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RADIATION OF A SURFACE WAVE FROM A CURVATURE DISCONTINUITY IN AN IMPEDANCE SURFACE
Part I. CONVEX BEND

by

Edward F. Kuester and David C. Chang

Electromagnetics Laboratory
Department of Electrical Engineering
University of Colorado, Boulder, Colorado 80309

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RADIATION OF A SURFACE WAVE FROM A CURVATURE DISCONTINUITY IN AN IMPEDANCE SURFACE. PART I. CONVEX BEND

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Abstract

An asymptotic solution for the problem of radiation of a surface wave incident on a convex curvature discontinuity in an impedance surface is found. The method employs a local approximate Green's function for a section of curved impedance surface which possesses the same formal structure as that of the straight surface. The radiation pattern thus obtained agrees with those of previous analyses, without the need to assume a second straight section terminating the curved section or to go to an extremely far field to extract it.

I. Introduction

Open structures such as optical dielectric waveguides are limited in usefulness by an inherent tendency to radiate at irregularities (bends of the guide axis, for example)[1]. An important problem in the design of guiding systems is thus to determine the radiation losses one may expect from such irregularities. The surface wave itself is known to radiate as it travels around a curved section of guide, and the solution for this "continuous" radiation has been carried out for a general surface waveguide in [2]. Radiation also occurs at the junction between straight and curved sections of guide (more generally, between sections of different curvature) which is of a more discrete, "scattering" nature. This type of radiation has been less well studied, with attention usually focused on very simple structures [3-13]. Of these, all except [3-5] are either completely formal or do not give the pattern of the radiated field.

As a first step towards the development of a general theory of scattering at a curvature discontinuity in an open waveguide (which will be complementary to the theory of continuous radiation given in [2]) we develop here the solution for the simple case when the guiding structure is an impedance surface. In subsequent works the theory will be extended first to more general two-dimensional structures, and then to arbitrary three-dimensional guides. In many ways, this represents a considerable extension of the various geometrical and asymptotic theories of surface waves, which heretofore have been developed only for the case when the axis of the guiding structure is smooth [14-19]. The present work will deal with the case of a convex bend.

II. Statement of the Problem

Consider the junction between straight and curved sections of a surface with impedance Z_s as shown in Fig. 1. To avoid the necessity of dealing with a second transition in the surface, the straight section is assumed to extend from $y = -\infty$ to $y = 0$ while the curved section ranges between $\phi = 0$ and $\phi = \infty$, in a kind of "Riemann surface" of which only a finite range of angles ϕ lies in the physical or Euclidean plane [20]. This artifice should not trouble the reader, since it is similar in intent to the typical assumption of infinite extent of a straight waveguide in an excitation problem, and assures that none of the scattered fields are re-reflected from some obstacle further along the guide. We consider TE waves, so that

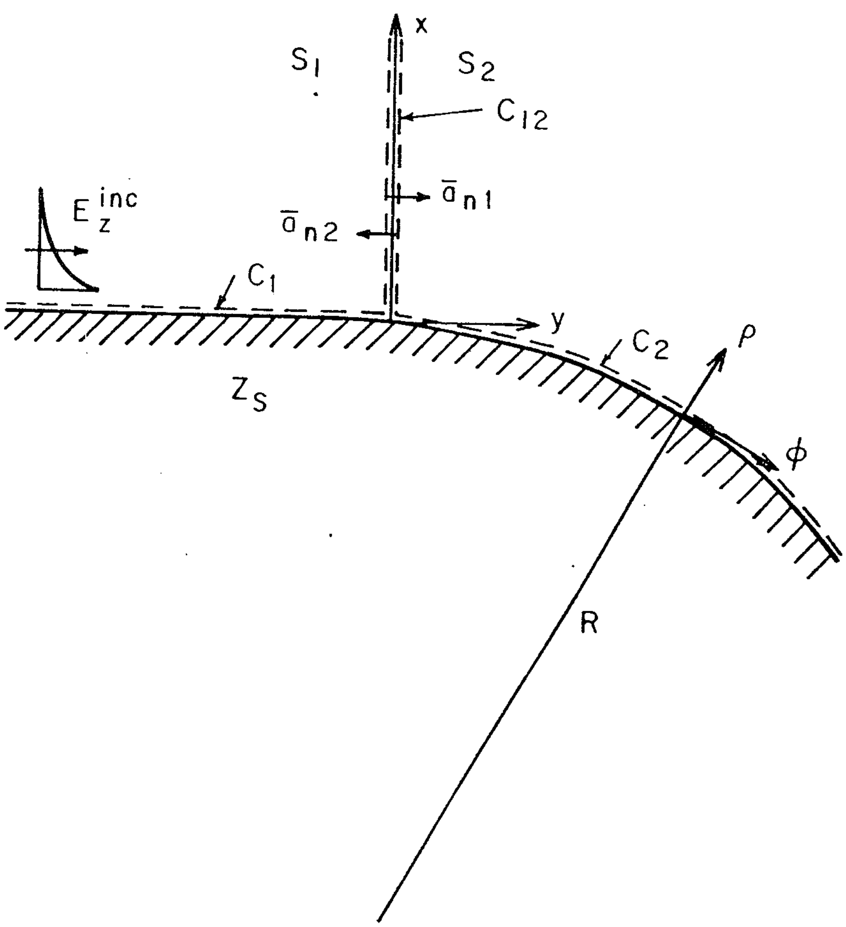

Fig. 1

Junction of straight and curved sections
of impedance surface.

$$(\nabla_t^2 + k^2)E_z = 0 \tag{1}$$

above the surface for $k^2 = \omega^2\mu_o\varepsilon_o$, and

$$\frac{\partial E_z}{\partial n} + k\gamma_o E_z = 0 \quad \text{on} \quad \left\{\begin{array}{l} x = 0,\ y \leqslant 0;\ n = x \\ \rho = R,\ \phi \geqslant 0;\ n = \rho \end{array}\right\} \tag{2}$$

on the surface, where R is the radius of curvature of the curved section, $\gamma_o = -i\, Z_o/Z_s$ is the normalized surface susceptance, $Z_o = (\mu_o/\varepsilon_o)^{\frac{1}{2}}$ is the characteristic impedance of free space, and n is the outward normal from the surface. With a time dependence $\exp(i\omega t)$, we impose additionally the radiation condition

$$r^{\frac{1}{2}}\left[\frac{\partial E_z}{\partial r} + ikE_z\right] \rightarrow 0$$

as $r = (x^2 + y^2)^{\frac{1}{2}} \rightarrow \infty$ above the surface.

The straight surface is known [4, p. 24] to support a single surface wave mode

$$\begin{aligned} E_z^{inc} &= \Psi_{so}(x)e^{-ik\nu_o y} \quad ; \quad \nu_o = (1+\gamma_o^2)^{\frac{1}{2}} \\ \Psi_{so}(x) &= e^{-k\gamma_o x} \end{aligned} \tag{3}$$

which we assume to be incident on the junction from the left. The subscript "s" here indicates the modal function for the surface-wave and "o" indicates the unperturbed straight waveguide section. With a surface wave mode of this kind as our excitation or incident field, we shall formulate the problem as an integral equation over the aperture, defined by $y = 0$, $0 \leqslant x < \infty$.

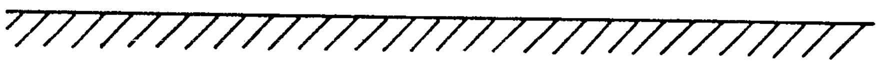

Fig. 2 Infinitely extended straight impedance plane.

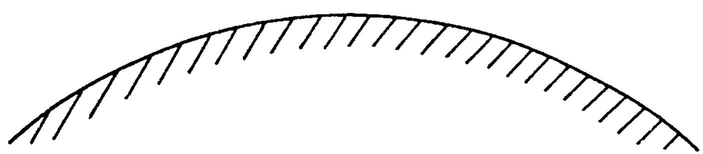

Fig. 3 Infinitely extended curved impedance surface.

First, we require the appropriate Green's functions. For region S_1 of Fig. 1, we use the Green's function G_1 for the infinitely extended straight impedance surface (Fig. 2) which satisfies the impedance boundary condition (2) at $x=0$ and the outgoing radiation condition as $r \to \infty$. Similarly, for region S_2, we use the Green's function G_2 for the infinitely extended $(-\infty < \phi < \infty)$ curved impedance surface (Fig. 3) which satisfies the impedance boundary condition at $\rho = R$ and the outgoing radiation condition as $(\rho^2 + R^2\phi^2)^{\frac{1}{2}} \to \infty$. Both G_1 and G_2 satisfy the wave equation

$$(\nabla_t^2 + k^2)G_{1,2}(\bar{\rho},\bar{\rho}') = -\delta(\bar{\rho}-\bar{\rho}')$$

and are easily constructed as Fourier integrals (see, e.g., [4, p.47] and [20, pp. 685-691]):

$$G_1 = -\frac{i}{4\pi} \int_{C_1} e^{-ik\nu|y-y'|} e^{-ikwx_>} \left\{ e^{ikwx_<} + \Gamma_o(\nu) e^{-ikwx_<} \right\} \frac{d\nu}{w}$$

$$w = (1-\nu^2)^{\frac{1}{2}} \; ; \quad x_{\gtrless} = {}^{\max}_{\min} \{x,x'\} \; ; \quad \Gamma_o(\nu) = \frac{w-i\gamma_o}{w+i\gamma_o} \tag{4}$$

$$G_2 = -\frac{ikR}{8} \int_{C_2} e^{-ik\nu|s-s'|} H^{(2)}_{\nu kR}(k\rho_>)\{H^{(1)}_{\nu kR}(k\rho_<) - H^{(2)}_{\nu kR}(k\rho_<)P(\nu,R)\}d\nu$$

$$s = R\phi; \quad \rho_{\gtrless} = {}^{\max}_{\min} \{\rho,\rho'\} \; ; \tag{5}$$

$$P(\nu,R) = \frac{H^{(1)'}_{\nu kR}(kR) + \gamma_o H^{(1)}_{\nu kR}(kR)}{H^{(2)'}_{\nu kR}(kR) + \gamma_o H^{(2)}_{\nu kR}(kR)}$$

Here $H^{(1)(2)}$ are Hankel functions of the first and second kind, respectively. The integration contours C_1 and C_2 are shown in Figs. 4

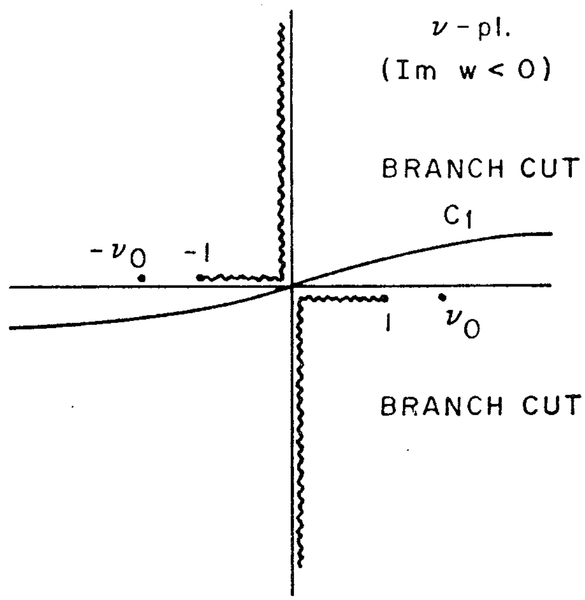

Fig. 4 Integration contour and singularities of integrand for G_1.

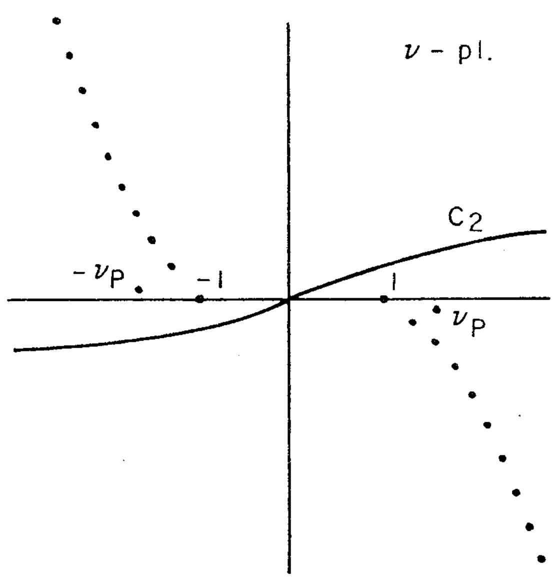

Fig. 5 Integration contour and singularities of integrand for G_2.

and 5 respectively, in relation to the singularities of the integrands. Note that in eqn. (4), C_1 must lie on the proper Riemann sheet where $\mathrm{Im}\, w < 0$.

To simplify the resulting integral equations we shall use slightly modified Green's functions whose normal derivatives vanish on C_{12}:

$$G^*_{1,2}(\bar{\rho},\bar{\rho}') = G_{1,2}(\bar{\rho},\bar{\rho}') + G_{1,2}(\bar{\rho},\bar{\rho}'_{im})$$

where if $\bar{\rho}' = (x',y')$ [resp. (ρ',ϕ')], $\rho'_{im} = (x', -y')$ [resp. $(\rho',-\phi')$]. Thus

$$\left.\frac{\partial G^*_{1,2}}{\partial y'}\right|_{y'=0} = \left[\frac{\partial G_{1,2}}{\partial y'} - \frac{\partial G_{1,2}}{\partial y'}\right]_{y'=0} = 0 \tag{6}$$

and

$$(\nabla_t^2 + k^2)G^*_{1,2}(\bar{\rho},\bar{\rho}') = -\delta(\bar{\rho}-\bar{\rho}') - \delta(\bar{\rho}-\bar{\rho}'_{im}) \tag{7}$$

Since S_1 and S_2 have infinite boundaries, we apply Green's theorem to outgoing fields in these regions: $E_z - E_z^{inc}$ and G_1 on S_1 and E_z and G_2 on S_2. Then, using boundary condition (2) and the radiation condition, we have

$$\int_{C_{12}} G_1^* \frac{\partial}{\partial n_1'} (E_z - E_z^{inc})d\ell' = \int_{S_1} [G_1^*\nabla_t'^2(E_z-E_z^{inc}) - (E_z-E_z^{inc})\nabla_t'^2 G_1^*]\ dS'$$

$$= \begin{cases} E_z(\bar{\rho})-E_z^{inc}(\bar{\rho}) & y \leqslant 0 \quad \text{(physical region)} \\ E_z(\bar{\rho}_{im})-E_z^{inc}(\bar{\rho}_{im}) & y \geqslant 0 \quad \text{(image region)} \end{cases} \tag{8}$$

and

$$\int_{C_{12}} G_2^* \frac{\partial}{\partial n_2'} E_z d\ell' = \int_{S_2} [G_2^*\nabla_t'^2 E_z - E_z\nabla_t'^2 G_2^*]dS' =$$

$$= \begin{cases} E_z(\bar{\rho}) & y \geq 0 \quad \text{(physical region)} \\ E_z(\bar{\rho}_{im}) & y \leq 0 \quad \text{(image region)} \end{cases} \tag{9}$$

It is easily shown by elementary integration [4,pp.29-31] that

$$\begin{aligned} -\int_{C_{12}} G_1^* \frac{\partial E_z^{inc}}{\partial n_1'} d\ell' &= -2\int_0^\infty G_1(x,y;x',0) \frac{\partial E_z^{inc}}{\partial y'} dx' \\ &= \Psi_{so}(x) e^{-ik\nu_o|y|} \end{aligned} \tag{10}$$

so that, setting $y = 0$ in (8)-(10) and enforcing continuity of tangential $\bar{E}$ and $\bar{H}$ (i.e. of E_z and its normal derivative) on the aperture C_{12}, we end up with the following Fredholm equation of the first kind for the unknown function $f(x)$ which gives the value of $\partial E_z/\partial y$ at the aperture:

$$\Psi_{so}(x) + \int_0^\infty (K_1 + K_2) f(x')dx' = 0 \tag{11}$$

where

$$K_{1,2}(x,x') = G_{1,2}(x,0;x',0); \quad f(x') = [\partial E_z/\partial y']_{y'=0}$$

Equations (8) and (9) in the physical regions will give E_z in terms of $f(x')$ once the latter function is determined. Equation (10) and the fact that $K_2 \to K_1$ as $kR \to \infty$ [21] verify that $f(x)$ reduces to $\partial E_z^{inc}/\partial y$ on the aperture and the incident wave is transmitted unaffected, as we expect. Thus it is convenient to rewrite (11) as

$$\Psi_{so}(x) + 2\int_0^\infty K_1 f(x')dx' + \int_0^\infty (K_2 - K_1) f(x')dx' = 0 \tag{12}$$

for the purpose of finding a perturbation solution for large kR.

III. Approximate Form of the Green's Function in the Curved Region

An effective perturbation solution of (11a) requires that we be able to express the difference between K_2 and K_1 as a relatively simple function of kR. Although it is known [21] that $G_2 \to G_1$ as $kR \to \infty$ for any fixed values of $\bar{\rho}$ and $\bar{\rho}'$, the eigenfunctions of the two regions are known to be vastly different from one another because the singularity structure of equation (4) is quite different than that of (5) [22] . As shown in Fig. 5, aside from the surface-wave pole ν_ρ (which is only slightly perturbed from ν_o [23]), the branch cuts in Fig. 4 have disappeared and an infinite set of "Watson poles" or "creeping wave poles" [20, pp. 685-698] has replaced them. Since the spectra of the two Green's functions are so different, and the residue series over the creeping wave modes is known to converge poorly in the geometrically illuminated region of the curved section [20, p. 691] , we desire some approximate expression for G_2 with singularities similar to those in Fig. 4.

Let us first recall how the straight waveguide Green's function G_1 is expressed as an expansion over the surface-wave mode and a continuous spectrum of radiation modes [4, pp.24-31, 44-48]. By deforming C_1 over the branch cut in the lower half-plane, picking up the residue at ν_o and changing integration variables, we have

$$G_1(\bar{\rho},\bar{\rho}') = -i\,\frac{\gamma_o}{\nu_o}\,\Psi_{so}(x)\Psi_{so}(x')e^{-ik\nu_o|y-y'|} - \frac{i}{4}\int_0^\infty \Psi_{co}(w,x)\Psi_{co}(w,x')e^{-ik\nu|y-y'|}\,\frac{dw}{\nu N^2(w)} \qquad (13)$$

which follow will be poor in these neighborhoods, we may make them on the contour of integration away from these points and subsequently deform it in any permissible fashion. The only restriction will be that the mean-square error over the contour be small compared to the magnitude of the integral itself.[2] We shall obtain more quantitative error criteria below.

Equation (5) now becomes

$$G_2 = G_{2s} + G_{2c} \tag{14}$$

where

$$G_{2s} = -i\frac{\gamma_o}{\nu_o} e^{-ik\nu_\rho|s-s'|} \Psi_s(x,R)\Psi_s(x',R) \tag{15}$$

$$\Psi_s(x,R) = \left\{ -\frac{i\nu_o \pi kR}{4\gamma_o} \; \frac{H^{(1)'}_{\nu_\rho kR}(kR) + \gamma_o H^{(1)}_{\nu_\rho kR}(kR)}{\frac{\partial}{\partial\nu}[H^{(2)'}_{\nu kR}(kR) + \gamma_o H^{(2)}_{\nu kR}(kR)]} \right\}^{\frac{1}{2}}_{\nu=\nu_\rho} H^{(2)}_{\nu_\rho kR}(k\rho) \tag{16}$$

$$G_{2c} = -\frac{ikR}{8} \int_{C_2'} e^{-ik\nu|s-s'|} H^{(2)}_{\nu kR}(k\rho_>)\{H^{(1)}_{\nu kR}(k\rho_<) - H^{(2)}_{\nu kR}(k\rho_<) P(\nu,R)\} \tag{17}$$

$$(x = \rho - R)$$

the square root in (16) being chosen so that $\Psi_s \to \Psi_{so}$ as $R \to \infty$. At present G_{2s} and Ψ_s are exact, and in particular retain the attenuative character of the surface-wave as it propagates around the bend. We shall, in solving the integral equation (12), find it convenient to use an approximation for Ψ_s in order to simplify the calculations. This is derived in Appendix A.

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[2] Compare, e.g. Rulf's treatment [24,25] of a similar problem, where the restrictions imposed seem somewhat more stringent than necessary.

On C_2', now, we replace the Hankel functions in (17) by a suitable asymptotic representation for $kR >> 1$. The Debye formulas are such a representation if we avoid circles of radius $O[(kR)^{-2/3}]$ around the turning points, ± 1, $\pm\alpha_{\gtrless}$ [20, pp. 710-712; 26; 27]. A representation which is continuous on all of C_2' and whose exponentially dominant part is correct at all points of this contour is[3]

$$H^{(2)}_{\nu kR}(kR\alpha) \sim \left(\frac{2}{\pi kR}\right)^{\frac{1}{2}} \frac{e^{-i[kR\alpha\zeta - \pi/4]}}{(\alpha^2-\nu^2)^{\frac{1}{4}}} \left\{L\left(\nu kR, \frac{\nu^2}{\alpha^2}\right) + \frac{i\nu}{(\alpha^2-\nu^2)^{\frac{1}{2}}} M\left(\nu kR, \frac{\nu^2}{\alpha^2}\right)\right\}$$

$$H^{(1)}_{kR}(kR\alpha) \sim \left(\frac{2}{\pi kR}\right)^{\frac{1}{2}} \frac{e^{i[kR\alpha\zeta - \pi/4]}}{(\alpha^2-\nu^2)^{\frac{1}{4}}} \left\{L\left(\nu kR, \frac{\nu^2}{\alpha^2}\right) - \frac{i\nu}{(\alpha^2-\nu^2)^{\frac{1}{2}}} M\left(\nu kR, \frac{\nu^2}{\alpha^2}\right)\right\} - H^{(2)}_{\nu kR}(kR\alpha)$$

where

$$\zeta = (1 - \nu^2/\alpha^2)^{\frac{1}{2}} - (\nu/\alpha)\arccos(\nu/\alpha)$$

$$= (1 - \nu^2/\alpha^2)^{\frac{1}{2}} + i(\nu/\alpha)\ln[(\nu/\alpha) + i(1 - \nu^2/\alpha^2)^{\frac{1}{2}}]$$

$$L(\nu kR, p^2) = 1 - \frac{81\left(\frac{p^2}{1-p^2}\right) + 462\left(\frac{p^2}{1-p^2}\right)^2 + 385\left(\frac{p^2}{1-p^2}\right)^3}{1152(\nu kR)^2} + \cdots$$

$$M(\nu kR, p^2) = \frac{3 + 5\left(\frac{p^2}{1-p^2}\right)}{24\nu KR} + \cdots$$

The branch cuts from $\nu = \pm\alpha$ are shown in Fig. 6, and are defined by $\mathrm{Im}\,\zeta = 0$ or $\mathrm{Im}(\zeta + \pi\nu/\alpha) = 0$ and $-\pi < \arg(1-\nu^2/\alpha^2)^{\frac{1}{2}} < -\pi/2$. The Riemann sheet is taken as the one where $\arg\zeta = \arg(1-\nu^2/\alpha^2)^{\frac{1}{2}} = -\pi/2$ on the real axis for $\nu > \alpha$. Neither L nor M exhibits any branch cut behavior. Let us repeat that only the <u>dominant</u> part of the asymptotic expansion is given correctly by the above expressions on all of C_2'; the

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[3]Note that it is at this point that the Watson poles disappear and are replaced by branch cuts.

subdominant part is incorrect, for example, below the real axis in the right half-plane. For purposes of constructing an approximate continuous mode spectrum, however, the dominant part will suffice.

The derivatives of the Hankel functions are similarly expanded as

$$H^{(2)'}_{\nu kR}(kR\alpha) \sim -i\left(\frac{2}{\pi kR}\right)^{\frac{1}{2}} \frac{(\alpha^2-\nu^2)^{\frac{1}{4}}}{\alpha} e^{-i[kR\alpha\zeta-\pi/4]}\left\{N\left(\nu kR,\frac{\nu^2}{\alpha^2}\right) - \frac{i\nu}{(\alpha^2-\nu^2)^{\frac{1}{2}}} Q\left(\nu kR,\frac{\nu^2}{\alpha^2}\right)\right\}$$

$$H^{(1)'}_{\nu kR}(kR\alpha) \sim i\left(\frac{2}{\pi kR}\right)^{\frac{1}{2}} \frac{(\alpha^2-\nu^2)^{\frac{1}{4}}}{\alpha} e^{i[kR\alpha\zeta-\pi/4]}\left\{N\left(\nu kR,\frac{\nu^2}{\alpha^2}\right) + \frac{i\nu}{(\alpha^2-\nu^2)^{\frac{1}{2}}} Q\left(\nu kR,\frac{\nu^2}{\alpha^2}\right)\right\} - H^{(2)'}_{\nu kR}(kR\alpha)$$

where

$$N(\nu kR,p^2) = 1 + \frac{135\left(\frac{p^2}{1-p^2}\right) + 594\left(\frac{p^2}{1-p^2}\right)^2 + 455\left(\frac{p^2}{1-p^2}\right)^3}{1152(\nu kR)^2} + \cdots$$

$$Q(\nu kR,p^2) = \frac{9+7\left(\frac{p^2}{1-p^2}\right)}{24\nu kR} + \cdots$$

and again, neither N nor Q exhibits any branch cut behavior. In all the above expressions, it has been assumed that α (which is equal to either 1, $\alpha_>$ or $\alpha_<$ here) is positive real.

Inserting all of the foregoing into equation (5) results in

$$G_{2c} \sim -\frac{i}{4\pi}\int_{C_2'} \frac{e^{-ik\nu|s-s'|-ikh_>}}{(\alpha_>^2-\nu^2)^{\frac{1}{4}}(\alpha_<^2-\nu^2)^{\frac{1}{4}}} S^+_{L_>}\{e^{ikh_<}S^-_{L_<} + \Gamma(\nu,R)e^{-ikh_<} S^+_{L_<}\}\,d\nu \qquad (18)$$

where

$$S^{\pm}_{L_{\gtrless}} = L\left(\nu kR,\frac{\nu^2}{\alpha_{\gtrless}^2}\right) \pm \frac{i\nu}{(\alpha_{\gtrless}^2-\nu^2)^{\frac{1}{2}}} M\left(\nu kR,\frac{\nu^2}{\alpha_{\gtrless}^2}\right)$$

$$\Gamma(\nu,R) = \frac{w[N+(i\nu/w)Q]-i\gamma_o[L-(i\nu/w)M]}{w[N-(i\nu/w)Q]+i\gamma_o[L+(i\nu/w)M]}$$

$$h_{\gtrless} = R\{(\alpha_{\gtrless}^2-\nu^2)^{\frac{1}{2}} - w - \nu\arccos(\nu/\alpha_{\gtrless}) + \nu\arccos\nu\}$$

$$= \int_R^{\rho_{\gtrless}} \frac{R}{\rho}\left(\frac{\rho^2}{R^2}-\nu^2\right)^{\frac{1}{2}} d\rho$$

Here, $\alpha_{\gtrless} = \rho_{\gtrless}/R$ and L,M,N and Q denote the above asymptotic series with $p^2 = \nu^2$. At this point, the effect of the Watson poles has been approximated by replacing them with a set of branch cuts [28], and the structure of G_2 now bears a great deal more similarity to that of G_1 in (4) than previously. Additionally, a number (dependent upon how many terms in the asymptotic series in the denominator of $\Gamma(\nu,R)$ have been retained) of spurious poles clustered in the neighborhood of $\nu = \pm 1$ have appeared, which must also be removed.

The branch points at $\nu = \pm\alpha_{\gtrless}$ can be removed by expanding all functions of $\alpha_{\gtrless}$ in the integrand of (18) in a Taylor series about the point $x_{\gtrless} = 0$ where $x_{\gtrless} = \rho_{\gtrless} - R$. For small enough x/R this is valid again outside the vicinity of the turning points $\nu = \pm 1$, $\pm\alpha_{\gtrless}$ which have already been avoided. Thus

$$(\alpha_{\gtrless}^2 - \nu^2)^{-\beta} = (1-\nu^2)^{-\beta}\{1 - \frac{2\beta k x_{\gtrless}}{kRw^2} + \cdots\} \tag{19}$$

and

$$e^{\pm ikh} = e^{\pm ikwx}\{1 \pm i\frac{k^2x^2\nu^2}{2kRw} + \cdots\} \tag{20}$$

Finally, avoiding the surface-wave pole ν_p (and the unperturbed pole ν_o) we develop $\Gamma(\nu,R)$ as an asymptotic series in $(kR)^{-1}$, thus losing the previously mentioned spurious poles:

$$\Gamma(\nu,R) \sim \Gamma_o(\nu) + \frac{1}{kR}\Gamma_1(\nu) + \cdots \tag{21}$$

where $\Gamma_o(\nu)$ is given in (4) and

$$\Gamma_1(\nu) = -\frac{i}{12}\frac{3+2\nu^2}{w^3}\Gamma_o(\nu) + \frac{i}{w(w+i\gamma_o)^2}$$

If in (18) we regard the expansions (19), (20) and (21) for these quantities as formally substituted into the integral, the integrand now possesses branch points only at $\nu = \pm 1$. If we move the branch cuts around as in Fig. 7, so that the Riemann sheet is the same as that (Fig. 4) used for G_1, the analogy with G_1 is then complete. It is now obvious that the analogy to the straight waveguide is possible when the radius of curvature is large, i.e. $kR \gg 1$, and when one is interested in a region where $x/R \ll 1$.

To arrive at a representation G_2 like that of (13), it remains to deform C_2' over the cut in the lower half-plane. Since all of the preceding series developments generate successively higher powers of w in the denominator, it may occur that a contour deformation over the cut produces a residue term or an apparently divergent branch cut integral. Although the $(kR)^{-1}$ terms determined below do not exhibit this phenomenon, it is possible that higher order terms may do so. Since the integral is known to be finite before the contour deformation, a method for dealing with this eventuality is presented in Appendix B.

With this understanding we perform the contour deformation in (18). Changing the integration variable to w (on the contour shown in Fig. 8),

$$G_{2c} \sim -\frac{i}{4\pi}\int_{-\infty}^{\infty} e^{-ik\nu|s-s'|} e^{-ikh_<} \left(\frac{1-\nu^2}{\alpha_>^2-\nu^2}\right)^{\frac{1}{4}} \left(\frac{1-\nu^2}{\alpha^2-\nu_<^2}\right)^{\frac{1}{4}} S_{L_>}^{+}\{e^{ikh_<} S_{L_<}^{-} + \\ + \Gamma(\nu,R)e^{-ikh_<} S_{L_<}^{+}\}\frac{dw}{\nu}$$

(recall that the expressions $e^{\pm ikh_<}$, $\Gamma(\nu,R)$, etc., are to be understood as series developments). Changing variables from w to $-w$,

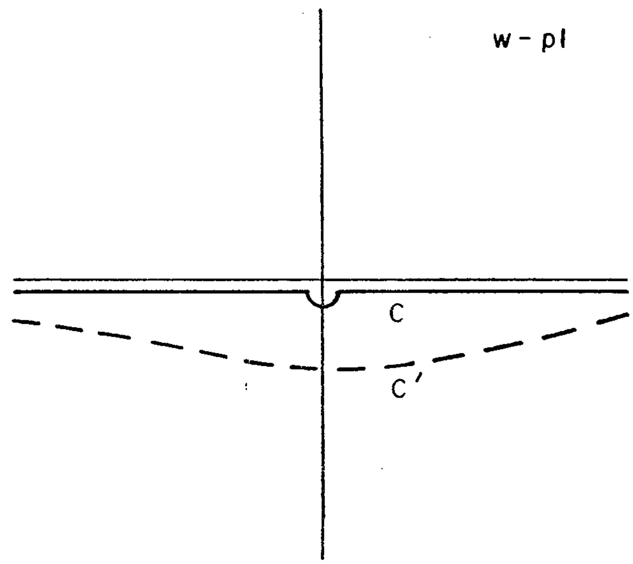

Fig. 8

Integration contour in w - plane. On contour C, $\nu = (1 - w^2)^{\frac{1}{2}} > 0$ for $|w| < 1$, $\nu = -i\,(w^2 - 1)^{\frac{1}{2}}$ for $|w| > 1$.

noting that $\Gamma(\nu,R) \to 1/\Gamma(\nu,R)$ under this transformation,

$$G_{2c} \sim -\frac{i}{4\pi}\int_{-\infty}^{\infty} e^{-ik\nu|s-s'|} e^{+ikh_>} \left(\frac{1-\nu^2}{\alpha_>^2-\nu^2}\right)^{\frac{1}{4}} \left(\frac{1-\nu^2}{\alpha_<^2-{}^2}\right)^{\frac{1}{4}} S_{L_>}^- \{e^{-ikh_<} S_{L_<}^+ + \frac{1}{\Gamma(\nu,R)} e^{+ikh_<} S_{L_<}^- \} \frac{dw}{\nu}$$

Adding the previous two expressions gives

$$G_{2c} \sim -\frac{i}{8\pi}\int_{-\infty}^{\infty} \frac{e^{-ik\nu|s-s'|}}{\Gamma(\nu,R)} \{\left(\frac{1-\nu^2}{\alpha_>^2-\nu^2}\right)^{\frac{1}{4}} [e^{ikh_>} S_{L_>}^- + \Gamma(\nu,R) e^{-ikh_>} S_{L_>}^+] \cdot \tag{22}$$

$$\cdot \left(\frac{1-\nu^2}{\alpha_<^2-\nu^2}\right)^{\frac{1}{4}} [e^{ikh_<} S_{L_<}^- + \Gamma(\nu,R) e^{-ikh_<} S_{L_<}^+]\} \frac{dw}{\nu}$$

where $1/\Gamma(\nu,R)$ is to be understood as the formal inverse of (21):

$$1/\Gamma(\nu,R) \sim [1/\Gamma_o(\nu)]\{1 - \frac{1}{kR}[\Gamma_1(\nu)/\Gamma_o(\nu)] + \cdots\} \tag{23}$$

From the manner of its construction it is clear that the integrand of (22) is an even function of w, and so cannot contribute a residue from a pole at $w = 0$, as was suggested above.

Finally, by splitting $\Gamma(\nu,R)$ into the square of its (once again, formal) square root, we can express (22) in the manner of (13) as an integral of a product of continuous "quasi-eigenfunctions"

$$\begin{aligned} G_{2c} &\sim -\frac{i}{4}\int_{-\infty}^{\infty} \frac{e^{-ik\nu|s-s'|}}{\nu N^2(w)} \Psi_c(w,x;R)\, \Psi_c(w,x';R)\, dw \\ &= -\frac{i}{2}\int_{-\infty}^{\infty} \frac{e^{-ik\nu|s-s'|}}{\nu N^2(w)} \Psi_c(w,x;R)\, \Psi_c(w,x';R)\, dw \end{aligned} \tag{24}$$

where

$$\Psi_c(w,x;R) = \frac{1}{2}\left(\frac{1-\nu^2}{\rho^2/R^2-\nu^2}\right)^{\frac{1}{4}} \left\{\left(\frac{\Gamma_o(\nu)}{\Gamma(\nu,R)}\right)^{\frac{1}{2}} (w+i\gamma_o) S_{L_\rho}^- e^{ikh} + \left(\frac{\Gamma(\nu,R)}{\Gamma_o(\nu)}\right)^{\frac{1}{2}} (w-i\gamma_o) S_{L_\rho}^+ e^{-ikh}\right\} \tag{25}$$

and we understand (from (21) and (23))

$$\left(\frac{\Gamma_o(\nu)}{\Gamma(\nu,R)}\right)^{\frac{1}{2}} \sim 1 - \frac{1}{kR}\,\frac{\Gamma_1(\nu)}{2\Gamma_o(\nu)} + \cdots \tag{26}$$

$$\left(\frac{\Gamma(\nu,R)}{\Gamma_o(\nu)}\right)^{\frac{1}{2}} \sim 1 + \frac{1}{kR}\,\frac{\Gamma_1(\nu)}{2\Gamma_o(\nu)} + \cdots \tag{27}$$

Combining (24) and (15) we have

$$G_2 \simeq -i\,\frac{\gamma_o}{\nu_o}\,e^{-ik\nu_o|s-s'|}\,\Psi_s(x;R)\,\Psi_s(x';R) - \\ -\frac{i}{2}\int_o^\infty \frac{e^{-ik\nu|s-s'|}}{\nu N^2(w)}\,\Psi_c(w,x;R)\,\Psi_c(w,x;R)\,dw \tag{28}$$

Equation (28) is now formally analogous to the modal expansion used in the straight waveguide section as given in (13). Thus, similar to the situation encountered in a straight waveguide, the field of a point source in the curved section can also be expanded in a modal representation consisting of a "pseudo"-surface wave mode $\Psi_s(x;R)$, and a continuous set of "pseudo" radiation modes, $\Psi_c(x;R)$. Here we note the word "pseudo" is used to emphasize the fact that these modes do not necessarily constitute the legitimate description of the field everywhere in a curved waveguide section like their counterparts do in a straight waveguide section. They are however a correct alternative representation of the exact formulation only in the region where the radial observation distance $x = \rho - R$ is much smaller than R, and provided that $kR \gg 1$. (This of course poses no practical restriction in the application of this theory to optical waveguiding systems.) Thus, we should stress that the theory to be developed in the following is an asymptotic one, valid in the sense that $kR \to \infty$ while $x/R \ll 1$.

In Appendix A it is shown that

$$\Psi_s(x;R) \sim \Psi_{so}(x) + \frac{1}{kR} \Psi_{s1}(x) + \cdots$$

where

$$\Psi_{s1}(x) = e^{-k\gamma_o x}\left[\frac{k^2x^2\nu_o^2}{2\gamma_o} - \frac{1}{2\nu_o^2\gamma_o} - \frac{1}{4\nu_o^2\gamma_o^3}\right] \tag{29}$$

Further, it is straightforward to show from (25) that

$$\Psi_c(w,x;R) \sim \Psi_{co}(w,x) + \frac{1}{kR}\Psi_{c1}(w,x) + \cdots$$

where (30)

$$\Psi_{c1}(w,x) = \tfrac{1}{2}\{e^{+ikwx}(w+i\gamma_o)[-\frac{kx}{2w^2} + \frac{ik^2x^2\nu^2}{2w} - \frac{i}{2w(w^1+\gamma_o^2)}] +$$

$$+ e^{-ikwx}(w-i\gamma_o)[-\frac{kx}{2w^2} - \frac{ik^2x^2\nu^2}{2w} + \frac{i}{2w(w^2+\gamma_o^2)}]\}$$

We now make the following remarks. The $(kR)^{-1}$ term of the integrand in (28) is

$$\frac{e^{-ik\nu|s-s'|}}{\nu N^2(w)}[\Psi_{co}(w,x)\Psi_{c1}(w,x') + \Psi_{co}(w,x')\Psi_{c1}(w,x)]$$

It is readily seen that this expression remains finite at $w = 0$, and so the precautions in Appendix B are not needed here. We end this section by presenting the following integrals which will be required in the next section.

$$\int_0^\infty \Psi_{so}(x)\Psi_{s1}(x)dx = 1/(8k\nu_o^2) \tag{31}$$

$$\int_0^\infty \Psi_{so}(x)\Psi_{c1}(x)dx = \frac{w(\nu^2+\nu_o^2)}{k(w^2+\gamma_o^2)^2} \tag{32}$$

$$\int_0^\infty \Psi_{s1}(x)\Psi_{co}(x)dx = -\frac{2w\nu_o^2}{k(w^2+\gamma_o^2)^2} \tag{33}$$

IV. Perturbation Solution of the Integral Equation

We return now to consideration of the solution of the integral equation given in (12). We may expand $f(x)$ in terms of the transverse cross-section modes of the straight guide:

$$f(x) = A_o\Psi_{so}(x) + \int_0^\infty A(w)\Psi_{co}(w,x)dw \tag{34}$$

Inserting (34) into (12), and using (13) along with the straight guide orthogonality relations gives

$$A_o = -ik\nu_o\{1 + 2k\gamma_o[A_oC_{ss} + \int_0^\infty A(w)C_s(w)dw]\} \tag{35}$$

$$A(w) = -\frac{ik^2\nu}{N^2(w)}\{A_oC_s(w) + \int_0^\infty A(w')C(w,w')dw'\} \tag{36}$$

where

$$C_{ss} = \int_0^\infty \int_0^\infty \Psi_{so}(x)\Psi_{so}(x')[K_2-K_1]\,dx\,dx'$$

$$C_s(w) = \int_0^\infty \int_0^\infty \Psi_{so}(x)\Psi_{co}(w,x')[K_2-K_1]\,dx\,dx'$$

$$C(w,w') = \int_0^\infty \int_0^\infty \Psi_{co}(w,x)\Psi_{co}(w',x')[K_2-K_1]dx\,dx'$$

From our study of G_2 in section III, it is clear that C_{ss}, $C_s(w)$ and $C(w,w')$ are all small quantities to order $(kR)^{-1}$, that A_o is a zeroth-order quantity, and $A(w)$ is of order $(kR)^{-1}$.

By formally expanding $[K_2 - K_1]$ in inverse powers of kR, we have

$$C_{ss} \sim \frac{1}{kR} C_{ss}^{(0)} + \frac{1}{(kR)^2} C_{ss}^{(1)} + \cdots$$

$$C_s(w) \sim \frac{1}{kR} C_s^{(0)}(w) + \frac{1}{(kR)^2} C_s^{(1)}(w) + \cdots$$

$$C(w,w') \sim \frac{1}{kR} C^{(0)}(w,w') + \frac{1}{(kR)^2} C^{(1)}(w,w') + \cdots$$

If one assumes corresponding expansions for the mode amplitudes

$$A_o \sim -ik\nu_o + \frac{1}{kR} A_o^{(0)} + \frac{1}{(kR)^2} A_o^{(1)} + \cdots$$

$$A(w) \sim \frac{1}{kR} A^{(0)}(w) + \frac{1}{(kR)^2} A^{(1)}(w) + \cdots$$

A solution of (35)-(36) can be obtained by matching coefficients of like inverse powers of kR. Thus in particular

$$A_o^{(0)} = -ik\nu_o \{-2ik^2 \nu_o \gamma_o C_{ss}^{(0)}\} \tag{37}$$

$$A^{(0)}(w) = -\frac{k^3 \nu\nu_o}{N^2(w)} C_s^{(0)}(w) \tag{38}$$

$$A_o^{(1)} = -ik\nu_o \{-2ik^2 \nu_o \gamma_o C_{ss}^{(1)} - 4k^4 \nu_o^2 \gamma_o^2 [C_{ss}^{(0)}]^2 - 2k^4 \nu_o \gamma_o \int_0^\infty \nu [C_s^{(0)}(w)]^2 \frac{dw}{N^2(w)}\} \tag{39}$$

$$A^{(1)}(w) = -\frac{k^3 \nu\nu_o}{N^2(w)} \{C_s^{(1)}(w) - 2ik^2 \nu_o \gamma_o C_{ss}^{(0)} C_s^{(0)}(w) - ik^2 \int_0^\infty \nu' C_s^{(0)}(w') C^{(0)}(w,w') \frac{dw}{N^2(w')}\} \tag{40}$$

Substitution of (13) and (34) into (8) and (10) now gives the field in the straight section as

$$E_z = \Psi_{so}(x)[e^{-ik\nu_o y} + \Omega e^{ik\nu_o y}] + \int_0^\infty B_-(w)\Psi_{co}(w,x)e^{ik\nu y}dw \qquad (y < 0) \tag{41}$$

where the reflection coefficient Ω is given by

$$\Omega = 1 - iA_o/k\nu_o \sim (kR)^{-1}[2ik^2\nu_o\gamma_o C_{ss}^{(0)}] + \cdots \tag{42}$$

and

$$B_-(w) = -iA(w)/k\nu \sim (kR)^{-1}[ik^2\nu_o C_s(w)/N^2(w)] + \cdots \tag{43}$$

On the other hand, the substitution of (28) and (34) into (9) gives the fields in the curved section as

$$E_z = T\Psi_s(x,R)e^{-ik\nu_\rho s} + \int_0^\infty B_+(w)\Psi_c(w,x,R)e^{-ik\nu s}dw \qquad (s > 0) \tag{44}$$

where the transmission coefficient T is given by

$$\begin{aligned} T &= 2i\frac{\gamma_o}{\nu_o}\left\{A^o \int_0^\infty \Psi_s(x,R)\Psi_{so}(x)dx + \int_0^\infty A(w) \int_0^\infty \Psi_s(x,R)\Psi_{co}(w,x)dx\, dw\right\} \\ &\sim 1 + \frac{1}{kR}\left[\frac{\gamma_o(1-8ik^2\nu_o^3 C_{ss}^{(0)})}{4\nu_o^2}\right] + \frac{1}{(kR)^2}T^{(1)} + \cdots \end{aligned} \tag{45}$$

and

$$\begin{aligned} T^{(1)} = 2k\gamma_o &\left\{\int_0^\infty \Psi_{s2}(x)\Psi_{so}(x)dx - \frac{ik\gamma_o C_{ss}^{(0)}}{4\nu_o} - ik\nu_o C_{ss}^{(1)} - 2k^3\nu_o^2\gamma_o[C_{ss}^{(0)}]^2\right. \\ &\left. -ik\nu_o \int_0^\infty \frac{\nu C_s^{(0)}(w)}{N^2(w)}\left[\frac{-2\nu_o w}{(w^2+\gamma_o^2)^2} - ik^2 C_s^{(0)}(w)\right]dw\right\} . \end{aligned}$$

In addition,

$$B_+(w) = \frac{i}{\nu N^2(w)} \{A^o \int_0^\infty \Psi_{so}(x)\Psi_c(w,x;R)\,dx + \int_0^\infty A(w') \int_0^\infty \Psi_{co}(w';x)\Psi_c(w,x;R)\,dx\,dw'\}$$

$$\sim (kR)^{-1} \left\{ -\frac{ik\nu_o}{N^2(w)} \left[\frac{iw(\nu^2+\nu_o^2)}{k\nu(w^2+\gamma_o^2)^2} + kC_s^{(0)}(w) \right] \right\} + \cdots \qquad (46)$$

In obtaining these expressions, (31)-(33) and (37)-(39) have been used.

Equation (41) is the proper modal representation of the electric field in the straight waveguide section. We see that, for an incident surface-wave field of unit amplitude, the scattered field due to the curvature discontinuity now consists of a reflected surface-wave of amplitude Ω, given by (42), and a reflected radiation field expressed in terms of a continuous modal spectrum $B_-(w)$, given by (43). Similarly, we have in the curved section, a transmitted pseudo-surface wave of amplitude T given by (45), and a radiation field expressed in terms of a pseudo-continuous modal spectrum $B_+(w)$ given by (46) which is analogous to the one obtained in the straight waveguide section. In order to obtain a more explicit expression, use is made of (13) and (28) for the evaluation of coefficients $C_s^{(0)}$ and $C_{ss}^{(0)}$ in a straightforward manner.

$$C_{ss}^{(0)} = -\frac{i}{8k^2\nu_o^3} \quad ; \quad C_s^{(0)}(w) = -\frac{iw}{2k^2\nu} \frac{(\nu_o-\nu)^2}{(w^2+\gamma_o^2)^2} \qquad (47)$$

so that

$$\left.\begin{aligned}
\Omega &\simeq (kR)^{-1}\left[\frac{\gamma_o}{4\nu_o^2}\right] \\
B_-(w) &\simeq (kR)^{-1}\left[\frac{w\nu_o}{2\nu N^2(w)} \frac{(\nu_o-\nu)^2}{(w^2+\gamma_o^2)^2}\right] \\
B_+(w) &\simeq (kR)^{-1}\left[\frac{w\nu_o}{2\nu N^2(w)} \frac{(\nu_o+\nu)^2}{(w^2+\gamma_o^2)^2}\right] \\
T &\simeq 1 + (kR)^{-2}T^{(1)}
\end{aligned}\right\} \qquad (48)$$

and $T^{(1)}$ has simplified to

$$T^{(1)} = 2k\gamma_o \left\{ \int_0^\infty \Psi_{s2}(x)\Psi_{so}(x)dx - ik\nu_o C_{ss}^{(1)} + \frac{\nu_o}{4k}\int_0^\infty \frac{w^2(\nu_o+\nu)^2(\nu_o-\nu)^2 dw}{\nu N^2(w)\,(w^2+\gamma_o^2)^4} \right\} \tag{49}$$

Further, since

$$C_{ss}^{(1)} = -\frac{i}{k\nu_o}\int_0^\infty \Psi_{s2}(x)\,\Psi_{so}(x)dx - \frac{i\gamma_o}{64k^2\nu_o^5} - \frac{i}{2k^2}\int_0^\infty \frac{w^2(\nu^2+\nu_o^2)^2 dw}{\nu N^2(w)\,(w^2+\gamma_o^2)^4} \tag{50}$$

we have finally

$$T^{(1)} = -\frac{\gamma_o^2}{32\nu_o^4} - \frac{\gamma_o\nu_o}{\pi}\int_0^\infty \frac{w^2(\nu_o^4+6\nu_o^2\nu^2+\nu^4)}{\nu(w^2+\gamma_o^2)^5}\,dw$$

$$= -\frac{\gamma_o^2}{32\nu_o^4} - \frac{\gamma_o\nu_o}{\pi}\int_0^{\pi/2} \frac{\sin^2\theta\,[\nu_o^4+6\nu_o^2\cos^2\theta + \cos^4\theta]d\theta}{(\sin^2\theta+\gamma_o^2)^5}$$

$$- i\,\frac{\gamma_o\nu_o}{\pi}\int_1^\infty \frac{w^2(\nu_o^4+6\nu_o^2\nu^2+\nu^4)}{\sqrt{w^2-1}\,(w^2+\gamma_o^2)^5}\,dw$$

For subsequent power conservation discussions, only the real part of $T^{(1)}$ will be needed:

$$\mathrm{Re}\,T^{(1)} = -\frac{\gamma_o^2}{32\nu_o^4} - \frac{8\nu_o^6-6\nu_o^4+4\nu_o^2-1}{32\nu_o^4\gamma_o^6} = -\frac{\nu_o^2(\nu_o^2+4)}{32\gamma_o^6} \tag{51}$$

Equation (48) represents the first non-vanishing corrections to all the relevant fields in the problem.

V. The Far-Field of the Scattered Radiation

Consider the radiation field

$$E_z = \int_0^\infty B^{\pm}(w)\Psi_{co}(w,x)e^{\mp ik\nu y}dw$$

on a section of straight guide. If $B^{\pm}$ has, at worst, singularities of the type ν^{-1} and is finite at $w=0$, the application of a saddle-point integration yields the following far-field expansion [4, p.27]:

$$E_z \sim \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} F(\theta)e^{-ikr+i\pi/4} \tag{52}$$

where

$$F(\theta) = \pm\cos\theta\,(\sin^2\theta+\gamma_o^2)^{\frac{1}{2}}B^{\pm}(w=\sin\theta\,;\ \nu=\pm\cos\theta)e^{-iq(\theta)}$$

and $\tan\theta = x/y$, $\tan q(\theta) = \gamma_o/\sin\theta$, and $kr = k(x^2+y^2)^{\frac{1}{2}} \gg 1$. This is the usual method for obtaining the radiation pattern $F(\theta)$.

Now consider the field

$$E_z = \int_0^\infty B^{\pm}(w)\Psi_c(w,x;R)e^{\mp ik\nu s}dw$$

on a section of curved guide. Restricting the singularities of $B^{\pm}$ as above, we obtain in the same manner as (52), the far-field expression on a curved section:

$$E_z \sim \left(\frac{\pi}{2k\tilde{r}}\right)^{\frac{1}{2}} F(\tilde{\theta})e^{-ik\tilde{r}+i\pi/4} \tag{53}$$

where F is formally the same as in (52), but with θ replaced by $\tilde{\theta}$, where $\tan\tilde{\theta} = x/s$, and $k\tilde{r} = k(x^2+s^2)^{\frac{1}{2}} \gg 1$. The substitution of s for y appears to distort the far-field in the curved section relative to the straight section, but recalling the condition $x/R \ll 1$, we arrive at the situation illustrated in Fig. 9. As long as $kR \gg k\tilde{r} \gg 1$ (region A),

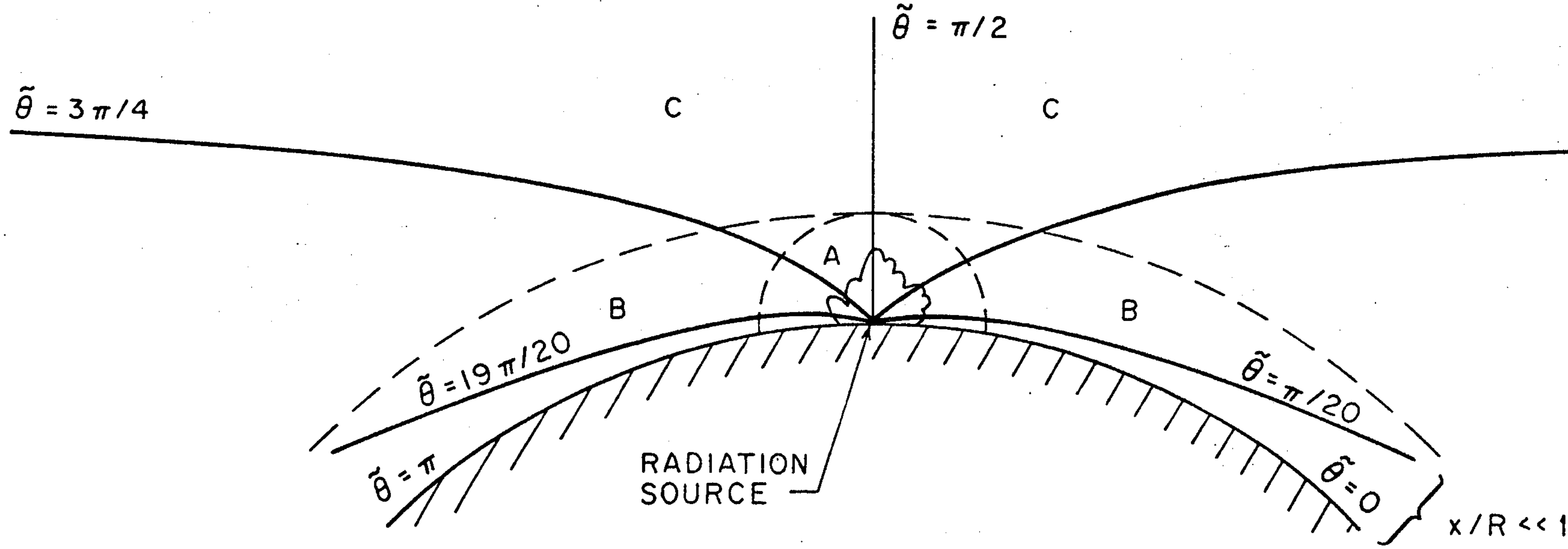

Fig. 9

Region of validity of far-field pattern on curved section: (A) complete radiation pattern, (B) $\tilde{\theta} \simeq 0$ and π.

the pattern (53) is valid for all observation angles $0 < \tilde{\theta} < \pi$, and furthermore within this range, $\tilde{\theta} \simeq \theta$ and $\tilde{r} \simeq r$, so that no distortion of the pattern appears. For $\tilde{\theta} \simeq 0$ or π, the field is small, and has negligible effect when another discontinuity (e.g. a transition back to a straight section) is encountered (see region B). Outside the boundary layer (region C) Ψ_c no longer accurately describes the radiated fields; however, since the fields near the surface are known accurately, some form of Kirchhoff-Huyghen's principle may be used to continue the fields beyond this boundary layer [3,5,33]. Physically, of course, the radiation lobes are not expected to bend along with the impedance surface as (53) would seem to indicate if x/R is not small, but should continue outward as if the impedance surface were absent.

From (48), (52) and (53) then, we can calculate the radiation pattern of the discontinuity in the present case as

$$|F(\theta)|^2 = \frac{\nu_o^2 \sin^2\theta}{\pi^2(\nu_o^2-\cos^2\theta)(\nu_o-\cos\theta)^4(kR)^2} \tag{54}$$

It is easily seen that this pattern consists of a single lobe whose maximum (for $\gamma_o/\nu_o << 1$, a typical situation in optical waveguides) is located at

$$\theta \simeq \gamma_o/2,$$

i.e., the pattern is strongly endfire with radiation in the backward direction nearly absent. Obviously, when two or more discontinuities are present on the same structure, the fields scattered from each one will interfere to produce an overall radiation of much more complicated structure.

We remark finally that both (52) and (53) must be modified in the presence of singularities of $B^{\pm}$ more severe than allowed above, to account for the possibility of their proximity to the saddle-point (especially in region B). This may be done in the manner described in [20, pp. 399-410], and assures the overall finiteness of the far-field expressions as does the Hadamard finite part in the spectral integral (Appendix B). Additionally, for any modification of the saddle-point technique to be valid, $B^{\pm}$ must have no exponential behavior of any consequence (i.e., comparable to kr or $k\tilde{r}$) which would substantially affect the location of the saddle point.

VI. Power and Orthogonality

One implication of the mode orthogonality statements following (13) for the straight waveguide is that the total power flow through a guide cross-section is simply the sum (or in the case of continuous modes, the integral) of the powers carried by the individual modes. Thus, taking positive power flow to be in the +y direction, the surface-wave mode

$$E_z = B_o^{\pm}\Psi_{so}\, e^{\pm ik\nu_o y}$$

carries the time-average power

$$\begin{aligned} P_{so}^{\pm} &= \tfrac{1}{2}\mathrm{Re}\int_0^{\infty} \overline{E}\times\overline{H}^{*}\cdot\, \overline{a}_y dx = \pm\frac{\nu_o}{2Z_o}\int_0^{\infty}|E_z|^2\, dx \\ &= \pm\frac{\nu_o/\gamma_o}{4\omega\mu_o}\, |B_o^{\pm}|^2 \end{aligned} \qquad (55)$$

whereas the collection of radiation modes

$$E_z = \int_0^{\infty} B^{\pm}(w)\,\Psi_{co}(w,x)e^{\mp ik\nu y}\, dw$$

carries the power

$$P^{\pm}_{co} = \pm \frac{1}{2Z_o} \mathrm{Re} \int_0^\infty \int_0^\infty \{B^{\pm}(w) B^{\pm *}(w') [e^{\mp ik\nu y}] [\nu' e^{\mp ik\nu' y}]^* \cdot$$

$$\cdot \int_0^\infty \Psi_{co}(w,x)\Psi_{co}(w',x)dx\} \; dw\, dw' \tag{56}$$

$$= \pm \frac{\pi}{4\omega\mu_o} \int_0^1 \nu(w^2 + \gamma_o^2)|B^{\pm}(w)|^2 \; dw$$

by virtue of the orthogonality properties of these modes.

It is useful to inquire whether, on the curved section, similar orthogonality statements might apply to the asymptotic mode functions Ψ_s and Ψ_c, in order that we be able to similarly discuss the power transport properties of the modes and pseudo-modes. We first consider the surface-wave:

$$E_z = B_o^{\pm} \Psi_s(x;R) e^{\mp ik\nu_p s}$$

Since Ψ_s (before it is expanded asymptotically) is an exact eigenfunction of the guide, it may be shown, using the relation [26, p. 484]

$$\int \mathcal{C}_{\nu kR}(k\rho) \mathcal{D}_{\mu kR}(k\rho) \frac{R}{\rho}\, d\rho = \frac{\rho}{(\mu^2-\nu^2)kR} [\mathcal{C}_{\nu kR}(k\rho) \mathcal{D}'_{\mu kR}(k\rho) - \mathcal{D}_{\mu kR}(k\rho) \mathcal{C}'_{\nu kR}(k\rho)] \tag{57}$$

where $\mathcal{C}$ and $\mathcal{D}$ are arbitrary cylinder functions, that

$$\int_R^\infty \left[\Psi_s(x;R)\right]^2 \frac{R}{\rho}\, d\rho = \frac{\nu_o/\nu_\rho}{2k\gamma_o} \tag{58}$$

where the limit $\mu \to \nu = \nu_\rho$ has been taken in (57), and (16), the boundary condtions, and the Wronskian of the relevant cylinder functions have been used.

Comparing this expression with (55), it is clear that (58) cannot give a statement of power orthogonality, inasmuch as the exact eigenfunction is known to be lossy because of radiation of the surface-wave [2]. In the asymptotic form of Ψ_s found in Appendix A, however, these losses have been neglected; hence, since in the approximation involved in equation (29) we have

$$[H^{(2)}_{\nu_\rho kR}(k\rho)]^* \simeq - H^{(2)}_{\nu_\rho kR}(k\rho)$$

we may write down the following "asymptotic" power relation in analogy with (55):

$$P_s^{\pm} = \tfrac{1}{2}\,\mathrm{Re}\int_R^\infty E \times H^* \cdot \bar{a}_\phi d_\rho = \pm \frac{h_p}{2Z_o}\int_R^\infty |E_z|^2 \frac{R}{\rho} d\rho \tag{59}$$

$$\simeq \frac{\nu_o/\gamma_o}{4\omega\mu_o} |B_o^{\pm}|^2 e^{\mp 2\alpha_p s}$$

where $\nu_p = h_p - i\alpha_p$; $h_p, \alpha_p > 0$. When Ψ_s is represented in the form (29), the relation (58) is to be understood as an asymptotic relation rather than an equality. That is, if N terms in the expression (29) are retained, expression (58) is valid to $0[(kR)^{-(N+1)}]$. It is important to note that, at $s = 0$, equation (59) reduces to (55) as $R \to \infty$ (as it must), and that it is <u>independent of kR</u>.

A similar orthogonality statement is derived in Appendix C for the pseudo-modes Ψ_c:

$$\int_R^\infty \Psi_c(w,x;R)\Psi_c(w',s;R)\,\frac{R}{\rho}\,d\rho \sim \frac{N^2(w)}{k}\{\delta(w-w') - \delta(w+w')\}$$
$$= \frac{N^2(w)}{k}\,\delta(w-w') \qquad (0 \leq w,\ w' < \infty) \tag{60}$$

The sense of (60) is similar to that of (58): if N terms in the development (30) are retained, the integration (60) will result in (say)

$$\frac{N^2(w)}{k}\,\delta(w-w') + \delta^{(N+1)}(w-w')0[(kR)^{-(N+1)}]$$

Since the sense of generalized functions is always that they be integrated with an infinitely differentiable "test function" $\phi(w')$, any such integration will produce the result

$$\frac{N^2(w)}{k}\ \phi(w) + \phi^{(N+1)}(w)O[(kR)^{-(N+1)}]$$

understood in the ordinary asymptotic sense. It may be similarly shown that the asymptotic relation

$$\int_R^\infty \Psi_s(x;R)\Psi_c(w,x;R)\frac{R}{\rho}\,d\rho \sim O[(kR)^{-N}]\ \ (\text{for any}\ \ N) \tag{61}$$

as $kR \to \infty$ in the above sense, that is, the result of an integration of (61) over w will be asymptotically smaller than any power of $(kR)^{-1}$.

Possessing these orthogonality statements, it is now meaningful to formulate a statement similar to (55) for the curved section. Thus a collection of radiation modes

$$E_z \simeq \int_0^\infty B^\pm(w)\Psi_c(w,x;R)e^{\mp ik\nu s}dw$$

corresponds to a total radiated power

$$P_o^\pm \simeq \frac{\pi}{4\omega\mu_o}\int_o^1 \nu(w^2+\gamma_o^2)|B^\pm(w)|^2dw \tag{62}$$

We remark that, for the curvature discontinuity problem, (55), (56), (59) and (62) imply that the condition of power conservation

$$1 - |T|^2 \simeq |\Omega|^2 + \frac{\pi\gamma_o}{\nu_o}\int_o^1 \nu(w^2+\gamma_o^2)\{|B^-(w)|^2 + |B^+(w)|^2\}dw \tag{63}$$

holds to the retained order of accuracy in kR. It is a straightforward manner to verify (63) to the order $(kR)^{-2}$ for the expressions given in (48) and (51).

VII. Discussion and Comparison with Other Results

Shevchenko [4] considers a two-junction problem in which the curved section is terminated with another straight section. If we solve the second junction problem in entirely the same manner as the first (but see also Appendix E), it is easy to show that Shevchenko's results are identical[4] to ours, and furthermore, to explicitly give physical significance to individual terms entering into the two-junction problem. A major advantage of the present method is that the effects of each junction can be examined separately, and the result proves to be much more general than initially perceived by Shevchenko.

With the various corrections which appear in this problem in mind, we can examine the result of Maley [5]. This solution accounts only for the first order change, ν_1 in phase velocity of the surface-wave on the curved section. We have seen that the reflected surface-wave, the geometrical distortion of the surface-wave fields, and continuous mode contributions are all of the same (1/kR) order as the phase velocity correction, and hence their neglect in [5] explains the difference in radiated fields. The radiation pattern of one of the discontinuities as found in [5] is

$$F(\theta) = \frac{\nu_o^2 - \cos^2\theta}{\pi^2 4\nu_o^2\gamma_o^2(\nu_0 - \cos\theta)^4 (kR)^2} \tag{64}$$

The discrepancy between (54) and (64) is rather serious near the point of maximum radiation $(\theta = \gamma_o/2)$.

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[4] Note, however, some misprints in [4], especially involving the reflection coefficient.

It is interesting to note that an analysis of the plane wave diffraction problem at a curvature discontinuity [34] gives an expression for the scattered field identical to the present one, if a complex angle of incidence (with $\cos\alpha = \nu_o$) is formally inserted into their result (even though only real angles of incidence were considered).

The approach outlined in this report can be considerably simplified by avoiding the derivation given in section III, and instead (and with considerable hindsight) making an ansatz regarding the existence and asymptotic expansions of Ψ_s and Ψ_c. We may then construct the terms in these series directly from the differential equation as shown in Appendix D. Furthermore, our investigation of the Green's function showed that the modal functions were (as always, asymptotically) complete. Once again regarding completeness along with orthogonality (see Appendix D) as an ansatz, we may avoid the use of Green's theorem and formulation of an integral equation by assuming a priori a field expansion in the curved section (Appendix E).

Let us then summarize the limitations we found to this approach. The Debye expansions were valid for kR large (more precisely $(kR)^{2/3} \gg 1$). The Taylor series required that $x/R \ll 1$. Thus, this technique is

a) asymptotic, and

b) local to the guide surface.

As such it bears a great deal of similarity to various asymptotic theories of surface-wave propagation [14-19], which have previously been developed only when the directrix of the guiding structure is smooth. Further work along these lines should prove quite useful.

The next important problem to be attacked by this method is that of the concave bend. The main difference presented by this problem is the possibility of non-attenuating, "whispering-gallery" type modes. Energy converted into such modes from the surface-wave must not only be considered as loss, but the possibility of interference with and consequently further degradation of the signal makes these modes potentially more destructive than the more straightforward losses considered in this report.

Finally, let us compare the scattered radiation loss determined above with the continuous attenuation suffered by the surface-wave propagating around the bend. The latter is proportional to the total angle Φ of bend, and is given by [23]

$$P_{surf} \simeq \frac{2}{\gamma_o} \exp(1/\nu_o^2) \exp[-\tfrac{2}{3}(\nu_o \delta)^{-1}] (\Phi/\delta) \tag{65}$$

for $\delta = \nu_o/(\gamma_o^3 kR) << 1$, whereas by equations (51) and (61), the scattered power (both radiated and in the reflected surface wave) is

$$P_{scat} \simeq -2 \frac{\mathrm{Re}\, T^{(1)}}{(kR)^2} = \frac{\nu_o^2(\nu_o^2+4)}{16\gamma_o^6 (kR)^2} = \frac{\delta^2(\nu_o^2+4)}{16} \tag{66}$$

In Fig. 10, P_{surf} and P_{scat} are compared for a bend angle of $\Phi = 30^o$. It is seen that for extremely small ($\lesssim .05$)δ, the scattering loss will dominate whereas for larger δ (but still small compared to 1), the exponential dependence of (65) causes a dramatic increase in the attenuation of the surface wave. Figure 11 shows the critical angle ϕ_c at which the two-power losses become equal. Though quite a sensitive function of δ near the transition point $\delta \simeq .05$, it is seen to be relatively insensitive to changes in γ_o, outside of its dependence in δ. Since $\gamma_o = 0.1$ would be

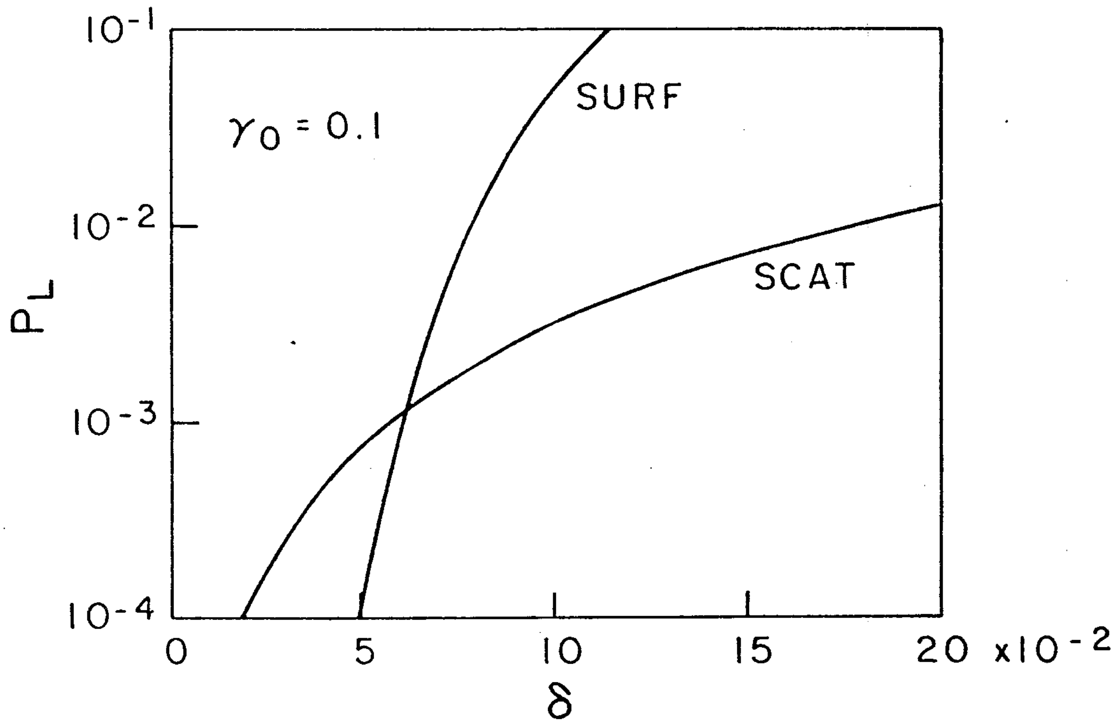

Fig. 10

Power lost by the surface-wave in a bend of $\Phi = 30^{o}$ versus the scattered power at a curvature discontinuity.

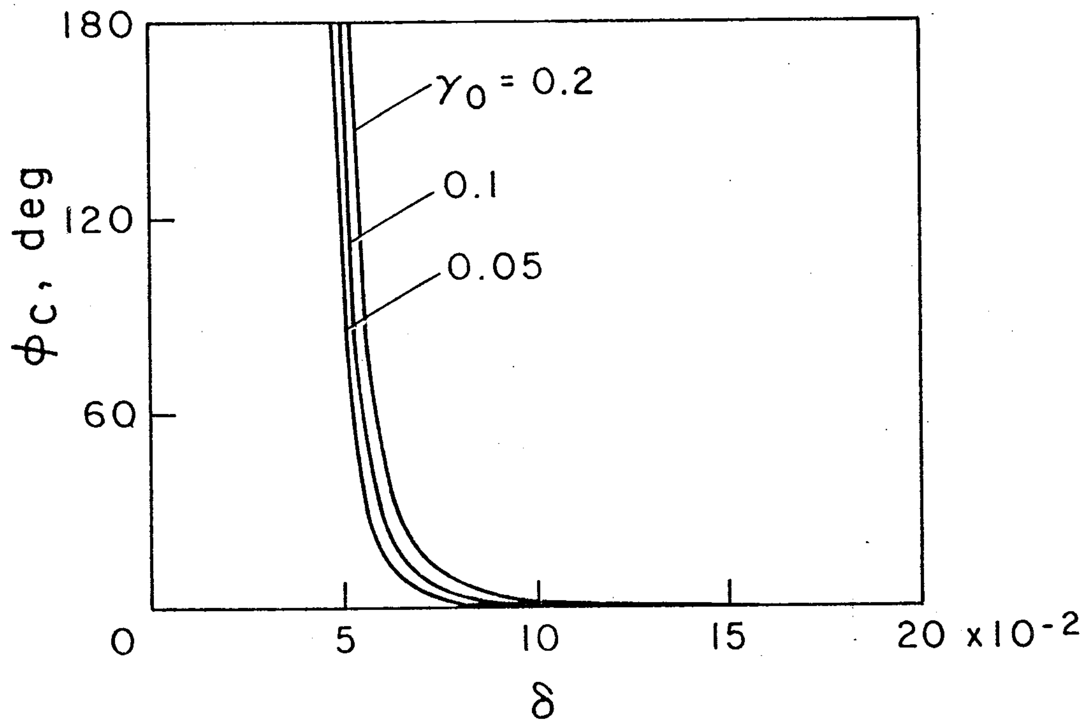

Fig. 11

Angle ϕ_c of bend at which $P_{surf} = P_{scat}$

a quite typical value in an optical waveguide situation, this indicates the possibility that either type of radiation loss might be most important. Indeed, the experimental situation of [35] for the dielectric slab turns out to be quite close to the cross-over point shown in Fig. 10, which is consistent with the fact that neither type of loss alone could explain the obtained results.

Acknowledgments

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Appendix A

Asymptotic form of the surface-wave mode function

In this Appendix an asymptotic expression for the mode function $\Psi_s(x,R)$ defined by (16) is derived. For purposes of solving the integral equation, such an expression need only be valid on the aperture, i.e., for $s = s'$. It is thus unnecessary to calculate the attenuative part of this mode, since this will be significant only after the surface-wave has traversed an appreciable distance around the bend. Once the excitation of the mode at the aperture has been determined, these corrections (which are calculable as in [2], [23]) may be made to properly account for the attenuation away from the aperture.

Since we may expand $H^{(2)}_{\nu_p kR}(k\rho)$ in much the same way as was done on the contour C_2', the main problem is to properly expand the term in brackets in (16). Using the Debye expansions given in section III for the Hankel functions, it is clear that the parts of $H^{(1)}$ and $H^{(1)'}$ which are proportional to $H^{(2)}$ and $H^{(2)'}$ respectively will contribute nothing to the residue, hence the bracket term in (16) becomes

$$\left\{\frac{\nu_o \pi kR}{4\gamma_o} \quad \frac{N(\nu_p, R)}{\frac{\partial}{\partial\nu}[D(\nu,R)]_{\nu=\nu_p}}\right\}^{\frac{1}{2}} e^{ikR\zeta}$$

where

$$N(\nu,R) \simeq N_o(\nu) + \frac{1}{kR} N_1(\nu) + \ldots$$

$$D(\nu,R) \simeq D_o(\nu) + \frac{1}{kR} D_1(\nu) + \ldots$$

and here

$$N_o(\nu) = w - i\gamma_o \quad ; \quad D_o(\nu) = w + i\gamma_o$$

$$N_1(\nu) = \frac{i}{2w^2} - \frac{i}{24}\frac{(3+2\nu^2)}{w^3} N_o(\nu) \tag{A.1}$$

$$D_1(\nu) = -\frac{i}{2w^2} + \frac{i}{24}\frac{(3+2\nu^2)}{w^3} D_o(\nu)$$

We can thus write

$$\nu_p \sim \nu_o + \frac{\nu_1}{kR} + \frac{\nu_2}{(kR)^2} + \ldots \tag{A.2}$$

which is determined as the zero of $D(\nu,R)$ which approaches ν_o as $kR \to \infty$. Thus [23] we have

$$\nu_1 = -D_1(\nu_o)/D'_o(\nu_o) \qquad \text{etc.}$$

by expanding the $D_i(\nu)$ in Taylor series about ν_o and equating coefficients of like powers of $(kR)^{-1}$ to zero. The derivative can be expanded similarly as

$$\frac{\partial}{\partial\nu}[D(\nu,R)]_{\nu=\nu_p} \sim D'_o(\nu_o)\ \{1 + \frac{1}{kR}\frac{\nu_1 D''_o(\nu_o) + D'_1(\nu_o)}{D'_o(\nu_o)} + \ldots\}$$

Thus

$$\left.\frac{N(\nu,R)}{\frac{\partial}{\partial\nu}[D(\nu,R)]}\right|_{\nu=\nu_p} \simeq \frac{N_o(\nu_o)}{D'_o(\nu_o)} + \frac{1}{kR}\left\{\frac{\nu_1 N'_o(\nu_o) + N_1(\nu_o)}{D'_o(\nu_o)} - \frac{N_o(\nu_o)[\nu_1 D''_o(\nu_o) + D'_1(\nu_o)]}{[D'_o(\nu_o)]^2}\right\} + \ldots$$

In the present situation the above results in

$$\nu_1 = 1/(2\nu_o\gamma_o)$$

$$\left. \frac{N(\nu,R)}{\frac{\partial}{\partial\nu}[D(\nu,R)]} \right|_{\nu=\nu_p} = \frac{2\gamma_o^2}{\nu_o} [S(R)]^2$$

where

$$S(R) \sim 1 - \frac{1}{kR} \left(\frac{2\gamma_o^4 + 13\gamma_o^2 + 5}{24\nu_o^2 \gamma_o^3} \right) + \ldots \tag{A.3}$$

Using the Debye expansion for $H^{(2)}_{\nu_p kR}(k\rho)$ in (16) then gives

$$\Psi_s(x,R) \simeq e^{-ikh_\rho} \left(\frac{1-\nu_o^2}{\alpha^2-\nu_\rho^2} \right)^{\frac{1}{4}} S(R) S^+_{L_\rho} \tag{A.4}$$

Finally, if the ρ-terms remaining are expanded in a Taylor series about $x = 0$, and remembering that ν_p is also a series in $(kR)^{-1}$, we have finally

$$\Psi_s(x,R) \simeq \Psi_{so}(x) + \frac{1}{kR} \Psi_{s1}(x) + \ldots$$

where (A.5)

$$\Psi_{s1}(x) = e^{-k\gamma_o x} \left[\frac{k^2\nu_o^2 x^2}{2\gamma_o} - \frac{\gamma_o^2 + \nu_o^2}{4\nu_o^2 \gamma_o^3} \right]$$

The form of Ψ_{so} and the x^2 term of Ψ_{s1} agree, with the exception of a sign error which appears to result from improperly evaluating the residue, with the result of Molotkov [17], who has obtained them using a constructive approach beginning with the differential equation. This approach is used in Appendix D to calculate both the surface-wave and continuous modal functions, thus avoiding the tedious procedures of this Appendix and section III.

Appendix B

Handling of singularities in the contour deformation

Terms of higher order in $(kR)^{-1}$ in (22) may require the evaluation of an integral of the form

$$\int_C \frac{A(w)\,dw}{w^{2n}} \tag{B.1}$$

where the integral is over the contour shown in Fig. 8, n is a positive integer, and $A(w)$ is an even analytic function of w. The integration is assumed to converge suitably at infinity.

By deforming the contour away from the real axis to some contour C' (the broken line in Fig. 8) and writing (B.1) as

$$\int_{C'} \frac{A(w)}{w^{2n}}\,dw = \int_{C'} \frac{A(w) - A(0) - \frac{w^2}{2!}A''(0) - \ldots - \frac{w^{2n-2}}{(2n-2)!}A^{[2n-2]}(0)}{w^{2n}}\,dw\ .$$

$$+ \sum_{j=0}^{n-1} \frac{A^{[2j]}(0)}{(2j)!} \int_{C'} \frac{dw}{w^{2(n-j)}} \tag{B.2}$$

All terms of the sum vanish since $w^{-p} \to 0$ as $w \to \infty$ for $p \geq 1$. But the integrand of the remaining integral is finite at $w = 0$ and so C' may be deformed back onto the real axis, and the integral is an ordinary one:

$$\int_C \frac{A(w)\,dw}{w^{2n}} = \int_{-\infty}^{\infty} \frac{A(w) - A(0) - \frac{w^2}{2!}A''(0) - \ldots - \frac{w^{2n-2}}{(2n-2)!}A^{[2n-2]}(0)}{w^{2n}}\,dw$$

$$\equiv \text{P.f.} \int_{-\infty}^{\infty} \frac{A(w)}{w^{2n}} \, dw \tag{B.3}$$

The integral in (B.3) is known as the "finite part" of the divergent integral

$$\int_{-\infty}^{\infty} \frac{A(w)}{w^{2n}} \, dw \ ,$$

a notion introduced by Hadamard [29,30]. It is a generalization of the notion of Cauchy principal values, and in many cases retains the desirable properties of ordinary integrals. The finite part may also be regarded as an evaluation of the integral in the sense of generalized functions [30,31].

Appendix C

Proof of the orthogonality of the pseudo-modes in the asymptotic sense

It will facilitate the proof if we note that we can rewrite (25) for $0 \leq w < \infty$ (i.e., on the left side of the lower cut in the ν-plane) as

$$\Psi_c(w,x;R) \sim \frac{(kRw)^{\frac{1}{2}}}{2} N(w)[-P(\nu,R)]^{-\frac{1}{2}} \{H^{(1)}_{\nu kR}(k\rho) - P(\nu,R)\, H^{(2)}_{\nu kR}(k\rho)\} \qquad (C.1)$$

where

$$P(\nu,R) \sim -\Gamma(\nu,R) \exp[2i(kR\zeta - \pi/4)]$$

To cover the case $-\infty < w \leq 0$, we utilize the relation

$$\Psi_c(w,x;R) = (\text{sgn } w)\, \Psi_c(|w|,x;R)$$

Note that it is not necessarily claimed that (C.1) represents a valid mode function for the structure, but only that we can utilize the asymptotic equivalence in proving a general (asymptotic) orthogonality relation.

Using formula (57), the boundary conditions at $\rho = R$ and the asymptotic forms of $H^{(1)}$ and $H^{(2)}$ for $k\rho \to \infty$, we obtain

$$\int_R^\infty \Psi_c(w,x;R)\, \Psi_c(w',x;R) \frac{R}{\rho} d\rho \sim (\text{sgn } w)(\text{sgn } w') \frac{i|ww'|^{\frac{1}{2}} N(w)\, N(w')}{k\pi(\nu'^2 - \nu^2)} \cdot$$

$$\cdot \{[-P(\nu,R)]^{\frac{1}{2}} [-P(\nu',R)]^{-\frac{1}{2}} e^{ikR\pi(\nu-\nu')/2} - [-P(\nu,R)]^{-\frac{1}{2}}[-P(\nu',R)]^{\frac{1}{2}} e^{-ikR\pi(\nu-\nu')/2}\} \qquad (c.2)$$

As $kR \to \infty$, (C.2) has the asymptotic sense of a generalized function. Reinserting the asymptotic form of $P(\nu,R)$ gives

$$\int_R^\infty \Psi_c(w,x;R)\ \Psi_c(w',x;R)\ \frac{R}{\rho}\,d\rho \sim (\operatorname{sgn} w)(\operatorname{sgn} w')\ \frac{i|ww'|^{\frac{1}{2}}\ N(w)\ N(w')}{k\pi(w^2-w'^2)}\ \cdot$$

$$\cdot\ \left\{\left[\frac{\Gamma(\nu,R)}{\Gamma(\nu',R)}\right]^{\frac{1}{2}} e^{ikR[f(w)-f(w')]} - \left[\frac{\Gamma(\nu',R)}{\Gamma(\nu,R)}\right]^{\frac{1}{2}} e^{-ikR[f(w)-f(w')]}\right. \tag{C.3}$$

where $f(w) = \zeta + \pi\nu/2 = |w| - \nu[\arccos\nu - \pi/2]$.

The sense of (C.3) is as follows. Consider, for example, the relation

$$\frac{e^{-i\alpha(x-x')}}{x-x'} \sim -\pi i\delta(x-x') \qquad (\alpha\to\infty) \tag{C.4}$$

This statement is valid in the sense that

$$\int_{-\infty}^{\infty} \frac{e^{-i\alpha(x-x')}}{x-x'}\ g(x')\ dx' \sim -\pi i g(x) + o(\alpha^{-N}) \qquad (\alpha\to\infty)$$

for <u>any</u> positive N , if $g(x)$ is an infinitely differentiable function which decays, along with its derivatives, suitable rapidly at $\pm\infty$ (i.e., a "good" function [31, p. 41]). This follows from the asymptotic properties of Fourier transforms of generalized functions [31, pp. 316-319].

Equation (C.3) can thus be written

$$\int_R^\infty \Psi_c(w,x;R)\ \Psi_c(w',x;R)\ \frac{R}{\rho}\,d\rho \sim -\frac{|ww'|^{\frac{1}{2}}N(w)N(w')(\operatorname{sgn} w)(\operatorname{sgn} w')[f(w)-f(w')]}{k(w^2-w'^2)}\ \cdot$$

$$\cdot\left\{\left[\frac{\Gamma(\nu,R)}{\Gamma(\nu',R)}\right]^{\frac{1}{2}} + \left[\frac{\Gamma(\nu',R)}{\Gamma(\nu,R)}\right]^{\frac{1}{2}}\right\}\ \delta[f(w)-f(w')] \tag{C.5}$$

This expression is zero unless $f(w) = f(w')$, which can occur only at $w' = \pm w$. Now the rules for change of variable in the delta-function state that, under suitable conditions on $h(x)$ [31, p. 256]:

$$\delta[h(x)] = \sum_m \frac{\delta(x-x_m)}{|h'(x_m)|}$$

where x_m are the (simple) zeroes of h. Since

$$f'(w) = -(w/\nu) \arcsin \nu$$

the right hand side of (C.5) becomes

$$-\frac{|ww'|^{\frac{1}{2}} \operatorname{sgn}(w) \operatorname{sgn}(w') N(w) N(w')}{k} \frac{f(w) - f(w')}{w^2-w'^2} \left\{ \left[\frac{\Gamma(\nu,R)}{\Gamma(\nu',R)}\right]^{\frac{1}{2}} + \left[\frac{\Gamma(\nu',R)}{\Gamma(\nu,R)}\right]^{\frac{1}{2}} \right\} \cdot$$

$$\cdot \left\{ \frac{\delta(w-w') + \delta(w+w')}{|(w/\nu) \arcsin \nu|} \right\}$$

$$= \frac{N^2(w)}{k} \{\delta(w-w') - \delta(w+w')\} \tag{C.6}$$

This is precisely the relation which obtains in the straight waveguide [4, p. 29]. When restricted to $0 \leq w, w' < \infty$, (C.6) becomes simply

$$\frac{N^2(w)}{k} \delta(w-w')$$

which is the form quoted following equation (13).

Appendix D

Constructive method for determining the modal functions

Once the modal functions are derived directly from the Green's function and their structure is realized, a much simpler method for their calculation can be used (that of Lewin [32], Jouguet [36], or Molotkov [17], with certain generalizations), permitting extension of this method to more complicated structures. We present this method for both discrete and continuous modal functions in the present Appendix.

The wave equation can be written as

$$(x + R) \frac{\partial}{\partial x} [(x + R)\frac{\partial \Psi}{\partial x}] + R^2 \frac{\partial^2 \Psi}{\partial s^2} + k^2(x + R)^2\Psi = 0 \tag{D.1}$$

We consider first the surface wave. Assume that $\Psi_s(x,R)$ and ν_p have the asymptotic expansions

$$\Psi_s \sim \Psi_{so} + \frac{1}{kR} \Psi_{s1} + \ldots \tag{D.2}$$

$$\nu_p \sim \nu_o + \frac{1}{kR} \nu_1 + \ldots \tag{D.3}$$

Substituting (D.2) and (D.3) into (D.1) and equating like powers of R^{-1} to zero gives

$$\Psi''_{so} + k^2(1 - \nu_o^2)\, \Psi_{so} = 0 \tag{D.4}$$

$$\Psi_{s1} + k^2(1 - \nu_o^2)\, \Psi_{s1} = -2kx\, \Psi''_{so} - k\, \Psi'_{so} + 2k^2(\nu_o \nu_1 - kx)\, \Psi_{so} \tag{D.5}$$

$$\begin{aligned}\Psi''_{s2} + k^2(1 - \nu_o^2)\, \Psi_{s2} = {} & -k^2x^2\Psi''_{so} - k^2x\, \Psi'_{so} + k^2(2\nu_o\nu_2 + \nu_1^2 - k^2x^2)\Psi_{so} \\ & -2kx\, \Psi''_{s1} - k\, \Psi'_{s1} + 2k^2(\nu_o\nu_1 - kx)\Psi_{s1}\end{aligned} \tag{D.6}$$

and so on. Similarly the boundary condition becomes

$$\{\Psi'_{sj} + k\gamma_o \Psi_{sj}\}_{x=0} = 0 \qquad j = 0,1,2, \ldots \tag{D.7}$$

In the above, primes denote differentiation with respect to x . Condition (D.7) for $j = 0$ and (D.4) now give

$$\Psi_{so} = e^{-k\gamma_0 x}$$

where the (arbitrary) amplitude has been set equal to one.

Proceeding to (D.5), we have the general solution

$$\Psi_{s2} = Ae^{-k\gamma_o x} + Be^{k\gamma_o x} + \frac{e^{-k\gamma_o x}}{(2k\gamma_o)^2} \{2k^3 x(1 - 2\gamma_o \nu_o \nu_1) + 2k^4 x^2 \nu_o^2 \gamma_o\} \tag{D.8}$$

Rejecting the exponentially growing part of the solution as incompatible with the surface wave, we have $B = 0$. Then applying the boundary condition (D.7) for $j = 1$, we find that

$$\nu_1 = 1/(2\nu_o \gamma_o)$$

and

$$\Psi_{s2} = e^{-k\gamma_o x} \{A + \frac{k^2 x^2 \nu_o^2}{2\gamma_o}\}$$

With no more conditions to apply, it would appear that the constant A must remain arbitrary. Lewin [32] has pointed out that this arbitrariness may be seen from the fact that a modal function may be multiplied by any constant which is itself developable in an asymptotic series in $(kR)^{-1}$. Thus, the calculation of each Ψ_{sj} will introduce a further arbitrary constant. We have seen in section V, however (see equation (57)) that an orthogonality property holds when the mode functions are derived in the Green's function:

$$\nu_p \int_R^\infty [\Psi_s(x,R)]^2 \frac{R}{\rho} d\rho \sim \nu_o/(2k\gamma_o) \tag{D.9}$$

or, developing the left-hand side in inverse powers of kR :

$$\nu_o \int_o^\infty \Psi^2 (x) \ dx + \frac{1}{kR} \left\{ \nu_1 \int_0^\infty \Psi_{so}^2(x) \ dx + 2\nu_o \int_0^\infty \Psi_{s1}(x) \ \Psi_{so}(x) \ dx \right.$$
$$\left. - \nu_o \int_0^\infty kx \ \Psi_{so}^2(x) \ dx \right\} + \ldots$$

$$\sim \nu_o/(2k\gamma_o) \tag{D.10}$$

i.e., all higher-order terms on the left must vanish identically. In particular, requiring the term in curly brackets in (D.10) to vanish results in

$$A = -(\nu_o^2 + \gamma_o^2)/(4\gamma_o^3\nu_o^2)$$

in agreement with equation (29). With condition (D.10), this process may be continued to find higher order terms of Ψ_s .

Now we consider the continuous spectrum modal functions. We now assume

$$\Psi_c(w,x,R) \sim \Psi_{co}(w,x) + \frac{1}{kR} \Psi_{c1}(w,x) + \ldots \tag{D.11}$$

but no perturbation on w , which is real and nonnegative. Proceeding as above, we obtain

$$\Psi''_{co} + k^2w^2 \ \Psi_{co} = 0 \tag{D.12}$$

$$\Psi''_{c1} + k^2w^2 \ \Psi_{c1} = -2kx \ \Psi''_{co} - k \ \Psi'_{co} - 2k^3x \ \Psi_{co} \tag{D.13}$$

$$\Psi''_{c2} + k^2w^2 \ \Psi_{c2} = -k^2x^2 \ \Psi''_{co} - k^2x \ \Psi'_{co} - k^4x^2 \ \Psi_{co} - 2kx \ \Psi''_{c1} - k \ \Psi'_{c1} - 2k^3x \ \Psi_{c1} \tag{D.14}$$

with boundary condition (D.7) holding now for the Ψ_{cj}. (D.7) and (D.12) now give

$$\Psi_{co} = \frac{1}{2}\,[(w - i\gamma_o)\, e^{-ikwx} + (w + i\gamma_o)\, e^{ikwx}]$$

where again the arbitrary amplitude has been chosen to match the straight waveguide modal function. From (D.13), now, we obtain the general solution

$$\begin{aligned}\Psi_{c1} = A\, e^{-ikwx} + B\, e^{ikwx} &+ \frac{1}{2}(w - i\gamma_o)\, e^{-ikwx}\, [-\frac{kx}{2w^2} - i\,\frac{k^2\nu^2x^2}{2w}] \\ &+ \frac{1}{2}(w + i\gamma_o)\, e^{ikwx}\, [-\frac{kx}{2w^2} + i\,\frac{k^2\nu^2x^2}{2w}]\end{aligned} \tag{D.15}$$

Use of the boundary conditions gives

$$A(w + i\gamma_o) - B(w - i\gamma_o) = i/2w$$

As before, we are apparently left with one arbitrary constant; however, if we enforce the orthogonality condition (59), we find that

$$\begin{aligned}\int_0^\infty \Psi_{co}(w,x)\,\Psi_{co}(w',x)\,dx + \frac{1}{kR}\,\{ &\int_0^\infty \Psi_{co}(w,x)\,\Psi_{c1}(w',x)\,dx + \int_0^\infty \Psi_{c1}(w,x)\,\Psi_{co}(w',x)dx \\ &- \int_0^\infty kx\Psi_{co}(w,x)\,\Psi_{co}(w',x)\,dx\} + \ldots \\ \sim \frac{N^2(w)}{k}\,[\delta(w - w') &- \delta(w + w')]\end{aligned} \tag{D.16}$$

whence we require the term in curly brackets to vanish. Now, it can be easily shown using standard results from the theory of generalized functions (see [4] or [31]) that

$$\int_0^\infty kx\Psi_{co}(w,x)\ \Psi_{co}(w',x)\ dx = \frac{\pi}{4k}\{-2i(ww' - \gamma_o^2)\ \Lambda'(w + w') - 2i(ww' + \gamma_o^2)\ \Delta'(w - w')$$

$$-2\gamma_o\delta(w + w') + 2\gamma_o\delta(w - w')\}$$

and $$\int_0^\infty \Psi_{co}(w,x)\ \Psi_{cl}(w',x)\ dx = \left(\frac{A(w')}{w' - i\gamma_o} + \frac{B(w')}{w' + i\gamma_o}\right)\frac{N^2(w)}{k}\ [\delta(w-w') - \delta(w+w')]$$

$$- \frac{ww'}{k(w^2 - w'^2)^3}\ [\nu^2(w'^2 + \gamma_o^2) + \nu'^2((w'^2 + \gamma_o^2) + 2(w^2 + \gamma_o^2))]$$

$$+ \frac{\pi\gamma_o}{4kw'^2}\ [\delta(w + w') - \delta(w - w')] - \frac{\pi\gamma_o\nu'^2}{2kw'}\ [\delta'(w + w') + \delta'(w - w')]$$

where

$$\Delta(u) = \text{V.p.}\ (\frac{i}{\pi u})$$

implies that the principal value is to be understood in an integration involving Λ, and similarly the higher order singularities in the above equations imply the Hadamard finite part (see Appendix B). Thus we have

$$\frac{N^2(w)}{k}\ [\delta(w - w') - \delta(w + w')]\left\{\frac{A(w')}{w' - i\gamma_o} + \frac{B(w')}{w' + i\gamma_o} + \frac{A(w)}{w - i\gamma_o} + \frac{B(w)}{w + i\gamma_o}\right\} = 0$$

or, simply

$$A(w + i\gamma_o) + B(w - i\gamma_o) = 0$$

We then finally have

$$A = \frac{i}{4w(w + i\gamma_o)} \qquad B = -\frac{i}{4w(w - i\gamma_o)}$$

which is in agreement with equation (30). Thus, we have shown that with proper enforcement of an orthogonality condition natural to the curved guide, it is possible to construct the asymptotic modal functions in a manner much simpler than that given in the text. As usual for this kind of construction, however, the restrictions on the validity of these constructed functions cannot be obtained from the construction alone, and we must rely on the more careful derivation in the text for such information.

Appendix E

Solution of the aperture field by a "mode-matching" method

Let us begin by assuming the expansions (41) and (44) with Ω, $B_\pm(w)$ and T as unknowns to be determined. We have in addition

$$\begin{aligned} -i\omega\mu_o\, H_x &= \partial E_z/\partial y \qquad (y < 0) \\ -i\omega\mu_o\, H_\rho &= (R/\rho)\ \partial E_z/\partial s \qquad (s > 0) \end{aligned} \tag{E.1}$$

Continuity of tangential E and H at the aperture $(y = s = 0)$ gives

$$\begin{aligned} (1+\Omega)\,\Psi_{so}(x) + \int_0^\infty B_-(w)\,\Psi_{co}(w,x)\,dw &= T\Psi_s(x,R) + \int_0^\infty B_+(w)\,\Psi_c(w,x,R)\,dw \\ \nu_o(1-\Omega)\,\Psi_{so}(x) - \int_0^\infty \nu B_-(w)\,\Psi_{co}(w,x)\,dw &= \Big[\nu_p T\Psi(x,R) + \int_0^\infty \nu B_+(w)\,\Psi_c(w,x,R)\,dw\Big]\frac{R}{\rho} \end{aligned} \tag{E.2}$$

Now we multiply successively by Ψ_{so} and Ψ_{co}, and integrate over x, utilizing the orthogonality properties:

$$(1+\Omega)\,\frac{1}{2k\gamma_o} = T\int_0^\infty \Psi_{so}(x)\,\Psi_s(x,R)\,dx + \int_0^\infty B_+(w)\,P(w,R)\,dw \qquad \text{(a)}$$

$$\nu_o(1-\Omega)\,\frac{1}{2k\gamma_o} = \nu_p T\int_0^\infty \frac{R}{R+x}\,\Psi_{so}(x)\,\Psi_s(x,R)\,dw + \int_0^\infty B_+(w)\,\nu Q(w,R)\,dw \qquad \text{(b)}$$

$$B_-(w)\,\frac{N^2(w)}{k} = T\,W(w,R) + \int_0^\infty B_+(w')\,S(w,w',R)\,dw' \qquad \text{(c)} \qquad \text{(E.3)}$$

$$-\nu B_-(w)\,\frac{N^2(w)}{k} = \nu_p\,T\,V(w,R) + \int_0^\infty B_+(w')\,\nu' U(w,w',R)\,dw' \qquad \text{(d)}$$

Similarly operating with Ψ_s and Ψ_c, and using (57) and (59):

$$T \frac{\nu_o}{\nu_p} \frac{1}{2k\gamma_o} = (1 + \Omega) \int_0^\infty \frac{R}{R+x} \Psi_{so}(x) \Psi_s(x,R)dx + \int_0^\infty B_-(w) V(w,R)dw \qquad (a)$$

$$T\nu_o \frac{1}{2k\gamma_o} = \nu_o(1 - \Omega) \int_0^\infty \Psi_{so}(x) \Psi_s(x,R)dx - \int_0^\infty B_-(w) \nu W(w,R)dw \qquad (b)$$

$$(E.4)$$

$$B_+(w) \frac{N^2(w)}{k} = (1 + \Omega) Q(w,R) + \int_0^\infty B_-(w') U(w',w,R)dw' \qquad (c)$$

$$\nu B_+(w) \frac{N^2(w)}{k} = \nu_o(1 - \Omega) P(w,R) - \int_0^\infty B_-(w') \nu' S(w',w,R)dw' \qquad (d)$$

In (E.3) and (E.4) we have abbreviated

$$P(w,R) = \int_0^\infty \Psi_c(w,x,R) \Psi_{so}(x)dx \sim \frac{1}{kR}\left[\frac{w(\nu^2 + \nu_o^2)}{k(w^2 + \gamma_o^2)^2}\right] + \ldots$$

$$Q(w,R) = \int_0^\infty \frac{R}{R+x} \Psi_c(w,x,R) \Psi_{so}(x)dx \sim \frac{1}{kR}\left[\frac{2w\nu_o^2}{k(w^2 + \gamma_o^2)^2}\right] + \ldots$$

$$S(w,w',R) = \int_0^\infty \Psi_{co}(w,x) \Psi_c(w',x,R)dx \sim \frac{N^2(w)}{k} \delta(w - w') + \ldots$$

$$U(w,w',R) = \int_0^\infty \frac{R}{R+x} \Psi_{co}(w,x) \Psi_c(w',x,R)dx \sim \frac{N^2(w)}{k} \delta(w - w') + \ldots$$

$$V(w,R) = \int_0^\infty \frac{R}{R+x} \Psi_{co}(w,x) \Psi_s(x,R)dx \sim \frac{1}{kR}\left[- \frac{w(\nu^2 + \nu_o^2)}{k(w^2 + \gamma_o^2)^2}\right] + \ldots$$

$$W(w,R) = \int_0^\infty \Psi_{co}(w,x) \Psi_s(x,R)dx \sim \frac{1}{kR}\left[- \frac{2w\nu_o^2}{k(w^2 + \gamma_o^2)^2}\right] + \ldots$$

Equations (32), (33) and (60) have been used in the above. Furthermore,

$$\int_0^\infty \Psi_{so}(x)\ \Psi_s(x,R)dx \sim \frac{1}{2k\gamma_o} + \frac{1}{kR}\left[\frac{1}{8k\nu_o^2}\right] + \ldots$$

$$\int_0^\infty \frac{R}{R+x}\ \Psi_{so}(x)\ \Psi_s(x,R) \sim \frac{1}{2k\gamma_o}\ \frac{\nu_o}{\nu_p} - \frac{1}{kR}\left[\frac{1}{8k\nu_o^2}\right] + \ldots$$

where (31) and (57) have been used. Then to the order of accuracy retained in the above, solving (E.3) or (E.4) (which are equivalent) yields

$$T = 1 + 0(\frac{1}{kR})^2$$

$$\Omega = \frac{1}{kR}\left(\frac{\gamma_o}{4\nu_o^2}\right) + 0(\frac{1}{kR})^2$$

$$B_-(w) = \frac{1}{kR}\left[\frac{w\nu_o}{2\nu N^2(w)\,(W^2 + \gamma_o^2)^2}\ (\nu_o - \nu)^2\right] + 0(\frac{1}{kR})^2$$

$$B_+(w) = \frac{1}{kR}\left[\frac{w\nu_o}{2\nu N^2(w)\,(w^2 + \gamma_o^2)^2}\ (\nu_o + \nu)^2\right] + 0(\frac{1}{kR})^2$$

agreeing with (48). Higher-order terms can, of course, be obtained by computing the integrals P, Q, S, U, V, W to the desired accuracy.

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