SHARPENING INTONATION: APPLYING JUST AND PYTHAGOREAN TUNING SYSTEMS FOR COLLEGIATE VIOLINISTS

by

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Violinists of all levels perpetually confront the onerous challenge of playing with accurate intonation. Movements as small as millimeters on the fingerboard can significantly alter pitch precision, which students dichotomously simplify as either correct or incorrect. Yet, notes on the violin do not exist in such an uncompromising manner. Pitch is strikingly flexible, allowing for an infinite spectrum of sonorities that are not as readily accessible on other instruments, such as the piano, where note frequencies are uniformly produced. Combined with the violin's ability to play more than one tone simultaneously, students quickly find the endeavor to play in tune to be frustrating and insurmountable.

Violin instructors are guilty of inadvertently instilling a fixed sense of pitch. A common scenario involves the teacher asking the student to isolate an erroneous note, determine whether it is too sharp or flat, and then move it until they are satisfied. At best, the student realizes their tendency for a particular note – such as playing an $F^{\#}$ too low – and will play the note slightly higher or lower than they did only for this one moment, but without a definable goal. At worst, the student assumes that they always play that $F^{\#}$ too low and will overcorrect by playing *every* $F^{\#}$ higher, regardless of its context. Unlike pianists and other fixed-pitch instrumentalists, where an $F^{\#}4$ would sound exactly the same if it were the leading tone of a G Major scale or the major third of a D Major triad, violinists have the capacity to alter notes to make them sound more pleasing either melodically or harmonically, but confusion often arises when music employs both concurrently. Violinists need a framework that specifies which notes to adjust, and, most importantly, how and when to adjust them.

A note in isolation cannot be defined as in-tune nor out-of-tune; a pitch's intonation is critically dependent on its context.[1] Harmony, both linearly and vertically, dictate intonation and the well-trained violinist is able to systematically accommodate either or both by employing Pythagorean and Just tuning systems. As advanced/early undergraduate violinists learn increasingly complex music that features both monophonic and homophonic textures, as in the Solo Sonatas and Partitas of J.S. Bach, students can quickly and methodically improve their intonation by discriminating between Pythagorean and Just intonation systems. The unequal steps of the Pythagorean scale lend itself to heightened expressivity in horizontal melodies, whereas the pure, mathematically simple intervals of a Justly tuned scale enable maximum resonance when playing chords; students must realize that these two systems are incompatible.

As a violinist who shares the lifelong struggle to improve intonation, I have sought a concise, specific, and practicable process to apply to both my own performance and teaching; framing intonation around Pythagorean and Just systems has proven to be the most concrete and practical. Yet, my experiences in studio lessons, chamber music, and orchestral playing have seldom specified the critical role that these two systems play in perceiving intonation. Chamber music has come the closest, with instructors addressing key facets of intonation such as resonance, open string tuning, and melodic inflection, but the instructional language has been inconsistent and imprecise, making it difficult to transfer these skills to different contexts. Intonation is, evidently, a revolving topic in string research but there is a similar absence – and need – for a tangible method to improve intonation. Students long for a map, or even a "cheat-sheet" that shows how to play more in tune.

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[1] Benjamin Whitcomb, "Improving Intonation," *ASTA* 57, no.4, (November 2007): 42, https://doi.org/10.1177/000313130705700406.

Benjamin Whitcomb and Michael Kimber are two string pedagogues who explicitly cite and compare the Pythagorean and Just systems, but the isolated examples used in their research do not provide enough of a repeatable process for students to easily extrapolate to their own practice.[2] However, their breakdown of the two systems in terms of their construction and relationship to string instruments forms a sufficient baseline to codify how to approach intonation. My research consolidates the numerical analyses of tuning systems by pedagogues such as Whitcomb and Kimber and reframes it into an accessible strategy for students to efficiently and consciously improve intonation.

While they are arguably the most relevant and applicable to violin playing, the Pythagorean and Just systems are not infallible systems. Moreover, there is not one system that is "perfectly" suited for any instrument for all contexts. Even Equal Temperament, often referred to as a compromise, is not a universal panacea to improve intonation. Understanding how these different systems divide the Western twelve-tone scale elucidates not only their utility and limitations, but also points to potential performance implications for violinists.

Overview and Construction of Tuning Systems

Around 550 BCE, the Greek philosopher Pythagoras experimented with a monochord, where a movable bridge was placed in between a single string fixed at two ends.[3] He discovered that dividing the string with the moveable bridge at certain points produced consistent ratios of the fundamental (open) string. With the string divided in half, each half would sound an octave higher than the open string, a ratio of 2:1. Likewise, placing the bridge at third of the length

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[2] Michael Kimber, "Teaching a Melodic Awareness of Intonation," *ASTA* 42, no. 4 (May 1992): 59-63; Benjamin Whitcomb, "Improving Intonation," 42-45.
[3] Donald A. Hodges, *Music in the Human Experience*, 2nd ed. (New York: Routledge, 2019), 91.

produces a pitch an octave plus a perfect fifth higher from the fundamental or a perfect fifth higher than the previous division (3:2) and a quarter length is two octaves higher from the open string or a perfect fourth higher from the previous (4:3); these rational divisions (1/2, 1/3, 1/4, 1/5…) form what is now recognized as the harmonic series.[4]

These tones – colloquially referred to as overtones, harmonics, or partials – are produced at varying strengths with any naturally occurring sound.[5] It is the unique spread of a tone's harmonics that distinguishes one instrument's timbre from another, like an auditory fingerprint. Moreover, the successive notes of the harmonic series define distinct intervals – the octave, perfect fifth, perfect fourth, major and minor third – and their "prevalence…in the lower part of the harmonic series contributed to the development of Western (European) harmony".[6] These intervals, which are arithmetic ratios of sound waves, are a natural acoustic phenomenon that are the foundation to Just intonation.

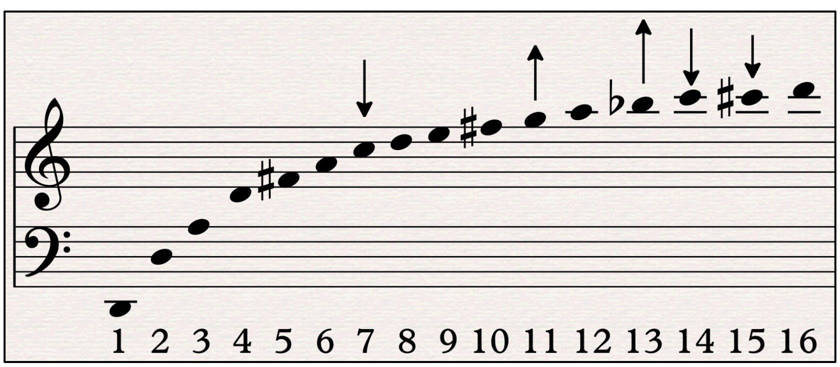

Figure 1. *Harmonic Series on the Fundamental Note 'D'.*

[4] Hodges, *Music in the Human Experience*, 91.
[5] The terms harmonic and partial are interchangeable and are numbered to include the fundamental. Overtones do not include the fundamental. The second harmonic/partial is the same as the first overtone.
[6] Ross W. Duffin, *How Equal Temperament Ruined Harmony (and Why You Should Care)*, (New York: W.W. Norton & Company, 2006), 14.

Violinists frequently utilize the harmonic series in a manner identical to Pythagoras's experimentation. With each of the four strings fixed at the nut and the bridge, the left hand functions as the moveable bridge that stops the string. Lightly touching the string at proportional divisions produces the notes in the harmonic series.

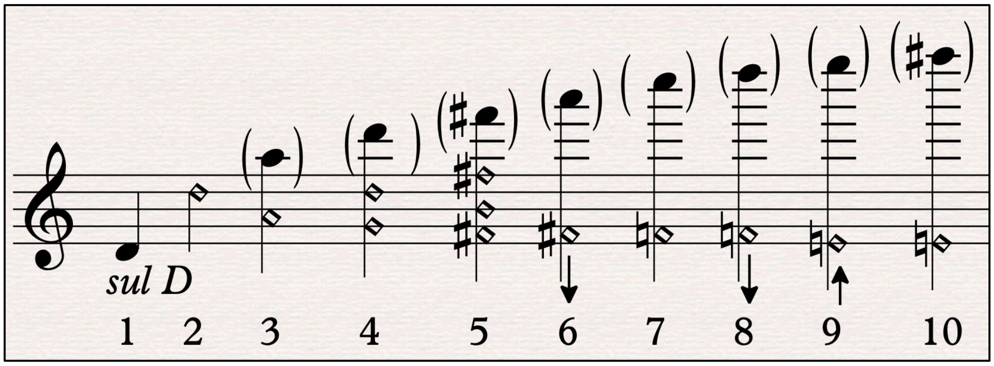

Figure 2. *Natural Harmonics on the D String of a Violin*.

Just as the lower-numbered harmonics are louder than those higher in the series with any given sound, the easiest harmonics to locate and play on the violin are those closest to the fundamental, namely the octave (half the string length), octave + fifth (third length), double octave (quarter length), and occasionally the double octave + major third (fifth length). Subsequent harmonics have more complex ratios and are both less resonant and consonant.

The most consonant harmonic (and simplest ratio) is the octave, at 2:1. The next simplest is the perfect fifth, 3:2, which is the first in the series to generate a new note. Given its pleasant sonority alongside its extensiveness throughout the early part of Western music, stacking acoustically pure fifths in succession would be a seemingly logical process to create a series of unrepeated notes.[7] As it turns out, doing so does indeed yield 12 unique pitches that divide an octave; this resulting scale, based on the 3:2 beatless fifth, is referred to as the Pythagorean scale.

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[7] Duffin, *How Equal Temperament Ruined Harmony*, 15.

After building perfect fifths one after another and placing them in within one octave, this scale, although appearing to have twelve equal chromatic pitches, has uneven steps (with noticeably narrow half-steps) and non-identical enharmonics (sharps are higher than enharmonic flats) – which may actually prove to be advantageous in monophonic contexts.[8] More problematically, a sequence of 12 fifths should return back to the starting pitch (e.g. D-A-E-B-F$^{\#}$-C$^{\#}$-G$^{\#}$-D$^{\#}$-A$^{\#}$-E$^{\#}$/F-C-G-D), but that twelfth fifth exceeds the original pitch. Comparing the ratios of seven perfect octaves ($(2/1)^7 = 128$) to twelve acoustically pure fifths ($(3/2)^{12} = 129.746$), Pythagorean fifths do not create a perfect circle of fifths, as it does in Equal Temperament. This difference is known as the Pythagorean comma, which Pythagoras understood but mainly disregarded since the range from the starting pitch (e.g. D1) to the end pitch (D7) was so wide and impractical at the time.[9] While this imperfection does not significantly impact violinists, it inevitably set forth a multitude of endeavors to correct this discrepancy, leading to different tuning systems.

Equal Temperament (popularized in the 18th century) accounts for the Pythagorean comma by narrowing each perfect fifth by 1/12 of the comma in order to return the series back to the starting pitch. Unfortunately, this not only undermines the purity of the perfect fifth (and perfect fourths, which are slightly wider than pure), but also ruins all other intervals except for the octave. Yet, the compositional advantage of having equal half-steps and identical enharmonics overshadowed the disparity of its intervals compared to the harmonic series. The perfect fifth in ET is not that far off from acoustical purity, being only about a fiftieth of a semitone flat (about 2 cents), but its major third (the next simplest ratio after the 4:3 perfect fourth) is markedly wider

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[8] Michael Kimber, "Teaching a Melodic Awareness of Intonation," *ASTA* 42, no. 4 (May 1992): 60, https://doi.org/10.1177/000313139204200220.
[9] Michael Halewood, "On Equal Temperament: Tuning, Modernity, and Compromise," *History of the Human Sciences* 28, no. 3 (February 2015): 11, https://doi.org/10.1177/0952695114567480.

than in Just intonation at "one-seventh of a semitone wider (14 cents) than acoustically pure 4:3 major thirds."[10] Pythagorean fifths are, of course, pure but its thirds deviate even further from Just intonation.

Interval	Pythagorean		Just		Equal Temperament
	Ratio	Decimal	Ratio	Decimal	Decimal
m2	256/243	1.053	16/15	1.067	1.059
M2	9/8	1.125	9/8	1.125	1.123
m3	32/27	1.185	6/5	1.200	1.189
M3	81/64	1.266	5/4	1.250	1.259
P4	4/3	1.333	4/3	1.333	1.335
TT	1024/729	1.405	7/5	1.400	1.414
P5	3/2	1.500	3/2	1.500	1.498
m6	128/81	1.580	8/5	1.600	1.587
M6	27/16	1.688	5/3	1.667	1.682
m7	16/9	1.778	16/9	1.778	1.782
M7	243/128	1.898	15/8	1.875	1.888
P8	2/1	2	2/1	2	2

Figure 3. *Comparison of Frequency Ratios of Pythagorean, Just, and ET Tuning.*[11]

The deviation of ratios of certain intervals of Pythagorean tuning and ET from the pure ratios is not inherently inferior, at least not when these intervals are played non-simultaneously. For instance, the perfect fifth in ET is only *slightly* narrower than pure and when played one note

[10] Duffin, *How Equal Temperament Ruined Harmony*, 20.
[11] Adapted from Whitcomb, "*Intonation on a String Instrument*" (2017), with several corrected ratios.

after another (e.g. D to A), the difference is imperceptible, but when played as a dyad, the impurity is audibly perceptible as a pulsing or beating – a faster pulsation indicates a greater difference in the ratio of the intervals which is perceived as less in-tune. Therefore, the ET perfect fifth, with both notes sounding simultaneously, is acceptably impure (a decimal of 1.498 vs 1.5, or about 2 cents wider than Just).[12] However, the thirds/sixths of both ET and Pythagorean are unmistakably discordant from Just tuning (see Figure 3), making them unsuitable to be played concurrently. These might not sound out-of-tune to an untrained ear – one that is accustomed to a piano and has not experienced the serenity of pure thirds. If two or more pitches are played together, the human ear strives for harmonic simplicity: the naturally occurring, small whole-number ratios found in Just intonation.[13] To summarize, improving intonation for chords – which are comprised of simultaneous intervals – is defined as bringing the notes closer to their simple ratios in order to remove beating (Just intonation). Notes played in succession – such as in monophonic melodies – are not strictly reliant on securing the purity of Just intervals because beating does not occur as audibly, meaning Pythagorean or ET tuning could be employed.

Meantone temperaments attempt to create as many pure thirds as possible by narrowing perfect fifths. The increasing use of thirds throughout the Renaissance until the late seventeenth century popularized these systems, with the quarter-comma meantone temperament being most favored (four consecutive fifths are narrowed by 1/4 of the difference between a Pythagorean and Just major third, known as the syntonic comma); other variations disperse the comma in smaller

[12] Cents are a unit that represents the difference between pitches (the human perception of frequencies). An ET semitone is 100 cents, and an octave comprises of 1200 cents.
[13] Derle Ray Long, "Coincidence Theory: Seeking a Perceptual Preference for Just Intonation, Equal Temperament, and Pythagorean Intonation in Excerpts for Wind Instruments" (PhD. diss., University of Southern Mississippi, 2008), 103, https://aquila.usm.edu/cgi/viewcontent.cgi?article=2179&context=dissertations

intervals across more fifths.[14] Because meantone temperaments result in unequal fifths and other intervals, they are decidedly not as practical for violinists who not only have a penchant for pure fifths (evident by the prevailing tuning of the open strings in pure or near-pure fifths), but can freely adjust intonation independent of the key (unlike keyboard instruments at the time). Nevertheless, the predilection of purely tuned thirds of these systems is worth valuing, intervals that are trivialized in Pythagorean tuning and ET.

Application to Violinists

Violinists should adhere to the following generalities: (1) double stops and chords are tuned using the pure ratios of Just intonation, (2) monophonic melodies, scales, and arpeggios are tuned with Pythagorean intonation, and (3) non-tonal repertoire and unisons with piano should use Equal Temperament.

Chords

Students are most likely to have unknowingly practiced in ET (using an electronic tuner) and Pythagorean intonation (instructors emphasizing high major thirds and sevenths, low minor thirds and sevenths), which are an augmentation of ET intervals – comparing the decimal values between the two systems (Figure 3) shows that the intervals are very closely aligned.[15] As such, when students encounter double stops, students will find that these two systems are incompatible with the more desirable pure intervals of Just intonation and will have to adjust these tendencies accordingly.

[14] Duffin, *How Equal Temperament Ruined Harmony*, 27.
[15] Benjamin Whitcomb, "Intonation on a String Instrument: Three Systems of Tuning and Temperament," *ASTA* 67, no. 2 (May 2017), 21, https://doi.org/10.1177/000313131706700215.

A classic exercise to demonstrate the variability of a single note when played in a double stop is to have students compare an E4 on the D string first with the open A string and then the open G string (Figure 4). The whole step from the open D (D4) to E4 is constructed using two perfect fifths (Pythagorean), D4 – A4 – E5, and using the E5 (open E) to match the fingered E4. This E4, when played with the open A as a double stop, creates a pure perfect fourth: in-tune! However, without relocating this E4, it is too sharp when played with the open G3; the player must significantly lower the E4 for it to remove its beating with the open G.

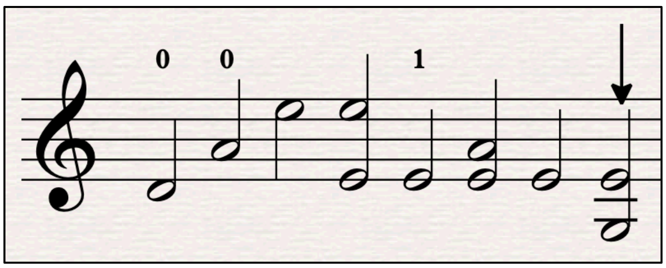

Figure 4. *Variability of the E4 in Two Double Stops*

Beyond indiscriminately moving a faulty note until it sounds pleasant, violinists can determine which direction to modify notes based on the disparity between Pythagorean/ET and Just intervals.

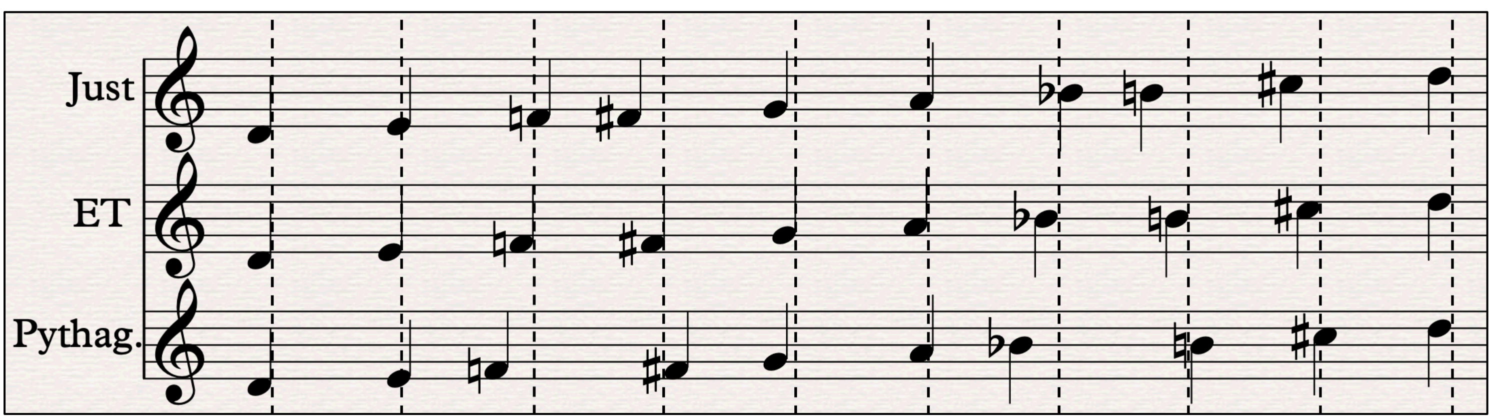

Figure 5. *Comparison of Just, ET, and Pythagorean Tuning in D.*

Figure 5 (using the values from Figure 3) juxtaposes the particularly egregious distances of ET and Pythagorean thirds and sixths compared to the purely tuned Just intervals. ET and Pythagorean major thirds and major sixths are wider than pure; minor thirds and minor sixths are narrower. Assuming that early collegiate violinists have been conditioned to place notes in a manner most consistent with ET or Pythagorean intonation, students must correct these habits by playing major thirds and major sixths *narrower* and, conversely, minor thirds and minor sixths *wider*. In the earlier case of the E4 played in a double stop with the open G, an astute violinist would identify this interval as a major sixth (using G as the bass and the key), deduce that a Justly tuned major sixth is narrower than ET or Pythagorean tuning, and appropriately lower the E. Moreover, an astute violinist should be prepared to adjust the E again if its harmonic context changes, such as if that E suddenly functions monophonically as a leading tone in F Minor which requires a departure from Just tuning.

Melodic Tuning

Renowned cellist Pablo Casals (1876-1973) is typically credited with exemplifying *expressive intonation*, where "leading notes lead" in order to intensify melodic direction. In other words, chromatic tendencies are exaggerated to clearly distinguish between major and minor sonorities: higher (wider) major intervals and lower (narrower) minor intervals.[16] Returning to Figure 3, this is plainly reflected in Pythagorean ratios, where minor intervals (i.e. minor thirds, sixths, and sevenths) are glaringly narrower than in Just intonation and the mirroring major intervals are wider. Violinists can extrapolate these proclivities by playing tighter half-steps in melodies: in a major scale, this would mean playing a higher major third to accentuate the mediant and subdominant scale degrees as well as a higher major seventh to highlight the leading-tone and

[16] Michael Kimber, "Teaching a Melodic Awareness of Intonation," 59.

the tonic while natural minor scales seek a lower minor third.

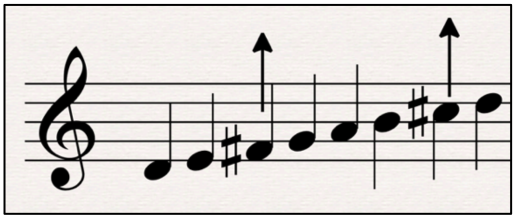

Figure 6.

Raised Major Third and Seventh in D

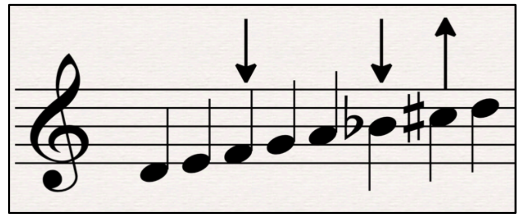

Figure 7.

Lowered Minor Third and Sixth, Raised Leading-Tone in D Harmonic

Especially in rapid, scalar passages, these exaggerated intervals not only increase perceived accuracy, but also function as a concrete and direct means of heightening tension as opposed to apathetically practicing scales or enforcing consistent left-hand frames and patterns without connecting them to melodic contexts.[17] Whereas Just intonation depends on local, vertical harmony, successful Pythagorean intonation is contingent on the ability to contextualize a group of horizontal notes into a scale.

Open Strings

While violinists now have a basic model for adjusting pitch based melodically or harmonically, some pitches are immovable in real-time performance, namely the four open strings. Furthermore, the open strings are susceptible to sympathetic vibration caused by overlapping harmonics of another note, also referred to as resultant, difference, or Tartini tones.[18] For instance, younger players are routinely taught to tune fingered notes such as a G, D, A, E by referencing the open string; if the fingered note is in tune, the corresponding open string will visibly vibrate (e.g. a fourth finger A on the D string will cause the adjacent open A string to

[17] David Sartiri, "Intonation Demystified," *ASTA* 59, no. 1 (February 2009), 42. https://doi.org/10.1177/000313130905900108.
[18] Sartiri, "Intonation Demystified," 41.

vibrate). What is actually occurring is that the first harmonic – and all harmonics, since this is a unison –of both the fingered note and the open string are aligned, resulting in enhanced resonance. If these harmonics are not in agreement, beating is produced due to incongruous ratios – out of tune.

Discrepancies between concurrent first partials are the easiest to diagnose as it is the strongest harmonic. Naturally, the same applies for other partials and reinforces the validity of Just intonation. Figure 8 depicts the partials of F4 and A4 (open A), a double stop that many students improperly tune. Both notes share an A6 in their harmonic series: the fifth harmonic for F and fourth for A. Because the open A (and its harmonics) are fixed, a violinist must place the F

Figure 8. *Aligning the Overlapping Harmonics of F and A.*

slightly higher (than in Pythagorean tuning or ET) to ensure that these harmonics match. Uncoincidentally, this accomplishes the same task of narrowing major thirds to satisfy Just intonation; in this case, the third itself (A4) cannot be lowered, thus the root (F4) must be raised.

The same process can be applied to all double stops where an open string is the third of a double stop: raised B^{b}/lowered B♮ + open D, raised F♮/lowered F$^{\#}$ + open A, and raised C♮/lowered C$^{\#}$ + open E. Any time – in nearly any context – these double stops appear in music, violinists should aim to follow these provisions to maximize resonance and, as such, intonation accuracy.

Paradoxically, violinists should not try to alter every possible note (when playing double stops) to resonate with open strings. One striking example is the double stop G4 + E5 on the D and A strings. Both of these notes could be tuned to match their respective open strings, but doing so would compromise the purity of the major sixth between the two notes. If the violin has been conventionally tuned in pure perfect fifths, the resulting major sixth is too wide; either the G should be raised (reducing the sympathetic vibration with the open G) or the E should be lowered (jeopardizing the open E). Melodic implications and preferences for a certain voice could influence a decision for one or the other (to be addressed in a forthcoming section), but both are not ideal if one overly relies on open strings for intonation, especially if the fingered notes in question can be played with an open string. Trade-offs have to be made to conform with Just intonation, melodic conscientiousness, or instrument resonance.

A violinist could intentionally lower the tuning of their E string to achieve the best of both worlds: both the fingered G4 and E5 conform with the harmonics of their respective open strings and the major sixth would be purely tuned, but at the cost of having an impure fifth between the open A and E strings. Recall that an ET fifth is slightly narrower than pure (Figure 3). Although violinists (and the vast majority of string players) revel in tuning their strings in perfect fifths, it is not uncommon to situationally tune strings in narrower fifths, closer to ET. This could be particularly advantageous when playing with piano and there are extended unison passages with the violin playing open strings – a violinist would sacrifice the objectively more in-tune perfect fifths of their open strings to match the "out-of-tune" fifths of the piano.

In string quartets, violinists regularly tune a narrower fifth between their A and E strings by lowering their E string to create a pure major third with a cellist's open C.[19] This is analogous to the adjustments made in meantone temperaments – stacking successive 3:2 fifths (Pythagorean)

[19] Timothy James Cuffman, "A Practical Introduction to Just Intonation Through String Quartet Playing," (DMA diss., University of Iowa, May 2016), 6.

creates too wide of a range for thirds to be pure, so some (not all) fifths are narrowed to make some thirds pure. Lowering the E string is still circumstantial; it accomplishes the one task of guaranteeing the pure third with an open C, which would be imperative if a quartet was playing a piece in C major and the final chord is the tonic triad (and an open E is unavoidable due to double stops). However, rehearsing a piece in F major (where C major chords functioning as a dominant harmony would be rife) may be problematic if an open E must serve as both a pure third of a C major triad (lower than ET) and as a leading tone to the tonic F (higher than ET).

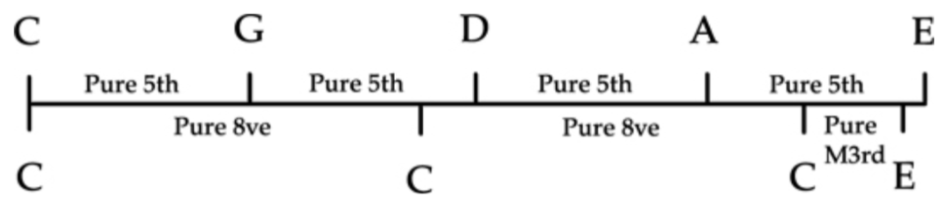

Figure 9. *Syntonic Comma, Produced by Comparing Four Pure Fifths with Two Octaves and a Pure Major Third.*[20]

Violinists must decisively consider the prevalence of harmonic and melodic implications when isolating certain notes and how these pitches can be manipulated by the tuning of the open strings, if they should be at all. Ultimately, students must recognize that specific notes can exist in more than one system and, more importantly, that they have the capability and obligation to decide which system to use.

Practice and Implementation

Just intonation can be practiced as soon as students begin playing double stops. Etude books such as Josephine Trott's *Melodious Double-Stops for Violin*, Book 1 (1925) and Rodolphe

[20] Duffin, *How Equal Temperament Ruined Harmony*, 25.

Kreutzer's *42 Studies* (1805) are indispensable for students as these provide a framework for students to practice listening to and playing in Just intonation. Trott's introductory exercises (#1-9) are particularly useful because all of the double-stops contain an open string, meaning students can identify the note tendencies with open strings described earlier.

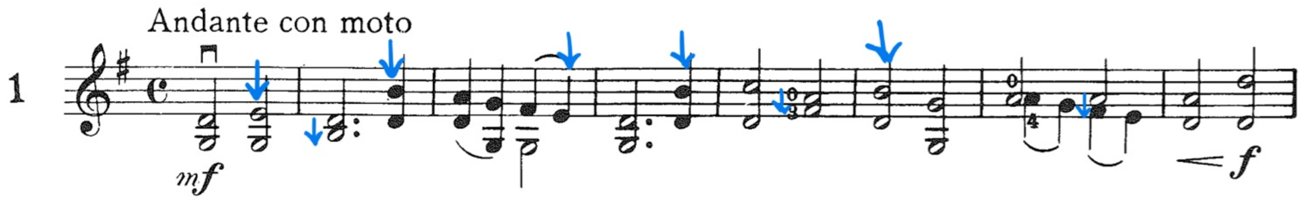

Figure 10. *Josephine Trott*, *Melodious Double-Stops, #1.*

Since pure intervals (especially thirds) are easiest to tune when there is a continuous reference pitch (i.e. the fundamental note), exercises such as Trott #17 or Kreutzer #34 are effective at drilling note tendencies based on intervallic structure. In Trott #17 (Figure 11), the placement of the F$^{\#}$4 is dictated by the open A string which functions as a drone for the measure. Since F$^{\#}$ is a major sixth in an A scale, it should be lowered. The next measure is slightly trickier because there is seemingly no open string as a fixed reference. However, the G4 can and should be tuned sympathetically to the open G (to maximize instrument resonance). Now with one note "solved" for the student, the droning B4 must be lowered to create a pure major third with the G. Likewise, in measure three, a student can decide to place the E4 to match their open E (which will conveniently create a perfect fourth with the A4 in that measure), thereby informing them to play a lower C$^{\#}$5 – a major sixth from E or a major third from A.

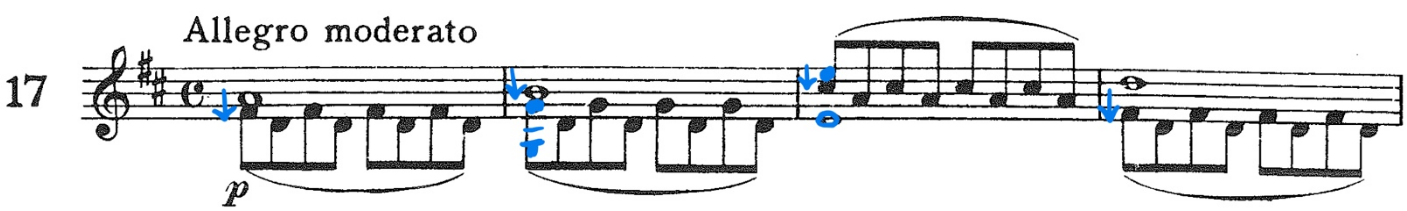

Figure 11. *Trott*, *Melodious Double-Stops, #17, mm. 1-4.*

Kreutzer #34 (Figure 12) is slightly more complex because there are three voices: bass (sounding predominantly on beats 1 and 3), an inner drone, and treble. If only the treble line or bass were played, Pythagorean intonation would be used. However, both of these voices must be tuned with the inner drone – at least for all consonant intervals (perfect octaves/fifths/fourths and all thirds and sixths). Following the procedure used for the Trott exercises, the D4 in the first measure is matched with the open D, causing the F$^{\#}$4 to be placed lower. Consequently, the B4 is lowered to create a perfect fourth with this F$^{\#}$, possibly uncomfortably low for a student who has always played this B in relation to the open E. In the second measure, tuning the G4 with the open G is advantageous for resonance which results in a lower C$^{\#}$ in the bass (students will want to play it too high since it is the leading tone in D major), higher B^{b}4 for a pure minor third, and lower E5 for a pure major sixth.

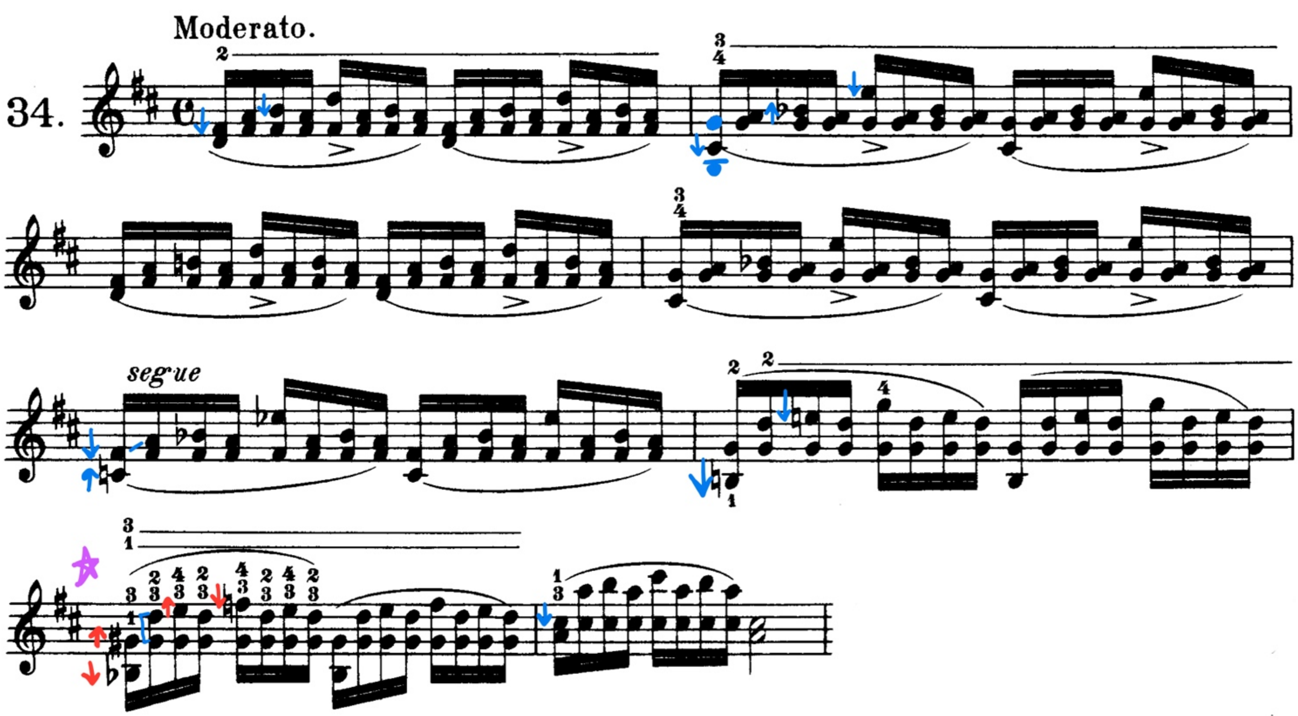

Figure 12. *Kreutzer, 42 Studies, #34, mm. 1-8.*

Measure seven presents a new complication: an augmented interval. While the B^b3 and $G^{\#}4$ would sound enharmonically as a minor seventh in ET, it should not be purely tuned as such because of the voice-leading tendencies of an augmented-sixth chord (all three types occur in this measure). To dramatize the harmonic implication of this chord, Pythagorean intonation can be substituted, resulting in a lower B^b (which resolves down to A) and a higher $G^{\#}$ (resolving up to A). The E5 and F5 could be tuned to match the higher $G^{\#}$, but it would not necessarily be discordant if these are played with a Pythagorean lens given the harmonic dissonance already occurring in this measure from the augmented-sixth. As violin students continue to progress into repertoire with multiple voices, they should expect to encounter situations like this one where they find that one system is more suitable than the other, and even conditions that enable brief interchangeability between Just and Pythagorean.

This invariably occurs throughout several movements of J.S. Bach's *Sonatas and Partitas for Solo Violin* (1720), a staple of the solo violin repertoire. Most, if not all, students are assigned to work on at least one sonata or partita during their undergraduate studies. Students tend to fear these pieces not only because of the exceptional technical and musical demands, but also because of how exposed intonation feels – there is no other instrument to hide behind. Fortunately, a student can transfer the tuning strategies practiced in their etudes to these pieces to systematically and effectively improve intonation by employing the Just and Pythagorean systems.

A striking example is the introduction of the first movement of Bach's Sonata No. 1 in G Minor for the violin (Figure 13). The first troublesome note is the B^b4 located in the chord and in the following descending scale. After tuning the Bb using just intonation to form a minor sixth with the open D and major sixth with the top G5 (that is tuned with the open G), the player must play a lower, “expressive” Bb in the scale as it now functions as a minor third in a monophonic, G Minor scale. The next potentially problematic note is the F$^{\#}$4 which is usually held as a diminished fifth with the C5. While the C is sustained, the F$^{\#}$ should be lowered, but since only the F$^{\#}$ is tied into the next beat, a violinist could choose to place the F$^{\#}$ higher in favor of its function as a leading-tone to the tonic G. The resulting tritone with the C$^{\#}$ would not be purely tuned, but this is arguably acceptable because tritones are already perceived as dissonant; an impurely tuned tritone is less “offensive” than an impure third. Its function as a leading-tone towards the end of the measure is indisputably Pythagorean however, as it resides in a monophonic scale.

Figure 13. *First Measure of Bach’s Sonata No. 1 in G Minor.*

In just the first measure, this process sets precedents for the violinist that echo some of the generalities outlined earlier: (1) notes that form thirds and sixths within a double-stop or chord that do not also have a dual function as an inflected note in a horizontal melody are tuned Justly, (2) inflected notes (usually scale degrees 3, 6 and 7) not played in a chord are tuned with Pythagorean ratios, (3) dissonant intervals in a double-stop or chord are not necessarily tuned Justly, and (4) notes that exist both in melodic and harmonic contexts can be tuned in either Just or Pythagorean, a decision that falls on the violinist. The first two principles provide a road-map, or even a loose answer-key, that will considerably aid a student in conceptualizing what is “in-tune,” especially if they have no plan for improving intonation. The third and fourth stipulations allow for an artistic judgment facilitated by intonation – a factor that is not usually considered to be artistic.

Still, this freedom does not license the violinist to willfully play disputable chords inharmoniously; some moments are decidedly more or less ambiguous than others. In fact, continuing along with the first movement of the G Minor Sonata reveals several more of such cases. One occurs in measure three (Figure 14): the F$^{\#}$5 should be tuned Justly with the D5 if played as a double stop, but this would result in a wider half-step between the F$^{\#}$ and G in the top voice. If a student wanted to preserve and draw attention to this leading-tone to tonic motion, the alternative would be to play a higher F$^{\#}$ (Pythagorean), but the eighth-note D would have to be dropped as sustaining it would improperly highlight an impure third (out-of-tune). An interval that could be deliberately played more "out-of-tune," however, is the major seventh between E^{b}4 and D5 in the following beat. If tuning the D5 with open D, the E^{b} should theoretically be raised to create a pure major seventh, but the interval is already quite dissonant. Increasing the dissonance by overly lowering the E^{b} would both beneficially spotlight the half-step motion in the bass (Pythagorean) and dramatize the dissonant arrival of this chord.

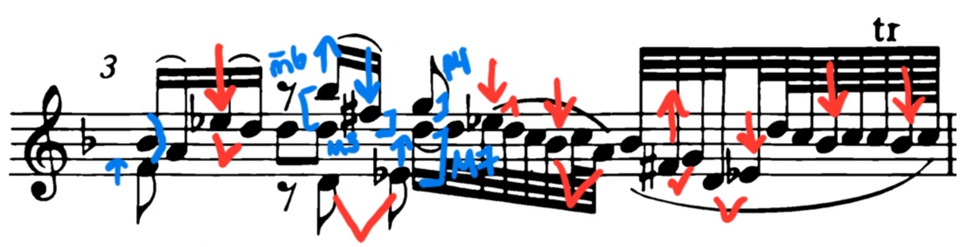

Figure 14. *Bach, Sonata No. 1 in G Minor, mm. 3.*

Another example that offers the violinist some controversial flexibility is the opening of the Loure in the E Major Partita (Figure 14). The phrase begins monophonically which would seem to imply Pythagorean tuning, meaning that the G$^{\#}$5 both on the downbeat and on beat four should be raised to create a wide major third from the tonic E and a narrow half-step with A. However, E major is not conclusively established as the key until the implied authentic cadence on beat five. The $G^{\#}$ that occurs in a double stop with the E in the last quarter note of the bar should be unquestionably tuned purely (lowered). In order to better contextualize the opening $G^{\#}$ as not the third of an E major scale but as the third of an E major triad, it could be advantageous to play the first two $G^{\#}$'s lower, in Just intonation, which would also align it with the fifth partial of the E string – the E string vibrates with more roundness with a lower $G^{\#}$. Both the Just and Pythagorean $G^{\#}$ are viable options available for the violinist, but they should be acutely aware that the option exists.

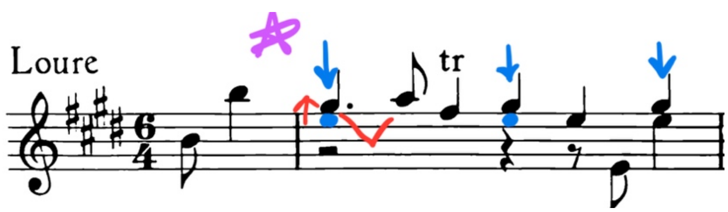

Figure 15. *Bach, Partita No. 3 in E Major, mm. 1.*

One movement that is distinctly more unyielding for the violinist is the Andante from Sonata No. 2 in A Minor. Essentially a duet for solo violin, each voice would be tuned with the Pythagorean system if independently played – high E's and B's (mediant and leading-tone in C major). As the two voices are played in conjunction, vertical alignment necessitates Just intonation, superseding many melodic inflections. In the first measure (Figure 16), the C4 is placed as a perfect fourth from the open G and the E4 is lowered to create a pure major third. Because the F4 must form a perfect fourth with the C, the resulting half step between the E and F is rather wide – wider than in ET or Pythagorean tuning. The next E in the measure (second sixteenth note of the third beat) should probably match the low E's played earlier, but because of the tapered shape of each of the notes in the lower voice, the C that was played with the F is not rearticulated, meaning a case could be made to play that E higher since it is a passing tone between the F4 and D4. However, it would be more straightforward for a student to play all of the E's in this measure similarly as a low major third because the first finger can be held down when playing the perfect fourth to easily reproduce the E.

Figure 16. *Bach, Sonata No. 2 in A Minor, Andante, mm. 1.*

Subsequent measures are also best tuned with Just intonation since there is a persistent voice providing a fundamental – note that this fundamental freely changes and is not restricted to the macro harmony (this applies to all previous examples utilizing Just intonation as well). Moreover, this fundamental, or reference note, is usually the bottom note(s) and the pitches above are the ones that are altered.

The exception is when the notes on top are open strings; the bottom notes would be tuned according to the open string, if a player privileges pure intonation. This notably occurs twice in measure eight (Figure 17). The tuning of the open E on the downbeat dictates the tuning of the lower three notes: remembering the open E is wider than ET when purely tuned with the open A, the C5 should be placed higher to create a tighter, pure major third. It then follows that the G4 should be nudged higher to create a perfect fourth with the C5 and, in turn, a higher C4 to match the higher G4 for a perfect fifth. However, after adjusting from the top down rather than the bottom up, the C4 is higher than a C played as a perfect fourth from the open G (and the higher G4 is sharper than the open G). Because of the varying ways that a violinist could execute the arpeggiation of this chord (e.g. breaking up the quadruple stop into two double stops, or rolling bottom two voices and landing on the top as a double stop), this discrepancy can be neglected –

the G4 could match the open G, resulting in an impure perfect fourth with the C5 but a pure, unaltered perfect fifth with the C4 in favor of ensuring that the top major third is pure.

Figure 17. *Bach, Sonata No. 2 in A Minor, Andante, mm. 7-8.*

The raised C4 could even be sneakily lowered after the chord in order to restore the original placement of the tonic C. Doing so would ensure that the major third between C and E (on the fourth eighth note of the measure) is not sharper than the same interval in the beginning. The next chord (on the third beat) is another that needs to be tuned from the top down given fixed open A in the top voice: the E4 should assuredly be raised from the previous chord to produce a pure perfect fourth with the open A which then requires the C4 to be raised. However, if that C4 was lowered after the downbeat chord where it was raised, this pitch would be altered three times in this one measure – not ideal since this is the bass voice. If this voice was isolated but retained all of these adjustments, it would sound quite dubious; imagine a cellist playing this voice but sharpening and flattening the note on each bow stroke.

Nonetheless, only a single violinist plays this measure, meaning that the rhythmically displaced nature of triple/quadruple stops (all of the notes will not sound simultaneously as it would on a piano, harpsichord, or in a duet) disguises the potential wavering of the C4. Still, after a student has come to terms with this predicament, they may decide to play the same C4 (tuned with the open G, lower than where it "should" be on the downbeat) for the entire measure, resigning themselves to accept some technically impure intervals (namely the perfect fourth on the downbeat and the major third on the third beat) but with the benefit of maintaining a consistent bass voice.

Conclusion

Violin teachers and students should integrate both a conceptual and practical understanding of tuning systems to systematically improve intonation accuracy. Implementation does not need to solely revolve around an inconsequential study of detailed mathematics; rather, it should use these theoretical mechanics to guide a musician's most powerful tool – the ear. For violin playing, Just and Pythagorean tuning remain the most utilitarian systems in the majority of contexts because they offer a balance between vertical/harmonic and horizontal/melodic playing. These systems facilitate proactive error detection and correction and provide students with a tangible, conscious process to develop sensitive intonation skills that are unique to string playing.

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