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LOW-FREQUENCY PROPERTIES OF TRANSVERSELY PERIODIC LOSSY WAVEGUIDES

Edward F. Kuester

Electromagnetics Laboratory
Department of Electrical and Computer Engineering
Campus Box 425
University of Colorado
Boulder, Colorado 80309

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I. INTRODUCTION

In [1], we outlined the basic properties of electromagnetic waveguide modes propagating on a transversely periodic array of lossy dielectric material. This laid the groundwork for a study of propagation in and reflection from arrays of pyramid-cone absorbers as used in anechoic measurement chambers. Indeed, the "tapered waveguide" way of looking at this problem was originally proposed by Katsenelenbaum [2] and carried out for special cases and in various approximations in [3] and [4]. In this report, we will extend the study of [1] to cover the low-frequency properties of these waveguide modes (i.e. for the case when the periods of the material are small compared to a wavelength).

The exact analysis (by numerical or other means) of modes propagating on such periodic structures does not seem to have been carried out for the case of electromagnetic waves. The analogous problem for elastic wave propagation in periodic composite media has received some attention but is quite computationally intensive [5]-[7]. When the periods of the structure become small (compared to a wavelength as well as to any other length scale in the problem) and we care only about averaged values of the fields over a period rather than their microstructure within a period, a technique known as homogenization can be used. In this method, the averaged fields are found to satisfy Maxwell's equations wherein the actual spatial distributions of permittivity and permeability are replaced by homogenized (or effective) values of these parameters respectively. The solution for averaged fields in a homogeneous medium is naturally much easier than for the exact fields in the periodically inhomogeneous one.

There are several books available describing the homogenization method [8]-[10],[14] and a large number of papers, the most relevant of which to our problem are [11]-[13],[15],[16]. Perhaps closest of these to the results obtained in this report is [16], wherein it is shown that effective permittivity and permeability obtained for a static problem may also be used when the period of the structure is small compared to a wavelength. We will prove the same thing here for the problem of interest to us.

Once this is done, we will be able to invoke one of several variational estimates available in the literature for calculating the static homogenized permittivity and permeability coefficients. This will be discussed in Section 3. Finally, in Section 4 we will evaluate the coupling coefficients for tapered arrays (as derived in [1]) for low frequency limit. By means of this calculation, the problem of low frequency reflection from such tapered arrays will have been reduced to that of reflection from an equivalent inhomogeneous, anisotropic plane layer. The analysis of this problem will be the subject of a separate report.

II. LOW-FREQUENCY BEHAVIOR OF QUASI-PLANE WAVE MODES

Let us recall some results of [1]. A uniform, transversely periodic waveguide as shown in Figure 1 supports waveguide modes of propagation which have the structure of Bloch waves in the transverse direction:

$$\left.\begin{aligned}\overline{E}(x,y,z) &= e^{-jk_0\overline{n}_e\cdot\overline{r}}\,\overline{F}_E(\overline{\rho}) \\ \overline{H}(x,y,z) &= e^{-jk_0\overline{n}_e\cdot\overline{r}}\,\overline{F}_H(\overline{\rho})\end{aligned}\right\} \tag{1}$$

Here, $\overline{F}_E$ and $\overline{F}_H$ are periodic functions of x and y with periods a and b respectively, $\overline{\rho} = x\overline{a}_x + y\overline{a}_y$ is a transverse position vector, $\overline{r} = \overline{\rho} + z\overline{a}_z$ is the position vector itself, and

$$k_0\overline{n}_e = \overline{k}_t + \beta\overline{a}_z \equiv \overline{\beta} \tag{2}$$

The transverse component $\overline{n}_t = \overline{k}_t / k_0$ of $\overline{n}_e$ is to be regarded as fixed and given. For each $\overline{k}_t$ and frequency ω (corresponding to an assumed time factor of $e^{j\omega t}$), there are only certain allowed values of the propagation constant β, corresponding to a discrete spectrum of modes.

It is reasonable to assume that there are certain of these modes for which β approaches zero as $k_0 \to 0$ with $\overline{n}_e$ held fixed. The fields of these modes will reduce to static ones in this limit, as we shall show below. For reasons to be made clear shortly, we will call these modes quasi-plane wave modes (in analogy with quasi-TEM modes which propagate at low frequencies on inhomogeneously-filled transmission lines).

Formally, let us take as a small parameter $\delta = k_0 d$, where $d = \sqrt{a^2 + b^2}$ is the diagonal length of a period cell. We now assume that $\overline{F}_E, \overline{F}_H$ and $n_z = \beta/k_0$ can all be expanded as power series in δ:

$$\left.\begin{aligned} \overline{F}_E &\sim \overline{F}_{E0} + \delta\overline{F}_{E1} + \dots \\ \overline{F}_H &\sim \overline{F}_{H0} + \delta\overline{F}_{H1} + \dots \\ n_z &\sim n_{z0} + \delta n_{z1} + \delta^2 n_{z2} + \dots \end{aligned}\right\} \tag{3}$$

Now, Maxwell's equations (equation (3) of [1]) can be written as

$$\left.\begin{aligned} d\nabla_t \times \overline{F}_E &= j\delta[\,\overline{n}_e \times \overline{F}_E - \zeta_0\mu_r\overline{F}_H] \\ d\nabla_t \times \overline{F}_H &= j\delta[\,\overline{n}_e \times \overline{F}_H - \frac{\varepsilon_r}{\zeta_0}\overline{F}_E] \end{aligned}\right\} \tag{4}$$

where $\zeta_0 = (\mu_0/\varepsilon_0)^{1/2}$. Substituting (3) into (4) and equating like powers of δ, we find:

$$\left.\begin{aligned} \nabla_t \times \overline{F}_{E0} &= 0 \\ \nabla_t \times \overline{F}_{H0} &= 0 \end{aligned}\right\} : \delta^0 \tag{5}$$

$$\left.\begin{aligned} \nabla_t \times \overline{F}_{E1} &= \frac{j}{d}[(\overline{n}_t + \overline{a}_z n_0) \times \overline{F}_{E0} - \zeta_0\mu_r\overline{F}_{H0}] \\ \nabla_t \times \overline{F}_{H1} &= \frac{j}{d}[(\overline{n}_t + \overline{a}_z n_0) \times \overline{F}_{H0} + \frac{\varepsilon_r}{\zeta_0}\overline{F}_{E0}] \end{aligned}\right\} : \delta^1 \tag{6}$$

$$\left.\begin{aligned} \nabla_t \times \overline{F}_{E2} &= \frac{j}{d}[(\overline{n}_t + \overline{a}_z n_0) \times \overline{F}_{E1} - \zeta_0\mu_r\overline{F}_{H1} + n_1\overline{a}_z \times \overline{F}_{E0}] \\ \nabla_t \times \overline{F}_{H2} &= \frac{j}{d}[(\overline{n}_t + \overline{a}_z n_0) \times \overline{F}_{H1} + \frac{\varepsilon_r}{\zeta_0}\overline{F}_{E1} + n_1\overline{a}_z \times \overline{F}_{H0}] \end{aligned}\right\} : \delta^2 \tag{7}$$

Consequences of (6) and (7) which follow after taking the transverse divergence of these equations are

$$\left.\begin{aligned} \nabla_t \cdot (\varepsilon_r \overline{F}_{E0}) &= 0 \\ \nabla_t \cdot (\mu_r \overline{F}_{H0}) &= 0 \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} \nabla_t \cdot (\varepsilon_r \overline{F}_{E1}) &= \frac{j\varepsilon_r}{d}(\overline{n}_t + \overline{a}_z n_0) \cdot \overline{F}_{E0} \\ \nabla_t \cdot (\mu_r \overline{F}_{H1}) &= \frac{j\mu_r}{d}(\overline{n}_t + \overline{a}_z n_0) \cdot \overline{F}_{H0} \end{aligned}\right\} \tag{9}$$

Of course, $\overline{F}_{Ei}$ and $\overline{F}_{Hi}$, i = 0,1,2,. . . , like their parent functions $\overline{F}_E$ and $\overline{F}_H$, are periodic in x and y with periods a and b respectively.

From (5) and (8), we observe that $\overline{F}_{E0}$ and $\overline{F}_{H0}$ are in fact source-free, spatially periodic (in x and y) <u>static</u> field solutions for the composite medium of Figure 1. For $\overline{F}_{E0}$, there will be two linearly independent solutions of this type from which any arbitrary solution follows by superposition (roughly speaking, two orthogonal polarizations). To discuss these solutions, we first introduce the averaging operation over a period cell C(x,y) centered at the point x,y (Fig. 2). For any function f(x,y), define

$$f_{av}(x,y) = \frac{1}{ab}\int_{C(x,y)} f(x',y')\, dS' \tag{10}$$

Using known results from vector calculus (see, e.g., [15]), we can show that

$$\nabla_t f_{av} = \frac{1}{ab} \oint_{B(x,y)} f(x',y')\overline{a}_n' \, dl'$$

$$= \frac{1}{ab} \int_{C(x,y)} \nabla_t' f(x',y') \, dS' \tag{11}$$

$$= (\nabla_t f)_{av}\big|_{x,y}$$

where B(x,y) is the boundary of C(x,y) and $\overline{a}_n'$ is the outward unit normal vector to B(x,y) in the xy-plane. Similarly,

$$\left.\begin{aligned} \nabla_t \times \overline{F}_{av} &= (\nabla_t \times \overline{F})_{av} \\ \nabla_t \cdot \overline{F}_{av} &= (\nabla_t \cdot \overline{F})_{av} \end{aligned}\right\} \tag{12}$$

and so on.

If f is a periodic function of periods a and b, then f_{av} = const, and by (11) the periodic function $\nabla_t f$ has average value zero:

$$(\nabla_t f)_{av} = 0 \tag{13}$$

On the other hand, if $\nabla_t f$ is a periodic function, then $(\nabla_t f)_{av} = \overline{A} = A_1\overline{a}_x + A_2\overline{a}_y$ = const and, by (11),

$$f_{av} = A_1 x + A_2 y + A_3 = \overline{A} \cdot \overline{\rho} + A_3 \tag{14}$$

where A_1, A_2 and A_3 are constants. Then f - f_{av} has average value zero, and following [11] we can show that $f - f_{av}$ is also periodic, for the components of $\nabla_t f_{av}$ can be calculated using only line integrals instead of integrals over C:

$$\left.\begin{aligned} \bar{a}_x \cdot \nabla_t f_{av} &= \frac{1}{a}\int_{\bar{\rho}}^{\bar{\rho}+a\bar{a}_x} \nabla_t' f(\bar{\rho}') \cdot d\bar{l}' \\ \bar{a}_y \cdot \nabla_t f_{av} &= \frac{1}{b}\int_{\bar{\rho}}^{\bar{\rho}+b\bar{a}_y} \nabla_t' f(\bar{\rho}') \cdot d\bar{l}' \end{aligned}\right\} \tag{15}$$

To see this, note that these integrals are constant (independent of ρ) due to the periodicity of $\nabla_t f$ and the fact that the integrals are independent of the choice of integration path. Because they are constants, one further averaging (in the y or x direction respectively) yields the same constant, namely the appropriate component of $\nabla_t f_{av}$. Then by (15) ,

$$\begin{aligned} [f - f_{av}]_{\bar{\rho}}^{\bar{\rho}+a\bar{a}_x} &= \int_{\bar{\rho}}^{\bar{\rho}+a\bar{a}_x} \nabla'_t(f - f_{av}) \cdot d\bar{l}' \\ &= a\bar{a}_x \cdot \nabla_t(f - f_{av})_{av} = 0 \end{aligned} \tag{16}$$

and similarly for $[f - f_{av}]_{\bar{\rho}}^{\bar{\rho}+b\bar{a}_y}$, and thus $f - f_{av}$ is periodic.

Finally (also as in [11]), if $\bar{F}$ and $\nabla_t f$ are bounded and $\nabla_t \cdot \bar{F} = 0$, the

$$\begin{aligned} (\bar{F} \cdot \nabla_t f)_{av} &= [\bar{F} \cdot \nabla_t (f - f_{av})]_{av} + (\bar{F} \cdot \nabla_t f_{av})_{av} \\ &= \frac{1}{ab}\int_c \bar{F} \cdot \nabla_t (f - f_{av})\, dS + \bar{F}_{av} \cdot \nabla_t f_{av} \end{aligned} \tag{17}$$

But by the divergence theorem, the first term on the right hand side is

$$\frac{1}{ab}\int_c \bar{F}\cdot\nabla_t(f-f_{av})dS = \frac{1}{ab}\oint_B (f-f_{av})\bar{a}_n\cdot\bar{F}\,dl = 0$$

because $\bar{F}$ and $f - f_{av}$ (see above) are periodic. Thus,

$$(F \cdot \nabla_t f)_{av} = F_{av} \cdot \nabla_t f_{av} \tag{18}$$

Returning to (5) and (8), we see that $\bar{F}_{E0}$ and $\bar{F}_{H0}$ must be gradients of scalar potentials and that their z-components are constant. Let us denote

$$\bar{F}_{E0,av} \equiv \bar{E}_0 \quad ; \quad \bar{F}_{H0,av} \equiv \bar{H}_0 \tag{19}$$

Then

$$\bar{F}_{E0t} = -\nabla_t \Phi_o \; ; \qquad \bar{F}_{H0t} = -\nabla_t \Psi_o \tag{20}$$

and

$$F_{Eoz} = E_{oz} = \text{const} \; ; \quad F_{Hoz} = H_{oz} = \text{const} \tag{21}$$

By (8) and (14), we see that Φ_0 and Ψ_0 satisfy the two-dimensional static potential equations

$$\nabla_t \cdot (\varepsilon_r \nabla_t \Phi_0) = 0 \; ; \quad \nabla_t \cdot (\mu_r \nabla_t \Psi_0) = 0 \tag{22}$$

with average values

$$\Phi_{o,av} = -(E_{ox}\, x + E_{oy}\, y); \qquad \Psi_{o,av} = -(H_{ox}\, x + H_{oy}\, y) \tag{23}$$

The functions $\Phi_0 - \Phi_{0,av}$ and $\Psi_0 - \Psi_{0,av}$ will be periodic (the arbitrary additive constants in (23) have been taken to be zero).

If our interest is only in average field values rather than their fine (periodic) structure, then only a single tensor quantity from each of the static problems is needed. These are the effective (homogenized) permittivity $[\varepsilon^h]$ and permeability $[\mu^h]$. They are defined as

$$\begin{aligned} \varepsilon_0(\varepsilon_r F_{Eo})_{av} &\equiv [\varepsilon^h] \cdot E_o \\ \mu_o(\mu_r F_{Ho})_{av} &\equiv [\mu^h] \cdot H_o \end{aligned} \tag{24}$$

for any average fields E_o and H_o, and can be calculated from a knowledge of the static fields F_{Eo} and F_{Ho}. Since ε_r, μ_r, F_{Eot} and F_{Hot} depend only on x and y, while F_{Eoz} and F_{Hoz} are constant, we see that $[\varepsilon^h]$ and $[\mu^h]$ have the structure:

$$[\varepsilon^h] = \begin{bmatrix} \left[\varepsilon_t^h\right] & & 0 \\ & & 0 \\ 0 & 0 & \varepsilon_{zz}^h \end{bmatrix}; \qquad [\mu^h] = \begin{bmatrix} \left[\mu_t^h\right] & & 0 \\ & & 0 \\ 0 & 0 & \mu_{zz}^h \end{bmatrix} \tag{25}$$

where

$$\left.\begin{aligned} \varepsilon_{zz}^{h} = \varepsilon_0(\varepsilon_r)_{av} = \frac{\varepsilon_0}{ab}\int_c \varepsilon_r(x,y)dS \\ \mu_{zz}^{h} = \mu_0(\mu_r)_{av} = \frac{\mu_0}{ab}\int_c \mu_r(x,y)dS \end{aligned}\right\} \tag{26}$$

while $[\varepsilon_t^h]$ and $[\mu_t^h]$ are symmetric 2×2 tensors. These latter can be approximated quite well by variational methods, as discussed in Section 3.

Now, averaging (6) and (9) over a period cell C results in

$$\left.\begin{aligned} \frac{\overline{n}_e}{c} \times \overline{E}_0 - [\mu^h] \cdot \overline{H}_0 = 0 \\ \frac{\overline{n}_e}{c} \times \overline{H}_0 + [\varepsilon^h] \cdot \overline{E}_0 = 0 \end{aligned}\right\} \tag{27}$$

and

$$\left.\begin{aligned} \overline{n}_e \cdot [\varepsilon^h] \cdot \overline{E}_0 = 0 \\ \overline{n}_e \cdot [\mu^h] \cdot \overline{H}_0 = 0 \end{aligned}\right\} \tag{28}$$

where $c = (\mu_0 \varepsilon_0)^{-1/2}$ is the speed of light in vacuum. Of course, (28) can also be obtained directly from (27). These are simply the relations obeyed by the vector amplitudes $\overline{E}_0$ and $\overline{H}_0$ of a plane wave propagating with wave vector $k_0 \overline{n}_e$ in a homogeneous but anisotropic medium characterized by $[\varepsilon^h]$ and $[\mu^h]$.

For our purposes, it is the transverse field components which are of interest, as they will obey ordinary transmission-line equations. From the transverse parts of (27) we have

$$\left.\begin{aligned} \frac{n_z}{c}\bar{a}_z\times\bar{E}_{ot}+\frac{\bar{n}_t}{c}\times\bar{a}_z E_{oz}-[\mu_t^h]\cdot\bar{H}_{ot}=0 \\ \frac{n_z}{c}\bar{a}_z\times\bar{H}_{ot}+\frac{\bar{n}_t}{c}\times\bar{a}_z H_{oz}-[\varepsilon_t^h]\cdot\bar{E}_{ot}=0 \end{aligned}\right\} \tag{29}$$

and from the z-components,

$$\left.\begin{aligned} \frac{\bar{n}_t}{c}\times\bar{E}_{ot}-\mu_{zz}^h H_{oz}\bar{a}_z=0 \\ \frac{\bar{n}_t}{c}\times\bar{H}_{ot}+\varepsilon_{zz}^h E_{oz}\bar{a}_z=0 \end{aligned}\right\} \tag{30}$$

Eliminating E_{oz} and H_{oz} from these equations gives

$$\frac{n_z}{c}\bar{a}_z\times\bar{E}_{ot}-[\mu_e]\cdot\bar{H}_{ot}=0 \tag{31}$$

$$\frac{n_z}{c}\bar{a}_z\times\bar{H}_{ot}+[\varepsilon_e]\cdot\bar{E}_{ot}=0 \tag{32}$$

where

$$\left.\begin{aligned} [\mu_e]=[\mu_t^h]+\frac{1}{c^2\varepsilon_{zz}^h}\begin{bmatrix} -n_{ty}^2 & n_{tx}n_{ty} \\ n_{tx}n_{ty} & -n_{tx}^2 \end{bmatrix} \\ [\varepsilon_e]=[\varepsilon_t^h]+\frac{1}{c^2\mu_{zz}^h}\begin{bmatrix} -n_{ty}^2 & n_{tx}n_{ty} \\ n_{tx}n_{ty} & -n_{tx}^2 \end{bmatrix} \end{aligned}\right\} \tag{33}$$

Since (as is shown by direct computation)

$$\bar{a}_z\times\left\{[D]\cdot\bar{A}_t\right\}=[D^c]\cdot(a_z\cdot\bar{A}_t) \tag{34}$$

where A_t is any transverse vector, [D] is a 2×2 matrix and $[D^c]$ is the matrix of cofactors of [D] (transpose of the classical adjoint or adjugate):

$$[D^c] = \{\det(D)\,[D^{-1}]\}^T \tag{35}$$

$$= \begin{bmatrix} D_{yy} & -D_{yx} \\ -D_{xy} & D_{xx} \end{bmatrix}$$

then we have by taking the cross-product of a_z with (31):

$$\frac{n_z}{c}\overline{E}_{ot} + [\mu_e^c] \cdot (\bar{a}_z \times \overline{H}_{ot}) = 0 \tag{36}$$

Equations (32) and (36) form a matrix analog of the transmission-line equations. We can eliminate $a_z \times H_{ot}$ between them to get

$$\left\{[\mu_e^c] \cdot [\varepsilon_e] - \frac{n_z^2}{c^2}\right\} \cdot \overline{E}_{ot} = 0 \tag{37}$$

That is, the values of n_z^2/c^2 are eigenvalues of the matrix $[\mu_e^c] \cdot [\varepsilon_e]$, and, if these eigenvalues are distinct, they correspond to two orthogonal vectors $\overline{E}_{ot}^{(1)}$ and $\overline{E}_{ot}^{(2)}$:

$$\overline{E}_{ot}^{(1)} \cdot \overline{E}_{ot}^{(2)} = 0 \tag{38}$$

These are polarizations of quasi-plane waves which will maintain their identity when propagating through the homogenized medium. Further, from (32) or (36), (38) also implies

$$\overline{E}_{ot}^{(1)} \cdot \bar{a}_z \times \overline{H}_{ot}^{(2)} = \overline{E}_{ot}^{(2)} \cdot \bar{a}_z \times \overline{H}_{ot}^{(1)} \tag{39}$$

This could also have been deduced from the orthogonality property (15) of[1], together with property (18) of the averaging operator, as well as (5) and (20).

Now the mode fields (in the notation of [1]), in their quasistatic limits, become (denoting mode indices by ±1 and ±2):

$$\left.\begin{aligned} \overline{\mathcal{E}}_{1,2t}(\pm\overline{k}_t) &\cong e^{\mp j\overline{k}_t\cdot\overline{\rho}}\,\overline{F}^{(1,2)}_{Eot} \\ \overline{\mathcal{H}}_{1,2t}(\pm\overline{k}_t) &\cong e^{\mp j\overline{k}_t\cdot\overline{\rho}}\,\overline{F}^{(1,2)}_{Hot} \end{aligned}\right\} \qquad (40)$$

and

$$\left.\begin{aligned} \overline{\mathcal{E}}_{-1,2t}(\pm\overline{k}_t) &= \mathcal{E}_{1,2t}(\pm\overline{k}_t) \\ \overline{\mathcal{H}}_{-1,2t}(\pm\overline{k}_t) &= -\,\mathcal{H}_{1,2t}(\pm\overline{k}_t) \end{aligned}\right\} \qquad (41)$$

since (5) and (8) are independent of $\overline{n}_t$. Then by formula (21) of [1], the norms of the fundamental modes are

$$\begin{aligned} N_{1,2}(\pm\overline{k}_t) &= -N_{-1,2}(\pm\overline{k}_t) \\ &\cong \int_c \overline{F}^{(1,2)}_{Eot}\times\overline{F}^{(1,2)}_{Hot}\cdot\overline{a}_z\,dS \\ &= ab\,\overline{E}^{(1,2)}_{ot}\times\overline{H}^{(1,2)}_{ot}\cdot\overline{a}_z \end{aligned} \qquad (42)$$

Once again we have used (18), (5) and (20).

3. APPROXIMATIONS FOR THE HOMOGENIZED PARAMETERS

For the uniform array of cylinders, our lowest order approximation problem can be regarded as solved if we can calculate $\left[\varepsilon_t^h\right]$ and $\left[\mu_t^h\right]$. This has been done numerically for some specific geometries using the finite element method [17] - [20]. However, there has also been some work done using variational methods to calculate upper and lower bounds for the effective parameters when ε_r and μ_r are purely real.* The earliest and most widely known set of bounds was derived originally by Hashin and Shtrikman [23] and later generalized in a number of ways [24] - [28], [50]. For the case of a two-phase medium (ε_r and μ_r are piecewise constant and take on only two distinct values) with appropriate symmetry in the x and y directions, the effective tensors are simply multiples of the identity tensor:

$$[\varepsilon_t^h] = \begin{bmatrix} \varepsilon_t & 0 \\ 0 & \varepsilon_t \end{bmatrix} \quad ; \qquad [\mu_t^h] = \begin{bmatrix} \mu_t & 0 \\ 0 & \mu_t \end{bmatrix} \quad ; \tag{43}$$

Then the Hashin-Shtrikman bounds are

$$\varepsilon_1 \left[1 + f_2 \frac{2(\varepsilon_2 - \varepsilon_1)}{(1+f_2)\varepsilon_1 + f_1 \varepsilon_2} \right] \le \varepsilon_t \le \varepsilon_1 \left[1 + f_2 \left(\frac{\varepsilon_2 + \varepsilon_1}{\varepsilon_1} \right) \frac{\varepsilon_2 - \varepsilon_1}{f_2 \varepsilon_1 + (1+f_1)\varepsilon_2} \right] \tag{44}$$

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*When ε_r and μ_r are complex, we can obtain separate upper and lower bounds for the real and imaginary parts of the effective parameters by replacing ε_r and μ_r in the following formulas by their real or imaginary parts respectively. Stricter bounds related to the Hashin-Shtrikman bounds (but in the complex plane) are given in [21], [22].

where ε_1, $\varepsilon_2 > \varepsilon_1$ are the permittivities and f_1, $f_2 = 1-f_1$, the fractional areas occupied by the phases 1 and 2 within a period cell C. These are the best bounds obtainable using only the parameters ε_1, ε_2, f_1 and f_2. For the case of a square array of circular rods, the Hashin-Shtrikman lower bound in (44) was originally obtained by Rayleigh [29] and rediscovered many years later by Corti et al [30].

A second set of bounds which used more geometrical information about the array was obtained shortly after the Hashin-Shtrikman bounds by Jackson and Coriell [31]-[33], and developed further by subsequent workers [34]-[38] and [10, p. 139]. When applied to a medium whose effective parameters are given by (43), the Jackson-Coriell bounds are:

$$\varepsilon_0 \int_0^b \frac{d_y}{\int_0^a \frac{dx}{\varepsilon_r(x,y)}} \leq \varepsilon_t \leq \varepsilon_0 \frac{1}{\int_0^a \frac{dx}{\int_0^b \varepsilon_r(x,y)dy}} \tag{45}$$

For the case of a rectangular array of rectangular rods, the upper bound in (45) had been given earlier by Collin [39] as an approximation based on the neglect of fringing fields.*

To assess the accuracy of these various bounds, we consider the case of a square array of square rods as shown in Fig. 3. This case will have practical application to the pyramid-cone absorbers used in anechoic chambers. Let $g = a_2/a$ be the ratio of the side of the rod to the period of

*The labellings of the dimensions $t_{1,2}$ and $d_{1,2}$ in Fig. 3 of [39] appear to have been mistakenly interchanged.

the array. Then $f_2 = g^2$, and we may adapt the bounds of (44) and (45) to this configuration. By symmetry, the transverse tensors have the form (43), and we have (for $\varepsilon_1 < \varepsilon_2$, and both real)

$$\varepsilon_t \geq \varepsilon_1 \left[1 + g^2 \frac{2(\varepsilon_2 - \varepsilon_1)}{(1 + g^2)\varepsilon_1 + (1 - g^2)\varepsilon_2}\right] \qquad \text{(HS)} \qquad (46)$$

$$\varepsilon_t \leq \varepsilon_1 \left[1 + g^2 \left(\frac{\varepsilon_2 + \varepsilon_1}{\varepsilon_1}\right) \frac{\varepsilon_2 - \varepsilon_1}{g^2\varepsilon_1 + (2 - g^2)\varepsilon_2}\right] \qquad \text{(HS)} \qquad (47)$$

$$\varepsilon_t \geq \varepsilon_1 \left[1 + g^2 \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 g + \varepsilon_2(1 - g)}\right] \qquad \text{(JC)} \qquad (48)$$

$$\varepsilon_t \leq \varepsilon_1 \left[1 + g^2 \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1(1 - g + g^2) + \varepsilon_2 g(1 - g)}\right] \qquad \text{(JC)} \qquad (49)$$

In Fig. 4, we plot the estimates (46) - (49) for ε_t vs g for the case when $\varepsilon_2 = 10\varepsilon_0$, $\varepsilon_1 = \varepsilon_0$. We also plot numerical results for this case, obtained from [18] using Keller's scaling theorem [40]. There is some error involved in reading these results from a graph, so that this curve in Fig. 4 occasionally exceeds an upper bound or falls below a lower bound by a small amount. We see from Fig. 4 that the HS upper bound (47) is not very accurate. Moreover, both JC bounds (48) and (49) are of moderately good accuracy (about 7.5% in error at worst case) and the HS lower bound (46) is in excellent agreement with numerical results (the worst error is 4%

which is comparable to the error in reading the graph in [18]). Based upon this evidence, we conjecture that (46) is the best of the approximate formulas for ε_t in this geometry and that its use can be extended to complex values of ε_1 and ε_2. However, no numerical results for this latter situation are available for comparison to settle the matter definitively. The problem must be left to future study.

We may also consider the case of the generalized checkerboard array shown in Fig. 5. For a filling fraction $f_2 < 1/2$, we have an array of square absorbers oriented at 45° relative to the array axes, while for $f_2 > 1/2$, we have the complementary structure of square air voids embedded in an absorbing backdrop. Numerical results for this configuration have been given in [17], [52] and [53]. The Hashin-Shtrikman bounds are, as ever, given by (44). The Jackson-Coriell bounds, upon evaluating (45), are found to be

$$\varepsilon_t \geq \varepsilon_1\left(1-\frac{h}{a}\sqrt{2}\right)+\frac{\varepsilon_1\varepsilon_2}{\varepsilon_1-\varepsilon_2}\,ln\left[1-\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2}\,\frac{h}{a}\sqrt{2}\right] \qquad \text{(JC)}\left(f_2=\frac{h^2}{a^2}<\frac{1}{2}\right) \tag{50}$$

$$\varepsilon_t^{-1} \geq \varepsilon_1^{-1}\left(1-\frac{h}{a}\sqrt{2}\right)+\frac{1}{\varepsilon_2-\varepsilon_1}\,ln\left[1+\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1}\,\frac{h}{a}\sqrt{2}\right] \qquad \text{(JC)}\left(f_2=\frac{h^2}{a^2}<\frac{1}{2}\right) \tag{51}$$

while for $f_2 = 1-\frac{l^2}{a^2}>\frac{1}{2}$, we let $h \rightarrow l$ and $\varepsilon_1 \rightarrow \varepsilon_2$ in (50) and (51). Here, h, l and a are defined as in Fig. 5. A check of these bounds against the numerical results of [17], [52] or [53] shows that no single one is as accurate an approximation here as (46) is for the square array of squares. If, however, we take the square root of the product of the Jackson-Coriell bounds:

$$\varepsilon_t \cong \varepsilon_m \left\{ \frac{(\Delta-1)(1-g) - \Delta\, ln[1+g(\frac{1}{\Delta}-1)]}{(\Delta-1)(1-g) + ln[1+g(\Delta-1)]} \right\}^{1/2} \tag{52}$$

where

$$\left.\begin{aligned} &\varepsilon_m = \varepsilon_1;\ \ g = \frac{h}{a}\sqrt{2};\ \ \Delta = \frac{\varepsilon_2}{\varepsilon_1} \text{ for } f_2 < \frac{1}{2} \\ &\varepsilon_m = \varepsilon_2;\ \ g = \frac{l}{a}\sqrt{2};\ \ \Delta = \frac{\varepsilon_1}{\varepsilon_2} \text{ for } f_2 > \frac{1}{2} \end{aligned}\right\} \tag{53}$$

we find a relatively good approximation for all values of g. When $\varepsilon_1/\varepsilon_2=10$, the maximum error of (53) is 8%. In Fig. 6, we illustrate these values graphically.

4. HOMOGENIZED DESCRIPTION OF SLOWLY-TAPERED ARRAYS

We now again permit (as in [1], section 4) ε and μ to be functions of z, so that n_z, F_E and F_H, as well as the quantities derived from them (like n_{z0}, E_0, H_0, etc.), will also depend parametrically on z (Fig. 7). If the variations in z are on a length scale which is large (compared with local wavelength, periods a and b, and so on), then the coupled-mode formalism developed in [1] will be especially suitable for describing propagation in this structure.

From [1], we find that the transverse (xy) fields in the tapered region can be written as a sum of modal Bloch waves of the type (1). The amplitudes $A_m(z)$ of these waves then satisfy coupled-mode equations ([1], eqn. (31)). When variation with z is sufficiently weak, we find that coupling between different modes is quite small, and that the set of coupled-mode equations may be truncated to include only the amplitudes $A_+(z)$ and $A_-(z)$ of the forward and backward traveling fundamental (quasi-plane wave) modes 1 <u>or</u> 2 alone. This truncated system has the form:

$$\left.\begin{aligned} A_+'(z) + j\beta(z)A_+(z) &= C_{++}(z)A_+(z) + C_{+-}(z)A_-(z) \\ A_-'(z) - j\beta(z)A_-(z) &= C_{-+}(z)A_+(z) + C_{--}(z)A_-(z) \end{aligned}\right\} \qquad (54)$$

where the coupling coefficients $C_{\pm\pm}$ are given by ([1], eqn. (32)). Here and afterwards, we may drop the explicit indication of polarization 1 or 2, since the coupling between them is neglected.

From ([1], eqn. (32)), and (40) - (42), and once again using properties (18), (15) and (20), we find that

$$C_{--} = - C_{++} = \frac{1}{2N_+}\frac{dN_+}{dz} \qquad (55)$$

and

$$C_{+-} = -C_{-+} = \frac{ab}{2N_+}\left[\overline{E}_{ot}\times\frac{d\overline{H}_{ot}}{dz} - \frac{d\overline{E}_{ot}}{dz}\times\overline{H}_{ot}\right]\cdot\overline{a}_z$$

$$= \frac{ab}{2N_+}\overline{E}_{ot}\cdot\frac{d}{d_z}\left\{\frac{c}{n_z}[\varepsilon_e]\right\}\cdot\overline{E}_{ot} \tag{56}$$

where (32) has been used to eliminate $\overline{a}_z\times\overline{H}_{ot}$ from (56).

We will hereafter specialize to the case of interest for the tapered absorber problem--when $\left[\varepsilon_t^h\right]$ and $\left[\mu_t^h\right]$ are diagonal and invariant to interchange of x and y, as in (43). Then with no loss of generality we can take

$$\overline{n}_t = \overline{a}_x \sin\theta \tag{57}$$

where θ can be thought of as an angle of incidence with respect to the z-axis in a free space region. Then $\overline{E}_{ot}$, by (37) and (33), is either x-polarized (parallel polarization to the xz-plane) or y-polarized (perpendicular polarization). We are now free to normalize $\overline{E}_{ot}=\overline{a}_x$ or $\overline{a}_y$ respectively (independent of z) for these cases, whence $d\overline{E}_{ot}/dz = 0$, and

$$C_{++} = -C_{--} = C_{-+}$$
$$= \frac{1}{2Z_c}\frac{dZ_c}{dz} \tag{58}$$

where the intrinsic wave impedance Z_c for each mode is defined as

$$Z_c = \frac{\overline{E}_{ot}\cdot\overline{E}_{ot}}{\overline{a}_z\cdot\overline{E}_{ot}\times\overline{H}_{ot}} = \frac{ab}{N_+} \tag{59}$$

Since by (33), we have

$$
\left.
\begin{aligned}
[\mu_e^c] &= \begin{bmatrix} \mu_t - \dfrac{\sin^2\theta}{c^2 \varepsilon_{zz}^h} & 0 \\ 0 & \mu_t \end{bmatrix} \\
[\varepsilon_e] &= \begin{bmatrix} \varepsilon_t & 0 \\ 0 & \varepsilon_t - \dfrac{\sin^2\theta}{c^2 \mu_{zz}^h} \end{bmatrix}
\end{aligned}
\right\} \tag{60}
$$

we can summarize the properties of the two polarizations as follows (cf. (32) and (37)):

Parallel Polarization:

$$
\frac{n_z}{c} = \pm \sqrt{\varepsilon_t \left(\mu_t - \frac{\sin^2\theta}{c^2 \varepsilon_{zz}^h} \right)} \tag{61}
$$

$$
Z_c = \sqrt{\frac{\mu_t - \dfrac{\sin^2\theta}{c^2 \varepsilon_{zz}^h}}{\varepsilon_t}} \tag{62}
$$

Perpendicular Polarization

$$\frac{n_z}{c} = \pm\sqrt{\left(\varepsilon_t - \frac{\sin^2\theta}{c^2\mu_{zz}^h}\right)\mu_t} \tag{63}$$

$$Z_c = \sqrt{\frac{\mu_t}{\varepsilon_t - \frac{\sin^2\theta}{c^2\mu_{zz}^h}}} \tag{64}$$

We now pass to a conventional transmission-line description from (54) by putting

$$\left.\begin{aligned} E(z) &= A_+(z) + A_-(z) \\ H(z) &= \frac{A_+(z) - A_-(z)}{Z_c} \end{aligned}\right\} \tag{65}$$

and using (58) to obtain

$$\left.\begin{aligned} E'(z) + j\beta Z_c H(z) &= Z_c' H(z) \\ H'(z) + \frac{j\beta}{Z_c} E(z) &= -\frac{Z_c'}{Z_c} H(z) \end{aligned}\right\} \tag{66}$$

But finally, the terms on the right sides of (66) are small because of the slowly-varying assumption which allowed us to set the other coupling coefficients to zero. We thus arrive at the ordinary transmission-like equations

$$\left.\begin{aligned} E'(z) + j\beta Z_c H(z) &= 0 \\ H'(z) + \frac{j\beta}{Z_c} E(z) &= 0 \end{aligned}\right\} \tag{67}$$

The gently tapered composite medium is, to this order of approximation, equivalent to an inhomogeneous (in z only) but anisotropic medium. The calculations of fields in such an equivalent medium are considerably simpler than are those of the original problem.

This result has previously been obtained for the wedge geometry by Bucci and Franceschetti [3] (for normal incidence only) and by Bell et al. [41], Bouchitte and Petit [42] and Borovskii and Khizhnyak [56] (for arbitrary incidence perpendicular to the edges of the wedges). An analogous result for the scalar problem and circular cone geometry was obtained earlier by Kohler et al. [43] and more generally for the scalar problem by Bakhvalov [51]. The present result constitutes the full generalization to the three-dimensional electromagnetic case.

5. Conclusion

In this report, we have demonstrated the low-frequency equivalence between an array of gently-tapered pyramid-cone absorbers (or other slowly varying transversely periodic structures) to a one-dimensionally inhomogeneous but anisotropic medium. The application of this equivalence to the calculation of reflection coefficients of such absorbers will be dealt with in a subsequent report.

It should be noted that the restriction to slowly varying structures in the z-direction may not be essential. There are a number of papers [44]-[49], [14], [54], [55], [57], [58] dealing with discontinuities between different periodic structures. These seem to indicate that the equivalent boundary conditions on the average field are also those dictated by the equivalent parameters of the bulk medium as found in section 2. A precise assertion of this fact, however, does not seem to have been made in the literature.

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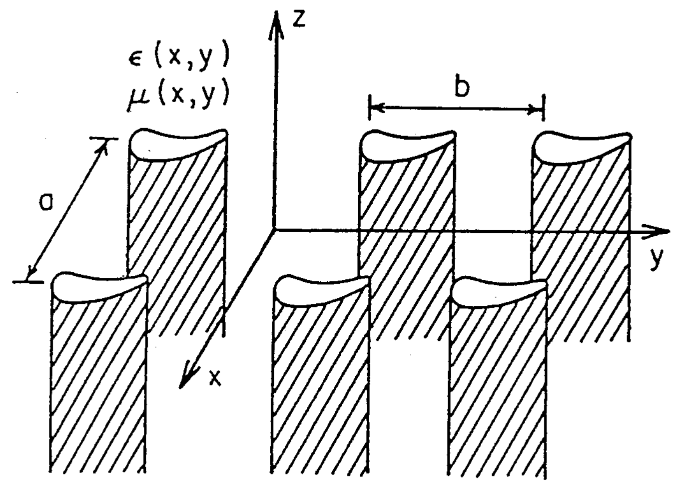

Fig. 1: Uniform transversely periodic waveguide.

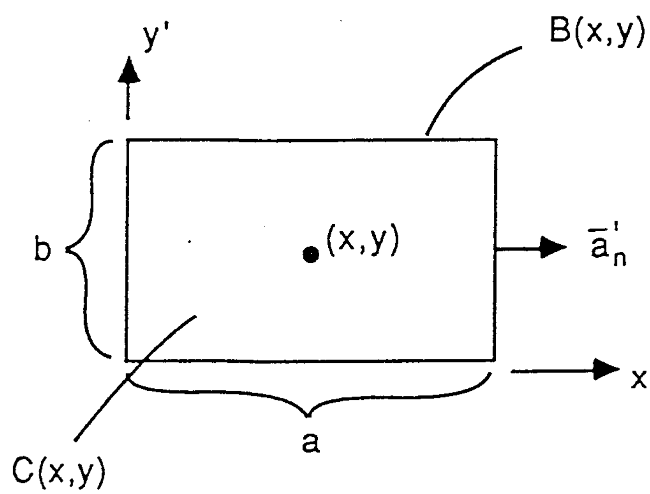

Fig. 2: Geometry of a period cell C(x,y).

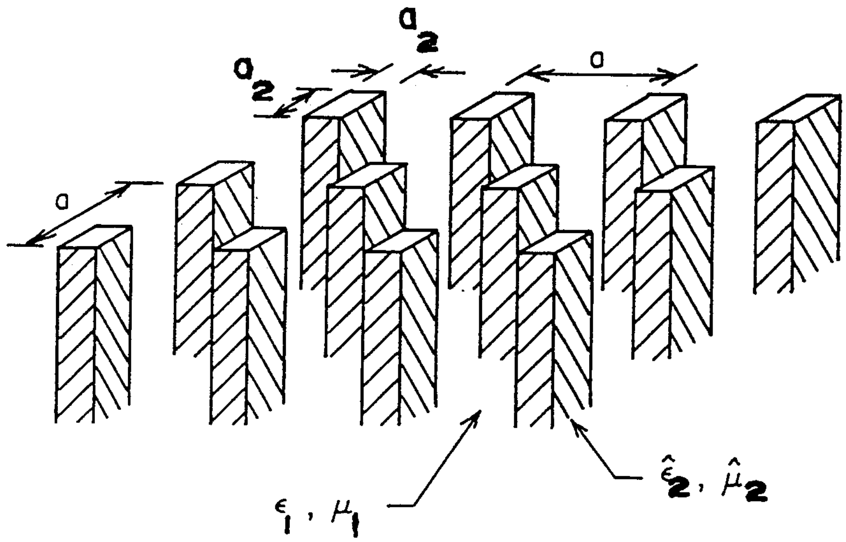

Fig. 3: Square array of square absorbing rods.

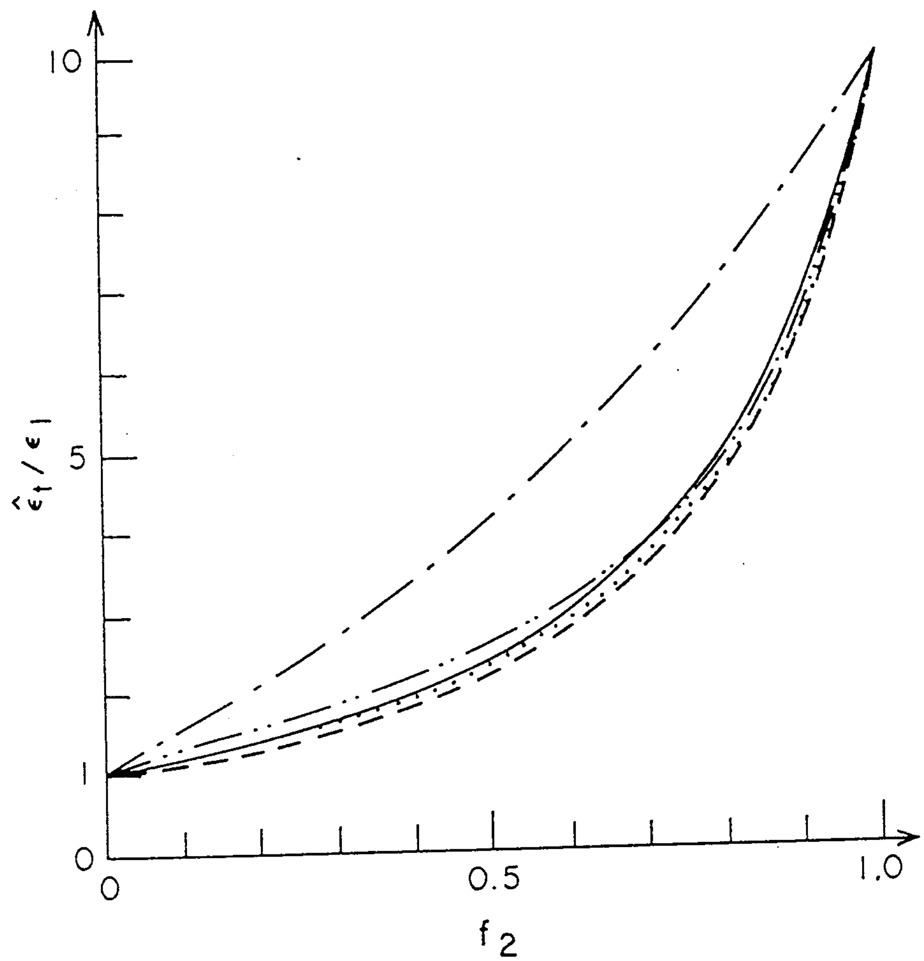

Fig. 4: Comparison of the approximate and numerical results for $\hat{\varepsilon}_t$ of the array of Fig. 3, with $\varepsilon_2 = 10\ \varepsilon_1$:
——— finite-element [18]. —— ·· —— eqn. (49),
—— —— —— eqn. (48), —— · —— eqn. (47), ·.··· eqn. (46).

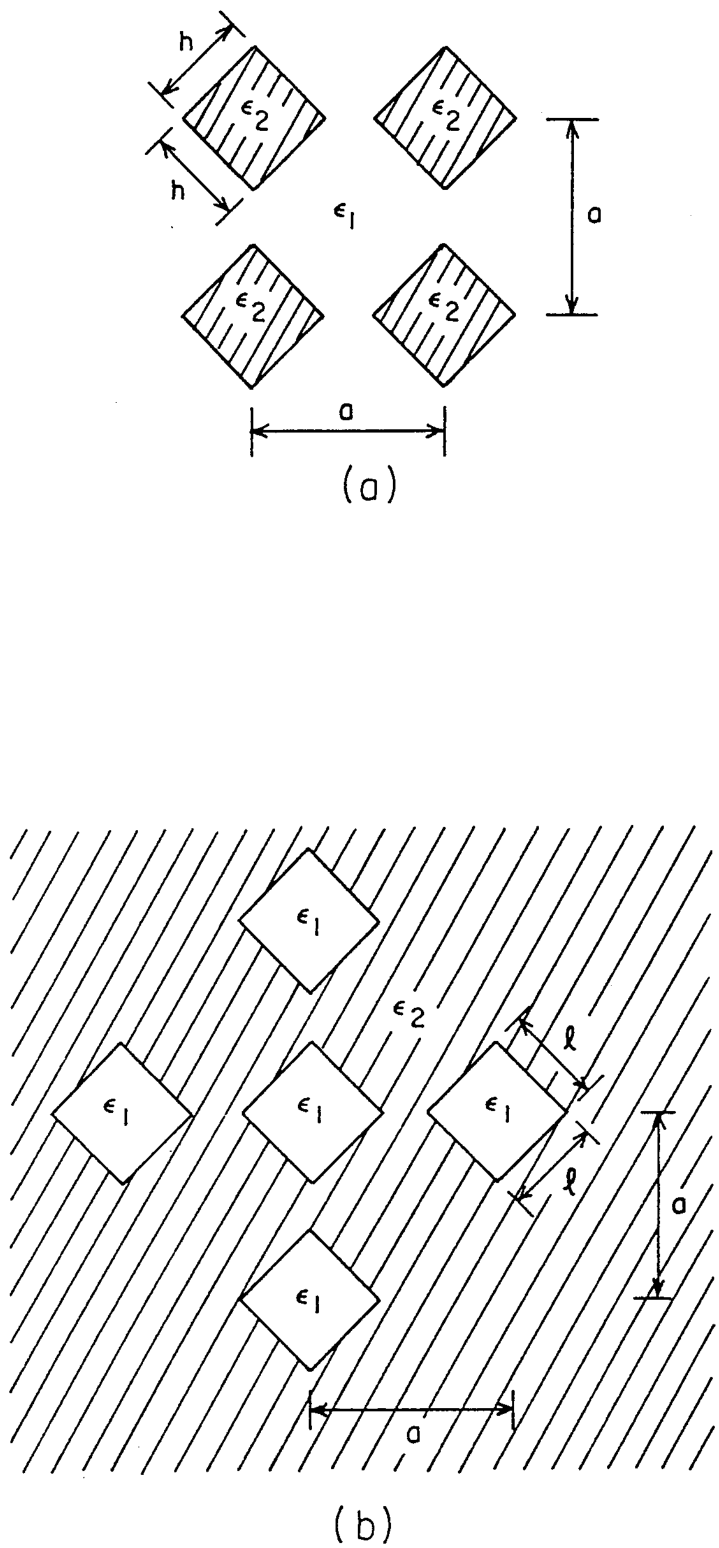

Fig. 5: Generalized checkerboard array: (a) $f_2 = h^2/a^2 < 1/2$; (b) $f_2 = 1 - l^2/a^2 > 1/2$.

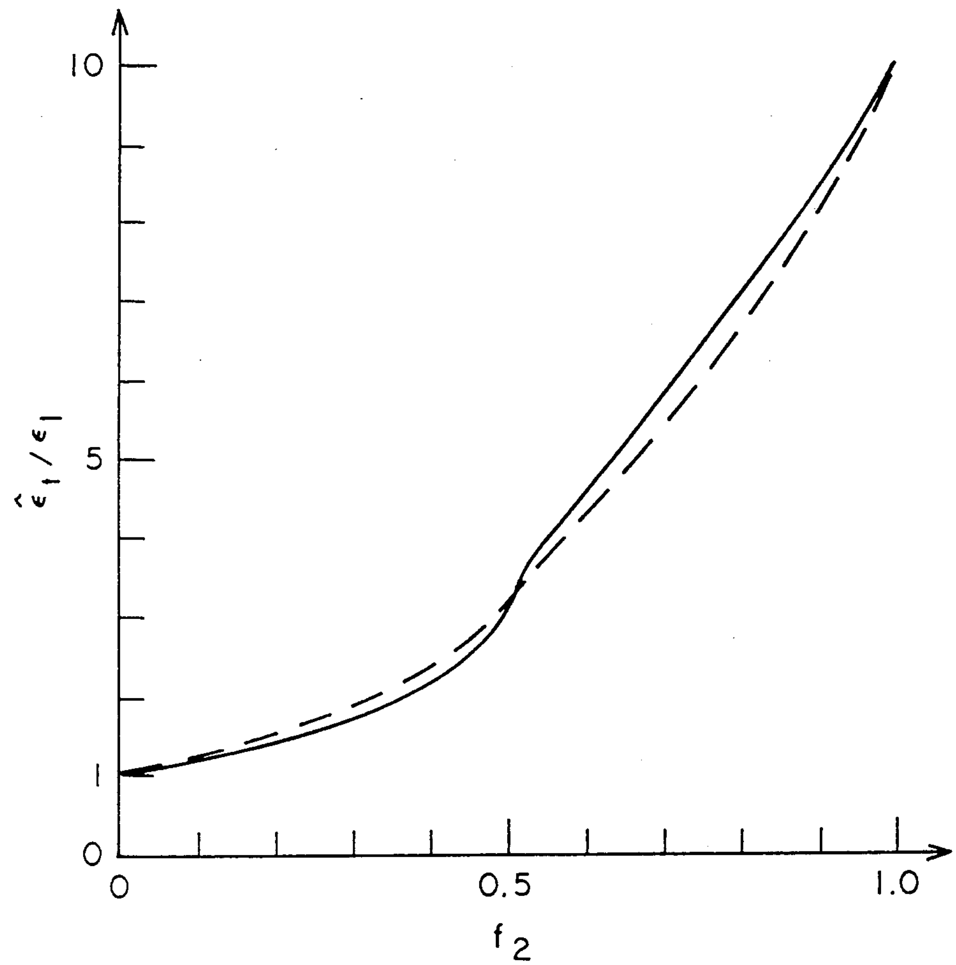

Fig. 6: Comparison of approximate (— — —, eqn. (52)) and numerical (———, [17], [52]) results for $\hat{\varepsilon}_t$ of the array of Fig. 5 with $\varepsilon_2 = 10\ \varepsilon_1$.

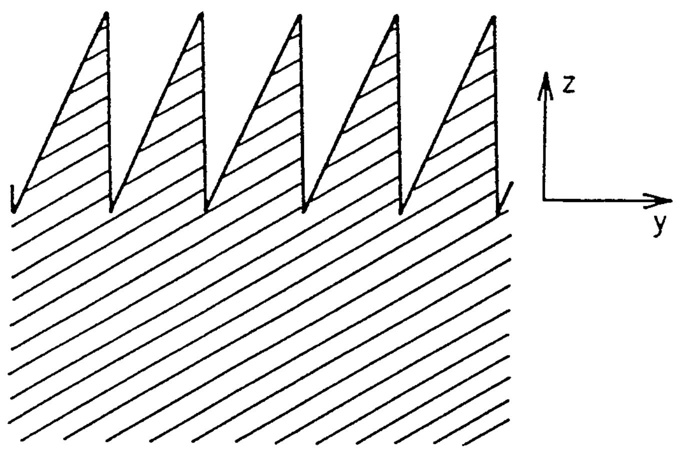

Fig. 7: Tapered transversely periodic waveguide.