A Classical Technique to Prove the h-Cobordism Theorem

by

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A Classical Technique to Prove the h-Cobordism Theorem

Thesis directed by Prof. Carla Farsi

Let W be an m dimensional compact smooth manifold with boundary $\partial W = \partial_- W \sqcup \partial_+ W$, where submanifolds $\partial_- W$ and $\partial_+ W$ are closed and disjoint. Then suppose that W, $\partial_- W$, and $\partial_+ W$ are all simply connected, $\dim W \geq 6$, and $H_*(W, \partial_- W) = 0$. The *h-Cobordism Theorem* states that W is diffeormorphic to a product cobordism.

In this paper we will follow a classical technique developed by John Milnor in his "Lectures on the h-Cobordism Theorem" half a century ago.

Dedication

To all people who believe Japan will recover from the heart-rending tragedy in no time.

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Contents

Chapter 1

Introduction

1.1 Differential Manifolds

We will start with preliminary definitions which are used throughout this paper.

Definition 1 (Manifolds)**.** *A topological manifold M of dimension $m \geq 0$ is a second countable Hausdorff topological space so that the following properties are satisfied:*

(1) A family of open sets $\{U_\alpha\}_{\alpha \in \Lambda}$ covers M, where Λ is an index set,

(2) For each $\alpha \in \Lambda$, (U_α, ϕ_α) is called a coordinate chart, where $\phi_\alpha U_\alpha \to \mathbb{R}^n$ is a homeomorphism, and

(3) For all α and β in Λ, there is a homeomorphism called a transition map $\phi_\alpha \circ \phi_\beta^{-1} : \phi(U_\beta) \to \mathbb{R}^n$ whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Note that if each transition map is in $C^p(\mathbb{R}^m, \mathbb{R}^m)$ for a nonnegative integer $p \geq 0$ or in $C^\infty(\mathbb{R}^m, \mathbb{R}^m)$, then M is called a C^p-manifold or *smooth* manifold, respectively. Throughout this paper we assume that manifolds are equipped with a smooth structure. See Milnor[6] for the detail.

Note also that an *atlas* $\mathcal{A}$, the set of coordinate charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ is called *maximal* if one adds an extra chart to $\mathcal{A}$, then the property (3) fails. Now let us observe that we can always determine a maximal atlas $\mathcal{M}$. For if we define a partial ordering on $\mathcal{A}$ by the set inclusion, then the *Zorn's Lemma* guarantees that a maximal atlas $\mathcal{M}$ is determined. Therefore, without loss of generality, we always suppose that a manifold M is equipped with a maximal atlas $\mathcal{M}$.

For two smooth manifolds M and N of dimension m and n, respectively, we say that a map $f : M \to N$ is smooth if $\psi_\alpha \circ f \circ \phi_\beta^{-1} : \phi_\beta(U_\beta) \to \psi_\alpha(U_\alpha)$ is a smooth map, where $\{(U_\alpha, \psi_\alpha)\}$ and $\{(U_\beta, \phi_\beta)\}$ are charts for N and M, respectively.

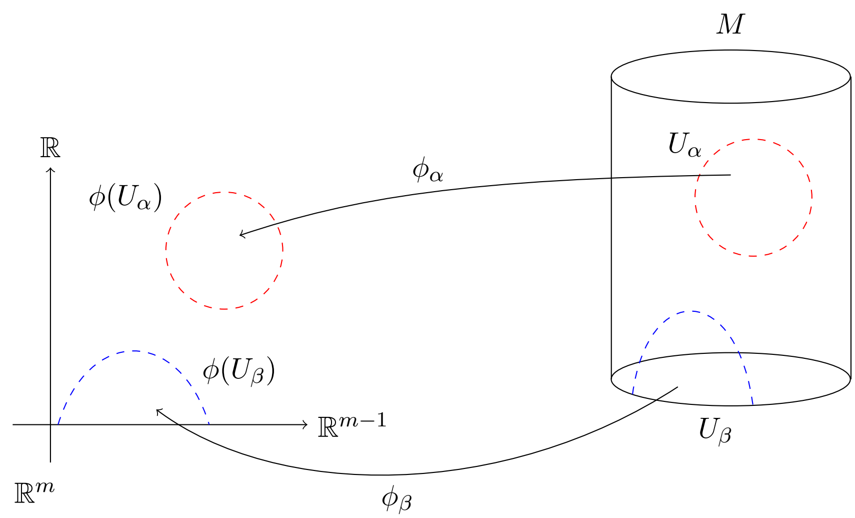

Figure 1.1: A smooth manifold M with boundary ∂M.

For an upper half m-dimensional Euclidean space $\mathbb{H}^m = \{(x_1, ..., x_m) \in \mathbb{R}^m | x_m \geq 0\}$, one defines a (smooth) manifold with *boundary* by allowing charts homeomorphic to $\mathbb{H}^m$. And the set of points $x \in M$, where each m-th coordinate of $\phi_\beta(x)$ is 0, is called the *boundary* of M and denoted by ∂M. And the boundary ∂M is of dimension $m - 1$. On the other hand, a manifold without a boundary is called *closed*.

Definition 2 (Tangent Spaces and Tangent Bundles)**.** *The tangent space T_pM to M at p is the set of all germs at p $X_p : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that for all $\alpha, \beta \in \mathbb{R}$ and all $f, g \in C^\infty(M, \mathbb{R})$*

(1) $X_p(\alpha f + \beta g) = \alpha X_p f + \beta X_p g$ (linearity),

(2) $X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$ (Leibnitz rule),

with the operations in T_pM defined by

$$(X_p + Y_p)f = X_p f + Y_p f \tag{1.1}$$

$$(\alpha X_p)f = \alpha(X_p f). \tag{1.2}$$

Then a tangent bundle is defined to be a collection of all tangent spaces, i.e.

$$TM = \{X_p \in T_pM | p \in M\} = \bigcup_{p \in M} T_pM. \tag{1.3}$$

Moreover a smooth map $X : M \to TM$ such that $\pi(X) = \mathrm{id}$, where π and id *denote the projection and identity map respectively, is called a vector field on M. And the push-forward of a vector field X on M by some diffeomorphism $f : M \to N$ is the vector field $f_* X_{f(p)}$ on N defined by $T_{f(p)}f(X_p)$.*

For a smooth manifold M of dimension m, recall that M is *orientable* if the determinants of the Jacobian of all transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are posititve whenever $U_\alpha \cap U_\beta \neq \emptyset$ for any charts (U_α, ϕ_α) and (U_β, ϕ_β). Or equivalently, the orientation $\langle M \rangle$ of M is given by an m-frame of vector fields $\langle \zeta_1, \ldots, \zeta_m \rangle$.

1.2 Cobordisms

Definition 3 (Cobordisms)**.** *For a smooth compact m-dimensional manifold W with a boundary $\partial W = \partial_- W \sqcup \partial_+ W$, where both $\partial_- W$ and $\partial_+ W$ are closed submanifolds, W is called a cobordism from $\partial_- W$ to $\partial_+ W$ and denoted by a triple $(W; \partial_- W, \partial_+ W)$.*

A compact manifold M with boundary ∂M has a neighbohood V of ∂M with a diffeomorphism $g : \partial M \times [0, 1) \to V$. Such a neighborhood V is called a *collar neighborhood* of the boundary ∂M.

For two cobordisms $(W; \partial_- W, \partial_+ W)$ and $(W'; \partial_- W', \partial_+ W')$, if there is a diffeomorphism $h : \partial_+ W \to \partial_- W'$ defined, we can glue or attach those two cobordisms via the map h. In other words, we have a new cobordism $W \cup_h W'$ from $\partial_- W$ to $\partial_+ W'$, where

$$W \cup_h W' = W \sqcup W' / \sim_h , \tag{1.4}$$

with an equivalence relation $\sim_h$ generated by $x \sim_h h(x)$ for every $x \in M$. Two boundaries $\partial_- W$ and $\partial_+ W$ are glued together via the diffeomorphism h along their collar neighborhoods.

Note that there exists a unique smooth structure $\mathcal{B}$, which is compatible with the given structures $\mathcal{A}$ and $\mathcal{A}'$ on W and W', respectively. See Milnor [6] for the detail.

Chapter 2

Morse Functions

2.1 Morse Functions

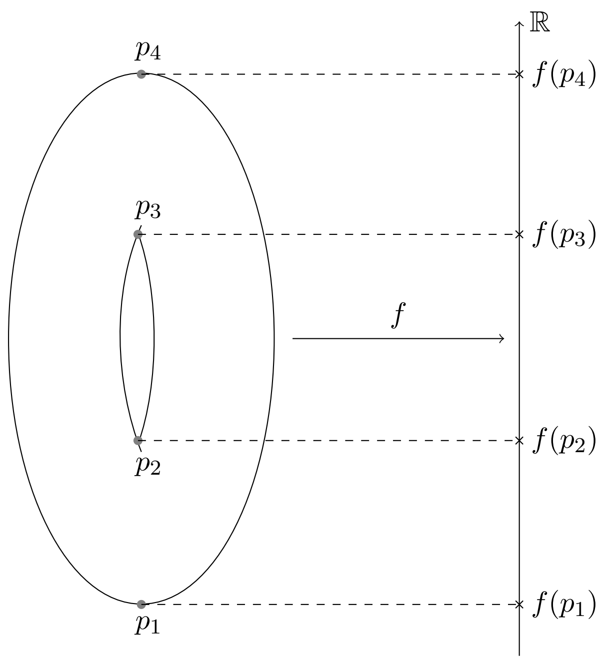

Figure 2.1: A torus of genus 1.

Let us begin with an example as Figure 2.1 shows. For a torus T^2 we can think of the height function f as a Morse function.

Definition 4 (Critical Points of A Smooth Function)**.** *Let M be an m-dimensional manifold without boundary and $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$ an atlas for M. Then for a smooth scalar function $f : M \to \mathbb{R}$, $p \in M$*

is called a critical point of f if

$$\frac{\partial(f \circ \phi_\alpha^{-1})}{\partial x_i}(\phi_\alpha(p)) = 0 \quad \textit{for} \quad i = 1, \ldots, m, \tag{2.1}$$

where (U_α, ϕ_α) is a coordinate chart that contains p.

Let us observe that a critical point $p \in M$ does not depend on the choice of a coordinate system. For two distinct charts (U_α, ϕ_α) and (U_β, ϕ_β), assume that $p \in U_\alpha \cap U_\beta$. Then $\phi_\alpha(p) = (x_1, \ldots, x_m)$ and $\phi_\beta(p) = (y_1, \ldots, y_m)$ and thus

$$\frac{\partial(f \circ \phi_\beta^{-1})}{\partial y_i}(\phi_\beta(p)) = \sum_{j=1}^{m} \frac{\partial x_j}{\partial y_i} \cdot \frac{\partial(f \circ \phi_\alpha^{-1})}{\partial x_j}(\phi_\alpha(p)) = 0. \tag{2.2}$$

So for simplicity, from now on, we write $\frac{\partial f}{\partial x_i}$ to denote each partial derivative $\frac{\partial(f \circ \phi_\alpha^{-1})}{\partial x_i}$.

Then we define a symmetric bilinear form of a smooth function f as follows.

Definition 5 (The Hessian Matrix)**.** *Let $f : M \to \mathbb{R}$ be a smooth function defined on a smooth manifold M equipped with an atlas $\{U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ and $p \in M$ a critical point of f. Then the Hessian $H_f(p)$ of f at p is an $m \times m$ matrix whose entries are the second order partial derivatives, i.e.*

$$H_f(p) = \Big[\frac{\partial^2(f \circ \phi_\alpha^{-1})}{\partial x_i \partial x_j}(\phi(p))\Big]. \tag{2.3}$$

Moreover, a critical point p is called non-degenerate if $\det(H_f(p)) \neq 0$. Otherwise, it is called degenerate.

Lemma 6. *A non-degenerate critical point $p \in M$ does not depend on the choice of a coordinate system at p.*

Proof. Let $(x_1, \ldots x_m)$ and $(y_1, \ldots y_m)$ be two local coordinate systems at p. Then

$$\frac{\partial^2 f}{\partial y_{i'} \partial y_{j'}} = \sum_{i,j=1}^{m} \frac{\partial x_j}{\partial y_{i'}} \cdot \frac{\partial x_i}{\partial y_{j'}} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \tag{2.4}$$

$$= \sum_{i,j=1}^{m} \frac{\partial x_j}{\partial y_{i'}} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot \frac{\partial x_i}{\partial y_{j'}}. \tag{2.5}$$

This implies that

$$H'_f(p) = J(p)^T H_f(p) J(p), \tag{2.6}$$

where $J(p) = \left[\frac{\partial x_i}{\partial y_j}\right]$ is the matrix of coordinate change, and $H_f(p)$ and $H'_f(p)$ are the Hessian matrices with respect to coordinate systems $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$, respectively. Since $J(p)$ is invertible, $\det(J(p)) \neq 0$ and

$$\det(H'_f(p)) = \det(J(p)^T H_f(p) J(p)) = \det(J(p)^T) \cdot \det(H_f(p)) \cdot \det(J(p)) \tag{2.7}$$

$$= \det(J(p))^2 \cdot \det(H_f(p)). \tag{2.8}$$

So $\det(H'_f(p)) \neq 0$ if and only if $\det(H_f(p)) \neq 0$. Therefore any non-degenerate critical point p is independent of the choice of coordinate systems. □

So from now on we write $H_f(p) = \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$ for simplicity.

Definition 7 (Morse Functions)**.** *A function $f : M \to \mathbb{R}$ is a Morse function if every critical point of f is non-degenerate.*

Next we observe that a Morse function is expressed as a quadratic form around small neighborhood of a non-degenerate critical point.

Lemma 8 (Morse's Lemma)**.** *If $f : M \to \mathbb{R}$ is a Morse function with a non-degenerate critical point $p \in M$, then there exists a coordinate system $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ of p such that the function f can be expressed as the form*

$$f(x) = -x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_m^2 + c \tag{2.9}$$

around a neighborhood U of p, where $c \in \mathbb{R}$ is a constant. Furthermore the uniquely determined non-negative integer λ is called the index of p.

Proof. Since $p \in M$, $\phi(p) = x = (x_1, \ldots, x_m)$ for some coordinate chart $\phi : U \to \mathbb{R}^m$. And without loss of generality, let us suppose that p corresponds to $(0, \ldots, 0)$ via ϕ and $f(p) = (0, \ldots, 0)$. Then by the *Fundamental Theorem of Calculus* (FTC),

$$f(x) = f(x) - f(p) = \int_0^1 \frac{df}{dt}(tx)dt \tag{2.10}$$

$$= \int_0^1 \frac{df}{dt}(tx_1, \ldots, tx_m)dt \tag{2.11}$$

$$= \int_0^1 \sum_{i=1}^{m} x_i \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_m)dt \tag{2.12}$$

$$= \sum_{i=1}^{m} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_m)dt. \tag{2.13}$$

Now let $g_i : M \to \mathbb{R}$ be smooth functions defined by $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt$. Then

$$f(x) = f(x_1, \ldots, x_m) = \sum_{i=1}^{m} x_i g_i(x_1, \ldots, x_m), \tag{2.14}$$

where $g_i(0, \ldots, 0) = \frac{\partial f}{\partial x_i}(0, \ldots, 0) = 0$. Moreover, by applying the FTC again, there are smooth functions $h_{ij} : M \to \mathbb{R}$ so that

$$g_i(x) = \sum_{i=1}^{m} x_j h_{ij}(x) \tag{2.15}$$

with $h_{ij}(0, \ldots, 0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0)$. Then

$$f(x) = f(x_1, \ldots, x_m) = \sum_{i,j=1}^{m} x_i x_j h_{ij}(x_1, \ldots, x_m). \tag{2.16}$$

Now let $H_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$, and we have

$$f(x) = f(x_1, \ldots, x_m) = \sum_{i,j=1}^{m} x_i x_j H_{ij}(x_1, \ldots, x_m) \tag{2.17}$$

and $H_{ij}(x_1, \ldots, x_m) = H_{ji}(x_1, \ldots, x_m)$. Moreover,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(0, \ldots, 0) = 2H_{ij}(0, \ldots, 0). \tag{2.18}$$

Now observe that the Hessian $H_f(p)$ is a bilinear, symmetric, and non-degenerate form. So $H_f(p)$ is diagonalizable by the *Gram-Schmidt orthonormalization process*, and moreover the diagonalized $H_f(p)$ contains a negative definite maximal submatrix. The size of such a submatrix corresponds to the index λ of the critical point. $\square$

We can immediately show the following corollary since each partial derivative $\frac{\partial f}{\partial x_i}$ of the standard form of a Morse function f is either $-2x_i$ or $2x_i$ on some open neighborhood U of a non-degenerate critical point p.

Corollary 9. *A non-degenerate critical point $p \in M$ of a Morse function $f : M \to \mathbb{R}$ is isolated.*

Observe also that every compact manifold M can be covered by finitely many charts $\{U_i, \phi_i\}_{i=1}^k$ and the following holds true.

Corollary 10. *If M is a compact manifold, then a Morse function $f : M \to \mathbb{R}$ has only finitely many non-degenerate critical points.*

For an open set $U \subset \mathbb{R}^m$, consider a smooth function $f : U \to \mathbb{R}$. Recall that the image $f(C) \subset \mathbb{R}$ has Lebesgue measure zero, where C is the set of critical points of f, by the *Sard's Lemma*. We will use the Sard's Lemma to prove the follwoing lemma that constructs a Morse function from an arbitrary smooth function f by perturbing f using an appropriate linear function on $\mathbb{R}^m$.

Lemma 11. *Let $U \subset \mathbb{R}^m$ be an open set and $f : U \to \mathbb{R}^m$ a smooth function. Then there exist some $a_1, \ldots, a_m \in \mathbb{R}$ so that*

$$f(x_1, \ldots, x_m) - (a_1 x_1 + \cdots + a_m x_m) \tag{2.19}$$

is a Morse function.

Proof. Define a map $h : U \to \mathbb{R}^m$ by

$$h(x_1, \cdots, x_m) = \Big(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_m}\Big)^T. \tag{2.20}$$

Then observe that the Jacobian $J_h(p)$ of h at $p \in \mathbb{R}^m$ is the Hessian $H_f(p)$ of f at p, *i.e.*

$$J_h(p) = \Big[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\Big] = H_f(p). \tag{2.21}$$

So $\det(J_h(p)) = \det(H_f(p))$ and p is a critical point of h if and only if $\det(H_f(p)) = 0$. Now by the *Sard's Lemma* let us choose $a = (a_1, \cdots, a_m)^T \in \mathbb{R}^m$ such that a is not a critical value of h.

Then define $\tilde{f} : U \to \mathbb{R}^m$ by

$$\tilde{f}(x_1, \cdots, x_m) = f(x_1, \cdots, x_m) - (a_1x_1 + \cdots + a_mx_m). \tag{2.22}$$

We claim that $\tilde{f}$ is a Morse function. So if p is a critical point of $\tilde{f}$, then

$$0 = \frac{\partial \tilde{f}}{\partial x_i}(p) = \frac{\partial f}{\partial x_i}(p) - a_i \tag{2.23}$$

for $i = 1, \cdots, m$. This implies that

$$h(p) = (a_1, \cdots, a_m)^T \tag{2.24}$$

and p is not the critical point of h. Therefore $\det(J_h(p)) \neq 0$ and thus $\det(H_{\tilde{f}}(p)) = \det(H_f(p)) \neq 0$. Hence p is a non-degenerate critical point of $\tilde{f}$ and $\tilde{f}$ is a Morse function. □

Definition 12. *Let $(W; \partial_- W, \partial_+ W)$ be a cobordism. A Morse function $f : W \to [0,1]$ on the cobordism is a smooth function such that*

(1) $\partial_- W = f^{-1}(0)$ and $\partial_+ W = f^{-1}(1)$, and

(2) Every non-degenerate critical point p of f exists in $\operatorname{Int} W = W - \partial W$.

The following lemma lets us construct a smooth function without any critical points near the boundary of W. Note that such a smooth function is constructed with the help of a partition of unity. *A partition of unity* is a family of non-negative smooth functions $\{\psi_i : W \to \mathbb{R}_{\geq 0}\}$ such that the closure $\overline{V_i}$ of $V_i = \{x \in W | \psi_i(x) > 0\}$ form a locally finite cover of W, and such that $\sum_i \psi_i(x) = 1$ for every $x \in W$.

Lemma 13. *There exists a smooth function $f : W \to [0,1]$ on a cobordism $(W; \partial_- W, \partial_+ W)$ such that f has no critical points in any neighborhood of $\partial W = \partial_- W \sqcup \partial_+ W$.*

Proof. Let $\{U_i, \phi_i\}_{i=1}^k$ be an atlas for W so that no U_i intersects with both $\partial_- W$ and $\partial_+ W$. Then define $f_i : U_i \to [0,1]$ by

$$f_i(p) = \begin{cases} \pi_m \circ \phi_i(p) = x_m & \text{if } U_i \cap \partial_- W \neq \emptyset \\ 1 - \pi_m \circ \phi_i(p) = 1 - x_m & \text{if } U_i \cap \partial_+ W \neq \emptyset \\ \frac{1}{2} & \text{if } U_i \cap \partial W = \emptyset, \end{cases} \tag{2.25}$$

where $\pi_m : \mathbb{R}^m \to \mathbb{R}$ is the projection onto the m-coordinate. Now choose a partition of unity $\{\psi_i : M \to \mathbb{R}\}_{i=1}^k$ dominated by the open cover $\{U_i\}_{i=1}^k$, *i.e.* the $\overline{V_i} \subset U_i$, where $V_i = \psi_i^{-1}((0,1])$. Then define a map $f : W \to [0,1]$ by

$$f(p) = \psi_1(p)\tilde{f}_1(p) + \cdots + \psi_k(p)\tilde{f}_k(p), \tag{2.26}$$

where each $\tilde{f}_i : W \to [0,1]$ is the smooth extension of $f_i : U_i \to [0,1]$ such that $\tilde{f}_i|U_i = f_i$ and $\tilde{f}_i|W - U_i = 0$. Observe that f is well-defined and smooth on W. Moreover $f^{-1}(0) = \partial_- W$ and $f^{-1}(1) = \partial_+ W$. Finally we claim that the derivative of f is non-zero on $\partial_- W \sqcup \partial_+ W$. So let $q \in \partial_- W$. Then $q \in U_j$ and $\phi_j(q) > 0$ for some j. Also let $\phi_j(p) = (x_1, \cdots, x_m)$ and consider the coordinate system. Then

$$\frac{\partial f}{\partial x_m} = \frac{\partial}{\partial x_m} \sum_{i=1}^k \phi_i \tilde{f}_i = \sum_{i=1}^k \frac{\partial}{\partial x_m} \phi_i \tilde{f}_i \tag{2.27}$$

$$= \sum_{i=1}^k (\tilde{f}_i \frac{\partial \phi_i}{\partial x_m} + \phi_i \frac{\partial \tilde{f}_i}{\partial x_m}). \tag{2.28}$$

Since $f_j(q) = 0$ and $f_j(q) = 0$ for $j \neq i$, the first summand is zero. So $\frac{\partial f}{\partial x_m}(q) = \sum_{i=1}^k \phi_i(q) \frac{\partial \tilde{f}_i}{\partial x_m}(q)$. Because of construction of $\tilde{f}_j$, $\frac{\partial \tilde{f}_j}{\partial x_m}(q) = 1$ or 0 for $q \in \partial_- W$. Therefore $\frac{\partial f}{\partial x_m}(q) \neq 0$. Similarly, if $q \in \partial_+ W$, then $\frac{\partial f}{\partial x_m}(q) \neq 0$. Hence the derivative of f is non-zero on $\partial_- W \sqcup \partial_+ W$. $\square$

Note that a polynomial of first and second order partial derivatives can be approximated by another polynomial, and the following lemmas hold true.

Lemma 14. *For an open set $U \subset \mathbb{R}^m$, a compact set $K \subset U$, and a smooth function $f : U \to \mathbb{R}$, if f has only non-degenerate critical points in K, then there is some constant $\delta > 0$ such that if*

$g : U \to \mathbb{R}$ *is smooth on* K *with*

$$\Big|\frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i}\Big| < \delta \quad \text{and} \quad \Big|\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 g}{\partial x_i \partial x_j}\Big| < \delta \tag{2.29}$$

for all $i, j = 1, \cdots, m$ *then* g *has only non-degenerate critical points.*

Lemma 15. *Let* U *and* U' *be open sets in* $\mathbb{R}^m$ *and* $h : U \to U'$ *be a diffeomorphism such that* $K' = h(K) \subset U$ *for some compact set* $K \subset U$*. Then for every* $\epsilon > 0$ *there is some* $\delta > 0$ *such that if* $f : U' \to \mathbb{R}$ *is smooth with*

$$|f| < \delta, \quad \Big|\frac{\partial f}{\partial x_i}\Big| < \delta, \quad \text{and} \quad \Big|\frac{\partial^2 f}{\partial x_i \partial x_j}\Big| < \delta \tag{2.30}$$

for all $i, j = 1, \cdots, m$ *on* $K' \subset U'$*, then*

$$|f \circ h| < \epsilon, \quad \Big|\frac{\partial}{\partial x_i} f \circ h\Big| < \epsilon, \quad \text{and} \quad \Big|\frac{\partial^2}{\partial x_i \partial x_j} f \circ h\Big| < \epsilon \tag{2.31}$$

on $K \subset U$*.*

Now let us define C^∞-topology on the set of smooth functions $C^\infty(M, \mathbb{R})$. So for a compact and closed manifold M let us consider a finitely many charts $\{(U_i, \phi_i)\}_{i=1}^k$ and a compact refinement $\{C_i\}_{i=1}^k$ of the open cover $\{U_i\}_{i=1}^k$, *i.e.* each compact C_i is contained in U_i and the refinement still covers W. Then for every $\delta > 0$ define a neighborhood $N(\delta)$ of $C^\infty(M, \mathbb{R})$ by

$$N(\delta) = \{g \in C^\infty(M, \mathbb{R}) | \; |g_l^{(n)}| < \delta\} \tag{2.32}$$

on $\phi_l(C_l)$, where $g_l = g \circ \phi_l^{-1}$, and $g_l^{(n)}$ denotes the partial derivative of order n for all $n \geq 0$. In other words, $N(\delta)$ is an open neighborhood of the zero function.

Definition 16. *The topology on* $C^\infty(M, \mathbb{R})$ *generated by* $N(f, \delta) = f + N(\delta)$*, where* $f \in C^\infty(M, \mathbb{R})$ *is called the* C^∞*-topology.*

Theorem 17. *On any cobordism* $(W; \partial_- W, \partial_+ W)$*, there exists a Morse function* $f : W \to \mathbb{R}$

Proof. Let U be an open neighborhood of $\partial W = \partial_- W \sqcup \partial_+ W$. Since W is compact and Hausdorff, there is an open set V such that $\partial_- W \subset \overline{V} \subset U$. Then let us take a finite open cover $\{U_i\}_{i=1}^k$

for W such that either $U_i \subset U$ or $U \subset W - \partial W$, and consider its compact refinement $\{C_i\}_{i=1}^k$ of $\{U_i\}_{i=1}^k$. Now let

$$C = \bigcup_{i' \in \{1,\cdots,k\}} C_{i'} \quad \text{such that} \quad C_{i'} \subset U_{i'} \subset U. \tag{2.33}$$

By Lemma 14 and Lemma 15, for a small neighborhood $N \subset C^\infty(W, \mathbb{R})$ of f, no function in N has critical points in C. Observe also that $0 < f < 1$ on $W - V$. Let $N' \subset C^\infty(M, \mathbb{R})$ be a neighborhood of f such that every $g \in N'$ is $0 < g < 1$ on $W - V$. Then let $N_0 = N \cap N'$ By Lemma 14 and 15, there exists $f_1 \in N_0$ such that f_1 has only non-degenerate critical points on C_1, and a neighborhood $N_1 \subset N_0$ of f_1 such that every $g_1 \in N_1$ has only non-degenerate critical points on C_1. By repeating this recursive procedure, at the k-stage, there is $f_k \in N_k \subset N_{k-1} \subset \cdots \subset N_0$ so that f_k has only non-degenerate critical points in

$$C \cup C_1 \cup \cdots \cup C_k = M. \tag{2.34}$$

Since $f_k \in N_0 \subset N'$, $f_k|V = f|V$, $f_k^{-1}(0) = \partial_- W$, $f_k^{-1}(1) = \partial_+ W$, and f_k has no critical points in any neighborhood of ∂W. Therefore f_k is a Morse function on $(W; \partial_- W, \partial_+ W)$. $\square$

We can slightly perturb a Morse function f to get another Morse function with distinct critical values as the following lemma states.

Corollary 18 (Non-resonant Morse Functions). *Let $(W; \partial_- W, \partial_+ W)$ be a cobordim and $f : W \to [0,1]$ a Morse function with finitely many critical points $p_1, \cdots, p_k$. Then there exists another Morse function $g : W \to [0,1]$ that approximates f and*

$$g(p_i) \neq g(p_j) \tag{2.35}$$

for $i \neq j$. And such a Morse function g is called non-resonant.

Corollary 19. *For a Morse function $f : W \to [0,1]$ on a cobordism $(W; \partial_- W, \partial_+ W)$, suppose that $c \in (0,1)$ is not a critical value of f. Then $f^{-1}([0,c])$ and $f^{-1}([c,1])$ are both smooth compact manifolds with boundary.*

Definition 20. *The Morse number μ of a cobordism $(W; \partial_- W, \partial_+ W)$ is the minimum number of critical points of all Morse functions $f : W \to [0, 1]$.*

Moreover we have the following corollary since Corollary 18 lets us choose a Morse function with distinct critical values.

Corollary 21. *Any cobordism $(W; \partial_- W, \partial_+ W)$ can be decomposed into a composition of cobordisms with Morse number $\mu = 1$.*

2.2 Gradient-like Vector Fields

In this section we define a vector field $X : W \to TW$ with special properties as follows.

Definition 22 (Gradient-like Vector Fields)**.** *For a cobordism $(W; \partial_- W, \partial_+ W)$, let $f : W \to \mathbb{R}$ be a Morse function with k critical points $p_1, \cdots, p_k$. Then a vector field $X : W \to TW$ is called a gradient-like vector field associated to f if*

$$Xf > 0 \text{ on } W - \{p_1, \cdots, p_k\}$$

and for each critical point $p \in W$ of index λ there exists an open neighborhood U of p such that the coordinate of the vector field X around U is given by

$$(-x_1, \cdots, -x_\lambda, x_{\lambda+1}, \cdots, x_n),$$

and such that X at $p \in W$ has the form

$$X_p = -2x_1 \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} \cdots + 2x_m \frac{\partial}{\partial x_m}.$$

Recall that an *integral curve* $\varphi_p : I \to W$ of X on a compact manifold W is a C^∞-diffeomorphism such that for every $t \in I$ and $p \in M$ $\varphi_p(0) = p$ and

$$\frac{d}{dt}(f \circ \varphi_p) = Xf, \tag{2.36}$$

where $I = [0, 1]$ denotes the unit interval. Then the lemma discussed below lets us construct a gradient-like vector field X from an arbitrary Morse function f.

Lemma 23. *For every Morse function $f : W \to \mathbb{R}$ there exists a gradient-like vector field $X : W \to TW$ for f.*

Proof. For simplicity we will show the case f has only one critical point $p \in W$ of index λ. Using the standard form of a Morse function f, there exists an open neighborhood U_0 of p and the coordinate system $(x_1, \cdots, x_\lambda, x_{\lambda+1}, \cdots, x_m)$ such that

$$f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \dots x_m^2 \tag{2.37}$$

on U_0. Now let $U \ni p$ be an open set so that $p \in \overline{U} \subset U_0$. Since each partial derivative is nonzero on $W - U_0$, the *Implicit Function Theorem* guarantees that for every non-critical point $p' \in W - U_0$ there exits a coordinate system $(x_1', \cdots, x_m')$ in an open neighborhood $U' \subset W - U_0$ such that

$$f(x) = x_1' + k,$$

where k_i is constant. Since $W - U_0$ is compact, it is covered by a finite open cover $\{U_1, \dots, U_k\}$ such that

(1) $U_0 \cap U_i = \emptyset$ for $i = 1, \dots, k$, and

(2) U_i has coordinates $x^i = (x_1^i, \dots, x_m^i)$ and $f(x^i) = x_1^i + k_i$, where each k_i is a constant, on U_i.

On the neighborhood U_0, there exists the gradient-like vector field $X^0 : U_0 \to TW$ associated to the Morse function f so that

$$X^0 = -2x_1 \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} \cdots + 2x_m \frac{\partial}{\partial x_m}.$$

Moreover on each U_i there is a vector field $X^i : U_i \to TW$ such that

$$X^i = \frac{\partial}{\partial x_1^i} \text{ for } i = 1, \dots k.$$

Now consider a partition of unity $\{\psi_i : W \to \mathbb{R}\}_{i=0}^k$ subordinate to the open cover $\{U_i\}_{i=0}^k$. Therefore $X : W \to TW$ defined by

$$X = \psi_0 X^0 + \psi_1 X^1 + \cdots + \psi_k X^k$$

is a desired gradient-like vector field for f. $\square$

Chapter 3

Product and Elementary Cobordisms

3.1 Product Cobordisms

We first investigate the simplest cobordism called a *product cobordism.*

Definition 24 (Product Cobordisms)**.** *A cobordism* $(W; \partial_- W, \partial_+ W)$ *is called a product cobordism if it is diffeomorphic to* $(\partial_- W \times I; \partial_- W, \partial_- W)$*, where* $I = [0, 1]$*.*

Theorem 25. *If the Morse number* μ *of* $(W; \partial_- W, \partial_+ W)$ *is zero, then* $(W; \partial_- W, \partial_+ W)$ *is a product cobordism.*

Proof. Note that every Morse function $f : W \to \mathbb{R}$ has no critical points since $\mu = 0$. Also observe that there is a gradient-like vector field $Y : W \to TW$ for f. So $Yf > 0$ on W. Then define another gradient-like vector field $X : W \to TW$ by rescaling

$$X = \frac{1}{Yf} Y.$$

Xf is constant on W, *i.e.*

$$Xf = \frac{1}{Yf} Yf = 1.$$

For $p \in \partial W = \partial_- W \sqcup \partial_+ W$ and an open neighborhood U of p, $f|U : U \to \mathbb{R}$ extends to a smooth map $\tilde{f}|\tilde{U} : \tilde{U} \to \mathbb{R}$, where $\tilde{U}$ is homeomorphic to an open set in $\mathbb{R}^m$. Then the gradient-like vector field $X|U : U \to TW$ also extends to $\tilde{X} : \tilde{U} \to TW$. Because $Xf = 1$ on a compact manifold W, there is an integral curve $\varphi_p : I \to W$ such that $f \circ \varphi_p(t) = t + c$, where c is a constant. Now define a C^∞-diffeomorphism $\psi_p : I \to W$ by $\psi_p = \varphi_p(t-c)$. Then

$$f \circ \psi_p(t) = t \;\forall t \in I$$

and $\psi_p(0) = p$. Therefore $h : \partial_- W \times I \to W$ defined by

$$h(p,t) = \psi_p(t)$$

is a diffeomorphism with the inverse $h^{-1}(p) = (\psi_p(0), f(p))$. □

3.2 Elementary Cobordisms and Surgery Theory

Definition 26 (Elementary Cobordisms). *An elementary cobordism is a cobordism* $(W; \partial_- W, \partial_+ W)$ *that admits a Morse function* $f : W \to \mathbb{R}$ *with exactly one critical point* $p \in W$.

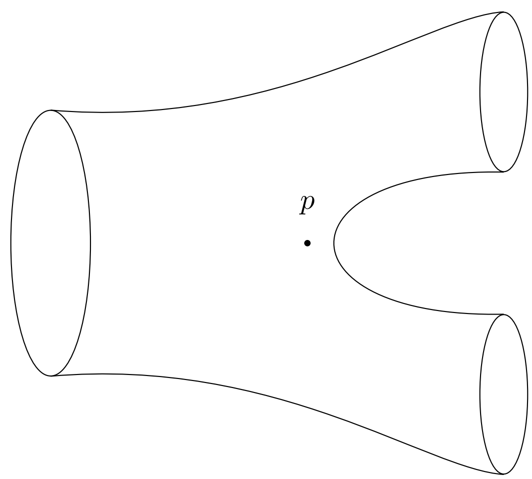

Figure 3.1: An elementary cobordism with a critical point p.

For an $(m-1)$-dimensional manifold M consider an embedding $\iota : S^{\lambda-1} \times \text{Int } D^{m-\lambda} \to M$. Let $\chi(M, \iota)$ denote the manifold obtained by

$$M - \iota(\partial D^\lambda \times \mathbf{0}) \sqcup \text{Int } D^\lambda \times S^{m-\lambda-1} = M - \iota(\partial D^\lambda \times \mathbf{0}) \cup \text{Int } D^\lambda \times S^{m-\lambda-1} / \sim_\iota ,$$

where $\sim_\iota$ is the equivalence relation generated by $\iota(u, \theta v) \sim_\iota (\theta u, v)$ for $u \in S^{\lambda-1}$ and $v \in S^{m-\lambda-1}$, where $\theta \in (0,1)$. Then $\chi(M, \iota)$ is obtained from M by *surgery of type* $(\lambda, m-\lambda)$.

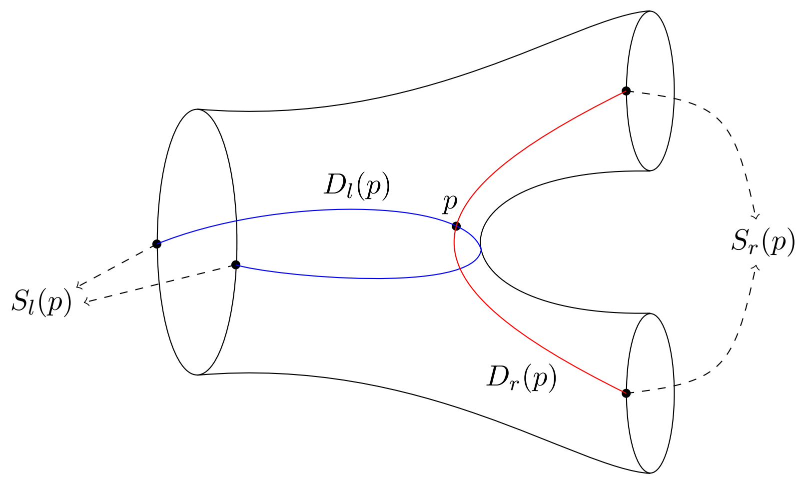

Figure 3.2: An elementary cobordism with embedded disks and spheres.

Theorem 27. *A cobordism $(W; \partial_- W, \partial_+ W)$ with $\partial_+ W = \chi(\partial_- W, \iota)$ is an elementary cobordism and there exists a Morse function $f : W \to \mathbb{R}$ with exactly one critical point $p \in W$ of index λ.*

Observe that $\phi : S^{\lambda-1} \times (D^{m-\lambda} - \mathbf{0}) \to (\text{int } D^{\lambda} - \mathbf{0}) \times S^{m-\lambda-1}$ that maps $(u, \theta v)$ to $(\theta u, v)$ is a diffeomorphism. So $\chi(M, \iota)$ is a smooth manifold since the smooth structure is inherited by the smooth structure of both $M - \iota(S^{\lambda-1} \times \mathbf{0})$ and $D^{\lambda} \times S^{m-\lambda-1}$.

Now let us construct a cobordism whose boundary is $M \sqcup \chi(M, \iota)$. So consider

$$L = \{(x, y) \in \mathbb{R}^{\lambda} \times \mathbb{R}^{m-\lambda} | -1 \leq -\|x\|^2 + \|y\|^2 \leq 1\} \cap$$
$$\{(x, y) \in \mathbb{R}^{\lambda} \times \mathbb{R}^{m-\lambda} | \|x\| \cdot \|y\| \leq \cosh 1 \cdot \sinh 1\}.$$

Because of being a subspace of $\mathbb{R}^m$, L is a smooth manifold whose boundary is given by $\partial_- L = \{(x, y) \in \mathbb{R}^{\lambda} \times \mathbb{R}^{m-\lambda} | -\|x\|^2 + \|y\|^2 = -1\}$ and $\partial_+ L = \{(x, y) \in \mathbb{R}^{\lambda} \times \mathbb{R}^{m-\lambda} | -\|x\|^2 + \|y\|^2 = 1\}$. Now observe that $\partial_- L$ is diffeomorphic to $S^{\lambda-1} \times \text{int } D^{m-\lambda}$ via correspondence between $(u \cosh \theta, v \sinh \theta)$ and $(u, \theta v)$ for $\theta \in [0, 1)$. Similarly, $\partial_+ L$ is diffeomorphic to int $D^{\lambda} \times S^{m-\lambda-1}$ via correspondence between $(u \sinh \theta, v \cosh \theta)$ and $(u, \theta v)$. Moreover, for the hypersurfaces $-\|x\|^2 + \|y\|^2 = c$ and the map $t \mapsto (e^{-t}x, e^t y)$ gives orthogonal trajectories through these hypersurfaces. If $x = 0 \in \mathbb{R}^{\lambda}$ or $y = 0 \in \mathbb{R}^{m-\lambda}$, then they are straight line segments. Otherwise, the maps parametrize a hyperbola from some $(u \cosh \theta, v \sinh \theta) \in \partial_- L$ to $(u \sinh \theta, v \cosh \theta) \in \partial_+ L$.

Next construct the manifold $\omega(M,\iota)$ as follows. So, for $u \in S^{\lambda-1}$, $v \in S^{m-\lambda-1}$, $\theta \in (0,1)$, and $c \in [-1,1]$, consider the disjoint union

$$(M - \iota(S^{\lambda-1} \times \mathbf{0})) \times [-1,1] \sqcup L/\sim_\iota, \tag{3.1}$$

where the equivalence relation $\sim_\iota$ generated by identification of $(\iota(u,\theta v),c)$ with the unique point $(x,y) \in L$ so that

$$-\|x\|^2 + \|y\|^2 = c$$

and

$$(x,y) \text{ is on the orthogonal trajectory through } (u\cosh\theta, v\sinh\theta).$$

This correspondence defines a diffeomorphism

$$\iota(S^{\lambda-1} \times (\text{int } D^{m-\lambda} - \mathbf{0})) \times [-1,1] \to L \cap ((\mathbb{R}^\lambda - \mathbf{0}) \times (\mathbb{R}^{m-\lambda} - \mathbf{0})),$$

and we get a differential manifold $\omega(M,\iota)$ with boundary. So observe the following two cases, $c=-1$ and $c=1$.

Case 1: $(c=-1)$. If $c=-1$, then we identify points of $(\iota(S^{\lambda-1} \times (\text{int } D^{m-\lambda} - \mathbf{0}))) \times \{-1\}$ with the points of $\partial_- L$ that corresponds to $S^{\lambda-1} \times (\text{int } D^{m-\lambda} - \mathbf{0})$. So one boundary of the manifold $\omega(M,\iota)$ is given by M.

Case 2: $(c=1)$. If $c=1$, then we identify the points of $(\iota(S^{\lambda-1} \times (\text{int } D^{m-\lambda} - \mathbf{0}))) \times \{1\}$ with the points of $\partial_+ L$ that corresponds to $(\text{int } D^\lambda - \mathbf{0}) \times S^{m-\lambda-1}$. So another boundary of $\omega(M,\iota)$ is $\chi(M,\iota)$.

Therefore $\omega(M,\iota)$ is a cobordism whose boundary is $M \sqcup \chi(M,\iota)$.

Proposition 28. *There exists a Morse function $f : \omega(M,\iota) \to [-1,1]$ with $f^{-1}(-1) = M$, $f^{-1}(1) = \chi(M,\iota)$, and there is exactly one critical point of index λ in the interior of $\omega(M,\iota)$.*

Proof. Define a map $f : \omega(M,\iota) \to [-1,1]$ by

$$f(u) = \begin{cases} c & \text{if } u = (z,c) \in (M - \iota(S^{\lambda-1} \times \mathbf{0})) \times [-1,1] \\ -\|x\|^2 + \|y\|^2 & \text{if } u \in L \end{cases}$$

By construction of the manifold $\omega(M,\iota)$, f agrees on the overlap and thus the map is *well-defined* on $\omega(M,\iota)$. Moreover observe that $x \in \mathbb{R}^\lambda$ and $y \in \mathbb{R}^{m-\lambda}$. So f is a Morse function with only one critical point of index λ as desired. $\square$

Theorem 29. *Let $(W;\partial_- W,\partial_+ W)$ be an elementary cobordism with an embedding $\iota : S^{\lambda-1} \times \mathrm{Int}\, D^{m-\lambda} \to \partial_- W$. Then $(W;\partial_- W,\partial_+ W)$ is diffeomorphic to $(\omega(\partial_- W,\iota);\partial_- W,\chi(\partial_- W,\iota))$.*

Proof. For simplicity we assume $f(p) = 0$, where p is the critical point of the Morse function f. By Morse Lemma, there is a small neighborhood U around p so that f can be written as

$$f(x,y) = -\|x\|^2 + \|y\|^2,$$

where $x \in \mathbb{R}^\lambda$ and $y \in \mathbb{R}^{m-\lambda}$. Now let $\epsilon > 0$ such that

$$L_\epsilon = \{(x,y) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} | -\epsilon \leq -\|x\|^2 + \|y\|^2 \leq \epsilon\} \cap$$
$$\{(x,y) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} | \|x\| \cdot \|y\| \leq \epsilon \cosh 1 \cdot \sinh 1\}.$$

satisfying $L_\epsilon \subset U'$, where $\phi : U \to U' \subset \mathbb{R}^m$ is the local coordinate chart near p. Then $i := \phi^{-1}|\partial_- L_\epsilon : \partial_- L_\epsilon \to f^{-1}(-\epsilon)$ is an embedding, and $W_\epsilon = f^{-1}([-\epsilon,\epsilon])$ is a cobordism that is identified as $\omega(f^{-1}(-\epsilon), \phi^{-1}|\partial_- L_\epsilon)$. Moreover $f^{-1}(-\epsilon)$ and $f^{-1}(\epsilon)$ are diffeomorphic to $\partial_- W$ and $\partial_+ W$, respectively. $\square$

For a topological space X and its subspace A with the inclusion $i : A \hookrightarrow X$, recall that A is called a *deformation retract* of X if there exists some continuous map $r : X \to A$ such that

$$r \circ i = \mathrm{id}_A \text{ and } i \circ r \simeq \mathrm{id}_X. \tag{3.2}$$

Observe by definition above that A and X have the same homotopy type if A is a deformation retract of X. Moreover if A and X have the same homotopy type, then the homology group $H_n(A)$ is isomorphic to $H_n(X)$.

Theorem 30. *For a cobordism $(W;\partial_- W,\partial_+ W)$ and a Morse function $f : W \to \mathbb{R}$ with one critical point $p \in W$ of index λ, let $D_l(p)$ denote the left-hand λ-disk associated to a gradient-like vector field $X : W \to TW$ for f. Then $\partial_- W \cup D_l(p)$ is a deformation retract of W.*

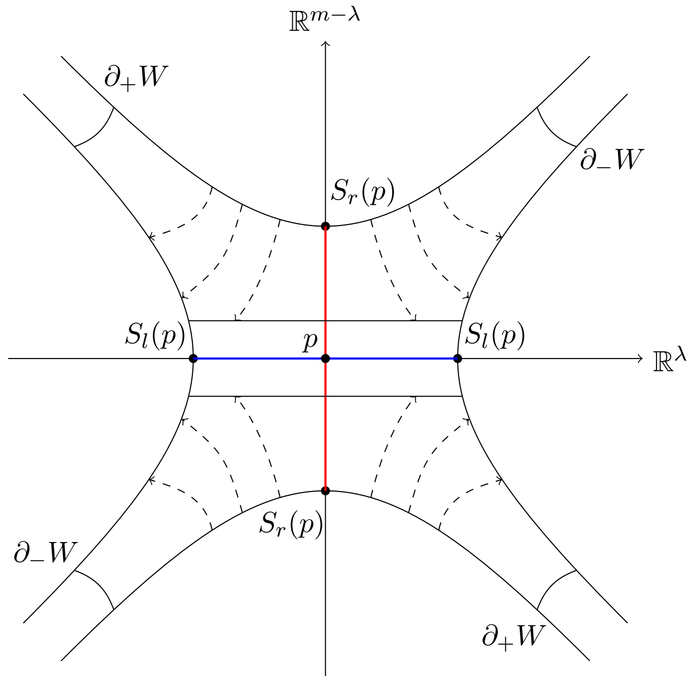

Figure 3.3: The first retraction r from W to $\partial_-W \cup C$.

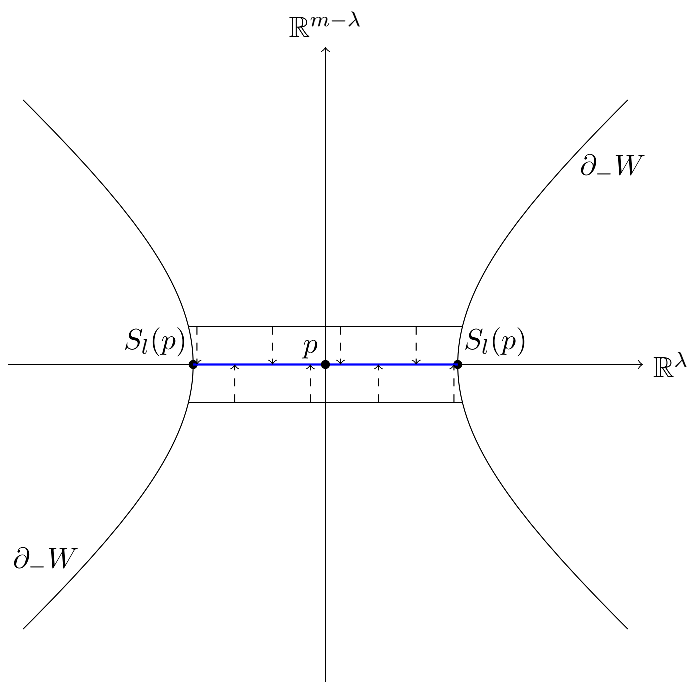

Figure 3.4: The second retraction r' from $\partial_-W \cup C$ to $\partial_-W \cup D_l(p)$.

Proof. By Theorem 29, observe that $(W;\partial_-W,\partial_+W)$ is diffeomorphic to

$$(\omega(\partial_-W,\iota);\partial_-W,\chi(\partial_-W,\iota)) \tag{3.3}$$

for some embedding $\iota : S^{\lambda-1}\times \text{Int } D^{m-\lambda}\to\partial_-W$. Also there is a Morse function $f:\omega(\partial_-W,\iota)\to[-1,1]$ on $\omega(\partial_-W,\iota)$. For L in **??**, let $D_l(p)=\{(x,y)\in L|\|y\|=0\}$ denote an embedded disk Also let $C=\{(x,y)\in L|\|y\|\leq\frac{1}{10}\}$ denote an open neighborhood of $D_l(p)$. Now define a retraction $r_t:\omega(\partial_-W,\iota)\to\partial_-W\cup C$ as follows. Recall from the equation 29 above that

$$(\partial_-W-\iota(S^{\lambda-1}\times\mathbf{0}))\times[-1,1]\cup_\iota L.$$

So for $(u,c)\in\partial_-W-\iota(S^{\lambda-1}\times\mathbf{0}))\times[-1,1]$ define

$$r_t(u,c)=(u,c-t(c+1))\ \forall t\in[0,1] \tag{3.4}$$

and for $(x,y)\in L$ define

$$r_t(x,y)=\begin{cases}(x,y)\text{ if }\|y\|\leq\frac{1}{10}\\ \\ (\frac{x}{\rho},\rho y)\text{ if }\|y\|\geq\frac{1}{10}\end{cases}, \tag{3.5}$$

where $\rho=\rho(x,y,t)=\max\{\frac{1}{10\|y\|},\zeta\}$ and ζ is the unique positive real solution of the equation

$$-\frac{\|x\|^2}{\zeta^2}+\zeta^2\|y\|^2=(-\|x\|^2+\|y\|^2)(1-t)-t. \tag{3.6}$$

So if $\|y\|\geq\frac{1}{10}$ then ρ determines a trajectory from (x,y) to some point in ∂_-W.

Next construct another retraction $r'_t:\partial_-W\cup C\hookrightarrow\partial_-W\cup D_l(p)$ as follows. For $(x,y)\in C$ define

$$r'_t(x,y)=\begin{cases}(x,(1-t)y)\text{ if }\|x\|^2\leq 1\\ \\ (x,\rho' y)\text{ if }1\leq\|x\|^2\leq 1+\frac{1}{100}\end{cases}, \tag{3.7}$$

where $\rho'=\rho'(x,y,t)=(1-t)+t\frac{\sqrt{x^2-1}}{\|y\|^2}$. Since $(x,(1-t)y)$ and $(x,\rho'y)$ coincide when $\|x\|^2=1$, r'_t is well-defined and thus a retraction of $\partial_-W\cup C$ to $\partial_-W\cup D_l(p)$. Therefore $r'_t\circ r_t$ is a deformation retract of W to $\partial_-W\cup D_l(p)$. □

Corollary 31. $H_n(W,\partial_-W)\cong\mathbb{Z}$ *if* $n=\lambda$*. Otherwise* $H_n(W,\partial_-W)\cong 0$.

Proof. By Theorem 30, there is an embedding

$$j : (D^\lambda, S^{\lambda-1}) \to (W, \partial_- W) \tag{3.8}$$

such that $j^{-1}(\partial_- W) = S^{\lambda-1}$ and $\partial_- W \cup j(D^\lambda)$ is a deformation retract of W. So

$$H_*(W, \partial_- W) \cong H_*(\partial_- W \cup j(D^\lambda), \partial_- W). \tag{3.9}$$

Moreover $j : S^{\lambda-1} \to \partial_- W$ extends to an embedding $\tilde{j} : S^{\lambda-1} \times \text{Int } D^{m-\lambda} \to \partial_- W$ and let $U = \text{im } \tilde{j}$. Therefore

$$\begin{aligned}
H_*(W, \partial_- W) &\cong H_*(\partial_- W \cup j(D^\lambda), \partial_- W) && (3.10)\\
&\cong H_*(\partial_- W \cup j(D^\lambda) - (\partial_- W - U), \partial_- W - (\partial_- W - U)) && (3.11)\\
&\cong H_*(U \cup j(D^\lambda), U) && (3.12)\\
&\cong H_*(j^{-1}(U \cup j(D^\lambda)), j^{-1}(U)) && (3.13)\\
&\cong H_*(D^\lambda, S^{\lambda-1}) && (3.14)
\end{aligned}$$

because of excision and homotopy invariance. Hence

$$H_n(W, \partial_- W) \cong H_n(D^\lambda, S^{\lambda-1}) \cong \begin{cases} \mathbb{Z} \text{ if } n = \lambda \\ \\ 0 \text{ if } n \neq \lambda \end{cases}. \tag{3.15}$$

□

Now we can generalize Theorem 30 and Corollary 31 as follows. Suppose that a cobordism $(W; \partial_- W, \partial_+ W)$ admits a Morse function $f : W \to \mathbb{R}$ with k critical points $p_1, \ldots, p_k$ of indices $\lambda_1, \ldots, \lambda_k$, respectively. Moreover suppose that those critical points have the same critical value, *i.e.* $f(p_1) = \cdots = f(p_k)$. By perturbing the Morse function f and the associated gradient-like vector field X, obtain k embeddings $\iota_i : S^{\lambda_i - 1} \times \text{Int } D^{m-\lambda_i} \to V$ and construct the smooth manifold $\omega(V; \iota_1, \ldots, \iota_k)$ as given above for each $i = 1, \ldots, k$. This manifold $\omega(\partial_- W; \iota_1, \ldots, \iota_k)$ is diffeomorphic to W and moreover $\partial_- W \cup D_l(p_1) \cup \cdots \cup D_l(p_k)$ is a deformation retract of W.

Therefore if $\lambda_1 = \cdots = \lambda_k = \lambda$, then $H_n(W, \partial_- W) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (k summands) for $n = \lambda$. Otherwise $H_n(W, \partial_- W) \cong 0$ for $n \neq \lambda$.

Chapter 4

Rearrangement of Cobordisms

4.1 Rearrangement of Critical Values

In this section we will consider a cobordism $(W; \partial_- W, \partial_+ W)$. with exactly two critical points p and p' of indices λ and λ', respectively. However their indices are not necessarily same or consecutive.

Let $\varphi : W \times \mathbb{R} \to W$ denote a *flow* of the vector field X. For a critical point $p \in W$ of a Morse function $f : W \to \mathbb{R}$ define

$$W^s(p; X) = \left\{ x \in W | \lim_{t \to +\infty} \varphi(x, t) = p \right\}$$

and

$$W^u(p; X) = \left\{ x \in W | \lim_{t \to -\infty} \varphi(x, t) = p \right\}$$

the *stable* and *unstable manifold* of p with respect to X. Observe that the stable manifold $W^s(p; X)$ is equivalent to the left-hand disk $D_l(p)$ of the critical point p. Similarly, $W^u(p; X)$ is equivalent to the right-hand disk $D_r(p)$.

The following theorem states that we can perturb a Morse function f by an isotopy so that two critical points of f change their critical values in some small neighborhoods of those two critical points.

Theorem 32 (Preliminary Rearrangement Theorem)**.** *For a cobordism $(W; \partial_- W, \partial_+ W)$ let $f : W \to [0, 1]$ be a non-resonant Morse function with exactly two critical points $p, p' \in W$. Suppose* *for simplicity that $f(W) = [0,1]$ and $f(p) < f(p')$. Moreover, for some gradient-like vector field X for f, suppose that*

$$K_p = W^s(p;X) \cup W^u(p;X) \tag{4.1}$$

and

$$K_{p'} = W^s(p';X) \cup W^u(p';X). \tag{4.2}$$

are disjoint. Then there exists another Morse function $g : W \to [0,1]$ satisfying the following properties:

(1) X is a gradient-like vector field for g,

(2) $g(p) = a$ and $g(p') = a'$ for some $a, a' \in [0,1]$,

(3) $f - g$ is a constant function in an open neighborhood of $\{p, p'\}$, and

(4) g coincides with f near $\partial_- W \cup \partial_+ W$.

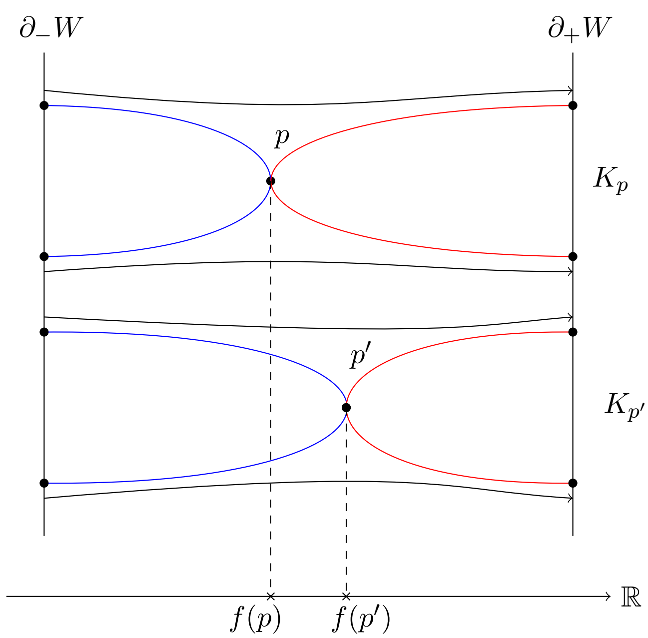

Figure 4.1: Rearrangement of critical values.

Proof. Note first that $\partial_- W = f^{-1}(0)$ and $\partial_+ W = f^{-1}(1)$, and let

$$S_l(p) = \partial_- W \cap W^s(p; X) \text{ and } S_l(p') = \partial_- W \cap W^s(p'; X) \tag{4.3}$$

denote the left-hand spheres of p and p', respectively.

Observe that all integral curves outside $K = K_p \cup K_{p'}$ proceed from $\partial_- W$ to $\partial_+ W$. For each $x \in W - K$ consider the uniquely determined integral curve $\varphi : W \times [0,1] \to W$ such that

$$\varphi(x, 0) = x. \tag{4.4}$$

Then let $\tau : M \to \mathbb{R}$ so that $\varphi(x, \tau(x)) \in \partial_- W$. So, by the *Immersion Theorem* the map $\pi : W - K \to \partial_- W$ that assigns $x \in W - K$ the unique point $\varphi(x, \tau(x))$ is well-defined and smooth. Moreover, if x is near K, then $\pi(x)$ is near K as well.

Now let $\mu : \partial_- W \to [0,1]$ be a smooth function defined by

$$\mu(x) = \begin{cases} 0 \text{ if } x \in U_0 \\ 1 \text{ if } x \in U_1, \end{cases} \tag{4.5}$$

where U_0 and U_1 are small open neighborhoods of x such that $U_0 \cap S_l(p') \neq \emptyset$ and $U_1 \cap S_l(p) \neq \emptyset$. Using $\pi : W - K \to \partial_- W$, μ extends to a smooth function $\tilde{\mu} : W \to [0,1]$ defined by

$$\tilde{\mu}(x) = \begin{cases} 0 \text{ if } x \in \tilde{U}_0 \\ 1 \text{ if } x \in \tilde{U}_1 \\ k \text{ elsewhere }, \end{cases} \tag{4.6}$$

where $k \in \mathbb{R}$ is a constant, and $\tilde{U}_0$ and $\tilde{U}_1$ are small open neighborhoods of x such that $\tilde{U}_0 \cap K_p' \neq \emptyset$ and $\tilde{U}_1 \cap K_p \neq \emptyset$.

Now define a smooth function $G : [0,1] \times [0,1] \to [0,1]$ with the following properties:

(1) For all x and y, $\frac{\partial G}{\partial x} > 0$ and G increases from 0 to 1 as x increases from 0 to 1.

(2) $G(f(p), 0) = a$ and $G(f(p'), 1) = a'$,

(3) $G(x,y) = x$ for x near 0 and 1 and for all y,

(4) $\frac{\partial G}{\partial x}(x,0) = 1$ for x near $f(p)$, and

(5) $\frac{\partial G}{\partial x}(x,1) = 1$ for x near $f(p')$.

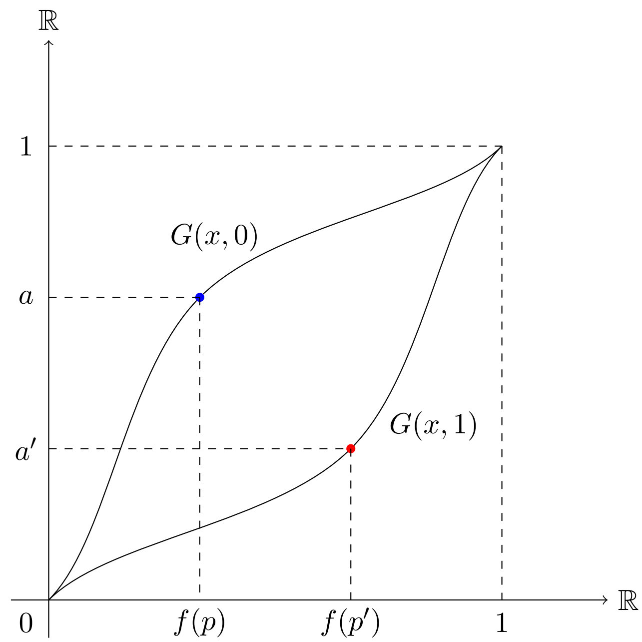

Figure 4.2: The graph of G.

Define a smooth function g by $g(x) = G(f(x), \tilde{\mu}(x))$. Observe that $f - g$ is constant near p and p' by properties (4) and (5). Thus X is also a gradient-like vector field for g as well near the critical points p and p'. By property (1) X is a gradient-like vector field for g away from critical points. Property (2) shows that $g(p) = a$ and $g(p') = a'$. Moreover by property (3) $g = f$ near the boundaries $\partial_- W$ and $\partial_+ W$. Therefore g is a Morse function with the desired property. $\square$

Corollary 33. *Suppose that the Morse function f has k critical points $p_1, \ldots, p_k$ such that*

$$\{p_1, \ldots, p_l\} \subset f^{-1}(b)$$

and

$$\{p_{l+1}, \ldots, p_k\} \subset f^{-1}(b'),$$

where $\{p_1, \ldots, p_l\}$ and $\{p_{l+1}, \ldots, p_k\}$ have indices λ and λ', respectively. Let $p = \{p_1, \ldots, p_l\}$ and $p' = \{p_{l+1}, \ldots, p_k\}$ for simplicity. Then the Theorem 32 is still valid.

4.2 Rearrangement of Spheres

Definition 34 (Product Neighborhoods)**.** *Let M be an m dimensional manifold and N an r dimensional submanifold of M. Then an open neighborhood U of N, which is diffeomorphic to $N \times \mathbb{R}^{m-n}$, is called a product neighborhood of N in M.*

Lemma 35. *Let N and N' be n and n' dimensional submanifolds of a manifold M of dimension m, respectively. Suppose that N has a product neighborhood in M and that $n + n' < m$. Then there exists some diffeomorphism $h : M \to M$ such that h is smoothly isotopic to the identity $\mathrm{id}_M : M \to M$ and $h(N) \cap N' = \emptyset$.*

Proof. Let $k : N \times \mathbb{R}^{m-n} \to U \subset M$ be a diffeomorphism so that $k(N \times 0) = N$ for $0 \in \mathbb{R}^{m-n}$. Moreover let $N_0 = U \cap N$, $\pi : N \times \mathbb{R}^{m-n} \to \mathbb{R}^{m-n}$ a canonical projection, and $g = \pi \circ k^{-1}|N_0 : N_0 \to \mathbb{R}^{m-n}$. Observe that $k(N \times x) \cap N' \neq \emptyset$ if and only if $x \in g(N_0)$. And if N_0 is nonempty, then $\dim N_0 = n < m - n'$. By Sard's Lemma $g(N_0)$ has the Lesbegue measure 0. So choose $u \in \mathbb{R}^{m-n} - g(N_0)$.

Now we construct a diffeomorphism h of M onto itself such that h is isotopic to the identity $\mathrm{id}_M : M \to M$ and $h(N)$ to $k(N \times u)$. So define a smooth vector field $X : \mathbb{R}^{m-n} \to \mathbb{R}^{m-n}$ such that

$$X = \begin{cases} u \text{ if } |x| \leq |u| \\ 0 \text{ if } |x| \geq 2|u| \end{cases} \tag{4.7}$$

for every $x \in \mathbb{R}^{n-m}$. By construction of the vector field, X has a compact support. Furthermore $\partial\mathbb{R}^{n-m} \neq \emptyset$. This implies that the integral curves $\psi(x,t)$ are defined for all $t \in [0,1]$. Then $\psi(x,0) = x = \mathrm{id}_M(x)$, $\psi(x,1)$ is a diffeomorphism that carries 0 to u, and thus $\phi : \mathbb{R}^{m-n} \times [0,1] \to \mathbb{R}^{m-n}$ is an isotopy from id_M to the diffeomorphism $\psi(x,1)$. This isotopy leaves all points fixed outside a bounded set U in $\mathbb{R}^{m-n}$, so define an isotopy $h_t : M \to M$ by

$$h_t(w) = \begin{cases} k(q, \psi(x,t)) \text{ if } w = k(q,x) \in U \\ w \text{ if } w = M - U \end{cases} . \tag{4.8}$$

Therefore $h = h_1$ is the desired diffeomorphism such that $h(N) \cap N' = \emptyset$. □

Theorem 36. *Suppose that $\lambda \geq \lambda'$ and let $h : W \to W$ be a diffeomorphism with the property given in Lemma 35. Then there exists a gradient-line vector field $\tilde{X}$ for f so that $h(S_r(p))$ is disjoint from $S_l(p')$ in some open neighborhood U of $V = f^{-1}(\frac{1}{2})$ and $\tilde{X}$ coincides with X outside U.*

Proof. Observe first that $S_r(p)$ has a product neighborhood in V. By Lemma 35 there exists a diffeomorphism $h : V \to V$ such that h is smoothly isotopic to the identity $\mathrm{id}_V : V \to V$ and $h(S_r(p)) \cap S_l(p') = \emptyset$. We use this diffeomorphism to construct new gradient-like vector field $\tilde{X}$. So choose $a < \frac{1}{2}$ so that $f^{-1}([a, \frac{1}{2}])$ does not contain p. Then the integral curves of $Y = \frac{1}{Xf}X$ determine a diffeomorphism $\varphi : [a, \frac{1}{2}] \to f^{-1}([a, \frac{1}{2}])$ such that $f(\varphi(q,t)) = t$ and $\varphi(\frac{1}{2}, q) = q \in V$. Now define a diffeomorphism $H : [a, \frac{1}{2}] \times V \to [a, \frac{1}{2}] \times V$ by $H(t,q) = (t, h_t(q))$, where h_t is a smooth map from $[a, \frac{1}{2}] \times V$ to V and isotopy from id_M to h such that

$$h_t = \begin{cases} \mathrm{id}_M \text{ for } t \text{ near } a \\ h \text{ for } t \text{ near } \frac{1}{2} \end{cases} . \tag{4.9}$$

Then $\tilde{Y} = (\varphi \circ H \circ \varphi^{-1})_* Y$ is a smooth vector filed defined on $f^{-1}([a, \frac{1}{2}])$, which coincides with Y near $f^{-1}(a)$ and V. Moreover $\tilde{Y} f = 1$ identically. Finally the vector field $\tilde{X}$ on W defined by

$$\tilde{X} = \begin{cases} (Xf)\tilde{Y} \text{ on } f^{-1}([a, \frac{1}{2}]) \\ X \text{ elsewhere} \end{cases} \tag{4.10}$$

is a new gradient-like vector field for f.

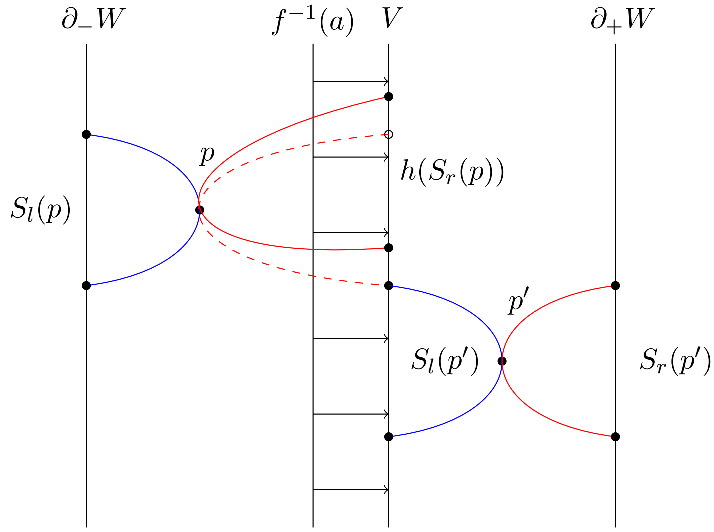

Figure 4.3: Construction of an isotopy to make $S_r(p)$ disjoint from $S_l(p')$.

Now for every fixed $q \in V$ $\varphi(t, h_t(q))$ is an integral curve of $\tilde{X}$ from $\varphi(a, q) \in f^{-1}(a)$ to $\varphi(\frac{1}{2}, h(q)) = h(q) \in V$. It follows that the right-hand sphere $\varphi(a \times S_r(p))$ in $f^{-1}(a)$ is carried to $h(S_r(p))$ in V. Thus $h(S_r(p)) \cap S_l(p') = \emptyset$. $\square$

Observe that Theorem 36 is generalized as follows. If c denotes a cobordism that admits a Morse function f with k critical points $p_1, \ldots, p_k$ of index λ, and c' a cobordism with l critical points $p'_1, \ldots p'_l$ of index λ' of f, then the new gradient-like vector field $\tilde{X}$ is constructed from X on a small open neighborhood of V such that right-hand spheres $S_r(p_i)$ and $S_l(p'_j)$ are disjoint by some isotopy of V.

Corollary 37. *If $\{p_1, \ldots, p_l\}$ and $\{p_{l+1}, \ldots, p_k\}$ are the sets of critical points of indices λ and λ', respectively, then the gradient-like vector field X can be changed so that $h(S_r(p_i)) \cap S_l(p'_j) = \emptyset$ for $i = 1, \ldots, l$ and $j = l+1, \ldots, k$.*

By Theorem 32 and 36, we should be able to construct a various types of isotopies to make two critical points have the same critical values, and to make spheres disjoint. So we can state the following theorem as a result.

Theorem 38 (Final Rearrangement Theorem)**.** *Any cobordism* $c = (W; \partial_- W, \partial_+ W)$ *factors into a composition*

$$c = c_0 c_1 \cdots c_m, \tag{4.11}$$

where $m = \dim W$*, and each* c_λ *admits a Morse function* f *possessing critical points of index* λ *with the same critical value.*

Chapter 5

First Cancellation Theorem

In this section we assume that a cobordism $(W; \partial_- W, \partial_+ W)$ has exactly two critical points p^λ and $p^{\lambda+1}$ in W of indices λ and $\lambda + 1$.

Next consider embedded submanifolds N and N' of M whose dimensions are n and n', respectively. Recall that N and N' intersect *transversely* if

$$T_p M = T_p N + T_p N'$$

for every $p \in N \cap N'$. In other words, two submanifolds N and N' intersect transversely if the tangent spaces at each point of the intersection span the tangent space of M.

Let N and N' be orientable submanifolds of M with dimensions r and s, respectively, so that $m = r + s$. Furthermore let f be a non-resonant Morse function defined on a cobordism $(W; \partial_- W, \partial_+ W)$ with two critical points p^λ and $p^{\lambda+1}$ of indices λ and $\lambda + 1$, respectively, such that $f(p^\lambda) < \frac{1}{2} < f(p^{\lambda+1})$. In the f-fiber $V = f^{-1}(\frac{1}{2})$, a gradient-like vector field X associated with f determines a right-hand sphere $S_r(p^\lambda)$ of p^λ and a left-hand sphere $S_l(p^{\lambda+1})$ of $p^{\lambda+1}$.

At the end of Chapter 4, Lemma 35 and 36 let us construct an isotopy of V to make two spheres $S_r(p)$ and $S_l(p')$ disjoint. In the similar way, we can define a isotopy of V so that those spheres have transverse intersection in V. Then we have the following theorem.

Theorem 39. *The gradient-like vector field X can be changed to another gradient-like vector field $\tilde{X}$ so that $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$ intersect transversely in V.*

Now observe that V is compact and

$$\dim S_r(p^\lambda) + \dim S_l(p^{\lambda+1}) = (m - \lambda - 1) + \lambda = m - 1 = \dim V. \tag{5.1}$$

So for each $q \in S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ there exists some local coordinate system $(x_1, \ldots, x_m)$ on an open neighborhood U of q in V such that q corresponds to $(0, \ldots, 0) \in \mathbb{R}^{m-1}$ and that

$$\begin{cases} x_1 = \cdots = x_\lambda = 0 \text{ on } U \cap S_r(p^\lambda) \\ x_{\lambda+1} = \cdots = x_{m-1} = 0 \text{ on } U \cap S_l(p^{\lambda+1}). \end{cases} \tag{5.2}$$

By this construction q a unique point contained in $S_r(p^\lambda) \cap S_l(p^{\lambda+1}) \cap U$. Therefore we should be able to assume that the intersection $S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ consists of finitely many points.

Theorem 40 (First Cancellation Theorem)**.** *Assume that $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$ intersect transversely in V and $S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ is a single point, $\{q\} = S_r(p^\lambda) \cap S_l(p^{\lambda+1})$. On an arbitrary small neighborhood U of the single integral curve $\varphi_q : \mathbb{R} \to W$ from p^λ to $p^{\lambda+1}$, a gradient-like vector field X can be altered to a nowhere zero vector field X' so that all integral curves proceed from $\partial_- W$ to $\partial_+ W$. Furthermore X' is a gradient-like vector field for another Morse function f' such that f' coincides with f near $\partial_- W \cup \partial_+ W$.*

Idea of Proof. Observe first by existence and uniqueness of the ODEs, there is a unique integral curve $\varphi_q : \mathbb{R} \to W$ such that $\varphi_q(0) = q$. Then φ_q has the following property: $\lim_{t \to -\infty} \varphi_q(t) = p^\lambda$ and $\lim_{t \to +\infty} \varphi_q(t) = p^{\lambda+1}$. Since the index of p^λ is λ by assumption, the gradient-like vector field X at p^λ is of the form

$$X_{p^\lambda} = -2x_1 \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} \cdots + 2x_m \frac{\partial}{\partial x_m} \tag{5.3}$$

on some open neighborhood U_p^λ of p^λ with a coordinate system $(x_1, \ldots, x_m)$. Now consider the appropriate coordinate change

$$(x_1, \ldots, x_\lambda, x_{\lambda+1}, \ldots, x_n) \mapsto (x_{\lambda+1}, \ldots, x_\lambda, x_1, \ldots, x_n). \tag{5.4}$$

Then X can be rewritten as

$$X = 2x_1 \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} - 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} \cdots + 2x_m \frac{\partial}{\partial x_m}. \tag{5.5}$$

Similarly, on some neighborhood $U_{p^{\lambda+1}}$ of $p^{\lambda+1}$, there is a coordinate system $(x'_1, \dots, x'_m)$ and X at $p^{\lambda+1}$ has the form

$$X = -2x'_1 \frac{\partial}{\partial x'_1} \cdots - 2x'_\lambda \frac{\partial}{\partial x'_\lambda} - 2x'_{\lambda+1} \frac{\partial}{\partial x'_{\lambda+1}} \cdots + 2x'_m \frac{\partial}{\partial x'_m}. \tag{5.6}$$

Then find a local coordinate system $(x_1, \dots, x_m)$ in a neighborhood U with the following two properties:

(1) The coordinates of p^λ and $p^{\lambda+1}$ correspond to $(0, 0, \dots, 0)$ and $(1, 0, \dots, 0)$, respectively, and

(2) X has the form

$$X = 2v_1(x_1) \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} - 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} + 2x_{\lambda+2} \frac{\partial}{\partial x_{\lambda+2}} + \cdots + 2x_m \frac{\partial}{\partial x_m} \tag{5.7}$$

on U, where $v_1 : [-2\delta, 1+2\delta] \to \mathbb{R}$ is a smooth scalar function defined as follows. For some small $\delta > 0$, v_1 is defined by

$$v_1(x_1) = \begin{cases} x_1 \text{ if } x_1 \in U_0, \\ 1 - x_1 \text{ if } x_1 \in U_1 \ , \end{cases} \tag{5.8}$$

where U_0 and U_1 are small open neighborhoods of 0 and 1, respectively. Also $v_1(x_1) > 0$ for $0 < x_1 < 1$.

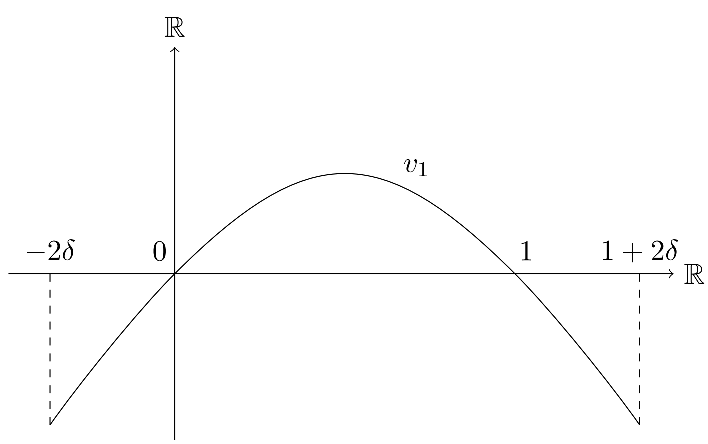

Figure 5.1: The graph of v_1.

By construction of the vector field, X in 5.7 coincides with 5.5 and 5.6 in open neighborhoods U_0 and U_1, respectively. Finally perturb the vector field X to construct a nowhere zero vector field $\tilde{X}$ in U. To do this consider the family of smooth scalar functions $\{v_{1-t} : [-2\delta, 1+2\delta] \to \mathbb{R}\}_{t\in[0,1]}$ with the following properties:

(1) $\{v_{1-t}\}_{t\in[0,1]}$ depends smoothly on t,

(2) v_1 is the function defined in Definition 5.8,

(3) $v_0 < 0$ on $[-2\delta, 1+2\delta]$, and

(4) $v_0(x_1) = v_1(x_1)$ if $x_1 < \frac{\delta}{2}$ or $x_1 > 1 + \frac{\delta}{2}$.

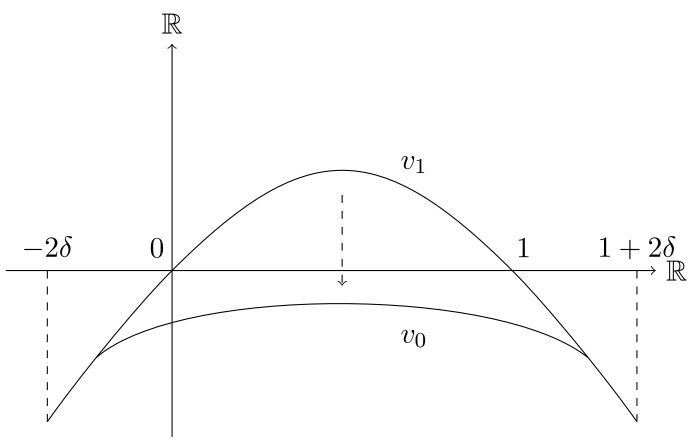

Figure 5.2: Deformation of v_1 to v_0.

This deforms v_1 to v_0 smoothly and the values of v_0 are all negative on $[-2\delta, 1+2\delta]$. Then define a vector field $\tilde{X} : W \to TW$ by

$$\tilde{X} = 2v_\rho(x_1)\frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} - 2x_{\lambda+1}\frac{\partial}{\partial x_{\lambda+1}} + 2x_{\lambda+2}\frac{\partial}{\partial x_{\lambda+2}} + \cdots + 2x_m\frac{\partial}{\partial x_m}, \tag{5.9}$$

where $\rho = x_2^2 + \cdots + x_n^2$. By its construction $\tilde{X}$ is a nowhere zero vector field on U, which coincides wtih X in 5.7 outside U. Moreover every integral curve φ proceeds from $\partial_- W$ to $\partial_+ W$. Hence $\tilde{X}$ and φ determine a smooth function $\tilde{f} : W \to \mathbb{R}$ such that $\tilde{f}$ coincides with f outside U and $\frac{d\tilde{f}}{dt}(\varphi) = \tilde{X}\tilde{f}$. Thus $\tilde{f}$ is a Morse function with no critical points in W. $\square$

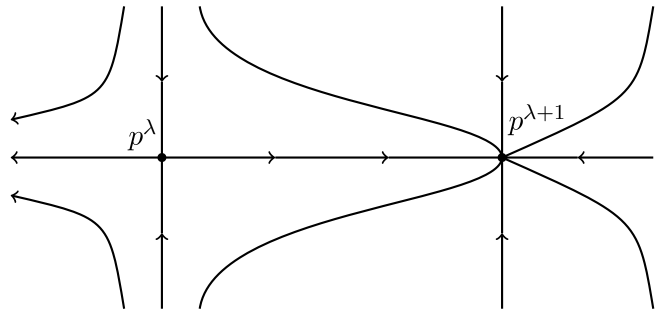

Figure 5.3: The two critical points p^λ and $p^{\lambda+1}$ in the vector field $\tilde{X}$.

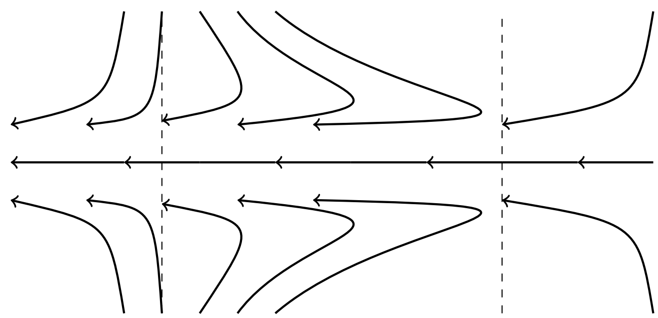

Figure 5.4: Cancellation of critical points p^λ and p^λ.

Chapter 6

Second Cancellation Theorem

Recall first from the *Whitney embedding theorem* that every compact smooth manifold M can be embedded in some Euclidean space $\mathbb{R}^n$. See Milnor [**?**, Milnor1997]or the proof.

Definition 41 (Normal Bundles)**.** *Suppose that an* m*-dimensional manifold* M *is embedded in* $\mathbb{R}^n$ *for some* n*. Then* $v \in \mathbb{R}^n$ *is called perpendicular to* M *at* $p \in M$ *if* $X_p \cdot v = 0$ *for all* $X_p \in T_pM$*, where* $\cdot$ *is the inner product defined on* $\mathbb{R}^n$*. Moreover the normal bundle* $\nu(M)$ *of* M *in* $\mathbb{R}^n$ *is defined by*

$$\nu(M) = \{(p, v) \in M \times \mathbb{R}^{n-m} | v \text{ is perpendicular to } M \text{ at } p\}. \tag{6.1}$$

For an m-dimensional smooth manifold M, let N and N' be submanifolds of dimension r and s, respectively, such that $r + s = m$. Suppose that N and N' intersect in finitely many points $p_1, \ldots, p_k \in M$, transversely. Suppose also that M and the normal bundle $\nu(N')$ of N' in M are both oriented. Because

$$T_{p_i}M = T_{p_i}N \oplus T_{p_i}N'$$

at each p_i, where $T_{p_i}N$ has a positively oriented r-frame $\langle \zeta_1, \ldots, \zeta_r \rangle$ of linearly independent vectors generating $T_{p_i}N$, $\langle \zeta_1, \ldots, \zeta_r \rangle$ is a basis for the fiber at p_i of $\nu(N')$.

Definition 42 (Intersection Numbers)**.** *The sign of intersection* $\epsilon(p_i)$ *at each* p_i *is defined to be either* $+1$ *or* -1 *according to a positively or negatively oriented basis for the fiber at* p_i *of* $\nu(N')$*.*

And the intersection number $\langle N\rangle \cdot \langle N'\rangle$ *is defined by*

$$\langle N\rangle \cdot \langle N'\rangle = \sum_{i=1}^{k} \epsilon(p_i).$$

Note that for an orientable manifold M every submanifold N of M is orientable if and only if $\nu(N)$ of N is orientable. Moreover, given an orientation for N, we have a canonical way to orient $\nu(N)$ and *vice versa*.

Lemma 43. *Let a manifold* M *and its submanifold* N' *be both compact and connected without boundary. Then there exists a isomorphism*

$$\psi : H_0(N') \to H_r(M, M - N').$$

We will use the lemma above without proof since it is easy to verify. Then the following theorem is based on the *Thom Isomorphism Theorem* and *Tubular Neighborhood Theorem*. Readers are encouraged to consult Kosinski [9] for the detail.

Lemma 44. *For the sequence*

$$H_r(N) \xrightarrow{i_*} H_r(M) \xrightarrow{j_*} H_r(M, M - N'), \tag{6.2}$$

where j_* *and* i_* *are both induced by the inclusion map,* $i_* \circ j_*(\langle M\rangle) = \langle N'\rangle \cdot \langle N\rangle\psi(\alpha)$.

Theorem 45. *Let* N *and* N' *be smooth closed submanifolds of dimensions* r *and* s*, respectively, such that* N *and* N' *intersect transversely in the smooth closed* $(r+s)$*-dimensional manifold* M*. Suppose that* N *and the normal bundle* $\nu(N')$ *in* M *are both oriented. Moreover suppose that* $r + s \geq 5$*,* $r \geq 3$*, and suppose that the inclusion* $i : M - N \hookrightarrow M$ *induces the injective map* $i_* : \pi_1(M - N) \to \pi_1(M)$ *if* $s = 1$ *or* $s = 2$.

Let $p, q \in N \cap N'$ *be a pair of intersection points with opposite intersection numbers such that there exists some loop* γ *connecting* p *and* q *so that* γ *is contractible in* M*. Suppose that* γ *does not contain any other intersection points in* $N \cap N' - \{p, q\}$.

Under the assumption given above, there exists some isotopy $h_t : M \to M$*, where* $t \in [0, 1]$*, such that*

(1) $h_0 = \mathrm{id}_M : M \to M$,

(2) h_t *fixes* id_M *near* $N \cap N' - \{p, q\}$, *and*

(3) $h_1(N) \cap N' = N \cap N' - \{p, q\}$.

Theorem 45 lets us construct an isotopy of M such that a pair of intersection points with opposite intersection numbers is cancelled. See Milnor[6] for the detail construction of such an isotopy.

Theorem 46 (Second Cancellation Theorem)**.** *For a cobordism* $(W; \partial_- W, \partial_+ W)$*, suppose* W*,* $\partial_- W$*, and* $\partial_+ W$ *are simply connected,* $\lambda \geq 2$*, and* $\lambda + 1 \leq n - 3$*. If* $\langle S_r(p^\lambda)\rangle \cdot \langle S_l(p^{\lambda+1})\rangle = \pm 1$*, then* X *can be altered near the fiber* V *so that* $S_r(p^\lambda)$ *and* $S_l(p^{\lambda+1})$ *in* V *intersect at a single point, transversely. Then the First Cancellation Theorem applies and* W *is therefore diffeomorphic to* $\partial_- W \times [0, 1]$*.*

Proof. Observe first that $\dim S_r(p^\lambda) = n - \lambda - 1$, $\dim S_l(p^{\lambda+1}) = \lambda$, and $\dim V = n - 1$. So $\dim S_r(p^\lambda) \geq 3$ and $\dim V = \lambda - 1$. Moreover $\pi_1(V) = 0$ by the *Seifert-van Kampen Theorem*. Then consider the following two cases.

Case 1: $\lambda \geq 3$. Then $\dim S_l(p^{\lambda+1}) \geq 3$ and $\dim V \geq 6$. Then the assumptions of Theorem 45 are satisfied.

Case 2: $\lambda = 2$. Then $\dim S_l(p^{\lambda+1}) = 2$ and $\dim V \geq 5$. Since $\{x \in W | a < f(x) < b\}$, where $a = f(\partial_- W - S_l(p^{\lambda+1}))$ and $b = f(V - S_r(p^\lambda))$, does not contain any critical points, the gradient-like vector field X for f determines a diffeomorphism between $\partial_- W - S_l(p^{\lambda+1})$ and $V - S_r(p^\lambda)$. Thus $\pi_1(V - S_r(p^\lambda)) \cong \pi_1(\partial_- W - S_l(p^{\lambda+1}))$. Now let U be a product neighborhood of S_l in $\partial_- W$. Since $\dim S_l(p^{\lambda+1}) = n - \lambda - 1 \geq 3$, $\pi_1(U - S_l(p^{\lambda+1})) \cong \mathbb{Z}$. Moreover there is a diagram for the fundamental groups given below. Thus $i_* : \pi_1(V - S_r(p^\lambda)) \to \pi_1(V)$ induced by the inclusion $i : V - S_r(p^\lambda) \hookrightarrow V$ is injective.

So for both cases the assumptions of Theorem 45 are satisfied. Then the *First Cancellation Theorem* directly applies if $S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ consists of a single point, for $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$

intersect transversely at a singleton in V. So suppose that $S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ is not a single point. Since $\langle S_r(p^\lambda)\rangle \cdot \langle S_l(p^{\lambda+1})\rangle = \pm 1$, the number of intersection points is odd, say $2k+1$. Then there are pairs of intersection points $\{p_1, q_1\}, \ldots,$ and $\{p_k, q_k\}$ so that $\epsilon(p_i) = +1$ and $\epsilon(q_i) = -1$ for $i = 1, \ldots, k$. By Theorem 45, each pair $\{p_i, q_i\}$ can be eliminated by isotopies of W. Thus W can be deformed so that $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$ have a transverse intersection at a single point in V. This completes the proof.

$$\begin{tikzcd}
 & \pi_1(V) = \mathbb{Z} & \\
\pi_1(V - S_l(p^{\lambda+1})) \arrow[ur] & & \pi_1(N) = \mathbb{Z} \arrow[ul] \\
 & \pi_1((V - S_l(p^{\lambda+1})) \cap N) = \mathbb{Z} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Figure 6.1: A diagram of fundamental groups induced by inclusions.

$\square$

Chapter 7

Cancellation of Critical Points of Indices λ with $2 \leq \lambda \leq m-2$

7.1 A Chain Complex and Homology of Manifolds

We focus our attention on homology groups with integer coefficients. So suppose first that M is an m dimensional compact smooth and orientable manifold with boundary ∂M. Observe that the orientation of M given by an orientation of its tangent bundle TM corresponds to an orientation of M specified its orientation generator $\langle M \rangle$ of $H_n(M)$.

Lemma 47. *Let M be an oriented closed smooth manifold of dimension λ embedded in $\partial_- W$ and $\langle M \rangle \in H_\lambda(M)$ the orientation generator. Let $i_* : H_\lambda(M) \to H_\lambda(W, \partial_+ W)$ be the map induced by the inclusion h :. Then*

$$i_*(\langle M \rangle) = (\langle S_r(p_1) \rangle \cdot \langle M \rangle)\langle D_l(p_1) \rangle + \cdots + (\langle S_r(p_l) \rangle \cdot \langle M \rangle)\langle D_l(p_l) \rangle. \tag{7.1}$$

Corollary 48. *With respect to the basis $\{\langle D_l(p_1) \rangle, \ldots, \langle D_l(p_l) \rangle\}$, the boundary map $\partial : H_{\lambda+1}(W \cup W', W) \to H_\lambda(W, \partial_- W)$ for the triple $\partial_- W \subset W \subset W \cup W'$ is a linear map, i.e. ∂ is represented by the $k \times l$ matrix A whose entries are $a_{ij} = \langle S_r(p_i) \rangle \cdot \langle S_l(p'_j) \rangle$.*

$$\begin{tikzcd}
 & & H_\lambda(S_l(p'_j)) \arrow[d, "i_*"'] \\
H_{\lambda+1}(W \cup W', W) \arrow[r] \arrow[drr, bend right] \arrow[ddrr, bend right, "\partial"'] & H_{\lambda+1}(W', \partial_+ W) \arrow[r] & H_\lambda(\partial_+ W) \arrow[d] \\
 & & H_\lambda(W) \arrow[d] \\
 & & H_\lambda(W, \partial_- W)
\end{tikzcd}$$

By Theorem 38 any cobordism $c = (W; \partial_- W, \partial_+ W)$ decomposes into a composition of c_λ, *i.e.*

$$c = c_0 c_1 \cdots c_m$$

such that each c_λ has a Morse function f with critical points of index λ on the fiber $f^{-1}(a)$ for some $a \in \mathbb{R}$. Or equivalently, we can think of a Morse function $f : W \to \mathbb{R}$ with the property called *self-indexing*, *i.e.* $f(p^\lambda) = \lambda \in \mathbb{Z}_{\geq 0}$ for every critical point p^λ of index λ. Let $W_k = f^{-1}([-\frac{1}{2}, k + \frac{1}{2}])$ and $V_{k+} = f^{-1}(k + \frac{1}{2})$ for a non-negative integer k. So setting $W_{-1} = \partial_- W$, we have the sequence

$$\partial_- W = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_m = W.$$

Now let $C_\lambda = H_\lambda(W_\lambda, W_{\lambda-1})$ and $\partial : C_\lambda \to C_{\lambda-1}$ the boundary homomorphism defined in Corollary 48, and consider the long exact sequence of the triple $(W_{\lambda+1}, W_\lambda, W_{\lambda-2})$

$$\cdots \to H_{\lambda+1}(W_{\lambda+1}, W_\lambda) \to H_\lambda(W_\lambda, W_{\lambda-2}) \to H_\lambda(W_{\lambda+1}, W_{\lambda-2}) \to 0$$

and another exact sequence of the triple $(W_\lambda, W_{\lambda-1}, W_{\lambda-2})$

$$0 \to H_\lambda(W_\lambda, W_{\lambda-2}) \to H_\lambda(W_\lambda, W_{\lambda-1}) \to H_{\lambda-1}(W_{\lambda-1}, W_{\lambda-2}) \to \cdots.$$

Observe that there is a commutative diagram of pairs of spaces given by two triples given above. Then define a λ-th homology group by

$$H_\lambda(W_\lambda, W_{\lambda-1}) = \ker \partial / \mathrm{im}\ \partial, \tag{7.2}$$

and it is well known that $C_* = (C_\lambda, \partial)$ is a chain complex and $H_\lambda(C_*) = H_\lambda(W, \partial_- W)$ for every λ with $0 \le \lambda \le m$.

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
C_{\lambda+} = H_{\lambda+1}(W_{\lambda+1}, W_\lambda) \arrow[r] \arrow[dr, bend right, "\partial"] & H_\lambda(W_\lambda, W_{\lambda-2}) \arrow[r] \arrow[d] & H_\lambda(W_{\lambda+1}, W_{\lambda-2}) \arrow[r] & 0 \\
 & C_\lambda = H_\lambda(W_\lambda, W_{\lambda-1}) \arrow[d, "\partial"] & & \\
 & C_{\lambda-1} = H_{\lambda-1}(W_{\lambda-1}, W_{\lambda-2}) & &
\end{tikzcd}$$

7.2 Cancellation of Critical Points in the Middle Dimensions

Construct a Morse function $\tilde{f}$ as follows. Using Theorem 32 perturb the Morse function f so that $\tilde{f}$ coincides with f outside a small open neighborhood U_1 of p_1, $\tilde{f}(p_1) > f(p_1)$, and $\tilde{f}$ has the same critical points $p_1, \ldots, p_k$ and the gradient-like vector field as f. Now let $t \in \mathbb{R}$ such that $\tilde{f}(p_1) > t > f(p_1)$, and let $V = \tilde{f}^{-1}(t)$.

The left-hand $(\lambda-1)$-sphere $S_l(p_1)$ and the right-hand $(m-\lambda-1)$-spheres $S_r(p_i)$ for $2 \le i \le k$ in V are disjoint. Let $a \in S_l(p_1)$ and $b \in S_r(p_2)$. Then since W is connected, V is connected and hence there exists some embedding $\iota : (0,3) \to V$ such that the image $\iota(0,3)$ intersects with $S_l(p_1)$ and $S_r(p_2)$ exactly once, transversely, in $a = \iota(1)$ and $b = \iota(2)$ respectively, and such that $\iota(0,3) \cap S_r(p_i) = \emptyset$ for $i = 3, \ldots, k$. Then extend the embedding ι as follows. See Milnor [6] for details.

Lemma 49. *There is an embedding $\tilde{\iota} : (0,3) \times \mathbb{R}^{\lambda-1} \times \mathbb{R}^{m-\lambda-1} \to V$ such that*

(1) $\tilde{\iota}(t, 0, 0) = \iota(t)\ \forall t \in (0,3)$,

(2) $\tilde{\iota}^{-1}(S_l(p_1)) = 1 \times \mathbb{R}^{\lambda-1} \times 0$, $\iota^{-1}(S_r(p_2)) = 2 \times 0 \times \mathbb{R}^{m-\lambda-1}$, and

(3) $\mathrm{im}\ \tilde{\iota} \cap S_r(p_i) = \emptyset$ *for* $i = 3, \ldots, k$. *Moreover* ι *maps* $1 \times \mathbb{R}^{\lambda-1} \times 0$ *into* $S_l(p_1)$ *with positive orientation and* $\iota((0,3) \times \mathbb{R}^{\lambda-1} \times 0)$ *intersects* $S_r(p_2)$ *at* $\iota(2,0,0) = b$ *with intersection number* $+1$.

Theorem 50 (Basis Theorem)**.** *On a cobordism* $(W; \partial_- W, \partial_+ W)$ *of dimenstion* m, *let* $f : W \to \mathbb{R}$ *be a Morse function with* k *critical points* $p_1, \ldots,$ *and* p_k *such that*

$$f(p_1) = \cdots = f(p_k), \tag{7.3}$$

and let X *be a gradient-like vector field for* f. *Suppose that all critical points are of the same index* λ *with* $2 \leq \lambda \leq m-2$. *Moreover suppose that* W *is connected. Then for every basis for* $H_\lambda(W, \partial_- W)$ *there exists another Morse function* f' *and another gradient-like vector field* X' *for* f' *with the following properties:*

(1) f' *and* X' *both coincide with* f *and* X *respectively in some open neighborhood* U *of* $\partial_- W \cup \partial_+ W$,

(2) f' *has the same critical points* $p_1, \ldots, p_k$ *with* $f'(p_1) = \cdots = f'(p_k)$, *and*

(3) the suitably oriented left-hand disks $D_l(p_1), \ldots, D_l(p_k)$ *forms a basis for* $H_\lambda(W, \partial_- W)$.

Proof. Let $\{b_1, \ldots, b_k\}$ be a basis for $H_\lambda(W, \partial_- W) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (k summands), where each b_j is homologous to the left-hand disk $D_l(p_j)$ with some fixed orientation. Then let $\nu(D_r(p_1)), \ldots, \nu(D_r(p_k))$ denote the normal bundles so that

$$\langle D_r(p_i) \rangle \cdot \langle D_l(p_j) \rangle = \delta_{ij}, \tag{7.4}$$

where δ_{ij} denotes the Kronecker delta sign. So for every λ-disk D embedded in W such that $D \subset \partial_- W$ its representation is given by

$$\alpha_1 b_1 + \cdots + \alpha_k b_k \in H_\lambda(W, \partial_- W) \tag{7.5}$$

for some integers $\alpha_1, \ldots, \alpha_k$. So D is homologous to $\alpha_1 D_l(p_1) + \cdots + \alpha_k D_l(p_k)$ and

$$\langle D_r(p_i)\rangle \cdot \langle D\rangle = \langle D_r(p_i)\rangle \cdot (\alpha_1\langle D_l(p_1)\rangle + \cdots + \alpha_k\langle D_l(p_k)\rangle) \tag{7.6}$$

$$= \alpha_1\langle D_r(p_i)\rangle \cdot \langle D_l(p_1)\rangle + \cdots + \alpha_k\langle D_l(p_k)\rangle \cdot \langle D_l(p_k)\rangle \tag{7.7}$$

$$= \sum_{j=1}^{k} \alpha_j \langle D_r(p_i)\rangle \cdot \langle D_l(p_j)\rangle \tag{7.8}$$

$$= \sum_{j=1}^{k} \alpha_j \delta_{ij} \tag{7.9}$$

$$= \alpha_i. \tag{7.10}$$

So $D = (\langle D_r(p_i)\rangle \cdot \langle D\rangle)b_1 + \cdots + (\langle D_r(p_k)\rangle \cdot \langle D\rangle)b_k$.

Now construct a Morse function f' and a gradient-like vector field X' such that $D_l'(p_1)$, $D_l(p_2)$, …, and $D_l(p_k)$ are the new left-hand disks with

$$\langle D_r(p_1)\rangle \cdot \langle D_l'(p_1)\rangle = 1,\ \langle D_r(p_2)\rangle \cdot \langle D_l'(p_1)\rangle = 1, \tag{7.11}$$

and

$$\langle D_r(p_i)\rangle \cdot \langle D_l'(p_1)\rangle = 0 \tag{7.12}$$

for $i = 3, \ldots, k$. This implies that $\{b_1 + b_2, b_2, \ldots, b_k\}$ is the new basis for $H_\lambda(W, \partial_- W)$. In order to verify this elementary row operation of the basis, we will need to construct an isotopy as followos.

For some fixed $\delta > 0$ let $\alpha : \mathbb{R} \to [1, \frac{3}{2}]$ be a smooth function such that

$$\alpha(x) = \begin{cases} \frac{9}{4} \text{ if } x \leq \delta \\ 1 \text{ if } \text{ if } x \geq 2\delta. \end{cases} \tag{7.13}$$

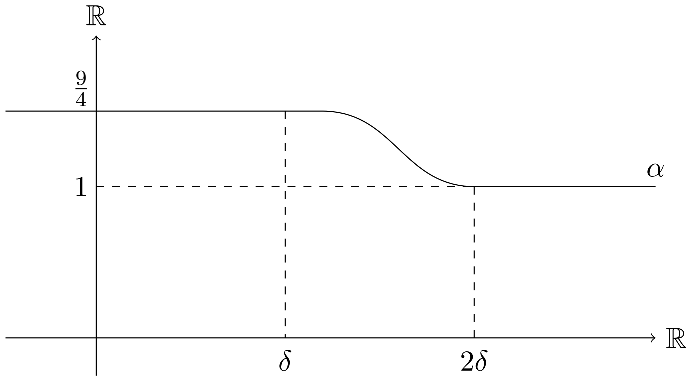

Figure 7.1: The graph of α.

Then construct a smooth isotopy $H_t : (0,3) \times \mathbb{R}^{\lambda-1} \times \mathbb{R}^{m-\lambda-1} \to (0,3) \times \mathbb{R}^{\lambda-1} \times \mathbb{R}^{m-\lambda-1}$ such that

(1) $H_t = \mathrm{id}$ outside some compact set, where $0 \le t \le 1$.

(2) $H_t(1, x, 0) = (t\alpha(\|x\|^2) + (1-t), x, 0)$ for $x \in \mathbb{R}^{\lambda-1}$.

Now define a new isotopy F_t of V by

$$F_t(v) = \begin{cases} \tilde{\imath} \circ H_t \circ \tilde{\imath}^{-1}(v) \text{ for } v \in \operatorname{im} \tilde{\imath} \\ v \text{ otherwise} \end{cases} . \tag{7.14}$$

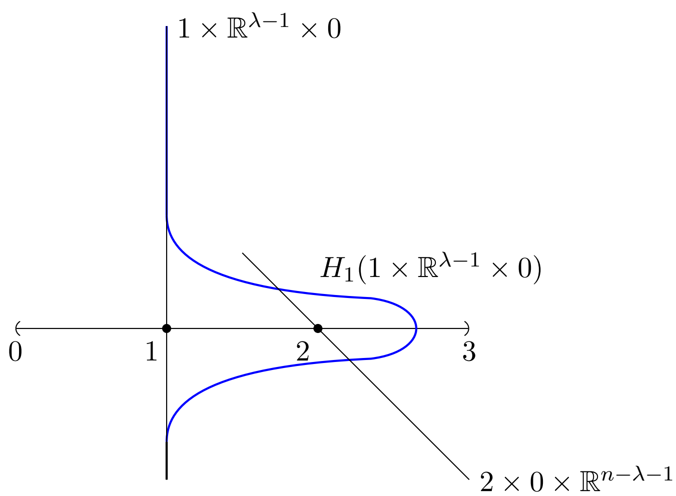

Figure 7.2: The graph of H_t.

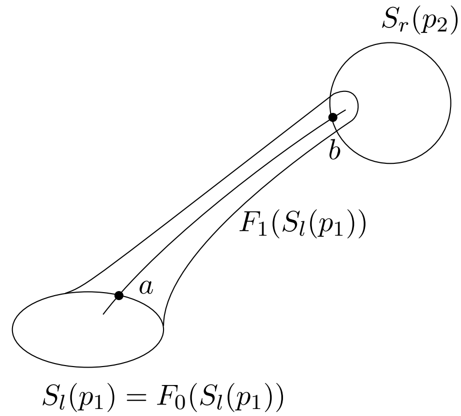

Figure 7.3: Deformation by the isotopy F_t.

$\square$

Theorem 51. *For a cobordism $(W; \partial_- W, \partial_+ W)$ of dimension $m \geq 6$, let f be a Morse function without any critical points of indices $0, 1, m-1$, or m. Suppose that W, $\partial_- W$, and $\partial_+ W$ are all simply connected. Moreover suppose that $H_*(W, \partial_- W)$ is trivial. Then $(W; \partial_- W, \partial_+ W)$ is a product cobordism.*

Proof. Let c denote the cobordism $(W; \partial_- W, \partial_+ W)$. By the Final Rearrangement Theorem it follows that c decomposes into the factors $c = c_2 c_3 \cdots c_{m-2}$, and there exists a Morse function f on c such that each restriction $f|c_\lambda$ contains all critical points of index λ with the same critical value. Then consider the sequence of the following chain complex

$$C_{m-2} \xrightarrow{\partial} C_{m-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_{\lambda+1} \xrightarrow{\partial} C_\lambda \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_2. \tag{7.15}$$

For each λ choose a basis $\{z_1^{\lambda+1}, \ldots, z_{k_{\lambda+1}}^{\lambda+1}\}$ for the cycle $\ker \partial$ of $\partial : C_{\lambda+1} \to C_\lambda$. Because $H_*(W, \partial_- W)$ is trivial by assumption, the sequence given above is exact and there exist k_λ elements $b_1^{\lambda+1}$, $\ldots$, and $b_{k_\lambda}^{\lambda+1}$ in C_λ such that $\partial(b_i^{\lambda+1}) = z_i^\lambda$ in $C_{\lambda+1}$ for $i = 1, \ldots k_\lambda$. Thus $\{z_1^{\lambda+1}, \ldots, z_{k_{\lambda+1}}^{\lambda+1}, b_1^{\lambda+1}, \ldots, b_{k_\lambda}^{\lambda+1}\}$ is a basis for $C_{\lambda+1}$.

Since $2 \leq \lambda \leq \lambda + 1 \leq m - 2$ by assumption, by Basis Theorem there exist a Morse function f' and gradient-like vector field X' defined on c such that the left-hand disks contained in c_λ and $c_{\lambda+1}$ are the bases for C_λ and $C_{\lambda+1}$, respectively.

Now let $p \in c_\lambda$ and $q \in c_{\lambda+1}$ be the critical points that correspond to the basis elements z_1^λ and $b_1^{\lambda+1}$. Perturb f' in open neighborhoods U_p of p and U_q of q so that

$$f(p^\lambda_{i_\lambda}) < f(p) < f(q) < f(p^{\lambda+1}_{i_{\lambda+1}}), \tag{7.16}$$

where $p^\lambda_{i_\lambda}$ and $p^{\lambda+1}_{i_{\lambda+1}}$ are the critical points of indices λ and $\lambda+1$, respectively. Then $c_\lambda c_{\lambda+1}$ factors into $c'_\lambda c_p c_q c'_{\lambda+1}$, where both c_p and c_q are elementary cobordisms containing critical points p and q, respectively. Let $t \in \mathbb{R}$ such that $f(p) < t < f(q)$ and let $W' = c_p c_q$. Then consider the fiber $V = f^{-1}(t)$. Observe that all W', $\partial_- W'$, and $\partial_+ W'$ are simply connected. Since $z_1^\lambda = \partial(b_1^\lambda)$ by construction of ∂, $S_r(p)$ and $S_l(q)$ have the intersection number ± 1, *i.e.* $\langle S_r(p)\rangle \cdot \langle S_l(q)\rangle = \pm 1$. By the Second Cancellation Theorem, the critical points p and q in $c_p c_q$ are eliminated and $c_p c_q$ is a product cobordism. Repeating this process and hence all critical points in the cobordism c are eliminated. Thus $(W; \partial_- W, \partial_+ W)$ is a product cobordism. $\square$

Chapter 8

Cancellation of Critical Points of index 0 and 1

8.1 Index 0 Cancellation

We first eliminate all critical points of 0 from a cobordism. The idea is to find out a transverse intersection between right-hand and left-hand spheres of critical points of indices 0 and 1, respectively, so that the First Cancellation Theorem applies.

Theorem 52. *Suppose that $H_0(W, \partial_- W) = 0$. Then the critical points of index 0 are cancelled against the same number critical points of index 1.*

Proof. Consider the homology groups with coefficients from $\mathbb{Z}_2 = \{0, 1\}$. Since $H_0(W, \partial_- W) = 0$, the boundary homomorphism $\partial : H_1(W_1, W_0) \to H_0(W_0, \partial_- W)$ is surjective. Observe that ∂ is given by the matrix whose (i, j)-th entries are

$$\langle S_r(p_i^0) \rangle \cdot \langle S_l(p_j^1) \rangle \mod 2, \tag{8.1}$$

where p_i^0 and p_j^1 denote critical points of indices 0 and 1, respectively. Observe that every right-hand $(m-1)$-sphere $S_r(p_i^0)$ has at least one left-hand 0-sphere $S_l(p_j^1)$ such that $\langle S_r(p_i^0) \rangle \cdot \langle S_l(p_j^1) \rangle \not\cong 0$ under modulo 2 because ∂ is surjective. $S_r(p_i^0) \cap S_l(p_j^1)$ cannot contain more than two points and thus it consists of an odd number of points. Therefore $S_r(p_i^0)$ and $S_l(p_j^1)$ intersects transversely in a single point and the First Cancellation Theorem applies. $\square$

8.2 Index 1 Cancellation

We will need the two lemmas introduces below to prove Theorem 56, which eliminates critical points of index 1.

Lemma 53. *For $0 \leq \lambda < m$, there exists a smooth function $f : \mathbb{R}^m \to \mathbb{R}$ such that*

(1) $f(x_1, \ldots, x_m) = x_1$ outside of a compact set, and

(2) f has exactly two non-degenerate critical points p^λ and $p^{\lambda+1}$ of indices λ and $\lambda+1$ respectively with $f(p^\lambda) < f(p^{\lambda+1})$.

Proof. Identify $\mathbb{R}^n$ with $\mathbb{R} \times \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda-1}$ and let (x, y, z) be a point in the product. For simplicity let y^2 and z^2 denote $\|y\|^2$ and $\|z\|^2$, respectively. Moreover let $s : \mathbb{R} \to \mathbb{R}$ be a smooth function with compact support such that $s(x) + x$ has exactly two non-degenerate critical points a_1 and a_2.

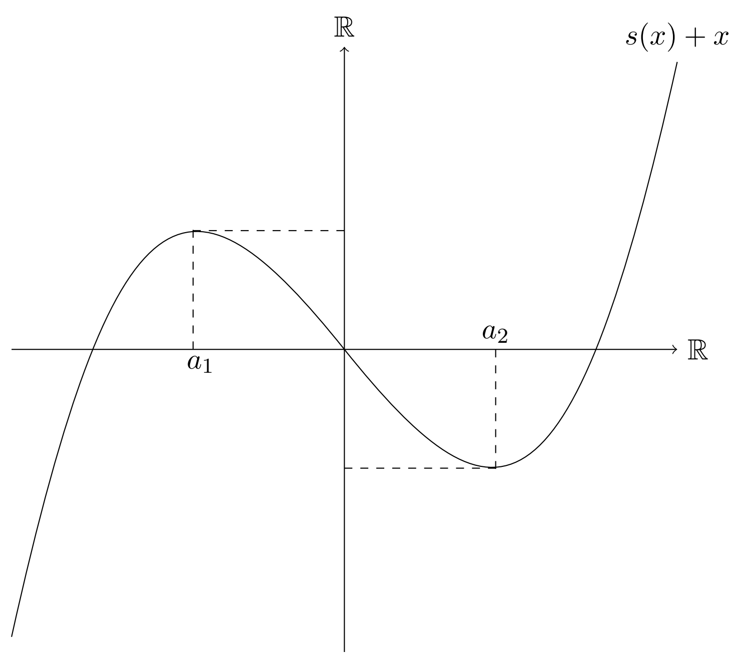

Figure 8.1: The graph of $s(x) + x$.

Next construct smooth functions $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with compact supports so that

(1) $\alpha(x) = 1$ if $|x| \leq 1$,

(2) $|\alpha'(x)| < \frac{1}{\sup|s(x)|}$ for all x,

(3) $\beta(x) = 1$ if $\alpha(x) \neq 0$,

(4) $\gamma(x) = 1$ if $s'(x) \neq 0$, and

(5) $|\gamma'(x)| < \frac{1}{\sup(x\beta(x))}$.

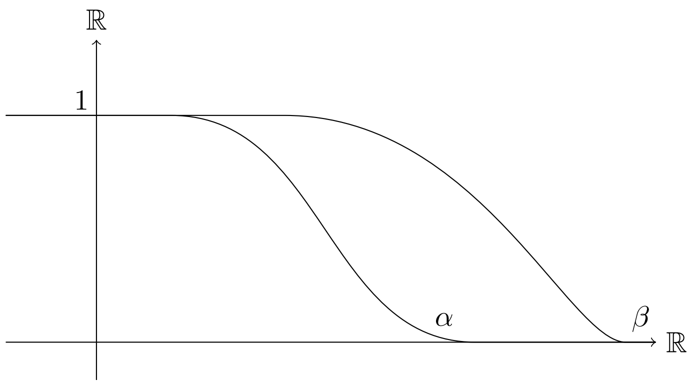

Figure 8.2: The graphs of α and β.

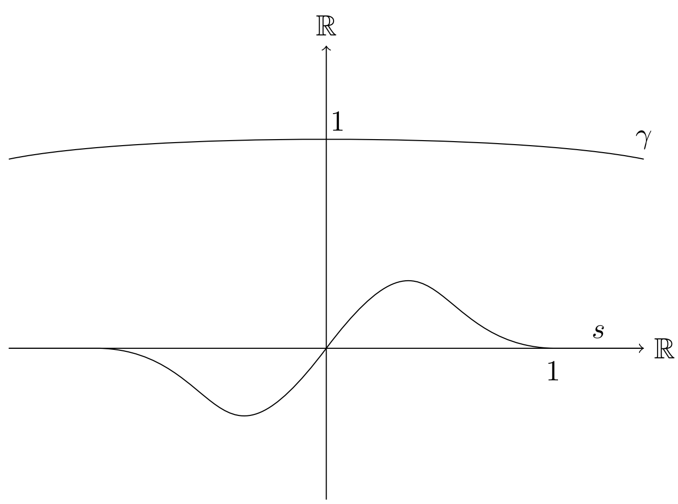

Figure 8.3: The graphs of s and γ.

Define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x, y, z) = x + s(x)\alpha(y^2 + z^2) + (-y^2 + z^2)\gamma(x)\beta(-y^2 + z^2)$. Observe by construction of f that

(1) $f(x) - x$ has compact support,

(2) On the open interval such that $\alpha = 1$ and $\gamma = 1$, f corresponds to $s(x) + x$.

Observe also that

$$\frac{\partial f}{\partial x} = 1 + s'(x)\alpha(y^2 + z^2) + (-y^2 + z^2)\gamma'(x)\beta(-y^2 + z^2) \neq 0 \tag{8.2}$$

if $s'(x) \neq 0$ or $\gamma'(x) \neq 0$. Observe also that the gradient of f vanishes only if $y = 0$ and $z = 0$. In this case $\alpha = 1$, and therefore $f(x)$ reduces to $s(x) + x$. □

Lemma 54. *If S_r^{m-2} is a right-hand sphere in V_{1+}, then there exits a 1-sphere embedded in V_{1+} such that it intersects with S_r^{m-2} transversely but does not meet with any other right-hand spheres in V_{+1}.*

Proof. Choose a sufficiently imbedded 1-disk $D^1 \subset V_{1+}$ so that D^1 intersects with S_r^{m-2} at the midpoint q of D^1 and that D^1 does not intersect with any other right-spheres in V_{1+}. Then translate the end points a and b of D^1 left along the integral curves φ and φ' of the gradient-like vector field X to $\varphi(a)$ and $\varphi'(b)$ in V_{0+}. $V_{0+} = \partial_- W$ is connected and of dimension $m - 1 \leq 2$, there exists a smooth path γ so that γ joins $\varphi(a)$ and $\varphi'(b)$, and that γ does not intersect with any left-hand 0-spheres in V_{0+}. Then $\varphi^{-1} \circ \gamma$ is a smooth path joining a and b in V_{0+} and avoids all right-hand spheres. Note that $\dim V_{0+} = m - 1 \geq 3$, and by Theorem define a smooth function $g : S^1 \to V_{1+}$ by the following properties:

(1) $g^{-1}(q) = a \in S^1$ and g embeds smoothly a closed set A containing a onto some neighborhood of q in D.

(2) $g(S^1 - a)$ does not intersect with any right-hand $(m - 2)$-spheres.

This completes the proof of theorem. □

Theorem 55. *Let M and N be smooth manifolds of dimension m and n such that $n \leq 2m+3$. If two smooth embeddings i and j of M into N are homotopic, then i and j are smoothly isotopic.*

See Whitney [14] for the proof of the theorem.

Theorem 56. *Suppose that W and $\partial_- W$ are simply connected, and $m \geq 5$. Moreover suppose that $(W; \partial_- W, \partial_+ W)$ has no critical points of index 0. Then for each critical point p_1 of index 1 there exist a pair of critical points q_2 of auxiliary index 2 and p_3 of index 3 such that p_1 is cancelled against q_2.*

Proof. Note first that V_{2+} is simply connected, *i.e.* $\pi_1(V_{2+}) = \pi_1(W) = 0$. Note also from generalization of Theorem 30 that

$$\mathcal{D}_r^{m-1} \cup \mathcal{D}_r^{m-2} \cup V_{2+} \cup \mathcal{D}_l^3 \cup \mathcal{D}_l^4 \cdots \cup \mathcal{D}_l^m, \tag{8.3}$$

where $\mathcal{D}_r^\lambda$ and $\mathcal{D}_l^\lambda$ denote the collection of λ-disks attached to V_{2+}, is a deformation retract of W.

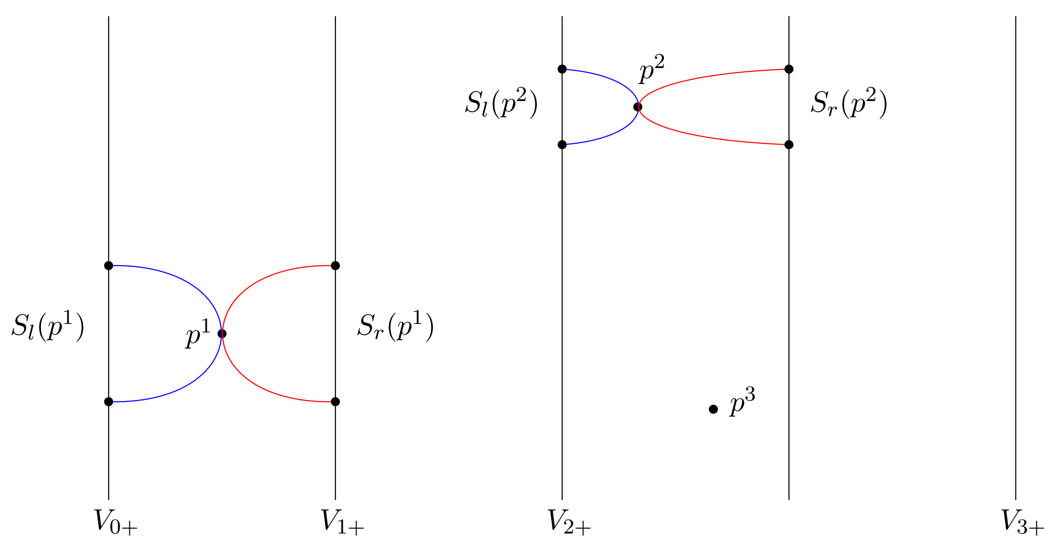

Figure 8.4: Auxiliary critical points p^2 and p^3.

Now for every critical point p^1 construct an 1-sphere $S(p^1)$ embedded in V_{1+} by Lemma 54. Moreover by Theorems 36 the gradient-like vector field X can be adjusted so that $S(p^1)$ does not intersect with any left-hand spheres in V_{1+}. So we can translate $S(p^1)$ right to V_{2+}. Extend a collar neighborhood to the right of V_2. Apply the *Implicit Function Theorem* to choose an embedded open set $U \subset \mathbb{R}^m$ and some coordinate system $(x_1, \ldots, x_m)$ such that $f(x_1, \ldots, x_m)|U = x_m$. Then by Lemma 53 construct another Morse function $\tilde{f}$ such that $\tilde{f}$ coincides with f outside U with extra non-degenerate critical points p^2 and p^3 of indices 2 and 3, respectively.

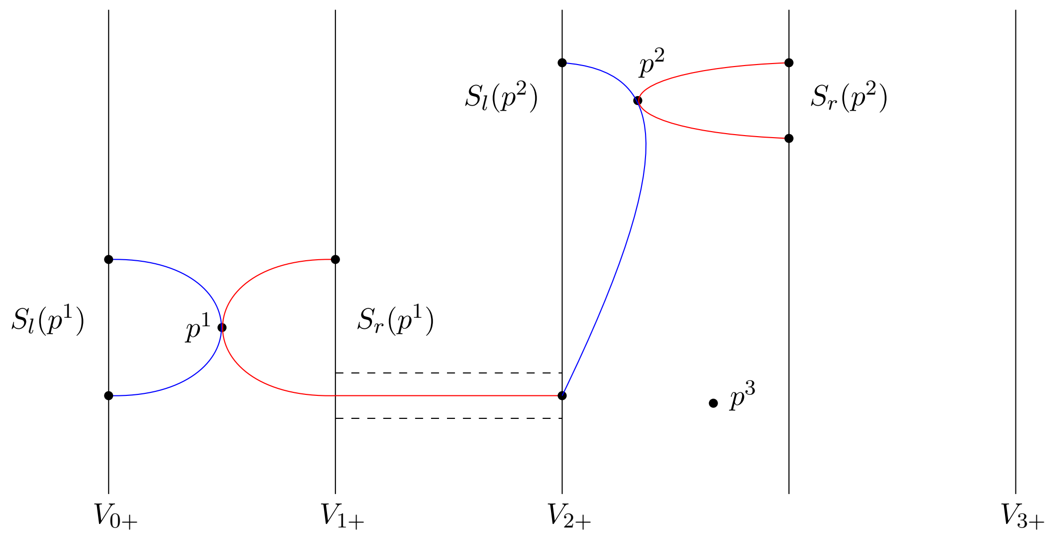

Figure 8.5: Auxiliary critical points p^2 and p^3.

Denote by $S_l(p^2)$ the left-hand sphere of p^2 in V_{2+}, and construct a smooth isotopy $h : V_{2+} \to V_{2+}$ such that $h(S_l(p^2)) = S(p^1)$. Then adjust X so that $S_l(p^2)$ and $S(p^1)$ coincide. Observe by construction that $S_l(p^2)$ intersects with $S(p)$ in a single point transversely. Therefore the *First Cancellation Theorem* applies and two critical points p^1 and p^2 are eliminated. Then perturb f so that p^3 is on the fiber $f^{-1}(3) \subset V_{3+}$, and this completes the proof of the theorem. $\square$

Chapter 9

The h-Cobordism Theorem

In Chapter 5, 6, and 7, we studied how to eliminate pairs of critical points of consecutive indices. Moreover Chapter 8 made us cancel critical points of indices 0 and 1. Finally we should claim that the following main theorem so called the *h-Cobordism Theorem* holds true.

Theorem 57 (The h-Cobordism Theorem)**.** *Let* $(W; \partial_- W, \partial_+ W)$ *be a cobordism such that*

(1) All W*,* $\partial_- W$*, and* $\partial_+ W$ *are simply connected.*

(2) $H_*(W, V) = 0$*.*

(3) $\dim W \geq 6$*.*

Then W *is diffeomorphic to* $\partial_- W \times [0, 1]$ *and thus a product cobordism.*

Proof. Let a Morse function $f : W \to \mathbb{R}$ be self-indexing, *i.e.* $f(p^\lambda) = \lambda \in \mathbb{Z}$ for each non-degenerate critical point $p^\lambda \in W$. By Theorem 52 and 56 all critical points of indices 0 and 1 are eliminated. Then consider the function $-f$. It is easy to observe that $-f$ is also a Morse function that contains the exactly same critical points with opposite indices. In other words a critical point p^λ of index λ of f is of index $m - \lambda$ of $-f$. So all critical points of indices m and $m - 1$ are also cancelled. Finally apply Theorem 51 and other critical points are all eliminated. $\square$

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