6b Single wire antenna input susceptance vs $k_0 h$

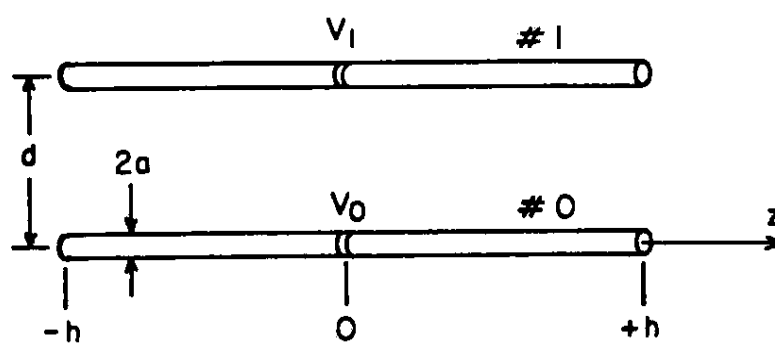


Figure 3.7 Two wire, finite length, center-fed antenna

(where the coefficients B_0 and C_0 are determined as $\bar{E}(h)$ and $\bar{E}(-h)$ were for the single dipole case in section 3.6, from equations (3.8) and (3.9)). The "0" subscripted variables are for the antenna mode and the "1" subscripted variables are for the transmission line mode.

The coefficients C_0 , B_0 , C_1 and B_1 are determined in Appendix B so that $C_0 = B_0 = Q_{00} [J_\infty(h)]_0 / [1 - Q_{00} (J_\infty(2h))_0]$.

Knowing this, equation (3.37) becomes

$$J_0(z) = (J_\infty(|z|))_0 + \frac{[J_\infty(h-z) + J_\infty(h+z)]_0 Q_{00} (J_\infty(h))_0}{[1 - Q_{00} (J_\infty(2h))_0]} \quad (3.38)$$

The resulting input admittance is then found to be

$$Y_a(0) = J_\infty(0)_0 - 2 \left[J_\infty(h)_0 \right]^2 Q_{00} / [1 - Q_{00} J_\infty(2h)_0] \quad (3.39)$$

Y_a and Y_t are compared with King's second order approximations⁸ for various $k_0 d$ in figures 3.8. The results are in good agreement for $\Omega = 2 \ln(2h/a) = 15$. Figures 3.9 show Y_a and Y_t for varied $k_0 h$, with $d = .5 h$ and $d = h$. As would be expected, for the antenna mode the admittance resonance peaks are greater for greater separation; for the transmission line mode, the admittance resonance peaks are smaller for greater separation. The slight variances in the resonances for $d = .5 h$ and $d = h$ are caused by end effects (differences in end capacitances).

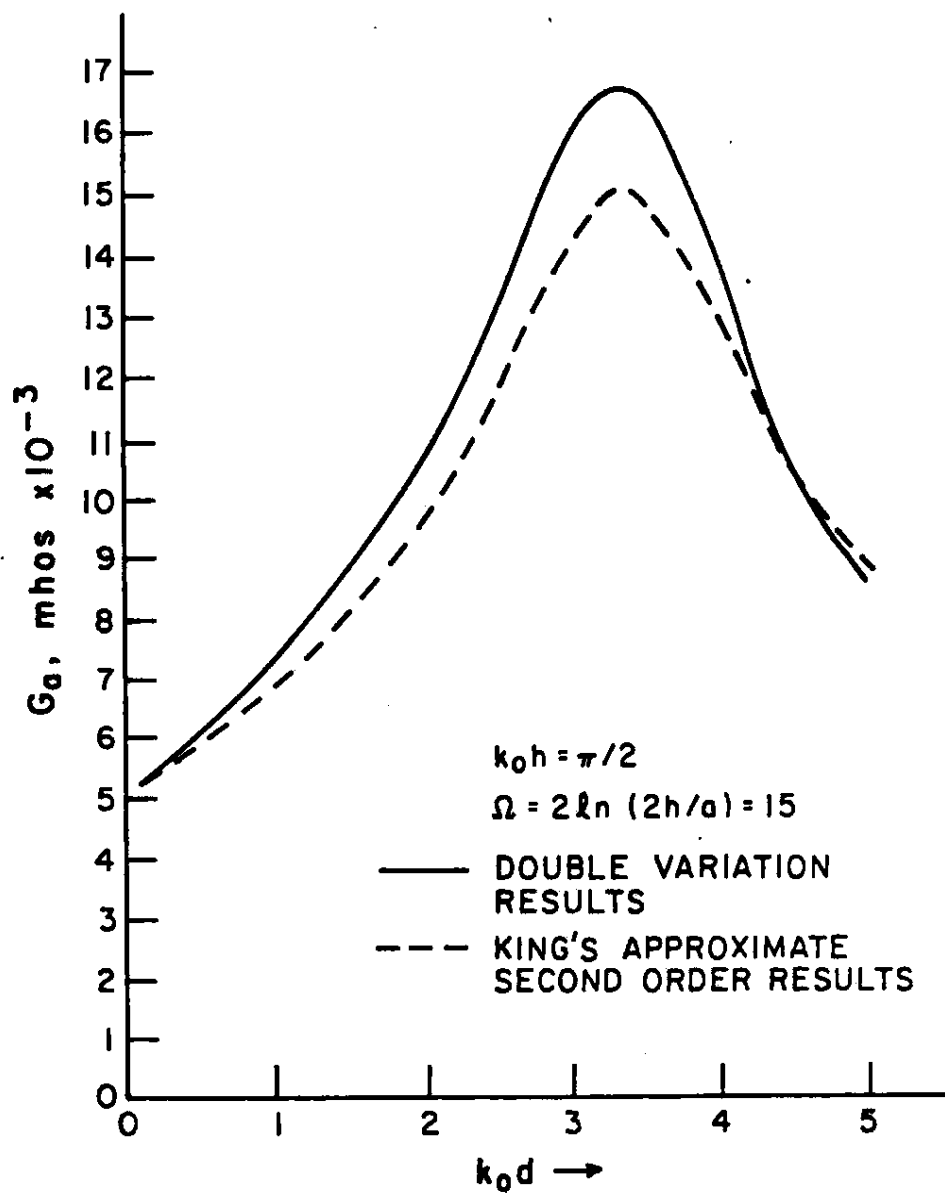


Figure 3.8a Two parallel dipole antennas antenna mode conductance vs $k_0 d$ (Distance between antennas varied)

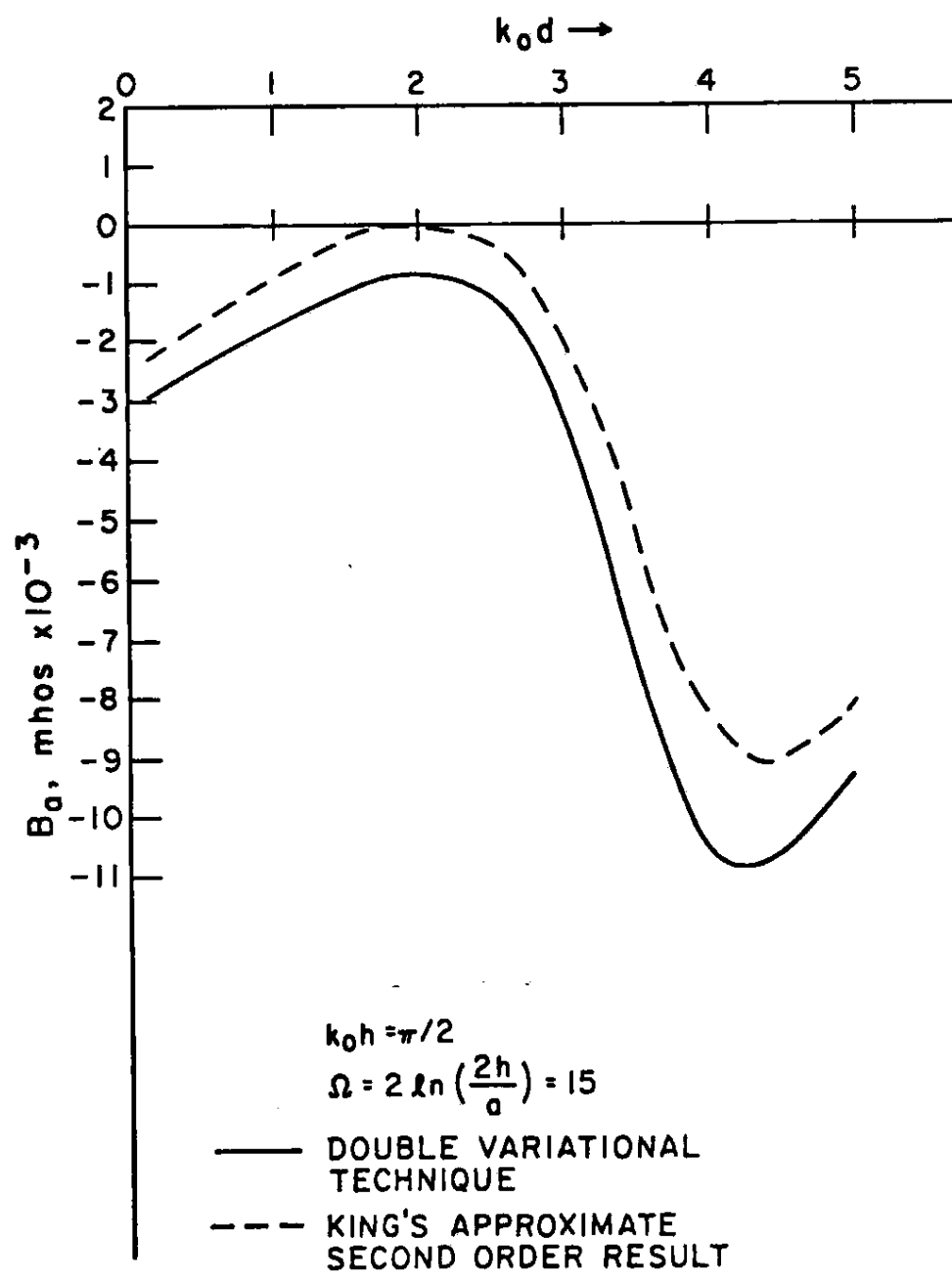


Figure 3.8b Two dipole antennas antenna mode susceptance vs $k_0 d$

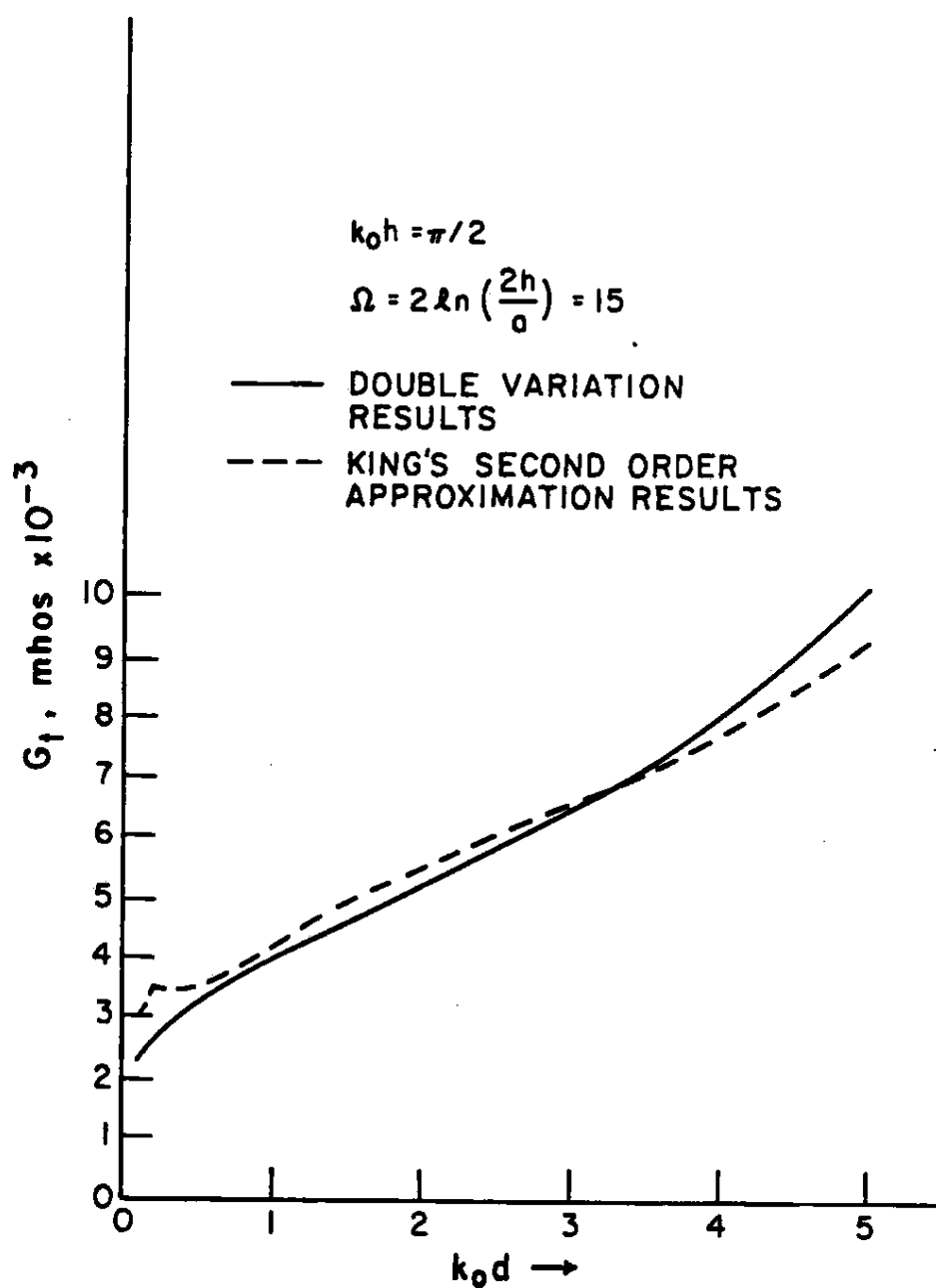


Figure 3.8c Two dipole antennas transmission line mode conductance vs $k_0 d$

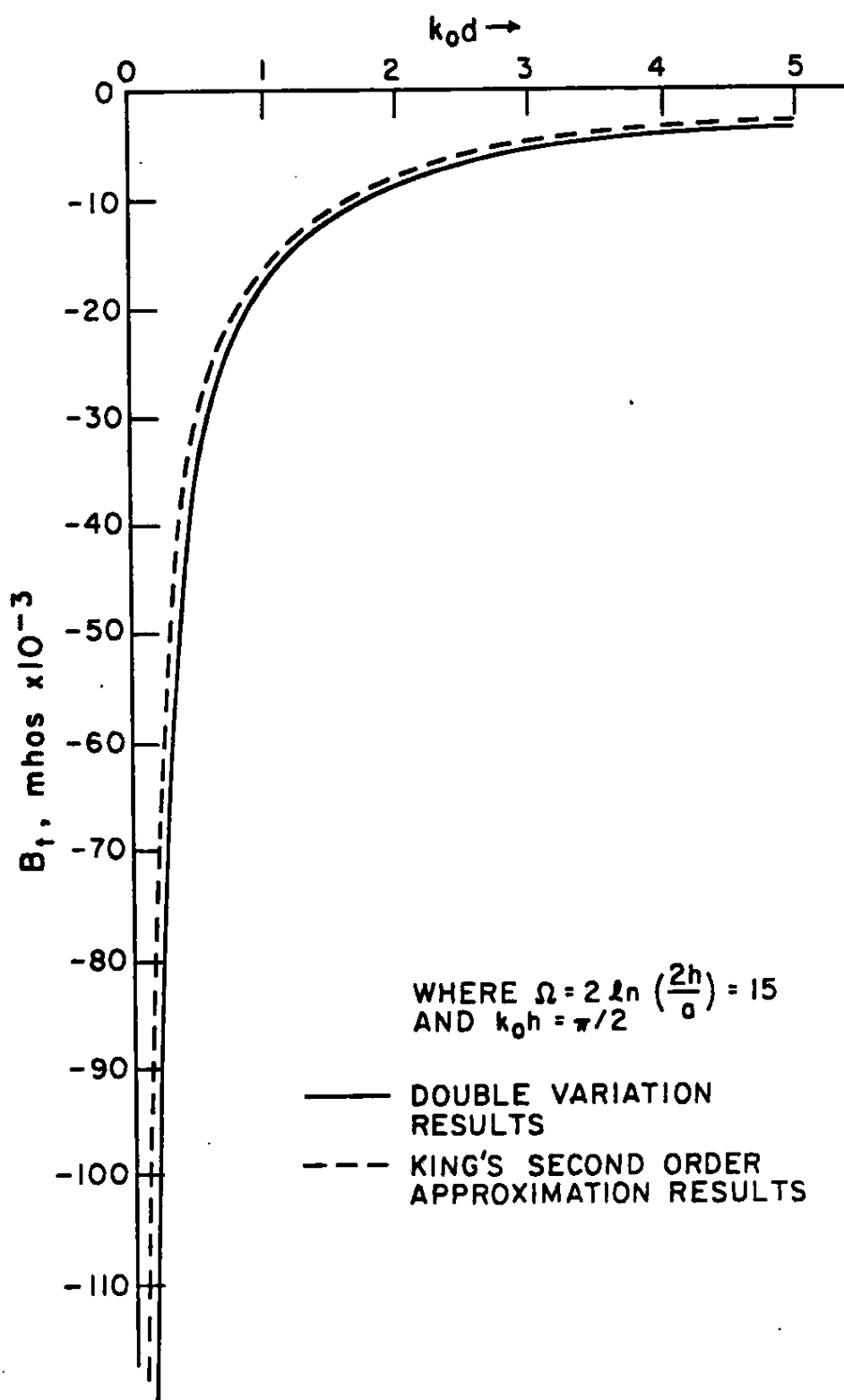


Figure 3.8d Two dipole antennas transmission line mode susceptance vs $k_0 d$

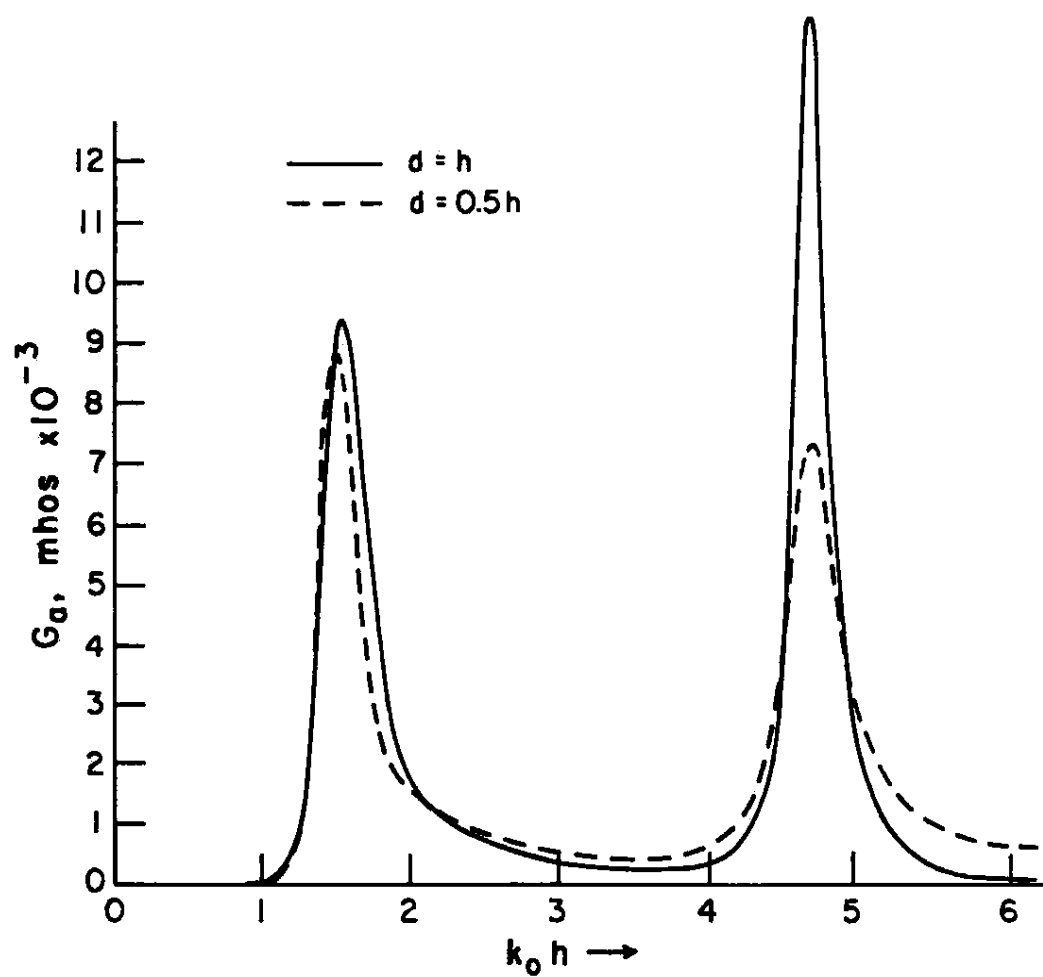


Figure 3.9a Two parallel dipole antennas antenna mode conductance vs $k_0 h$ (frequency varied)

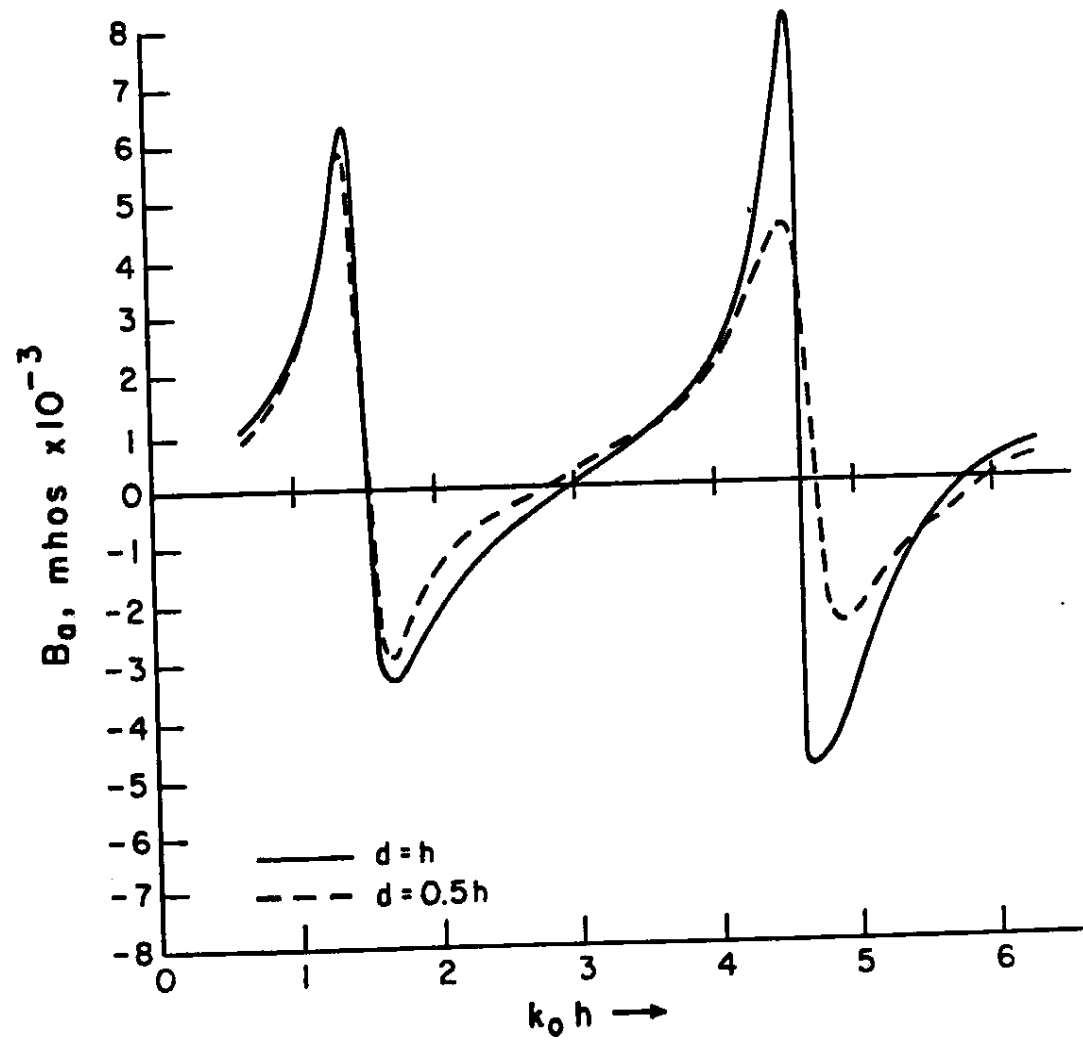


Figure 3.9b Two dipole antennas antenna mode susceptance vs $k_0 h$

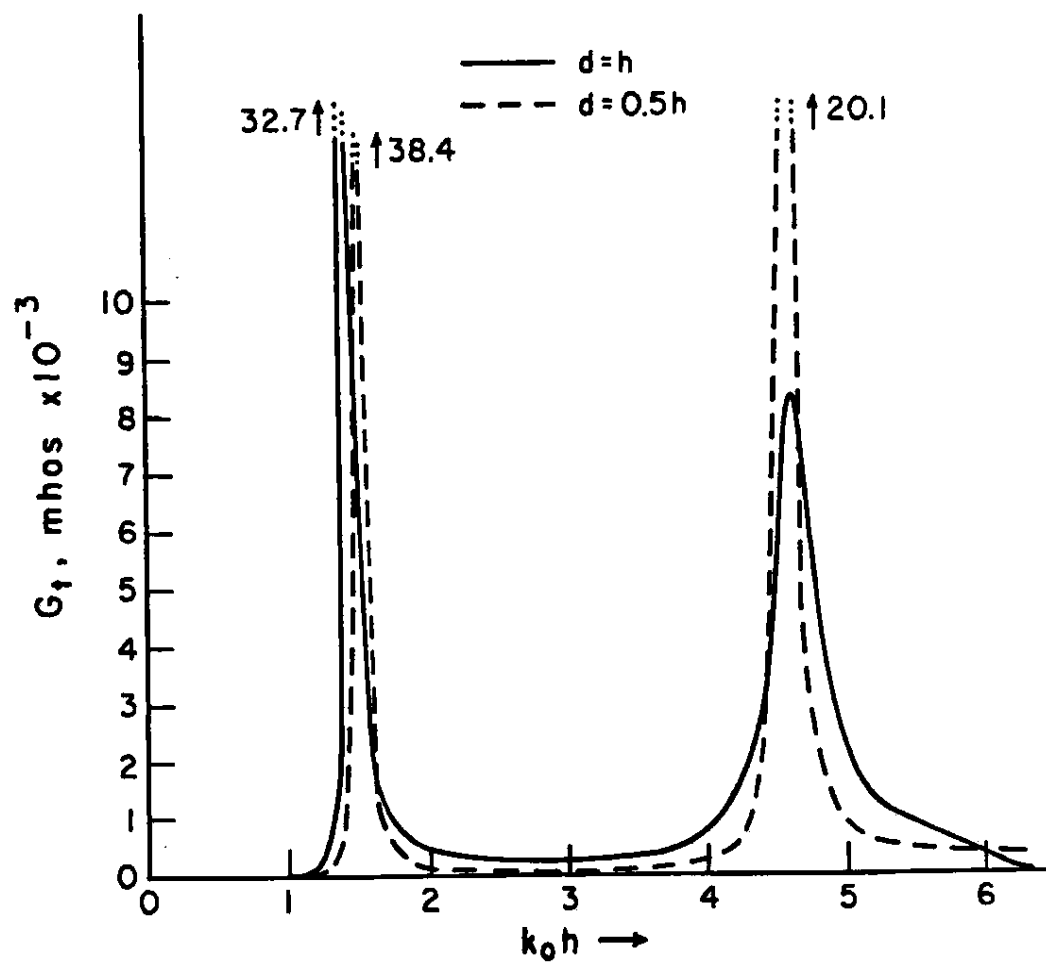


Figure 3.9c Two dipole antennas transmission line mode conductance vs $k_0 h$

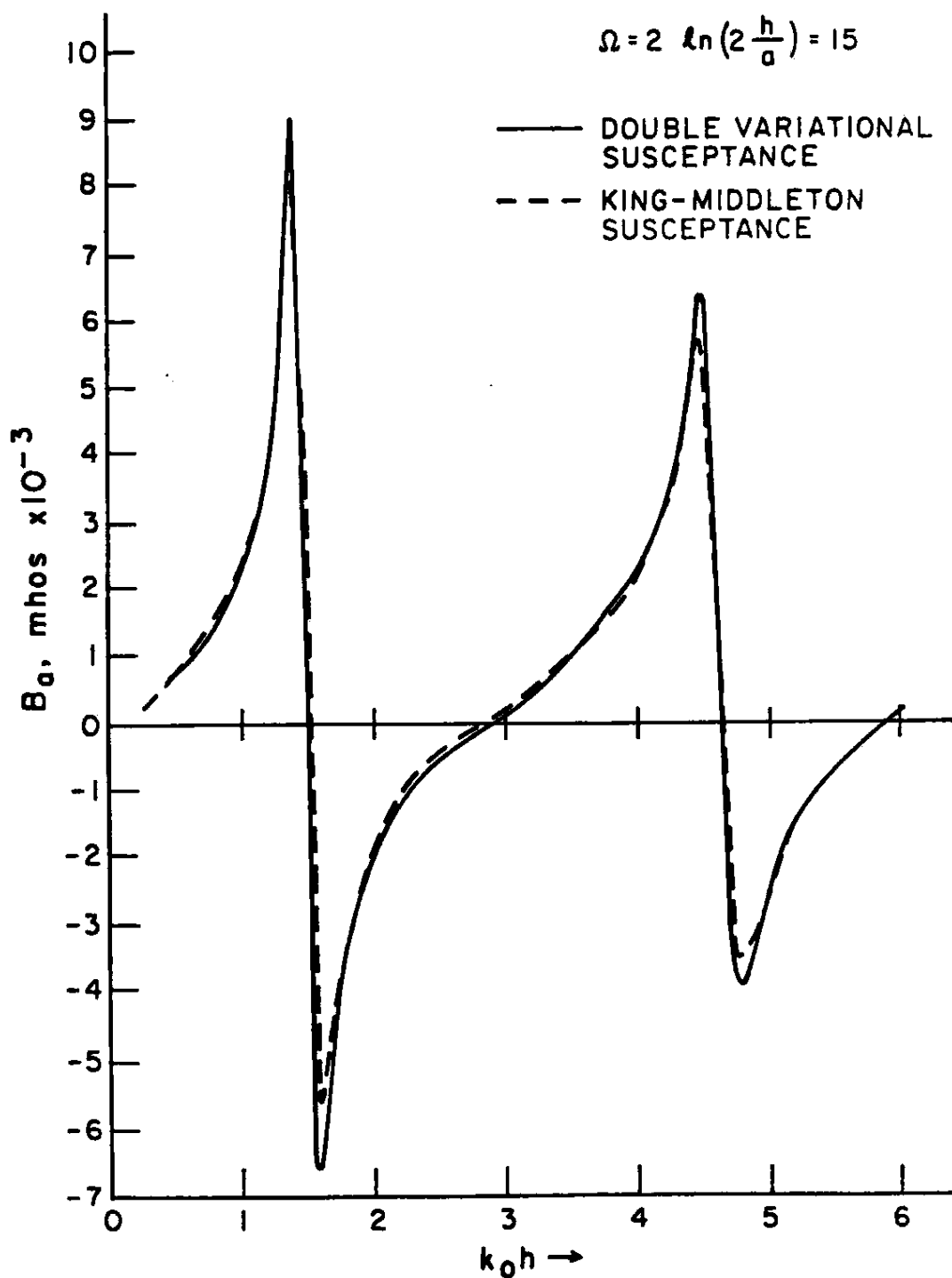


Figure 3.9d Two dipole antennas transmission line mode susceptance vs $k_0 h$

3.8 Circular Array of Three Dipole Antennas System

Mode Evaluation

As another example of a finite length antenna array, we examine the three dipole antenna circular array (each element is of equal length and is center fed) shown in figure 3.10. The development for this geometry is identical to that for the antennas in a plane, except that the kernel is defined by

$$K_{pq}(z-z') = (i\zeta_0/4\pi) \exp(-ik_0 R_{pq})/R_{pq}$$

$$\text{where } R_{pq} = ((z-z')^2 + a_{pq}^2)^{1/2} \text{ and } a_{pq} = \begin{cases} a & \text{if } p = q \\ d & \text{if } p \neq q \end{cases}.$$

To use equations (3.1) and (3.17), we first determine the P_{mn} coefficients from (3.16):

$$\bar{\bar{P}}(z) = \begin{bmatrix} P_{00} & 0 & 0 \\ 0 & 0 & P_{12} \\ 0 & P_{21} & 0 \end{bmatrix}$$

$$\begin{aligned} \text{where } P_{00}(z) = & -(\zeta_0/2\pi) \exp(-ik_0 z) \{E_1(ik_0 r_a^-) + 2E_1(ik_0 r_d^-) + \\ & \exp(2ik_0 z) [E_1(ikr_a^+) + 2E_1(ikr_d^+)]\} \end{aligned} \quad (3.40a)$$

$$\begin{aligned} \text{and } P_{12}(z) = P_{21}(z) = & -(\zeta_0/2\pi) \exp(-ik_0 z) \\ & \{E_1(ikr_a^-) - E_1(ikr_d^-) + \exp(2ik_0 z) [E_1(ikr_a^+) - E_1(ikr_d^+)]\} \end{aligned} \quad (3.40b)$$

The next step is to determine the coefficients A_m and A_n in $F_{\alpha\beta}$ and $J_{S_\beta}^\alpha(z)$. These are done as the coefficients were found

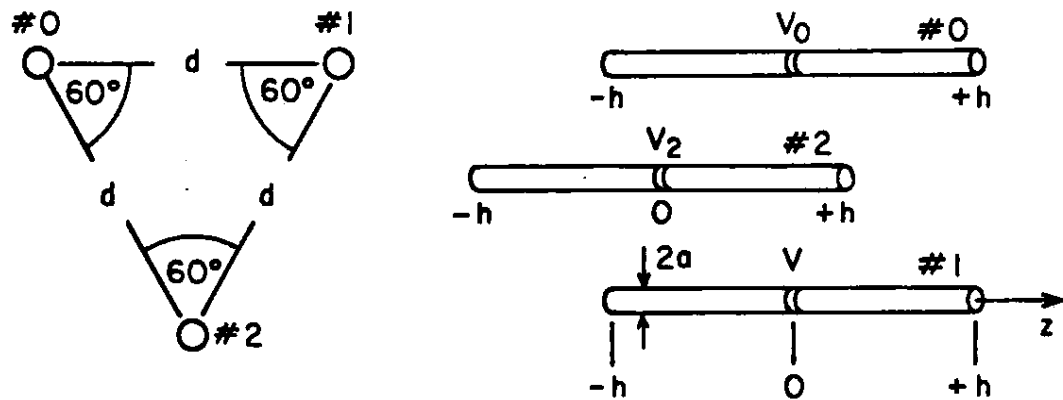


Figure 3.10 Three dipole circular array antenna

in section 3.3; details are in Appendix B. The resulting incident current is

$$\bar{J}_\infty(z_\beta) = \bar{J}_{s_\beta}^\alpha(z_\beta) = -(1/P_{00}, 1/P_{12}, 1/P_{12})^T \quad (3.41)$$

Since there again is no coupling between the antenna mode ($J_{\infty 00}$) and the transmission line modes ($J_{\infty 12}$ and $J_{\infty 21}$), it is possible to use the scalar equation (3.6) as we did for the two dipole case in section 3.7:

$$J_Y^\alpha(z) = J_{\infty Y}(|z|) + J_{\infty Y}(h+z)B_Y + C_Y J_{\infty Y}(h-z) \quad (3.42)$$

for $m = n = 0$ or $mn = 12$ or 21 . The subscript γ refers to either antenna mode ($\gamma = a$ for $m = n = 0$) or transmission line modes ($\gamma = t$ for $mn = 12$ or 21).

The transfer admittance matrix is found using equation (3.29) and is just $\bar{Q} = \bar{P}(0)$. Now B_Y and C_Y can be determined as they were in section 3.7 so that

$$C_Y = B_Y = Q_Y J_{\infty Y}(h) / [1 - Q_Y J_{\infty Y}(2h)] \quad (3.44)$$

Equation (3.42) becomes now

$$J_Y^\alpha(z) = J_{\infty Y}^\alpha(|z|) - J_{\infty Y}^\alpha(h) Q_Y \left[J_Y^\alpha(h+z) + J_{\infty Y}^\alpha(h-z) \right] / [1 - Q_Y J_{\infty Y}^\alpha(2h)] \quad (3.45)$$

and the input admittance is

$$Y_Y = \{J_{\infty Y}(0) - 2[J_{\infty Y}(h)]^2 Q_Y / [1 - Q_Y J_{\infty Y}(2h)]\} / V_Y. \quad (3.46)$$

King states that the input admittance for the transmission line mode of the three wire antenna is identical to the transmission line mode of the two wire finite length done in section 3.7.⁹ Equation (3.46) for $\gamma = t$ is identical to the transmission line mode admittance for the finite length two wire given in equation (3.39).

CHAPTER IV

UNEVEN LENGTH, TWO ELEMENT DIPOLE ANTENNA

In the previous examples of the double variational analysis of parallel dipole antennas, the antennas were of identical length. The total current and the input admittance derived from equation (3.7) are therefore simply evaluated without matrix arithmetic since the Q matrices reduce to simple matrices with only one nonzero element per row. When the wires are not truncated at the same z , however, mutual impedance terms make \bar{Q} more complicated.

For the two wire, unevenly truncated, finite length antenna shown in figure 4.1, the matrix equation (3.7) becomes necessary. The \bar{Y}_∞ is gotten from the infinitely long, two wire antenna analysis of section 3.3; the junction impedance \bar{Q}_B and \bar{Q}_C however, must include terms to compensate for the extra length of wire #1.

4.1 The Reflected Current Double Variational Expression

The first step in the determination of the junction impedance matrix for the antenna shown in figure 4.1, is to determine the variational expression for the reflected currents. Since the reflected currents on each conductor is affected by the uneven termination, we will first examine the wire to wire current

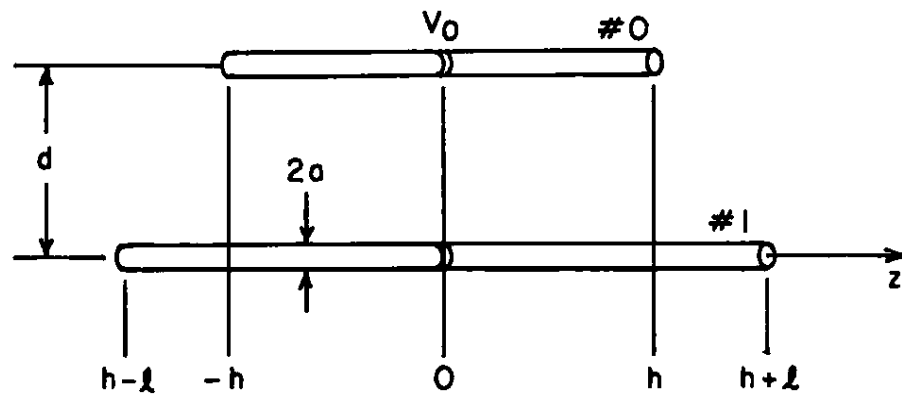


Figure 4.1 Two wire, unevenly truncated, finite length antenna

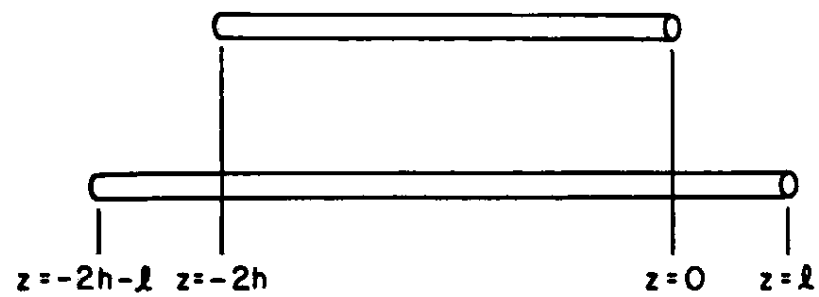


Figure 4.2 Redefinition of origin for figure 4.1

expressions and then determine the system mode currents for use in (3.7).

To start with, the electric field on the p^{th} wire is defined by

$$e_{z_p}(z) = D \sum_{q=0}^1 \int_{-\infty}^{z_q} I_q(z') K_{pq}(z, z') dz' ; \quad p = 0, 1 \quad (4.1)$$

where $K_{pq}(z, z') = -(i\epsilon_0/4\pi k_0 R_{pq}) \exp(-ik_0 R_{pq})$, $R_{pq} = ((z-z')^2 + a_{pq}^2)^{1/2}$, $D = (a^2/\partial z^2 + k_0^2)$, and $I_q(z')$ is the current on the q^{th} conductor. By defining

$$L_{pq} I_q = D \int_{-\infty}^{\infty} I_q(z') K_{pq}(z, z') dz \quad (4.2)$$

where $I_q(z) = 0$ if $z > z_q$ we can then write the vector electric field on the conductors as

$$e_z(z) = \begin{pmatrix} e_{z_0} \\ e_{z_1} \end{pmatrix} = \sum_{q=0}^1 \bar{L} \bar{I}_q \quad (4.3)$$

We consider two different states of excitation, α and β , whose electric field sources are

$$\bar{e}_z^\alpha(z) = - \begin{pmatrix} V_0^\alpha \\ V_1^\alpha \end{pmatrix} \delta(z + z_\alpha) \quad (4.4a)$$

for the source at $z = -z_\alpha$, and

$$\bar{e}_z^\beta(z) = - \begin{pmatrix} V_0^\beta \\ V_1^\beta \end{pmatrix} \delta(z + z_\beta) \quad (4.4b)$$

for the source at $z = -z_\beta$. The two possible states we examine are $V_0 = V_1 = 1$ (the even state) and $V_0 = -V_1 = 1$ (the odd state).

As we did in section 3.4, we define scattered currents

$$i_p^\alpha(z) = \begin{cases} I_p^{\alpha,r}(r) , & z < z_p \\ -I_p^{\alpha,i}(z) , & z > z_p \end{cases} \quad (4.5)$$

where again $I_p^{\alpha,r}$ is the reflected current and $I_p^{\alpha,i}$ is the incident current ($I_{\infty p}(z)$) on conductor p ; and similarly for $i^\beta(z)$. We also define

$$\bar{i}^\alpha = \begin{pmatrix} i_0^\alpha \\ i_1^\alpha \end{pmatrix} \quad (4.6a)$$

and

$$\bar{I}^{\alpha,i} = \begin{pmatrix} I_0^{\alpha,i} \\ I_1^{\alpha,i} \end{pmatrix} \quad (4.6b)$$

so that for all z

$$\bar{I}^\alpha = \bar{I}^{\alpha,i} + \bar{i}^\alpha \quad (4.7)$$

Now by following the procedure of section 3.4 except using the offset from $z = 0$ to $z = z_p$, we get a double variational expression for the reflected current similar to that of equation (3.25):

$$V_q^\beta I_1^{\alpha,r}(-z_\beta) = \sum_{p=1}^1 \langle \bar{I}_\beta, \bar{I}_\alpha \rangle \quad (4.8)$$

where \bar{I}_β is defined by (4.5) and L is defined by (3.23). This will be true, if only one of V_0^β and V_1^β , namely V_q^β , is not equal to 0.

4.2 Trial Functions for the Uneven Length Dipoles

Now that a double variational expression for the reflected current (wire to wire expression) for the antenna of figure 4.1 has been found, it is necessary to select trial functions. Since the two wires do not end at the same z , approximate trial functions are

$$i_p^\beta = -I_p^{\beta i}(z_\beta + z_p) \exp(-ik_0|z - z_p|) \quad (4.9a)$$

and

$$i_q^\alpha = -I_q^{\alpha i}(z_\alpha + z_q) \exp(-ik_0|z - z_q|) \quad (4.9b)$$

where $I_q^{\alpha i}(z_\alpha + z_q)$ and $I_p^{\beta i}(z_\beta + z_p)$ are the amplitudes of the incident current at the end of the q^{th} and p^{th} conductors, respectively. Substituting equations (4.9) into (4.8) gives

$$\begin{aligned} V_q^{\beta i \alpha, r}(-z_\beta) = & -(i\zeta_0/4\pi k_0) \sum_{p=1}^1 \int_{-\infty}^{\infty} dz I_p^{\beta i}(z_\beta + z_p) \exp(-ik_0|z - z_p|) \\ & D(z) \int_{-\infty}^{\infty} dz' K_{pq}(z, z') I_q^{\alpha i}(z_\alpha + z_q) \exp(-ik_0|z' - z_q|) \end{aligned} \quad (4.10)$$

where $K_{pq}(z, z')$ and D are as defined in equation (4.1). The integrals in (4.10) are integrated in Appendix A so that

$$V_q^{\beta i \alpha, r}(-z_\beta) = \sum_{p=1}^1 I_p^{\beta i}(z_\beta + z_p) I_q^{\alpha i}(z_\alpha + z_q) r_{pq} \quad (4.11)$$

where

$$\begin{aligned} r_{pq} = & -(\zeta_0/2\pi) \{ \exp(-ik_0 z_{pq}) E_1(ik_0 r_{pq}^-) + \exp(ik_0 z_{pq}) E_1(ik_0 r_{pq}^+) \} , \\ r_{pq}^\pm = & (z_{pq}^2 + a_{pq}^2)^{1/2} \pm z_{pq} , \quad z_{pq} = z_p - z_q \quad \text{and} \\ a_{pq} = & \begin{cases} a , & \text{if } p = q \\ d , & \text{if } p \neq q . \end{cases} \end{aligned} \quad (4.12)$$

Since all z distances are relative distances, without loss of generality, we can change the zero reference to that shown in figure 4.2. Now a logical choice for the functions $I_p^{\beta i}(z_\beta + z_p)$ and $I_q^{\alpha i}(z_\alpha + z_q)$ since we are working with thin wires is

$$I_p^{\beta i}(z_\beta + z_p) = I_p^{\beta inc}(z_\beta) \exp(-ik_0 z_p) \quad (4.13a)$$

$$I_q^{\alpha i}(z_\alpha + z_q) = I_q^{\alpha inc}(z_\alpha) \exp(-ik_0 z_q) \quad (4.13b)$$

where $I_p^{\beta inc}(z_\beta)$ and $I_q^{\alpha inc}(z_\alpha)$ are currents incident on the p^{th} and q^{th} wires (respectively) at $z = 0$ position (the reference point). Substituting equations (4.13) into (4.11) gives

$$V_q^{\beta \alpha, r}(-z_\beta) = \sum_{p=0}^1 I_p^{\beta, inc}(z_\beta) I_q^{\alpha, inc}(z_\alpha) \exp(-ik_0(z_p + z_q)) \Gamma_{pq} \quad (4.14)$$

4.3 Transforming $I_q^{\alpha, r}$ to the System Mode Reflection Currents

In order to determine the relation between the system mode currents and the conductor to conductor currents used so far in this chapter, we use the discrete Fourier transform expression for the system mode current, equation (3.1) for $N = 2$:

$$J_m(z) = (1/2) \sum_{\mu=0}^1 I_\mu(z) \exp(-i\mu m\pi)$$

so that

$$J_0^\alpha(z) = [I_0^\alpha(z) + I_1^\alpha(z)]/2 \quad (4.15a)$$

and $J_1^\alpha(z) = [I_0^\alpha(z) - I_1^\alpha(z)]/2 \quad (4.15b)$

for both reflected and incident currents. Using these expressions for $I_0^{\alpha,r}$ and $I_1^{\alpha,r}$ given in equations (4.14), this gives

$$J_0^{\alpha,r}(-z_\beta) = (1/2) \sum_{p=0}^1 \{ I_p^{\beta\text{inc}}(z_\beta) I_0^{\alpha\text{inc}}(z_\alpha) \exp(-ik_0(z_0 + z_p)) r_{p0} + I_p^{\beta\text{inc}}(z_\beta) I_1^{\alpha\text{inc}}(z_\alpha) \exp(-ik_0(z_1 + z_p)) r_{p1} \} \quad (4.16a)$$

for $V_0^\beta = V_1^\beta = 1$ (β , even) and

$$J_1^{\alpha,r}(-z_\beta) = (1/2) \sum_{p=0}^1 \{ I_p^{\beta\text{inc}}(z_\beta) I_0^{\alpha\text{inc}}(z_\alpha) \exp(-ik_0(z_0 + z_p)) r_{p0} + I_p^{\beta\text{inc}}(z_\beta) I_1^{\alpha\text{inc}}(z_\alpha) \exp(-ik_0(z_1 + z_p)) r_{p1} \} \quad (4.16b)$$

for $V_0^\beta = -V_1^\beta = 1$ (β , odd).

Now it is necessary to use equations (4.15) for the β incident currents. For β even, $V_0^\beta = V_1^\beta = 1$ so that $I_0^{\beta\text{inc}}(z) = I_1^{\beta\text{inc}}(z)$, then

$$J_0^{\beta\text{inc}}(z) = (1/2) [I_0^{\beta\text{inc}}(z) + I_1^{\beta\text{inc}}(z)] = I_0^{\beta\text{inc}}(z) \quad (4.17a)$$

$$\text{and } J_1^{\beta\text{inc}}(z) = (1/2) [I_0^{\beta\text{inc}}(z) - I_1^{\beta\text{inc}}(z)] = 0 \quad (4.17b)$$

Similarly, if β is odd, $V_0^\beta = -V_1^\beta = 1$ so that $I_0^{\beta\text{inc}}(z) = -I_1^{\beta\text{inc}}(z)$ then

$$J_0^{\beta\text{inc}}(z) = (1/2) [I_0^{\beta\text{inc}}(z) + I_1^{\beta\text{inc}}(z)] = 0 \quad (4.18a)$$

$$J_1^{\beta\text{inc}}(z) = (1/2) [I_0^{\beta\text{inc}}(z) - I_1^{\beta\text{inc}}(z)] = I_0^{\beta\text{inc}}(z) \quad (4.18b)$$

Using equation (4.15) on $I_0^{\alpha\text{inc}}(z_\alpha)$ and $I_1^{\alpha\text{inc}}(z_\alpha)$, knowing that $z_0 = 0$ and $z_1 = \ell$, using equations (4.17) on (4.16a) and equations (4.18) on (4.16b) we get

$$J_0^{\alpha,r}(-z_\beta) = J_0^{\beta,inc}(z_\beta) \{ J_0^{\alpha,inc}(z_\alpha) Q_{00} + J_0^{\alpha,inc}(z_\alpha) Q_{01} \} \quad (4.19a)$$

$$\text{and } J_1^{\alpha,r}(-z_\beta) = J_1^{\beta,inc}(z_\beta) \{ J_0^{\alpha,inc}(z_\alpha) Q_{10} + J_1^{\alpha,inc}(z_\alpha) Q_{11} \} \quad (4.19b)$$

$$\text{where } Q_{01} = Q_{10} = r_{00}(1 - \exp(-2ik\ell))/2 \quad (4.20a)$$

$$Q_{00} = r_{00}(1 + \exp(-2ik\ell))/2 + r_{01} \exp(-ik\ell) \quad (4.20b)$$

$$\text{and } Q_{11} = r_{00}(1 + \exp(-2ik\ell))/2 - r_{01} \exp(-ik\ell) \quad (4.20c)$$

for r_{00} and r_{01} given in equation (4.12). (Details given in Appendix C.) It can readily be seen that if $\ell = 0$, this \bar{Q} reduces to the \bar{Q} for the evenly truncated two dipole antenna given in section 3.5.

4.4 Using the System Mode Reflected Current Expression (3.7)

Since the transfer admittance matrix, \bar{Y}_∞ , and the junction impedance matrix have been determined, all that remains to be done is to determine the system mode voltage $\bar{E}(0)$, since we will calculate $\bar{E}(h)$ and $\bar{E}(-h)$ in our computer calculations from equations (3.8) and (3.9). From equation (3.5) we have that $\bar{E} = (\bar{S}^*)^{-1} \bar{V}$, so we must find $(\bar{S}^*)^{-1}$ before we can determine $\bar{E}(0)$. We also have from (3.5) that

$$S_{mn} = \exp(\pi i m n) \quad \text{so that}$$

$$S_{00} = \exp(0) = 1$$

$$S_{01} = S_{10} = 1$$

$$\text{and } S_{11} = -1.$$

for this case $(S^*)^{-1} = \bar{S}/2$.

For the two element Yagi antenna of figure 4.3 where conductor #1 is the parasitic element, we have $\bar{E}(0) = (.5, .5)^T$. Figures 4.4 and 4.5 show the total current, $I_0(0)$, on conductor #0 and the total current, $I_1(0)$, on conductor #1--the parasite, respectively, for ℓ varied between -0.15 meters and 0.15 meters. It is interesting to note the relative phases of I_0 and I_1 when the parasite is longer ($\ell > 0$) or shorter ($\ell < 0$) than the driven element. When the parasite (#1) is longer, say $\ell = .05$ m for $d = .4$ h, then $I_1/I_0 \approx 0.5 \exp(2.4 \text{ radians})$. However, where I_1 is shorter, say $\ell = -.10$ meters again for $d = .4$ h, then $I_1/I_0 \approx 0.5 \exp(-2.1 \text{ radians})$. The elements are therefore phased nearly oppositely for the two cases. From antenna array theory, the radiation pattern for these two cases can be drastically different.¹⁰ This is consistent with the behavior of the "director" and "reflector" elements of yagi arrays with larger numbers of elements.¹¹

Figures 4.6 show the total currents on conductor #0 and conductor #1 for ℓ constant at .09 meters but for varied frequency. There is a pronounced primary resonance peak near $k_0 h = 1.45$ for both I_0 and I_1 which corresponds to the resonance of the shorter driven element. However, in I_0 a small secondary resonance near 1.25 can also be seen. This lower frequency resonance results from weak coupling with the longer parasitic element (#1).

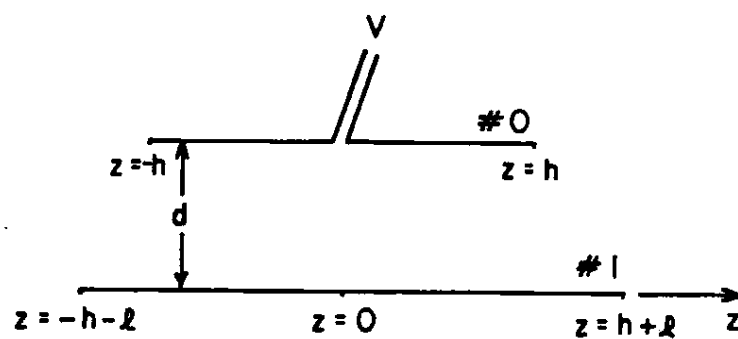


Figure 4.3 Two element Yagi antenna

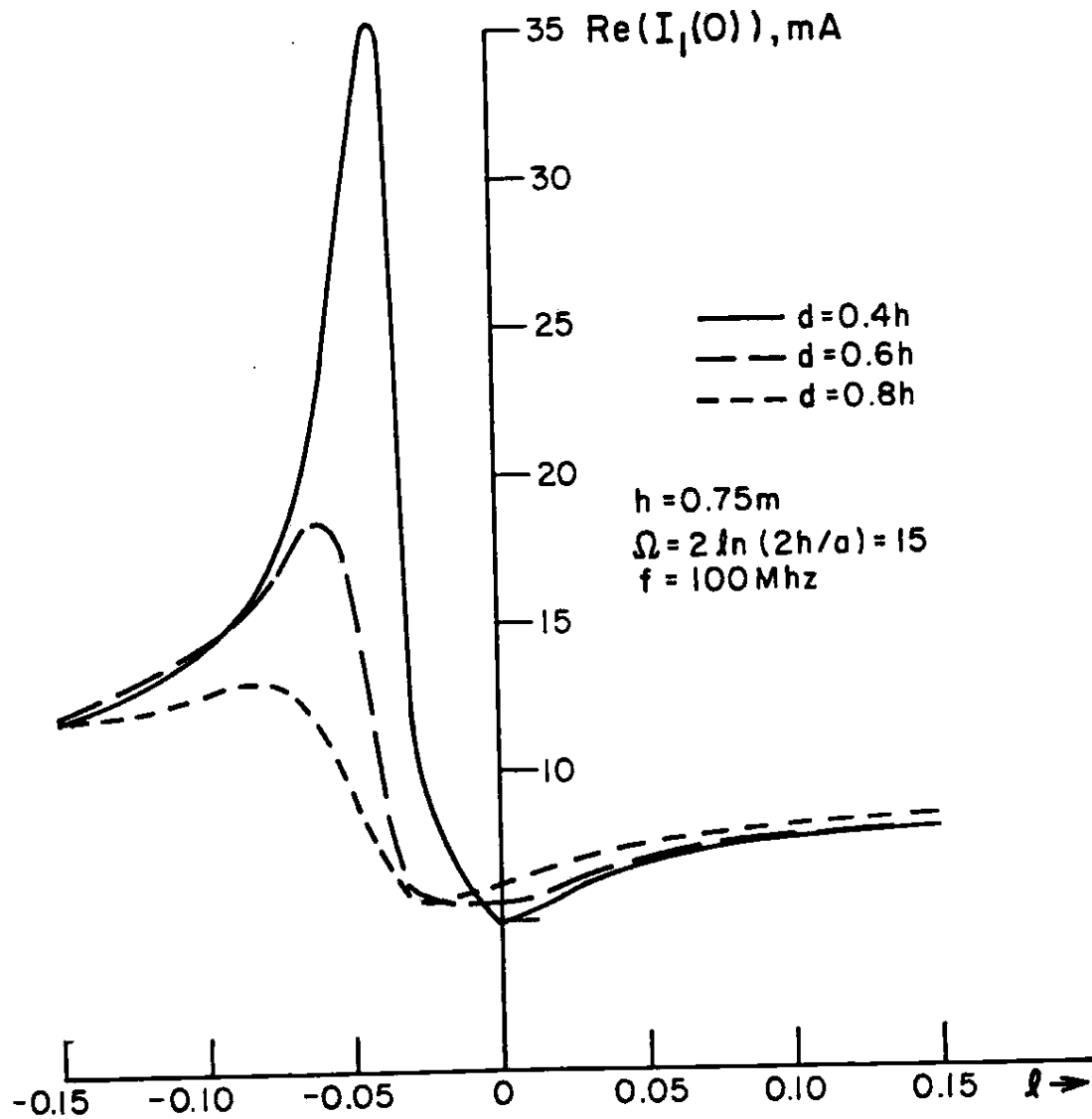


Figure 4.4a Two element Yagi antenna of figure 4.3 real part of the driven element current, $I_0(0)$, vs varied differential length, l

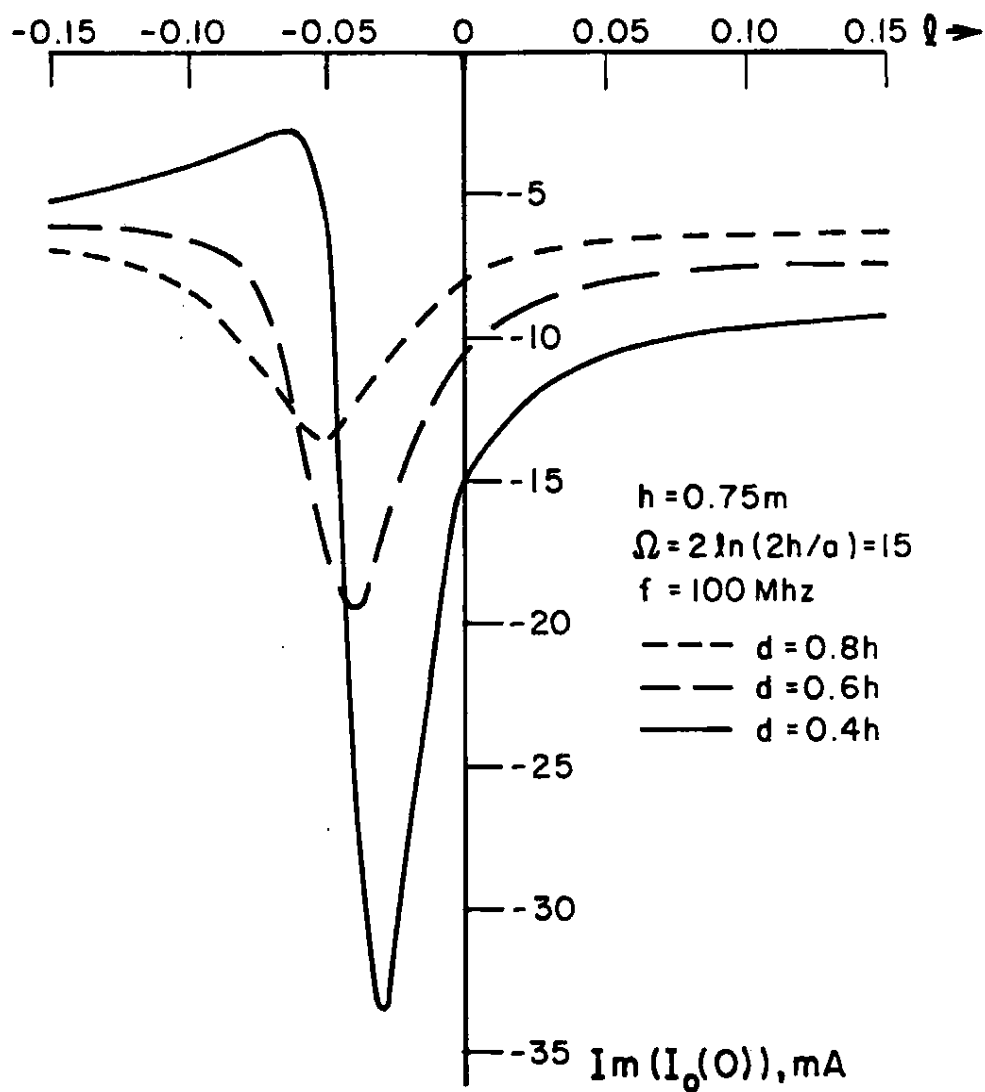


Figure 4.4b Two element Yagi antenna imaginary part of the driven element current, $I_0(0)$, vs varied length differential, l

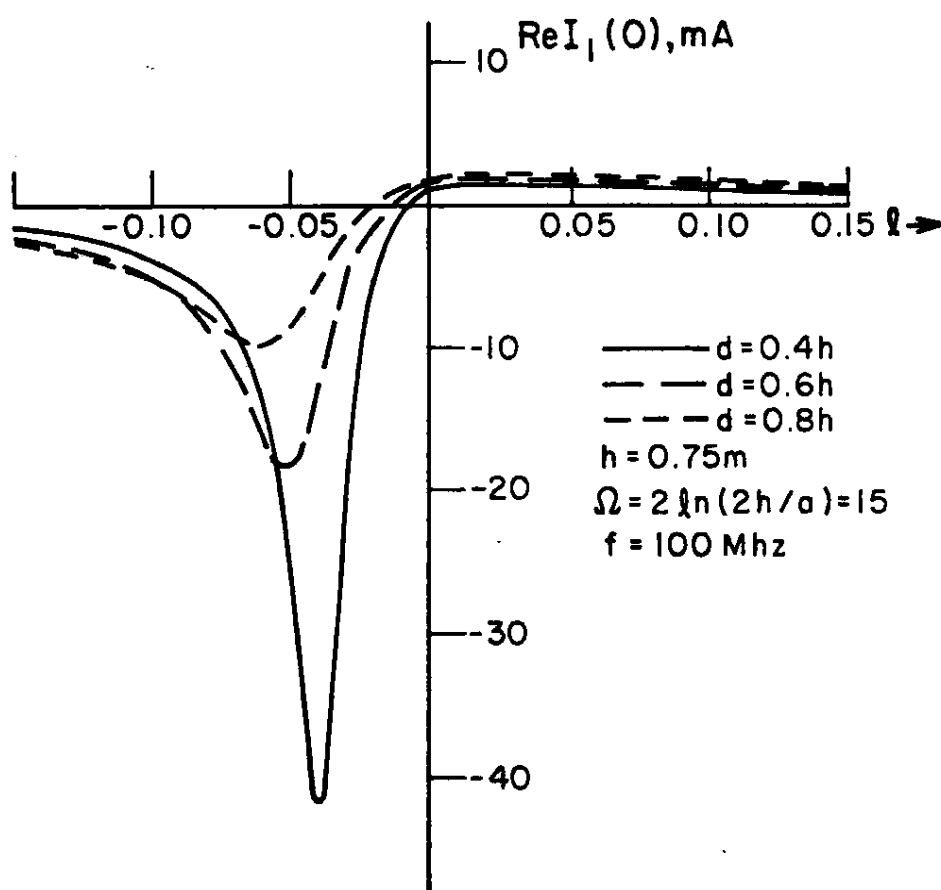


Figure 4.5a Two element Yagi antenna real part of the parasitic element current, $I_1(0)$, vs varied length differential, l

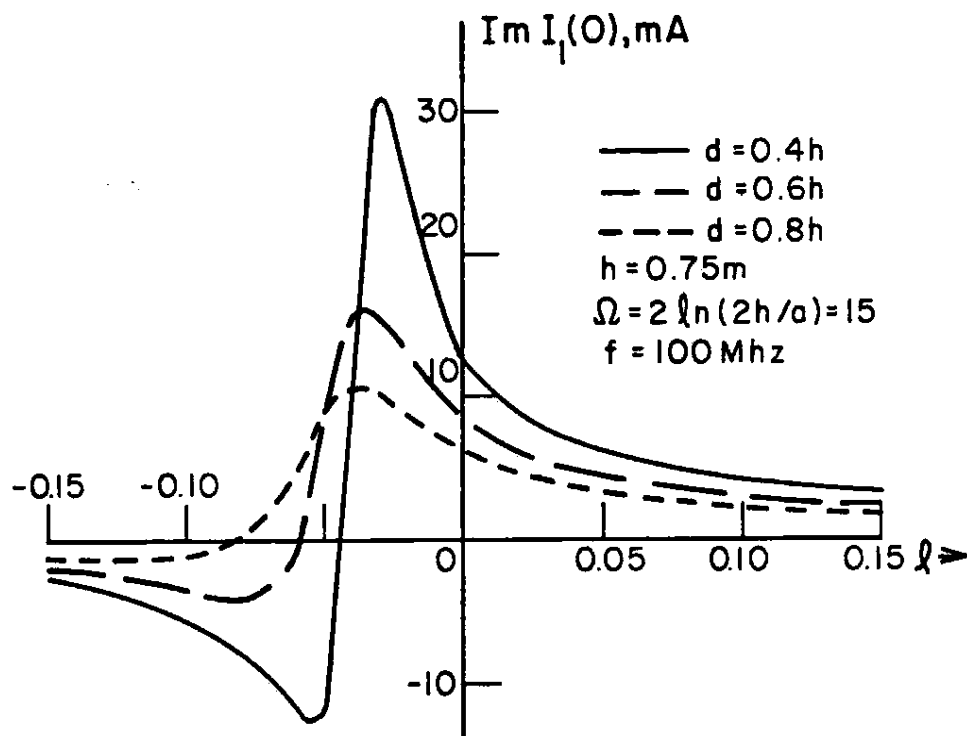


Figure 4.5b Two element Yagi antenna imaginary part of the parasitic element current, $I_1(0)$, vs varied length differential, l

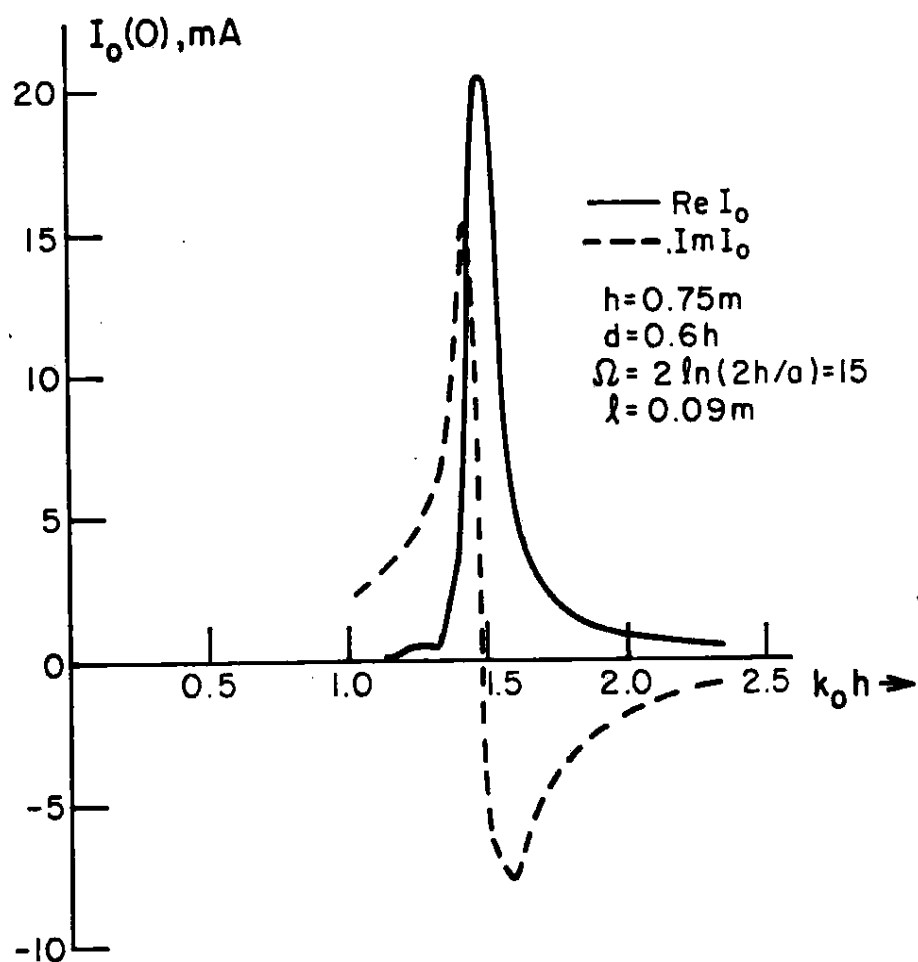


Figure 4.6a Two element Yagi antenna driven element current, $I_0(0)$, vs $k_0 h$ for constant λ but varied frequency

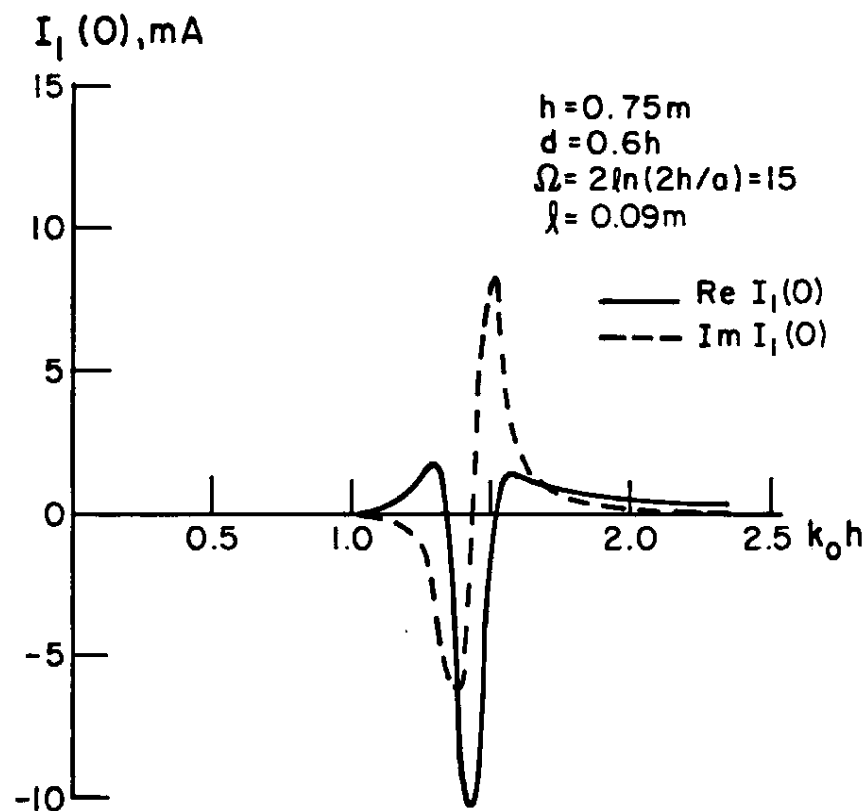


Figure 4.6b Two element Yagi antenna parasitic element current, $I_1(0)$, vs $k_0 h$ for constant λ but varied frequency

CHAPTER V

CONCLUSIONS

We have seen that Vainshtein's double variational technique can be extended from the single infinitely long, thin wire to the N element dipole antenna array with good comparison for the one element case to the method of Rispin and Chang, and for the two element case to the method of King-Middleton. We have also extended this technique to a two element Yagi antenna, something not possible with the current Wiener Hopf work. We also used much less computer time and space than would have been necessary for the moment method.

Using a straightforward extension of our procedure followed in Chapter IV, the double variational method could furthermore be extended to solve an N element Yagi antenna using computer calculations to determine the terms for the junction impedance, the \bar{Q} , and the conversion from wire to wire currents to system mode currents. The double variational procedure could also readily be used to find currents on other devices such as microstrip resonators and junctions if good trial functions for the currents were used.

We could improve the results gotten from this procedure by going to a second order approximation using the results from the first approximation for the incident and reflected currents

as trial functions. The integration required for the second order approximation would be more difficult than the first order approximation integration, however.

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APPENDIX A

A.1 Equality of $\langle I_\alpha, LI_\beta \rangle = \langle I_\beta, LI_\alpha \rangle$

One equality that is necessary for the manipulations of expressions for the incident current (the current for the infinitely long wires) is that of equation (2.4) in Chapter II:

$$\langle I_\alpha, LI_\beta \rangle = \langle I_\beta, LI_\alpha \rangle \quad (\text{A.1})$$

where L , I , and the expression \langle , \rangle are as defined in equations (2.6), (2.8), and (2.3). Substituting these equations into the right hand side of equation (A.1) gives

$$\langle I_\alpha, LI_\beta \rangle = \int_{-\infty}^{\infty} dz I_\alpha(z) \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-\infty}^{\infty} dz' I_\beta(z') K(z, z') . \quad (\text{A.2})$$

Assuming by reciprocity that $K(z, z') = K(z', z)$ and knowing that the kernel K (as defined by equation (1.2)) is a function of $(z-z')^2$ only so that $\partial K / \partial z = -\partial K / \partial z'$ equation (A.2) becomes

$$\begin{aligned} \langle I_\beta, LI_\alpha \rangle &= \int_{-\infty}^{\infty} dz' I_\alpha(z') \left(\frac{\partial^2}{\partial z'^2} + k_0^2 \right) \int_{-\infty}^{\infty} dz I_\beta(z) K(z, z') \\ &= \langle I_\alpha, LI_\beta \rangle . \end{aligned}$$

thereby demonstrating equation (A.1).

A.2 Integration of the integral $\mathcal{J}(z) =$

$$\frac{\int_{-\infty}^{\infty} dz \exp(-ik_0|z-c_1|) \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-\infty}^{\infty} dz' \exp(-ik_0|z'-c_2|) K_{pq}(z, z')}{}$$

An integral of this form is used in Chapters II-IV where $K_{pq}(z, z') = -i\rho_0 \exp(-ik_0 R_{pq}) / (R_{pq} 4\pi)$ and $R_{pq} = ((z-z')^2 + a_{pq}^2)^{1/2}$. To evaluate it we first substitute

$$\mathcal{F}_{pq} = \int_{-\infty}^{\infty} dz' \exp(-ik_0|z'-c_2|) K_{pq}(z, z') \quad (A.3)$$

The integral now becomes

$$\mathcal{J}(z) = \int_{-\infty}^{\infty} dz \exp(-ik_0|z-c_1|) \left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) \mathcal{F}_{pq} \quad (A.4)$$

Using integration by parts

$$\int_{-\infty}^{\infty} f' \phi dx = f \phi \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f \phi' dx$$

(Gradshtyn and Ryzhik, equation 2.02.5), we reduce

$$\int_{-\infty}^{\infty} dz \exp(-ik_0|z-c_1|) (\partial^2 \mathcal{F}_{pq} / \partial z^2) \text{ to } \left(\frac{\partial \mathcal{F}_{pq}}{\partial z} \exp(-ik_0|z-c_1|) \right) \Big|_{-\infty}^{\infty} + ik_0 \int_{-\infty}^{\infty} \left(\frac{\partial \mathcal{F}_{pq}}{\partial z} \right) H(z) \exp(-ik_0|z-c_1|) dz \quad (A.5)$$

where $H(z) = \begin{cases} 1, & z - c_1 < 0 \\ -1, & z - c_1 > 0 \end{cases}$. Using integration by parts

again on the second term in (A.5) gives

$$\left[(\partial \mathcal{F}_{pq} / \partial z) \exp(-ik_0|z-c_1|) + ik_0 H(z) \mathcal{F}_{pq} \exp(-ik_0|z-c_1|) \right] \Big|_{-\infty}^{\infty} - ik_0 \int_{-\infty}^{\infty} dz \mathcal{F}_{pq} \left[(-ik_0) + 2\delta(z-c_1) \right] \exp(-ik_0|z-c_1|).$$

Assuming k_0 has a small lossy part, $\exp(-ik_0|z-c_1|)$ goes to zero for the limits $-\infty$ and ∞ so that this reduces to

$$-k_0^2 \int_{-\infty}^{\infty} dz F_{pq} \exp(-ik_0|z-c_1|) - 2ik_0 F_{pq}(c_1) \exp(-ik_0|c_1-c_1|) .$$

Putting this back into $\mathcal{J}(z)$ of equation (A.4) gives

$$\begin{aligned} \mathcal{J}(z) = & \int_{-\infty}^{\infty} dz k_0^2 \exp(-ik_0|z-c_1|) F_{pq} + (-k_0^2 \int_{-\infty}^{\infty} dz F_{pq} \exp(-ik_0|z-c_1|)) \\ & - 2ik_0 F_{pq}(c_1) . \end{aligned}$$

This simplifies to

$$\begin{aligned} \mathcal{J}(z) = & -2ik_0 F_{pq}(c_1) \\ = & -2ik_0 \int_{-\infty}^{\infty} dz' \exp(-ik_0|z'-c_2|) K_{pq}(c_1, z') . \end{aligned}$$

Using the substitution $u = z' - c_2$, this becomes

$$\begin{aligned} \mathcal{J}(z) = & -\epsilon_0 \int_{-\infty}^{\infty} du \exp(-ik_0|u|) \exp(-ik_0[(c_1-u-c_2)^2 + a_{pq}^2]^{\frac{1}{2}}) / \\ & ([(c_1-u-c_2)^2 + a_{pq}^2]^{\frac{1}{2}} 2\pi) . \end{aligned}$$

By setting $c_2 - c_1 = c$, $\mathcal{J}(z)$ becomes

$$\begin{aligned} \mathcal{J}(z) = & -(\epsilon_0/2\pi) \{ \int_0^{\infty} du \exp(-ik_0 u) \exp(-ik_0[(u+c)^2 + a_{pq}^2]^{\frac{1}{2}}) / [(u+c)^2 + a_{pq}^2]^{\frac{1}{2}} \\ & + \int_0^{\infty} du \exp(-ik_0 u) \exp(-ik_0[(u-c)^2 + a_{pq}^2]^{\frac{1}{2}}) / [(u-c)^2 + a_{pq}^2]^{\frac{1}{2}} \} . \end{aligned}$$

Considering first

$$\begin{aligned} \mathcal{J}(z) = & \int_0^{\infty} dz \exp(-ik_0|z \pm c|) \exp(-ik_0(\pm c)) \exp(-ik_0[(z \pm c)^2 + a_{pq}^2]^{\frac{1}{2}}) / \\ & ([z \pm c]^2 + a_{pq}^2)^{\frac{1}{2}} \end{aligned}$$

with the substitutions $t = ([z \pm c]^2 + a_{pq}^2)^{\frac{1}{2}} + (z \pm c)$ and $dt = t dz / [(z \pm c)^2 + a_{pq}^2]^{\frac{1}{2}}$, we get

$$\hat{g}(z) = \int_{t_0}^{\infty} dt \exp(-ik_0 t) \exp(\mp c i k_0) / t$$

where $t_0 = [c^2 + a_{pq}^2]^{\frac{1}{2}} \pm c = r_{pq}^{\pm}$. Abramowitz and Segun in their equation 5.1.1 state

$$E_1(z) = \int_z^{\infty} (\exp(-t)/t) dt ,$$

so that

$$\hat{g}(z) = \exp(\mp i k_0 c) E_1(i k_0 r_{pq}^{\pm}) .$$

Therefore,

$$\begin{aligned} g(z) = & -(\zeta_0/2\pi) \{ \exp(-ik_0|c_2-c_1|) E_1(ik_0 r_{pq}^-) \\ & + \exp(-ik_0|c_2-c_1|) E_1(ik_0 r_{pq}^+) \} \end{aligned} \quad (A.6)$$

where $r_{pq}^{\pm} = (c^2 + a_{pq}^2)^{\frac{1}{2}} \pm c$, and $c = c_2 + c_1$.

A.3 $I_t^{\infty}(z)$ for Two Infinitely Long, Thin Wires

Far Field Approximations

To determine the far field expression for the transmission line mode current for the infinitely long, two thin wire antenna of section 2.3, we begin with equation (2.9)

$$I_t^{\infty}(z) = 2\pi \exp(-ik_0 z) / \zeta_0 f_t(z) \quad (A.7)$$

where $f_t(z) = E_1(ik_0 r_a^-) - E_1(ik_0 r_d^-) + \exp(2ik_0 z) [E_1(ik_0 r_a^+) - E_1(ik_0 r_d^+)]$ where $r_a^{\pm} = (z^2 + a^2)^{\frac{1}{2}} \pm z$ and $r_d^{\pm} = (z^2 + d^2)^{\frac{1}{2}} \pm z$.

Far from the source we can assume that $z^2 \gg a^2$, $z^2 \gg d^2$ and $(k_0 z) \gg 1$. This implies then that $r_a^+ \approx r_d^+ \approx 2z$ and

$$\begin{aligned} r_a^- &= z(1 + a^2/z^2)^{\frac{1}{2}} - z \\ &\approx z(1 + a^2/(2z^2)) - z \\ &= a^2/2z. \end{aligned}$$

Similarly $r_d^- = d^2/2z$. The series expansion for the exponential integral $E_1(z)$ is

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} (-1)^n z^n / (n n!)$$

where γ is Euler's constant. Using this series expansion for $E_1(ikr_a^-)$ and $E_1(ikr_d^-)$ gives

$$E_1(ikr_a^-) \approx -\gamma - \ln(ika^2/2z)$$

and

$$E_1(ikr_d^-) \approx -\gamma - \ln(ikd^2/2z). \quad (\text{A.8})$$

Using the asymptotic expansion for the exponential integral:

$E_1(z) = (\exp(-z)/z)\{1 - n/z + n(n+1)/z^2 - \dots\}$ (Abramowitz and Segun equation (5.1.51)) for $E_1(ik_0 r_a^+)$ and $E_1(ikr_d^+)$ gives

$$\begin{aligned} E_1(ikr_a^+) &\approx \exp(-ik_0 2z)\{1 - 1/i2kz\}/ik_0 2z \\ &\approx \exp(-ik_0 2z)/ik_0 2z \approx 0. \end{aligned}$$

Similarly $E_1(ikr_d^+) \approx 0$. Substituting this and (A.8) back into the expression for $f_t(z)$ in (A.7) gives

$$\begin{aligned}
 f_t(z) &\approx -\ln(ik_0 a^2/2z) + \ln(ik_0 d^2/2z) \\
 &= \ln(d^2/a^2) = 2 \ln d/a
 \end{aligned}$$

which means that

$$\begin{aligned}
 I_t^\infty(z) &\approx \exp(-ik_0 z)/(\rho_0 \ln(d/a)/\pi) \\
 &= \exp(-ik_0 z)/Z_c
 \end{aligned}$$

where Z_c is the characteristic impedance:

$$Z_c = \rho_0 \ln(d/a)/\pi .$$

APPENDIX B

B.1 Equality of $\langle I_\beta^i, L_- I_\alpha \rangle = \langle I_\alpha, LI_\beta^i \rangle_-$

One equality that is necessary for manipulations of the equations for reflected current is

$$\langle I_\beta^i, L_- I_\alpha \rangle = \langle I_\alpha, LI_\beta^i \rangle_- \quad (B.1)$$

where $L_- I_\alpha$, LI_β , $\langle \cdot \rangle$, and $\langle \cdot \rangle_-$ are as defined in equations (3.23), (2.3), and (3.24). By definition

$$\begin{aligned} \langle I_\beta^i, L_- I_\alpha \rangle &= \int_{-\infty}^{\infty} I_\beta^i L_- I_\alpha dz \\ &= \int_{-\infty}^{\infty} I_\beta^i(z) D(z) \int_{-\infty}^0 I_\alpha(z') K(z, z') dz' dz \end{aligned} \quad (B.2)$$

where $D(z) = (\partial^2 / \partial z^2 + k_0^2)$.

Now assuming by reciprocity that $K(z, z') = K(z', z)$ and knowing that K is a function of $(z-z')^2$ only so that $\partial K(z, z') / \partial z = -\partial K(z, z') / \partial z'$ equation (B.2) becomes

$$\langle I_\beta^i, L_- I_\alpha \rangle = \int_{-\infty}^0 dz' I_\alpha(z') D(z') \int_{-\infty}^{\infty} dz I_\beta^i(z) K(z, z') = \langle I_\alpha, LI_\beta^i \rangle_- \quad (B.3)$$

thereby demonstrating equation (B.1).

B.2 Development of $VI_\alpha^{\beta, r}(-z_\alpha) = -\langle i^\beta, L_+ I_\alpha^{\alpha, i} \rangle + \langle i^\beta, L_- I_\alpha^{\alpha, r} \rangle$

To develop a functional for the reflected current we begin with the functional $\langle I_\beta, L_- I_\alpha \rangle_-$.

$$\begin{aligned}
\langle I_\beta, L_- I_\alpha \rangle_- &= \langle I_\beta^i, L_- I_\alpha \rangle - \langle I_\beta^i, L_- I_\alpha \rangle_+ + \langle I_\beta^r, L_- I_\alpha^i \rangle_- \\
&\quad + \langle I_\beta^r, L_- I_\alpha^r \rangle
\end{aligned} \tag{B.4}$$

since $I_x = I_x^i + I_x^r$ and $\langle \cdot \rangle_- = \langle \cdot \rangle_+ - \langle \cdot \rangle_-$. Using equation (B.1) on $\langle I_\beta^i, L_- I_\alpha \rangle$, knowing that $LI_\alpha = \mathcal{E}_\alpha = -V_\alpha \delta(z+z_\alpha)$, that $\langle I_\beta^i, LI_\alpha^i \rangle = \langle I_\beta^i, -V_\alpha \delta(z+z_\alpha) \rangle_+ = 0$ and that $\langle I_\beta^r, LI_\alpha^i \rangle = \langle I_\beta^r, -V_\alpha \delta(z+z_\alpha) \rangle_- = -V_\alpha I_\beta^r(-z_\alpha)$, we obtain

$$\begin{aligned}
\langle I_\beta, L_- I_\alpha \rangle_- &= -V_\alpha \left[I_\alpha(-z_\beta) + I_\beta^r(-z_\alpha) \right] + \langle I_\beta^i, L_+ I_\alpha^i \rangle - \langle I_\beta^i, L_- I_\alpha^r \rangle_+ \\
&\quad - \langle I_\beta^r, L_+ I_\alpha^i \rangle_- + \langle I_\beta^r, L_- I_\alpha^r \rangle_- .
\end{aligned} \tag{B.5}$$

But $\langle I_\beta, L_- I_\alpha \rangle_- = \langle I_\alpha, LI_\beta \rangle_- = \langle I_\alpha, \mathcal{E}_\beta \rangle = -V_\beta I_\alpha(-z_\beta)$. Substituting this into (B.5) gives

$$V_\alpha I_\beta^r(-z_\alpha) = -\langle i_\beta, L_+ I_\alpha^i \rangle + \langle i_\beta, L_- I_\alpha^r \rangle = \langle i_\beta, LI_\alpha \rangle \tag{B.6}$$

where $i_\beta = \begin{cases} I_\beta^r, & z < 0 \\ -I_\beta^i, & z > 0 \end{cases}$ are the scattered currents.

B.3 Demonstrating that the Reflected Current Functional is Stationary

To see if I_β^r and I_α^r are actually double variational it is necessary to add a small perturbation to both I_β^r and I_α^r and use this in equation (B.6); therefore we set

$$I_\beta^r = I_{\beta 0}^r + \delta I_\beta^r \quad \text{and}$$

$$I_\alpha^r = I_{\alpha 0}^r + \delta I_\alpha^r .$$

Substituting these into the terms of the right-hand side of (B.6), the reflected current expression yields

$$\begin{aligned}\langle I_{\beta}^r, L_+ I_{\alpha}^i \rangle_- &= \langle I_{\beta_0}^r, L_+ I_{\alpha}^i \rangle_- + \langle \delta I_{\beta}^r, L_+ I_{\alpha}^i \rangle_- \\ \langle I_{\beta}^r, L_- I_{\alpha}^r \rangle_- &= \langle I_{\beta_0}^r, L_- I_{\alpha_0}^r \rangle_- + \langle \delta I_{\beta}^r, L_- I_{\alpha_0}^r \rangle_- + \langle I_{\beta_0}^r, L_- I_{\alpha_0}^r \rangle_- \\ &\quad + \langle \delta I_{\beta}^r, L_- I_{\alpha_0}^r \rangle_- \end{aligned}$$

and

$$\langle I_{\beta}^i, L_- I_{\alpha}^r \rangle_+ = \langle I_{\beta}^i, L_- I_{\alpha_0}^r \rangle_+ + \langle I_{\beta}^i, L_- \delta I_{\alpha}^r \rangle_- .$$

Assuming the double δ terms are approximately zero, using the equality $LI_{\beta} = V_{\beta} \delta(z+z_{\beta})$, and knowing (similar to equation (B.1)) $\langle I_{\beta}, L_- I_{\alpha} \rangle_- = \langle I_{\alpha}, L_- I_{\beta} \rangle_-$, these terms reduce, respectively, to

$$\langle I_{\beta}^r, L_+ I_{\alpha}^i \rangle_- \approx \langle I_{\beta_0}^r, L_+ I_{\alpha}^i \rangle_- + \langle \delta I_{\beta}^r, L_- I_{\alpha}^r \rangle_- \quad (B.7)$$

$$\langle I_{\beta}^r, L_- I_{\alpha}^r \rangle_- \approx \langle I_{\beta_0}^r, L_- I_{\alpha_0}^r \rangle_- + \langle \delta I_{\beta}^r, L_- I_{\alpha_0}^r \rangle_- + \langle \delta I_{\alpha}^r, L_- I_{\beta_0}^r \rangle_- \quad (B.8)$$

and

$$\langle I_{\beta}^i, L_- I_{\alpha}^r \rangle_+ = \langle I_{\beta}^i, L_- I_{\alpha_0}^r \rangle_+ + \langle \delta I_{\alpha}^r, L_- I_{\beta_0}^r \rangle_- \quad (B.9)$$

Substituting equations (B.7)-(B.9) into (B.6)

$$\begin{aligned}\langle i_{\beta}, GI_{\alpha} \rangle &= \langle I_{\beta}^i, L_+ I_{\alpha}^i \rangle - \langle I_{\beta}^r, L_+ I_{\alpha}^i \rangle_- - \langle I_{\beta}^i, L_- I_{\alpha}^r \rangle_+ + \langle I_{\beta}^r, L_- I_{\alpha}^r \rangle_- \\ &\approx \langle I_{\beta}^i, L_+ I_{\alpha}^i \rangle - \langle I_{\beta_0}^r, L_+ I_{\alpha}^i \rangle_- - \langle \delta I_{\beta}^r, L_- I_{\alpha}^r \rangle_- - \langle I_{\beta}^i, L_- I_{\alpha_0}^r \rangle_+ \\ &\quad - \langle \delta I_{\alpha}^r, L_- I_{\beta_0}^r \rangle_- + \langle I_{\beta_0}^r, L_- I_{\alpha_0}^r \rangle_- + \langle \delta I_{\beta}^r, L_- I_{\alpha_0}^r \rangle_- \\ &\quad + \langle \delta I_{\alpha}^r, L_- I_{\beta_0}^r \rangle_- \end{aligned}$$

The δ terms cancel each other, in so doing only the steady-state terms remain so that $\delta_{\alpha,\beta} I_{\beta}^r(-z_{\alpha}) = 0$. Thereby it is seen that I_{β}^r is stationary.

B.4 Truncated Two Wire Reflection Coefficient Thin Wire Approximations

In section 3.5 we used thin wire approximations ($a \ll d$) to arrive at a simplified expression for the transmission line mode current reflection coefficient. Here we show further mathematical details of this approximation.

Starting with equation (3.31)

$$R_{11} = \lim_{z_{\beta} \rightarrow \infty} \left[Y_{\infty}(z_{\beta}) \right]_{11} \exp(ik_0 z_{\beta}) Q_{11} \quad (B.14)$$

we substitute in equation (3.21) for $\left[Y_{\infty}(z_{\beta}) \right]_{11}$ so that

$$R_{11} = Q_{11} \lim_{z_{\beta} \rightarrow \infty} \{-\exp(ik_0 z_{\beta}) / P_{11}(z_{\beta})\} \quad (B.15)$$

where $P_{11}(z_{\beta}) = -(\zeta_0/2\pi)\exp(-ik_0 z_{\beta}) \left[E_1(ik_0 r_a^-) - E_1(ik_0 r_d^-) + \exp(i2k_0 z_{\beta}) \{E_1(ik_0 r_a^+) - E_1(ik_0 r_d^+)\} \right]$ and $r_d^{\pm} = (z_{\beta}^2 + d^2)^{\frac{1}{2}} \pm z_{\beta}$ and $r_a^{\pm} = (z_{\beta}^2 + a^2)^{\frac{1}{2}} \pm z_{\beta}$. Using the thin wire far field result for $\left[Y_{\infty}(z) \right]_{11}$ found in Appendix A, section A.3 and using equation (3.29) for Q_{11} , R_{11} becomes

$$\begin{aligned} R_{11} &\approx \pi Q_{11} / \{\zeta_0 \ln(d/a)\} \\ &\approx -\left[E_1(ik_0 a) - E_1(ik_0 d) \right] / \ln(d/a) . \end{aligned} \quad (B.16)$$

Now using the series expansion for the exponential integral

$$E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} (-1)^n z^n / (n n!)$$

where γ is Euler's constant, equation (B.16) becomes

$$R_{11} \approx \{ \ln(d/a) + [-ik_0 d + ik_0 a - (k_0 d)^2/4 + (k_0 d^2)/4] / \ln(d/a) \} \quad (B.17)$$

We would like to get the right hand side of (B.17) into the form of $\exp(-ik_0 d K_1 - (k_0 d)^2 K_2)$ so that we can write R_{11} in terms of exponential functions, so we use the series expansion for the exponential function

$$\exp x = 1 + x + x^2/2 + \dots$$

$$\approx 1 + x + x^2/2 \quad \text{for small } x.$$

We can now see that we need R_{11} in the form of

$$R_{11} \approx -\{1 - ik_0 d K_1 - (k_0 d)^2 K_2 - \frac{1}{2} (k_0 d)^2 K_1^2 + \mathcal{O}[(k_0 d)^3]\}.$$

Rewriting equation (B.17) into this form we have

$$R_{11} \approx -\{1 - [ik_0 d + (k_0 d)^2/4] / \ln(d/a) + \mathcal{O}[(k_0 a)]\}.$$

Now assuming that $a \ll d$, it can be seen that $K_1 = 1/\ln(d/a)$ so that

$$K_2 + K_1^2/2 = 1/4 \ln(d/a) \quad \text{or} \quad K_2 = [1 - 2/\ln(d/a)]/4 \ln(d/a) \quad \text{so}$$

$$\text{that } R_{11} \approx -\exp\{-ik_0 d / \ln(d/a)\} \exp\{-i(k_0 d)^2 [1 - 2/\ln(d/a)]/4 \ln(d/a)\}.$$

B.5 Three Wire Circular Array Antenna of Finite Length

J^{inc} Coefficient Derivation

In equation (3.17) we saw that

$$J_{s_\beta}^\alpha(z) = -1/F_{\alpha\beta} = -A_{s_\alpha}^\beta A_{s_\beta}^\alpha / \left\{ \sum_{mn=0}^{N-1} A_m^\alpha A_n^\beta p_{mn} \right\}$$

With \bar{P} as stated in equations (3.16) this becomes

$$F_{\alpha\beta} = -1/J_{s_\beta}^\alpha(z) = -\{A_0^\alpha A_0^\beta p_{00} + A_1^\alpha A_2^\beta p_{12} + A_2^\alpha A_1^\beta p_{12}\} / (A_{s_\alpha}^\beta A_{s_\beta}^\alpha)$$

To correctly select the A_m^α and A_n^β to equal one, it is necessary to remember the physical problem. We have a voltage sequence phased as

$$V_{\text{wire}\#0} = 1/3$$

$$V_{\text{wire}\#1} = \exp(-i2\pi s_\alpha/3)/3$$

and
$$V_{\text{wire}\#2} = \exp(-i4\pi s_\alpha/3)/3$$

while we have currents (due to the definition of the discrete Fourier transform) of

$$I_{\text{wire}\#0} = 1/3$$

$$I_{\text{wire}\#1} = \exp(i2\pi s_\alpha/3)/3$$

$$I_{\text{wire}\#2} = \exp(i4\pi s_\alpha/3)/3$$

This means the phase of the current and voltage on each wire is as shown in Figure B.1. The most direct method to get the correct A_m^α and A_n^β to set to 1 is to use

$$A_{N-s_\alpha}^\alpha = A_{N-s_\beta}^\beta = 1 \quad (\text{B.13})$$

to avoid phase errors.

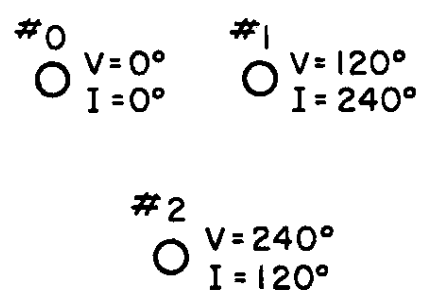


Figure B.1 Phasing differences for three dipole circular array antenna voltages and currents

For the case $s_\alpha = s_\beta = 0$,

$$F_{\alpha\beta}|_{s_\alpha=s_\beta=0} = \{A_0^\alpha A_0^\beta p_{00} + A_1^\alpha A_2^\beta p_{12} + A_2^\alpha A_1^\beta p_{12}\} / (A_0^\alpha A_0^\beta)$$

Using equation (B.13) we set $A_0^\alpha = A_0^\beta = 1$, so that

$$F_{\alpha\beta}|_{00} = (p_{00} + A_1^\alpha A_2^\beta p_{12} + A_2^\alpha A_1^\beta p_{12}) .$$

The variational property gives

$$\partial F_{\alpha\beta} / \partial A_1^\alpha = A_2^\beta p_{12} = 0$$

So that $A_2^\beta = 0$. Using the variational property again on $F_{\alpha\beta}$ gives

$$\partial F_{\alpha\beta} / \partial A_1^\beta = A_2^\alpha p_{11} = 0$$

resulting in $F_{\alpha\beta}|_\infty = p_{00}$, and $(Y_\infty)_{00} = -1/p_\infty$.

Now for $s_\alpha = 0$ and $s_\beta = 1$

$$J_{s_\beta}^\alpha(z)|_{01} = -A_1^\alpha A_0^\beta / \{A_0^\alpha A_0^\beta p_{00} + A_1^\alpha A_2^\beta p_{12} + A_2^\alpha A_1^\beta p_{12}\}$$

We want the coefficients $A_{3-0}^\alpha = A_0^\alpha = 1 = A_{3-1}^\beta = A_2^\beta$ according to equation (B.13) so that

$$J_{s_\beta}^\alpha|_{01} = -A_1^\alpha A_0^\beta / \{A_0^\beta p_{00} + A_1^\alpha p_{12} + A_2^\alpha A_1^\beta p_{12}\}$$

The double variational property gives

$$\partial J_{s_\beta}^\alpha / \partial A_0^\beta = A_1^\alpha A_0^\beta p_{00} / \{A_0^\beta p_{00} + A_1^\alpha p_{12} + A_2^\alpha A_1^\beta p_{12}\}^2 = 0$$

so that $A_1^\alpha A_0^\beta = 0$, therefore $J_{s_\beta}^\alpha|_{01} = (Y_\infty)_{01} = 0$.

For the $s_\alpha = 0$ and $s_\beta = 2$

Table B.1

Three dipole antenna circular array coefficients

| s_α | s_β | $A_m^\alpha = 1$ | $A_n^\beta = 1$ | $J_{s_\beta}^\alpha$ | y_{mn}^β |
|------------|-----------|------------------|-----------------|----------------------|----------------|
| 0 | 0 | A_0^α | A_0^β | $-1/P_{00}$ | $-1/P_{00}$ |
| 0 | 1 | A_0^α | A_2^β | 0 | 0 |
| 0 | 2 | A_0^α | A_1^β | 0 | 0 |
| 1 | 0 | A_2^α | A_0^β | 0 | 0 |
| 1 | 1 | A_2^α | A_2^β | 0 | 0 |
| 1 | 2 | A_2^α | A_1^β | $-1/P_{12}$ | $-1/P_{12}$ |
| 2 | 0 | A_1^α | A_0^β | 0 | 0 |
| 2 | 1 | A_1^α | A_2^β | $-1/P_{12}$ | $-1/P_{12}$ |
| 2 | 2 | A_1^α | A_1^β | 0 | 0 |

APPENDIX C

C.1 Calculations for Converting Wire to Wire Current to System Mode Current for the Two Element Dipole Antenna

To use equation (3.7) we need to convert the conductor to conductor reflected currents of equation (4.14) to system mode reflected currents. Here we show the details of obtaining equation (4.19) from equations (4.16):

$$J_0^{\alpha,r}(-z_\beta) = \left(\frac{1}{2}\right) \sum_{p=0}^1 \{ I_p^{\beta\text{inc}}(-z_\beta) I_0^{\alpha\text{inc}}(z_\alpha) \exp(-ik_0(z_0+z_p)) \Gamma_{p0} + I_p^{\beta\text{inc}}(z_\beta) I_1^{\alpha\text{inc}}(z_\alpha) \exp(-ik_0(z_1+z_p)) \Gamma_{p1} \} \quad (\text{C.1a})$$

$$\text{and } J_1^{\alpha,r}(-z_\beta) = \left(\frac{1}{2}\right) \sum_{p=0}^1 \{ I_p^{\beta\text{inc}}(-z_\beta) I_0^{\alpha\text{inc}}(z_\alpha) \exp(-ik_0(z_0+z_p)) \Gamma_{p0} + I_p^{\beta\text{inc}}(z_\beta) I_1^{\alpha\text{inc}}(z_\alpha) \exp(-ik_0(z_1+z_p)) \Gamma_{p1} \} \quad (\text{C.1b})$$

Equation (C.1a) was gotten for β even ($V_0 = V_1 = 1$) and equation (C.1b) was gotten for β odd ($V_0 = -V_1 = 1$). For β even we also found in equations (4.17a) that $I_0^{\beta\text{inc}}(z) = I_1^{\beta\text{inc}}(z) = J_0^{\beta\text{inc}}(z)$ but $J_1^{\beta\text{inc}}(z) = 0$. Using this in (C.1a) results in

$$J_0^{\alpha,r}(-z_\beta) = (J_0^{\beta\text{inc}}(z_\beta)/2) \sum_{p=0}^1 \{ [J_0^{\alpha\text{inc}}(z_\alpha) + J_1^{\alpha\text{inc}}(z_\alpha)] \cdot \exp(-ik_0(z_1-z_p)) \Gamma_{p0} + [J_0^{\alpha\text{inc}}(z_\alpha) - J_1^{\alpha\text{inc}}(z_\alpha)] \cdot \exp(-ik_0(z_1-z_p)) \Gamma_{p1} \}$$

where equation (3.1) was used on $I_0^{\alpha \text{inc}}(z_\alpha)$ and $I_1^{\alpha \text{inc}}(z_\alpha)$.

Using the fact that $\Gamma_{00} = \Gamma_{11}$ and $\Gamma_{01} = \Gamma_{10}$ (from equation (4.12)), and that $z_0 = 0$ and $z_1 = \ell$ we get

$$J_0^{\alpha, r}(-z_\beta) = \left(\frac{1}{2}\right) J_0^{\beta \text{inc}}(z_\beta) \sum_{p=0}^1 \{ J_0^{\alpha \text{inc}}(z_\alpha) [\Gamma_{p0} \exp(-ik_0(z_0+z_p)) + \Gamma_{p1} \exp(-ik_0(z_1+z_p))] + J_1^{\alpha \text{inc}}(z_\alpha) [\Gamma_{p0} \exp(-ik_0(z_0+z_p)) + \Gamma_{p1} \exp(-ik_0(z_1+z_p))] \}$$

so that

$$J_0^{\alpha, r}(-z_\beta) = \frac{1}{2} J_0^{\beta \text{inc}}(z_\beta) \{ J_0^{\alpha \text{inc}}(z_\alpha) [\Gamma_{00}(1 + \exp(-2ik\ell)) + 2\Gamma_{10} \exp(-ik\ell)] + J_1^{\alpha \text{inc}}(z_\alpha) [\Gamma_{00}(1 - \exp(-2ik\ell))] \} \quad (\text{C.2})$$

For β odd we found in equation (4.17b) that

$I_0^{\beta \text{inc}}(z) = -I_1^{\beta \text{inc}}(z) = J_1^{\beta \text{inc}}(z)$ but $J_0^{\beta \text{inc}} = 0$. Equation (C.1b) can now be written as

$$\begin{aligned} J_1^{\alpha, r}(-z_\beta) &= \left(\frac{1}{2}\right) \{ I_0^{\beta \text{inc}}(z_\beta) I_0^{\alpha \text{inc}}(z_\alpha) \Gamma_{00} + I_1^{\beta \text{inc}}(z_\beta) I_0^{\alpha \text{inc}}(z_\alpha) \Gamma_{10} \\ &\quad \cdot \exp(-ik\ell) - I_0^{\beta \text{inc}}(z_\beta) I_1^{\alpha \text{inc}}(z_\alpha) \Gamma_{10} \\ &\quad - I_1^{\beta \text{inc}}(z_\beta) I_1^{\alpha \text{inc}}(z_\alpha) \exp(-2ik\ell) \Gamma_{11} \} \\ &= \left(\frac{1}{2}\right) J_1^{\beta \text{inc}}(z_\beta) \{ I_0^{\alpha \text{inc}}(z_\alpha) \Gamma_{00} - I_0^{\alpha \text{inc}}(z_\alpha) \exp(-ik\ell) \Gamma_{01} \\ &\quad + I_1^{\alpha \text{inc}}(z_\alpha) \exp(-ik\ell) \Gamma_{10} - I_1^{\alpha \text{inc}}(z_\alpha) \exp(-2ik\ell) \Gamma_{11} \} \\ &= \left(\frac{1}{2}\right) J_1^{\beta \text{inc}}(z_\beta) \{ (J_0^{\alpha \text{inc}}(z_\alpha) + J_2^{\alpha \text{inc}}(z_\alpha)) (\Gamma_{00} - \Gamma_{01} \exp(-ik\ell)) \} \end{aligned}$$

$$\begin{aligned}
& + (J_0^{\alpha \text{inc}}(z_\alpha) - J_1^{\alpha \text{inc}}(z_\alpha))(\exp(-ik\ell)\Gamma_{10} \\
& - \exp(-2ik\ell)\Gamma_{11})) \} .
\end{aligned}$$

$$\begin{aligned}
J_1^{\alpha, r}(-z_\beta) = & \frac{1}{2} J_1^{\beta \text{inc}}(z_\beta) \{ J_0^{\alpha \text{inc}}(z_\alpha) \left[\Gamma_{00}(1 - \exp(-2ik\ell)) \right] \\
& + J_1^{\alpha \text{inc}}(z_\alpha) \left[\Gamma_{00}(1 + \exp(-2ik\ell)) - 2 \exp(ik\ell)\Gamma_{01} \right] \} \\
& \text{(C.2)}
\end{aligned}$$