The Log Product Formula and Deformations of Modules

by

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B.S., Stony Brook University, 2014

A thesis submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

2019

This thesis entitled:
The Log Product Formula and Deformations of Modules
written by L. S. Herr
has been approved for the Department of Mathematics

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Date

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Herr, L. S. (Ph.D., Mathematics)

The Log Product Formula and Deformations of Modules

Thesis directed by Prof. Prof. Jonathan Wise

The material of this thesis is drawn from two distinct papers. One part concerns deformations of modules. We pose an extension problem, the possibility and number of solutions of which are encoded in a banded gerbe on a topos of modules. This object represents a class $\underline{\mathrm{Def}} \in H^2(A\text{-mod}/M, \hbar_K)$ in cohomology.

Work of L. Illusie [29] produced a class $f \smile \omega \in \mathrm{Ext}^2_A(M, K)$ with similar properties, and questioned whether an approach along the lines of the present work was possible. We show that the two groups $H^2(A\text{-mod}/M, \hbar_K)$ and $\mathrm{Ext}^2_A(M, K)$ are isomorphic in such a way that our class is mapped to the inverse of Illusie's.

Our topology is an analogue for modules of that found in [71], [70]. These papers contain an erroneous argument; the present work is logically independent. Our work depends nevertheless on the ideas of those papers, and similarly circumvents the cotangent complex in the hopes of greater concreteness and simplicity.

The other portion of this thesis concerns logarithmic, or "log" geometry and its relationship with intersection theory. The standard intersection theory toolkit of normal cones, virtual fundamental classes, and Gysin Maps was introduced implicitly for the log context in the seminal work [23]. These definitions depend on the stacks and deformation theory developed by Olsson [60], [61]. Our techniques also rely on Artin Fans [2], Log Blowups [37], and a strictifying factorization found in [33].

To the author's knowledge, the present work is the first to explicitly examine and define log normal cones and their relatives. We aspire to develop this new technology further in future work, and regard the paper as a survey of foundational results in this direction. As an application, we prove a log version of a well-known theorem which computes the Gromov-Witten Invariants of a

product in terms of those of its factors. This extends results proved in [42].

Dedication

To my parents and grandparents, without whom this thesis wouldn't have an author.

Acknowledgements

I am enormously grateful to my advisor, Jonathan Wise, for a nearly constant stream of good ideas and exactly the right amount of patience. Thanks go also to the rest of my beneficent defense committee: Sebastian "Yano" Casalaina-Martin, Agnès Beaudry, Markus Pflaum, and Mark Shoemaker. Every Math Professor at the University of Colorado Boulder who has ever taught me anything deserves recognition, notably: David Grant, Katherine Stange, Alexander Gorokhovsky, and Su-ion Ih. The anonymous referee who edited the paper on which Chapter 5 is based provided invaluable conceptual feedback. The assiduous faculty of the Boulder Math Department deserve acknowledgment as well. I would also like to thank ...

- Bitsy "Sarah" Arpin: for positive reinforcement, diminutive stature, and Lily. "···!"
- Noah Williams, B.M.: for notes, moderation, and studying tactics.
- Carlos Pinilla, gourmand and audiophile: for sharing food, music, and messy roommates.
- Robert Hines: for his energizing zest for life during periods of weariness and anxiety.
- Paul Lessard: for finding the simple signal in the face of infinite noise.
- Sebastian Bozlee: for canonicity, concreteness, and reading this thesis.
- Patrick McFaddin: for encouragement when I struggled to digest foreign concepts.
- Tim Triggs and Mike Hendricks: for interesting counting problems with foggy expectations.
- Christian Walters and Stefan Evans: for broadening cultural experiences.
- Lucy "fatcan" Reid: for receiving all the recessive genes with Grace.

Contents

Chapter 0

How to Read This

> But where to start? he said Boston? College? 12th, 11th, 10th, 9th, 8th! 7th, 6th, 5th, 4th, 3rd, 2nd, 1st Grade? Kindergarten? Supervised play area? *Birth?* No! I am blameless. If for no other reason than that my initial error can't be pinpointed. Whose can be? Fisher shut his mouth for a while, as he was gradually overcome with an old fear[:] That of all his misfortunes owing to his complete ignorance of the multiplication table.
>
> [51, pg. 223]

We implore the reader to skip the first three chapters of the present volume. The original mathematical content is entirely contained in Chapters 4 and 5. The rest is meant as an expedient introduction to concepts used therein, and will have served its purpose if it temporarily relieves the readers of the burden of familiarizing themselves with the broader literature surrounding topoi, intersection theory, and logarithmic geometry.

The reader who wishes to gain any reasonable grasp on them should take up [52], [20], [63, 02P3], [38], [37], and [59] in earnest. Knowing full well that these references are irreplaceable, we have followed them as unerringly and completely as possible. We apologize to the original sources for not quoting them in full and to the reader for the illusion that such sources may be skipped.

Chapter 4 precedes Chapter 5 because the results are more important. We will make many citations to the stacks project [63] and use only the 4-character tag as reference.

Chapter 1

Categories and Deformations

> In mathematical cognition, insight is an activity external to the thing; it follows that the true thing is altered by it. The means employed, construction and proof, no doubt contain true propositions, but it must none the less be said that the content is false. In the [Pythagorean Theorem] the triangle is dismembered, and its parts consigned to other figures, whose origin is allowed by the construction upon the triangle. Only at the end is the triangle we are actually dealing with reinstated. During the procedure it was lost to view, appearing only in fragments belonging to other figures. Here, then, we see the negativity of the content coming in as well; this could just as much have been called a 'falsity' of the content as is the disappearance of supposedly fixed conceptions in the movement of the Notion.
>
> [10, Preface, # 43, pg. 25]

1.1 Categorical Preliminaries

Fix a universe without mention and consider only "small" sets lying in the universe, (Set). Likewise, restrict attention to the 2-category of small categories (Cat), the category of small topological spaces (Top), the category of small schemes (Sch), etc. We reserve the right to emphasize or omit the adjective "small."

1.1.1 Topoi

The goal of topoi and sites is to generalize intuition from sheaves on a topological space to more exotic categories. We replace "open covers" by more general notions of cover which can include covering spaces, smooth submersions, flat and surjective maps, etc.

Recall that a presheaf F on a category C is a contravariant functor

$$F : C^{op} \to (Set).$$

Morphisms are natural transformations. The category of presheaves on a fixed category C is denoted by $\widehat{C}$. The reader is left to formulate notions of presheaves of abelian groups, rings, etc. on their own.

Definition 1.1.1. A full subcategory $D \subseteq C$ is a *sieve* if, whenever $X \in D$ and there exists an arrow $Y \to X$, then $Y \in D$. We may identify sieves D with subpresheaves of the final presheaf. If $X \in C$, a *sieve on* X is a sieve of the "slice" or "over"-category C/X. We can generate a sieve from any subset of C. Write $\hbar_X$ for the Yoneda Presheaf:

$$\hbar_X(T) := \mathrm{Hom}(T, X).$$

A *(Grothendieck) Topology* on a category C is a distinguished collection of sieves $J(X)$ on each object $X \in C$ called *covering sieves* satisfying

- The maximal sieve $\hbar_X$ is in $J(X)$,
- If $D \in J(X)$ and $f : Y \to X$, the sieve f^*D of all $Z \to Y$ such that $Z \to Y \to X \in D$ is covering for Y, and
- If $D \in J(X)$ and E any sieve on X. If the pullback sieve f^*E is in $J(Y)$ for all $f : Y \to X \in D$, then E is in $J(X)$.

A category with a topology is called a *site*. A *sheaf* on a site (C, J) is a presheaf F on C such that the restriction map

$$F(X) = \mathrm{Hom}_{\widehat{C}}(\hbar_X, F) \to \mathrm{Hom}_{\widehat{C}}(D, F)$$

is a bijection for all covering sieves $D \in J(X)$. A *(Grothendieck) Topos* is a category E which is equivalent to the category $\widetilde{C}$ of sheaves on some site (C, J).

Given two topologies J, J' on a category C, we say that J' *is finer than* J if, for any $X \in C$, $J(X) \subseteq J'(X)$. This endows the set of topologies on C with a poset structure. This poset structure admits suprema [52, II.1.1.3].

A common way to specify a Grothendieck Topology on a category C with fiber products is to provide, for each $X \in C$, a collection of distinguished families of arrows $\{Y_i \to X\}$ called *covering families*. This assignment must satisfy:

- All isomorphisms are covering families:

$$\{X' \simeq X\}.$$

- Covering families are stable under pullback: if $\{Y_i \to X\}$ is covering and $Z \to X$ is any morphism, then

$$\{Y_i \times_X Z \to Z\}$$

  is covering.

- Transitivity: If $\{Y_i \to X\}$ is a covering family and we choose a covering family $\{Z_{ij} \to Y_i\}$ for each Y_i, then the set of composites

$$\{Z_{ij} \to Y_i \to X\}$$

  is covering.

A family of arrows $\{Y_i \to X\}$ defines a sieve D on X consisting of all $T \to X$ which factor through some $Y_i \to X$. In this way, a collection of covering families yields a Grothendieck Topology.

If a topology is described by a collection of covering families, the sheaf condition for a presheaf F may be expressed as the exactness of the sequences

$$F(X) \to \prod F(Y_i) \rightrightarrows \prod F(Y_i \times_X Y_j)$$

for any covering family $\{Y_i \to X\}$ [52, Corollaire 2.4].

Example 1.1.2. We collect a handful of examples of sites [58, Grothendieck pretopology – Revision 20], [63, 00UZ].

- The example which motivated the development of Grothendieck Topologies is the étale site. Let $S \in (Sch)$ be a scheme. Define the *big étale site* $\mathrm{Big}_{\text{ét}}(Sch)/S$ of S to be the category $(Sch)/S$ with the topology generated by covering families

$$\{Y_i \to X\}$$

  in which every arrow is étale and the family is jointly surjective.

- For set-theoretic reasons, it is often preferable to work with a modification of the first example. Consider the category of S-schemes $X \to S$ whose structure map is étale. Endow this with the same topology; namely, that covering families are families of étale maps which are jointly surjective. This is called the *small étale site*, denoted $(Sch/S)_{\text{ét}}$.

- An ordinary topology $\mathcal{T} \subseteq 2^X$ on a set X is naturally a poset with Grothendieck Topology, the covers being given by jointly surjective inclusions of open sets. In particular, $\{U_i \subseteq X\}$ is a covering family if and only if $\bigcup U_i = X$.

- A topology on the category of topological spaces itself may be generated by surjections $\{Y \to X\}$ which admit sections over a cover $\bigcup U_i = X$:

$$\begin{tikzcd} & Y \arrow[d] \\ U_i \arrow[ur] \arrow[r, hook] & X. \end{tikzcd}$$

- The category of $\mathbb{R}$-manifolds has a topology given by surjective submersions.

- The *chaotic* or *indiscrete* topology on any category is the topology whose covering families are just the isomorphisms $\{X' \simeq X\}$. Equivalently, the only covering sieve of an object X is $\mathcal{h}_X$. Sheaves for this topology are the same as presheaves.

Beware that seemingly different systems of covering families may induce equivalent topoi. E.g., consider the topologies generated by jointly surjective smooth morphisms and jointly surjective étale morphisms [63, 055V].

Definition 1.1.3. Consider the set of covering sieves $J(X)$ of an object X in a site C. Order $J(X)$ by inclusion. If F is a presheaf on C, define

$$LF(X) := \operatorname{colim}_{D \in J(X)} \operatorname{Hom}_{\widehat{C}}(D, F).$$

The collection of covering sieves $J(X)$ is natural in X due to pullbacks of covers remaining covers; via these maps, the colimits over them obtain a functoriality, and $LF(\cdot)$ is a presheaf.

The natural inclusion $\hbar_X \in J(X)$ gives rise to a map

$$\ell : F \to LF,$$

which may be thought of as a morphism of endofunctors $id \Rightarrow L$ on $\widehat{C}$.

The presheaf LF isn't quite a sheaf, but $L(L(F))$ always is. If F is already a sheaf, then $\ell : F \to L(L(F))$ is an isomorphism. The functor L is left-exact [52, Proposition II.3.2].

In other words, the inclusion of the category of sheaves into that of presheaves $i : \widetilde{C} \subseteq \widehat{C}$ admits an exact left adjoint $L \circ L$. Given a presheaf F, we write

$$F^{sh} := L(L(F))$$

and call this object the "sheafification" of F.

Remark 1.1.4. Warning: There are presheaves on certain sites with no sheafification! See [67, Theorem 5.5] for an example for the fpqc topology. The issue is set-theoretic, and disappears if the site possesses a set of topological generators or if one is careful with Grothendieck Universes.

Remark 1.1.5. Warning: The inclusion $i : \widetilde{C} \subseteq \widehat{C}$ is left-exact but not right-exact in general!

Example 1.1.6. The presheaves $\hbar_X$ representing objects $X \in C$ are not sheaves in general. In other words, the Yoneda Embedding

$$C \to \widehat{C}; \qquad X \mapsto \hbar_X$$

needn't factor through the subcategory $\widetilde{C} \subseteq \widehat{C}$. We consider the sheafified Yoneda Embedding

$$C \to \widehat{C} \to \widetilde{C}; \qquad X \mapsto (h_X)^{sh}$$

in bad cases.

The topology of C is said to be *subcanonical* if the presheaves h_X are already sheaves. Because the poset of topologies admits suprema, there is a finest subcanonical topology, called the *canonical topology.*

In "nice" categories [58, regular category – Revision 50], the canonical topology has a simple description: A morphism $\{Y \to X\}$ is covering if the diagram

$$Y \times_X Y \rightrightarrows Y \to X$$

is a coequalizer. The map $Y \to X$ is then called an *effective epimorphism.*

Functoriality among topoi boils down to the observation: Restriction of functors may have left and right adjoints, provided the target category has sufficient limits or colimits.

Definition 1.1.7. Suppose $u : C \to D$ is a functor and E is a category. Precomposition defines a functor

$$u^* : \underline{\mathrm{Hom}}(D, E) \to \underline{\mathrm{Hom}}(C, E).$$

If this functor has adjoints, the right adjoint Ran_u is called the "Right Kan Extension" and the left adjoint Lan_u is called the "Left Kan Extension." Explicitly, consider $F : D \to E$ and $G : C \to E$. We get equivalences between natural transformations:

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}(C,E)}(G, F \circ u) = \mathrm{Hom}_{\underline{\mathrm{Hom}}(D,E)}(\mathrm{Lan}_u G, F),$$

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}(C,E)}(F \circ u, G) = \mathrm{Hom}_{\underline{\mathrm{Hom}}(D,E)}(F, \mathrm{Ran}_u G).$$

The picture:

$$\begin{tikzcd} C \arrow[r, "G"] \arrow[d, "u"'] & E \\ D \arrow[ur, bend right] \arrow[u, phantom, ""] & \end{tikzcd} \arrow[phantom, "\Downarrow"] \quad \rightsquigarrow \quad \begin{tikzcd} & E. \\ D \arrow[ur, bend left, "\mathrm{Lan}_u G"] \arrow[ur, bend right] \arrow[ur, phantom, "\Downarrow"] & \end{tikzcd}$$

Remark 1.1.8. There is a dictionary at the heart of category theory which defines Kan Extensions, limits/colimits, adjoints, etc. all in terms of each other. For example, the colimit of a diagram $G : C \to E$ is its left Kan Extension along the functor $C \to *$ to the punctual category.

If E in the diagram

$$\begin{tikzcd} C \arrow[r, "G"] \arrow[d, "u"] & E \\ D & \end{tikzcd}$$

has suitable limits/colimits, we show how to construct the Kan Extensions of G [52, Proposition I.5.1].

For $d \in D$, let $d\backslash u$ denote the category where

- Objects are $c \in C$ together with $d \to u(c)$.
- Morphisms are morphisms $c \to c'$ which, after applying u, commute with the maps from d.

There is a natural diagram in E for any $d \in D$ given by

$$d\backslash u \to C \stackrel{G}{\to} E.$$

Then the Right Kan Extension can be specified on an object d by

$$\mathrm{Ran}_u G(d) := \lim(d\backslash u \to C \stackrel{G}{\to} E).$$

Likewise, define a category u/d for $d \in D$ to be objects $c \in C$ together with a map $u(c) \to d$. The left Kan Extension has the opposite formula

$$\mathrm{Lan}_u G(d) := \mathrm{colim}(u/d \to C \stackrel{G}{\to} E).$$

We will be primarily interested in the case $E = (Set)$ and our sources are opposite categories $u : C^{op} \to D^{op}$. Our formulas become:

$$\mathrm{Lan}_u G(d) := \mathrm{colim}(d\backslash u \to C^{op} \to G),$$

$$\mathrm{Ran}_u G(d) := \lim(u/d \to C^{op} \to G).$$

In this case both always exist, and we write $u_! G := \mathrm{Lan}_u G$, $u_* G := \mathrm{Ran}_u G$.

Note that $u_! \hbar_X = \hbar_{u(X)}$.

Definition 1.1.9. Let C, D be sites. A functor $u : C \to D$ "continuous" if restriction along u sends sheaves to sheaves:

$$\begin{tikzcd} \widetilde{D} \arrow[r, dashed] \arrow[d, hook] & \widetilde{C} \arrow[d, hook] \\ \widehat{D} \arrow[r, "u^*"] & \widehat{C}. \end{tikzcd}$$

This means that if $G \in \widetilde{D}$ is a sheaf on E, the precomposition $G \circ u$ is a sheaf on C.

A functor $u : C \to D$ is called "cocontinuous" if the right adjoint u_* of the restriction u^* sends sheaves to sheaves.

If we want to distinguish between the functors $u_!, u^*, u_*$ on presheaves or on sheaves, we may write

$$\widehat{u}_!, \widehat{u}^*, \widehat{u}_*$$

for presheaves and

$$\widetilde{u}_!, \widetilde{u}^*, \widetilde{u}_*$$

for sheaves.

Proposition 1.1.10. We recall intrinsic definitions of continuous and cocontinuous for functors $u : C \to D$ between sites.

- The functor u is continous if and only if [52, Proposition III.1.2]:

  * For all covering sieves $R \subseteq \hbar_X$ in C, the sheafification of the map of presheaves $u_! R \subseteq u(X)$ on D is an isomorphism.

  * There exists a functor $u_! : \widetilde{C} \to \widetilde{D}$ commuting with colimits which extends u:

    $$\begin{tikzcd} C \arrow[r, "u"] \arrow[d] & D \arrow[d] \\ \widetilde{C} \arrow[r, dashed, "u_!"] & \widetilde{D}. \end{tikzcd}$$

    The functors $(u_!, u^*)$ form an adjoint pair.

  * If both topologies are defined by covering families and u commutes with fiber products, then continuity is equivalent to:

– If $\{Y_i \to X\}$ is covering in C, then $\{u(Y_i) \to u(X)\}$ is covering in D.

- The functor u is cocontinuous if and only if

  * For every covering sieve $R \subseteq \hbar_{u(Y)}$ in D, the sieve generated by all $Z \to Y$ such that $u(Z) \to u(Z)$ is in R covers Y in C [52, Proposition III.2.2]. The sieve described is

$$u^*R \times_{u^*\hbar_{u(Y)}} \hbar_Y.$$

.

Example 1.1.11.

If (u, v) is an adjoint pair of functors, then u is cocontinuous if and only if v is continuous [52, Proposition III.2.5].

If $u : C \to D$ is a functor and D is equipped with a topology, there is a finest topology on C making u continuous. It is named the "induced topology." A sieve $R \subseteq \hbar_X$ is covering for the induced topology on C if and only if, for all $Y \to X$, the sieve

$$u_!(R \times_{\hbar_X} \hbar_Y) \to \hbar_{u(Y)}$$

becomes an isomorphism after sheafification on D. If fiber products exist in both categories and u commutes with them, a family $\{Y_i \to X\}$ is covering for the induced topology on C if and only if $\{u(Y_i) \to u(X)\}$ is covering on D. For any site C, the topology on C is the induced topology from the Yoneda Embedding

$$C \to \widetilde{C}$$

and the canonical topology on the target [52, Propositions III.3.2, III.3.5, Corollaire III.3.3].

If F is a presheaf on a site C, we endow the overcategory C/F with the topology induced from the functor $j_F : C/F \to C$. The functor j_F is also cocontinuous. If $Y \in C$, $X \in C/Y$, then sieves of X in C/Y are the same as those of $X \in C$ and this bijection takes covering sieves to covering sieves [**?**].

Theorem 1.1.12 (Comparison Lemma [52, Théorème III.4.1])**.** Let $i : C \subseteq D$ be a full subcategory of a site, endowed with the induced topology. If every object $Y \in D$ can be covered by objects belonging to C, then restriction of sheaves

$$i^* : \widetilde{D} \to \widetilde{C}$$

is an equivalence. If the topology of D is subcanonical, the converse is also true.

Example 1.1.13. Consider the category of affine schemes $C = (Aff)$ and the category of all schemed $D = (Sch)$. Every scheme may be covered by affines as part of the definition. The Comparison Lemma implies that the restriction

$$\widetilde{(Sch)} \to \widetilde{(Aff)}$$

is an equivalence.

Consider a fixed topological space X and the poset of its open subsets $\mathrm{Ouv}(X)$. Let B be a basis of open sets, and consider it with its natural sub-poset structure of $\mathrm{Ouv}(X)$. Then the Comparison Lemma says restriction

$$\widetilde{\mathrm{Ouv}(X)} \simeq \widetilde{B}$$

is an equivalence.

1.1.2 Simplicial Objects

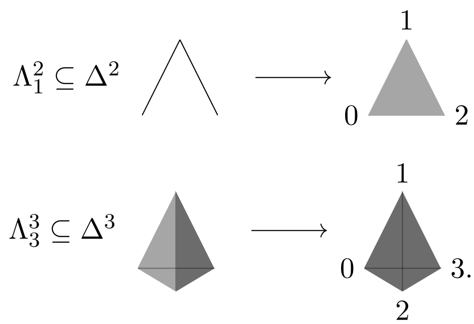
Definition 1.1.14. The category Δ has objects $[n]$ given by the poset $0 \leq 1 \leq \cdots \leq n$ with morphisms respecting the partial order $\leq$. It is equivalent to the category of linearly ordered finite nonempty sets. All morphisms may be expressed as composites of the codegeneracy $\sigma_i : [n] \to [n-1]$ and coface maps $\partial_i : [n] \to [n+1]$: the former is surjective and sends $i, i+1$ both to i, while the latter is injective and omits i.

Denote by $\widehat{\Delta}$ the category of presheaves on Δ, referred to as "simplicial sets." Given a simplicial set X, write X_n instead of $X([n])$ and write the restriction maps as $s_i = \sigma_i^* : X_{n-1} \to X_n$

and $d_i = \partial_i^* : X_{n+1} \to X_n$. The Yoneda embedding

$$\Delta \to \widehat{\Delta}$$

gives us standard simplicial sets that we will write $\Delta^n := \hbar_{[n]}$. The kth horn $\Lambda^n_k \subseteq \Delta^n$ is the subpresheaf corresponding to the sieve generated by the coface maps ∂_i for $i \neq k$:



We call a functor $\Delta \to C$ a *cosimplicial object* in C and $\Delta^{op} \to C$ a *simplicial object* in C. We denote simplicial objects in C as $C^{\Delta^{op}}$.

Remark 1.1.15. For C, D two categories, let $\underline{\mathrm{Hom}}_\star(C, D)$ temporarily denote the full subcategory of the functor category whose objects are functors $f : C \to D$ which preserve colimits.

The Yoneda embedding into the presheaf category $C \to \widehat{C}$ is the "free cocompletion" of C in the sense that restriction induces an equivalence of categories:

$$\underline{\mathrm{Hom}}_\star(\widehat{C}, D) = \underline{\mathrm{Hom}}(C, D).$$

There is a canonical cosimplicial category

$$[\cdot] : \Delta \to (Cat).$$

defined by $[\cdot](n) := [n]$, regarded as a poset category. This gives rise to a fully faithful *nerve* functor

$$(Cat) \longrightarrow \widehat{\Delta}$$

$$C \longmapsto N(C),$$

where n simplices of $N(C)$ are given by

$$N(C)_n := \mathrm{Hom}_{(Cat)}([n], C).$$

By the above free cocompletion property, we have

$$\underline{\mathrm{Hom}}_\star(\widehat{\Delta}, (Cat)) = \underline{\mathrm{Hom}}(\Delta, (Cat)).$$

Therefore our cosimplicial category $[\cdot]$ extends naturally to a colimit-preserving functor

$$\widehat{\Delta} \to (Cat).$$

This functor is a left adjoint to the nerve N called the "homotopy category" [47, Definition 1.1.5.14].

As in [47, Proposition 1.1.2.2], the fully faithful functor $N : (Cat) \to \widehat{\Delta}$ has essential image those simplicial sets S which satisfy a horn-filling condition: for any bold diagram and $0 < k < n$, there exists a unique lift

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & S \\ \Delta^n \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

This condition implies in particular that the simplicial set S is 2-coskeletal; i.e., that the functor $S : \Delta \to (Set)$ is determined by its values S_1, S_0 and the structure maps between them. We may identify categories with their nerve simplicial sets.

The 2-coskeletal condition on a small category C means it is determined by a diagram of sets

$$\begin{tikzcd} \mathrm{Mor}\ C \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathrm{Ob}\ C. \arrow[l, "id"'] \end{tikzcd}$$

The horn-filling condition for $\Lambda^2_1 \subseteq \Delta^2$ gives us the composition law

$$\mathrm{Mor}\ C \times_{s, \mathrm{Ob}\ C, t} \mathrm{Mor}\ C \to \mathrm{Mor}\ C.$$

The required associativity and compatibility with identity come from the horn-filling condition for Δ^3.

Recall that a category is called a *groupoid* if every arrow is an isomorphism. Equivalently, there is a function

$$i : \mathrm{Mor}\ C \to \mathrm{Mor}\ C$$

such that $i(f)$ is the inverse to f [63, 0230].

1.1.3 Generalities on Categories

Definition 1.1.16. A *2-category C* is a category enriched in categories [62, Appendix B], [63, 003D]. This means C has objects $\mathrm{Ob}\, C$ and, given any pair $x, y \in \mathrm{Ob}\, C$, there's a category $\mathrm{Hom}_C(x, y)$ of morphisms between them. The objects of the category $\mathrm{Hom}_C(x, y)$ are referred to as 1-morphisms and the morphisms are 2-morphisms. The diagram

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\eta"] & y \end{tikzcd}$$

means f, g are objects of $\mathrm{Hom}_C(x, y)$ and η is a morphism between them.

In this setting, "composition" of 2-morphisms can have two distinct meanings:

- Composition within a single category $\mathrm{Hom}_C(x, y)$. This is called "vertical composition:"

$$\begin{tikzcd} x \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow\zeta"] \arrow[r, phantom, shift right=3, "\Downarrow\eta"] & y. \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\eta\circ\zeta"] & y \end{tikzcd}$$

- Composition via the composition functor from the enriched category C, called "horizontal composition:"

$$\circ : \mathrm{Hom}_C(x, y) \times \mathrm{Hom}_C(y, z) \to \mathrm{Hom}_C(x, z).$$

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow\eta"] & y \arrow[r, bend left, "g'"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\zeta"] & z \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} x \arrow[r, bend left, "g'\circ g"] \arrow[r, bend right, "f'\circ f"'] \arrow[r, phantom, "\Downarrow\eta\star\zeta"] & y \end{tikzcd}$$

A *(2, 1)-category* is a 2-category in which all 2-morphisms are invertible with respect to vertical composition.

We remark that there is a sense in which one can compose 2-morphisms with 1-morphisms in a 2-category C sometimes called "whiskering." Let $f \in \mathrm{Hom}_C(x, y)$ and $\eta : g \Rightarrow g' \in \mathrm{Hom}_C(y, z)$:

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\eta"] & z. \end{tikzcd}$$

These data yield a 2-morphism $\eta \star id_f$:

$$\begin{tikzcd} x \arrow[r, bend left, "g\circ f"] \arrow[r, bend right, "g'\circ f"'] \arrow[r, phantom, "\Downarrow\eta\star id_f"] & z. \end{tikzcd}$$

We consider the 2-category (Cat) as a (2, 1)-category. That is, it has

- Objects: small categories C.
- Morphisms: Functors between categories.
- 2-Morphisms: Natural *isomorphisms* between functors.

We disallow general natural transformations in (Cat) to cohere with [47] and the conventions for algebraic stacks in [63]. It also simplifies the discussion of 2-fiber products:

Definition 1.1.17. A *2-commutative square* is a diagram of functors between categories:

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "g"] & B \arrow[d, "f"] \\ C \arrow[r, "p"] & D \end{tikzcd}$$

together with a natural isomorphism $\eta : f \circ q \simeq p \circ g$.

The *2-fiber product* of a cospan $C \xrightarrow{p} D \xleftarrow{f} B$ is the category $C \times_D B$ with:

- Objects given by pairs of objects $c \in \mathrm{Ob}\, C, b \in \mathrm{Ob}\, B$ together with an isomorphism $p(c) \simeq f(b)$.
- Morphisms $(c, b) \to (c', b')$ are pairs of morphisms $c \to c' \in C$, $b \to b' \in B$ such that the square

$$\begin{tikzcd} p(c) \arrow[r] \arrow[d, dash, "\sim"] & p(c') \arrow[d, dash, "\sim"] \\ f(b) \arrow[r] & f(b') \end{tikzcd}$$

  commutes in D.

The 2-fiber product sits in a 2-commutative diagram

$$\begin{tikzcd} C \times_D B \arrow[r, "p'"] \arrow[d, "f'"] & B \arrow[d, "f"] \\ C \arrow[r, "p"] & D \end{tikzcd} \qquad \eta_0 : f \circ p' \simeq p \circ f',$$

with f', p' given by projection onto c, b and the natural isomorphism η given by the structure isomorphisms from each object.

This category satisfies an important property: given any 2-commutative square

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "g"] & B \arrow[d, "f"] \\ C \arrow[r, "p"] & D \end{tikzcd} \qquad \eta : f \circ q \simeq p \circ g,$$

there exists a functor $z : A \to C \times_D B$ and a natural transformation η_0 such that the horizontal composition

$$\begin{tikzcd} A \arrow[r, "z"] & C \times_D B \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \eta_0"] & D \end{tikzcd}$$

gives the arrow $\eta = \eta_0 \star id_z$. Moreover, any two such functors $z : A \to C \times_D B$ are isomorphic.

Remark 1.1.18. The "important property" of the 2-fiber product should be taken as a definition. All 2-commutative squares with that property are equivalent.

For more general limits of categories, one should resort to the "homotopy limit" which serves as a notion of limit for ∞-categories. If C is a category enriched in nice topological spaces and $p : I \to C$ is a diagram, the map

$$\mathrm{Hom}_C(x, \lim_I p) \to \mathrm{holim}_I \mathrm{Hom}_C(x, p(\cdot))$$

should be a homotopy equivalence. We won't use these homotopy limits explicitly, so we don't discuss them further.

Say a category I is *finite* if Ob I and Mor I are both finite (possibly empty). Given a category C, define a category $C^{\triangleleft}$ to have

- Objects $\{``-\infty"\} \coprod \mathrm{Ob} C$

- Morphisms

$$\mathrm{Hom}_{C^{\triangleleft}}(x,y) := \begin{cases} \mathrm{Hom}_C(x,y) & \text{if } x,y \in C \\ \\ \{*\} & \text{if } x = -\infty \\ \\ \emptyset & \text{if } y = -\infty, x \neq -\infty \end{cases}$$

As a simplicial set, $C^{\triangleleft}$ is the join $\Delta^0 \star C$. Write $C^{\triangleright} := ((C^{op})^{\triangleleft})^{op}$.

Definition 1.1.19. A category C is called *cofiltered* if, for every diagram $d : I \to C$ indexed by a finite category I, there is an extension of d to a functor $d^{\triangleleft} : I^{\triangleleft} \to C$. In other words, one can find another object of C with maps to every object in the image of d such that these maps commute with the maps in d.

We say C is *filtered* if its opposite category is cofiltered; equivalently, every finite diagram $d : I \to C$ can be extended to $d^{\triangleright} : I^{\triangleright} \to C$.

We say the limit or colimit of a functor $F : C \to D$ is cofiltered or filtered if C has that property.

Remark 1.1.20. A category C is cofiltered if and only if it satisfies:

(1) The category C is nonempty.

(2) Given a cospan depicted by solid arrows:

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d] \\ Y \arrow[r] & Z, \end{tikzcd}$$

one can extend it to the commutative square with the dashed arrows.

(3) Given two arrows $f, g : X \rightrightarrows Y$, there is another $h : W \to X$ such that $f \circ h = g \circ h$.

Definition 1.1.21 ([52, Définition 2.4.1])**.** Let C be a category with finite limits and colimits. A functor $f : C \to D$ is *left-exact* if the natural map

$$f(\lim_I p(i)) \to \lim_I f(p(i))$$

is an isomorphism, for any finite diagram $p : I \to C$. It is called *right-exact* if the maps

$$\mathrm{colim}_I\, f(p(i)) \to f(\mathrm{colim}_I\, p(i))$$

are isomorphisms, for all finite diagrams $p : I \to C$. The functor is called *exact* if it is both left exact and right exact.

Given an adjoint pair (u, v) of functors, u is right-exact and v is left-exact.

Remark 1.1.22. Filtered colimits and cofiltered limits are often exact, meaning they commute with all finite limits and colimits. This is almost Grothendieck's AB5 condition [63, 079A]. This property holds for (Set) as well as any topos.

Definition 1.1.23. Fix a category C.

A map $X \to Y$ is an *epimorphism* if, for all $T \in C$, the restriction map

$$\mathrm{Hom}(Y, T) \to \mathrm{Hom}(X, T)$$

is injective. We say a family of arrows $\{X_i \to Y\}$ is an epimorphism if the restriction map as above to the product $\prod \mathrm{Hom}(X_i, T)$ is injective.

If C has products, we say that an epimorphism $f : X \to Y$ is

- *universal* if any pullback of f is also an epimorphism,

- *effective* if the diagram

$$X \times_Y X \rightrightarrows X \to Y$$

  is a coequalizer.

A *zero object* $0_C \in C$ in a category is an object which is both initial and final. If $f : X \to Y \in C$ is a morphism, the *kernel* of f is the pullback $0_C \times_Y X$. Likewise, the *cokernel* of f is the pushout $X \sqcup_Y 0_C$.

A functor $F : C \to D$ is called *conservative* if it "detects isomorphisms:" whenever $f : X \to Y \in C$ is such that $F(f)$ is an isomorphism in D, f is an isomorphism in C.

Definition 1.1.24. Let $D \subseteq C$ be a full subcategory. Consider the Yoneda embedding composed with restriction to D:

$$\varphi : C \to \widehat{C} \to \widehat{D}$$

$$X \mapsto \mathrm{Hom}(\,\cdot\,, X)$$

We say D *generates* C if φ is conservative. We say D generates C

- *by epimorphisms* if φ is faithful.
- *by strict epimorphisms* if φ is fully faithful.

We have the implications:

$$\begin{tikzcd} & \text{(generating by strict epimorphisms)} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \text{(generating)} & & \text{(generating by epimorphisms)} \end{tikzcd}$$

Remark 1.1.25. The reader may check that the conditions of Definition 1.1.24 may be simply expressed on objects:

- D generates C if and only if, given any morphism $X \to Y$ in C such that

  $$\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y)$$

  is a bijection for all $T \in D$, $X \to Y$ is an isomorphism.
- D generates C by epimorphisms if and only if, for all $Y \in C$, the family $\{T \to Y\}$ of all arrows with target Y and source $T \in D$ is an epimorphic family.
- D generates C by strict epimorphisms if that same family is not only epimorphic but strictly epimorphic.

Definition 1.1.26 ([47, A.1.1.2])**.** An object X in a category C which admits small colimits is called *finitely presentable* if the natural map

$$\mathrm{colim}_I \,\mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \mathrm{colim}_I\, Y_i)$$

is an isomorphism for I filtered.

A category C is called *presentable* if it admits small colimits and there is a subset $S \subseteq \mathrm{Ob}C$ such that every object in S is finitely presentable and every object in C may be obtained as the colimit of objects in S.

Example 1.1.27. The abelian group $\mathbb{Z}\left[\frac{1}{2}\right] = \mathrm{colim}_{n\in\mathbb{N}} \frac{1}{2^n}\mathbb{Z}$ is not finitely presented because the identity map doesn't lie in the image of any finite stage

$$\mathrm{Hom}_{abgrp}(\mathbb{Z}\left[\frac{1}{2}\right], \frac{1}{2^n}\mathbb{Z}).$$

Abelian groups and R-modules in general are presentable however, using the set of finitely presentable objects

$$\{R/I \,|\, I \subseteq R \text{ an ideal.}\}$$

A stack is intuitively defined as a sheaf of categories. We make sense of a (contravariantly) fibered category and then require gluing for both objects and morphisms, much the same as one defines a sheaf [65].

Definition 1.1.28. Consider a functor $p : F \to C$. Let $\varphi : \alpha \to \beta$ be an arrow in F with image $f : U \to V$. We say φ is *cartesian* if, for all $\psi : \gamma \to \beta$ lying over an arrow $g : W \to V$ and $h : W \to U$ such that $g = f \circ h$, there exists a unique arrow $\chi : \gamma \to \alpha$ such that $\varphi \circ \chi = \psi$ and $p(\chi) = h$. Pictorially,

$$\begin{tikzcd}
 & \alpha \arrow[r, "\varphi"] \arrow[dd, mapsto] & \beta \arrow[dd, mapsto] & & F \arrow[dd, mapsto, "p"] \\
\gamma \arrow[ur, dashed, "\exists!\chi"] \arrow[urr, bend right, "\psi"'] \arrow[dd, mapsto] & & & & \\
 & U \arrow[r, "f"] & V & & C. \\
W \arrow[ur, "h"] \arrow[urr, bend right, "g"'] & & & &
\end{tikzcd}$$

This amounts to the functor

$$F/\varphi \to F/\beta \times_{C/V} C/f$$

being an equivalence of categories [47, Remark 2.4.1.2].

A *fibered category* $p : F \to C$ is a functor such that, for any $f : U \to V$ in C and $\beta \in C$ with $p(\beta) = V$, there exists a cartesian arrow $\varphi : \alpha \to \beta$ with $p(\varphi) = f$. The *fiber* of p over some object $U \in C$ is the category F_U with:

- Objects: $\alpha \in F$ such that $p(\alpha) = U$.
- Morphisms: $\varphi : \alpha \to \beta$ such that $p(\varphi) = id_U$; in particular, $p(\alpha) = p(\beta) = U$.

Given an arrow $\varphi : \alpha \to \beta \in F$, we can record the pair $(p(\varphi), \beta)$. There is a contractible space of cartesian arrows ("unique up to unique isomorphism") φ inducing a fixed (f, β) which is nonempty precisely if the functor p is a fibered category. A *cleavage* of $p : F \to C$ is a choice of a cartesian arrow inducing every such $(f, \beta) \in C^{[1]} \times_{t,C,p} F$. A cleavage exists by the axiom of choice. Having fixed a cleavage, we write $\beta|_U$ or $f^*\beta$ and say "pullback of β along f" for the source of the cartesian arrow in the cleavage inducing (f, β).

We record a few examples of fibered categories from [65, 3.2].

Example 1.1.29. These functors all give fibered categories:

- Consider a category C admitting fiber products. Let $C^{[1]}$ be the "category of arrows" in C:
  - $*$ Objects: Arrows $f : X \to Y$ in C and
  - $*$ Morphisms: Commutative Squares

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y'. \end{array}$$

  The functor $C^{[1]} \to C$ sending $f : X \to Y$ to Y yields a fibered category. The cartesian arrows are precisely cartesian squares. This justifies the nomenclature.

- The functor $(Top) \to (Set)$ assigning to each topological space its underlying set is a fibered category. Cartesian arrows are continuous functions $f : X \to Y$ for which X has the induced topology from f.

- Let G be a group object in a topos E. A G-torsor in E is

We may identify fibered categories $p : F \to C$ endowed with a cleavage with "pseudo-functors" $q : C \to (Cat)$. To produce a pseudo-functor from a fibered category with cleavage, send $U \in C$ to the fiber F_U and send $f : U \to V$ in C to the "functor" $F_V \to F_U$ which sends $\beta \in F_V$ to a source of a cartesian arrow $\varphi : \alpha \to \beta$ lying over f.

For the other direction, send a pseudo-functor $q : C \to (Cat)$ to the category of pairs (U, α), with $U \in C$ and $\alpha \in q(U)$. This lies over C via projection onto the first factor. There is a general construction called "straightening" and "unstraightening" at play here which is analogous to the identification between functions and sections of trivial bundles [47, §3.2].

In analogy with the procession from presheaves to sheaves, a fibered category is sometimes called a "pseudo-functor." We think of a fibered category as a "functor" $C \to (Cat)$ sending U to F_U.

For such a "presheaf of categories" to be a stack, it must satisfy a "gluing" or "descent" axiom. Consider a fibered category $F \to C$ over a *site* C. The gluing axiom requires an equivalence

$$F_X \to \mathrm{holim}_{U \in D} F_U$$

for any $X \in C$ and $D \subseteq \hbar_X$ covering sieve [58, descent – Revision 42]. Because this definition lacks concreteness, we spell out the category on the right when C has fiber products.

Definition 1.1.30. Let C be a site with fiber products and $F \to C$ be a fibered category. Consider a covering family $\mathcal{U} = \{U_i \to U\}$ and write $U_{ij} := U_i \times_U U_j$, $U_{ijk} := U_i \times_U U_j \times_U U_k$, etc.. A *descent datum* for $\mathcal{U}$ [65, Definition 4.2] is a pair of sets $(\{\xi_i\}, \{\varphi_{ij}\})$ of

- Objects $\xi_i \in F_{U_i}$ and

- Isomorphisms $\varphi_{ij} : \xi_i|_{U_{ij}} \simeq \xi_j|_{U_{ij}}$ beween restrctions to each intersection U_{ij} such that

- The isomorphisms $\{\varphi_{ij}\}$ satisfy the "cocycle condition" that the triangle

$$\begin{tikzcd} & \xi_j|_{U_{ijk}} \arrow[dr, dash, "\varphi_{jk}\sim"] & \\ \xi_i|_{U_{ijk}} \arrow[ur, dash, "\varphi_{ij}\sim"] \arrow[rr, dash, "\varphi_{ik}\sim"'] \arrow[r, phantom, "\circ" near end] & & \xi_k|_{U_{ijk}} \end{tikzcd}$$

  should commute.

Descent Data for $\mathcal{U}$ form a category $F_{\mathcal{U}}$ whose objects are descent data and morphisms $(\{\xi_i\}, \{\varphi_{ij}\}) \xrightarrow{f} (\{\eta_i\}, \{\psi_{ij}\})$ are

- Collections of arrows $f_i : \xi_i \to \eta_i$ such that

- The diagrams

$$\begin{tikzcd} \xi_i|_{U_{ij}} \arrow[r, "f_i"] \arrow[d, "\varphi_{ij}"'] \arrow[dr, phantom, "\circ"] & \eta_i|_{U_{ij}} \arrow[d, "\psi_{ij}"] \\ \xi_j|_{U_{ij}} \arrow[r, "f_j"'] & \eta_j|_{U_{ij}} \end{tikzcd}$$

  commute.

There is a natural map

$$F_U \to F_{\mathcal{U}}$$

which associates to an object $\xi \in F_U$ all its pullbacks to U_i, U_{ij}, U_{ijk}, etc. The fibered category F is a *stack* if this natural map is an equivalence.

We sketch a way to do without the choice of cleavage in our definition of descent data [65, page 73]. Define categories of diagrams in F

$$F_{\mathcal{U}}^{\square} := \left\{ \begin{tikzcd} & \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\ \xi_{ik} \arrow[rr] \arrow[dd] & & \xi_k \arrow[dd] & \\ & \xi_{ij} \arrow[rr] \arrow[dl] & & \xi_j \arrow[dl] \\ \xi_i \arrow[rr] & & \xi & \end{tikzcd} \right\} \qquad F_{\mathcal{U}}^{\triangle} := \left\{ \begin{tikzcd} & \xi_{ijk} \arrow[rr] \arrow[dl] \arrow[dd] & & \xi_{jk} \arrow[dl] \arrow[dd] \\ \xi_{ik} \arrow[rr] \arrow[dd] & & \xi_k & \\ & \xi_{ij} \arrow[rr] \arrow[dl] & & \xi_j \\ \xi_i & & & \end{tikzcd} \right\}$$

where such diagrams lie over the diagram of structure maps between U_{ijk}, U_{ij}, U_i, etc. in such a way that every arrow is cartesian. Because the space of choices of a cartesian arrow is contractible the morphism $F_{\mathcal{U}}^{\square} \to F_U$ sending such a cube to ξ is an equivalence. Choose an inverse to this equivalence and compose with the forgetful functor:

$$F_U \to F_{\mathcal{U}}^{\square} \to F_{\mathcal{U}}^{\Delta}$$

to obtain a functor which is naturally isomorphic to any natural map $F_U \to F_{\mathcal{U}}$ coming from a chosen cleavage as in Definition 1.1.30. A "coordinate-free" way to formulate the stack condition is therefore that the forgetful map

$$F_{\mathcal{U}}^{\square} \to F_{\mathcal{U}}^{\Delta}$$

is an equivalence.

We will be primarily interested in stacks $p : F \to C$ such that each fiber F_U is a groupoid. To emphasize this property, we may say F is a stack in groupoids.

The (2, 1)-category of stacks over a fixed site C has

- Objects given by stacks over C,
- Morphisms given by functors which commute with the structure functors to C,
- Natural transformations are required to be natural isomorphisms.

1.1.4 Cohomology in a Topos

Definition 1.1.31. A category $\mathscr{A}$ is

- *additive* if it is enriched over abelian groups and has finite coproducts.
- *abelian* if it is additive, it admits kernels and cokernels, and, for every map $f : X \to Y$, the canonical morphism

$$\operatorname{coker}(\ker f \to X) \to \ker(Y \to \operatorname{coker} f)$$

is an isomorphism.

A functor $f : \mathscr{A} \to \mathscr{B}$ between categories enriched over abelian groups is called *additive* if the map induced by f

$$\mathrm{Hom}_{\mathscr{A}}(x, y) \to \mathrm{Hom}_{\mathscr{B}}(f(x), f(y))$$

is a group homomorphism for every $x, y \in \mathscr{A}$.

Remark 1.1.32. Consider objects $X_1, \dots X_n \in \mathscr{A}$ in an additive category. By Proposition 2.1 of [58, additive category – Revision 33], the map

$$\coprod X_i \to \prod X_i$$

given by the collection of arrows

$$X_i \to X_j = \begin{cases} id_{X_i} & \text{if } i = j \\ \\ 0 & \text{if } i \neq j \end{cases}$$

is an isomorphism. To emphasize the symmetry in this situation, we write

$$\bigoplus X_i := \coprod X_i \simeq \prod X_i.$$

In particular, the zero object is given by the empty product/coproduct.

Definition 1.1.33. An object $I \in \mathscr{A}$ in an abelian category is *injective* if, for all spans

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & Y \\ I \arrow[ur, dashed] & \end{tikzcd}$$

with i a monomorphism, there exists a dashed arrow making the diagram commute.

An abelian category $\mathscr{A}$ is said to have *enough injectives* if every object may be embedded into an injective object.

An object $P \in \mathscr{A}$ in an abelian category is *projective* if, for all cospans

$$\begin{tikzcd} & P \arrow[d] \\ X \arrow[ur, dashed] \arrow[r] & Y, \end{tikzcd}$$

in which $X \to Y$ is an epimorphism, there exists a dashed arrow making the diagram commute. We say $\mathscr{A}$ has *enough projectives* if every object X has an epimorphism $P \to X$ from a projective object P.

Remark 1.1.34. If an abelian category $\mathscr{A}$ has a generator, admits direct sums, and filtered colimits are exact, then it has enough injectives [24, Theorème 1.10.1]. Under these hypotheses, $\mathscr{A}$ is sometimes called a *Grothendieck* abelian category.

If (u, v) is an adjoint pair and u is left-exact, then v sends injectives to injectives [52, Proposition V.0.2].

Any abelian category $\mathscr{A}$ admits a fully faithful and exact functor into A-mod, for some ring A. If $\mathscr{A}$ has small coproducts and a generator $x \in \mathscr{A}$ which is projective and of finite presentation, then it's equivalent to A-mod for $A = \mathrm{Hom}(x, x)^{op}$ [58, Freyd-Mitchell embedding theorem – Revision 7].

See [24] for elaboration on abelian categories, chain complexes, injective resolutions, quasi-isomorphisms, spectral sequences, etc.

Proposition 1.1.35. Consider $\mathscr{A} \xrightarrow{f} \mathscr{B} \xrightarrow{g} \mathscr{C}$ two additive, left-exact functors between abelian categories. Suppose $\mathscr{A}$ and $\mathscr{B}$ have enough injectives and that f takes injectives to objects which are acyclic for g. Then there exists a spectral sequence of functors [24, Theorème 2.4.1]

$$E_2^{p,q} := R^p g \circ R^q f \Rightarrow R^{p+q} g \circ f.$$

This goes by the name of the "Grothendieck-Leray" Spectral Sequence.

Now fix (E, A) a topos with a commutative ring object $A \in E$. Let B be another ring in E.

Proposition 1.1.36 ([52, Proposition 1.2])**.** This pair of statements are equivalent for a $B - A$-bimodule M:

- M is left-A-flat: the functor

  $$P \mapsto P \otimes_A M$$

  is exact.

- For every injective B-module I, the left A-module $\mathrm{Hom}_B(M, I)$ is injective.

The categories of left and right A-modules are equivalent by commutativity. We denote this category by A-mod. If $X \in E$, and N is an A-module in E, define the cohomology groups:

$$H^p(X, N) := \mathrm{Ext}^p_A(A^{(X)}, N).$$

The module A isn't included in the notation because the cohomology groups are independent of the sheaf of rings. That is, they only depend on the underlying sheaf of abelian groups of N [52, Corollaire 3.5]. The ext-groups are, by definition, the derived functors of

$$\mathrm{Hom}_A(A^{(X)}, N) = \mathrm{Hom}_E(X, N) = \Gamma(X, N) = N(X).$$

1.2 Differentials and Deformations

1.2.1 Kähler Differentials

Definition 1.2.1. Let $A \to B$ be a ring map and J a B-module. An A-*derivation* $\delta : B \to J$ is an A-linear map satisfying the "Leibniz Rule:"

$$\delta(fg) = f\delta(g) + g\delta(f).$$

From this, we see $\delta(A) = 0$. The set of A-derivations from B to J is denoted $\mathrm{Der}_A(B, J)$ and possesses a natural B-module structure.

In the same context, an A-*extension* of B by J is a surjection $B' \to B$ of A-algebras with kernel J such that $J^2 = 0$ via the multiplication of B'. In this context, J also has the structure of a B-module. The set of A-extensions of B by J up to equivalence is denoted $\mathrm{Exal}_A(B, J)$ and also possesses a natural B-module structure.

These groups are functorial in a sense we make explicit.

- A morphism of B-modules $J \to J'$ yields maps

$$\mathrm{Der}_A(B, J) \to \mathrm{Der}_A(B, J')$$

and

$$\mathrm{Exal}_A(B, J) \to \mathrm{Exal}_A(B, J')$$

coming from postcomposition and the pushout of extensions:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] \arrow[dl, phantom, "\lrcorner"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J' \arrow[r] & B' \oplus_J J' \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

- A commutative square of rings

$$\begin{tikzcd}
B_0 \arrow[r] & B_1 \\
A_0 \arrow[r] \arrow[u] & A_1 \arrow[u]
\end{tikzcd}$$

begets maps

$$\mathrm{Der}_{A_1}(B_1, J) \to \mathrm{Der}_{A_0}(B_0, J)$$

and

$$\mathrm{Exal}_{A_1}(B_1, J) \to \mathrm{Exal}_{A_0}(B_0, J)$$

coming from precomposition and the pullback of extensions:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, equal] & B_1' \times_{B_1} B_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] & B_1' \arrow[r] & B_1 \arrow[r] & 0.
\end{tikzcd}$$

The reader is invited to verify the claimed exactness. These functorialities commute and give rise to group structures:

$$\mathrm{Der}_A(B, J) \oplus \mathrm{Der}_A(B, J) \longrightarrow \mathrm{Der}_{A\times A}(B \times B, J \oplus J) \longrightarrow \mathrm{Der}_A(B, J)$$

$$\mathrm{Exal}_A(B, J) \oplus \mathrm{Exal}_A(B, J) \longrightarrow \mathrm{Exal}_{A\times A}(B \times B, J \oplus J) \longrightarrow \mathrm{Exal}_A(B, J).$$

The reader can check that the group structure on derivations amounts to pointwise addition. The group structure on extensions sends a pair of extensions

$$0 \longrightarrow J \longrightarrow B' \longrightarrow B \longrightarrow 0$$

$$0 \longrightarrow J \longrightarrow B'' \longrightarrow B \longrightarrow 0$$

to the extension given by the pushout:

$$\begin{tikzcd}
0 \arrow[r] & J \oplus J \arrow[r] \arrow[d, "id+id"] & B' \times_B B'' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \widetilde{B} \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

There is a universal A-derivation $d : B \to \Omega_{B/A}$ in the sense that

$$\mathrm{Der}_A(B, J) = \mathrm{Hom}_B(\Omega_{B/A}, J)$$

via precomposition with d. The target $\Omega_{B/A}$ is called the module of *Kähler Differentials*. One may form $\Omega_{B/A}$ as a quotient of the free B-module on symbols $\{db | b \in B\}$ under the relations:

- $da = 0$ if $a \in A$,
- $d(b + b') = db + db'$, and
- $d(b \cdot b') = bdb' + b'db$.

Define an A-algebra $B + \epsilon J$ called the "trivial extension" or the "algebra of dual numbers" to have underlying A-module $B \oplus J$ and multiplication

$$(b, j) \cdot (b', j') := (bb', bj' + b'j).$$

The projection onto the first coordinate defines an A-algebra map $B + \epsilon J \to B$. This yields an A-extension of B by J. The multiplication is defined so that

$$\mathrm{Hom}_{A-\mathrm{alg}/B}(B, B + \epsilon J) = \mathrm{Der}_A(B, J),$$

where the lefthand side refers to A-algebra sections of the map $B + \epsilon J \to B$.

A crucial simple case of Kähler Differentials is when $B = A[S]$ is the free A-algebra on some not necessarily finite set S. Then $\Omega_{A[S]/A} = A[S]^S$; that is, they are the free $A[S]$-module on S. This is because the Leibniz Rule determines the value of a derivation out of $A[S]$ on monomials in S based on its values on the elements of S.

In classical Algebraic Geometry, a composable pair of ring maps $A \to B \to C$ gives rise to an exact sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0,$$

or equivalently

$$0 \to \mathrm{Der}_B(C, J) \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J).$$

If we assume $B \to C$ is surjective, then $\Omega_{C/B} = 0$. Let $I = \ker(B \to C)$. Then this sequence extends to

$$I/I^2 \xrightarrow{d} \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0.$$

This "conormal sequence" gives the most practical way to compute Kähler Differentials, via a trick which is we will reuse later to define the cotangent complex. Given $A \to C$ a ring map, choose a free A-algebra $B = A[S]$ and a surjection $B \to C$. Then $\Omega_{B/A} \otimes_B C = C^S$ and $\Omega_{C/A}$ is seen to be the quotient of this free module by the Jacobian map d.

The conormal sequence asks a question: is it possible to extend the "functor of Kähler Differentials" to the left and get a long exact sequence? The answer is yes, although we offer a modular interpretation for the more immediate terms to the left before extending all the way.

Proposition 1.2.2. The exact sequence of derivations may be extended to a longer exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_B(C,J) \arrow[r, "u"] & \mathrm{Der}_A(C,J) \arrow[r, "v"] & \mathrm{Der}_A(B,J) \arrow[dll, out=-10, in=170, "\partial"] \\
& \mathrm{Exal}_B(C,J) \arrow[r, "u'"] & \mathrm{Exal}_A(C,J) \arrow[r, "v'"] & \mathrm{Exal}_A(B,J).
\end{tikzcd}$$

Proof. The map u is the natural inclusion and v is the restriction. It is then clear that they compose to zero and the sequence is exact at that step. The map ∂ sends a derivation $\delta : B \to J$ to the extension

$$0 \to J \to C + \epsilon J \to C \to 0,$$

with $C + \epsilon J$ regarded as a B algebra via $B \xrightarrow{id+\delta} B + \epsilon J \to C + \epsilon J$. Because δ was an A-derivation, it's clear that $u' \circ \partial = 0$. Moreover, if u' sends an extension to 0, it's clear that the extension splits as an extension of modules as above. For this to be a splitting of A-algebras means the B-derivation $B \to C + \epsilon J \to J$ is 0 on A, which gives us exactness of the composable pair ∂, u'.

The map v' simply pulls an extension back along the map $B \to C$:

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r] \arrow[d, equal] & B \times_C C' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r] \arrow[d] \arrow[l, dashed, bend right] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & J \arrow[r] & C' \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

The cartesian square attests to the fact that splittings of the pullback are equivalent to B-algebra structures on the original extension, which is the content of exactness of the composable pair v', u'.

□

1.2.2 The Cotangent Complex

We observe without further mention that everything we've said so far can be done in the context of sheaves of algebras and modules in a topos, as opposed to merely in the category of sets. Fix such a topos E. We may identify sheaves on E for the canonical topology with objects in E that represent them.

Consider a map of rings in E. If $S \in E$, define the free A-algebra on S to be the sheafification of the presheaf:

$$U \to A(U)[S(U)].$$

This yields a functor $F : E \to A\text{-alg}$, which is the right adjoint of the forgetful functor $U : A\text{-alg} \to E$. Likewise, the free A-module on $S \in E$ is denoted $A^{(S)}$ and defined as the sheafification of the presheaf

$$U \mapsto \bigoplus_{S(U)} A(U).$$

Consider an A-algebra B, and apply FU repeatedly to B to get an augmented simplicial object:

$$P_\bullet : \quad \cdots A[A[A[B]]] \overset{\rightarrow}{\rightrightarrows} A[A[B]] \rightrightarrows A[B] \to B.$$

That is, $P_n := A[A[\cdots[B]\cdots]] = (FU)^{n+1}B$ and there's an augmentation $P_0 \to B$. This is an example of the bar construction coming from the monad FU, and is sometimes called the standard resolution [63, 08PM]. It is patently a simplicial A-algebra, and we use it to define a simplicial B-module ${\mathbb{L}_{B/A}}^{\Delta}_{\bullet}$ via

$${\mathbb{L}_{B/A}}^{\Delta}_{n} := \text{“}(\Omega_{P_\bullet/A} \otimes_{P_\bullet} B)_n\text{”} := P_n^{(P_{n-1})} \otimes_{P_n} B = B^{(P_{n-1})}.$$

To be clear, the 0th simplex is ${\mathbb{L}_{B/A}}^{\Delta}_{0} = B^{(B)}$. This simplicial object admits an augmentation ${\mathbb{L}_{B/A}}^{\Delta}_{0} \to \Omega_{B/A}$. Every face map is surjective, and so is the augmentation.

Recall the unnormalized chain complex [48, Definition 1.2.3.8]: consider a simplicial object $P_\bullet \in \mathscr{A}^{\Delta^{op}}$ in an additive category $\mathscr{A}$. Define the unnormalized chain complex $C_*(P_\bullet)$ to have the same objects, and chain map the alternating sum of the face maps:

$$C_n(P_\bullet) = P_n$$

$$d : C_n(P_\bullet) \to C_{n-1}(P_\bullet); \quad d := \sum (-1)^i d_i.$$

In the case of an abelian category, we can also define a normalized chain complex [48, Definition 1.2.3.9] which is quasi-isomorphic to the unnormalized chain complex. If $\mathscr{A}$ is idempotent complete, the unnormalized chain complex forms part of the well-known Dold-Kan Equivalence:

$$\mathrm{Ch}_{\geq 0}(\mathscr{A}) \simeq \mathscr{A}^{\Delta^{op}}.$$

Definition 1.2.3. The *cotangent complex* $\mathbb{L}_{B/A}$ is the unnormalized chain complex of the simplicial object ${\mathbb{L}_{B/A}}^{\Delta}_{\bullet}$. It is considered as an object in the derived category of B-modules.

Given $A \to B \to C$, we get a transitivity triangle by choosing simplicial A-algebras compatibly:

$$\begin{tikzcd} A \arrow[r] \arrow[d, equal] & P_\bullet \arrow[r] \arrow[d] & Q_\bullet \arrow[d] \\ A \arrow[r] & B \arrow[r] & C. \end{tikzcd}$$

Because these simplicial algebras are free and smooth, we get a triangle in the derived category [29, II.2.1]:

$$\mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B} \xrightarrow{+1} . \tag{1.1}$$

The “Fundamental Theorem” [29, Theorème 1.2.3] of the cotangent complex is that

$$\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, J) = \mathrm{Exal}_A(B, J).$$

One also computes that $h_0(\mathbb{L}_{B/A}) = \Omega_{B/A}$ using the augmentation and the fact that the Dold-Kan Correspondance identifies homotopy groups of the simplicial objects to homology groups of the complex.

These isomorphisms and the exact triangle (1.1) produce a long exact sequence from a C-module J:

$$\begin{aligned} 0 \longrightarrow \mathrm{Der}_B(C, J) \longrightarrow \mathrm{Der}_A(C, J) \longrightarrow \mathrm{Der}_A(B, J) &\longrightarrow \\ \mathrm{Exal}_B(C, J) \longrightarrow \mathrm{Exal}_A(C, J) \longrightarrow \mathrm{Exal}_A(B, J) &\xrightarrow{\partial} \\ \mathrm{Ext}^2_C(\mathbb{L}_{C/A}, J) \longrightarrow \cdots . \end{aligned}$$

In and around Proposition 1.2.5.4 [29], Illusie proves that this sequence is an extension of the long exact sequence we described in Proposition 1.2.2.

In the case $A \to B \to C$ with $B \to C$ surjective with kernel I and B a smooth A-algebra, the cotangent complex is of perfect amplitude in [-1, 0] and is quasi-isomorphic to

$$I/I^2 \xrightarrow{d} \Omega_{B/A} \otimes_B C.$$

This corresponds on schemes to morphisms $f : X \to Y$ which factor as a closed immersion composed with a smooth map. If $A \to B$ is smooth in particular, then

$$\mathbb{L}_{B/A} = \Omega_{B/A}[0].$$

If $A \to B$ is surjective with kernel I, then

$$h_1(\mathbb{L}_{B/A}) = I/I^2, \quad h_0(\mathbb{L}_{B/A}) = 0.$$

This means that [29, Lemme 1.2.8]

$$\mathrm{Hom}_B(I/I^2, J) = \mathrm{Exal}_A(B, J).$$

This equality is obtained by pushing out the extension

$$0 \to I/I^2 \to A/I^2 \to B \to 0.$$

1.2.3 Deformations of Algebras

Fix an extension of rings

$$0 \to J \to A' \to A \to 0.$$

It may be possible to obtain a morphism between two extensions of algebras

$$\eta : \quad \begin{tikzcd}
0 \arrow[r] & I \arrow[r] & B' \arrow[r] & B \arrow[r] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Such an extension η yields a morphism $u : J \otimes_A B \to I$ of B-modules and includes the data of a ring map $A \to B$. As in [29, Problème 2.1.2.1], the question is whether one can find such an η with a fixed $u(\eta) : J \otimes_A B \to I$ extending a fixed map $A \to B$.

In this case, the long exact sequence corresponding to the composable pair $A' \to A \to B$ becomes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Der}_A(B, I) \arrow[r, "\sim"] & \mathrm{Der}_{A'}(B, I) \arrow[r] & 0 \arrow[dll, out=-10, in=170] \\
& \mathrm{Exal}_A(B, I) \arrow[r] & \mathrm{Exal}_{A'}(B, I) \arrow[r, "u"] & \mathrm{Hom}_B(J \otimes_A B, I) \arrow[dll, "\partial", out=-10, in=170] \\
& \mathrm{Ext}^2_B(\mathbb{L}_{B/A}, I) \arrow[r] & \cdots .
\end{tikzcd}$$

Here, we've used the calculation of algebra extensions in the surjective case: $\mathrm{Exal}_{A'}(A, J) = \mathrm{Hom}_A(J/J^2, I)$ and $J^2 = 0$.

Then one can check [29, 2.1.2.2] that the map marked u in this sequence sends an extension of algebras η to the induced map $u(\eta) : J \otimes_A B \to I$. This long exact sequence the claim that, for any $u \in \mathrm{Hom}_B(J \otimes_A B, I)$, there is an obstruction $\partial u \in \mathrm{Ext}^2_B(\mathbb{L}_{B/A}, I)$ to the possibility of coming up with an extension η which induces the given map $u = u(\eta)$. Provided such an obstruction vanishes and there does exist such an extension η, the sequence also articulates that the set of all such extensions is naturally a torsor under the group $\mathrm{Exal}_A(B, I)$ [29, Proposition 2.1.2.3]. This answers the question of classifying such η posed above.

We pause to mention that everything above has an analogue for graded algebras. There is a graded cotangent complex $\mathbb{L}_{B/A}{}^{gr}$ which is isomorphic without grading to the original cotangent complex $\mathbb{L}_{B/A}$. Graded derived maps out of such a graded cotangent complex classify graded derivations and algebra extensions.

Illusie's strategy for the deformations of modules is to embed the module deformation problem into a deformation problem of graded algebras, so this will be important to us in the next section. Complements are also explored in [29] concerning the deformations of maps between algebras, modules, etc., but we will have no need for such constructions.

1.2.4 Deformations of Modules

This section summarizes results of [29, §IV.3].

Fix a ringed topos (E, A) and a squarezero extension of sheaves of A-algebras:

$$0 \to J \to A' \to A \to 0.$$

Fix A-modules M and K, naturally considered as A'-modules which J annihilates. We consider extensions

$$\xi : 0 \to K \to M' \to M \to 0$$

of A'-modules – J need not annihilate M'. From an extension ξ, we get an invariant map given on local sections by:

$$\theta(\xi) : J \otimes_A M \to K$$

$$j \otimes m \mapsto jm'$$

where $m' \in M'$ is a lift of $m \in M$.

Illusie develops a notion of cotangent complex for graded algebras, and then obtains an exact sequence which illuminates obstructions and classifications of extensions ξ as above with fixed map $\theta(\xi)$. We sketch this approach before defining the sequence in an ad-hoc manner.

Write $C' := A + \epsilon M'$, $C = A + \epsilon M$, and endow them with the grading placing A in degree zero and M, M' in degree one. Consider K as a module in degree one. The extension ξ of modules begets an extension of graded algebras

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & C' \arrow[r] & C \arrow[r] & 0 \\ 0 \arrow[r] & J \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & 0, \end{tikzcd}$$

with the lower row all in degree zero. This construction produces an isomorphism

$$\mathrm{Ext}^1_{A'}(M, K) \simeq \mathrm{Exal}_{A'}(C, K)_{gr},$$

where the right hand side refers to graded extensions of algebras. In this case, we see $J \otimes_A C \to K$ is precisely $\theta(\xi) : J \otimes_A M \to K$.

In this context, the long exact sequence arising from the composable pair $A' \to A \to C$ is of the form:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}_A(C, J) \arrow[r] & \mathrm{Der}_{A'}(C, J) \arrow[r] & 0 \arrow[dll, out=0, in=180] \\ & \mathrm{Ext}^1_A(M, J) \arrow[r] & \mathrm{Ext}^1_{A'}(M, J) \arrow[r] & \mathrm{Hom}_A(J \otimes_A M, K) \arrow[dll, out=0, in=180, "\partial"] \\ & \mathrm{Ext}^2_C(\mathbb{L}_{C/A}, J) \arrow[r] & \cdots & \end{tikzcd}$$

Chapter 2

Intersection Theory and Virtual Fundamental Classes

Assumption 2.0.1. For this section and any using intersection theory, we assume that schemes are finite type and Noetherian over a ground field K. The Stacks Project develops these ideas in greater generality [63, 02QL].

Remark 2.0.2. Assumption 2.0.1 gives us a dimension function on points of our schemes X defined by the transcendence degree: $\dim x = \operatorname{tr.deg}_K k(x)$. It also gives us finite irreducible decompositions. By [63, 02QO], the dimension of an integral subscheme Z with generic point ξ and a closed point $z \in Z$ is equivalently given by

- $\dim_K \xi$,
- $\dim Z$, or
- $\dim \mathcal{O}_{Z,z}$.

Definition 2.0.3. Let $X \subseteq Y$ be the closed immersion of affine schemes coming from a surjection $B \twoheadrightarrow A$, and write I for the kernel. Apply Sym_B to the inclusion $I \subseteq B$ and take the epi-mono factorization to get the blowup algebra:

$$\mathrm{Sym}_B I \twoheadrightarrow \bigoplus_{n \geq 0} I^n \cdot t^n \subseteq B[t].$$

Now tensor $\cdot \otimes_B A$ to restrict to X:

$$\operatorname{Sym}_A I/I^2 \twoheadrightarrow \bigoplus_{n\geq 0} I^n/I^{n+1} \cdot t^n \subseteq A[t].$$

Taking Spec defines the closed immersion of the *normal cone* into the *normal sheaf*:

$$C_{X/Y} \subseteq N_{X/Y}.$$

This construction didn't depend on X, Y being affine, and we define the normal cone and normal sheaf more broadly in the same way.

Remark 2.0.4. Let F be a coherent module on a scheme X. Then $\operatorname{Sym}_X F$ is an $\mathbb{A}^1$-module scheme.

The normal cone and normal sheaf correspond to sheaves of graded rings $S^\bullet = \bigoplus_{n\geq 0} S^i$ on a scheme X with two important properties:

- $S^0 = \mathcal{O}_X$,
- S^1 is coherent as a module and locally generates $S^\bullet$.

Assumption 2.0.5. We assume that all our sheaves of graded rings satisfy the preceding two conditions.

Remark 2.0.6. What is the geometric meaning of our assumptions on graded rings? A grading $S^\bullet = \bigoplus_{n\geq 0} S^n$ induces

$$\gamma : S^\bullet \to S^\bullet[t], \qquad s \mapsto s \cdot t^{\deg s}$$

which is a section of the map $S^\bullet[t] \to S^\bullet$ sending t to 1. If we compose γ with the map sending t to 0, we get the map $S^\bullet \to \mathcal{O}_X$ sending everything with nonzero degree to 0.

On schemes, this entails an action of the monoid scheme $\mathbb{A}^1$ on $C = \operatorname{Spec} S^\bullet$ over X such that the action of 0 factors through the vertex. The condition $S^0 = \mathcal{O}_X$ means killing off everything in nonzero degrees is an isomorphism, and corresponds to the section of $C \to X$ called the *vertex*. For S^1 to locally generate $S^\bullet$ means

$$\operatorname{Sym}_X S^1 \to S^\bullet$$

is a surjection of sheaves, so every cone C embeds into an $\mathbb{A}^1$-module $C^{ab} := \operatorname{Spec}\operatorname{Sym}_X S^1$.

Remark 2.0.7. A map $f : X \to Y$ is a local complete intersection if and only if $C_{X/Y} = N_{X/Y}$ [8, Proposition 3.12].

2.1 Chow Groups

This section is well-known. We provide a condensed summary of the results in [63, 02P3] and [20].

2.1.1 Cycles and Multiplicities

Definition 2.1.1. A *prime k-cycle* of a scheme X is an integral, k-dimensional closed subscheme $Z \subseteq X$. The group of *k-cycles* $Z_k X$ is the free abelian group generated by prime k-cycles. Note that the k-cycles of X are the same as those of the reduction $Z_k X = Z_k X_{red}$. Write $Z_* X := \bigoplus_{k \geq 0} Z_k X$.

Remark 2.1.2. If X is of equidimension n, the group $Z_{n-1} X$ is nothing but the group Div$X$ of Weil Divisors. We sometimes say *prime divisor* for a prime $(n-1)$-cycle in this setting.

Definition 2.1.3. Let R be a ring and M a finitely generated R-module. Given a finite, strictly increasing filtration

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M,$$

its *length* is defined to be n. The *length* of M $\operatorname{len}_A M$ is defined to be the supremum over all such filtrations.

Remark 2.1.4. We collect a handful of standard properties of length, collected from [63, 00IU].

- Given a short exact sequence of R-modules

  $$0 \to M' \to M \to M'' \to 0,$$

  if any two of the three are of finite length, the third is as well. In addition, length is additive:

  $$\operatorname{len}_R M = \operatorname{len}_R M' + \operatorname{len}_R M''.$$

This implies that the length is similarly additive over all exact sequences.

- Length sends a map $R \to S$ of rings and an S-module N to an inequality

$$\operatorname{len}_R N \geq \operatorname{len}_S N$$

which is equality when $R \to S$ is surjective. In particular, if M is a module supported at a point, the length is the dimension of the vector space over the fraction field. Moreover, localizing the ring R can only decrease the length of M.

- Any maximal chain will have the same length, and the successive grades quotients are of the form $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

- Let $(A, \mathfrak{m})$ be a local ring and $A \to B$ a map to a ring B with finitely many maximal ideals $\mathfrak{m}_i$ all lying over $\mathfrak{m}$. Then $[\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})]$ is finite, and for any B-module M of finite length,

$$\operatorname{len}_A M = \sum_{\mathfrak{m}_i} [\kappa(\mathfrak{m}_i) : \kappa(\mathfrak{m})] \operatorname{len}_{B_{\mathfrak{m}_i}} M_{\mathfrak{m}_i}.$$

- If $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is a flat, local homomorphism of local rings, then

$$\operatorname{len}_A M \cdot \operatorname{len}_B(B/\mathfrak{m}_A B) = \operatorname{len}_B(B \otimes_A M).$$

Definition 2.1.5. An integral, k-dimensional closed subscheme $Z \subseteq X$ has an associated k-cycle $[Z] \in Z_k X$. We extend this to equidimensional closed subschemes. Let $Y \subseteq X$ be a closed subscheme all of whose irreducible components are dimension k. Let $Y' \subseteq Y$ be an irreducible component with generic point ξ. Define the *multiplicity* of Y' in Y to be

$$m_{Y',Y} := \operatorname{len}_{\mathcal{O}_{Y,\xi}} \mathcal{O}_{Y',\xi}.$$

Then the cycle class of Y is the sum

$$\sum_{Y' \subseteq Y} m_{Y',Y} [Y']$$

over irreducible components of Y.

2.1.2 Pushforward and Pullback

Definition 2.1.6. If X is a scheme with irreducible components $X_1, \dots X_n$ whose generic points are $\eta_1, \dots \eta_n$, we define [63, 01RV] the function field

$$R(X) := \mathcal{O}_{X,\eta_1} \times \cdots \times \mathcal{O}_{X,\eta_n}.$$

This may be equivalently defined as the ring of rational functions on X. If X is integral, $R(X) = k(\eta)$.

Definition 2.1.7. If $f : X \to Y$ is a proper map of schemes and $Z \subseteq X$ is an integral, closed subscheme of dimension k, then the scheme-theoretic image has lesser dimension $\dim f(Z) \leq \dim Z$ [63, 02R1]. If this inequality is in fact an equality, the extension of function fields $[R(Z) : R(f(Z))]$ is finite. Define the proper pushforward $f_* : Z_k X \to Z_k Y$ on primitive k-cycles as

$$f_*[Z] := \begin{cases} [R(Z) : R(f(Z))] \cdot [f(Z)] & \text{if } \dim f(Z) = \dim Z \\ \\ 0 & \text{otherwise} \end{cases}$$

Remark 2.1.8. A morphism $f : X \to Y$ is said to have relative dimension r if every nonempty fiber is equidimensional of dimension r. If $f : X \to Y$ is flat of relative dimension r and $Z \subseteq Y$ is closed, then either

$$\dim f^{-1}(Z) = \dim Z + r$$

or $f^{-1}(Z)$ is empty [63, 02R8]. Further, "flat of relative dimension r" is stable under pullback.

Consider a map $f : X \to Y$. If $Z \subseteq Y$ is defined by the vanishing by a quasi-coherent ideal I, the scheme-theoretic preimage $f^{-1}Z$ is the vanishing locus of the image of f^*I in $\mathcal{O}_X$.

Definition 2.1.9. Let $f : X \to Y$ be flat of relative dimension r. Consider a k-dimensional integral closed subscheme $Z \subseteq Y$. Define the flat pullback:

$$f^*[Z] := [f^{-1}Z].$$

This is a $(k+r)$-cycle on X. Extend the definition to all k-cycles $f^* : Z_k Y \to Z_{k+r} X$ by additivity.

Remark 2.1.10. We collect some basic results on pushforward and pullback:

- Let $j : U \subseteq X$ be an open immersion with complement $i : Z \subseteq X$. The maps

  $$Z_k Z \xrightarrow{i_*} Z_k X \xrightarrow{j^*} Z_k U \to 0$$

  form an exact sequence.

- Consider a pullback square

  $$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

  with q flat of relative dimension r and f proper. By [63, 02RG],

  $$q^* f_* = p^* f'_* : Z_k X \to Z_{k+r} Y'.$$

- If f is finite locally free of degree r, then it is both proper and flat and $f_* f^*$ amounts to multiplication by r.

2.1.3 Divisors and Chow Groups

If $Z \subseteq X$ is a codimension-one closed subscheme of an equidimensional scheme,

Definition 2.1.11. The *order of vanishing* of $f \in R(X)^*$ along a prime divisor $Z \subseteq X$ with generic point η is defined

$$\operatorname{ord}_Z f := \operatorname{len}_R R/(f)$$

where $R = \mathcal{O}_{X,\eta}$. Define the *divisor of f* via the sum over all prime divisors

$$\operatorname{div} f = \operatorname{div}_X f := \sum_{Z \subseteq X} \operatorname{ord}_Z f \cdot [Z].$$

This sum is finite by [63, 02RL]. Define the *class group* of X as the cokernel $Cl(X) = A_{n-1}X := \operatorname{Div}(X)/R(X)^*$ by the divisor function.

Define the *group of rational equivalences* $W_k X$ analogously as the direct sum of $R(T)^*$ over $T \subseteq X$ an integral, closed subscheme of dimension $(k+1)$. The *Chow Group* of k-cycles is then defined as the cokernel

$$A_k X := Z_k X / W_k X.$$

of the divisor function. Write $A_* X := \bigoplus_{k \geq 0} A_k X$.

Remark 2.1.12. The case where $\mathcal{O}_{X,\eta}$ is regular for any generic point in a prime divisor $\eta \in Z$ has been described as "regularity in codimension one" [28]. In this situation, the one-dimensional, regular local ring $R = \mathcal{O}_{X,\eta}$ is a DVR [63, 00PD]. Let $t \in R$ be a uniformizer. Then $\operatorname{len}_R R/(t) = 1$, so $\operatorname{ord} f$ coincides with the valuation of R.

Consider X integral and not necessarily regular, but with normalization $\widetilde{X} \to X$. The order of $f \in R(X)^* = R(\widetilde{X})^*$ is given by the summation

$$\operatorname{ord}_Z f = \sum_{\widetilde{Z} \mapsto Z} [R(\widetilde{Z}) : R(Z)] \cdot \operatorname{ord}_{\widetilde{Z}} f$$

as in [20, Example 1.2.3]. This reassures us that ord in our generality is determined by the situation where X is regular in codimension one.

Remark 2.1.13. Consider a map of integral schemes $f : X \to Y$ and elements $r \in R(X)$, $s \in R(Y)$. If f is flat of some constant relative dimension, then [63, 02RR]

$$f^* \operatorname{div}_Y s = \operatorname{div}_X s.$$

If f is instead dominant, proper and $Nm : R(X) \to R(Y)$ is the norm, then [63, 02RT]

$$f_* \operatorname{div}_X r = \operatorname{div}_Y Nm(r).$$

From this, we see that flat pullback f^* and proper pushforward f_* descend to maps on Chow Groups.

2.2 Algebraic Stacks

2.2.1 Algebraic Spaces

This subsection makes sense of the statement: "Schemes admit descent for the Zariski topology, but not for the étale topology."

Let $p : F \to F'$ be a map between presheaves on a site C. We say p is "covering" if, for any $\hbar_X \to F'$ with $X \in C$, the image of

$$\hbar_X \times_{F'} F \to \hbar_X$$

is a covering sieve on X. By [52, Proposition 5.1], this is equivalent to the sheafification

$$F^{sh} \to F'^{sh}$$

being an epimorphism. This topology on $\widehat{C}$ is the finest subcanonical topology (every epimorphic family is covering) for which all covering families $\{Y_i \to X\}$ in C induce covering families $\{\hbar_{Y_i} \to \hbar_X\}$ in $\widehat{C}$ [52, Proposition 5.4].

2.2.2 Algebraic Stacks

As in Remark 1.1.15, a groupoid G is given by a diagram of sets

$$\begin{tikzcd} \mathrm{Mor}\ G \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \mathrm{Ob}\ G \arrow[l, "id"'] \end{tikzcd}$$

with a composition law

$$m : \mathrm{Mor}\ G \times_{s, \mathrm{Ob}\ G, t} \mathrm{Mor}\ G \to \mathrm{Mor}\ G$$

which is compatible with a bunch of identities.

Enrich the diagram to one of sheaves $R, U \in \widetilde{C}$ on a site C:

$$\begin{tikzcd} R \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & U, \arrow[l, "id"'] \end{tikzcd}$$

again with composition law and inverses. This groupoid object in $\widetilde{C}$ gives us a bona fide functor $C^{op} \to (Gpoid)$ corresponding to a fibered category. This fibered category needn't be a stack, but its stackification is called the "quotient stack" and denoted $[U/R]$.

So far, nothing we've said about stacks is of any geometric import. The most important context in the sequel will be:

- the étale site of schemes: $C = (Sch)_{\text{ét}}$,

- the sheaves R, U should be representable, either by schemes or algebraic spaces, and

- the structure morphisms of the groupoids (R, U, s, t, m, i) should be smooth.

In this context, we call a groupoid object in the category of algebraic spaces a *smooth groupoid*. Often, we abbreviate the data (R, U, s, t, m, i) to $R \rightrightarrows U$.

Definition 2.2.1. Let $p : F \to (Sch)_{\text{ét}}$ be a stack. A *presentation* for F is a smooth groupoid $R \rightrightarrows U$ and an equivalence

$$[U/R] \simeq F.$$

Such a stack F is called *algebraic* or *Artin* if it admits such a presentation by a smooth groupoid.

An algebraic stack is called *DM* or *Deligne-Mumford* if there exists a surjective, étale map $U \to F$ from a scheme U.

The 2-category of algebraic stacks is the full sub-2-category of stacks over $(Sch)_{\text{ét}}$ on those which are algebraic.

Remark 2.2.2. The sections [63, 04T3, 04TJ] show how to make our definition intrinsically. Let $F \to (Sch)_{\text{ét}}$ be a stack in groupoids. It is called *algebraic* if

- The diagonal $F \to F \times F$ is representable by algebraic spaces,

- There exists a scheme U and a surjective, smooth map $U \to F$.

Remark 2.2.3. Our references to the Stacks Project work in the fppf topology $(Sch)_{fppf}$ instead. As explained after their definition of algebraic stack [63, 026O], the resulting 2-categories of algebraic stacks are equivalent.

The seminal reference [41] works over the site $(Aff)_{\text{ét}}$ of affine schemes only. This also yields an equivalent 2-category, because each scheme may be covered by its affine open subschemes.

The reader is also invited to make sense of algebraic stacks over $S \in (Sch)$: similar stacks over $(Sch/S)_{\text{ét}}$.

Remark 2.2.4. We collect some basic facts about algebraic stacks we will use without mention.

- A stack over $(Sch)_{\text{ét}}$ is algebraic if and only if it is equivalent to an algebraic stack.
- The sub-2-category of $(Cat)/(Sch)$ on the algebraic stacks is closed under 2-fiber products.

2.3 Virtual Fundamental Classes

2.3.1 Kresch's Chow Groups

We continue to apply Assumption 2.0.1 to algebraic stacks. We recall the definition of the Chow Groups of an algebraic stack [40]:

Definition 2.3.1. Let X be an algebraic stack. Write $Z_k X$ for the free abelian group of integral closed substacks of X. Write $W_k X$ for the direct sum of $R(T)^*$ for all closed integral subschemes $T \subseteq X$ of dimension $(k+1)$. These are both sheaves for the smooth topology.

Define the *naïve Chow Groups* $A_k^\circ X$ via the cokernel

$$A_k^\circ X := Z_k X / W_k X$$

of the map which is induced smooth-locally by the divisor function. This is the presheaf quotient.

We need two categories from each algebraic stack Y:

- $\mathfrak{A}_Y$ is the category of projective Y-stacks with maps $X \to X'$ given by inclusion of connected components over Y. A projective morphism $p : Y' \to Y$ induces a functor $p_* : \mathfrak{A}_{Y'} \to \mathfrak{A}_Y$
- $\mathfrak{B}_Y^{op}$ is the category of vector bundles over Y with morphisms given by surjections $F \twoheadrightarrow E$. A morphism $f : Y' \to Y$ gives rise to covariant functors in the other direction $f^* : \mathfrak{B}_Y^{op} \to \mathfrak{B}_{Y'}^{op}$.

If Y is connected, define the Edidin-Graham-Totaro Chow Groups as the limit over flat pullbacks along surjections of vector bundles:

$$\widehat{A}_k Y := \lim_{E \in \mathfrak{B}_Y^{op}} A_{k+\operatorname{rank} E}^\circ E.$$

For Y not necessarily connected, define them as the sum $\bigoplus \widehat{A}_k Y'$ over the connected components $Y' \subseteq Y$.

If $p : X \to Y$ is a projective morphism with X connected, define the restricted EGCT groups

$$\widehat{A}^p X := \lim_{E \in \mathfrak{B}_Y^{op}} A^\circ_{k+\operatorname{rank} E} p^* E.$$

For X not necessarily connected, define them again as the sum over its connected components. We have natural maps

$$\widehat{A}_k X \xleftarrow{\iota_p} \widehat{A}^p_k X \xrightarrow{p_*} \widehat{A}_k Y.$$

Let T and X be Y-algebraic stacks and consider a pair $p, q : T \rightrightarrows X$ of projective morphisms whose morphisms to Y are naturally isomorphic. Let $\widehat{B}^{p,q}_k X$ denote the image of the difference of the two natural maps

$$\widehat{A}^p_k T \times_{\widehat{A}_k T} \widehat{A}^q_k T \to \widehat{A}^p_k T \times \widehat{A}^q_k T \rightrightarrows \widehat{A}_k X.$$

This image is the set of differences between pairs $(\beta_1, \beta_2) \in \widehat{A}^p_k T \times \widehat{A}^q_k T$ with the same image in $\widehat{A}_k T$:

$$\{p_* \beta_1 - q_* \beta_2 \mid \iota_p \beta_1 = \iota_q \beta_2\}.$$

Write $\widehat{B}_k X$ for the subgroup of $\widehat{A}_k X$ generated by all such T and pairs $p, q : T \rightrightarrows X$. Then Kresch's Chow Groups [40, Definition 2.1.11] are defined as

$$A_k Y := \operatorname{colim}_{\mathfrak{A}_Y} (\widehat{A}_k X / \widehat{B}_k X).$$

Remark 2.3.2. Consider two inclusions of components $i, j : X' \rightrightarrows X$ over Y. This yields a pair of projective morphisms over Y and hence a subgroup $\widehat{B}^{i,j}_k X \subseteq \widehat{B}_k X$. This means the colimit used to define Kresch's Chow Groups may be taken over the poset of equivalence classes in $\mathfrak{A}_Y$ instead [40, Remark 2.1.10].

Remark 2.3.3. Kresch's Chow Groups amount to considering naive Chow Groups of vector bundles and enforcing flat pullbacks and projective pushforwards to be well-defined before summing over connected components.

We need a result on these Chow Groups to make our definition of virtual fundamental class in the next section make sense.

Definition 2.3.4. If X is isomorphic to a quotient stack $[U/G]$ for U an algebraic space and G a linear algebraic group, then we call it a *global quotient stack*.

Now consider an algebraic stack X locally of finite type over $\mathbb{C}$ which admits a stratification by locally closed substacks U_i which are each global quotient stacks. One says X *admits a stratification by (global) quotient stacks*.

Proposition 2.3.5 ([40, Proposition 5.3.2])**.** Given a vector bundle stack $\pi : E \to X$ on an algebraic stack X locally of finite type over $\mathbb{C}$, assume X admits a stratification by global quotient stacks as above. Then the pullback

$$\pi^* : A_*X \to A_*E$$

is an isomorphism.

2.3.2 Virtual Fundamental Classes

The most important Chow classes for us will be "Virtual Fundamental Classes," which arise from normal cones. First generalize our definition of normal cone to stacks. We need a lemma:

Lemma 2.3.6 ([49, Lemmas 2.27, 2.28])**.** Given a DM-type morphism $f : X \to Y$ of algebraic stacks locally of finite type over $\mathbb{C}$, one can find a nonunique commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & S \arrow[d, shift left] \arrow[d, shift right] \\ U \arrow[r] \arrow[d] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

with U, V schemes, $U \to X$ and $V \to Y$ smooth surjections, $R = U \times_X U, S = V \times_Y V$ and the vertical maps the projections, $U \to V$ a closed immersion and $R \to S$ a locally closed immersion.

Given a chosen diagram as in Lemma 2.3.6, write $Q = U \times_Y U$. We have a cartesian square

$$\begin{tikzcd} R \arrow[r] & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & S \arrow[d] \\ & U \times U \arrow[r] & V \times V. \end{tikzcd}$$

which induces morphisms

$$C_{R/S} \to C_{Q/S} \to C_{U\times U/V\times V} \rightrightarrows C_{U/V}$$

which can be augmented to a smooth groupoid of schemes [49, Proposition 2.29]. This smooth groupoid results in an algebraic stack

$$“C''_{X/Y}$$

over X.

Definition 2.3.7 ((Intrinsic) Normal Cone)**.** The algebraic stack just defined is well-defined in that it is independent of choices. It is called the *(Intrinsic)* Normal Cone.

This normal cone is simply a stack lying above the source of the morphism. To get a Chow Class out of this, we need a “perfect obstruction theory:”

Definition 2.3.8. A *perfect obstruction theory* [49, Condition 3.3 (*)], [8, Definition 4.4] for a DM-type map $f : X \to Y$ of algebraic stacks locally of finite type over $\mathbb{C}$ is a closed embedding of cone stacks

$$C_{X/Y} \subseteq E$$

into a vector bundle stack E on X.

An equivalent definition of perfect obstruction theory is a map $\mathcal{E} \to \mathbb{L}_{X/Y}$ in the derived category which is an isomorphism on h^0 and an epimorphism on h^{-1}. Truncating in degrees $[-1, 0]$ and applying the “$h^1/h^0(\cdot^\vee)$” construction [8, §2] to this morphism yields a closed embedding of cone stacks

$$N_{X/Y} \subseteq E := h^1/h^0(\mathcal{E}^\vee)$$

which we compose with the natural closed immersion $C_{X/Y} \subseteq N_{X/Y}$ to get a perfect obstruction theory in the sense of Definition 2.3.8. See [69] for more on this equivalence.

Kresch's notion of the Chow Groups of X entails equivalence classes of closed immersions into vector bundles, not vector bundle *stacks*, over X. We need an additional assumption on X for this to be okay:

Definition 2.3.9. Suppose given a map $f : X \to Y$ with a perfect obstruction theory $C_{X/Y} \subseteq E$. Assume X admits a stratification by global quotient stacks as in Definition 2.3.4. The *Virtual Fundamental Class* of (f, E) is the unique class

$$[f, E]^{vfc} \in A_* X$$

which pulls back to $C_{X/Y} \subseteq E$ under the projection $E \to X$. Both uniqueness and existence of such a class rely on Proposition 2.3.5.

Chapter 3

Logarithmic Algebraic Geometry

> How to conceive what is outside a text? That which is more or less than a text's *own, proper* margin? For example, what is other than the text of Western metaphysics? It is certain that the trace which 'quickly vanishes in the destiny of Being (and) which unfolds ... as Western metaphysics' escapes every determination, every name it might receive in the metaphysical text. It is sheltered, and therefore dissimulated, in these names. It does not appear in them as the trace 'itself.' But this is because it could never appear itself, *as such*. Heidegger also says that difference cannot appear as such: 'Lichtung des Unterschiedes kann deshalb auch nicht bedeuten, dass der Unterschied als der Unterschied erscheint.' There is no essence of *différance*; it (is) that which not only could never be appropriated in the *as such* of its name or its appearing, but also that which threatens the authority of the *as such* in general, of the presence of the thing itself in its essence. That there is not a proper essence of *différance* at this point, implies that there is neither a Being nor truth of the play of writing such as it engages *différance*.
>
> [11, Différance, pg. 25-6]

The historical motivation for the concept of log structures is the case of a "simple-normal-crossings divisor" [59, III.1.8], [17]. Semistable reductions over a discrete valuation ring are of this form at the special fiber by definition.

The union of the first r coordinate hyperplanes $Y \subseteq \mathbb{A}^n$ is given by the vanishing of the principal ideal

$$I = (x_1 \dots x_r) \subseteq \mathbb{Z}[x_1, \dots, x_n] = \Gamma(\mathbb{A}^n).$$

Consider a scheme $X \to S$ over a base S. A divisor $Y \subseteq X$ is called *simple normal crossings* if, locally on X, there's an étale map $f = (f_1, \dots f_n) : X \to \mathbb{A}^n_S$ from which Y is obtained as the vanishing of the product $f_1 \cdots f_r$ of the first $r \leq n$ coordinates.

The module of Kähler Differentials for X/S is

$$\Omega_{X/S} = \mathcal{O}_X \cdot df_1 \oplus \cdots \oplus \mathcal{O}_X \cdot df_n.$$

Write $U := X \setminus Y$ and $j : U \subseteq X$. The sheaf $j_* j^* \Omega_{X/S} = \Omega_{X/S}[f_1^{-1}, \cdots, f_r^{-1}]$. Define the sheaf of *differentials with log poles along* Y to be the sub-$\mathcal{O}_X$-module

$$\Omega^\ell_{X/S} = \Omega_{X/S}(\log Y) \subseteq j_* j^* \Omega_{X/S}$$

generated by

$$\{\frac{df_1}{f_1}, \dots, \frac{df_r}{f_r}, df_{r+1}, \dots, df_n\}.$$

We may write $d\log f := \dfrac{df}{f}$.

Use the Jacobian map to make sense of $\Omega^\ell_{Y/S}$:

$$I/I^2 \to \Omega^\ell_{X/S}|_Y \to \Omega^\ell_{Y/S} \to 0.$$

We argue that if $r = n$, $\Omega^\ell_{Y/S}$ is free. The union of the coordinate hyperplanes is therefore "smooth" in some log sense.

The principal ideal $I = (f_1 \cdots f_n)$ has image under the differential:

$$\begin{aligned} d(f_1 \cdots f_n) &= (f_2 \cdots f_n) df_1 + \cdots (f_1 \cdots f_{n-1}) df_n \\ &= (f_1 \cdots f_n)(\frac{df_1}{f_1} + \frac{df_2}{f_2} + \cdots + \frac{df_n}{f_n}) \\ &\in I \cdot \Omega^\ell_{X/S} \end{aligned}$$

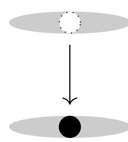
and we mod out by $I \cdot \Omega^\ell_{X/S}$ to restrict to Y. Therefore $dI = 0$ and $\Omega^\ell_{X/S}|_Y = \Omega^\ell_{Y/S}$ is free.

The union of the coordinate hyperplanes is clearly not smooth as a scheme, and we need to remember

- the locally given étale map to $\mathbb{A}^n_S$ and
- which coordinate hyperplanes $r \leq n$ are included in Y.

The general concept of a log structure replaces $\mathbb{A}^n$ with Spec of other monoid rings. The second bullet point specifies the log structure on $\mathbb{A}^n$.

Why is this one example so important that it gives rise to a whole field of inquiry? Recall that a noetherian scheme X is *proper* if, for all discrete valuation rings R with fraction field K, the diagrams



$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \\ \operatorname{Spec} R \arrow[ur, dashed] & \end{tikzcd}$$

may be filled uniquely. Discrete valuation rings are the noetherian, regular, local rings of dimension one, so $\operatorname{Spec} R$ is a local ring on a curve and $\operatorname{Spec} K$ is the complement of the closed point. The valuative criterion above therefore says that proper schemes admit unique limits along one-dimensional families.

If $\ell \subseteq L$ is an extension of fields, one can define a "universal properification" of $\operatorname{Spec} L$ as the set of valuation rings R intermediate between ℓ and L. This is called the "Riemann-Zariski Space." For L a function field over ℓ, this space is given by the inverse limit of all projective schemes over ℓ with function field L.

These ideas can be expanded to show that any $X \to S$ finite type, separated over a quasicompact, quasiseparated S factors as a quasicompact open immersion $X \to \overline{X}$ composed with a proper map $\overline{X} \to S$ [63, 0F41]. In good situations, the boundary $\overline{X} \setminus X$ will be a simple normal crossings divisor.

These universal compactifications have spiritual similarity with the Stone-Čech compactification or the small object argument. More concretely, the points of the universal compactification should be some sort of rigid analytic space obtained by taking all possible blowups. The points of this limit come from valuations, and Berkovich Spaces are seen as one universal way to compactify.

We will first review essential facts about monoids from [59]. We will define log structures intrinsically, and then see that the cases of interest are all given by "charts" like the map to $\mathbb{A}^n_S$.

3.1 Monoids

We remind the reader that all our monoids are commutative with unit. We reserve the right to use additive notation $(P, +, 0)$ or multiplicative $(P, \cdot, 1)$ for a monoid and to refer to either as "multiplication" or "addition." A morphism $\theta : P \to Q$ of monoids must not only satisfy $\theta(p_1 + p_2) = \theta(p_1) + \theta(p_2)$, but also that $\theta(0) = 0$. Arbitrary limits and colimits exist in the category of monoids. Write . . .

- P^{gp} for the Grothendieck Groupification.
- P^* for the subgroup of units.
- P^+ for the nonunits.
- $\overline{P}$ for the quotient P/P^*, called the *characteristic monoid*.

For groups, there is a standard anti-equivalence:

$$\left\{ \begin{array}{c} \text{Quotients of groups} \\ \\ G \twoheadrightarrow K \end{array} \right\}^{op} \leftrightsquigarrow \left\{ \begin{array}{c} \text{Normal Subgroups} \\ \\ N \trianglelefteq G \end{array} \right\}$$

given by taking quotients by normal subgroups and finding the kernel of a quotient. In monoids, the kernel is often useless. The preimage of zero in the addition map $\mathbb{N} \times \mathbb{N} \xrightarrow{+} \mathbb{N}$ is just 0, and yet this map is far from an injection. Nevertheless, the antiequivalence may be salvaged in modified form.

What use is the kernel? The fibers of the quotient $G \twoheadrightarrow K$ all differ by addition of elements coming from N. In other words,

$$G \oplus N \xrightarrow{\sim} G \times_K G$$

via the shearing map $(g, n) \mapsto (g, g + n)$. To this end, we can just use $G \times_K G$ in place of $N \subseteq G$.

Because of the free-forgetful adjunction between monoids and sets, fiber products are the same in both. Thus $G \times_K G \rightrightarrows G$ is an equivalence relation. It is also a monoid. If both, we say it is a *congruence relation*.

In any topos, epimorphisms are effective as part of Giraud's axioms. This means that maps out of the diagram

$$[G \times_K G \rightrightarrows G]$$

are the same as maps out of the quotient K. This exemplifies the principle of the Dold-Kan Equivalence that simplicial sets are a replacement of chain complexes in nonabelian situations. We get an equivalence

$$\left\{ \begin{array}{c} \text{Quotients of groups} \\ \\ G \twoheadrightarrow K \end{array} \right\}^{op} \leftrightsquigarrow \left\{ \begin{array}{c} \text{Congruences} \\ \\ E \rightrightarrows G. \end{array} \right\}$$

We can obtain the pushout of a diagram

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "v"] & R \\ Q & \end{tikzcd}$$

by coequalizing the maps from P to $R \oplus Q$. This congruence relation may be complicated, but admits a simpler description in the case where any of P, Q, or R is a group [59, Proposition I.1.1.5]:

$$(r, q) \simeq (r', q')$$

precisely when there are $p, p' \in P$ such that

$$q + v(p') = q' + v(p)$$

and

$$r + u(p) = r' + u(p').$$

If P is a group, this is the quotient by the antidiagonal action of P:

$$(r, q) \simeq (r + v(p), q + u(-p)), \qquad p \in P, q \in Q, r \in R.$$

Example 3.1.1. Define the cokernel of a map $P \to Q$ of monoids via the congruence for pairs $p, p' \in P$:

$$q \sim q' \qquad \text{if } q + p = q' + p'.$$

For example,

$$\mathrm{coker}(\mathbb{N} \xrightarrow{\Delta} \mathbb{N}^2) = \mathbb{Z}.$$

Cokernels may be ill-behaved:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = f : \mathbb{N}^2 \to \mathbb{N}^2; \qquad \mathrm{coker}(f) = 0.$$

Remark 3.1.2. Pushouts $R \oplus_P Q$ are sometimes written $R \otimes_P Q$ because they also represent "bilinear" maps out of $R \times Q$:

$$f : R \times Q \to T; \qquad f(r + p, q) = f(r, q + p).$$

Tensor products commute with colimits because they're a right adjoint. They don't commute with products.

Example 3.1.3. Groupification $P \mapsto P^{gp}$ is not exact. It commutes with products, but not fiber products. For example, consider the kernel of

$$\mathbb{N}^2 \xrightarrow{+} \mathbb{N}$$

compared with the kernel of the groupification.

Most often, our monoids are *integral*, meaning $P \subseteq P^{gp}$. A monoid P is integral if and only if it satisfies the *cancellation property*:

$$\text{if } p + r = q + r \in P, \text{ then } p = q.$$

A monoid P is *finitely generated* if a fixed finite set $S \subseteq P$ may be chosen so that all elements of P are combinations of those of S. Equivalently, there is a surjection:

$$\mathbb{N}^{(S)} \to P.$$

We say a monoid P is *fine* if it is finitely generated and integral.

Integral monoids are equivalent to ordered groups in a way we presently describe:

Definition 3.1.4. A *P-set* is a set S together with a map

$$P \times S \to S$$

satisfying the usual identities. Maps between P-sets should be equivariant with respect to the action of P.

A P-set S defines a category $\mathcal{T}_P S$ called the *Transporter* with underlying digraph described by the action and projection:

$$[P \times S \rightrightarrows S].$$

In other words,

- the objects are $\mathrm{Ob}\mathcal{T}_P S = S$, and
- the morphisms are $\mathrm{Hom}_{\mathcal{T}_P S}(s,t) := \{p \in P \,|\, p.s = t\}$.

Composition is given by multiplication in P.

The transporter category is an enrichment of a natural poset structure on S. Namely, $s \leq t \in S$ if there exists an element $p \in P$ such that $p.s = t$.

The cases $P \circlearrowright P$ and $P \circlearrowright P^{gp}$ are of primary interest. We may write $\mathcal{T}P := \mathcal{T}_P P$.

Remark 3.1.5. The poset structure on $\mathcal{T}P$ induces a congruence relation $\sim$: $p \sim q$ if $p \leq q$ and $q \leq p$. The monoid $M/\sim$ is always sharp, and the map

$$\overline{M} \to M/\sim$$

is an isomorphism if the action $P \circlearrowright P$ is free [59, Around Definition 1.3.1].

The groupification P^{gp} is the analogue of the fraction field of an integral domain. We also have analogues of less extreme localizations.

Definition 3.1.6. Let $P \circlearrowright S$ be a P-set and choose a subset $F \subseteq P$. Then there exists a P-set $F^{-1}S$ on which F acts bijectively and a map $S \to F^{-1}S$.

This map satisfies a universal property: let T be another P-set on which the elements of F act through bijections. A map $S \to T$ factors through a unique map $S \to F^{-1}S \overset{\exists!}{\dashrightarrow} T$.

The map $S \to F^{-1}S$ and its target are referred to as the *localization of S by F* [59, I.1.4.4].

We are ready to define scheme-like contraptions made out of monoids that form one approach to the relationship between schemes and monoids exemplified at the start of this chapter.

Definition 3.1.7. Let P be a monoid. An *ideal* $I \subseteq P$ is a subset which is closed under the action of P: $PI \subseteq I$. The ideal is *prime* if $I \neq P$ and, whenever $p + q \in I$, either p or q is in I.

A *face* of a monoid is a sub*monoid* $F \subseteq P$ where, if $p + q \in F$, *both* p and q are in F. We write $F \prec P$.

Write $\operatorname{Spec} P$ for the set of prime ideals of P. The complement of a prime ideal is a face and vice versa:

$$\operatorname{Spec} P := \left\{ \begin{array}{c} \text{Prime Ideals} \\ \\ \pi \subseteq P \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{Faces} \\ \\ F \prec P \end{array} \right\}^{op}.$$

We endow the set $\operatorname{Spec} P$ with the *Zariski Topology* with closed sets given by

$$V(S) := \{\pi \in \operatorname{Spec} P \mid \pi \supseteq S\}, \qquad S \subseteq P.$$

The *dimension* of $\operatorname{Spec} P$ is the supremum of lengths n of all chains of strict inclusions of primes of P:

$$\emptyset = \pi_0 \subsetneq \pi_1 \subsetneq \cdots \subsetneq \pi_n = P^+.$$

The *height* of a prime $\pi \subseteq P$ is the maximum such chain with $\pi = \pi_n$ in the above instead of P^+.

If $f \in P$ and $\pi \subseteq P$ is a prime, we traditionally write $P\left[\frac{1}{f}\right] = \langle f\rangle^{-1}P$ and $P_\pi = (P \setminus \pi)^{-1}P$.

Remark 3.1.8. Every monoid has a largest and smallest prime: $\emptyset, P^+ \subseteq P$. In this way, $\operatorname{Spec} P$ behaves like that of a local ring – there's a unique generic point $\emptyset$ and a unique closed point P^+. These two coincide precisely when P is a group.

Likewise, $M^*, M \subseteq M$ are the smallest and largest faces. Faces of the multiplicative monoid of a ring are usually called "saturated multiplicative subsets," but we avoid this terminology to reserve "saturated" for monoids. Observe that, if S is a P-set, $F \subseteq P$ is a subset, and $\langle F \rangle$ is the face it generates, then the localizations are the same:

$$F^{-1}S \simeq \langle F \rangle^{-1} S.$$

The union of a family of prime ideals is prime, and the intersection of a family of faces is a face. The interior ideal of P is the set of elements which don't lie in a proper face; equivalently, elements in the intersection of all nonempty primes. This corresponds to the nilradical.

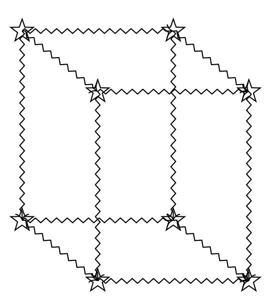
Example 3.1.9. The zero submonoid $0 \in \mathbb{N}$ is a face. This property says exactly that $\mathbb{N}$ is sharp. The only other face is all of $\mathbb{N}$.

Likewise, the only two prime ideals of $\mathbb{N}$ are $\mathbb{N}_{>0}$ and $\emptyset$. The picture depicts a single specialization:

$$\operatorname{Spec} \mathbb{N} : \qquad ☆ \rightsquigarrow \bullet$$

with $☆ = \emptyset$ and $\bullet = \mathbb{N}_{>0}$.

If we look at products $\mathbb{N}^k$ of the natural numbers, we get cubes of specializations in k different directions:



Example 3.1.10. Let $\pm$ denote the multiplicative monoid on $\{0, 1\} \subseteq \mathbb{Z}$. Given a morphism $\theta : P \to \pm$, the preimage $\theta^{-1}(1)$ is a face of P. All faces of P are obtained this way. This means

$1 \in \pm$ is the universal face. Likewise, $0 \in \pm$ is the universal prime.

Each monoid has a morphism $P \to \pm$ called the *vertex* classifying the minimal face P^*. Each morphism $P \to \pm$ classifying a face $F \subseteq P$ factors as:

$$P \to F^{-1}P \to \pm.$$

A morphism $\theta : P \to Q$ of monoids is called *local* if $\theta^{-1}(Q^+) = P^+$. This is equivalent to $\theta^{-1}(Q^*) = P^*$. Geometrically, this corresponds to the preimage of the closed point being only the closed point. A morphism θ is local if and only if it lies over the vertex maps 3.1.10:

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & Q. \arrow[dl] \\ & \pm & \end{tikzcd}$$

Proposition 3.1.11. We collect some essential facts about ideals, primes, faces, etc.

- If P is fine, then $\operatorname{Spec} P$ is a finite set [59, I.1.4.7].
- Let $S \subseteq P$ be a subset [59, I.1.4.2]:
  * The ideal PS generated by a set $S \subseteq P$ is given by all $p \in P$ such that $p \geq s$ for some $s \in S$.
  * Let $Q \subseteq P$ be the submonoid generated by S. The face $\langle S \rangle \subseteq P$ generated by S is the set of $p \in P$ such that $p \leq q$ for some $q \in S$.
  * In particular, if $S = \{p\}$ is a singleton, the face generated is all $q \in P$ such that $q \leq np$.
- If P is integral, all faces are exact. If P is fine, all faces generated by a single element: $F = \langle f \rangle$ [59, I.2.1.17].

The affine scheme $\operatorname{Spec} A$ of a ring A is not just a topological space, but a ringed one. A *monoided space* is (the topos of) a topological space or an étale site with a sheaf of monoids [59, II.1.1.1]. A *sharply* monoided space is one whose sheaf of monoids has sharp sections. Write $(MonSp)$, $(MonSp^{\#})$ for the categories of such topoi.

Define a sheaf $M_{\operatorname{Spec} P}$ on $\operatorname{Spec} P$ by its values on distinguished opens $D(f)$:

$$D(f) \mapsto P\left[\frac{1}{f}\right] = \langle f \rangle^{-1} P.$$

We get a sharply monoided space $(\operatorname{Spec}^{\#} P, M^{\#}_{\operatorname{Spec} P})$ by

$$D(f) \mapsto \overline{P\left[\frac{1}{f}\right]}.$$

Remark that the functor of points equates

$$\overline{P\left[\frac{1}{f}\right]} = \overline{P}/f.$$

In either category, $\operatorname{Spec}^{(\#)} P$ represents the functor

$$(X, M_X) \mapsto \operatorname{Hom}(P, \Gamma(M_X)).$$

The stalks at a prime $\pi \in \operatorname{Spec} P$ are given by P_π and $\overline{P_\pi} = \overline{P}/(\overline{P} \setminus \pi)$, respectively. If P is integral, P_π is the subset of P^{gp} whose "denominators" aren't in π. The subset $\operatorname{Spec} P_\pi \subseteq \operatorname{Spec} P$ is the set of $\rho \in \operatorname{Spec} P$ specializing to π.

If P is fine, we've seen that all faces are generated by a single element in Proposition 3.1.11. This is the biggest difference between the usual story of schemes and their stalks: the complement of a prime is generated as a face by a single element $P \setminus \pi = \langle f \rangle$, so

$$M_{\operatorname{Spec} P, \pi} = P_\pi = P\left[\frac{1}{f}\right] = D(f).$$

The set of generizations of π is open, so an open set is specified by a set of generizations. The topology on $\operatorname{Spec} P$ is thereby determined by the preordered set of generizations. This is always the case for finite topological spaces.

A map $\theta : P \to Q$ induces a map

$$\theta^* : \operatorname{Spec} Q \to \operatorname{Spec} P; \qquad \rho \mapsto \theta^{-1}\rho.$$

Crucially, the induced maps on stalks

$$P_{\theta^*\rho} \to Q_\rho$$

are local. Indeed, this is tautological from the diagram

$$P \to Q \to \pm$$

classifying ρ.

The reader familiar with schemes knows what comes next:

Definition 3.1.12. A monoided space (X, M_X) which is locally isomorphic to $\operatorname{Spec} P$ for some monoid is called a *monoscheme* [**?**]. A sharply monoided space which is locally isomorphic to $\operatorname{Spec}^{\#} P$ is called a *fan* [33, 2.2.6]. A monoscheme has an induced fan, but fans are not monoschemes.

Maps $(X, M_X) \to (Y, M_Y)$ between monoschemes are maps of monoided spaces for which the stalks

$$M_{Y,f(x)} \to M_{X,x}$$

are local maps of monoids for all $x \in X$.

The nomenclature "monoscheme" evokes "monoids" as well as "schemes over $\mathbb{F}_1$" in the sense of [64].

Remark 3.1.13. Given $\theta : P \to Q$, we get a map

$$\operatorname{Spec} Q \to \operatorname{Spec} P; \qquad \pi \mapsto \theta^{-1} I.$$

This map necessarily sends $\emptyset$ to itself. This means that generic points are sent to generic points, so all maps are "dominant." If $\operatorname{Spec} \theta$ is a closed immersion, it is surjective.

Ogus introduces pairs of a monoid and an ideal called "acceptably idealized monoids" as a way to add in meaningful closed submonoschemes [59, I.1.5.1]. We won't need this abstraction.

Proposition 3.1.14. A monoid P is finitely generated if and only if it is finitely presented. In fact, every congruence relation on a finitely generated monoid is finitely generated [59, I.2.1.9].

Remark 3.1.15. The proposition is proven by demonstrating a "noetherian" property for monoids holds for $\mathbb{N}$. This is a consequence of "Hilbert's Basis Theorem."

There are many such monoidal parallels to classical commutative algebra. Here are a handful of them in order to whet the reader's appetite:

- Nakayama's Lemma
- Primary Decomposition
- Fractional Ideals

3.2 Log Schemes and Monoschemes

For our purposes, monoids are a tool that vastly simplifies computations involving ordinary schemes. This section makes the connections between monoids and schemes explicit in three ways.

Definition 3.2.1. Let $(X, \mathcal{O}_X)$ be a scheme. A *prelog structure* on X is a sheaf of monoids M_X on the étale site of X equipped with a map

$$\alpha : M_X \to (\mathcal{O}_X, \cdot)$$

to the multiplicative monoid of the structure sheaf. Maps between prelog structures are maps of sheaves of monoids which commute with the structure maps to $\mathcal{O}_X$.

This map is a *log structure* if α restricts to an isomorphism

$$\alpha^{-1}\mathcal{O}_X^* \simeq \mathcal{O}_X^*.$$

We identify the two via this isomorphism. A scheme together with a log structure is called a *log scheme*. Maps between log structures are the same as maps between their underlying prelog structures.

Example 3.2.2. Every scheme admits two log structures: $M_X = \mathcal{O}_X^*$ and $M_X = \mathcal{O}_X$. These are called the *trivial* and *empty* log structure, respectively. Both are interesting, but we use the trivial log structure to identify schemes as a full subcategory of log schemes; write X° for a scheme X endowed with the trivial log structure $M_X = \mathcal{O}_X^*$.

Example 3.2.3. Let k be a field. The *standard log point* is the scheme $\operatorname{Spec} k$ together with the log structure

$$\alpha : k^* \oplus \mathbb{N} \to k; \qquad (t,n) \mapsto \begin{cases} 0 & \text{if } n \neq 0 \\ t & \text{if } n = 0 \end{cases}$$

We think of this map as $(t,n) \mapsto t \cdot 0^n$. If k is algebraically closed, all log structures on $\operatorname{Spec} k$ are of this form.

Let R be a discrete valuation ring; $\operatorname{Spec} R$ is then called a "trait" or a "dash." Let $R' = R \backslash 0$ and write k for the residue field of R. The *standard log dash* is the scheme $\operatorname{Spec} R$ with the log structure $R' \to R$. At the other extreme, the *hollow log structure* on R is given by

$$\alpha : V' \to V; \qquad v \mapsto \begin{cases} 0 & \text{if } v \in V^+ \\ v & \text{if } v \in V^* \end{cases}$$

Since V'/V^* is isomorphic to the value group $\mathbb{N}$, the standard log structure on the dash restricts to the standard log structure on k. The hollow log structure does as well.

The isomorphism $V'/V^* \simeq \mathbb{N}$ depends on a choice of uniformizer, and is not functorial. If we fix a uniformizer $\pi \in V'$, then we obtain infinitely many log structures associated to

$$\mathbb{N} \to V'; \qquad 1 \mapsto \pi^n.$$

The case $n = 1$ is the standard log structure, and one can make precise the statement that the log structures "converge" to the hollow one in the limit [59, III.1.5.6].

Example 3.2.4. Let $i : Y \subseteq X$ be a closed subscheme with complementary open $j : U \subseteq X$. Consider the subsheaf $M \subseteq \mathcal{O}_X$ with sections

$$M(U) := \{f \in \mathcal{O}_X(U) \mid f|_U \in \mathcal{O}_U^*\}.$$

This is the sheaf of regular functions whose restriction to U is invertible: $M = \mathcal{O}_X \times_{j_*\mathcal{O}_U} j_*\mathcal{O}_U^*$. Sums of invertible elements needn't be invertible, but M is a monoid under multiplication. In fact, M is a sheaf of faces in the multiplicative monoid $\mathcal{O}_X$ necessarily containing $\mathcal{O}_X^*$.

The inclusion $M \subseteq \mathcal{O}_X$ *compactifying* log structure.

There is an initial map from a prelog structure $\alpha : M_X \to \mathcal{O}_X$ to a log structure, given by the *associated log structure* M_X^{ass}:

$$\begin{tikzcd}
\alpha^{-1}\mathcal{O}_X^* \arrow[r] \arrow[d] & M_X \arrow[d] \arrow[ddr, bend left, "\alpha"] & \\
\mathcal{O}_X^* \arrow[r] \arrow[drr, bend right] & M_X^{ass} \arrow[dr, dashed] & \\
 & & \mathcal{O}_X.
\end{tikzcd}$$

We need to make sense of morphisms between different log schemes, not just morphisms of log structures on one fixed scheme. A *morphism of log schemes* $(X, M_X) \to (Y, M_Y)$ is a map $X \to Y$ of schemes and a morphism

$$f^{-1}M_Y \to M_X$$

that fits into a commutative square

$$\begin{tikzcd}
f^{-1}M_Y \arrow[r] \arrow[d] & M_X \arrow[d] \\
\mathcal{O}_Y|_X \arrow[r] & \mathcal{O}_X.
\end{tikzcd}$$

We can equivalently reformulate the definition as:

$$\begin{tikzcd}
M_Y \arrow[r] \arrow[d] & f_*M_X \arrow[d] \\
\mathcal{O}_Y \arrow[r] & f_*\mathcal{O}_X.
\end{tikzcd}$$

We do not expect $f^{-1}M_Y \to \mathcal{O}_X$ to be a log structure. Write $f^*M_Y = M_Y|_X$ for its associated log structure, the *pullback* log structure. Likewise, denote by $f_*^\ell M_X$ the *pushforward* log structure given by the pullback

$$\begin{tikzcd}
f_*^\ell M_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & f_*M_X \arrow[d] \\
\mathcal{O}_Y \arrow[r] & f_*\mathcal{O}_X.
\end{tikzcd}$$

We often abuse notation and simply write f_*M_X instead of $f_*^\ell M_X$. We see that a morphism $(X, M_X) \to (Y, M_Y)$ of log schemes is a morphism $X \to Y$ of schemes, together with either of the equivalent data:

- A map $M_Y \to f_*M_X$ of log structures on Y, or
- A map $M_Y|_X \to M_X$ of log structures on X.

Example 3.2.5. Given a log scheme X, we write X° also for the underlying scheme of X endowed instead with the trivial log structure. There is a morphism

$$X \to X^\circ$$

which is initial for all maps from X to a scheme, considered as a log scheme with trivial log structure.

On the other hand, the only log schemes with a map

$$X^\circ \to X$$

lifting the identity on underlying schemes have trivial log structure. There is a similar construction to X° but for the empty log structure, but we don't need it.

Example 3.2.6. If a morphism of schemes $X \to Y$ is compatible with fixed open sets U, V:

$$\begin{tikzcd} U \arrow[r, dashed] \arrow[d, hook] & V \arrow[d, hook] \\ X \arrow[r] & Y, \end{tikzcd}$$

then it induces a map between the log schemes X and Y endowed with the compactifying log structures.

3.2.1 Monoid Rings and Charts

There are many ways to describe a log structure. The first approach of K. Kato [38] is given by "charts." A map $P \to \Gamma(M_X)$ is equivalent to the data of a map out of the constant sheaf $\underline{P}_X \to M_X$ by the adjunction $(\underline{\cdot}_X, \Gamma(\cdot))$. We say $P \to \Gamma(M_X)$ furnishes a *chart* if the corresponding map

$$\underline{P}_X^{ass} \to M_X$$

is an isomorphism.

A log scheme which locally admits a chart is called *quasi-coherent.* Log structures with charts are similar to quasi-coherent sheaves.

The map $P \to \Gamma(M_X) \to \Gamma(\mathcal{O}_X)$ which becomes the structure map of the log structure after sheafification is classified by a map

$$X \to \operatorname{Spec} \mathbb{Z}[P].$$

We remind the reader of these rings and their features.

Given a ring R and a monoid P, the *monoid ring* is

$$R[P] := \{f : P \to R \mid f(p) \neq 0 \text{ for only finitely many } p\}.$$

The values $f(p)$ are the *coefficients* of p, and we write such a finitely supported function as a polynomial

$$\sum f(p)[p].$$

The addition is pointwise and the multiplication is determined by linearity and

$$[p] \cdot [q] = [p + q].$$

Example 3.2.7. Write $\sigma(f)$ for the *support* of f: the set of $p \in P$ for which $f(p) \neq 0$. Then

- $\sigma([p] \cdot f) = p + \sigma(f)$,
- $\sigma(f + g) \subseteq \sigma(f) \cup \sigma(g)$, and
- $\sigma(fg) \subseteq \sigma(f) + \sigma(g)$ [59, I.3.5.4].

The convex hull of $\sigma(f)$ in $C(P)$ is sometimes called the *Newton Polyhedron.*

Proposition 3.2.8. Let R be a ring and P an integral monoid.

- If P^{gp} is torsion-free and R an integral domain, $R[P]$ is an integral domain.
- If P is fine, R normal, then

  $$R[P^{sat}] \to R[P]$$

  is the normalization map.

- If P is fine, $R \to R[P]$ is fppf [59, I.3.4.1].
- If R is noetherian and Cohen-Macaulay and P is fs, then $R[P]$ is Cohen-Macaulay [59, I.3.4.3].

We write

$$\mathbb{A}_P := \operatorname{Spec} \mathbb{Z}[P].$$

Our study of intersection theory will be over $\mathbb{C}$, and we will use the notation $\mathbb{A}_P := \operatorname{Spec} \mathbb{C}[P]$. This scheme has a monoid structure coming from the maps

$$\mathbb{Z}[P] \to \mathbb{Z}[P] \otimes \mathbb{Z}[P]; \qquad [p] \mapsto [p] \otimes [p],$$

$$\mathbb{Z}[P] \to \mathbb{Z}; \qquad [p] \mapsto 1.$$

We endow $\mathbb{A}_P$ with the log structure associated to

$$P \to \mathbb{Z}[P].$$

Maps of schemes from T to $\mathbb{A}_P$ correspond to maps $\mathbb{Z}[P] \to \Gamma(\mathcal{O}_T)$. The log structure rigidifies this data: Maps of *log* schemes are given by:

$$\mathrm{Hom}_{log}(T, \mathbb{A}_P) = \mathrm{Hom}_{mon}(P, \Gamma(M_T)).$$

In particular [59, I.3.1.2],

$$\mathrm{Hom}_{log}(\mathbb{A}_P, \mathbb{A}_Q) = \mathrm{Hom}_{mon}(Q, P).$$

Observe

$$H(P) = \mathrm{Hom}_{log}(\mathbb{A}_{\mathbb{N}}, \mathbb{A}_P).$$

The monoid schemes also have a vertex map, coming from the vertex

$$P \mapsto \pm \subseteq \mathbb{Z}.$$

We write $\mathbb{A}_P^* := \mathbb{A}_{P^{gp}}$, in analogy with the traditional dense open torus of a toric variety. Given a morphism $\theta : P \to Q$, write

$$\mathbb{A}_{Q/P}^* := \mathbb{A}_{\operatorname{coker} \theta^{gp}} = \ker(\mathbb{A}_Q^* \to \mathbb{A}_P^*).$$

We get an action $\mathbb{A}^*_{Q/P} \circlearrowright \mathbb{A}_Q$ over P. The shearing map

$$\mathbb{A}^*_{Q/P} \times \mathbb{A}_Q \to \mathbb{A}_Q \times_{\mathbb{A}_P} \mathbb{A}_Q$$

extends the $\mathbb{A}^*_{Q/P}$-torsor $\mathbb{A}^*_Q \to \mathbb{A}^*_P$. This action stabilizes the closed immersions

$$\mathbb{A}_{Q_F} \subseteq \mathbb{A}_Q, \quad \mathbb{A}_F \subseteq \mathbb{A}_Q$$

for any face $F \prec Q$ [59, I.3.3.5].

3.2.2 Fans and Log Stacks

The choice of a chart is something very non-canonical. For example, suppose $\theta : P \to Q$ of quasi-integral monoids induces an isomorphism on sharpenings: $\overline{\theta} : \overline{P} \simeq \overline{Q}$. Given a map $\alpha : Q \to \Gamma(\mathcal{O}_X)$, the log structures associated to α and $\alpha \circ \theta$ are the same.

It may be better to consider charts given by sharp monoids $\overline{P}$. An example will reassure us that this is often locally possible – we can choose a chart $\overline{P} \to \Gamma(\mathcal{O}_X)$ instead of $P \to \Gamma(\mathcal{O}_X)$:

Example 3.2.9. Let (X, M_X) be a monoided space. A chart $P \to \Gamma(M_X)$ with P integral and M_X a sheaf of integral monoids is called *neat* at a point $x \in X$ if the composite

$$P \to M_{X,x} \to \overline{M}_{X,x}$$

is an isomorphism.

If M_X is fine, then it admits a local chart that is neat at $x \in X$ if and only if

$$0 \to M^*_{X,x} \to M^{gp}_{X,x} \to \overline{M}^{gp}_{X,x} \to 0$$

splits. In particular, it splits if M_X is saturated [59, II.2.3.7].

We have already discussed $\mathbb{A}_P$, which represents the functor

$$X \mapsto \mathrm{Hom}(P, \Gamma(M_X)).$$

We can describe this as a composite:

$$(LSch) \to (MonSch) \overset{h_{\mathrm{Spec}\,P}}{\to} (Set).$$

Instead, consider the sharp version of $\operatorname{Spec}^{\#} P$:

$$(LSch) \to (MonSch^{\#}) \overset{h_{\operatorname{Spec}^{\#} P}}{\to} (Set); \qquad (X, M_X) \mapsto (X, \overline{M}_X) \mapsto \operatorname{Hom}(P, \Gamma(\overline{M}_X)).$$

Let P be an fs monoid. The action $\mathbb{A}_P^* \circlearrowright \mathbb{A}_P$ presents a monoidal quotient stack

$$A_P := [\mathbb{A}_P/\mathbb{A}_P^*]$$

called an "Artin Cone." This stack represents a *functor* on fine log schemes – this means it represents a sheaf of *sets* just like an ordinary log scheme even though it is a complicated stack without log structures. This behaves like $\mathbb{A}_P^* \circlearrowright \mathbb{A}_P$ is a free action!

The study of these stacks goes back to [60], where they're attributed to Illusie and Lafforgue. We describe the functor of points of A_P separately for schemes and log schemes.

Proposition 3.2.10. Let P be an fs monoid. The fibers $A_P(T)$ over a test scheme T are given by a choice of fs log structure N_T on T, together with a map $P \to \overline{N}_T$ which étale-locally lifts to a chart [60, Proposition 5.14, 5.15].

The log structure on A_P descends from that of $\mathbb{A}_P$ [60, After 5.13]. With this natural log structure, the stack represents a functor we've seen before:

Proposition 3.2.11 ([60, Proposition 5.17])**.** Let T be a fine log scheme. Then

$$\operatorname{Hom}_{fs}(T, A_P) = \operatorname{Hom}(P, \Gamma(\overline{M}_T)) = \operatorname{Hom}(T, \operatorname{Spec}^{\#} P).$$

We are mostly interested in *quasi-integral* monoids P, for which $P^* \circlearrowright P$ acts freely and thereby becomes a torsor over $\overline{P}$. Quasi-integral log structures admit a description similar to Deligne and Faltings' original works, due to [46], [9].

The diagram

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{m} & \mathbb{G}_m \\ \circlearrowright & & \circlearrowright \\ \mathbb{A}^1 \times \mathbb{A}^1 & \xrightarrow{m} & \mathbb{A}^1 \end{array}$$

presents the monoid

3.2.3 Log Blowups and Exactification

Let $Y \subseteq X$ be a closed immersion of schemes corresponding to an ideal sheaf I. The ordinary blowup $Bl_Y X \to X$ has a kind of "functor of points:"

Proposition 3.2.12 ([28, Proposition 7.14])**.** Let X is noetherian, I is a coherent sheaf of ideals. If $f : Z \to X$ is such that $f^*I \cdot \mathcal{O}_Z$ is an invertible sheaf, then there exists a unique factorization

$$\begin{tikzcd} & Bl_Y X \arrow[d] \\ Z \arrow[ur, dashed] \arrow[r, "f"] & X. \end{tikzcd}$$

This statement is not if and only if: maps to the blowup are not the same as maps to X satisfying the property, but certain maps are given that way. In particular, maps into the exceptional divisor aren't of this form.

If we take the blowup of an ideal of monoids in a suitable sense, we will see that it has a *bona fide functor of points*. Log maps to such a blowup will be uniquely identified with maps satisfying a similar property; in particular, the blowup map will be a *monomorphism* among integral schemes.

Definition 3.2.13. Let I be a coherent sheaf of monoid ideals in a fine (resp. fs) log scheme X. The *log blowup* of X at I is the sieve $D \subseteq h_X$ on $(LSch)^{fine}$ (resp. (fs)) of $f : T \to X$ such that f^*I is invertible.

Log blowups are representable by a fine (resp. fs) log scheme $B\ell_I X$ [59, III.2.6.3].

Proposition 3.2.14. Let I be a coherent sheaf of monoid ideals in a fine (resp. fs) log scheme X and refer to [59, III.2.6.3].

- Log blowups are compatible with base change:

$$B\ell_I X {\times_X}^{fine} T = B\ell_{I|_T} T.$$

  Here, it is imperative that the fiber product take place in the fine or fs category.

- Log blowups are proper monomorphisms which are universally surjective.

- Log blowups commute: let J be another coherent sheaf on X. Then

$$B\ell_I X \times_X B\ell_J X \simeq B\ell_{I+J} X \simeq B\ell_I(B\ell_J X) \simeq B\ell_J(B\ell_I X).$$

One of the main uses of a log blowup is to render a morphism "exact:"

Definition 3.2.15. A map $\theta : P \to Q$ of monoids is called *exact* if the square

$$\begin{tikzcd} P \arrow[r, "\theta"] \arrow[d] & Q \arrow[d] \\ P^{gp} \arrow[r, "\theta"] & Q^{gp} \end{tikzcd}$$

is a pullback.

Proposition 3.2.16. Let $\theta : P \to Q$ be a map of monoids.

- The map θ is exact if and only if $\overline{\theta}$ is [59, I.4.2.1].
- If θ is exact, then $\operatorname{Spec}\theta$ is surjective. The converse holds if P is fs [59, I.4.2.2].
- If θ is exact, then it is universally local. The converse holds if P is saturated [59, I.4.2.3].
- The integral pullback of an exact morphism is exact [59, I.2.1.16].
- If $\theta : P \to Q$ is a homomorphism of integral monoids, then it is exact if and only if, whenever $\theta(p) \leq \theta(p')$, then $p \leq p'$ [59, I.2.1.16].
- M is integral if and only if $\Delta : M \to M \times M$ is exact [59, I.2.1.16].

Proposition 3.2.17. If $\theta : P \to Q$ is injective, the bullets are equivalent [59, I.4.2.7, I.2.1.16]:

- θ is exact,
- the induced action $P \circlearrowright Q$ preserves $Q \setminus P$,
- $\mathbb{Z}[\theta] : \mathbb{Z}[P] \to \mathbb{Z}[Q]$ splits as a morphism of $\mathbb{Z}[P]$-modules,
- $\mathbb{Z}[\theta]$ is universally injective.

In the case where θ is injective and exact, the map $\mathbb{Z}[\theta]$ "universally descends flatness:" given any pushout of rings:

$$\begin{tikzcd} \mathbb{Z}[P] \arrow[r] \arrow[d] & \mathbb{Z}[Q] \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & B \end{tikzcd}$$

and any A-module M, the module M is flat over A if and only if the module $M \otimes_A B$ is flat over B [59, I.4.2.8].

Example 3.2.18. Let $\theta : P \to Q$ be a morphism of monoids. Form the pullback

$$\begin{tikzcd} P^\theta \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Q \arrow[d] \\ P^{gp} \arrow[r] & Q^{gp}. \end{tikzcd}$$

Then $P^\theta \to Q$ is exact and $P \to P^\theta$ becomes an isomorphism upon groupification. This factorization is natural in morphisms θ [**?**].

Proposition 3.2.19. Let $f : X \to Y$ be a quasicompact morphism of fine log schemes. Locally on Y, one can blow up Y so that f pulls back in the category of integral monoids to an exact morphism [59, III.2.6.7].

This is a form of [33, Proposition 2.3.12].

A related, important use of log blowups is to achieve "valuativization." A *valuative monoid* P is an integral monoid such that, for all $x \in P^{gp}$, at least one of $x, -x$ is in P. The poset structure on $\mathcal{I}_P P^{gp}$ for an integral monoid P is a total order if and only if P is valuative.

Example 3.2.20. The natural numbers $\mathbb{N}$ form a valuative monoid, as does any group. Given any totally ordered group, the nonnegative elements form a valuative monoid. The additive monoid

$$\mathbb{N}[\epsilon] := \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ or } (x = 0 \text{ and } y \geq 0)\}$$

is valuative. The nomenclature reflects the use of this monoid as an "infinitesimal extension" in the definition of the smooth topology on monoids.

We will see than any saturated monoid is the intersection of all valuative submonoids in its groupification containing it [37, Lemma 1.1.9].

Let L be an abelian group. We say of two submonoids $Q, Q' \subseteq L$ that Q' *dominates* Q if $Q \subseteq Q'$ and the inclusion is local. The maximal submonoids with respect to the partial order of domination are valuative submonoids P such that $P^{gp} = L$ [59, I.2.4.1]. Our next proposition illustrates that most valuative monoids aren't finitely generated.

Proposition 3.2.21. Let P be sharp, fine, and valuative. Then $P = \mathbb{N}$ or $P = 0$. The same is true if P is only assumed to be fs, sharp, and of dimension one [59, I.2.4.2].

Moreover, every fs monoid of dimension one is $\mathbb{N} \times \Gamma$ for some finite abelian group Γ.

Example 3.2.22 (Divisorial Valuations)**.** Let P be fine and consider a prime $\mathfrak{p} \subseteq P$ of height one. Then $P_{\mathfrak{p}}^{sat}$ is valuative and there's a unique valuation

$$v_{\mathfrak{p}} : P^{gp} \to \mathbb{Z}$$

such that

$$v_{\mathfrak{p}}^{-1}\mathbb{N}_{>0} \cap P = \mathfrak{p}.$$

Moreover,

$$P_{\mathfrak{p}}^{sat} = v_{\mathfrak{p}}^{-1}(\mathbb{N})$$

[59, I.2.4.4].

If P is fs, then

$$P = \bigcap_{v_{\mathfrak{p}},\, \mathrm{ht}\mathfrak{p}=1} v_{\mathfrak{p}}^{-1}\mathbb{N}$$

and

$$P^* = \bigcap_{v_{\mathfrak{p}},\, \mathrm{ht}\mathfrak{p}=1} v_{\mathfrak{p}}^{-1}0.$$

That is to say, P is the intersection of its localizations at height-one primes [59, I.2.4.5].

A finitely generated, nonempty ideal of a valuative log scheme is principal for the same reason noetherian valuation rings are principal ideal domains. The idea behind the "valuativization" of a log scheme is that the converse also holds:

Definition 3.2.23. Let X be a log scheme with integral, quasi-coherent log structure. Order the coherent, nonempty ideal sheaves $I \subseteq M_X$ by

$$I \leq J \text{ if there is an ideal } K \subseteq M_X \text{ such that } IK = J.$$

Define

$$X^{val} := \lim_{I \subseteq M_X} B\ell_I X.$$

Such a limit of schemes will likely not be a scheme. It has an incarnation as a "log space" [37], a locally ringed space with log structure. Moreover, any map $Y \to X$ with Y a locally ringed space with valuative log structure factors uniquely through $X^{val} \to X$.

This important but subtle construction is attributed to Ofer Gabber. According to [37], we offer a direct description of the underlying topological space.

Fix a chart $X \to \operatorname{Spec} \mathbb{Z}[P]$. If $P \to V$ is a map of monoids, write

$$X_V := X \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[V].$$

A point of X^{val} is a pair (V, x) with $V \subseteq P^{gp}$ a valuative submonoid containing P and $x \in X_V$ such that the map

$$V \to \mathcal{O}_{X_V,x} = (\mathcal{O}_X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[V])_x$$

is local.

We identify the point corresponding to a pair (V, x). If $I \subseteq P$ is a finitely generated ideal, it becomes principal after extension to V: $IV = (a)$. This gives $P[a^{-1}I] \subseteq V$. Then $\operatorname{Spec} \mathbb{Z}[P[a^{-1}I]]$ is an open subset $D_+(a) \subseteq B\ell_I P$, and we have a map

$$\varphi : \operatorname{Spec} \mathbb{Z}[V] \to \operatorname{Spec} \mathbb{Z}[P[a^{-1}I]] \subseteq B\ell_I P.$$

The point in X^{val} corresponding to (V, x) is the limit of the images of $x \in X_V$ under $\varphi \times_{\operatorname{Spec} \mathbb{Z}[P]} X : X_V \to B\ell_I X$.

The stalks of the structure sheaves are:

$$\mathcal{O}_{X^{val},(V,x)} = \mathcal{O}_{X_V,x},$$

$$M_{X^{val},(V,x)} = V \oplus_{V^*} \mathcal{O}^*_{X_V,x}.$$

We can also describe the topology on X^{val}. If $Q \subseteq P^{gp}$ contains P and is finitely generated over it, consider a open immersion $U \subseteq X_Q$. The map $U^{val} \to X^{val}$ is a strict open immersion of log spaces. Opens of the form U^{val} form a basis for the topology on X^{val}. A pair (V, x) is in U^{val} if and only if $Q \subseteq V$ and the image of x lands in $U \subseteq X_Q$:

$$\begin{tikzcd} U^{val} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X^{val} \arrow[d] \\ U \arrow[r] & X_Q \arrow[d] \\ & X. \end{tikzcd}$$

3.3 Toric Geometry

If P is a fine monoid, write $H(P) := \mathrm{Hom}(P, \mathbb{N})$.

Proposition 3.3.1.

- The monoid $H(P)$ is fs and sharp.
- The reflexivity map
$$P \to H(H(P))$$
factors through an isomorphism $\overline{P^{sat}} \simeq H(H(P))$. The functor H is therefore an anti-autoequivalence of sharp, toric monoids [59, I.2.2.3].
- All faces $F \prec P$ are given by $h^{-1}(0)$ for some $h : P \to \mathbb{N}$ [59, I.2.2.1]. That is, $0 \in \mathbb{N}$ is the universal face among fine monoids.
- The pairing $P \times H(P) \to \mathbb{N}$ induces an antiequivalence
$$\left\{ \begin{array}{c} \text{Faces} \\ \\ F \prec P \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{Faces} \\ \\ F \prec H(P) \end{array} \right\}^{op}.$$
- Any fine, sharp monoid P is a submonoid of $\mathbb{N}^r \times \Gamma$ for some finite group Γ. If P^{gp} is torsion-free, one may take $\Gamma = 0$. These facts proceed from a splitting of
$$P^{gp} \to HHP^{gp},$$

which has kernel the torsion subgroup of P^{gp} [59, I.2.2.7].

- The interior of $H(P)$ consists in the *local* homomorphisms $P \to \mathbb{N}$.

Our discussion of cones is valid over any archimedean, ordered field. We stick to $\mathbb{R}$ for simplicity.

Definition 3.3.2. A *cone* C is a monoid with an $\mathbb{R}$-vector space structure on C^{gp} such that C is closed under the action of $\mathbb{R}_{\geq 0}$. We usually consider sharp cones, $C^* = 0$. These are called "strongly convex rational cones" in the literature.

Let $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ be vector spaces with a perfect pairing

$$M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}; \qquad (m, n) \mapsto \langle m, n \rangle,$$

and consider $C \subseteq M_{\mathbb{R}}$ a finitely generated cone. Write

$$C^{\vee} := \{h : M_{\mathbb{R}} \to \mathbb{R} \mid h|_C \geq 0\},$$

$$C^{\perp} := \{h : M_{\mathbb{R}} \to \mathbb{R} \mid h|_C = 0\}.$$

Given a subset $S \subseteq V$ of a vector space, we can take the *conical hull*

$$C(S) := \mathbb{R}_{\geq 0}\langle S \rangle \subseteq V$$

to be the subset of positive linear combinations of elements of S. We use this often when $S \subseteq V$ is given by $P \subseteq P^{gp} \otimes \mathbb{R}$ for an integral monoid.

Proposition 3.3.3. If P is integral, we have a bijection

$$\left\{ \begin{array}{c} \text{Faces} \\ \\ F \prec P \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{c} \text{Faces} \\ \\ G \prec C(P) \end{array} \right\}$$

given by $F \mapsto C(F)$ and $G \mapsto G \cap_{C(P)} P$ [59, I.2.3.7]. We have $C(P) \simeq C(P^{sat})$, and a larger square of homeomorphisms of spectra [**?**]:

$$\begin{array}{ccc} \operatorname{Spec} C(P^{sat}) & \longrightarrow & \operatorname{Spec} C(P) \\ \downarrow & & \downarrow \\ \operatorname{Spec} P^{sat} & \longrightarrow & \operatorname{Spec} P. \end{array}$$

Proposition 3.3.4. Let $C, C' \subseteq M_{\mathbb{R}}$ be finitely generated cones.

- Any face $F \prec C$ is given by $h^{-1}(0)$ for some map $h : M_{\mathbb{R}} \to \mathbb{N}$.

- We have an antiequivalence:

$$\left\{ \begin{array}{c} \text{Faces} \\ \\ F \prec C \end{array} \right\}^{op} \leftrightsquigarrow \left\{ \begin{array}{c} \text{Faces} \\ \\ G \prec C^{\vee} \end{array} \right\}$$

  given by $F \mapsto F^{\vee}$ and $G \mapsto G^{\vee}$ [59, I.2.3.12].

- We have the identities

$$(C + C')^{\vee} = C^{\vee} \cap C'^{\vee}$$

  and

$$C^{\vee} + C'^{\vee} = (C \cap C')^{\vee}.$$

- *Gordan's Lemma:* Let L be a finitely generated abelian group, $C \subseteq L \otimes \mathbb{R}$ a finitely generated monoid. Then $C \cap L$ is finitely generated.

A cone $C \subseteq V$ is said to be *simplicial* if $C \simeq \mathbb{R}^n_{\geq 0}$. Every sharp, finitely generated cone in $\mathbb{R}$ or $\mathbb{R}^2$ is simplicial. A theorem of Carathéorodory [59, I.2.5.1] states that every cone C generated by a finite set S is the union of its simplicial subcones generated by linearly independent subsets of S. In fact, they can all be taken to be of the same dimension.

3.4 Log Differentials

The point of log structures is to qualify in what way mildly singular schemes can be "log smooth." We began the section on log structures with the historically original notion of differential forms with log poles along a normal crossings divisor; we now expand this notion to general log structures. Not only will normal crossings divisors become log smooth, but we will also see that log smooth schemes over a point are "toric varieties with torsion." Log smooth maps are a class of non-flat families of these toric varieties.

Refer to the parallel notions of Kähler Differentials, Derivations, and the Cotangent Complex recalled in Chapter 1.

Instead of just fixing a ring homomorphism $A \to B$, we need to fix a commutative square $\theta : \alpha \to \beta$:

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow \\ P & \longrightarrow & Q \end{array}$$

with $A \to B$ a ring map and the rest of the diagram in the category of monoids. Consider a B-module I. A *log derivation* of θ with values in I is a pair

$$D : B \to I$$

$$\delta : Q \to I$$

such that D is an ordinary A-derivation and δ satisfies:

- $\delta(P) = 0$ and
- $D(\beta(q)) = \beta(q) \cdot \delta(q)$ for all $q \in Q$.

The set $\mathrm{Der}_\alpha(\beta, I)$ of all log derivations of θ with values in I has a pointwise B-module structure. This B-module of log derivations is covariantly functorial in I and contravariantly functorial in morphisms of commutative squares $\alpha \to \beta$ [59, IV.1.1.3]. The compatibility $D \circ \beta = \beta \cdot \delta$ conceptually defines δ as

$$\text{“}d\log\beta(q) = \frac{D\beta(q)}{\beta(q)}.\text{”}$$

This is literally correct if $\beta(q) \in B^*$ is a unit.

Just as for ordinary derivations, there's a universal log derivation.

Proposition 3.4.1. Fix a commutative diagram $\theta : \alpha \to \beta$ and a B-module I as above. There is a B-module Ω^ℓ_θ equipped with a “universal θ-derivation:”

$$D : B \to \Omega^\ell_\theta$$

$$\delta : Q \to \Omega^\ell_\theta$$

forming an element $(D,\delta) \in \mathrm{Der}_\alpha(\beta, \Omega^\ell_\theta)$ such that all other θ-derivations valued in I are the composite of (D,δ) with a unique map $\Omega^\ell_\theta \to I$ [**?**].

Write $\pi : Q \to Q^{gp}/P^{gp}$. The universal derivation is given as the pushout:

$$\begin{tikzcd}
Q \arrow[r, "\beta\otimes\pi"] \arrow[d, "\beta"] & B \otimes Q^{gp}/P^{gp} \arrow[dd] \\
B \arrow[d, "D"] & \\
\Omega_{B/A} \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \Omega^\ell_\theta.
\end{tikzcd}$$

The functorialities of log derivations and $\Omega^\ell_\cdot$ may be seen in the functoriality of the rest of the diagram. Moreover, we can readily compute the most important case where $\alpha : P \to R[P]$, $\beta : Q \to R[Q]$ are monoid rings with their canonical log structures:

Proposition 3.4.2. Let $\theta : \alpha \to \beta$ be a commutative square of monoid rings

$$\begin{tikzcd}
R[P] \arrow[r] & R[Q] \\
P \arrow[r] \arrow[u, "\alpha"] & Q \arrow[u, "\beta"']
\end{tikzcd}$$

and continue to write $\pi : Q \to Q^{gp}/P^{gp}$. Let I be an $R[Q]$-module. Then restriction yields an isomorphism

$$\mathrm{Der}_\alpha(\beta, I) \simeq \mathrm{Hom}_{abgp}(Q^{gp}/P^{gp}, I),$$

which means

$$\Omega^\ell_\theta = Q^{gp}/P^{gp} \otimes R[Q].$$

The universal derivation is

$$\delta : Q \to Q^{gp}/P^{gp} \otimes R[Q]; \qquad q \mapsto \pi(q) \otimes q$$

and D is the map on $R[Q]$ induced by δ [59, IV.1.1.4].

In particular, if Q^{gp}/P^{gp} is a finitely generated group with torsion a finite group of order invertible in R, then Ω^ℓ_θ is free.

These notions may be applied to sheaves. If $X \to Y$ is a morphism of schemes and E is a sheaf of $\mathcal{O}_X$-modules, log derivations of $X \to Y$ with values in E entail similar pairs of maps for the diagram

$$\begin{array}{ccc} \mathcal{O}_Y|_X & \longrightarrow & \mathcal{O}_X \\ \uparrow & & \uparrow \\ M_Y|_X & \longrightarrow & M_X \end{array}$$

valued in E. These log derivations also come from an $\mathcal{O}_X$-module $\Omega^\ell_{X/Y}$ in the same way.

There is an alternative description of the Log Kähler Differentials which hinges on a lemma:

Lemma 3.4.3. The inclusion $\mathcal{O}_X^* \subseteq \mathcal{O}_X$ generates $\mathcal{O}_X$ as a sheaf of additive monoids. So does M_X, for any log structure on X.

If $a \in \mathcal{O}_X(U)$ is some local section, it restricts to any stalk $\mathcal{O}_{X,x}$, $x \in X$ to either a unit or an element of the maximal ideal. In the latter case, $a - 1$ is a unit.

This lemma implies that derivations (D, δ) are determined by δ. However δ, as a map of multiplicative monoids, doesn't see the additive structure of the ring.

Let $R \subseteq \mathcal{O}_X \otimes M_X^{gp}$ be generated by local sections given by

$$\sum \alpha(m_i) \otimes m_i - \sum \alpha(m_i') \otimes m_i'$$

for $\sum \alpha(m_i) = \sum \alpha(m_i')$, $m_i, m_i' \in M_X$.

We want to mod out by R, which is the same as taking the coequalizer of

$$\mathbb{Z}[M_X] \times_{\mathcal{O}_X} \mathbb{Z}[M_X] \rightrightarrows \mathcal{O}_X \otimes M_X^{gp}.$$

Together with quotienting out by the image of $M_Y|_X$, this gives the Log Kähler Differentials [59, IV.1.2.11]:

Proposition 3.4.4. The unique derivation extending

$$\delta : M_X \to \mathcal{O}_X \otimes (M_X^{gp}/M_Y^{gp}|_X)/R; \qquad m \mapsto 1 \otimes m$$

is also universal. It thereby enjoys a unique isomorphism

$$\Omega^\ell_{X/Y} \simeq \mathcal{O}_X \otimes (M_X^{gp}/M_Y^{gp}|_X)/R.$$

Proposition 3.4.5. If $\theta : P \to Q$ is injective, the bullets are equivalent:

- The inclusion θ is exact.
- The complement $Q \setminus P$ is closed under the action of P.
- The map $\theta_{\mathbb{Z}} : \mathbb{Z}[P] \to \mathbb{Z}[Q]$ splits as a map of $\mathbb{Z}[P]$-modules.
- The map $\theta_{\mathbb{Z}} : \mathbb{Z}[P] \to \mathbb{Z}[Q]$ "universally descends flatness:" For any $\mathbb{Z}[P]$-algebra A form the pushout

$$\begin{tikzcd} A \arrow[r] & B \\ \mathbb{Z}[P] \arrow[u] \arrow[r] & \mathbb{Z}[Q]. \arrow[u] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

  We say θ universally descends flatness if an A-module M is flat if and only if $M \otimes_A B$ is a flat B-module.

[59, I.4.2.8]

Chapter 4

The Log Product Formula

If there were even a trifle nonempty, Emptiness itself would be but a trifle. But not even a trifle is nonempty. How could emptiness be an entity?

...

Action and misery having ceased, there is nirvana. Action and misery come from conceptual thought. This comes from mental fabrication. Fabrication ceases through emptiness.

...

For him to whom emptiness is clear, everything becomes clear. For him to whom emptiness is not clear, nothing becomes clear.

[12, XIII # 7, XVIII # 5, XXIV # 14]

4.1 Introduction

4.1.1 The Log Product Formula

The purpose of the present paper is to prove the "Product Formula" for Log Gromov-Witten Invariants. For ordinary Gromov-Witten Invariants, the analogous formula was established by K. Behrend in [7].

Let V, W be log smooth, quasiprojective log schemes. Let Q be the fs fiber product

$$\overline{M}_{g,n}(V) \times^{\ell}_{\overline{M}_{g,n}} \overline{M}_{g,n}(W),$$

with maps

$$\overline{M}_{g,n}(V \times W) \xrightarrow{h} Q \xrightarrow{\widetilde{\Delta}} \overline{M}_{g,n}(V) \times \overline{M}_{g,n}(W).$$

One can naturally endow Q with a "log virtual fundamental class" in two ways: pushing forward that of $\overline{M}_{g,n}(V \times W)$ or pulling back that of $\overline{M}_{g,n}(V) \times \overline{M}_{g,n}(W)$. The Product Formula equates these:

Theorem 4.1.1 (The "Log Gromov-Witten Product Formula")**.** The two log virtual fundamental classes are equal in A_*Q:

$$h_*[V \times W, E(V \times W)]^{\ell vir} = \Delta^![V, E(V)]^{\ell vir} \times [W, E(W)]^{\ell vir}.$$

The symbol $\Delta^!$ refers to a "Log Gysin Map" for which we offer a definition, along with "Log Virtual Fundamental Classes."

This theorem was formulated for ordinary virtual fundamental classes in [42] and proved under the assumption that one of V or W has trivial log structure. Like their work and [7] before it, our proof centers on this cartesian diagram (Situation 4.6.5):

$$\begin{tikzcd}
\mathscr{M}^\ell_{g,n}(V \times W) \arrow[r, "h"] \arrow[d, "c"] \arrow[dr, phantom, "\ulcorner \ell"] & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & \mathscr{M}^\ell_{g,n}(V) \times \mathscr{M}^\ell_{g,n}(W) \arrow[d, "a"] \\
\mathfrak{D} \arrow[r, "l"] & Q' \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & \mathfrak{M}^\ell_{g,n} \times \mathfrak{M}^\ell_{g,n} \arrow[d, "s \times s"] \\
 & \overline{M}_{g,n} \arrow[r, "\Delta"] & \overline{M}_{g,n} \times \overline{M}_{g,n}.
\end{tikzcd}$$

One applies Costello's Formula [15, Theorem 5.0.1] and commutativity of the Gysin Map to this diagram to compare virtual fundamental classes.

In the log setting, one requires this diagram to be cartesian in the 2-category of *fs log algebraic stacks* in order to preserve modular interpretations. The assumption of [42] that V or W have trivial log structure ensures that these squares are *also* cartesian as underlying algebraic stacks.

These fs pullback squares in question likely aren't cartesian on underlying algebraic stacks. Therefore, none of the standard machinery of ordinary Gysin Maps and Normal Cones is valid. This quandary forced us to prove the log analogues of Costello's Formula and commutativity for our "Log Gysin Map." With these modifications, the original proof of K. Behrend essentially still works. We pause to comment on the new technology.

4.1.2 Log Normal Cones

The *Log Normal Cone* $C^{\ell}_{X/Y} = C_{X/\mathscr{L}Y}$ of a map $f : X \to Y$ of log algebraic stacks is the central object of the present paper. Every log map factors as the composition of a strict and an étale map $X \to \mathscr{L}Y \to Y$, so the cone is determined by two properties:

- It agrees with the ordinary normal cone for strict maps.
- If one can factor f as $X \to Y' \to Y$ with $Y' \to Y$ log étale, the cones are canonically isomorphic:
$$C^{\ell}_{X/Y} \simeq C^{\ell}_{X/Y'}.$$

This object becomes simpler in the presence of charts. Locally, we may assume the map $X \to Y$ has a chart given by a map of Artin Cones $A_P \to A_Q$. The map $A_P \to A_Q$ is log étale, so we can base change across it to get a strict map without altering the log normal cone.

Because this method can lead to radical alterations of the target Y, we recall another strategy that we learned from [33, Proposition 2.3.12]. For ordinary schemes, one locally factors a map as a closed immersion composed with a smooth map to get a presentation for the normal cone [8]. We obtain a similar local factorization (Construction 4.2.1) into a *strict* closed immersion composed with a log smooth map, and the same presentation exists for the log normal cone.

The above is made more precise in Remark 4.3.7. The charts and factorizations these techniques require are only locally possible, so we need to know how log normal cones change after étale localization. We encounter a well-known subtlety noticed by W. Bauer [61, §7]: The log normal cone isn't invariant under base-changes by log étale maps (Remark 4.3.14). Our workaround is somewhat different from that of Olsson. These results are at the service of log intersection theory, and we outline a standard package of log virtual fundamental classes and Log Gysin Maps.

4.1.3 Pushforward and Gysin Pullback

The proof of the Product Formula needs two ingredients: commutativity of Gysin maps and compatibility of pushforward with Gysin maps. The commutativity of Gysin Maps readily generalizes to the log setting in Theorem 4.4.12; on the other hand, compatibility with pushforward simply fails!

Nevertheless, the original proof of the product formula depends on a weak form of this compatibility first introduced by Costello [15, Theorem 5.0.1]. We prove a log version of this theorem and will offer further complements in [16].

We obtain another partial result towards compatibility of pushforward and Gysin Pullback. For a log blowup $p : \widehat{X} \to X$ with a log smoothness assumption, we show $p_*[\widehat{X}]^{\ell vir} = [X]^{\ell vir}$ in Theorem 4.4.10. The alternative approach of [6] may extend our results by modifying the notions of dimension, degree, pushforward, chow goups, etc. in the log setting. See also [**?**] for an insightful approach to Log Chow Groups.

We hope the technology and the strategy of reducing statements about log normal cones to the strict, ordinary case will be of interest.

4.1.4 Conventions

- We *only consider fs log structures.* We therefore use $\mathscr{L}, \mathscr{L}Y$ to refer to Olsson's stacks $\mathscr{T}or, \mathscr{T}orY$.

- We work over the base field $\mathbb{C}$.

- We adhere to the convention of [60] regarding the use of the term "algebraic stack": we mean a stack in the sense of [41, 3.1] such that

  $*$ the diagonal is representable and of finite presentation, and

  $*$ there exists a surjective, smooth morphism to it from a scheme.

  We do not require the diagonal morphism to be separated.

- By "log algebraic stack," we mean an algebraic stack with a map to $\mathcal{L}$. Maps between them need not lie over $\mathcal{L}$.

- The name "DM stack" means Deligne-Mumford stack and a morphism $f : X \to Y$ of algebraic stacks is (of) "DM-type" or simply "DM" if every Y-scheme $T \to Y$ pulls back to a DM stack $T \times_{f,Y} X$ [49].

- The word "cone" in "log normal cone" refers to a cone stack in the sense of [8].

- Let P be a sharp fs monoid. Write

$$A_P = [\operatorname{Spec} \mathbb{C}[P] / \operatorname{Spec} \mathbb{C}[P^{gp}]]$$

for the stack quotient in the étale topology endowed with its natural log structure [2], [13], [60]. Beware that some of these sources first take the dual monoid. This log stack has a notable functor of points for fs log schemes:

$$\operatorname{Hom}_{fs}(T, A_P) = \operatorname{Hom}_{mon}(P, \Gamma(\overline{M}_T)).$$

In particular,

$$\operatorname{Hom}_{fs}(A_P, A_Q) = \operatorname{Hom}_{mon}(Q, P).$$

We write A for $A_{\mathbb{N}} = [\mathbb{A}^1/\mathbb{G}_m]$. Log algebraic stacks of this form are called "Artin Cones." "Artin Fans" are log algebraic stacks which admit a strict étale cover by Artin Cones. The 2-category of Artin Fans is equivalent to a category of "cone stacks" [13, Theorem 6.11].

- The present paper concerns analogues of normal cones and pullbacks in the logarithmic category. We use the notation $\ulcorner$, $\times$, C for pullbacks and normal cones of ordinary stacks, and write $\ulcorner\ell$, $\times^{\ell}$, C^{ℓ} to distinguish the fs pullbacks and log normal cones. When they happen to coincide, we write $\not\ell$, $\ulcorner\not\ell$, $\times^{\not\ell}$, $C^{\not\ell}$ to emphasize this coincidence.

- Many of our citations could be made to original sources, often written by K. Kato, but we have opted for the book [59]. We have doubled references to Costello's Formula [15, Theorem 5.0.1], [16] where appropriate because we will have more to say building on future work.

4.2 Preliminaries and the Log Normal Sheaf

The present paper originated with one central construction, which we learned from [33, Lemma 2.3.12].

Construction 4.2.1. The normal cone of a morphism $f : B \to A$ of finite type is constructed by choosing a factorization $B \to B[x_1, \ldots, x_r] \twoheadrightarrow A$ inducing a closed immersion into affine r-space:

$$\operatorname{Spec} A \hookrightarrow \mathbb{A}^r_B \to \operatorname{Spec} B.$$

The normal cone of f may then be expressed as the quotient of the ordinary normal cone of the closed immersion by the action of the tangent bundle of $\mathbb{A}^r_B \to \operatorname{Spec} B$.

Let $P \to A$ and $Q \to B$ be morphisms from fs monoids to the multiplicative monoids of rings ("prelog rings"). A commutative square:

$$\begin{tikzcd} B \arrow[r, "f"] & A \\ Q \arrow[u] \arrow[r, "\theta"] & P \arrow[u] \end{tikzcd}$$

is a chart of a map between affine log schemes. Assume f is of finite type; θ automatically is by the fs assumption. We will obtain a factorization of the induced log schemes into a *strict closed immersion* followed by a *log smooth map*.

Start with a similar factorization

$$\begin{tikzcd} B \arrow[r] & B[x_1, \cdots, x_r, y_1, \ldots, y_s] \arrow[r, two heads] & A \\ Q \arrow[u] \arrow[r] & Q_s \arrow[u] \arrow[r, two heads] & P \arrow[u] \end{tikzcd}$$

with $Q_s = Q \oplus \mathbb{N}^s$ mapping to $B[x_1, \ldots x_r, y_1, \ldots y_s]$ by sending the generators of $\mathbb{N}^s$ to the algebra generators $y_1, \dots y_s$. Define Q_s^θ via the cartesian product

$$\begin{tikzcd}
Q_s \arrow[r, hook] \arrow[d, hook] & Q_s^\theta \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & P \arrow[d, hook] \\
Q_s^{gp} \arrow[r, equal] & Q_s^{gp} \arrow[r] & P^{gp}.
\end{tikzcd}$$

By definition, $Q_s^\theta \to P$ is exact, and $Q_s \to Q_s^\theta$ is a "log modification:" an isomorphism on groupifications. Witness also that $Q_s^\theta \to P$ is surjective, so the characteristic monoid map $\overline{Q_s^\theta} \xrightarrow{\sim} \overline{P}$ is an isomorphism [59, Proposition I.4.2.1(5)] and $\operatorname{Spec} P \to \operatorname{Spec} Q_s^\theta$ is strict. Take Spec of both rings and monoids [59, §II] to obtain a diagram with strict vertical arrows:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & X_\theta \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell", very near start] & \mathbb{A}_Y^{r+s} \arrow[r] \arrow[d] & Y \arrow[d] \\
\operatorname{Spec} P \arrow[r] & \operatorname{Spec} Q_s^\theta \arrow[r] & \operatorname{Spec} Q_s \arrow[r] & \operatorname{Spec} Q
\end{tikzcd}$$

We've written $Y = \operatorname{Spec} B$, $X = \operatorname{Spec} A$ and introduced the fs pullback X_θ in the diagram. The top row expresses our original map $\operatorname{Spec} f$ as the composition of a strict closed immersion, a log modification, and a smooth and log smooth morphism. The log modification $\operatorname{Spec} Q_s^\theta \to \operatorname{Spec} Q_s$ and hence $X_\theta \to \mathbb{A}_Y^{r+s}$ may be expressed as a (strict) open immersion into a log blowup as in [59, Lemma II.1.8.2, Remark II.1.8.5]. Hence $X \subseteq X_\theta$ is a strict closed immersion and $X_\theta \to Y$ is log smooth.

Remark 4.2.2. Continue in the notation of Construction 4.2.1. If we began with a morphism of fs log rings with f and θ both surjective, we could omit $Q_s \to B[x_1, \dots, x_r, y_1, \dots y_s]$. In that case, we obtain a factorization

$$X \subseteq X_\theta \to Y$$

where $X_\theta \to Y$ is not only log smooth but log étale.

As in [8], we will present the log normal cone locally as $C^\ell_{X/Y} = [C_{X/X_\theta}/T^\ell_{X_\theta/Y}]$ using these factorizations. The difficulty is then piecing together the local descriptions and checking compatibility. In this sense, the heavy lifting has already been done for us by [49]. We spend the rest of this section collecting relevant properties of the log normal sheaf $N^\ell_{X/Y}$. When we define the log normal cone $C^{\ell}_{X/Y} \subseteq N^{\ell}_{X/Y}$, its important properties will be locally deduced from such factorizations.

Remark 4.2.3. An algebraic stack X is DM if and only if the map $X \to \operatorname{Spec} k$ to the base field is of DM-type. If $X \to Y$ is a morphism of DM type and Y admits a stratification by global quotients, then so does X [49, Remark 3.2]. A morphism $f : X \to Y$ of algebraic stacks is of DM type if and only if its diagonal $\Delta_{X/Y} : X \to X \times_Y X$ is unramified [63, 06N3].

Lemma 4.2.4. Let $f : X \to Y$ be a morphism of log algebraic stacks. If the map on underlying stacks is of DM-type, then the induced maps $\mathscr{L}X \to \mathscr{L}Y$ and $X \to \mathscr{L}X$ are DM-type.

Proof. The inclusion $X \subseteq \mathscr{L}X$ representing strict maps is open, so it suffices to show that $\mathscr{L}X \to \mathscr{L}Y$ is DM-type.

We will argue that the diagonal of $\mathscr{L}X \to \mathscr{L}Y$ is unramified [63, 04YW]. The isomorphism $\mathscr{L}X \times_{\mathscr{L}Y} \mathscr{L}X \simeq \mathscr{L}(X \times^{\ell}_Y X)$ identifies the diagonal $\Delta_{\mathscr{L}X/\mathscr{L}Y}$ with the result of $\mathscr{L}$ applied to the fs diagonal

$$\Delta^{\ell}_{X/Y} : X \to X \times^{\ell}_Y X.$$

Any diagram:

$$\begin{tikzcd} S_0 \arrow[r] \arrow[d, hook] & \mathscr{L}X \arrow[d, "\mathscr{L}\Delta^{\ell}_{X/Y}"] \\ S_0' \arrow[r] \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] & \mathscr{L}(X \times^{\ell}_Y X) \end{tikzcd}$$

with $S_0 \subseteq S_0'$ a squarezero closed immersion of schemes is equivalent to a diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & X \arrow[d, "\Delta^{\ell}_{X/Y}"] \\ S' \arrow[r] \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] & X \times^{\ell}_Y X \end{tikzcd}$$

with $S \subseteq S'$ an exact closed immersion of log schemes. Composing with the fsification map $X \times^{\ell}_Y X \to X \times_Y X$ sends this square to

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & X \arrow[d, "\Delta^{\ell}_{X/Y}"] \\ S' \arrow[r] \arrow[ur, dashed, shift left] \arrow[ur, dashed, shift right] & X \times_Y X, \end{tikzcd}$$

in which case the two dashed arrows have the same underlying scheme map because $X \to X \times_Y X$ is unramified by hypothesis. Then the maps on log structure must be the same as well, because

$$(M_X \oplus^{\ell}_{M_Y} M_X)|_{S'} \to (M_X)|_{S'}$$

is an epimorphism.

$\square$

Recall the functor of points of the normal sheaf.

Definition 4.2.5 (Normal Sheaf Functor of Points)**.** Let $f : X \to Y$ be a DM morphism of algebraic stacks. Define a stack $N_{X/Y}$ over X named the *log normal sheaf* via its functor of points:

$$\left\{ \begin{tikzcd} & N_{X/Y} \arrow[d] \\ T \arrow[ur, dashed] \arrow[r] & X \end{tikzcd} \right\} := \left\{ \begin{tikzcd} (T, \mathcal{O}_X|_T) \arrow[r] \arrow[d, hook, "i"] & X \arrow[d] \\ (T, \mathcal{A}) \arrow[r] & Y \end{tikzcd} \quad \begin{array}{c} i \text{ is a square-zero closed} \\ \text{immersion with kernel } \mathcal{O}_T \end{array} \right\}$$

$$= \left\{ \begin{tikzcd} & & \mathcal{O}_Y|_T \arrow[d] \arrow[dr, bend left] & & \\ 0 \arrow[r] & \mathcal{O}_T \arrow[r] & \mathcal{A} \arrow[r] & \mathcal{O}_X|_T \arrow[r] & 0 \end{tikzcd} \quad \begin{array}{c} \text{a squarezero algebra} \\ \\ \text{extension on ét}(T) \end{array} \right\}$$

An obstruction theory for f is a fully faithful functor $N_{X/Y} \subseteq E$ into a vector bundle stack as in [68, Corollary 3.8].

The notion of "square-zero closed immersion" in the definition demands elaboration, since the objects involved are étale-locally ringed spaces. See [**?**] for details.

Remark 4.2.6. Suppose we specified an obstruction theory $E_\bullet \to \mathbb{L}_{X/Y}$ in the sense of [8]. The associated obstruction theory according to Definition 4.2.5 on T-points is given by:

$$N_{X/Y}(T) = \underline{\mathrm{Ext}}(\mathbb{L}_{X/Y}|_T, \mathcal{O}_T) \longrightarrow E = \underline{\mathrm{Ext}}(E_\bullet|_T, \mathcal{O}_T).$$

See [68, Corollary 4.9] and [26, Chapitre VIII: Biextension de faisceaux de groupes] for comparison and elaboration on $\underline{\mathrm{Ext}}(E_\bullet, J) = \Psi_{E_\bullet}(J)$. In particular, our obstruction theories are all representable by obstruction theories in the sense of [8].

Definition 4.2.7 (The Log Normal Sheaf)**.** Let $f : X \to Y$ be a DM morphism of log algebraic stacks. Let $T \to X$ be an X-scheme. A *deformation of log structures along f on T* is a log structure $M_{\mathcal{A}} \to \mathcal{A}$ on the étale site ét(T) of T with maps $(\mathcal{O}_Y|_T, M_Y|_T) \to (\mathcal{A}, M_{\mathcal{A}}) \to (\mathcal{O}_X|_T, M_X|_T)$ of log structures such that:

- The kernel $\ker(\mathcal{A} \to \mathcal{O}_X|_T) \simeq \mathcal{O}_T$ and the diagram

$$\begin{tikzcd} & & \mathcal{O}_Y|_T \arrow[d] \arrow[dr, bend left] & & \\ 0 \arrow[r] & \mathcal{O}_T \arrow[r] & \mathcal{A} \arrow[r] & \mathcal{O}_X|_T \arrow[r] & 0 \end{tikzcd}$$

  constitutes a squarezero algebra extension.

- The diagram

$$\begin{tikzcd} \mathcal{A}^* \arrow[r] \arrow[d] & \mathcal{O}_X^*|_T \arrow[d] \\ M_{\mathcal{A}} \arrow[r] & M_X|_T \end{tikzcd}$$

  is a pushout.

The second bullet says that $(\mathcal{A}, M_{\mathcal{A}})$ is a *strict* squarezero extension of $(\mathcal{O}_X|_T, M_X|_T)$; compare with "deformations of log structures" [**?**]. The square in the second bullet is also a pullback, and $M_{\mathcal{A}} \to M_X|_T$ is also a torsor under $1 + \mathcal{O}_T$.

Define the *log normal sheaf* to represent the deformations of log structures just defined:

$$\left\{ \begin{tikzcd} & N^{\ell}_{X/Y} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & X \end{tikzcd} \right\} := \{\text{Deformations of log structures along } f \text{ on } T\}.$$

We show that this definition agrees with Definition 4.2.5 in [**?**]: $N^{\ell}_{X/Y} = N_{X/\mathcal{L}Y}$.

To write down the functoriality of the log normal sheaf, we need to recall some of the machinery of log stacks found in [61].

We denote $\mathscr{L}^i := \mathscr{L}^{[i]}$, the stack of i-simplices of fs log structures. The jth face map d_j sends

$$(M_0 \to M_1 \to \cdots \to M_{i+1}) \mapsto \begin{cases} (M_1 \to M_2 \to \cdots \to M_{i+1}) & \text{if } j = 0 \\ (M_0 \to \cdots M_{j-1} \to M_{j+1} \cdots \to M_{i+1}) & \text{if } j \neq 0, i+1 \\ (M_0 \to \cdots \to M_i) & \text{if } j = i+1. \end{cases}$$

We write $s, t : \mathscr{L}^1 \to \mathscr{L}^0 = \mathscr{L}$ for the "source" d_1 and "target" d_0 maps, respectively. We have an isomorphism $\mathscr{L}^i = \mathscr{L}^1 \times_{t,\mathscr{L},s} \mathscr{L}^1 \times_{t,\mathscr{L},s} \cdots \times_{t,\mathscr{L},s} \mathscr{L}^1$ (i factors).

Endow $\mathscr{L}^i$ with the final tautological log structure, M_{i+1} in the above. All the face maps d_j are strict except $j = i+1$.

We continue [61] to use "$\square$" to denote the category with these objects, arrows, and relations:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 1 \arrow[d] \arrow[dl, phantom, "\circ"] \\ 2 \arrow[r] \arrow[ur, phantom, "\circ"] & 3 \end{tikzcd}$$

We adopt pictorial mnemonics for fully faithful morphisms of these finite diagrams: means the functor $[2] \subseteq \square$ avoiding 2, etc.

Definition 4.2.8 (Compare [61, Lemma 3.12])**.** Define $\mathcal{V} := \mathscr{L}^1 \times^{\ell}_{t,\mathscr{L},t} \mathscr{L}^1$. Given a scheme T, the points of this stack are cocartesian squares of *fs* log structures:

$$\mathcal{V}(T) := \left\{ \begin{tikzcd} M_0 \arrow[r] \arrow[d] & M_1 \arrow[d] \\ M_2 \arrow[r] \arrow[ur, phantom, "\ell\lrcorner"] & M_3. \end{tikzcd} \right\}$$

This is the "fsification" of the ordinary pullback $\mathscr{L}^1 \times_{t,\mathscr{L},t} \mathscr{L}^1$, endowed with the non-fs pushout $M_1 \oplus^{mon}_{M_0} M_2$ of the universal log structures.

The natural embedding $\mathcal{V} \to \mathscr{L}^{\square}$ exhibits the squares which are cocartesian as an open substack, as we'll record in Lemma 4.2.10.

For a morphism $q : Y' \to Y$ of log algebraic stacks, we obtain relative variants:

$$\mathcal{V}_q := \mathcal{V} \times_{\,, \mathscr{L}^1} Y', \qquad \mathscr{L}^{\square}_q := \mathscr{L}^{\square} \times_{\,, \mathscr{L}^1} Y'.$$

The fs pullback here agrees with the ordinary one because $Y' \to \mathcal{L}^1$ is strict. The points of these stacks over some scheme T are squares

$$\begin{tikzcd} M_Y|_T \arrow[r] \arrow[d] & M_{Y'}|_T \arrow[d] \\ M_0 \arrow[r] & M_1, \end{tikzcd}$$

with those of $\mathcal{V}_q$ required to be cocartesian.

Lemma 4.2.9. Let $\mathcal{L}^{arbfine}$ denote the stack of log structures which are fine but not necessarily saturated. The natural monomorphism

$$\mathcal{L} \hookrightarrow \mathcal{L}^{arbfine}$$

is an open immersion.

Proof. Consider some scheme X and pullback diagram

$$\begin{tikzcd} X^{fs} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{L} \arrow[d] \\ X \arrow[r] & \mathcal{L}^{arbfine} \end{tikzcd}$$

Then $X^{fs} \hookrightarrow X$ is a monomorphism, the locus where the stalks of M_X are saturated. After passing to an open cover of X, [59, Theorem II.2.5.4] provides us with a locally finite stratification $X = \bigsqcup_{\sigma \in \Sigma} X_\sigma$ where

- For each $\sigma \in \Sigma$, $\overline{M}_X|_\sigma$ is constant.
- The cospecialization maps for $x \in \overline{\{\xi\}} \subseteq X$

$$\overline{M}_x \to \overline{M}_\xi$$

  are localizations at faces.

The localization of a saturated monoid remains saturated [59, Remark I.1.4.5] and a monoid is saturated if and only if its characteristic monoid is [59, Proposition I.1.3.5]. We then have that $X^{fs} \subseteq X$ is locally a constructible subset which is closed under generization, and hence open [63, Tag 0542].

$\square$

We collect several results of [61] adapted to the fs setting:

Lemma 4.2.10 ([61, Theorem 2.4, Proposition 2.11, Lemma 3.12])**.** These statements remain true in the *fs* context:

(1) For any finite category Γ, the fibered category $\mathscr{L}^\Gamma$ of diagrams of fs log structures indexed by Γ is an algebraic stack.

(2) The simplicial face maps $d_j : \mathscr{L}^{i+1} \to \mathscr{L}^i$ are strict, étale, and DM-type for $j \leq i$.

(3) If $[1] \to \square$ avoids the initial object 0 (or), it induces a strict étale, DM-type morphism

$$\mathscr{L}^\square \to \mathscr{L}^1.$$

(4) If $[2] \to \square$ omits either 1 or 2 (or), it induces an étale, DM-type morphism

$$\mathscr{L}^\square \to \mathscr{L}^2.$$

(5) The map $\mathscr{V} \subseteq \mathscr{L}^\square$ is an open embedding.

(6) Given an fs pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell", very near start] & X \arrow[d] \\ Y' \arrow[r, "q"] & Y, \end{tikzcd}$$

the associated square of stacks

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ \mathscr{V}_q \arrow[r] & \mathscr{L}Y \end{tikzcd}$$

is a pullback.

Proof. Facts (1) through (4) are immediate by Lemma 4.2.9 and the analogous facts in [61]. The last two follow by the same arguments applied in the fs category.

$\square$

Remark 4.2.11. Apply $\mathscr{L}$ once more to the map $\mathscr{L}Y \to Y$: one gets

$$d_1 : \mathscr{L}^2 Y \to \mathscr{L}Y \quad (M_Y \to M_0 \to M_1) \mapsto (M_Y \to M_1).$$

The result is étale, so the original $d_1 : \mathscr{L}Y \to Y$ is log étale [60, Theorem 4.6 (ii)]. The same reasoning concludes $d_{i+1} : \mathscr{L}^{i+1}Y \to \mathscr{L}^i Y$ is log étale in general. In summary, all the face maps are log étale and all but $j = i+1$ are furthermore strict étale.

Remark 4.2.12. Given $q : Y' \to Y$ DM, the natural maps

$$\mathcal{V}_q \subseteq \mathscr{L}_q^{\square} \to \mathscr{L}Y'$$

are étale. The second map is the product of the étale map

$$\boxbslash^* : \mathscr{L}^{\square} \to \mathscr{L}^2$$

over $\mathscr{L}^1$ (via $\square$) with Y'.

Definition 4.2.13. Use Lemma 4.2.10, bullet (6) to turn one commutative square of DM maps into another:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "q"] & Y. \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ \mathscr{L}_q^{\square} \arrow[r] & \mathscr{L}Y \end{tikzcd}$$

Maps of normal sheaves

$$\varphi : N^{\ell}_{X'/Y'} \simeq N_{X/\mathscr{L}_q^{\square}} \to N^{\ell}_{X/Y}$$

arise from Remark 4.2.12 and the second square. We call the composite φ *Olsson's Morphism.*

Remark 4.2.14. In Definition 4.2.13, if the first square was an fs pullback square, the second factors:

$$\begin{tikzcd} X' \arrow[rr] \arrow[d] \arrow[drr, phantom, "\ulcorner", very near start] & & X \arrow[d] \\ \mathcal{V}_q \arrow[r, hook] & \mathscr{L}_q^{\square} \arrow[r] & \mathscr{L}Y. \end{tikzcd}$$

Since this square is a pullback, Olsson's morphism

$$\varphi : N^{\ell}_{X'/Y'} \simeq N_{X'/\mathscr{L}_q^{\square}} \simeq N_{X'/\mathcal{V}_q} \hookrightarrow N^{\ell}_{X/Y}|_{X'}$$

is then a closed immersion.

If q or f is also log flat, φ might not be an isomorphism. See Lemmas 4.3.15, 4.3.16 for the strict case.

Remark 4.2.15. A commutative square of DM maps may be factored:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y. \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{L}_q^{\square} \arrow[r] \arrow[d] & \mathcal{L}Y \arrow[d] \\ Y' \arrow[r] & Y. \end{tikzcd} \tag{4.1}$$

This induces a commutative square of normal sheaves:

$$\begin{tikzcd} N^{\ell}_{X'/Y'} \simeq N_{X'/\mathcal{L}_q^{\square}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & N^{\ell}_{X/Y} \arrow[d] \\ N_{X'/Y'} \arrow[r] & N_{X/Y}. \end{tikzcd} \tag{4.2}$$

The Olsson morphisms are thereby seen to be compatible with the ordinary functoriality of the normal sheaf via the forgetful maps $N^{\ell}_{X/Y} \to N_{X/Y}$.

Now suppose the original square (4.1) is an fs pullback:

- If q is strict, then $\mathcal{V}_q \simeq \mathcal{L}Y'$, and our fs pullback square factors as

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ \mathcal{V}_q \simeq \mathcal{L}Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{L}Y \arrow[d] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

  and the functor of points witnesses that (4.2) is cartesian.

- If instead f is strict, then $X' \to \mathcal{V}_q$ factors through Y', and the factorization

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, hook] \\ \mathcal{V}_q \arrow[r] & \mathcal{L}Y \end{tikzcd}$$

shows that the vertical arrows of (4.2) are isomorphisms and the Olsson Morphism is the same as the ordinary functoriality of the Normal Sheaf.

Remark 4.2.16. Given a commutative square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "q"] & Y, \end{tikzcd}$$

of DM maps we can form two other commutative squares out of it:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{L}_q^{\square} \arrow[r] & \mathcal{L}Y, \end{tikzcd} \quad \begin{tikzcd} X' \arrow[r] \arrow[d] & \mathcal{L}X \arrow[d] \\ \mathcal{L}Y' \arrow[r] & \mathcal{L}Y. \end{tikzcd}$$

They induce morphisms

$$N^{\ell}_{X'/Y'} \simeq N^{\ell}_{X'/\mathcal{L}_q^{\square}} \to N^{\ell}_{X/Y}|_{X'},$$

$$N^{\ell}_{X'/Y'} \to N_{\mathcal{L}X/\mathcal{L}Y}|_{X'}.$$

Form the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \mathcal{L}X \arrow[r, "s"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ \mathcal{L}_q^{\square} \arrow[r] \arrow[d] \arrow[rr, bend right] & \mathcal{L}^2Y \arrow[r, "d_0"] & \mathcal{L}Y \\ \mathcal{L}Y' & & \end{tikzcd}$$

to see that the two morphisms of normal sheaves are compatible:

$$N^{\ell}_{X'/Y'} \simeq N_{X'/\mathcal{L}_q^{\square}} \to N_{\mathcal{L}X/\mathcal{L}^2Y}|_{X'} \subseteq N^{\ell}_{X/Y}|_{X'}.$$

Lemma 4.2.17. Suppose given a pair of commutative squares:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

of DM-type maps. The diagram

$$\begin{tikzcd} & N^{\ell}_{Y/Y'} \arrow[dr] & \\ N^{\ell}_{X/X'} \arrow[ur] \arrow[rr] & & N^{\ell}_{Z/Z'} \end{tikzcd}$$

commutes, where all the arrows are Olsson's morphisms.

Proof. Introduce an algebraic X-stack $\mathscr{W}$, with functor of points:

$$\left\{ \begin{tikzcd} & \mathscr{W} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed] & X \end{tikzcd} \right\} := \left\{ \begin{tikzcd} M_2 & M_1 \arrow[l] & M_0 \arrow[l] \\ M_X|_T \arrow[u] & M_Y|_T \arrow[l] \arrow[u] & M_Z|_T \arrow[l] \arrow[u] \end{tikzcd} \quad \begin{array}{c} \text{commutative diagrams of} \\ \text{fs log structures on } T \end{array} \right\}$$

In other words, $\mathscr{W} := (\mathcal{L}^{\square} \times_{\mathcal{L}^1} \mathcal{L}^{\square}) \times_{\mathcal{L}^2} X$.

All the triangles in this diagram commute because of the definition of Olsson morphisms and the functor of points of N:

$$\begin{tikzcd}
 & N_{X/\mathcal{L}^{\square}_{g\circ f}} \arrow[drr, bend left] & & \\
N^{\ell}_{X'/X} \arrow[ur, dash, bend left, "\sim"] \arrow[r] \arrow[d, dash, "\sim"] & N_{X/\mathscr{W}} \arrow[u] \arrow[rr] \arrow[dl] \arrow[dd] & & N^{\ell}_{Z'/Z} \\
N_{X'/\mathcal{L}^{\square}_f} \arrow[dr, bend right] & & & \\
 & N_{Y/\mathcal{L}^{\square}_g} \arrow[uurr] \arrow[r, dash, "\sim"'] & N^{\ell}_{Y'/Y} \arrow[uur, bend right] &
\end{tikzcd}$$

Restricting the diagram to $N^{\ell}_{X'/X}$, $N^{\ell}_{Y'/Y}$, and $N^{\ell}_{Z'/Z}$, we get the result.

□

Proposition 4.2.18. Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ DM-type maps of log algebraic stacks, the Olsson Morphisms yield a complex of stacks

$$N^{\ell}_{X/Y} \to N^{\ell}_{X/Z} \to N^{\ell}_{Y/Z}|_X,$$

in that the composite factors through the vertex.

If h is smooth, $N^{\ell}_{Y/Z} = BT^{\ell}_{Y/Z}$ and rotating the triangle in the derived category yields an exact sequence of cone stacks:

$$T^{\ell}_{Y/Z}|_X \to N^{\ell}_{X/Y} \to N^{\ell}_{X/Z}.$$

Proof. The Olsson Morphisms come about from the commutative diagram

$$\begin{array}{ccccc} X & = & X & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Z & = & Z \end{array} \rightsquigarrow \begin{array}{ccc} N^{\ell}_{X/Y} & \longrightarrow & N^{\ell}_{Y/Y}|_X = X \\ \downarrow & & \downarrow{\scriptstyle 0} \\ N^{\ell}_{X/Z} & \longrightarrow & N^{\ell}_{Y/Z}|_X. \end{array}$$

The surjectivity, smoothness, and calculation of the fiber of $N^{\ell}_{X/Y} \to N^{\ell}_{X/Z}$ may all be checked routinely using the functor of points.

□

Remark 4.2.19. Suppose given a (not necessarily commutative) finite diagram of cones. If the diagram induced by taking abelian hulls is commutative, so was the original.

4.3 Properties of the Log Normal Cone

We are ready to define the log normal cone. We recall the essential properties of the ordinary normal cone; the rest of the section establishes analogous properties in the log context.

Remark 4.3.1. Consider a DM-type morphism $f : X \to Y$ of algebraic stacks. K. Behrend and B. Fantechi defined the Intrinsic Normal Cone [8]

$$C_f = C_{X/Y} \subseteq N_{X/Y};$$

C. Manolache [49] removed their assumptions of smooth Y and DM X. This cone has the following basic properties:

(1) A commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{q} & Y \end{array}$$

yields a morphism of cones $\varphi : C_{X'/Y'} \to C_{X/Y} \times_X X'$.

- if the square was cartesian, φ is a closed embedding.
- if also f or q was flat, φ is an isomorphism.

(2) For a composite

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

- if g is l.c.i., $C_{X/Y} = N_{X/Y}$ and the sequence

$$N_{X/Y} \to C_{X/Z} \to C_{Y/Z}|_X$$

  of cone stacks is exact.

- if h is smooth, the sequence

$$T_{Y/Z}|_X \to C_{X/Y} \to C_{X/Z}$$

  is exact.

(3) Obstruction Theories and Gysin Pullbacks are obtained by placing the cone in a vector bundle stack $C_{X/Y} \subseteq E$ (see [49], [68], [40]).

Definition 4.3.2 (Log Intrinsic Normal Cone, Olsson Morphisms)**.** Let $f : X \to Y$ be a DM-type morphism of log algebraic stacks. We define the *Log (Intrinsic) Normal Cone*

$$C^{\ell}_{X/Y} := C_{X/\mathcal{L}Y} \subseteq N^{\ell}_{X/Y}$$

after [23]. Endow it with the log structure pulled back from X. Given a commutative square of log algebraic stacks and its partner

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{L}^{\square}_q \arrow[r] & \mathcal{L}Y, \end{tikzcd}$$

the latter induces

$$\varphi : C^{\ell}_{X'/Y'} \simeq C_{X'/\mathcal{L}^{\square}_q} \to C^{\ell}_{X/Y}.$$

This is again called the *Olsson Morphism*.

Remark 4.3.3. The map $\mathcal{L}Y \to Y$ has a section $Y \subseteq \mathcal{L}Y$ which is an open immersion. This open immersion represents strict log maps to Y.

As a result, if $X \to Y$ is DM and strict, $C^\ell_{X/Y} = C_{X/Y}$ and $N^\ell_{X/Y} = N_{X/Y}$. In addition, the Olsson Morphisms are the same as the ordinary functoriality of the normal cone (Remarks 4.2.19 and 4.2.15).

The Olsson Morphism of any fs pullback square is a closed immersion, because it fits into a commutative square of closed immersions from Remark 4.2.14:

$$\begin{tikzcd} C^\ell_{X'/Y'} \arrow[r] \arrow[d, hook] & C^\ell_{X/Y}|_{X'} \arrow[d, hook] \\ N^\ell_{X'/Y'} \arrow[r, hook] & N^\ell_{X/Y}|_{X'}. \end{tikzcd}$$

Remark 4.3.4 (Short Exact Sequences of Cone Stacks)**.** Recall [8, Definition 1.12]. Let E be a vector bundle stack and C, D cone stacks all on some base algebraic stack X. A composable pair of morphisms of cone stacks

$$E \to C \to D$$

is called a *short exact sequence* if

- $C \to D$ is a smooth epimorphism.
- The square
$$\begin{tikzcd} E \times C \arrow[r, "pr_2"] \arrow[d, "\sigma"] & C \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$
  where pr_2 is the projection and σ the action, is cartesian.

These are equivalent to having $C \simeq E \times_X D$ locally in X.

Note that this definition is *fpqc*-local in the base X [63, 02VL]. Another reduction we will need applies in case there is a commutative diagram of cone stacks

$$\begin{tikzcd} E \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "s"] & D \arrow[d, "t"] \\ E' \arrow[r] & C' \arrow[r] & D' \end{tikzcd}$$

with E, E' vector bundles. If the top sequence is exact and the arrows labeled s, t are smooth and surjective, then the bottom is exact. To see this, pushout along $E \to E'$ so as to assume $E = E'$

(s, t remain smooth and surjective). The diagram on the left is the pullback along the smooth surjection $D' \to D$ of the one on the right:

$$\begin{tikzcd} E \times C' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C' \arrow[d] \\ C' \arrow[r] & D' \end{tikzcd} \quad \begin{tikzcd} E \times C \arrow[r] \arrow[d] & C \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

and we can verify that $E \times C$ is the pullback after smooth-localizing.

Proposition 4.3.5. Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ are DM maps between log algebraic stacks, and g is log smooth. Then

$$T^{\ell}_{Y/Z}|_X \to C^{\ell}_{X/Y} \to C^{\ell}_{X/Z}$$

is an exact sequence of cone stacks.

Proof. Encode the log structures on the maps via the top row of the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathcal{L}Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{L}^2 Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{L}^2 \arrow[d] \\ & Y \arrow[r] & \mathcal{L}Z \arrow[r] & \mathcal{L}^1. \end{tikzcd}$$

Since $Y \to \mathcal{L}Z$ is smooth, $\mathcal{L}Y \to \mathcal{L}^2 Z$ is. Moreover, they have the same tangent bundle:

$$T^{\ell}_{Y/Z}|_{\mathcal{L}Y} = T_{Y/\mathcal{L}Z}|_{\mathcal{L}Y} = T_{\mathcal{L}Y/\mathcal{L}^2 Z}$$

since the vertical maps are log étale [59, Corollary IV.3.2.4].

Together with the isomorphism $C^{\ell}_{X/Z} \simeq C_{X/\mathcal{L}^2 Z}$, we obtain the exact sequence.

$\square$

Remark 4.3.6. In the proof, the composite

$$C^{\ell}_{X/Y} \to C_{X/\mathcal{L}^2 Z} \simeq C^{\ell}_{X/Z}$$

is precisely the Olsson Morphism. This is immediate from the diagram:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d] & X \arrow[d] \\ \mathcal{L}^{\square}_q \arrow[r] \arrow[d] & \mathcal{L}^2 Z \arrow[d] \\ \mathcal{L}Y \arrow[ur] & \mathcal{L}Z. \end{tikzcd}$$

Remark 4.3.7. The introduction promised three characterizations of $C^{\ell}_{X/Y}$.

The log intrinsic normal cone is characterized by the strict case of Remark 4.3.3 and the log étale case of Proposition 4.3.5. This is because any map $X \to Y$ factors into the strict map $X \to \mathcal{L}Y$ composed with the log étale map $\mathcal{L}Y \to Y$ (Remark 4.2.11).

We can unpack this definition locally using charts. Suppose a morphism has a global fs chart by Artin Cones:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A_P & \longrightarrow & A_Q. \end{array}$$

The morphism $A_P \to A_Q$ is log étale [60, Corollary 5.23]. Let $W = A_P \times^{\ell}_{A_Q} Y$ denote the fs pullback, so that $X \to Y$ factors through a strict map to W and W is log étale over Y. We immediately get

$$C^{\ell}_{X/Y} = C_{X/W}.$$

The reader may be reassured by working locally with this definition. If the reader wants instead to work with charts $\mathrm{Spec}(P \to \mathbb{C}[P])$ in the traditional sense, then log étaleness is no longer immediate and we must check Kato's Criteria [59, Corollary IV.3.1.10].

Recall Construction 4.2.1 – after localizing in the étale topology, we obtain a factorization of any map $X \to Y$ as a *strict* closed immersion followed by a log smooth map

$$X \subseteq X_\theta \to Y.$$

Proposition 4.3.5 therefore locally provides a presentation of the log normal cone:

$$C^{\ell}_{X/Y} = [C_{X/X_\theta}/T^{\ell}_{X_\theta/Y}].$$

Lemma 4.3.8. Given a DM map $f : X \to Y$ of log algebraic stacks with X quasicompact, the map

$$X \to \mathcal{L}Y$$

factors through an open quasicompact subset $U \subseteq \mathcal{L}Y$.

Our applications require openness; otherwise the lemma is trivial.

Proof. The claim is étale-local in Y and X because X is quasicompact. We can thereby assume we have a global chart

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ A_P \arrow[r] & A_Q. \end{tikzcd}$$

The map $A_P \times_{A_Q} Y \to \mathcal{L}Y$ is étale [60, Corollary 5.25] and X factors through its open, quasicompact image.

$\square$

Remark 4.3.9. This lemma ensures that any DM map $X \to Y$ of log stacks with X quasicompact factors through $X \to U \to Y$ with $X \to U$ strict, U quasicompact, and $U \to Y$ log étale.

Example 4.3.10. We provide an example of Construction 4.2.1 and Remark 4.2.2.

Consider the diagonal morphism $\mathbb{A}^1 \xrightarrow{\Delta} \mathbb{A}^2$. The addition map $\mathbb{N}^2 \xrightarrow{+} \mathbb{N}$ gives a chart for Δ.

Denote by B the log blowup of $\mathbb{A}^2$ at the ideal $I \subseteq M_{\mathbb{A}^2}$ generated by $\mathbb{N}^2 \setminus \{0\} \subseteq \mathbb{N}^2$. The pullback $\Delta^* I$ is generated by the image of the composite

$$\mathbb{N}^2 \setminus \{0\} \subseteq \mathbb{N}^2 \xrightarrow{+} \mathbb{N}.$$

The pullback is generated globally by a single element and so Δ factors through the log blowup B.

Name the generators $\mathbb{N}^2 = \mathbb{N}e \oplus \mathbb{N}f$. The log blowup B is covered by two affine opens $D_+(e)$ and $D_+(f)$, on which e and f are invertible.

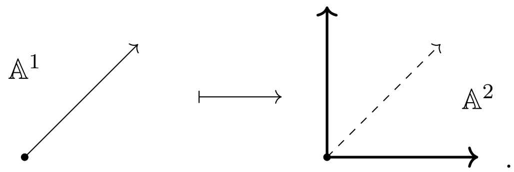
On the chart $D_+(e)$, the morphism $\mathbb{A}^1 \to B$ looks like

$$\begin{tikzcd} \mathbb{N} \arrow[d] & \mathbb{N}e \oplus \mathbb{N}(f-e) \arrow[l] \arrow[d] \\ \mathbb{C}[t] & \mathbb{C}[x, \frac{y}{x}]. \arrow[l] \end{tikzcd}$$

The horizontal morphisms send $f - e \mapsto 0$ and $\frac{y}{x} \mapsto 1$. Because $(f-e)$ maps to $1 \in \mathbb{C}[t]$, the composite

$$\mathbb{N}e \oplus \mathbb{N}(f-e) \to \mathbb{N} \to \mathbb{C}[t]$$

is another chart for the same log structure on $\mathbb{A}^1$. This means that $\mathbb{A}^1 \to D_+(e)$ is strict. The same discussion applies to $D_+(f)$. In the tropical picture [13, §2], we subdivided $\mathbb{A}^2$ at the image of the ray corresponding to $\mathbb{A}^1$:



Proposition 4.3.11. Consider DM-type morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ between log algebraic stacks. If $C^\ell_{X/Y} = N^\ell_{X/Y}$, then

$$N^\ell_{X/Y} \to C^\ell_{X/Z} \to C_{\mathcal{L}Y/\mathcal{L}Z}|_X$$

is an exact sequence of cone stacks.

Proof. Compare [8, Proposition 3.14].

By Proposition 4.2.18 and Remark 4.2.16, this sequence composes to zero. Remark 4.3.4 allows us to repeatedly *fpqc*-localize in X to check exactness of such a sequence. Localization reduces to the case where X, Y, and Z are affine log schemes and the map $Y \to Z$ admits a global fs chart. We are therefore in the situation of Construction 4.2.1.

Reduction to $g : Y \to Z$ Strict

Factor $Y \to Z$ into a strict closed immersion composed with a log smooth map:

$$Y \subseteq W \twoheadrightarrow Z.$$

We obtain a diagram

$$\begin{tikzcd}
 & T^\ell_{W/Z}|_X \arrow[r, equal] \arrow[d] & T_{\mathcal{L}W/\mathcal{L}Z}|_X \arrow[d] \\
N_{X/Y} \arrow[r] \arrow[d, equal] & C^\ell_{X/W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C_{\mathcal{L}Y/\mathcal{L}W}|_X \arrow[d] \\
N_{X/Y} \arrow[r] & C^\ell_{X/Z} \arrow[r] & C_{\mathcal{L}Y/\mathcal{L}Z}|_X.
\end{tikzcd}$$

Observe that the diagram commutes – the morphism $T^{\ell}_{W/Z}|_X \to C^{\ell}_{X/W}$ in the proof of Proposition 4.3.5 factors through an identification $T^{\ell}_{W/Z}|_{\mathcal{L}W} \simeq T^{\ell}_{\mathcal{L}W/\mathcal{L}^2 Z}$. Because $\mathcal{L}W \to W$ is log étale, the two tangent spaces are isomorphic [59, IV.3.2.4]. Thus the right square is a pullback. The vertical maps of cones are smooth surjections, so it suffices to show the middle row is exact as in Remark 4.3.4. We may thereby assume $W = Z$ and $g : Y \to Z$ is a strict closed immersion.

Reduction to $f : X \to Y$ Strict

Use Construction 4.2.1 again to factor $X \to Z$ as a strict closed immersion composed with a log smooth map $X \subseteq W \twoheadrightarrow Z$. The map $X \to W' := W \times_Z Y$ is again a strict closed immersion:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, bend right] & W' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & W \arrow[d] \\ & Y \arrow[r, hook] & Z. \end{tikzcd} \tag{4.3}$$

Because the top row is strict, $X \to \mathcal{L}W'$ factors through the open subset $W' \subseteq \mathcal{L}W'$ and

$$C_{\mathcal{L}W'/\mathcal{L}W}|_X = C_{\mathcal{L}W'/\mathcal{L}W}|_{W'}|_X = C^{\ell}_{W'/W}|_X = C_{W'/W}|_X.$$

The fs pullback square in (4.3) also induces a cartesian square of stacks:

$$\begin{tikzcd} \mathcal{L}W' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{L}W \arrow[d] \\ \mathcal{L}Y \arrow[r] & \mathcal{L}Z \end{tikzcd}$$

with $\mathcal{L}W \to \mathcal{L}Z$ smooth. This reveals that

$$C_{\mathcal{L}Y/\mathcal{L}Z}|_{\mathcal{L}W'} = C_{\mathcal{L}W'/\mathcal{L}W}.$$

Putting this together with the above, we have computed

$$C_{\mathcal{L}Y/\mathcal{L}Z}|_X = C_{W'/W}|_X.$$

The factorization (4.3) gives a diagram

$$\begin{tikzcd} T^{\ell}_{W'/Y}|_X \arrow[r, equal] \arrow[d] & T^{\ell}_{W/Z}|_X \arrow[d] & \\ N_{X/W'} \arrow[r] \arrow[d] & C^{\ell}_{X/W} \arrow[r] \arrow[d] & C_{W'/W}|_X \arrow[d, equal] \\ N_{X/Y} \arrow[r] & C^{\ell}_{X/Z} \arrow[r] & C_{\mathcal{L}Y/\mathcal{L}Z}|_X \end{tikzcd}$$

The composable vertical arrows are the quotients of Proposition 4.3.5, so the bottom row will be exact if we show the middle row is. The middle row is exact by a relative form of the original [8, Proposition 3.14].

□

Remark 4.3.12. The exact sequences of cone stacks in Propositions 4.3.5, 4.3.11 are natural in morphisms of composable pairs of arrows.

In the next example, the log normal cone differs from the ordinary scheme-theoretic one.

Example 4.3.13. In Example 4.3.10, we considered the log blowup B of $\mathbb{A}^2$ at the origin and the diagonal map. Pull back to get the identity log blowup of $\mathbb{A}^1$:

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner \ell", very near start] & B \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{A}^2. \end{tikzcd}$$

Let $_{\mathbb{N}}, _{\mathbb{N}^2}$ both be $\operatorname{Spec} \mathbb{C}$, with log structures coming from $\mathbb{N}$ and $\mathbb{N}^2$, respectively. Then the inclusions of the origins $_{\mathbb{N}} \in \mathbb{A}^1$ and $_{\mathbb{N}^2} \in \mathbb{A}^2$ are strict.

Take the pullback of the above diagram along the inclusion $_{\mathbb{N}^2} \in \mathbb{A}^2$:

$$\begin{tikzcd} _{\mathbb{N}} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner \ell", very near start] & D \arrow[d] \\ _{\mathbb{N}} \arrow[r] & _{\mathbb{N}^2}. \end{tikzcd}$$

The map $D \to _{\mathbb{N}^2}$ is the exceptional divisor of B, which is $\mathbb{P}^1$ with log structure $\overline{M}_x = \mathbb{N}^2$ at the intersections with the axes and $\overline{M}_x = \mathbb{N}$ elsewhere.

To see the log normal cone differ from the ordinary one, compute the normal cones of the arrows in this square: $C^\ell_{_{\mathbb{N}}/_{\mathbb{N}}} =$, $C^\ell_{_{\mathbb{N}}/_{\mathbb{N}^2}} = C^\ell_{_{\mathbb{N}}/D} = \mathbb{A}^1$, and $C^\ell_{D/_{\mathbb{N}^2}} = \mathbb{P}^1$. Although $_{\mathbb{N}}$ and $_{\mathbb{N}^2}$ have the same underlying scheme, the log normal cones of $_{\mathbb{N}}$ over them are different.

Remark 4.3.14. A handy consequence of Proposition 4.3.11 is that, if $Y \to Z$ is a DM-type morphism between log algebraic stacks and $Y' \to Y$ is a *strict* étale map, then

$$C^\ell_{Y'/Z} \simeq C^\ell_{Y/Z}|_{Y'}.$$

This is *not* true without the strictness assumption. This is the observation of W. Bauer precluding the existence of a log cotangent complex with all its desiderata (see [61, §7]).

In general, it need only be a closed immersion. This is because

$$C^{\ell}_{Y'/Z} \simeq C_{\mathscr{L}Y/\mathscr{L}Z}|_{Y'} \subseteq N_{\mathscr{L}Y/\mathscr{L}Z}|_{Y'} \subseteq N^{\ell}_{Y/Z}|_{Y'}$$

is a closed immersion which factors through $C^{\ell}_{Y/Z}|_{Y'}$, as in Remark 4.2.16.

For a single example, take the log blowup $B \to \mathbb{A}^2$ of the origin $\in \mathbb{A}^2$. The pullback defines a strict pullback square:

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell" very near start] & B \arrow[d] \\ {} \arrow[r] & \mathbb{A}^2. \end{tikzcd}$$

Because the horizontal morphisms are strict, their log normal cones coincide with the ordinary ones. Log blowups are log étale, so we would erroneously be led to conclude that

$$C_{D/B} \stackrel{?}{=} C_{/\mathbb{A}^2}|_D.$$

The inclusion $D \subseteq B$ is regular, and so is $\in \mathbb{A}^2$, so the normal cones and normal sheaves agree:

$$N_{D/B} = \mathcal{O}_B(D)|_B$$

$$N_{/\mathbb{A}^2}|_D = \mathbb{A}^2_D.$$

The dimensions are different, so they can't be equal.

Lemma 4.3.15. Suppose given a strict pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell" very near start] & X \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

of DM-type morphisms between log algebraic stacks for which q is strict and smooth. Then the Olsson Morphism

$$C^{\ell}_{X'/Y'} \xrightarrow{\sim} C^{\ell}_{X/Y}|_{X'}$$

is an isomorphism.

Proof. We first note that the Olsson Morphism $N^\ell_{X'/Y'} \to N^\ell_{X/Y}|_{X'}$ on log normal sheaves is an isomorphism. This is clear from the q strict pullback part of Remark 4.2.15 and the fact that the ordinary normal sheaves are isomorphic.

Now we know that the morphism of cones $C^\ell_{X'/Y'} \to C^\ell_{X/Y}|_{X'}$ is a closed immersion, and it suffices to show that it is moreover smooth and surjective. We express this map as a composite

$$C^\ell_{X'/Y'} \to C^\ell_{X'/Y} \to C^\ell_{X/Y}|_{X'}.$$

Proposition 4.3.5 asserts that the first map is smooth and surjective and Proposition 4.3.11 says the same for the second.

∎

Lemma 4.3.16. Suppose given a pair of fs pullback squares

$$\begin{tikzcd} \widetilde{X'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell"] & \widetilde{X} \arrow[d, "z"] \\ X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell"] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of DM-type morphisms between log algebraic stacks for which z is strict and smooth. Then the diagram of log normal cones

$$\begin{tikzcd} C^\ell_{\widetilde{X'}/Y'} \arrow[r] \arrow[d, "s'"] \arrow[dr, phantom, "\ulcorner"] & C^\ell_{\widetilde{X}/Y} \arrow[d, "s"] \\ C^\ell_{X'/Y'} \arrow[r] & C^\ell_{X/Y} \end{tikzcd}$$

is cartesian and the arrows s, s' are smooth epimorphisms.

Proof. Proposition 4.3.11 provides a map of short exact sequences of cone stacks:

$$\begin{tikzcd} BT^\ell_{\widetilde{X'}/X'} \arrow[r] \arrow[d, equal] & C^\ell_{\widetilde{X'}/Y'} \arrow[r, "t'"] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & C^\ell_{X'/Y'}|_{\widetilde{X'}} \arrow[d, hook] \\ BT^\ell_{\widetilde{X}/X}|_{\widetilde{X'}} \arrow[r] & C^\ell_{\widetilde{X}/Y}|_{\widetilde{X'}} \arrow[r, "\widetilde{t}"] & C^\ell_{X/Y}|_{\widetilde{X'}}. \end{tikzcd}$$

Witness that the right square is cartesian because [61]

$$T^\ell_{\widetilde{X'}/X'} = T^\ell_{\widetilde{X}/X}|_{\widetilde{X'}}$$

and that the arrows $t', \widetilde{t}$ are clearly smooth epimorphisms. The arrow $\widetilde{t}$ is pulled back from the smooth epimorphism $t : C^{\ell}_{\widetilde{X}/Y} \to C^{\ell}_{X/Y}|_{\widetilde{X}}$, so we have the top pullback square

$$\begin{tikzcd}
C^{\ell}_{\widetilde{X'}/Y'} \arrow[r] \arrow[d, "t'"] \arrow[dd, bend right=60, "s'"'] \arrow[dr, phantom, "\ulcorner", very near start] & C^{\ell}_{\widetilde{X}/Y} \arrow[d, "t"] \arrow[dd, bend left=60, "s"] & \\
C^{\ell}_{X'/Y'}|_{\widetilde{X'}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & C^{\ell}_{X/Y}|_{\widetilde{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \widetilde{X} \arrow[d] \\
C^{\ell}_{X'/Y'} \arrow[r] & C^{\ell}_{X/Y} \arrow[r] & X
\end{tikzcd}$$

The composite vertical rectangle of cones is the diagram we are after, and so the fact that this square is cartesian is clear. It remains only to note the bent arrows s, s' are smooth epimorphisms because they are the composites of t, t' with pullbacks of the smooth epimorphism $\widetilde{X} \to X$.

□

4.4 Logarithmic Intersection Theory

The Log Intersection Theory package is defined the same way as usual [49], mutatis mutandis.

Definition 4.4.1 (Log Perfect Obstruction Theory)**.** Define a *Log Perfect Obstruction Theory* (hereafter "Log POT") for a DM-type morphism $f : X \to Y$ to be a closed immersion of cone stacks

$$C^{\ell}_{X/Y} \subseteq E \qquad (\text{equiv. } N^{\ell}_{X/Y} \subseteq E)$$

of the log normal cone into a vector bundle stack E.

Given an fs pullback square

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner\ell", very near start] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

and a Log POT $C^{\ell}_{X/Y} \subseteq E$ for f, the Olsson Morphism

$$C^{\ell}_{X'/Y'} \overset{\varphi}{\hookrightarrow} C^{\ell}_{X/Y}|_{X'} \subseteq E|_{X'}$$

defines a "Pullback" Log POT.

A related notion of "Pullback" Log POT arises when $X' \to X$ is log étale and $f : X \to Y$ any DM-type map. Then Remark 4.3.14 shows the map

$$C^{\ell}_{X'/Y} \to C^{\ell}_{X/Y}|_{X'}$$

is a closed immersion, and we can compose with an obstruction theory for f to get one for the composite $X' \to X \to Y$.

Given a Log POT $C^{\ell}_{X/Y} \subseteq E$ for some f, suppose X has a stratification by global quotient stacks and Y is log smooth and equidimensional. Then [40, Proposition 5.3.2] gives us a unique cycle

$$[X, E]^{\ell vir} \in A_* X$$

which pulls back to the class $[C^{\ell}_{X/Y}] \in A_* E$. This class is called the *Log Virtual Fundamental Class* (hereafter "Log VFC").

Remark 4.4.2. When $\mathscr{L}Y$ is equidimensional, so is $C^{\ell}_{X/Y}$. The correct definition of the Log VFC requires that the cone be equidimensional. If Y is log smooth, $Y \subseteq \mathscr{L}Y$ is dense. If Y is also equidimensional, we get that $\mathscr{L}Y$ is. This explains our assumptions in Definition 4.4.1. We don't include these assumptions in the definition of a Log POT only because we may have Log Gysin maps more generally.

Definition 4.4.3 (Log Gysin Map)**.** Suppose a DM-type $f : X \to Y$ has a Log POT $C^{\ell}_{X/Y} \subseteq E$. Given a DM-type log map $k : V \to Y$ with V log smooth and equidimensional, form the fs pullback:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & V \arrow[d, "k"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The embedding

$$C^{\ell}_{W/V} \subseteq C^{\ell}_{X/Y}|_W \subseteq E|_W$$

results in a class

$$[C^{\ell}_{W/V}, E] \in A_* W.$$

Mimicking [49], we call this "map"

$$f^! = f^!_E$$

the *Log Gysin Map*.

Remark 4.4.4. Consider a DM-type morphism $f : X \to Y$ of log algebraic stacks. The cartesian square

$$\begin{tikzcd} \mathcal{L}X \arrow[r, "s"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ \mathcal{L}^2Y \arrow[r, "d_0"] & \mathcal{L}Y \end{tikzcd}$$

from Remark 4.2.16 results in a closed embedding

$$C_{\mathcal{L}X/\mathcal{L}Y} \simeq C_{\mathcal{L}X/\mathcal{L}^2Y} \subseteq C^\ell_{X/Y}|_{\mathcal{L}X}$$

which we use to canonically extend an obstruction theory $C^\ell_{X/Y} \subseteq E$ to a closed embedding

$$C_{\mathcal{L}X/\mathcal{L}Y} \subseteq E|_{\mathcal{L}X}.$$

Now suppose given a composable pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ as above and equip f, g with Log POT 's:

$$C^\ell_{X/Y} \subseteq F, \qquad C^\ell_{Y/Z} \subseteq G.$$

Define a *compatibility datum* for such a pair to be a traditional compatibility datum [49, Definition 4.5] for

$$X \xrightarrow{f} \mathcal{L}Y \xrightarrow{g} \mathcal{L}^2Z,$$

endowing $\mathcal{L}Y \to \mathcal{L}^2Z$ with the extended obstruction theory

$$C_{\mathcal{L}Y/\mathcal{L}^2Z} \simeq C^\ell_{Y/Z}|_{\mathcal{L}Y} \subseteq G|_{\mathcal{L}Y}.$$

We offer a couple of basic remarks about our definitions before the examples and theorems.

Remark 4.4.5. The map $f^!$ just defined takes in log smooth equidimensional stacks DM over Y and produces classes in certain Chow Groups. We do not know whether this operation may be extended to the "Log Chow" groups of [6].

Remark 4.4.6. Given an fs pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner \ell", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of DM maps where f has a Log POT $C^{\ell}_{X/Y} \subseteq E$, endow f' with the Pullback Log POT. Then

$$f^! = f'^!$$

when applied to log smooth, equidimensional log schemes over Y'.

Remark 4.4.7. If $C^{\ell}_{X/Y} = N^{\ell}_{X/Y}$ for a DM morphism $f : X \to Y$, we can take $E = N^{\ell}_{X/Y}$ as our obstruction theory. If X, Y are equidimensional and Y is log smooth, unwinding definitions shows

$$f^!(Y) = [X],$$

where $[X]$ is the fundamental class of X.

Remark 4.4.8. Log Gysin Maps don't commute with pushforward: Let

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner \ell", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

be an fs pullback square. Endow $f : X \to Y$ with a Log POT $C^{\ell}_{X/Y} \subseteq E$ and give f' the pullback obstruction theory. Then the usual equality [49, Theorem 4.1 (i)] can fail:

$$f^! q_* \neq p_* f'^!.$$

Take the square of Example 4.3.13

$$\begin{tikzcd} \mathbb{N} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\ulcorner \ell", very near start] & D \arrow[d] \\ \mathbb{N} \arrow[r] & \mathbb{N}^2 \end{tikzcd}$$

and apply both operations to $[\mathbb{N}]$ for a counterexample.

Remark 4.4.9. Virtual Fundamental Classes don't push forward along log blowups: Let $X \to F$ be the morphism from a stack X to its Artin Fan (the reader may take a traditional chart instead of F). Choose a finite subdivision $\widehat{F} \to F$, and form the fs pullback:

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner \ell", very near start] & \widehat{F} \arrow[d] \\ X \arrow[r] & F. \end{tikzcd}$$

Suppose given a map $f : X \to Y$ with a Log POT $C^{\ell}_{X/Y} \subseteq E$ and equip $f \circ p : \widehat{X} \to Y$ with the pullback obstruction theory

$$C^{\ell}_{\widehat{X}/Y} \subseteq C^{\ell}_{X/Y}|_{\widehat{X}} \subseteq E|_{\widehat{X}}.$$

Then possibly

$$p_*[\widehat{X}, E]^{\ell vir} \neq [X, E]^{\ell vir}.$$

A counterexample is again given by $p : D \to_{\mathbb{N}^2}$, $f :_{\mathbb{N}^2} =\!=_{\mathbb{N}^2}$ as in Example 4.3.13: $p_*[\mathbb{P}^1] = 0$ for dimension reasons.

The rest of this section and the next should reassure the disheartened reader that common-sense fomulas of ordinary intersection theory do remain true in the log setting. We regard Remarks 4.4.8, 4.4.9 as defects of the usual notion of pushforward p_* in the log setting. The morphisms $_{\mathbb{N}} \to D$, $D \to_{\mathbb{N}^2}$ of Example 4.3.13 are monomorphisms in the fs category, and $_{\mathbb{N}} \to_{\mathbb{N}^2}$ should be a cycle of *dimension one* in the "two dimensional" log point $_{\mathbb{N}^2}$.

The paper [6] introduces log chow groups to correct this defect, in particular via suitable notions of dimension and degree. See also [54]. We are eager to see which of our results may be extended using this improved technology.

For now, we content ourselves to use the observation of [57, Proposition 4.3] that log blowups are birational if the target is log smooth. We will use it to prove that weaker forms of the naïve guesses of Remarks 4.4.8, 4.4.9 do hold true, as well as straightforward commutativity of the Gysin Maps.

We will need to use Costello's notion of "pure degree d" [15, before Theorem 5.0.1] to make sense of pushforward on the level of cycles, given by cones embedded in vector bundles. The next theorem allows us to check statements about Log VFC's after a log blowup if the target is log smooth. Its statement and proof are similar to [4].

Theorem 4.4.10. Suppose given a DM-type map $f : X \to Y$ between locally noetherian algebraic stacks locally of finite type over $\mathbb{C}$ where Y is log smooth and equidimensional. Endow f with a Log POT E and let $X \to F$ be any DM morphism to an Artin Fan. Take the fs pullback along a finite subdivision

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner \ell"] & \widehat{F} \arrow[d] \\ X \arrow[r] & F. \end{tikzcd} \tag{4.4}$$

Endow $f \circ p$ with the pullback Log POT

$$C^{\ell}_{\widehat{X}/Y} \subseteq C^{\ell}_{X/Y}|_{\widehat{X}} \subseteq E|_{\widehat{X}}.$$

Then

$$p_*[\widehat{X}, E]^{\ell vir} = [X, E]^{\ell vir}$$

Proof. We will actually show that the map

$$t : C^{\ell}_{\widehat{X}/Y} \to C^{\ell}_{X/Y}$$

is of pure degree one. Then the pushforward $A_*E|_{\widehat{X}} \to A_*E$ sends the class of one cone to the other, and "intersecting with the zero section" gives the equality of VFC's.

We will reduce to the case where $X \to F$ is strict. The statement "t is of pure degree one" may be verified étale-locally in X, as we now argue.

Given a strict étale cover $X' \to X$, write $\widehat{X'} := \widehat{X} \times_X X'$. We have a pullback diagram

$$\begin{tikzcd} C^{\ell}_{\widehat{X'}/F} \arrow[r, "t'"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & C^{\ell}_{X'/F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X' \arrow[d] \\ C^{\ell}_{\widehat{X}/F} \arrow[r, "t"] & C^{\ell}_{X/F} \arrow[r] & X, \end{tikzcd}$$

as in Remark 4.3.14. Since $X' \to X$ is étale, the other vertical arrows are as well. The property "pure degree one" is smooth-local in the target, so t has it if t' does.

Now étale-localize in X so that $X \to F$ factors through a chart $X \to F_X \to F$ for X. Take the fs pullback along the subdivision $\widehat{F} \to F$:

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell", very near start] & \widehat{F}_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell", very near start] & \widehat{F} \arrow[d] \\ X \arrow[r] & F_X \arrow[r] & F. \end{tikzcd}$$

We can then replace F by F_X in the proof of the theorem and assume $X \to F$ is strict.

Apply the proof of Costello's Formula [15, Theorem 5.0.1] to (4.4) to conclude

$$t : C^{\ell}_{\widehat{X}/\widehat{F}} \to C^{\ell}_{X/F}$$

is of pure degree one, since $\widehat{F} \to F$ is birational.

Expanding upon (4.4):

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell", very near start] & \widehat{F} \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell", very near start] & \widehat{F} \arrow[d] \\ X \arrow[r] & F \times Y \arrow[r] \arrow[d] & F \\ & Y, & \end{tikzcd}$$

we get a map of exact sequences of cone stacks:

$$\begin{tikzcd} T^{\ell}_{Y}|_{\widehat{X}} \arrow[r] \arrow[d] & C^{\ell}_{\widehat{X}/\widehat{F}\times Y} \arrow[r] \arrow[d, "\widehat{t}"] \arrow[dr, phantom, "\ulcorner", very near start] & C^{\ell}_{\widehat{X}/\widehat{F}} \arrow[d, "t"] \\ T^{\ell}_{Y}|_{X} \arrow[r] & C^{\ell}_{X/F\times Y} \arrow[r] & C^{\ell}_{X/F}. \end{tikzcd}$$

After pulling the bottom row back to $\widehat{X}$, we get the identity on tangent bundles and see that the right square is a pullback. Since the property"of pure degree one" pulls back along smooth maps, the quotient maps in exact sequences of cone stacks are smooth, and t is pure degree one, $\widehat{t}$ is also pure degree one. Because $F, \widehat{F}$ are log étale over a point, $C^{\ell}_{\widehat{X}/\widehat{F}\times Y} = C^{\ell}_{\widehat{X}/\widehat{Y}}$ and $C^{\ell}_{X/F\times Y} = C^{\ell}_{X/Y}$, so the claim is proven.

□

Example 4.4.11. One must be cautious, for Theorem 4.4.10 is false without the assumption that Y is log smooth. Recall the exceptional divisor $D \to$ of the blowup of $\mathbb{A}^2$ at the origin $= \operatorname{Spec} \mathbb{C}$ from Example 4.3.13 and its normal cone $C^{\ell}_{D/} = \mathbb{P}^1$.

For the sake of contradiction, let $\widehat{X} = \mathbb{P}^1$ and $X = Y =$ as in the theorem. Endow $C^{\ell}_{/} =$ with the initial Log POT , $E =$. Then

$$[\widehat{X}, E]^{\ell vir} = [D, E]^{\ell vir} = [\mathbb{P}^1]$$

and

$$[X, E]^{\ell vir} = [, E]^{\ell vir} = [],$$

but again $p_*[\mathbb{P}^1] = 0$ for dimension reasons.

Theorem 4.4.12 (Commutativity of Log Gysin Map)**.** Given a composable pair of DM-type maps between log algebraic stacks

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

outfit f, g, and $g \circ f$ with log obstruction theories F, G, E and a compatibility datum (Remark 4.4.4). Require X to admit stratifications by global quotients.

If $k : V \to Z$ is a log smooth and equidimensional Z-stack and k is DM-type, take fs pullbacks:

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & U \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & V \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z. \end{tikzcd}$$

Then the equality

$$[C^{\ell}_{g \circ f} \subseteq E] = [C^{\ell}_{C^{\ell}_g|_X / C^{\ell}_g} \subseteq F \oplus G|_X] \tag{4.5}$$

holds on X.

Proof. Pullback via k all obstruction theories and their compatibility datum to reduce to showing the theorem for $k : V = Z$. We essentially apply [49, Theorem 4.8] to $X \to \mathcal{L}Y \to \mathcal{L}^2 Z$, endowed with the compatible triple F, G, E by composing with an isomorphism of distinguished triangles:

$$\begin{tikzcd} G|_X \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & E \arrow[d] \\ \mathbb{L}_{\mathcal{L}Y/\mathcal{L}Z}|_X \arrow[r] \arrow[d, "\sim"] & \mathbb{L}^{\ell}_{X/Z} \arrow[r] \arrow[d, equal] & \mathbb{L}^{\ell}_{X/Y} \arrow[d, "\sim"] \\ \mathbb{L}_{\mathcal{L}Y/\mathcal{L}^2 Z}|_X \arrow[r] & \mathbb{L}_{X/\mathcal{L}Z} \arrow[r] & \mathbb{L}_{X/\mathcal{L}^2 Z}. \end{tikzcd}$$

Use Lemma 4.3.8 repeatedly to obtain a strict diagram with U, V quasicompact and étale over the stacks $\mathcal{L}Y, \mathcal{L}^2Z$:

$$\begin{tikzcd} X \arrow[r] \arrow[dr, bend right] & U \arrow[r] \arrow[d] & V \arrow[d] \\ & \mathcal{L}Y \arrow[r] & \mathcal{L}^2Z. \end{tikzcd}$$

Endow the cone $C_{\mathcal{L}Y/\mathcal{L}Z}$ with the pullback log structure from $\mathcal{L}Y$ and pull it back along the part of the diagram above $\mathcal{L}Y$:

$$\begin{tikzcd} C_{\mathcal{L}Y/\mathcal{L}Z}|_X = C_{U/V}|_X \arrow[r] \arrow[dr, bend right] & C_{U/V} \arrow[d] \\ & C_{\mathcal{L}Y/\mathcal{L}Z}. \end{tikzcd}$$

The triangle is strict and the map $C_{U/V} \to C_{\mathcal{L}Y/\mathcal{L}Z}$ is pulled back from the étale $U \to \mathcal{L}Y$, so

$$C^\ell_{C_{\mathcal{L}Y/\mathcal{L}Z}|_X/C_{\mathcal{L}Y/\mathcal{L}Z}} = C_{C_{U/V}|_X/C_{U/V}}.$$

Write $i : X \to U$ $j : U \to V$ for the maps. Then the compatibility datum pulls back and [49, Theorem 4.8] gives us

$$(j \circ i)^!_E([V]) = i^!_F \circ j^!_G([V]).$$

Unwinding definitions, this becomes

$$[C_{X/V} \subseteq E] = [C_{C_{U/V}|_X/C_{U/V}} \subseteq F \oplus G|_X]. \tag{4.6}$$

This may be rewritten as

$$[C^\ell_{X/Z} \subseteq E] = [C^\ell_{C_{\mathcal{L}Y/\mathcal{L}Z}|_X/C_{\mathcal{L}Y/\mathcal{L}Z}} \subseteq F \oplus G|_X],$$

the claimed equality of classes.

□

Remark 4.4.13. Theorem 4.4.12 says that

$$(g \circ f)^! = f^! g^!$$

in the sense that any log smooth, equidimensional log stack over Z has rationally equivalent images under these two operations.

Remark 4.4.14. Consider an fs pullback of DM-type morphisms between log algebraic stacks:

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner\ell", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y. \end{tikzcd}$$

Write $r : X' \to Y$ for the composite $f \circ p = q \circ f'$. If f, q are endowed with Log POT's $C^\ell_{X/Y} \subseteq F$, $C^\ell_{Y'/Y} \subseteq E$, how should we give r a Log POT?

The fs pullback square induces a pullback of stacks, which may be reexpressed as a "magic square:"

$$\begin{tikzcd} \mathcal{L}X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{L}X \arrow[d] \\ \mathcal{L}Y' \arrow[r] & \mathcal{L}Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{L}X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{L}X \times \mathcal{L}Y' \arrow[d] \\ \mathcal{L}Y \arrow[r] & \mathcal{L}Y \times \mathcal{L}Y. \end{tikzcd}$$

The magic square induces a closed immersion

$$C_{\mathcal{L}X'/\mathcal{L}Y} \subseteq C_{\mathcal{L}X/\mathcal{L}Y}|_{\mathcal{L}X'} \times_{\mathcal{L}X'} C_{\mathcal{L}Y'/\mathcal{L}Y}|_{\mathcal{L}X'}$$

which pulls back to a closed immersion

$$C^\ell_{X'/Y} \subseteq C_{\mathcal{L}X/\mathcal{L}Y}|_{X'} \times_{X'} C_{\mathcal{L}Y'/\mathcal{L}Y}|_{X'}$$

on X'. As in Remark 4.4.4, we have closed embeddings $C_{\mathcal{L}X/\mathcal{L}Y} \subseteq C^\ell_{X/Y}|_{\mathcal{L}X}$, $C_{\mathcal{L}Y'/\mathcal{L}Y} \subseteq C^\ell_{Y'/Y}|_{\mathcal{L}Y'}$. We endow r with the Log POT given by the composite:

$$C^\ell_{X'/Y} \subseteq C_{\mathcal{L}X/\mathcal{L}Y}|_{X'} \times_{X'} C_{\mathcal{L}Y'/\mathcal{L}Y}|_{X'} \subseteq C^\ell_{X/Y}|_{X'} \times_{X'} C^\ell_{Y'/Y}|_{X'} \subseteq F|_{X'} \times_{X'} E|_{X'}.$$

We now construct a compatibility datum for the triangle $r = q \circ f'$, leaving the reader to apply the same argument to the other triangle $r = f \circ p$. By the definitions of the Log POT's, we have a commutative diagram:

$$\begin{tikzcd} C^\ell_{X'/Y'} \arrow[r] \arrow[d, hook] & C^\ell_{X'/Y} \arrow[r] \arrow[d, hook] & C^\ell_{Y'/Y}|_{X'} \arrow[d, hook] \\ F|_{X'} \arrow[r, "(0\times id)"] & E|_{X'} \times_{X'} F|_{X'} \arrow[r] & E|_{X'}. \end{tikzcd}$$

To be clear, the morphism $F|_{X'} \to E|_{X'} \times_{X'} F|_{X'}$ is the vertex map times the identity. It's clear the bottom row comes from a distinguished triangle in the derived category. The top row likewise comes from the distinguished triangle [61, Theorem 8.14] of Gabber's Log Cotangent Complexes:

$$Y'/Y|_{X'} \to X'/Y \to X'/Y' \to .$$

Corollary 4.4.15. Suppose given an fs pullback square

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner \ell", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

of DM-type morphisms between log algebraic stacks which admit stratifications by quotient stacks. Outfit q with a Log POT E and f with a Log POT F; give p, f' the pullback obstruction theories. Then

$$f'^! \circ q^! = p^! \circ f^!$$

in the sense that the operations send any log smooth equidimensional input stack to the same class in $A_* X'$.

Proof. Denote by $r : X' \to Y$ the map $f \circ p = q \circ f'$. Apply Theorem 4.4.12 to both commutative triangles using the compatibility datum constructed in Remark 4.4.14 to see that

$$p^! \circ f^! = r^! = f'^! \circ q^!.$$

□

4.5 The Log Costello Formula

This section proves a log analogue of the Costello Formula [15, Theorem 5.0.1]. We will have more to say building on future work [16].

Theorem 4.5.1. Consider an fs pullback square of DM-type maps between algebraic stacks:

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d, "f'"] \arrow[dr, phantom, "\ulcorner \ell", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "q"] & Y. \end{tikzcd}$$

Assume

- $Y' \to Y$ is of some pure degree $d \in \mathbb{Q}$ as in [15, Theorem 5.0.1],
- Y', Y are both log smooth and equidimensional,

- all arrows are DM-type and all stacks are locally noetherian and locally finite type over $\mathbb{C}$,
- X', X admit stratifications by global quotient stacks [40]
- q is proper.

Endow f with a log perfect obstruction theory E and give f' the pullback obstruction theory. Then

$$p_*[X', E|_{X'}]^{\ell vir} = d \cdot [X, E]^{\ell vir}$$

in the Chow Ring of X.

Remark 4.5.2. Let $Y' \to Y$ be a map between log smooth, equidimensional stacks which is of pure degree d. Let $W \to Y$ be a smooth, log smooth, integral, and saturated morphism and $\widetilde{W} \to W$ a log blowup. Form the fs pullback diagram:

$$\begin{tikzcd} \widetilde{W'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & \widetilde{W} \arrow[d] \\ W' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \not\ell"] & W \arrow[d] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

The property "of pure degree d" pulls back along smooth morphisms, so it applies to $W' \to W$. Then [57, Proposition 4.3] shows that $\widetilde{W} \to W$ is birational, so $\widetilde{W'} \to \widetilde{W}$ is also of pure degree d.

Proof of Theorem 4.5.1. Consider the morphism

$$s : C^{\ell}_{X'/Y'} \to C^{\ell}_{X/Y}.$$

We will prove that s is of pure degree d. Both "of pure degree" and the specific degree d can be checked after pulling back s along a strict, smooth cover of $C^{\ell}_{X/Y}$. Lemmas 4.3.15, 4.3.16 show that replacing Y or X by a smooth cover results in such a smooth cover of cones.

We may thereby assume X and Y are log schemes and the map f globally factors as in Construction 4.2.1:

$$X \to X_\theta \to \mathbb{A}^{r+s}_Y \to Y.$$

Note $\mathbb{A}_Y^{r+s} \to Y$ is smooth, log smooth, integral, and saturated, and $X_\theta \to \mathbb{A}_Y^{r+s}$ is a log blowup. We are in the situation of Remark 4.5.2, so pulling back:

$$\begin{tikzcd} X' \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner\ell", very near start] & X \arrow[d, hook] \\ X'_\theta \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell", very near start] & X_\theta \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

results in a map $X'_\theta \to X_\theta$ which is pure of degree d along $X \to X_\theta$. The proof of Costello's Formula [15, Theorem 5.0.1] then asserts that

$$C^\ell_{X'/X'_\theta} \to C^\ell_{X/X_\theta}$$

is of pure degree d. The short exact sequences of Proposition 4.3.5

$$\begin{tikzcd} T^\ell_{X'_\theta/Y'} \arrow[r] \arrow[d] & C^\ell_{X'/X'_\theta} \arrow[r] \arrow[d, "t"] & C^\ell_{X'/Y'} \arrow[d, "s"] \\ T^\ell_{X_\theta/Y} \arrow[r] & C^\ell_{X/X_\theta} \arrow[r] & C^\ell_{X/Y} \end{tikzcd}$$

let us conclude that s is as well.

$\square$

4.6 The Product Formula

Let V, W be log smooth, quasiprojective schemes throughout this section. We denote the stacks of *prestable curves* and *stable curves* which have n-markings and genus g by $\mathfrak{M}^\ell_{g,n}, \overline{M}_{g,n}$, respectively [63, 0DMG]. They are endowed with divisorial log structures coming from the locus of singular curves [23, 1.5, Appendix A], [35].

Definition 4.6.1 (Log Stable Maps)**.** The stack of log stable maps $\mathscr{M}^\ell_{g,n}(V)$ has fiber over an fs log scheme T the category of diagrams of fs log schemes

$$\begin{tikzcd} C \arrow[r] \arrow[d] & V \\ T & \end{tikzcd}$$

with $C \to T$ a log smooth curve [35, Definition 1.2] of genus g and n marked points, such that the underlying diagram of schemes is a stable map of curves.

Remarkably, the log algebraic stack $\mathscr{M}^{\ell}_{g,n}(\operatorname{Spec}\mathbb{C})$ of log curves without a map is isomorphic to the ordinary stack of stable curves $\overline{M}_{g,n}$ with log structure induced by the boundary of degenerate curves [35, Theorem 4.5]. The log structures of $\mathscr{M}^{\ell}_{g,n}(V)$ for a general fs target may be more complicated, as they have to do with the "tropical deformation space" of the curve [23].

Construction 4.6.2 ([23, Section 5])**.** We recall the construction [23, Section 5] of the natural Log POT for $\mathscr{M}^{\ell}_{g,n}(V) \to \mathfrak{M}^{\ell}_{g,n}$ to clarify differences in notation.

Write $\mathcal{U} \to \mathfrak{M}^{\ell}_{g,n}$ for the universal curve. Define $\mathcal{U}_V$ as the fs pullback, naturally equipped with a tautological map to V:

$$\begin{tikzcd} V & \mathcal{U}_V \arrow[l] \arrow[r, "\pi_V"] \arrow[d] \arrow[dr, phantom, "\ulcorner \ell"] & \mathscr{M}^{\ell}_{g,n}(V) \arrow[d] \\ & \mathcal{U} \arrow[r] & \mathfrak{M}^{\ell}_{g,n} \end{tikzcd}$$

This diagram induces maps between log cotangent complexes

$$\mathbb{L}^{\ell}_V|_{\mathcal{U}_V} \longrightarrow \mathbb{L}^{\ell}_{\mathcal{U}_V/\mathcal{U}} \xleftarrow{t} \mathbb{L}^{\ell}_{\mathscr{M}^{\ell}_{g,n}(V)/\mathfrak{M}^{\ell}_{g,n}}|_{\mathcal{U}_V}.$$

The map $\mathcal{U} \to \mathfrak{M}^{\ell}_{g,n}$ is integral, saturated, and log smooth according to its functor of points, so its underlying map of stacks is flat and the fs pullback square is also an ordinary pullback.

Then t is an isomorphism [61, 1.1 (iv)], and the log cotangent complex of V is [61, 1.1 (iii)]

$$\mathbb{L}^{\ell}_V = \Omega^{\ell}_V[0].$$

We've written [0] to consider a coherent sheaf as a chain complex concentrated in degree 0. Via the isomorphism t and this identification, we have obtained a map

$$\Omega^{\ell}_V[0]|_{\mathcal{U}_V} \to \mathbb{L}^{\ell}_{\mathcal{U}/\mathfrak{M}^{\ell}_{g,n}}|_{\mathcal{U}_V}. \tag{4.7}$$

We need the ordinary relative dualizing sheaf $\omega_{\pi_V^\circ}$ and the identification

$$L\pi_V^!(\cdot) = \omega_{\pi_V^\circ} \overset{L}{\otimes} L\pi_V^*(\cdot).$$

Tensor (4.7) by $\omega_{\pi_V^\circ}$ and use the adjunction:

$$\Omega_V^\ell[0]\,|_{\mathcal{U}_V} \overset{L}{\otimes} \omega_{\pi_V^\circ} \longrightarrow L\pi_V^! \mathbb{L}^\ell_{\mathscr{M}^\ell_{g,n}(V)/\mathfrak{M}^\ell_{g,n}},$$

$$E(V) := R\pi_{V*}(\Omega_V^\ell[0]\,|_{\mathcal{U}_V} \overset{L}{\otimes} \omega_{\pi_V^\circ}) \longrightarrow \mathbb{L}^\ell_{\mathscr{M}^\ell_{g,n}(V)/\mathfrak{M}^\ell_{g,n}}.$$

We won't repeat the verification [23, Proposition 5.1] that $E(V)$ is a Log POT.

Remark 4.6.3. The map (4.7) comes from the map on normal cones

$$C^\ell_{\mathscr{M}^\ell_{g,n}(V)/\mathfrak{M}^\ell_{g,n}}|_{\mathcal{U}_V} \xleftarrow{\sim} C^\ell_{\mathcal{U}_V/\mathcal{U}} \longrightarrow BT_V^\ell|_{\mathcal{U}_V}.$$

We needed duality, so we opted for the other perspective.

Remark 4.6.4 (Variants)**.** The reader may choose to work in the relative setting of a log smooth and quasiprojective map $V \to S$. Obstruction Theories are obtained in the same way.

We can naturally impose "contact order" conditions [2] in the log setting, but we only fix genus and number of markings to be consistent with [42]. The reader may readily vary the numerical type conditions in our formulas.

We need one more stack, $\mathfrak{D}$: Points of $\mathfrak{D}$ over T are diagrams $(C' \leftarrow C \to C'')$ of genus g, n-pointed prestable curves over T whose maps are partial stabilizations (they lie over the identities in $\overline{M}_{g,n}$) that don't both contract any component. In other words, $C \to C' \times C''$ itself is a stable map. This stack is only necessary to form an fs pullback square:

Situation 4.6.5 ([42, Section 2])**.** Recall the fs pullback square:

$$\begin{tikzcd}
\mathscr{M}^\ell_{g,n}(V \times W) \arrow[r] \arrow[d, "c"] \arrow[dr, phantom, "\ulcorner \ell"] & \mathscr{M}^\ell_{g,n}(V) \times \mathscr{M}^\ell_{g,n}(W) \arrow[d, "a"] \\
\mathfrak{D} \arrow[r, "\widetilde{\Delta}"] & \mathfrak{M}^\ell_{g,n} \times \mathfrak{M}^\ell_{g,n}
\end{tikzcd} \tag{4.8}$$

Let $C \to V \times W$ be a log stable map over a base T. The maps $(C \to V)$, $(C \to W)$ needn't be stable; denote their stabilizations by $(C' \to V)$, $(C'' \to W)$, respectively.

The top horizontal arrow in (4.8) sends $(C \to V \times W)$ to the induced log stable maps $(C' \to V, C'' \to W)$. The vertical arrow c sends $(C \to V \times W)$ to the partial stabilizations

$(C' \leftarrow C \rightarrow C'')$. The map $\widetilde{\Delta}$ sends a diagram $(C' \leftarrow C \rightarrow C'')$ to the pair of prestable curves C', C''. Finally, a sends a pair of log stable maps $(C' \rightarrow V, C'' \rightarrow W)$ to the prestable curves (C', C'').

This square has a factorization:

$$\begin{tikzcd}
\mathscr{M}^\ell_{g,n}(V \times W) \arrow[r, "h"] \arrow[d, "c"] \arrow[dr, phantom, "\ulcorner\ell"] & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell"] & \mathscr{M}^\ell_{g,n}(V) \times \mathscr{M}^\ell_{g,n}(W) \arrow[d, "a"] \\
\mathfrak{D} \arrow[r, "l"] & Q' \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\ulcorner\ell"] & \mathfrak{M}^\ell_{g,n} \times \mathfrak{M}^\ell_{g,n} \arrow[d, "s\times s"] \\
& \overline{M}_{g,n} \arrow[r, "\Delta"] & \overline{M}_{g,n} \times \overline{M}_{g,n},
\end{tikzcd} \tag{4.9}$$

where $s : \mathfrak{M}^\ell_{g,n} \rightarrow \overline{M}_{g,n}$ stabilizes a prestable curve.

To be clear, $Q = \mathscr{M}^\ell_{g,n}(V) \times^\ell_{\overline{M}_{g,n}} \mathscr{M}^\ell_{g,n}(W)$ and $Q' = \mathfrak{M}^\ell_{g,n} \times^\ell_{\overline{M}_{g,n}} \mathfrak{M}^\ell_{g,n}$ are the analogues of [42]'s P, $\mathfrak{P}$, etc.

Theorem 4.6.6 (The "Log Gromov-Witten Product Formula")**.** With V, W log smooth, quasiprojective schemes,

$$h_*[\mathscr{M}^\ell_{g,n}(V \times W), E(V \times W)]^{\ell vir} = \Delta^!([\mathscr{M}^\ell_{g,n}(V), E(V)]^{\ell vir} \times [\mathscr{M}^\ell_{g,n}(W), E(W)]^{\ell vir}).$$

Our proof will be the same as K. Behrend's [7]: we compute the log normal cone of the map $Q \rightarrow Q'$ in two different ways.

Remark 4.6.7 (On Diagram (4.9))**.** We equip a with the product $E(V) \boxplus E(W)$ of the natural Log POT 's of Construction 4.6.2, adopting the notation

$$E \boxplus E' := E|_{V\times W} \oplus E'|_{V\times W}.$$

The cotangent complex $\mathbb{L}^\ell_\Delta$ is of perfect amplitude in [-1, 0] because its source and target are log smooth. Therefore $C^\ell_\Delta = N^\ell_\Delta$ serves as a natural Log POT for itself. We equip ϕ with the pullback obstruction theory, resulting in

$$\Delta^! = \phi^!$$

by Remark 4.4.6. We endow the square bounded by ϕ and a with the natural compatibility datum afforded all such squares as in Remark 4.4.14.

All of the arrows in Diagrams (4.8) and (4.9) are of DM-type.

Lemma 4.6.8. The stabilization map $s : \mathfrak{M}^{\ell}_{g,n} \to \overline{M}_{g,n}$ is log smooth.

Proof. The cover $\bigsqcup_m \overline{M}_{g,n+m} \to \mathfrak{M}^{\ell}_{g,n}$ given by forgetting marked points and not stabilizing is strict smooth [42, 1.2.1]. This map is in particular kummer and surjective, and [**?**, Theorem 0.2] applies with $\mathbb{P} =$ "log smooth" once we argue that the composite $\bigsqcup_m \overline{M}_{g,n+m} \to \overline{M}_{g,n}$ is log smooth.

The forgetful map $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is the universal curve, so it is tautologically log smooth. We see the map $\overline{M}_{g,n+m} \to \overline{M}_{g,n}$ is log smooth by iterating this forgetfulness, and this completes the argument.

□

Remark 4.6.9. The map $\mathfrak{D} \to \mathfrak{M}^{\ell}_{g,n}$ which records the initial curve is log étale since the original map was étale [7, Lemma 4] and ours is the fsification thereof. The stack Q' is log smooth because the map $Q' \to \overline{M}_{g,n}$ is pulled back from $s \times s$.

Given a log étale map $X' \to X$ of log smooth log algebraic stacks with X equidimensional, we claim X' must be as well. The maps $X' \subseteq \mathcal{L}X'$, $X \subseteq \mathcal{L}X$ are dense because of the log smoothness assumption and the map $\mathcal{L}X' \to \mathcal{L}X$ is étale. Thus $\mathcal{L}X$ and $\mathcal{L}X'$ are equidimensional, as well as $X' \subseteq \mathcal{L}X'$. This argument shows that fsification preserves equidimensionality of log smooth stacks, so our fs versions of $\mathfrak{D}$, Q' are equidimensional because the original versions [7] were.

Lemma 4.6.10. The obstruction theories $E(V)$, $E(W)$, $E(V \times W)$ are compatible in the sense that

$$\widetilde{\Delta}^*(E(V) \boxplus E(W)) \simeq E(V \times W).$$

Proof. We completely echo the proof of [7, Proposition 6].

Consider the diagram of universal log curves and tautological maps with the notation:

$$\begin{tikzcd}
V & & V \times W \arrow[ll] \\
\mathcal{U}_V \arrow[u, "f_V"] \arrow[d, "\pi_V"'] & \widetilde{\mathcal{U}}_V \arrow[l, "s_V"'] \arrow[dr, "\widetilde{\pi}_V"'] \arrow[dl, phantom, "\ell\lrcorner"] & \mathcal{U}_{V\times W} \arrow[l, "q_V"'] \arrow[u, "f_{V\times W}"'] \arrow[d, "\pi_{V\times W}"] \\
\mathscr{M}^{\ell}_{g,n}(V) & & \mathscr{M}^{\ell}_{g,n}(V \times W). \arrow[ll, "r_V"]
\end{tikzcd}$$

We claim $F \to Rq_{V*}q_V^*F$ is an isomorphism for any vector bundle F on $\mathcal{U}_V$. The map q_V represents partial stabilization. We make the argument for contracting one $\mathbb{P}^1$ at a time.

We first compute that $R^p q_{V*}q_V^*F = 0$ for $p \neq 0$. This claim is local in $\mathcal{U}_V$, so assume F is trivial. The fiber of $R^p q_{V*}q_V^*F$ at a point x is $H^p(q_V^{-1}(x), q_V^*F)$. Hence the fibers $q_V^{-1}(x)$ are either a point or $\mathbb{P}^1$. On each fiber, the cohomology of the trivial vector bundle is concentrated in degree 0 [63, 01XS]. Not only are F and $q_{V*}q_V^*F$ abstractly isomorphic in that case, but the natural map is an isomorphism [19, Exercise 9.3.11].

The universal curve π_V is tautologically flat, integral, and saturated. The fs pullback square it belongs to is therefore also an ordinary flat pullback, subject to cohomology and base change [63, Tag 08IB]. This gives:

$$\begin{aligned}
Lr_V^* R\pi_{V*} Lf_V^* \Omega_V &= R\widetilde{\pi}_{V*} Ls_V^* Lf_V^* \Omega_V \\
&= R\widetilde{\pi}_{V*} Rq_{V*} q_V^* Ls_V^* Lf_V^* \Omega_V \\
&= R\pi_{V\times W*} Lf_{V\times W}^* (\Omega_V|_{V\times W}).
\end{aligned}$$

All the same goes for W. Add the two together to get

$$Lr_V^* R\pi_{V*} Lf_V^* \Omega_V \boxplus Lr_W^* R\pi_{W*} Lf_W^* \Omega_W \ = \ R\pi_{V\times W*} Lf_{V\times W}^* (\Omega_V \boxplus \Omega_W).$$

This is dual to the compatibility we set out to prove, so we are through.

$\square$

Proof of Theorem 4.6.6. Compute the log virtual fundamental class $[Q, E(V) \boxplus E(W)]^{vir}$ in two different ways:

$$\begin{aligned}
[Q, E(V) \boxplus E(W)]^{vir} &:= [C^{\ell}_{Q/Q'} \subseteq E(V) \boxplus E(W)] \\
&= a^!(Q') \\
&= a^! \phi^! (\mathfrak{M}^{\ell}_{g,n} \times \mathfrak{M}^{\ell}_{g,n}) \\
&= \phi^! a^! (\mathfrak{M}^{\ell}_{g,n} \times \mathfrak{M}^{\ell}_{g,n}) \\
&= \Delta^! [\mathscr{M}^{\ell}_{g,n}(V) \times \mathscr{M}^{\ell}_{g,n}(W), E(V) \boxplus E(W)]^{vir}.
\end{aligned}$$

On the other hand,

$$[Q, E(V) \boxplus E(W)]^{vir} = h_* [\mathscr{M}^{\ell}_{g,n}(V \times W), E(V \times W)]^{vir}$$

by the Log Costello Formula 4.5.1.

$\square$

Chapter 5

Deformations of Modules

> What is truth? a mobile army of metaphors, metonyms, anthropomorphisms, in short, a sum of human relations which were poetically and rhetorically heightened, transferred, and adorned, and after long use seem solid, canonical, and binding to a nation. Truths are illusions about which it has been forgotten that they are illusions, worn-out metaphors without sensory impact, coins which have lost their image and now can be used only as metal, and no longer as coins.
>
> [21]

5.1 Introduction

Consider a topos E and a squarezero extension of sheaves of rings

$$0 \to J \to A' \to A \to 0 \tag{5.1}$$

Fix A-modules M and K, naturally endowed with A'-module structures. The central ambition of this paper is to provide another answer to the following question, studied in [29]:

Question 5.1.1. Is there an extension of A'-modules

$$\xi : \quad 0 \to K \to M' \to M \to 0 \tag{5.2}$$

and, if so, how many are there?

We refine Question 5.1.1 in two ways. An important invariant of the extension (5.2) is the homomorphism

$$u : J \otimes_{A'} M \to K \qquad j \otimes m \mapsto j \cdot m' \tag{5.3}$$

where j and m are sections of J and M and m' is any preimage of m.

By sending an extension (5.2) to the induced map $u : J \otimes_{A'} M \to K$, we obtain a group homomorphism

$$\theta : \mathrm{Ext}^1_{A'}(M, K) \to \mathrm{Hom}_A(J \otimes_{A'} M, K)$$

Extensions may be classified according to their image under θ. In practice, we consider only those extensions which induce a fixed map u.

Given an extension (5.2), we may pull back along a map $N \to M$ of A-modules.

$$\begin{tikzcd}
\xi|_N : & 0 \arrow[r] & K \arrow[r, dashed] \arrow[d, equal] \arrow[rr, bend left, "0"] & M' \times_M N \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & N \arrow[r] \arrow[d] & 0 \\
\xi : & 0 \arrow[r] & K \arrow[r] & M' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

The dashed arrow makes the diagram commute, and the top row is a short exact sequence. This map of extensions is clearly cartesian, forming a fibered category $\underline{\mathrm{Ext}}^1_{A'}(\_\_, K) \to A\text{-mod}$ over the category of A-modules. The fiber over M is extensions of M by K, where *both are considered in* A'-mod.

The pullback $\xi|_N$ will map under θ to the composition $J \otimes_{A'} N \to J \otimes_{A'} M \xrightarrow{u} K$. This entails a morphism of fibered categories (in fact, *gr*-stacks) $\theta : \underline{\mathrm{Ext}}^1_{A'}(\_\_, K) \to \mathrm{Hom}_A(J \otimes_{A'} \_\_, K)$ over A-mod$/M$, the latter presheaf considered as a fibered category.

The strict pullback

$$\begin{tikzcd}
\underline{\mathrm{Def}}(__, u, K) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \underline{\mathrm{Ext}}^1_{A'}(__, K) \arrow[d] \\
\{*\} \arrow[r, "u"] & \mathrm{Hom}_A(J \otimes __, K)
\end{tikzcd} \tag{5.4}$$

defines a full fibered subcategory $\underline{\mathrm{Def}}(\_\_, u, K) \subseteq \underline{\mathrm{Ext}}^1_{A'}(\_\_, K)$ of the category of A'-module extensions over the category of A-modules over M, A-mod$/M$. Its sections are extensions ξ, with $\theta(\xi)$ a fixed map $u : J \otimes_{A'} M \to K$. These are referred to as "deformations."

Even when it is not possible to construct a deformation of u between M and K, one may always find a map $N \to M$ of A-modules and an extension

$$0 \to K \to N' \to N \to 0$$

of A'-modules which is a deformation of $J \otimes_{A'} N \to J \otimes_{A'} M \xrightarrow{u} K$.

In order to piece together the abundant extensions over $N \to M$ into one over M, we equip A-mod with a topology.

Definition 5.1.2 (The Topology on A-mod)**.** A family of maps $\{N_i \to M\}_I$ of modules is deemed covering if, for all finite sets of sections $\Lambda \subseteq M(X)$ over some $X \in E$, there exists a covering $\{U_j \to X\}_J$ in E such that, for each j, there is a single i and a lift of $\Lambda|_{U_j}$ to $N_i(U_j)$.

This site is simpler than, but directly analogous to, the site $\mathcal{O}_Y\text{–}\underline{\text{Alg}}/\mathcal{O}_X$ of [71]. The topology is subcanonical. In particular, we write $\hbar_K$ for the sheaf $N \mapsto \operatorname{Hom}_A(N, K)$. The topology is designed to achieve the next theorem, proved in Section **??**.

Theorem 5.1.3. The fibered category $\underline{\text{Def}}(\_\_, u, K) \to A\text{-mod}/M$ is a gerbe banded by $\hbar_K$.

The fact that $\underline{\text{Def}}(\_\_, u, K)$ forms a gerbe answers a few questions for free – in particular, the "how many?" of Question 5.1.1:

Corollary 5.1.4. The class of the $\hbar_K$-gerbe $\underline{\text{Def}}(\_\_, u, K)$ in $H^2(A\text{-mod}/M, \hbar_K)$ obstructs the existence of a deformation ξ with $\theta(\xi) = u$. Provided this class vanishes, the set of such ξ is naturally a torsor under $H^1(A\text{-mod}/M, \hbar_K)$. The automorphisms of any given extension are in canonical bijection with $H^0(A\text{-mod}/M, \hbar_K)$.

For any sheaf F on a site X, we identify $H^1(X, F)$ with F-torsors and $H^2(X, F)$ with gerbes banded by F (up to equivalence). There is a choice of sign hidden in this identification – ours is specified in the appendix.

The following theorem allows us to compute $H^p(A\text{-mod}/M, \hbar_K)$.

Theorem 5.1.5. The groups of p-extensions are all isomorphic to the cohomology of $\hbar_K$ on the site $A\text{-mod}/M$: $\operatorname{Ext}^p_A(M,K) \simeq H^p(A\text{-mod}/M, \hbar_K)$.

This theorem is proved in Section **??**. We describe the isomorphism of Theorem 5.1.5 explicitly in the cases $p = 1, 2$ of greatest interest in Propositions 5.4.1 and 5.4.5. As a result of this description, we can identify which 2-extension corresponds to our gerbe $\underline{\mathrm{Def}}(\_\_, u, K)$ in Section **??**:

Theorem 5.1.6. The diagram

$$\begin{tikzcd} \operatorname{Hom}_A(J \otimes_{A'} M, K) \arrow[r, "\smile\omega"] \arrow[dr, "\underline{\mathrm{Def}}"'] & \operatorname{Ext}^2_A(M,K) \arrow[d, "\mathrm{Split}"] \\ & H^2(A\text{-mod}/M, \hbar_K) \end{tikzcd} \tag{5.5}$$

anti-commutes.

The isomorphism Split of Theorem 5.1.5 for $p = 2$ sends a 2-extension to its $\hbar_K$-gerbe of splittings (Definition 5.4.2 and Proposition 5.4.5). The 2-extension ω is constructed before the theorem after choosing a resolution of M; however, the cup product $f \mapsto f \smile \omega$ is independent of that choice up to isomorphism. The map $\underline{\mathrm{Def}}$ sends a morphism $f : J \otimes_{A'} M \to K$ to the $\hbar_K$-gerbe $\underline{\mathrm{Def}}(\_\_, f, K)$ over $A\text{-mod}/M$. Anti-commutativity signifies that $\underline{\mathrm{Def}}(\_\_, f, K)$ and $\mathrm{Split}(f \smile \omega)$ represent additive-inverse cohomology classes. In other words, our obstruction and Illusie's are inverses.

The classification of deformations found in [29] produces the complex

$$0 \to \operatorname{Ext}^1_A(M,K) \to \operatorname{Ext}^1_{A'}(M,K) \xrightarrow{\theta} \operatorname{Hom}_A(J \otimes_{A'} M, K) \xrightarrow{\smile\omega} \operatorname{Ext}^2_A(M,K) \tag{5.6}$$

Lemma 5.1.7. The sequence of maps (5.6) is an exact sequence.

Proof. Choose an extension

$$\xi : \quad 0 \to K \to M' \to M \to 0 \quad \in \operatorname{Ext}^1_{A'}(M,K)$$

The action of J on M' factors as $J \otimes M' \twoheadrightarrow J \otimes M \xrightarrow{\theta(\xi)} K \hookrightarrow M'$ by the definition of θ. Since the first map is surjective and the last is injective, the composite is zero precisely when $\theta(\xi)$ is.

Observe that J annihilates M' if and only if M' is an A-module if and only if $\xi \in \mathrm{Ext}^1_A(M, K)$. This proves exactness at the domain of θ.

By Theorem 5.1.6, $\underline{\mathrm{Def}}(\_\_, u, K)$ and $\mathrm{Split}(u \smile \omega)$ are inverse cohomology classes. One gerbe has a section when the other does. For the gerbe $\underline{\mathrm{Def}}(\_\_, u, K)$ to have a section, u must be in the image under θ of some extension. For $\mathrm{Split}(u \smile \omega)$, this means that the 2-extension $u \smile \omega$ is equivalent to zero in $\mathrm{Ext}^2_A(M, K)$.

□

Exactness entails that the pushout $f \smile \omega$ is equivalent to the zero 2-extension precisely when f is in the image of θ. Under this light, Theorem 5.1.6 says Illusie's obstruction $f \smile \omega$ is identified with the inverse of our $\underline{\mathrm{Def}}(\_\_, f, K)$ under the isomorphism Split. This answers a generalization of Question 3.1.10 in [29].

The exact sequence (5.6) originates in the transitivity triangle for the graded cotangent complex produced in [29]. Our concrete descriptions of the maps augment those found in [63, Tag 08L8]. We can also construct the sequence without reference to the cotangent complex as follows.

Restrict scalars along the map $A' \to A$ to get a fully faithful embedding $r : A\text{-mod}/M \to A'\text{-mod}/M$. This is how we consider M and K as A'-modules, and we often continue to suppress the notation r. The functor r is cover-preserving and left exact, yielding a morphism of sites

$$\pi : A'\text{-mod}/M \to A\text{-mod}/M$$

To avoid ambiguity, we write cohomology on A-mod as $H^p(A/M, \hbar_K)$ and that on A'-mod as $H^p(A'/M, \hbar_K)$ (and similarly for global sections). The equality $\Gamma(A'/M, \hbar_K) = \Gamma(A/M, \pi_* \hbar_K)$ witnesses that the two global section maps to $(Sets)$ commute. The Grothendieck-Leray Spectral Sequence

$$E_2^{p,q} : \quad H^p(A/M, R^q \pi_* \hbar_K) \Rightarrow H^{p+q}(A'/M, \hbar_K) \tag{5.7}$$

yields a 5-term exact sequence. The concern of Section **??** is the next theorem.

Theorem 5.1.8. Illusie's exact sequence (5.6) and the 5-term exact sequence from the Grothendieck-Leray spectral sequence are isomorphic. By this, we mean that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_A(M,K) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^1_{A'}(M,K) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_A(J \otimes_{A'} M, K) \arrow[r] \arrow[d, dashed, "\sim"] & \mathrm{Ext}^2_A(M,K) \arrow[d, "\sim"] \\
0 \arrow[r] & H^1(A/M, \hbar_K) \arrow[r] & H^1(A'/M, \hbar_K) \arrow[r] & H^0(A/M, R^1\pi_*\hbar_K) \arrow[r] & H^2(A/M, \hbar_K),
\end{tikzcd} \tag{5.8}$$

where Illusie's exact sequence is the top row and the 5-term exact sequence is the bottom.

The solid vertical arrows of (5.8) are the isomorphisms of Corollary 5.1.5, except the last one has a minus sign. Once we show the diagram is natural in M and K in Lemma 5.5.3, we obtain the dashed arrow by sheafifying in M. Immediately from this identification, we may extend Illusie's exact sequence to the right via $\mathrm{Ext}^2_A(M,K) \to \mathrm{Ext}^2_{A'}(M,K)$.

Remark 5.1.9. The homotopy sheaves of *gr*-stacks allow for a slick interpretation of our work. Diagram (5.4) induces exact sequences of sheaves of homotopy groups:

$$\begin{tikzcd}
0 \arrow[r] & \pi_1\underline{\mathrm{Def}}(__, u, K) \arrow[r] & \pi_1\underline{\mathrm{Ext}}^1_{A'}(__, K) \arrow[r] & \pi_1\underline{\mathrm{Hom}}(J \otimes_{A'} __, K) \arrow[dll, out=-10, in=170] \\
& \pi_0\underline{\mathrm{Def}}(__, u, K) \arrow[r] & \pi_0\underline{\mathrm{Ext}}^1_{A'}(__, K) \arrow[r] & \pi_0\underline{\mathrm{Hom}}(J \otimes_{A'} __, K) \arrow[r] & 0
\end{tikzcd}$$

Exactness follows from [5, Proposition 6.2.6] when $u = 0$. The final surjectivity is checked by hand in the proof of Theorem 5.1.3, and local preimages ξ of an arbitrary map u under θ allows for a local, non-canonical equivalence $\underline{\mathrm{Def}}(\_\_, 0, K) \overset{+\xi}{\simeq} \underline{\mathrm{Def}}(\_\_, u, K)$. This verifies exactness of the sequence of homotopy sheaves for nonzero u.

Theorem 5.1.3 ensures that $\pi_0\underline{\mathrm{Def}}(\_\_, u, K) = 0$, and we immediately get $\pi_1\underline{\mathrm{Hom}}(J \otimes_{A'} \_\_, K) = 0$. The resulting isomorphisms

$$\pi_1\underline{\mathrm{Def}}(\_\_, u, K) \simeq \pi_1\underline{\mathrm{Ext}}^1_{A'}(\_\_, K)$$

and

$$\pi_0\underline{\mathrm{Ext}}^1_{A'}(\_\_, K) \simeq \pi_0\underline{\mathrm{Hom}}(J \otimes_{A'} \_\_, K)$$

correspond to the identification of $\hbar_K$ as the band of $\underline{\mathrm{Def}}(\_\_, u, K)$ and the inverse of the vertical dashed arrow in Theorem 5.1.8, respectively. To see the latter, describe the sheaf $\underline{\pi_0}\underline{\mathrm{Ext}}^1_{A'}(\_\_, K)$ as the sheafification of the presheaf $N \mapsto \mathrm{Ext}^1_{A'}(N, K) = H^1(A'/N, \hbar_K)$. This sheafification is, by definition, $R^1\pi_*\hbar_K$.

Our present work most heavily relies on the paper [71]. However, there is an error in the proof which we will correct in later work (see Remark 5.2.5). This paper is logically independent of [71] and none of the present article depends on the mistaken assertions therein.

In the body of the paper, we omit the subscript $\otimes_{A'}$ and write $\pi_*\underline{\mathrm{Ext}}^1_{A'}(\_\_, K)$ for the strict pullback of $\underline{\mathrm{Ext}}^1_{A'}(\_\_, K) \to A'$-mod along the restriction of scalars to A-mod, as in [63, 04WA] (for brevity and clarity, respectively).

5.2 The Topology on A-mod

In this section, we prove that $\underline{\mathrm{Ext}}^1_A(\_\_, K) \to A$-mod is a stack. We collect a number of convenient properties of A-mod along the way.

For an object $S \in E$, write A^S for the sheafification of the presheaf $U \mapsto \bigoplus_{\Gamma(U,S)} \Gamma(U, A)$. It deserves the title "free module" via universal property.

Suppose $j : U \to *_E$ is a map to the final object, and $\Lambda \subseteq \Gamma(U, M)$ is a finite subset. The constant sheaf $\underline{\Lambda}$ on U has an adjoint map $\underline{\Lambda} \to M|_U$, and another adjunction furnishes $j_!\underline{\Lambda} \to M$ in E.

One particularly useful tautological cover by free modules is $\{A^{j_!\underline{\Lambda}} \to M\}$, ranging over all such finite subsets of sections. Another is the single element cover $A^M \to M$. Such covers by free modules are good examples of those which arise in practice.

Lemma 5.2.1. The topology on A-mod is subcanonical.

Proof. Let $\{M_i \to M\}_I$ be a cover. We check by hand that

$$\bigoplus_{I\times I}(M_i \times_M M_j) \to \bigoplus_I M_i \to M \to 0 \tag{5.9}$$

is exact. The leftmost arrow is the difference of the two projections. Apply $\hbar_K$ to get the Čech Complex; this is exact precisely when $\hbar_K$ is a sheaf. Thus it suffices to show (5.9) is exact, and left exactness of $\hbar_K$ will give us the result. It is clear that the sequence is a complex and that $\bigoplus_I M_i \to M$ is surjective.

First assume the cover consists of a single element, $\{T \to M\}$. Let S be the kernel, fitting into a short exact sequence

$$0 \to S \to T \to M \to 0$$

Remark that $T \oplus S \to T \times_M T$ sending local sections $(t, s) \mapsto (t + s, t)$ is an isomorphism. The composite of this isomorphism with the map $T \times_M T \to T$ of (5.9) is projection onto S and inclusion. Then (5.9) takes the form

$$T \oplus S \to T \to M \to 0,$$

which is exact.

Now return to the general case of an arbitrary cover. In order to verify (5.9) is exact, we may freely localize in E. We argue that, after localization in E and addition of coboundaries, any local section of $\bigoplus_I M_i$ is equivalent to a section of a single M_0 (among the M_i). This reduces verifying exactness to the special case considered above.

After localization in E, all sections of $\bigoplus_I M_i$ are represented by finite sums of sections from various M_i. Choose $\sum_{k=1}^{n} m_{i_k} \in \Gamma(U, \bigoplus_I M_i)$ and consider the images $\overline{m}_{i_k}$ of m_{i_k} in M. Localize in E again and use the covering condition to lift the finite set of sections $\{\overline{m}_{i_k}\}_{k=1}^{n} \subseteq \Gamma(U, M)$ to some single M_0 among the M_i. Let m'_{i_k} be a chosen preimage in M_0 of $\overline{m}_{i_k}$.

Consider the section $\sum_{k=1}^{n} (m'_{i_k}, m_{i_k})$ of $\bigoplus_{I \times I} (M_i \times_M M_j)$. The second projection maps this section to the one we started with; the first yields a sum of elements of M_0. Therefore, our original section is equivalent to one in M_0 up to a coboundary.

□

Our next goal is to show the topology makes the fibered category $\underline{\mathrm{Ext}}^1_A(\_\_, K) \to A$-mod into a stack. Recall that extensions up to isomorphism form a group, with identity given by the trivial extension

$$\underline{0}: \quad 0 \to K \to K \oplus M \to M \to 0$$

The trivial extension is isomorphic to any extension whose epimorphism admits a section. Addition (the "Baer Sum") of two extensions

$$\xi: \quad 0 \to K \to M' \to M \to 0 \qquad \eta: \quad 0 \to K \to M'' \to M \to 0 \tag{5.10}$$

is defined by pulling back and pushing out the product of the two extensions along the maps in the diagram:

$$\begin{tikzcd} & & & M \arrow[d, "(id,id)"] & \\ 0 \arrow[r] & K \oplus K \arrow[r] \arrow[d, "id+id"] & M' \oplus M'' \arrow[r] & M \oplus M \arrow[r] & 0 \\ & K & & & \end{tikzcd}$$

In other words, the group law is defined by biadditivity and functoriality:

$$\mathrm{Ext}^1_A(M,K) \times \mathrm{Ext}^1_A(M,K) \to \mathrm{Ext}^1_A(M \oplus M, K \oplus K) \to \mathrm{Ext}^1_A(M,K)$$

We fix the notation ξ and η for the extensions above throughout this section.

Remark 5.2.2. We collect a few basic properties of the topology on A-mod.

- For $N, N' \in A\text{-mod}/M$, the presheaf $\underline{\mathrm{Hom}}_A(N, N')$ sending $P \mapsto \mathrm{Hom}_A(P \times_M N, P \times_M N')$ and the subpresheaf of isomorphisms $\underline{\mathrm{Isom}}_A(N, N')$ are both sheaves.
- Extensions ξ as in (5.10) are locally isomorphic to the trivial extension over A-mod.
- Given two families of maps $\{N_i \to M\}$ and $\{P_j \to M\}$, if the latter is covering and refines the former via maps $\{P_j \to N_{i_j}\}$ over M, then the former is also covering.

The first point follows formally from the subcanonicity of the topology. The second is shown by pullback along $M' \to M$ and the third follows from the definition of the topology.

Recall the trivial gerbe $B\hbar_K \to A$-mod whose sections over some M are $\hbar_K$-torsors: sheaves P on A-mod$/M$ which carry a free and transitive action of $\hbar_K|_M$ and locally admit sections. We often write $\hbar_{M'|M}$ for the sheaf some $M' \in A\text{-mod}/M$ represents to emphasize the structure map $M' \to M$, as opposed to the sheaf $\hbar_{M'}$ on A-mod.

Definition 5.2.3. The functor $\rho : \underline{\mathrm{Ext}}^1_A(\_\_, K) \to B\hbar_K$ between fibered categories sends an extension ξ to $\hbar_{M'|M}$. This sheaf becomes a $\hbar_K$-torsor via addition

$$K \times M' \to M' \times M' \xrightarrow{+} M'$$

and the Yoneda Embedding. Morphisms of extensions induce morphisms of representable sheaves which are $\hbar_K$-equivariant. By Remark 5.2.2, the sheaves $\hbar_{M'|M}$ are isomorphic to their structure group $\hbar_{K\oplus M} \simeq \hbar_K|_M$ after pullback along a cover in $A\text{-mod}/M$. The Yoneda Lemma verifies that ρ is fully faithful.

We will show ρ is an equivalence. Our construction relies on the free module functor

$$\lambda : E \to A\text{-mod}$$

sending $S \mapsto A^S$.

Lemma 5.2.4. The free module functor λ sends fiber products to covers. That is, the natural map

$$A^{S\times_R T} \to A^S \times_{A^R} A^T$$

is covering, for $S, R, T \in E$.

Proof. Since the family we wish to show is a cover consists of a single map, it suffices to show it is a cover in E instead of A-mod. Choose a section $\alpha \in \Gamma(U, A^S \times_{A^R} A^T)$; we wish to find a lift of α to $A^{S\times_R T}$ locally in E. Locally, we may assume $\alpha = (\sum x_k s_k, \sum y_k t_k)$ is a pair of finite sums with $x_k, y_k \in A(U)$, $s_k \in S(U)$, $t_k \in T(U)$ with the same image in $A^R(U)$.

Fix $r \in R(U)$ and suppose the s_k, t_k mapping to r are numbered $\{s_1, \cdots, s_n\}$, $\{t_1, \cdots, t_n\}$. In order for the two sums to have the same image, we must have

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k \in A(U)$$

Define z to be the value of either sum.

Consider the section β_r of $A^{S\times_R T}$ given by the sum of (s_i, t_j) with coefficients

$$\begin{cases} z & i = j = 1 \\ x_i + y_j & i = j \neq 1 \\ -y_i & j = 1 \neq i \\ -x_j & i = 1 \neq j \\ 0 & \text{otherwise} \end{cases}$$

Writing the coefficients as a matrix yields

$$\left(\begin{array}{cccc|c} z & -y_2 & \cdots & -y_n & t_1 \\ -x_2 & x_2 + y_2 & & 0 & t_2 \\ \vdots & & \ddots & & \vdots \\ -x_n & 0 & & x_n + y_n & t_n \\ \hline s_1 & s_2 & \cdots & s_n & \end{array}\right)$$

Adding up the rows and columns shows β_r projects to $\sum_{k=1}^{n} x_k s_k$ and $\sum_{k=1}^{n} y_k t_k$. Define

$$\beta = \sum_{r \in R(U)} \beta_r$$

Only finitely many of the terms of this sum are nonzero, and β indeed maps to α.

$\square$

Remark 5.2.5. The map $A^{S\times_R T} \to A^S \times_{A^R} A^T$ is *not* an isomorphism, in general. A counterexample is found already when $E = (Sets)$, $A = \mathbb{Q}$ and $R = \{r\}$.

Consider $S := \{x, y\}$ and $T := \{x', y'\}$ with their unique maps to R. Then $\mathbb{Q}^{S\times_R T} \to \mathbb{Q}^S \times_{\mathbb{Q}^R} \mathbb{Q}^T$ is surjective but not injective. For example, $(x, y) - (x, y') + (x', y') - (x', y)$ goes to zero. Hence the functor $S \mapsto A^S$ needn't commute with finite limits and is not left exact.

Applying Sym to the above counterexample shows the free algebra functor $S \mapsto A[S]$ isn't left exact either, contradicting a claim made in [71]. Forthcoming work will show the conclusions in [71] which rest on this erroneous claim remain true. For their proof, an analogue of Lemma 5.2.4 suffices.

Remark 5.2.6. Because the functor $\lambda : E \to A$-mod of Lemma 5.2.4 is not left exact, it doesn't induce a morphism of sites in the other direction. It is cocontinuous nonetheless, inducing a morphism of sites

$$E \to A\text{-mod}$$

The left exact left adjoint belonging to this morphism is precisely $F \mapsto (\lambda^* F)^{sh}$, the sheafification of precomposition by λ. We note that $\lambda^* F$ is already a sheaf.

For $\{S_i \to T\}$ a cover in E, the projections $A^{S_i \times_T S_j} \to A^{S_i}$ factor through the product $A^{S_i} \times_{A^T} A^{S_j}$. Taking sections over F, we get a sequence

$$F(A^T) \to \prod F(A^{S_i}) \rightrightarrows \prod F(A^{S_i} \times_{A^T} A^{S_j}) \hookrightarrow \prod F(A^{S_i \times_T S_j})$$

The last map is injective because the map $A^{S_i \times_T S_j} \to A^{S_i} \times_{A^T} A^{S_j}$ is covering. The sheaf condition for $\lambda^* F$ is that the diagram formed by the first arrow and the composites of the pair of arrows with the injection should be an equalizer. However, the sheaf condition on A-mod ensures that the diagram without the final injection is an equalizer. Postcomposing by an injection preserves such an equalizer diagram.

Proposition 5.2.7. The functor $\rho : \underline{\mathrm{Ext}}^1_A(\_\_, K) \to B\hbar_K$ of Definition 5.2.3 is an equivalence.

Proof. In the process of defining ρ, we remarked that it is fully faithful. It remains to show essential surjectivity.

For any $N \in A$-mod, write $j_N : A\text{-mod}/N \to A$-mod for the localization morphism of topoi. Write $\lambda : E \to A$-mod for the functor $S \mapsto A^S$ and λ^* for the induced functor on sheaves $F \mapsto F \circ \lambda$. Write α_N for the map $A \times N^{\times 2} \to N$ which is $(a, n, n') \mapsto a.(n + n')$ on sections.

Let P be an $\hbar_K|_M = \hbar_{K \oplus M}$-torsor on $A\text{-mod}/M$. Let $\{M_i \to M\}_I$ be a cover on which P is trivial. For $N \in A\text{-mod}/M$, define

$$L_N := \lambda^* j_{N!} P|_N$$

This is the sheafification of the functor $U \mapsto \coprod_{f \in \mathrm{Hom}_A(A^U, N)} P|_N(f)$.

Write $L_i := L_{M_i}$ and $L := L_M$ for brevity. There are maps $p_N : L_N \to N$ sending $P(f)$ to the section of N corresponding to f. If $N \to N' \in A\text{-mod}/M$, then $L_N = L_{N'} \times_{N'} N$ and p_N is the projection.

We will show $p_M : L \to M$ fits into an extension of modules which maps to P under ρ. First, we must augment the sheaf of sets L with an A-module structure.

Define $R \subseteq A\text{-mod}/M$ to be the full subcategory on those N whose structural morphism to M factors through some M_i (the sieve generated by the cover). We will produce an A-module structure on L_N for each $N \in R$ and then descend to L.

Remark that $\lambda^* j_{N!} \hbar_{N'|N} = N'$ functorially in N'. Choose an $\hbar_K$-isomorphism $f : P|_N \simeq \hbar_K|_N = \hbar_{K \oplus N}$. Give L_N the induced A-module structure from the K-isomorphism $\lambda^* j_{N!} f : L_N \simeq K \oplus N$. We claim this A-module structure is independent of the choice of f.

To that end, let $g : P|_N \simeq \hbar_{K \oplus N}$ be another $\hbar_K$-isomorphism. For the map $g \circ f^{-1}$ to be an $\hbar_K$-equivariant map of representable sheaves on $A\text{-mod}/N$, it must come from a map of A-modules. Since $g \circ f^{-1}$ comes from an A-module homomorphism, f and g endow $L_N(U)$ with the same A-module structure.

Since the A-module structure on each is well-defined, the equality of sheaves $L_N = L_{N'} \times_{N'} N$ is promoted to one of modules. In particular, the projection maps $L_N \to L_{N'}$ are each A-module maps.

We want to construct α_M using the cover. By the definition of the topology on A-mod, if $\{M_i \to M\}_I$ is a cover, then $\{M_i^{\times 2} \to M^{\times 2}\}_I$ is also a cover. It follows that the pullback $\{A \times L_i^{\times 2} \to A \times L^{\times 2}\}$ is covering in E.

Since the topology on E is subcanonical,

$$\hbar_L(A \times L^{\times 2}) \to \prod \hbar_L(A \times L_i^{\times 2}) \rightrightarrows \prod \hbar_L(A \times (L_i \times_L L_j)^{\times 2})$$

is an equalizer. The commutativity of the following diagram

$$\begin{tikzcd}
A \times (L_i \times_L L_j)^{\times 2} \arrow[r] \arrow[d, "\alpha_{L_i \times_L L_j}"] \arrow[dr, phantom, "\circ"] & A \times L_i^{\times 2} \arrow[r] \arrow[d, "\alpha_{L_i}"] & A \times L^{\times 2} \arrow[d, dashed] \\
L_i \times_L L_j \arrow[r] & L_i \arrow[r] & L
\end{tikzcd} \tag{5.11}$$

ensures the existence of the dashed arrow. Commutativity is the statement that the projections $L_i \times_L L_j = L_{M_i \times_M M_j} \to L_i$ are A-module maps.

The dashed arrow $A \times L^{\times 2} \dashrightarrow L$ *defines* addition and scalar multiplication for L. Since $\hbar_L$ is a sheaf, equality between two arrows to L may be checked after pulling back along a cover of M. This guarantees commutativity, associativity, etc.

The epimorphism $\bigoplus L_i \to \bigoplus M_i \to M$ factors through L, guaranteeing $L \to M$ to be epimorphic. The kernel is seen to be K by pulling back $L \to M$ along any $M_i \to M$.

It remains to show the extension

$$\xi : 0 \to K \to L \to M \to 0$$

represents P; that $\rho(\xi) \simeq P$. We build an isomorphism for any N in the sieve R and show it is independent of choices. Choose two $\hbar_K$-isomorphisms $f, g : P|_N \simeq \hbar_K|_N$. Apply $\hbar_{(\lambda^* j_{N!}\_\_)|N}$ to both and form the commutative diagram:

$$\begin{tikzcd}
 & \hbar_K|_N \arrow[dr, bend left, dash, "\sim f"] & \\
\hbar_{L_N|N} \arrow[ur, bend left, dash, "\sim f"] \arrow[rr, dashed] \arrow[dr, bend right, dash, "\sim g"'] & & P|_N \\
 & \hbar_K|_N \arrow[ur, bend right, dash, "\sim g"'] &
\end{tikzcd}$$

This verifies compatibility of the locally defined isomorphisms $\hbar_{L_N|N} \simeq P|_N$ and we obtain a global $\hbar_{L|M} \simeq P$. Hence ρ is essentially surjective.

$\square$

Remark 5.2.8. Lemma 5.2.7 implies that $\underline{\mathrm{Ext}}^1_A(\_\_, K) \to A$-mod is a stack, in fact a form of the trivial $\hbar_K$-gerbe. The proof checked descent by relying heavily on ρ.

In the same way, $\underline{\mathrm{Ext}}^1_{A'}(\_\_, K) \to A'$-mod is an $\hbar_K$-gerbe. However, $\pi_* \underline{\mathrm{Ext}}^1_{A'}(\_\_, K) \to A$-mod is a stack but no longer an $\hbar_K$-gerbe.

Diagram (5.4) defining $\underline{\mathrm{Def}}(\_\_, u, K)$ describes it as a strict fiber product of stacks over A-mod$/M$, so it is also a stack.

Remark 5.2.9. The topology was used in the proof only to the extent that, if $\{M_i \to M\}_I$ is a cover, then $\{M_i^{\times 2} \to M^{\times 2}\}_I$ is also a cover in E. We could vary the topology so that the Λ in the definition of the topology could only have at most two elements, and the proof would still work.

We speculate that allowing Λ to have at most three elements would suffice for $\underline{\mathrm{Ext}}^2_A(\_\_, K)$ to form a 2-gerbe, and consider this infinite hierarchy of topologies curious. In particular, descent for the dashed arrow in Diagram (5.11) seems to be possible by the same method for algebras for other operads and their modules.

We finish the section with a few more basic properties of the site A-mod. Define $\mathscr{P}$ as the presheaf on A-mod (resp. define a sheaf $\mathscr{P}_E$ on E) whose value on M is the set of submodules (resp. subsheaves) of M. Precomposing by the forgetful functor $A\text{-mod} \to E$, we regard $\mathscr{P}_E$ as a sheaf on A-mod and $\mathscr{P}$ as a subpresheaf.

Since E has a set of generators, $\mathscr{P}_E(M)$ and $\mathscr{P}(M)$ are indeed sets. Restriction maps are pullbacks of subobjects.

Lemma 5.2.10. The presheaf $\mathscr{P}$ on A-mod is a sheaf.

Proof. Let $\{M_i \to M\}_I$ be a cover, with submodules $N_i \subseteq M_i$. Write $M_{ij} := M_i \times_M M_j$. Suppose the pullbacks $N_i|_{M_{ij}} = N_j|_{M_{ij}} \subseteq M_{ij}$ are equal. We want to exhibit a submodule $N \subseteq M$ whose pullbacks to each M_i are precisely N_i.

Since $\mathscr{P}_E$ is a sheaf, the above descent data furnishes a subsheaf of sets $N \subseteq M$ on E; we must endow N with a submodule structure. We get a diagram as in (5.11) by replacing L by N, and the same argument produces the submodule structure.

$\square$

Corollary 5.2.11. The arrow category $q : \mathrm{Arr}(A\text{-mod}) \to A\text{-mod}$ [65, 3.15] is a stack, the functor sending an arrow to its codomain.

Proof. Isomorphisms form a sheaf for Arr(A-mod) because $\underline{\mathrm{Isom}}_A(N, N')$ is a sheaf.

Any arrow $N \to M \in A$-mod factors as $N \twoheadrightarrow P \hookrightarrow M$, an epimorphism composed with a monomorphism. Considering $\mathscr{P}$ as a fibered category, factor the functor q as

$$\mathrm{Arr}(A\text{-mod}) \to \mathscr{P} \to A\text{-mod}$$

The first arrow sends $N \to M$ to the image $P \subseteq M$. Since $\mathscr{P}$ is a sheaf, we need only show Arr(A-mod) $\to \mathscr{P}$ satisfies descent for the induced topology [63, 06NU, 09WX]. The corollary follows.

The induced topology refers to cartesian arrows over a cover in the base site. In other words, a cover in $\mathscr{P}$ is a cover $\{M_i \to M\}_I$ in A-mod together with a choice of subobject $N \subseteq M$ and its pullbacks to M_i.

Descent data for Arr(A-mod) here refers to a choice of epimorphism $M_i' \twoheadrightarrow N|_{M_i}$, isomorphisms between the pullbacks of M_i' and M_j' along M_{ij}'s two projections compatible with the epimorphisms, and compatibility of those isomorphisms on M_{ijk}. Remark that the kernel of each epimorphism must be the same, say K. This is precisely a descent datum for $\underline{\mathrm{Ext}}^1_A(\_\_, K) \to A$-mod, necessarily effective by Remark 5.2.8. We obtain an epimorphism $M' \twoheadrightarrow N$, also with kernel K, which pulls back to each $M_i' \twoheadrightarrow N|_{M_i}$ and verifies descent.

□

We have now developed enough technology to solve the deformation problem.

5.3 Cohomology on A-mod

We can quickly solve the deformation problem with an algebraic statement. This theorem yields an obstruction in degree-two cohomology. The rest of the section is devoted to the proof of Theorem 5.1.5.

Theorem 5.1.1. The fibered category $\underline{\mathrm{Def}}(\_\_, u, K) \to A\text{-mod}/M$ is a gerbe banded by $\mathscr{h}_K$.

Proof. We've seen already in Remark 5.2.8 that $\underline{\mathrm{Def}}(\_\_, u, K)$ is a stack.

Given an automorphism

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d, "\sim"] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K \arrow[r] & M' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

of a global section of $\underline{\mathrm{Def}}(\_\_, u, K) \to A\text{-mod}/M$, subtract the identity. The resulting morphism of chain complexes is zero on K and M, and the map $M' \to M'$ factors through M and K. Automorphisms of deformations are thereby in bijection with $\hbar_K(M)$.

It remains to show $\underline{\mathrm{Def}}(\_\_, u, K)$ is locally nonempty and two sections are locally isomorphic.

For both, we may assume $M = A^S$ for some sheaf of sets $S \in E$ by localizing in A-mod. Tensor the short exact sequence (5.1) by $\otimes A'^S$ to get a canonical deformation of $id_{J\otimes A^S}$:

$$\underline{\alpha} : \quad 0 \to J \otimes A^S \to A'^S \to A^S \to 0$$

To deform an arbitrary map $u : J \otimes M \to K$, simply localize in M and pushout $\underline{\alpha}$ by u. Observe $\theta(u \smile \underline{\alpha}) = u$.

Now we show sections of $\underline{\mathrm{Def}}(\_\_, u, K)$ are locally isomorphic. Choose an extension

$$\xi : \quad 0 \to K \to M' \to M \to 0$$

with $\theta(\xi) = u$. Since extensions are locally trivial, we may choose a cover $\{A^S \to M\}$ so that each S lifts to M'. We obtain a morphism of extensions

$$\begin{tikzcd} 0 \arrow[r] & J \otimes A^S \arrow[r] \arrow[d, dashed, "u"] & A'^S \arrow[r] \arrow[d, dashed] & A^S \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K \arrow[r] & M' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

witnessing that $\xi|_{A^S} \simeq u \smile \underline{\alpha}$. The induced map on the kernel is forced to be u.

$\square$

Now that we have a degree-two cohomological obstruction, we must work explicitly with the cohomology groups $H^p(A\text{-mod}/M, \hbar_K)$. The remainder of the section proves Theorem 5.1.5.

Lemma 5.3.1. A complex of A-modules

$$C_\bullet : \quad \cdots \to C_{p+1} \to C_p \to C_{p-1} \to \cdots$$

is exact if and only if, for any injective A-module K, the complex of homomorphisms into K is:

$$\hbar_K(C_\bullet) : \quad \cdots \leftarrow \mathrm{Hom}_A(C_{p+1}, K) \leftarrow \mathrm{Hom}_A(C_p, K) \leftarrow \mathrm{Hom}_A(C_{p-1}, K) \leftarrow \cdots$$

Proof. Standard.

□

Proposition 5.3.2. Given an injective A-module K, the higher cohomology of K all vanishes. That is, $H^p(A\text{-mod}/M, \hbar_K) = 0$ for $p \geq 1$.

Abstract properties of derived functors turn our main theorem into an immediate consequence of the previous proposition. We show how before providing the proof, the most complicated in this paper.

Theorem 5.1.5. The groups of p-extensions are all equivalent to the cohomology of $\hbar_K$ on the site $A\text{-mod}/M$: $\mathrm{Ext}^p_A(M, K) \simeq H^p(A\text{-mod}/M, \hbar_K)$.

Proof. Proposition 5.3.2 shows $H^p(A\text{-mod}/M, \hbar_K)$ is a universal δ-functor in K. Since $\mathrm{Ext}^p_A(M, K)$ is defined to be a universal δ-functor in K and $H^0(A\text{-mod}/M, \hbar_K) = \mathrm{Hom}_A(M, K)$, we get a unique isomorphism $H^p(A\text{-mod}/M, \hbar_K) \simeq \mathrm{Ext}^p_A(M, K)$ of δ-functors by [28, III.1.2.1].

□

Proof of Proposition 5.3.2. We will prove exactness of the Čech Complex in a series of lemmas to follow. We recall a well-known reduction to the vanishing of Čech Cohomology in the meantime ([63, 01EV], usually attributed to Cartan), as we will need the details in Lemma 5.3.6.

Assume inductively that $H^i(M, \hbar_K) = 0$ for $0 < i < p$ and any injective K. Proposition 5.2.7 yields the base case:

$$H^1(M, \hbar_K) = \mathrm{Ext}^1_A(M, K) = 0$$

for K injective. Consider a cover $\{M_i \to M\}_I$ and an injective A-module K.

The Čech Spectral Sequence [52, V.3.3] is:

$$H^j(M_\bullet, \underline{H}^k \hbar_K) \Rightarrow H^{j+k}(M, \hbar_K)$$

Here $\underline{H}^k \hbar_K$ is the presheaf $N \to H^k(N, \hbar_K)$ and is zero for $0 < k < p$ by inductive assumption. The only possibly nonzero terms on the diagonal $j+k=p$ are $H^p(M_\bullet, \underline{H}^0 \hbar_K)$ and $H^0(M_\bullet, \underline{H}^p \hbar_K)$. The filtration on degree p cohomology is expressed by the exact sequence

$$0 \to H^p(M_\bullet, \underline{H}^0 \hbar_K) \to H^p(M, \hbar_K) \to H^0(M_\bullet, \underline{H}^p \hbar_K) \to \cdots \tag{5.12}$$

The Čech spectral sequence and therefore this short exact sequence are natural with respect to refinement of the cover. The map on the right arises from the restriction of the presheaf $\underline{H}^p$.

In order to show $H^p(M, \hbar_K)$ vanishes, pick an element α. Then [63, 01FW] allows us to choose a cover $\{M_i \to M\}_I$ so that $\alpha|_{M_i} = 0$, so $\alpha \in H^p(M_\bullet, \underline{H}^0 \hbar_K)$ by the exactness of (5.12). It suffices therefore to show Čech Cohomology vanishes.

To that end, fix a total ordering on I. Write $M_{i_0 \cdots i_p} := M_{i_0} \times_M \cdots \times_M M_{i_p}$. Form the Čech Nerve, a simplicial object whose pth simplices are $\bigoplus M_{i_0 \cdots i_p}$, the sum ranging over ordered $(p+1)$-tuples of indices $i_0 \leq i_1 \leq \cdots \leq i_p$. The jth face map projects away from i_j, and we won't need the degeneracies. Take alternating sums of face maps to get the unnormalized chain complex:

$$\cdots \to \bigoplus M_{i_0 \cdots i_{p+1}} \to \bigoplus M_{i_0 \cdots i_p} \to \bigoplus M_{i_0 \cdots i_{p-1}} \to \cdots \to \bigoplus M_i \to M \to 0 \tag{5.13}$$

Čech Cohomology results from applying $\hbar_K$ to this sequence and taking cohomology. Complex (5.13) is exact precisely when Čech Cohomology vanishes by Lemma 5.3.1. We've reduced the proof to the following Lemma 5.3.3.

□

Lemma 5.3.3. Complex (5.13) is exact.

We prove this lemma after first handling a few special cases.

Lemma 5.3.4. Case 1 of Lemma 5.3.3: If $\{T \to M\}$ is a cover consisting of a single element (i.e. $I = \{*\}$), then the complex (5.13) is exact.

Proof. Remark in particular that $T \to M$ is an epimorphism. Write S for its kernel. We describe maps of sheaves of modules on sections $t \in \Gamma(U,T)$ and $s_i \in \Gamma(U,S)$ over some $U \in E$ for convenience.

The shearing map $T \oplus S^{\oplus p} \xrightarrow{\sim} T \times_M \cdots \times_M T$ sending $(t, s_1, \cdots s_p)$ to the partial sums $(t, t+s_1, t+s_1+s_2, \cdots, t+\Sigma s_i)$ is an isomorphism. Under this isomorphism, the simplicial module yielding (5.13) has p-simplices $T \oplus S^{\oplus p}$ and face maps given by

$$d_i(t, s_1, \cdots, s_p) := \begin{cases} (t, s_1, \cdots, s_{i-1}, s_i + s_{i+1}, s_{i+2}, \cdots, s_p) & \text{if } i \neq 0, p \\ (t+s_1, s_2, \cdots, s_p) & i = 0 \\ (t, s_1, \cdots, s_{p-1}) & i = p \end{cases}$$

The reader is invited to verify the simplicial axioms and verify this assignment yields an isomorphism of simplicial modules with $T \times_M \cdots \times_M T$.

Now we check by hand that the normalized chain complex associated to $T \oplus S^{\oplus p}$ is exact. The normalized chain complex in degree p is the intersection of all the kernels of the d_i, for $i \neq 0$ – its differentials are precisely d_0. Consider a local section $(t, s_1, \cdots, s_p)$ of the p-th degree of the normalized chain complex.

In order to be in the kernel of d_i for $i \neq 0, p$, $t = 0$, $s_i = -s_{i+1}$ and all $s_j = 0$ except $j = i, i+1$. Consider a few cases:

- $p \geq 3$: Varying i implies $t = s_1 = \cdots = s_p = 0$.
- $p = 2$: For d_2 to vanish we must also have $s_1 = 0$, and again $t = s_1 = s_2 = 0$.
- $p = 1$: For d_1 to vanish, $t = 0$.
- There are no requirements for $p = 0$.

The augmented normalized chain complex is thereby seen to be

$$0 \to S \to T \to M \to 0$$

with the natural maps. This is exact by assumption. The normalized chain complex is well known [29, I.1.3.3] to be quasi-isomorphic to the unnormalized chain complex (5.13).

□

Remark 5.3.5. We caution the reader that $\bigoplus$ and products over M do not commute, and hence (5.13) is not a series of fiber products of $\bigoplus M_i \to M$. That is, the problem does not reduce entirely to Lemma 5.3.4.

Lemma 5.3.6. Case 2 of Lemma 5.3.3: Suppose M has a finite set of global sections $\Lambda \subseteq \Gamma(E, M)$ so that the induced map from the free module on the constant sheaf $A^\Lambda \to M$ is covering. Then complex (5.13) is exact.

Proof. The covering condition allows us to localize in E so that Λ lifts to some M_i, say M_0. Then the hypothesized cover factors as $A^\Lambda \to M_0 \to M$ and $M_0 \to M$ is a cover.

The inclusion $M_0 \subseteq \{M_i\}_I$ is a refinement of covers. The short exact sequence (5.12) is contravariant under refinements:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^p(M_\bullet, \underline{H}^0 \hbar_K) & \longrightarrow & H^p(M, \hbar_K) & \longrightarrow & \cdots \\ & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & H^p(\{M_0\}, \underline{H}^0 \hbar_K) & \longrightarrow & H^p(M, \hbar_K) & \longrightarrow & \cdots \end{array}$$

By Lemma 5.3.4, the group $H^p(\{M_0\}, \underline{H}^0 \hbar_K)$ vanishes, but the injection $H^p(M_\bullet, \underline{H}^0 \hbar_K) \hookrightarrow H^p(M, \hbar_K)$ factors through this group; this implies $H^p(M_\bullet, \underline{H}^0 \hbar_K) = 0$. Equivalently, $\hbar_K(M_\bullet)$ is exact. Since K was any injective module, (5.13) is exact by Lemma 5.3.1.

□

We are finally ready to complete the proof of Lemma 5.3.3

Proof of Lemma 5.3.3. We often use the observation that, in order to verify exactness of the sequence of sheaves (5.13), we may freely localize in E.

To show Čech Cohomology vanishes, suppose some section $\beta = \sum m_j \in \Gamma(U, \bigoplus M_{i_0 \cdots i_p})$ maps to zero. Localize so that β is a global section. Define N as the image of the map $A^{\{m_j\}} \to M$ adjoint to the map from the constant sheaf $\{m_j\} \subseteq M_{i_0 \cdots i_p} \to M$.

Write $N_i := M_i \times_M N$ and form the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus N_{i_0\cdots i_{p+1}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \bigoplus N_{i_0\cdots i_p} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \bigoplus N_{i_0\cdots i_{p-1}} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner", very near start] & \cdots \arrow[r] & N \arrow[d, hook] \\
\cdots \arrow[r] & \bigoplus M_{i_0\cdots i_{p+1}} \arrow[r] & \bigoplus M_{i_0\cdots i_p} \arrow[r] & \bigoplus M_{i_0\cdots i_{p-1}} \arrow[r] & \cdots \arrow[r] & M
\end{tikzcd}$$

Each map $N_{i_0 \cdots i_q} \to M_{i_0 \cdots i_q}$ is the pullback of $N \hookrightarrow M$, so the vertical arrows are monomorphisms.

By construction, there is a preimage $\tilde{\beta} \in \Gamma(E, \bigoplus N_{i_0 \cdots i_p})$ of β. Moreover, $\tilde{\beta}$ maps to zero under the differential by injectivity of the vertical maps. The module N was defined as the image of $A^{\{m_j\}}$, so it falls under the jurisdiction of Lemma 5.3.6, and the top row is exact. Then $\tilde{\beta}$ is a boundary. This concludes the proof.

□

Armed with the isomorphism of Theorem 5.1.5, we now undertake its study.

5.4 Extensions and Cohomology

This section describes the isomorphisms of Theorem 5.1.5 in degrees $p = 1, 2$. We use this description to prove Theorem 5.1.6.

Lemma 5.4.1. The isomorphism $\mathrm{Ext}^p_A(M, K) \simeq H^p(A\text{-mod}/M, \hbar_K)$ of Theorem 5.1.5 in degree $p = 1$ is the restriction of the functor ρ of Definition 5.2.3 to isomorphism classes.

Proof. Given a short exact sequence

$$\gamma : 0 \to K \to N' \to N \to 0,$$

The diagram with horizontal arrows the boundary maps for the long exact sequences of Ext^p_A and H^p

$$\begin{tikzcd}
\mathrm{Hom}_A(M,N) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_A(M,K) \arrow[d, "\rho"] \\
H^0(A\text{-mod}/M, \hbar_N) \arrow[r] & H^1(A\text{-mod}/M, \hbar_K)
\end{tikzcd}$$

commutes. Indeed, a map $M \to N$ is sent via the boundary map for H^p to the $\hbar_K$-torsor of sections of $N' \to N$ (see the appendix):

$$P(T) := \left\{ \begin{tikzcd} & & N' \arrow[d] \\ T \arrow[r] \arrow[urr, dashed] & M \arrow[r] & N \end{tikzcd} \right\}$$

The boundary map for Ext^p_A sends the map $M \to N$ to the pullback $\gamma|_M$ of the extension along the map. Under ρ, this extension is sent to the $\hbar_K$-torsor represented by $M \times_N N'$ in A-mod$/M$. Sections of $M \times_N N' \to M$ and elements of P are in a canonical bijection which respects the action of $\hbar_K$.

Suppose now that N' is an injective A-module. Then the horizontal boundary maps are epimorphisms, and we see that ρ is the same as the isomorphism provided by Theorem 5.1.5.

□

We must now develop a considerable amount of technology to deal with the $p = 2$ case. Fix notation for two 2-extensions for the rest of the section:

$$\begin{array}{ll} \xi: & 0 \longrightarrow K \longrightarrow X \longrightarrow Y \longrightarrow M \longrightarrow 0 \\ \eta: & 0 \longrightarrow K \longrightarrow X' \longrightarrow Y' \longrightarrow M \longrightarrow 0 \end{array} \tag{5.14}$$

Write P for the module $\mathrm{coker}(K \to X) \simeq \ker(Y \to M)$ and P' likewise for $\mathrm{coker}(K \to X') \simeq \ker(Y' \to M)$.

Define the trivial 2-extension as

$$\underline{0}: \qquad 0 \longrightarrow K = K \xrightarrow{0} M = M \longrightarrow 0$$

Definition 5.4.2. A *butterfly* $\xi \simeq \eta$ between two 2-extensions is a completion of (5.15) or of the equivalent diagram (5.16). They form the isomorphisms in a 2-groupoid $\underline{\mathrm{Ext}}^2_A(M,K)$. The

2-isomorphisms are given by isomorphisms of the completions $Q \simeq Q'$ which commute with all of the structure maps.

We owe the concept to [5] and [27]. As in the latter, we'll be concerned only with the abelian case.

The fibered category $\underline{\mathrm{Isom}}(\xi, \eta) \to A\text{-mod}/M$ has fiber over $N \to M$ given by the category of butterflies between $\xi|_N$ and $\eta|_N$. Write $\mathrm{Split}(\xi)$ for $\underline{\mathrm{Isom}}(\xi, \underline{0})$.

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & & 0 & \\
0 \arrow[r] & K \arrow[r] \arrow[dd, equal] & X \arrow[rr] \arrow[dr, dashed] & & Y \arrow[r] \arrow[ur] & M \arrow[r] \arrow[dd, equal] & 0 \\
 & & & Q \arrow[ur, dashed] \arrow[dr, dashed] & & & \\
0 \arrow[r] & K \arrow[r] & X' \arrow[rr] \arrow[ur, dashed] & & Y' \arrow[r] \arrow[dr] & M \arrow[r] & 0 \\
 & 0 \arrow[ur] & & & & 0 &
\end{tikzcd} \tag{5.15}$$

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & 0 & & \\
 & 0 \arrow[dr] & & P \arrow[ur] \arrow[dr] & & 0 & \\
0 \arrow[dr] & & X \arrow[ur] \arrow[dr, dashed] & & Y \arrow[ur] \arrow[dr] & & 0 \\
 & K \arrow[ur] \arrow[dr] & & Q \arrow[ur, dashed] \arrow[dr, dashed] & & M \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & X' \arrow[ur, dashed] \arrow[dr] & & Y' \arrow[ur] \arrow[dr] & & 0 \\
 & 0 \arrow[ur] & & P' \arrow[ur] \arrow[dr] & & 0 & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{5.16}$$

In the first diagram, the NW-SE and SW-NE diagonals in the interior are short exact sequences. In the diamond-shaped diagram, each line is a short exact sequence. We will show $\underline{\mathrm{Isom}}(\xi, \eta)$ is an $\hbar_K$-banded gerbe even though we are particularly interested in $\mathrm{Split}(\xi)$.

The next lemma is similar to [5, Propositions 4.6.1 and 8.5.1]. However, the topology along which butterflies descend is different: we consider our topology on M, as opposed to the topology on the underlying site E on which all the modules are sheaves.

Lemma 5.4.3. The fibered category $\underline{\mathrm{Isom}}(\xi, \eta) \to A\text{-mod}/M$ is a stack.

Proof. All the data are local, so $\underline{\mathrm{Isom}}(\xi, \eta)$ is bound to be a stack. To show descent data are effective for example, one can locally

- Build an arrow $Q \to M$ ($\mathrm{Arr}(A\text{-mod}/M)$ is a stack – Corollary 5.2.11).
- Build factorizations $Q \to Y \to M$ and $Q \to Y' \to M$ ($\hbar_M$ is a sheaf).
- Check exactness of the short exact sequences

  $$0 \to X' \to Q \to Y \to 0$$

  and

  $$0 \to X \to Q \to Y' \to 0$$

  (the composite $\underline{\mathrm{Ext}}_A(\_\_, X') \to A\text{-mod}/Y \xrightarrow{j_!} A\text{-mod}/M$ is a stack, if j is the localization morphism of topoi).
- Check commutativity of the North, West, and South diamonds in Diagram (5.16) ($\underline{\mathrm{Hom}}_A(X, Y)$ is a sheaf on $A\text{-mod}/M$).

$\square$

Lemma 5.4.4. The stack $\underline{\mathrm{Isom}}(\xi, \eta) \to A\text{-mod}/M$ is a gerbe banded by $\hbar_K$.

Proof. This lemma is deduced from general principles about Picard Stacks and butterflies. Since we may localize to split our butterflies, we get local connectivity: $\pi_0 \underline{\mathrm{Isom}}(\xi, \eta) = *$. For a morphism $f : A \to B$ between strict Picard Stacks, the isomorphism $\underline{\mathrm{Aut}}(f) \simeq \pi_1(B)$ identifies the band of the gerbe as $\hbar_K$. We present a hands-on proof for the reader unfamiliar with these generalities.

Let a map $M \to K$ act on a butterfly as the maps $Y \to M \to K \to X'$ and $Y' \to M \to K \to X$ compatibly act on the two extensions in the above product.

Consider an automorphism of a butterfly (5.16). Subtracting the identity yields a map between the entire diagram which is zero except for a map $Q \to Q$. Each such map must factor uniquely as $Q \to M \to K \to Q$. This shows that 2-isomorphisms between two fixed butterflies in $\underline{\mathrm{Isom}}(\xi, \eta)$ are a pseudo-torsor under $\hbar_K$.

We show local existence. Localizing in M, we assume $Y \simeq M \oplus P$ and $Y' \simeq M \oplus P'$ are split extensions. Define $Q := (X' \oplus_K X) \oplus M$, with butterfly diagram:

$$\begin{tikzcd}
 & & P \arrow[dr] & & \\
 & X \arrow[ur] \arrow[dr] & & M \oplus P \arrow[dr] & \\
K \arrow[ur] \arrow[dr] & & Q \arrow[ur] \arrow[dr] & & M \\
 & X' \arrow[ur] \arrow[dr] & & M \oplus P' \arrow[ur] & \\
 & & P' \arrow[ur] & &
\end{tikzcd} \tag{5.17}$$

In order to show butterflies are pairwise locally isomorphic, pick an arbitrary butterfly Q' filling in the above diagram. Localizing in M sufficiently, Q' splits as $(X \oplus_K X') \oplus M$; we may choose an isomorphism of Q' with the above Q compatible with all the structure maps.

We leave the verification that all relevant composites in (5.17) are short exact sequences and that the diagrams formed by our map of butterflies commute to the dedicated reader.

□

We can finally describe the isomorphism of Theorem 5.1.5 in the case $p = 2$.

Proposition 5.4.5. The map Split : $\mathrm{Ext}^2_A(M, K) \simeq H^2(A\text{-mod}/M, \hbar_K)$ furnished by Theorem 5.1.5 sends a 2-extension to its $\hbar_K$-gerbe of splittings.

Proof. Write

$$m: \quad 0 \to P \to Y \to M \to 0$$

$$\gamma: \quad 0 \to K \to X \to P \to 0$$

so that $\xi = \gamma \smile m$

Consider the long exact sequence in Ext^p and H^p coming from γ. The isomorphism of Theorem 5.1.5 is one of universal δ-functors, so we get a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ext}^1_A(M,X) \arrow[r] \arrow[d, "\sim"] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}^1_A(M,P) \arrow[r, "\gamma\smile"] \arrow[d, "\sim"] \arrow[dr, phantom, "\circ"] & \mathrm{Ext}^2_A(M,K) \arrow[d, "\sim"] \\
\cdots \arrow[r] & H^1(M, \hbar_X) \arrow[r] & H^1(M, \hbar_P) \arrow[r] & H^2(M, \hbar_K)
\end{tikzcd}$$

The extension m maps to the 2-extension ξ under the boundary map $\gamma \smile$ by definition. To see what ξ maps to in H^2, send m around the bottom corner of the square. The boundary map on cohomology sends the torsor $\hbar_Y$ associated to m to its $\hbar_K$-gerbe of lifts to an $\hbar_X$-torsor. By commutativity of the left square, this is equivalent to the $\hbar_K$-gerbe of lifts of the extension m to an extension by X. As depicted in the rearranged butterfly diagram below, this gerbe is identical to $\mathrm{Split}(\xi)$.

$$\begin{tikzcd}
 & & P \arrow[dr] & & \\
 & X \arrow[ur] \arrow[dr, dashed] & & Y \arrow[dr] & \\
K \arrow[ur] \arrow[dr, equal] & & Q \arrow[ur, dashed] \arrow[dr, dashed] & & M \\
 & K \arrow[ur, dashed] \arrow[dr] & & M \arrow[ur, equal] & \\
 & & 0 \arrow[ur] & &
\end{tikzcd} \tag{5.18}$$

$\square$

The final ingredient in Theorem 5.1.6 is the map

$$\mathrm{Hom}_A(J \otimes M, K) \overset{\smile\omega}{\to} \mathrm{Ext}^2_A(M,K)$$

of Diagram (5.5) and (5.6). This homomorphism sends $u : J \otimes M \to K$ to its pushout $u \smile \omega$ along a fixed 2-extension ω.

In order to construct ω, take a flat A'-module mapping surjectively onto M with kernel L:

$$0 \to L \to H \to M \to 0.$$

When we tensor with A, we obtain

$$\omega : \quad 0 \to J \otimes M \to \overline{L} \to \overline{H} \to M \to 0.$$

We write $\overline{L}$ for $L \otimes A = L/JL$. Since ω computes $\mathscr{T}_0^{A'}(A, M)$ and $\mathscr{T}_1^{A'}(A, M)$, it is the canonical obstruction $\omega(A', M)$ in Illusie's work by [29, IV.3.1.9].

We must check ω is well-defined up to isomorphism. Given two flat surjections onto M, we can always choose a third surjecting onto both (e.g., the direct sum of a cover by free A'-modules trivializing both extensions). We may assume there is a map between the two flat resolutions:

$$\begin{tikzcd} 0 \arrow[r] & L' \arrow[r] \arrow[d] & H' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r] & H \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

In this case, simply tensor the whole diagram by A to get a map of complexes between the two definitions of ω. A morphism between 2-extensions as chain complexes induces a butterfly as in the appendix.

Hence ω is sufficiently well-defined to define a morphism to the group of connected components $\mathrm{Ext}^2_A(M, K)$, even though there's no canonical complex-level representative. The reader is free to fix one representative ω and transpose to a given one via the above.

Theorem 5.1.6. The diagram

$$\begin{tikzcd} \mathrm{Hom}_A(J \otimes M, K) \arrow[r, "\smile\omega"] \arrow[dr, "\underline{\mathrm{Def}}"'] & \mathrm{Ext}^2_A(M, K) \arrow[d, "\mathrm{Split}"] \\ & H^2(A\text{-mod}/M, \hbar_K) \end{tikzcd} \tag{5.19}$$

anti-commutes.

Fix $u \in \mathrm{Hom}_A(J \otimes M, K)$ and continue to write $\overline{L} := L \otimes A$. Our proof consists of two lemmas: one exhibits a functor $\beta : \mathrm{Split}(u \smile \omega) \to \underline{\mathrm{Def}}(\_\_, u, K)$ over A-mod$/M$, and the other shows it is an $\hbar_K$-anti-equivalence.

Lemma 5.4.6. There is a natural functor $\beta : \mathrm{Split}(u \smile \omega) \to \underline{\mathrm{Def}}(\_\_, u, K)$ over A-mod$/M$.

Proof. Given a splitting:

$$\begin{tikzcd}
0 \arrow[r] & J \otimes M \arrow[r] \arrow[dd, "u"] & \overline{L} \arrow[rr] \arrow[dr] & & \overline{H} \arrow[r] & M \arrow[r] \arrow[dd, equal] & 0 \\
& & & Q \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & K \arrow[r, equal] & K \arrow[rr, "0"] \arrow[ur] & & M \arrow[r, equal] & M \arrow[r] & 0
\end{tikzcd} \tag{5.20}$$

of $u \smile \omega$, consider the pushout

$$\begin{tikzcd}
\eta : \arrow[dd] & 0 \arrow[r] & L \arrow[r] \arrow[d] & H \arrow[r] \arrow[dd] & M \arrow[r] \arrow[dd, equal] & 0 \\
& & \overline{L} \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & & \\
\zeta : & 0 \arrow[r] & Q \arrow[r] & H \oplus_L Q \arrow[r, "h"] & M \arrow[r] & 0
\end{tikzcd}$$

Distinguish between three natural maps $H \oplus_L Q \to M$:

- $H \oplus_L Q \xrightarrow{h} M$ is the structure map $H \to M$ and zero on Q.
- $H \oplus_L Q \xrightarrow{q} M$ is the structure map $Q \to M$ and zero on H.
- $H \oplus_L Q \xrightarrow{h+q} M$ is the sum of the two maps above, given by both structure maps. It factors through $\overline{H}$.

Define an extension ξ by taking the kernel

$$\begin{tikzcd}
\xi : \arrow[d] & 0 \arrow[r] & K \arrow[r, dashed] \arrow[d] & M' \arrow[r, dashed, "g"] \arrow[d, dashed] & M \arrow[r] \arrow[d, equal] & 0 \\
\zeta : & 0 \arrow[r] & Q \arrow[r] \arrow[d] & H \oplus_L Q \arrow[r, "h"] \arrow[d, "h+q"] & M \arrow[r] & 0 \\
& 0 \arrow[r] & \overline{H} \arrow[r, equal] & \overline{H} \arrow[r] & 0 &
\end{tikzcd} \tag{5.21}$$

To show $\theta(\xi) = u$, tensor the diagram $\eta \to \zeta \leftarrow \xi$ by $\otimes_{A'} A$. We get a diagram containing

$$\begin{tikzcd}
\eta \otimes A: \arrow[d] & 0 \arrow[r] & J \otimes M \arrow[r] \arrow[d, equal] & \overline{L} \arrow[r] \arrow[d] & \cdots \\
\zeta \otimes A: & \cdots \arrow[r] & J \otimes M \arrow[r] \arrow[d, equal] & Q \arrow[r] & \cdots \\
\xi \otimes A: \arrow[u] & \cdots \arrow[r] & J \otimes M \arrow[r, "\theta(\xi)"] & K \arrow[r] \arrow[u, hook] & \cdots
\end{tikzcd}$$

The left pentagon of the original butterfly verifies that the map $J \otimes M \to \overline{L} \to Q$ factors as $J \otimes M \xrightarrow{u} K \hookrightarrow Q$. Combine this with the above diagram into

$$\begin{tikzcd}
& J \otimes M \arrow[dl, "\theta(\xi)"'] \arrow[d] \arrow[dr, "u"] & \\
K \arrow[dr] & \overline{L} \arrow[d] & K \arrow[dl] \\
& Q &
\end{tikzcd}$$

Hence $J \otimes M \xrightarrow{u} K \to Q$ and $J \otimes M \xrightarrow{\theta(\xi)} K \to Q$ are the same. Since $K \hookrightarrow Q$ is a monomorphism, this confirms $u = \theta(\xi)$.

An isomorphism $Q \simeq Q'$ of butterflies induces a unique isomorphism $H \oplus_L Q \simeq H \oplus_L Q'$ fixing H and L. These isomorphisms are compatible with a functorial isomorphism of the whole diagram (5.21) inducing the identity on K, M, and $\overline{H}$, whence a unique isomorphism on kernels $M' \simeq M''$. Let this be the action of β on arrows.

□

Lemma 5.4.7. The functor β of Lemma 5.4.6 is an anti-equivalence.

Proof. We continue to use terminology from the proof of Lemma 5.4.6.

A morphism of gerbes which is banded by an isomorphism is an equivalence by [22, IV.2.2.7]. We claim that β is banded not by the identity, but by $-id_K$.

A map $M \xrightarrow{\varphi} K$ acts on a butterfly (5.20) by adding the map $Q \to \overline{H} \to M \xrightarrow{\varphi} K \to Q$ to the identity on Q. Then the induced automorphism of $H \oplus_L Q$ is obtained by adding the identity to

$$H \oplus_L Q \xrightarrow{q} M \xrightarrow{\varphi} K \to H \oplus_L Q \tag{5.22}$$

We claim “a,” “b,” and “c” in the following solid diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[dr, "-g"'] \arrow[ddd, dashed, "z"'] \arrow[r, phantom, shift right=2ex, "a"] & H \oplus_L Q \arrow[r] \arrow[d, "q"] & \overline{H} \arrow[r] \arrow[ddd, "0"] & 0 \\
& & M \arrow[d, "\varphi"] \arrow[r, phantom, shift right=3ex, "c"] & & \\
& & K \arrow[d] \arrow[dl] & & \\
0 \arrow[r] & M' \arrow[r] \arrow[r, phantom, shift left=2ex, "b"] & H \oplus_L Q \arrow[r] & \overline{H} \arrow[r] & 0
\end{tikzcd} \tag{5.23}$$

The map $-g$ is the additive inverse of $g : M' \to M$ defined in (5.21), and z is the composite.

Diagram (5.21) witnesses the commutativity of rectangle “c” and triangle “b.” The same diagram also asserts $M' \to H \oplus_L Q \overset{h+q}{\to} \overline{H} \to M$ is zero; equivalently, that

$$-g : M' \to H \oplus_L Q \overset{-h}{\to} M$$

and

$$M' \to H \oplus_L Q \overset{q}{\to} M$$

are equal. This confirms commutativity of triangle “a.”

Diagram (5.23) defines a morphism of complexes; add the identity morphism to obtain a morphism of complexes given by

- id on $\overline{H}$.
- id+(5.22) on $H \oplus_L Q$.
- $id + z$ on M'.

By unwinding the definition of β on arrows, the action of φ on Q is sent by β to the automorphism of ξ which is $id + z$ on M' and the identity on M and K. This is precisely the action of $-\varphi$ on ξ.

$\square$

5.5 Illusie's Exact Sequence

This section describes Illusie's Exact Sequence

$$0 \to \operatorname{Ext}^1_A(M,K) \to \operatorname{Ext}^1_{A'}(M,K) \xrightarrow{\theta} \operatorname{Hom}_A(J \otimes M, K) \xrightarrow{\smile \omega} \operatorname{Ext}^2_A(M,K) \tag{5.6}$$

and proves Theorem 5.1.8.

We need naturality to construct the comparison diagram (5.8).

Lemma 5.5.1. The maps in (5.6) are all natural in M and K.

Proof. Naturality of the first arrow in (5.6) is clear.

Consider the pushout and pullback of an extension $\xi \in \operatorname{Ext}^1_{A'}(M,K)$ along maps $K \to L$ and $N \to M$.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & M' \times_M N \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & M' \oplus_K L \arrow[r] \arrow[ul, phantom, very near start, "\lrcorner"] & M \arrow[r] & 0
\end{tikzcd}$$

Tensoring $\_\_ \otimes A$, we get

$$\begin{tikzcd}
J \otimes N \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, equal] & \cdots \\
J \otimes M \arrow[r, "\theta(\xi)"] \arrow[d, equal] & K \arrow[r] \arrow[d] & \cdots \\
J \otimes M \arrow[r] & L \arrow[r] & \cdots
\end{tikzcd}$$

Commutativity of this diagram implies that $\theta(\xi|_N) = \theta(\xi) \circ (J \otimes N \to J \otimes M)$ and $\theta((K \to L) \smile \xi) = (K \to L) \circ \theta(\xi)$. Conclude the arrow θ in Diagram (5.6) is natural.

If $\theta(\xi) = 0$, or equivalently if M' is an A-module, then $\theta(\xi|_N) = \theta((K \to L) \smile \xi) = 0$ and $M' \times_M N$ and $M' \oplus_K L$ are both A-modules. The inclusion $\operatorname{Ext}^1_A(M,K) \subseteq \operatorname{Ext}^1_{A'}(M,K)$ beginning the sequence is natural.

The associativity of pushing out and pulling back 2-extensions furnishes the naturality of the last arrow, $\smile \omega$.

$\square$

Remark 5.5.2. Let $A+\epsilon M$ be the trivial squarezero algebra extension of A by M: the A-module $A \oplus M$ endowed with multiplication given by A's action and M squaring to zero. It may be graded by placing M in degree 1, A in degree 0.

Illusie defined the exact sequence (5.6) using the first graded piece of the cotangent complex transitivity triangle $\mathbb{L}^{\mathrm{gr}}_{A+\epsilon M/A/A'}$. The compatibility of that approach with this more direct one was verified already by Illusie as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_A(M,K) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_A(\mathbb{L}^{\mathrm{gr}}_{A+\epsilon M/A'},K) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^1_A(\mathbb{L}_{A/A'}\otimes_A M,K) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^2_A(M,K) \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Ext}^1_A(M,K) \arrow[r] & \mathrm{Ext}^1_{A'}(M,K) \arrow[r] & \mathrm{Hom}_A(J\otimes M,K) \arrow[r, "\smile\omega"] & \mathrm{Ext}^2_A(M,K)
\end{tikzcd}$$

The commutativity of the leftmost square was observed immediately before Proposition 3.1.5 [29, pg. 248], and the middle square is equivalent to diagram (3.1.3) on the previous page.

In the rightmost square, we are cupping with ω. Its two middle terms are $\tau_{[-1}(M \overset{L}{\otimes} A)$ by construction. By the naturality of both sequences in K (the top is obtained by applying the functor $\mathrm{Ext}^1_A(\_\_,K)$ to the transitivity triangle), it suffices to reduce to the case where $K := J\otimes M$ and $u = id_{J\otimes M}$. Since ω is the "canonical obstruction" by IV.3.1.9 of [29, pg. 250], the square commutes. (For us, the ground ring Υ is A'.)

It remains to show the following diagram commutes, and to describe the dashed arrow.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_A(M,K) \arrow[r] \arrow[d, "\sim"] & \mathrm{Ext}^1_{A'}(M,K) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_A(J\otimes M,K) \arrow[r] \arrow[d, dashed, "\sim"] & \mathrm{Ext}^2_A(M,K) \arrow[d, "\mathrm{Split}"] \\
0 \arrow[r] & H^1(A/M,\hbar_K) \arrow[r] & H^1(A'/M,\hbar_K) \arrow[r] & H^0(A/M,R^1\pi_*\hbar_K) \arrow[r] & H^2(A/M,\hbar_K)
\end{tikzcd} \tag{5.8}$$

Lemma 5.5.3. The solid arrows in Diagram (5.8) are natural in the A-modules M and K.

Proof. The whole solid diagram is natural in K by Lemma 5.5.1 and Theorem 5.1.5. The same is true of M, except possibly the vertical isomorphisms. Choose $N \to M$ in A-mod.

Given an extension

$$0 \to K \to M' \to M \to 0$$

representing the torsor $\hbar_K \circlearrowright \hbar_{M'|M}$, the pulled back extension represents $\hbar_{M' \times_M N|N} \simeq j^* \hbar_{M'}$. The naturality in H^1 comes from pullback of torsors. This trick shows the naturality of the first two vertical isomorphisms.

Now examine a 2-extension

$$\xi : \quad 0 \to K \to X \to Y \to M \to 0$$

A section of $\mathrm{Split}(\xi|_N)$ over $T \to N$ is simply a section of $\mathrm{Split}(\xi)$ over $T \to N \to M$, so we have a (strict) 2-fiber product

$$\begin{tikzcd} \mathrm{Split}(\xi|_N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Split}(\xi) \arrow[d] \\ A\text{-mod}/N \arrow[r, "j_!"] & A\text{-mod}/M \end{tikzcd}$$

Accordingly, $\mathrm{Split}(\xi|_N) \simeq \mathrm{Split}(\xi)|_N$, where the first pullback belongs to Ext^2_A and the second to H^2. Hence Split is natural in M.

$\square$

Lemma 5.5.4. The dashed arrow in Diagram (5.8) exists and is an isomorphism.

Proof. Lemma 5.5.3 shows that the diagram is natural in $M \in A$-mod. Sheafify to obtain

$$\begin{tikzcd} \mathrm{Ext}^1_{A'}(\_\_, K)^{sh} \arrow[r] \arrow[d] & \mathrm{Hom}_A(J \otimes \_\_, K) \arrow[d, dashed] \\ H^1(A'/\_\_, \hbar_K)^{sh} \arrow[r] & R^1\pi_* \hbar_K \end{tikzcd}$$

All of the arrows are isomorphisms, since the outer terms go to zero. Define the sought-after isomorphism as the composition of the other three.

$\square$

Remark 5.5.5. This last argument describes the maps $\mathrm{Ext}^1_{A'}(M,K) \xrightarrow{\theta} \mathrm{Hom}_A(J \otimes M, K)$ and $H^1(A'/M, \hbar_K) \to H^0(A/M, R^1\pi_*\hbar_K)$ as sheafification.

Theorem 5.1.8. Diagram (5.8) commutes. Scilicet, Illusie's exact sequence (5.6) and the 5-term exact sequence from the Grothendieck-Leray spectral sequence are isomorphic.

Proof. Take an extension $\xi : 0 \to K \to M' \to M \to 0$ of A-modules. Whether one first considers K, M', and M as A'-modules and then forms the torsor $\hbar_K \circlearrowleft \hbar_{M'}$ on A'-mod$/M$ or forms the torsor $\hbar_K \circlearrowleft \hbar_{M'}$ on A-mod$/M$ and then applies π^* makes no difference: $\pi^*\hbar_K = \hbar_K$ functorially. The left square commutes.

To verify that the two maps to a sheaf $\mathrm{Ext}^1_{A'}(\_\_, K) \rightrightarrows R^1\pi_*\hbar_K$ agree, we may sheafify. Then the arrow $\mathrm{Hom}_A(J \otimes \_\_, K) \to R^1\pi_*\hbar_K$ was defined to make this square commute.

The rightmost square remains. We show

$$\begin{tikzcd}
\mathrm{Hom}_A(J \otimes M, K) \arrow[r] \arrow[d] \arrow[dr, "-\underline{\mathrm{Def}}"] & \mathrm{Ext}^2_A(M,K) \arrow[d, "\mathrm{Split}"] \\
H^0(A/M, R^1\pi_*\hbar_K) \arrow[r] & H^2(A/M, \hbar_K)
\end{tikzcd}$$

commutes. The upper right triangle commutes by Theorem 5.1.6.

Under the lower left triangle, consider the image of $u \in \mathrm{Hom}_A(J \otimes M, K)$ under the two maps. The bottom horizontal arrow sends a global section to the inverse of (the class of) its gerbe of lifts to an $\hbar_K$-torsor in A'-mod$/M$ by conventions specified in the appendix.

By the commutativity of the leftmost square of diagram (5.8), we see that this gerbe is equivalent to the gerbe of lifts of the corresponding map $\mathrm{Hom}_A(J \otimes M, K)$ to an A'-module extension. This was the definition of $\underline{\mathrm{Def}}(\_\_, u, K)$.

$\square$

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