An Analysis of Reduced Hessian Methods for Constrained Optimization

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ABSTRACT

We study the convergence properties of reduced Hessian successive quadratic programming for equality constrained optimization. The method uses a backtracking line search, and updates an approximation to the reduced Hessian of the Lagrangian by means of the BFGS formula. Two merit functions are considered for the line seach: the ℓ_1 function and the Fletcher exact penalty function. We give conditions under which local and superlinear convergence is obtained, and also prove a global convergence result. The analysis allows the initial reduced Hessian approximation to be any positive definite matrix, and does not assume that the iterates converge, or that the matrices are bounded. The effects of a second order correction step, a watchdog procedure and of the choice of null space basis are considered. This work can been seen as an extension of the well known results of Powell (1976) for unconstrained optimization to reduced Hessian methods.

Key words. constrained optimization, reduced Hessian methods, quasi-Newton methods, successive quadratic programming, nonlinear programming

AMS(MOS) subject classification. 65, 49

1. Introduction.

In this paper we analyze reduced Hessian successive quadratic programming methods for solving the equality constrained optimization problem

$$\min_{x\in \mathbf{R}^n} f(x)$$
$$\text{subject to } \; c(x)=0, \tag{1.1}$$

where $f : \mathbf{R^n} \to \mathbf{R}$, and $c : \mathbf{R^n} \to \mathbf{R^t}$ are smooth nonlinear functions. These methods, which we also refer to as reduced Hessian methods, generate at x_k a search direction by solving the quadratic program

$$\min_{d \in \mathbf{R^n}} \; g(x_k)^T d + \frac{1}{2} d^T Z_k B_k Z_k^T d$$
$$\text{subject to } c(x_k) + A(x_k)^T d = 0, \tag{1.2}$$

where g is the gradient of f, $A(x) = [\nabla c_1(x), ..., \nabla c_t(x)]$ is the $n \times t$ matrix of constraint gradients, Z_k is a matrix whose columns form an orthonormal basis for the null space of $A(x_k)^T$, and B_k is a matrix that approximates the reduced Hessian of the Lagrangian function. The new iterate is given by

$$x_{k+1} = x_k + \alpha_k d_k,$$

where the steplength α_k is chosen to force progress towards the solution of (1.1). Our goal in this paper is to develop some practical convergence results for reduced Hessian methods in which B_k is updated by the BFGS formula and the initial matrix B_0 is an arbitrary positive definite matrix.

Reduced Hessian methods are a special case of successive quadratic programming (SQP) methods, which are based on the subproblem

$$\min_{d \in \mathbf{R^n}} \; g(x_k)^T d + \frac{1}{2} d^T M_k d$$
$$\text{subject to } c(x_k) + A(x_k)^T d = 0. \tag{1.3}$$

Specifically, problem (1.2) is equivalent to a problem of the form (1.3) with $M_k = Z_k B_k Z_k^T$. The general equality constrained quadratic program (1.3) is equivalent to a problem of the form (1.2) if and only if $Z_k^T M_k A(x_k) = 0$.

Solving problem (1.1) by iterative solution of (1.3) is an old idea since, if $M_k = \nabla^2_{xx} L(x_k, \lambda_k)$ and λ_k is the multiplier vector of the quadratic program at iteration $k-1$, this is equivalent to Newton's method on the Kuhn-Tucker conditions for (1.1). An alternative is to try to make M_k a secant approximation to the Hessian of the Lagrangian, using a positive definite secant update such as BFGS or DFP. That is, M_k would be updated so that $M_{k+1}\tilde{s}_k = \tilde{y}_k$, where $\tilde{s}_k = x_{k+1} - x_k$, and $\tilde{y}_k$ is some vector approximately equal to $\nabla^2_{xx} L(x_k, \lambda_k)\tilde{s}_k$, such as $\nabla_x L(x_{k+1}, \lambda_k) - \nabla_x L(x_k, \lambda_k)$. This idea cannot be carried out in a straightforward fashion since the Hessian of the Lagrangian at a solution of (1.1) is not necessarily positive definite. Several approaches have been proposed for coping with this difficulty, and reduced Hessian SQP is one of these. Before discussing reduced Hessian methods, we briefly mention some other approaches which instead solve a problem of the form (1.3) with M_k an $n \times n$ positive definite matrix.

An early proposal is to update M_k so as to approximate the Hessian of the augmented Lagrangian, $\nabla^2_{xx}L(x_k, \lambda_k) + \rho A_k A_k^T$, which is positive definite near the solution if the scalar ρ is chosen sufficiently large. This was analyzed by Han (1976), Tapia (1977), and Glad (1979), who showed that if a sufficiently large value of the augmentation parameter is used, and if x_0 and M_0 are good enough approximations to the solution and to the Hessian of the augmented Lagrangian, respectively, then the iterates converge Q-superlinearly to the solution. A different approach, due to Powell, is to update the matrix only part way so that $M_{k+1}\tilde{s}_k = \theta\tilde{y}_k + (1-\theta)M_k\tilde{s}_k$, where $\theta \in [0,1]$ is chosen to preserve a degree of positive definiteness. Powell (1978) proves that if $\{x_k\}$ converges to the solution, and if the sequences $\{\|M_k\|\}$ and $\{\|(Z_k^T M_k Z_k)^{-1}\|\}$ are bounded, then the convergence rate is R-superlinear. The same result is proved by Fenyes (1987) for his updating scheme, which preserves positive definiteness only of $Z_k^T M_k Z_k$. Boggs and Tolle (1985) suggest that M_k simply be left unchanged in cases when updating would cause a loss of positive definiteness. They prove that if $\{x_k\}$ converges to the solution Q-linearly, and if the directions produced by the algorithm converge sufficiently fast to the null space of the constraint derivatives, then $\{x_k\}$ converges Q-superlinearly.

The reduced Hessian approach is motivated by the fact that near the solution $Z_k^T\nabla^2_{xx}L(x_k,\lambda_k)Z_k$ is usually positive definite, and thus it is reasonable to approximate this matrix using a positive definite update formula. In this case the matrix B_k of (1.2) would be updated so that $B_{k+1}s_k = y_k$, where $s_k = Z_k^T(x_{k+1} - x_k)$ and y_k is a secant approximation to $Z_k^T\nabla^2_{xx}L(x_k,\lambda_k)Z_k s_k$. The approach also has the advantage that, when $n-t$ is small relative to n, the Hessian approximation that needs to be stored is smaller. Reduced Hessian updating methods have been proposed by Murray and Wright (1978), Gabay (1982), Gilbert (1987), Coleman and Conn (1984), and Nocedal and Overton (1985). For the last two approaches, their proposers prove that if x_0 and B_0 are good enough approximations to the solution and to the reduced Hessian of the Lagrangian, respectively, then the iterates converge 2-step Q-superlinearly to the solution. These two approaches differ primarily in the choice of y_k; that of Coleman and Conn is more costly in function evaluations, but is probably more robust than that of Nocedal and Overton (which is closer to the first two approaches mentioned). Actually, Coleman and Conn consider two versions of their algorithm; here we are referring to the version that uses only one constraint evaluation in the step computation. We also note that Fontecilla (1988) proposes a full Hessian method analogous to the algorithm of Coleman and Conn and proves a similar convergence result.

Most of these methods work reasonably well in most cases, but none of them is regarded as completely satisfactory in theory or in practice (see Powell (1987)). Note that all the above mentioned analyses either assume a good initial approximation to the solution and to the Hessian of the Lagrangian at the solution, or they assume that the iterates converge and that the Hessian approximations are bounded in some way. We regard these assumptions as undesirable since it is not known when they will be satisfied in practice. The objective of this work is to develop a convergence theory for reduced Hessian successive quadratic programming that only assumes of the matrices that the initial one is positive definite, and does not assume that the iterates converge. Since we are making no assumptions on B_k or on the convergence of the iterates, there is no guarantee that $x_k + d_k$ is closer to the solution x_* than x_k is. In practice a line search is usually relied on to force progress towards the solution. This is done by using a merit function $\varphi(x)$, and by computing the steplength α_k so that $\varphi(x_k + \alpha_k d_k)$ is significantly less than $\varphi(x_k)$.

We will analyze a procedure of this type and show that, under certain conditions, if x_1 is within a neighborhood of x_* this decrease in the merit function will force $\{x_k\}$ to converge to x_* R-linearly, whereupon known results will imply that the convergence is superlinear. Thus our work will be somewhat analogous to the well known paper of Powell (1976) on the convergence of the BFGS method with inexact line search for a convex objective function. We have chosen to consider reduced Hessian approaches here primarily because the issues we are interested in are simpler to deal with than for full Hessian approaches. Also for simplicity we have chosen to analyze an updating strategy like that of Coleman and Conn, but many of our results can probably be extended to the more complex Nocedal and Overton strategy.

The algorithm to be studied is defined in Section 2, and the methods for updating B_k and for performing the line search are laid out precisely. We consider two merit functions, the ℓ_1 function proposed as a merit function in Han (1977), and the Fletcher (1970), (1973) exact penalty function.

In Section 3 general results of Byrd and Nocedal (1987) on the BFGS update are used to show that, if an adequate line search is done, then the merit function is decreased significantly for at least a fraction of iterates. This fact is then used to prove a somewhat weak global convergence result. The effect of choice of the weight in the merit function is taken into consideration.

In Section 4 we consider the local behavior of the algorithm near a point satisfying the standard strong sufficiency conditions. We prove that, once the algorithm gets close enough to such a point it will converge R-linearly. The convergence results here and in Section 3 are somewhat more satisfactory for the ℓ_1 merit function than for the Fletcher function.

In Section 5 we study superlinear convergence. We consider the effect of the choice of null space basis Z_k on convergence rate, and look for conditions under which the algorithm takes unit steplenghts near the solution. This is not a problem for the Fletcher function, but for the ℓ_1 function the algorithm needs to be modified. We consider two modifications, the correction step and the watchdog technique, and show that they allow for unit steplenghts near the solution, which ensures a two-step Q-superlinear rate of convergence.

Notation. The Lagrangian function will be defined by

$$L(x, \lambda) = f(x) + \lambda^T c(x), \tag{1.4}$$

and we denote the reduced Hessian of the Lagrangian by G, i.e.

$$G_k = Z_k^T \nabla_{xx}^2 L(x_k, \lambda_k) Z_k. \tag{1.5}$$

Throughout the paper $\|\cdot\|$ denotes the l_2 vector norm or the corresponding induced matrix norm. When using the l_1 or l_∞ norms we will indicate it explicitly by writing $\|\cdot\|_1$ or $\|\cdot\|_\infty$. We recall that the l_1 and l_∞ norms are duals of each other, so that $\lambda^T c \leq \|\lambda\|_\infty \|c\|_1$. A solution of the problem (1.1) is denoted by x_*, and we let $e_k = x_k - x_*$.

2. Reduced Hessian Methods with Line Search

Now we describe a general reduced Hessian SQP algorithm of the type discussed in §1. We denote the merit function by φ, and its directional derivative at x in the direction d, by $D\varphi(x; d)$. The precise form of φ will be discussed later.

Algorithm 2.1

The constants $\eta \in (0, \frac{1}{2})$ and τ, τ' with $0 < \tau < \tau' < 1$ are given.

(1) Set $k = 1$ and choose a starting point x_1 and a symmetric and positive definite starting matrix B_1.

(2) Compute Z_k and obtain d_k by solving the quadratic program

$$\begin{aligned} &\min_{d \in \mathbf{R}^n} g_k^T d + \frac{1}{2} d^T Z_k B_k Z_k^T d \\ &\text{subject to } c_k + A_k^T d = 0. \end{aligned} \tag{2.1}$$

(3) Set $\alpha_k = 1$.

(4) Test the line search condition

$$\varphi(x_k + \alpha_k d_k) \leq \varphi(x_k) + \eta \alpha_k D\varphi(x_k; d_k). \tag{2.2}$$

(5) If (2.2) is not satisfied, choose a new α_k in $[\tau\alpha_k, \tau'\alpha_k]$ and go to (4); otherwise set

$$x_{k+1} = x_k + \alpha_k d_k. \tag{2.3}$$

(6) Compute

$$s_k = Z_k^T (x_{k+1} - x_k), \tag{2.4}$$

$$y_k = Z_k^T [\nabla_x L(x_k + \alpha_k h_k, \lambda_k) - \nabla_x L(x_k, \lambda_k)], \tag{2.5}$$

where λ_k is chosen so that (2.12) is satisfied. If $s_k \neq 0$ update B_k using the BFGS formula

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k y_k^T}{y_k^T s_k}. \tag{2.6}$$

(7) Set $k := k + 1$, and go to (2).

The solution to subproblem (2.1), which gives the step direction, may be expressed as

$$d_k = h_k + v_k, \tag{2.7}$$

where

$$h_k = -Z_k B_k^{-1} Z_k^T g_k, \tag{2.8}$$

and

$$v_k = -A_k [A_k^T A_k]^{-1} c_k, \tag{2.9}$$

give an orthogonal decomposition of d_k, and where g_k stands for $g(x_k)$, etc. The vector v_k is in the range space of A_k and may be regarded as a minimum norm Newton step on the equation $c(x) = 0$. The vector h_k lies in the null space of A_k^T, tends to move toward a stationary point of the Lagrangian and, to first order, leaves the value of c unchanged. Note that the approximation matrix B_k only affects the null space component h.

The procedure for choosing a new value of α in step (5) is not specified precisely so that our analysis can cover a variety of line search strategies. There are several procedures, such as a safeguarded interpolatory line search algorithm or simple multiplication by a constant, that would give a new α_k in the specified interval. Note that the line search always reduces the steplength and thus $\alpha_k \leq 1$ for all k. This is common in successive quadratic programming algorithms, and is due to the condition $c(x_k) + A(x_k)^T d_k = 0$.

In the algorithm, Z_k refers to an $n \times (n-t)$ matrix satisfying $A_k^T Z_k = 0$ and $Z_k^T Z_k = I$. These conditions do not specify Z_k uniquely, and the iteration does depend on our choice of Z_k. It turns out, however, that the results in Sections 3 and 4 are true for any choice of Z_k, and that only to prove superlinear convergence do we need to place additional restrictions on Z_k.

Let us now discuss the choice of the vectors s_k and y_k needed in step (6). Since B_k is meant to be an approximation to the reduced Hessian of the Lagrangian $Z_k^T \nabla_{xx}^2 L(x_k, \lambda_k) Z_k$ based on information at x_k and x_{k+1}, it is reasonable to define s_k by (2.4), or equivalently by

$$s_k = \alpha_k Z_k^T h_k, \tag{2.10}$$

but we could have replaced Z_k by Z_{k+1} in these expressions. The choice of y_k is less obvious. The formula we use in Algorithm 2.1 is that proposed and analyzed by Coleman and Conn (1984). To motivate this formula for y_k note from (2.10), and from the fact that $Z_k Z_k^T h_k = h_k$, that

$$\begin{aligned} Z_k^T \nabla_{xx}^2 L(x_k, \lambda_k) Z_k s_k &= Z_k^T [\nabla_{xx}^2 L(x_k, \lambda_k) \alpha_k h_k] \\ &\approx Z_k^T [\nabla_x L(x_k + \alpha_k h_k, \lambda_k) - \nabla_x L(x_k, \lambda_k)]. \end{aligned}$$

Since we want to impose the secant condition $B_{k+1} s_k = y_k$ it is natural to define y_k by (2.5). There are several slight variations of the formula for y_k that could be used. For example we could define

$$y_k = Z_{k+1}^T [\nabla_x L(x_{k+1}, \lambda_{k+1}) - \nabla_x L(x_{k+1} - \alpha_k h_k, \lambda_{k+1})],$$

thereby using the most recent information available. We will only consider the definition (2.5), but the results of this paper also hold for several of these variations.

A significantly different formula for y_k is

$$y_k = Z_{k+1}^T[\nabla_x L(x_{k+1}, \lambda_{k+1}) - \nabla_x L(x_k, \lambda_{k+1})]. \tag{2.11}$$

Formulas of this type have been suggested by Murray and Wright (1978), Gabay (1982), and Nocedal and Overton (1985). An advantage of using (2.11) is that it requires only one evaluation of the derivatives of f and c per iteration as opposed to two evaluations for (2.5). However, Nocedal and Overton note that (2.11) can be subject to instability in some cases, and in their analysis they stipulate that under certain conditions the update be skipped. In this paper we will analyze only the choice (2.5), and leave the formulas like (2.11), whose analysis is more complicated, for subsequent study.

There are several effective ways to estimate the Lagrange multiplier in the Hessian of the Lagrangian. We require only that λ_k be chosen so that

$$\|\lambda_k - \lambda_*\| \le \gamma_\lambda \|x_k - x_*\| \tag{2.12}$$

is satisfied for some constant γ_λ. This condition is satisfied by several formulas including

$$\lambda_k = -\left[A_k^T A_k\right]^{-1} A_k^T g_k \tag{2.13}$$

and

$$\lambda_k = -\left[A_k^T A_k\right]^{-1} [A_k^T g_k - c_k]. \tag{2.14}$$

Powell (1976) has shown that the BFGS method for unconstrained minimization has strong convergence properties if $y_k^T s_k > 0$ for all k, and if the sequence $\{y_k^T y_k / y_k^T s_k\}$ is uniformly bounded above. In this paper we will show that these two conditions are also crucial in the analysis of Algorithm 2.1. The following lemma shows that the definition (2.5) of y_k ensures that these two conditions hold near the solution.

Lemma 2.1 *Given an iterate x_k, a step $\alpha_k h_k$ and a Lagrange multiplier estimate λ_k, assume that there exist positive constants m, M such that*

$$m\|w\|^2 \le w^T \left[Z_k^T \nabla_{xx}^2 L(x, \lambda_k) Z_k\right] w \le M\|w\|^2, \tag{2.15}$$

for all $w \in \mathbf{R}^{n-t}$, and for all x in the line segment joining x_k and $x_k + \alpha_k h_k$. Then

$$\frac{y_k^T s_k}{\|s_k\|^2} \ge m \tag{2.16}$$

and

$$\frac{\|y_k\|^2}{y_k^T s_k} \le M. \tag{2.17}$$

Proof: If we define

$$\overline{G}_k = Z_k^T \int_0^1 \nabla_{xx}^2 L(x_k + \tau\alpha_k h_k, \lambda_k) d\tau Z_k,$$

then we have from (2.5)

$$y_k = \overline{G}_k s_k. \tag{2.18}$$

Thus (2.16) and (2.17) can be shown to follow from (2.15).

□

We now consider some merit functions to be used in step (4) of the algorithm. The first merit function used in a successive quadratic programming algorithm was the ℓ_1 merit function (cf. Han (1976))

$$\phi_\mu(x) = f(x) + \mu\|c(x)\|_1. \tag{2.19}$$

Han used the ℓ_1 norm of $c(x)$, but other choice of norms are possible. An alternative is the differentiable function proposed by Fletcher (1973). It is given by

$$\Phi_\nu(x) = f(x) + \hat{\lambda}(x)^T c(x) + \frac{1}{2}\nu\|c(x)\|^2, \tag{2.20}$$

where

$$\hat{\lambda}(x) = -\left[A(x)^T A(x)\right]^{-1} A(x)^T g(x) \tag{2.21}$$

is the least squares Lagrange multiplier estimate at x. To compute the derivative of this merit function requires second order information, due to the term $\hat{\lambda}(x)$. However Powell and Yuan (1986) describe a procedure that uses finite differences to approximate these second order terms with no extra evaluation of $\hat{\lambda}(x)$. In this paper we will assume, for simplicity, that the derivative of $\hat{\lambda}(x)$ is computed exactly.

Boggs and Tolle (1984) propose a merit function similar to (2.20), and most of our results for the Fletcher function can be extended to their merit function, if some additional assumptions are made. Other merit functions have been proposed by di Pillo and Grippo, and by Schittkowski (see Powell (1987) for a review), but they will not be studied in this paper.

It is essential that the step generated by Algorithm 2.1 define a descent direction for the merit function φ used, i.e. that $D\varphi(x_k; d_k) \leq 0$. Indeed, in order to establish a linear convergence rate, that quantity must be significantly negative. Therefore, we now calculate these directional derivatives, starting with the ℓ_1 merit function. Although this merit function is not differentiable everywhere, it does always have a one-sided directional derivative, and for the direction d_k generated by Algorithm 2.1, this takes a particularly simple form, as we now show.

From Taylor's theorem we have

$$\begin{aligned} \phi_{\mu_k}(x_k + \alpha d_k) - \phi_{\mu_k}(x_k) &= f(x_k + \alpha d_k) - f_k + \mu_k\|c(x_k + \alpha d_k)\|_1 - \mu_k\|c_k\|_1 \\ &\leq \alpha g_k^T d_k + \mu_k\|c_k + \alpha A_k^T d_k\|_1 + b_1\alpha^2\|d_k\|^2 \\ &\quad -\mu_k\|c_k\|_1, \end{aligned}$$

for some positive constant b_1. (Note that b_1 actually depends on the weight μ_k.) From (2.1) we have that $A_k^T d_k = -c_k$, and therefore, assuming $\alpha \leq 1$, we have

$$\phi_{\mu_k}(x_k + \alpha d_k) - \phi_{\mu_k}(x_k) \leq \alpha \left[g_k^T d_k - \mu_k \|c_k\|_1 \right] + \alpha^2 b_1 \|d_k\|^2. \tag{2.22}$$

Similarly, we obtain the lower bound

$$\phi_{\mu_k}(x_k + \alpha d_k) - \phi_{\mu_k}(x_k) \geq \alpha \left[g_k^T d_k - \mu_k \|c_k\|_1 \right] - \alpha^2 b_1 \|d_k\|^2. \tag{2.23}$$

Taking limits it is therefore clear that

$$D\phi_{\mu_k}(x_k; d_k) = g_k^T d_k - \mu_k \|c_k\|_1. \tag{2.24}$$

In order to separate out the effects on the merit function of the null space and range space components of the step we recall the decomposition $d_k = h_k + v_k$, given by (2.7)-(2.9). By (2.9), we have

$$g_k^T v_k = \hat{\lambda}_k^T c_k, \tag{2.25}$$

where $\hat{\lambda}_k = \hat{\lambda}(x_k)$ is given by (2.21) so that

$$D\phi_{\mu_k}(x_k; d_k) = g_k^T h_k - \mu_k \|c_k\|_1 + \hat{\lambda}_k^T c_k. \tag{2.26}$$

By (2.8) $g_k^T h_k = -g_k^T Z_k B_k^{-1} Z_k^T g_k$, and since the matrices $\{B_k\}$ will be forced to be positive definite, this term is always less than or equal to zero. Therefore to ensure that d_k is a descent direction for ϕ_{μ_k} it is sufficient to require that $\mu_k > \|\hat{\lambda}_k\|_\infty$. Such a condition is very common when using merit functions with sequential quadratic programming methods, and appears for example in the global analysis of Han (1977). If the sequence $\{\hat{\lambda}_k\}$ is bounded, then a sufficiently large μ exists satisfying $\mu > \|\hat{\lambda}_k\|_\infty$ for all k. Since, however, this value is not known in advance, at each step the weight $\mu_k > \|\hat{\lambda}_k\|_\infty$ should be chosen in such a way that it eventually becomes fixed. One way to do this is to choose μ_k at each iterate as follows:

$$\mu_k = \begin{cases} \mu_{k-1} & \text{if } \mu_{k-1} \geq \|\hat{\lambda}_k\|_\infty + \rho \\ \|\hat{\lambda}_k\|_\infty + 2\rho & \text{otherwise,} \end{cases} \tag{2.27}$$

where ρ is some positive constant.

From now on we will assume that when the ℓ_1 merit function ϕ_{μ_k} is used in Algorithm 2.1, the weight μ_k is chosen by (2.27). Therefore, for any x_k, $D\phi_{\mu_k}(x_k; d_k) < 0$, unless $Z_k^T g_k = 0$ and $c_k = 0$, which can occur only at a stationary point of problem (1.1).

As mentioned above, one could use other norms than the ℓ_1 norm in this merit function. In fact, all of the results and proofs in this paper involving the merit function (2.19) remain valid if the ℓ_1 norm is replaced with the ℓ_p norm for $p \in [1, \infty]$, provided that the ℓ_∞ norm in (2.27) and elsewhere is replaced with the dual norm ℓ_q, where $\frac{1}{p} + \frac{1}{q} = 1$. However, we will continue to write ℓ_1 norm for simplicity.

We now consider Fletcher's merit function (2.20). Since this function is differentiable we have

$$\nabla \Phi_{\nu_k}(x_k) = g_k + A_k \hat{\lambda}_k + (\hat{\lambda}'_k)^T c_k + \nu_k A_k c_k, \tag{2.28}$$

where $\hat{\lambda}'_k$ is the $t \times n$ matrix whose rows are the gradients of the Lagrange multiplier estimates. Thus, using (2.1) and (2.25) we have

$$\begin{aligned} D\Phi_{\nu_k}(x_k; d_k) &= g_k^T d_k - \hat{\lambda}_k^T c_k + c_k^T \hat{\lambda}'_k d_k - \nu_k \|c_k\|^2 \\ &= g_k^T h_k + c_k^T \hat{\lambda}'_k d_k - \nu_k \|c_k\|^2. \end{aligned} \tag{2.29}$$

Again, as with the ℓ_1 merit function, the first term is non-positive. It is also clear that, for any k, ν_k can be chosen large enough so that (2.29) is less than or equal to zero. However the algorithm for choosing ρ is more complex than (2.27), and we defer discussion of this issue to the next section, where we analyze the convergence of the algorithm.

3. Global Behavior of the Algorithm

We now consider the convergence properties of the reduced Hessian SQP algorithm defined in Section 2. We will show that, for a fraction of the steps, significant decrease in the merit function can be obtained, and that under appropriate assumptions this implies global convergence.

Equations (2.26) and (2.29) indicate that the direction generated by the algorithm is a descent direction for the two merit functions if μ_k and ν_k are sufficiently large and if $g_k^T h_k = g_k^T Z_k Z_k^T h_k < 0$. Therefore the null space component h_k must make an acute angle with the projection of $-g_k$ onto the null space, $-Z_k Z_k^T g_k$. In order to quantify the decrease in the merit function obtained in a step of the algorithm, we will consider closely this angle, which is defined by

$$\begin{aligned} \cos\theta_k &= \frac{-\left(Z_k Z_k^T g_k\right)^T h_k}{\|Z_k Z_k^T g_k\| \|h_k\|}, \\ &= \frac{-g_k^T h_k}{\|Z_k^T g_k\| \|h_k\|}, \end{aligned} \tag{3.1}$$

since $\|Z_k Z_k^T g_k\| = \|Z_k^T g_k\|$. Therefore, from (2.26) and (2.29) we have

$$D\phi_{\mu_k}(x_k; d_k) = -\|Z_k^T g_k\| \|h_k\| \cos\theta_k - \mu_k \|c_k\|_1 + \hat{\lambda}_k^T c_k, \tag{3.2}$$

and

$$D\Phi_{\nu_k}(x_k; d_k) = -\|Z_k^T g_k\| \|h_k\| \cos\theta_k + c_k^T \hat{\lambda}'_k d_k - \nu_k \|c_k\|^2. \tag{3.3}$$

From these relations it can be seen that for h_k to provide significant descent we must require that $\cos\theta_k$ not be too close to zero and that h_k not be too small in norm. Both these quantities depend very strongly on the reduced Hessian approximation B_k. By equation (2.8), h_k is computed so that $B_k Z_k^T h_k = -Z_k^T g_k$, and so by (2.10) we have that $B_k s_k = -\alpha_k Z_k^T g_k$. Therefore $\cos\theta_k$ can also be written as

$$\cos\theta_k = \frac{s_k^T B_k s_k}{\|s_k\| \, \|B_k s_k\|}, \tag{3.4}$$

and we have that

$$\frac{\|h_k\|}{\|Z_k^T g_k\|} = \frac{\|s_k\|}{\|B_k s_k\|}. \tag{3.5}$$

The following theorem, which is proved by Byrd and Nocedal (1987), establishes bounds on these quantities that hold for a fraction of the iterates.

Theorem 3.1 *Let $\{B_k\}$ be generated by the BFGS formula (2.6) where, for all $k \geq 1$, $s_k \neq 0$ and*

$$\begin{aligned} \frac{y_k^T s_k}{s_k^T s_k} &\geq m > 0 \\ \frac{\|y_k\|^2}{y_k^T s_k} &\leq M. \end{aligned}$$

Then, for any $p \in (0,1)$, there exist constants $\beta_1, \beta_2, \beta_3 > 0$ such that, for any $k \geq 1$, the relations

$$\cos\theta_j \geq \beta_1 \tag{3.6}$$

$$\beta_2 \leq \frac{\|B_j s_j\|}{\|s_j\|} \leq \beta_3 \tag{3.7}$$

hold for at least $\lceil pk \rceil$ values of $j \in [1,k]$.

This theorem, which is basic for the analysis of this paper, implies that a fraction p of the iterates with $s_k \neq 0$ are such that the null space component h_k gives a significant reduction in the merit function. Later we will see that the iterates with $s_k = 0$ also contribute significantly to the decrease in the merit fucntion. Since it will be useful to refer easily to these two classes of iterates, we will assign a value to p and make the following definition.

Definition 3.1 *Let p of Theorem 3.1 have the value value $p = \frac{5}{6}$. We define J to be the set of iterates for which (3.6) and (3.7) hold, or for which $s_k = 0$. We will call J the set of "good" iterates.*

This definition and Theorem 3.1 imply that, $J \cap [1,k]$ contains at least $\lceil \frac{5}{6}k \rceil$ iterates.

We are now ready to analyze the global behavior of the algorithm. We use the term global because we do not explicitly assume that the iterates are near the solution, but only make the following assumptions.

Assumptions 3.1 The sequence $\{x_k\}$ generated by the algorithm is contained in a convex set D with the following properties.

(1) The functions $f : \mathbf{R}^n \to \mathbf{R}$, and $c : \mathbf{R}^n \to \mathbf{R}^t$ and their first and second derivatives are uniformly bounded in norm over D.

(2) The matrix $A(x)$ has full column rank for all $x \in D$, and there is a constant γ_0 such that
$$||A(x)[A(x)^T A(x)]^{-1}|| \leq \gamma_0 \tag{3.8}$$
for all $x \in D$.

(3) For all $k \geq 1$ for which $s_k \neq 0$ we have
$$\frac{y_k^T s_k}{s_k^T s_k} \geq m > 0 \tag{3.9}$$
$$\frac{||y_k||^2}{y_k^T s_k} \leq M. \tag{3.10}$$

The following lemma on the relation between $||h||$ and $||Z^T g||$, for the good iterates, will be useful in deriving bounds on the directional derivative of the merit functions in the SQP direction. This lemma does not depend on the merit function used.

Lemma 3.2 *Suppose that the iterates $\{x_k\}$ generated by Algorithm 2.1 satisfy Assumptions 3.1. Then for any $j \in J$*
$$\frac{1}{\beta_3}||Z_j^T g_j|| \leq ||h_j|| \leq \frac{1}{\beta_2}||Z_j^T g_j||, \tag{3.11}$$
$$||d_j||^2 \leq \frac{1}{\beta_2^2}||Z_j^T g_j||^2 + \gamma_0^2||c_j||^2. \tag{3.12}$$

Proof: Let $j \in J$, and first assume that $s_j \neq 0$. From (3.5) and (3.7), we have that for $j \in J$
$$\frac{1}{\beta_3} \leq \frac{||h_j||}{||Z_j^T g_j||} \leq \frac{1}{\beta_2}, \tag{3.13}$$
which gives (3.11). Using (3.11), (2.9) and (3.8) we have
$$\begin{aligned} ||d_j||^2 &= ||h_j||^2 + ||v_j||^2 \\ &\leq \frac{1}{\beta_2^2}||Z_j^T g_j||^2 + \gamma_0^2||c_j||^2. \end{aligned}$$

If $s_j = 0$ then $Z_j^T g_j = h_j = 0$ and the result clearly holds.

□

3.1 The ℓ_1 Merit Function

We now establish some useful results about the behavior of Algorithm 2.1 with the ℓ_1 merit function, and use these results to establish a global convergence theorem. The following lemma shows that all the steps d_k generated by Algorithm 2.1 define descent directions for the ℓ_1 merit function, and that a significant reduction in this merit function is obtained for the good steps.

Lemma 3.3 *Let the iterates $\{x_k\}$ be generated by Algorithm 2.1 using the ℓ_1 merit function (2.19) with the weights chosen so that*

$$\mu_k \geq \|\hat{\lambda}(x_k)\|_\infty + \rho, \tag{3.14}$$

for all $k > 0$, where $\rho > 0$. Suppose that Assumptions 3.1 are satisfied. Then for all $k \geq 1$

$$D\phi_{\mu_k}(x_k; d_k) \leq -\|Z_k^T g_k\| \|h_k\| \cos\theta_k - \rho\|c_k\|_1, \tag{3.15}$$

and there is a positive constant b_2 such that for all $j \in J$

$$D\phi_{\mu_j}(x_j; d_j) \leq -b_2 \left[\|Z_j^T g_j\|^2 + \|c_j\|_1\right]. \tag{3.16}$$

Moreover for any value μ there is a positive constant γ'_μ such that if $j \in J$ and $\mu_j = \mu$ then

$$\phi_\mu(x_j) - \phi_\mu(x_{j+1}) \geq \gamma'_\mu \left[\|Z_j^T g_j\|^2 + \|c_j\|_1\right]. \tag{3.17}$$

Proof: From (3.2) and (3.14) it follows immediately that (3.15) holds for all $k \geq 1$. Now suppose $j \in J$. We can apply (3.11) and (3.6) to (3.15) and obtain inequality (3.16) with $b_2 = \min(\beta_1/\beta_3, \rho)$.

To consider the decrease in ϕ_μ in one iteration, for $j \in J$, note that the line search enforces the condition (2.2),

$$\phi_{\mu_j}(x_j) - \phi_{\mu_j}(x_{j+1}) \geq -\eta\alpha_j D\phi_{\mu_j}(x_j; d_j). \tag{3.18}$$

It is then clear from (3.16) that (3.17) holds, provided the α_j can be bounded from below. Suppose that $\alpha_j < 1$, which means that (2.2) failed for a steplength $\tilde{\alpha}$:

$$\phi_{\mu_j}(x_j + \tilde{\alpha}d_j) - \phi_{\mu_j}(x_j) > \eta\tilde{\alpha}D\phi_{\mu_j}(x_j; d_j), \tag{3.19}$$

where

$$\tau\tilde{\alpha} \leq \alpha_j \tag{3.20}$$

(see step 5 of Algorithm 2.1). From (2.22) and (2.24) we have

$$\phi_{\mu_j}(x_j + \tilde{\alpha}d_j) - \phi_{\mu_j}(x_j) \leq \tilde{\alpha}D\phi_{\mu_j}(x_j; d_j) + \tilde{\alpha}^2 b_1\|d_j\|^2, \tag{3.21}$$

where b_1 is a function of μ. Combining (3.19) and (3.21) we have

$$(\eta - 1)\tilde{\alpha}D\phi_{\mu_j}(x_j; d_j) < \tilde{\alpha}^2 b_1\|d_j\|^2 \tag{3.22}$$

From (3.12) and the fact that $\|c_j\|$ is uniformly bounded above we have

$$\|d_j\|^2 \le b_3[\|Z_j^T g_j\|^2 + \|c_j\|_1], \tag{3.23}$$

for $b_3 = \max(1/\beta_2^2, \gamma_0^2 \sup_{x\in D} \|c(x)\|)$. Combining (3.22), (3.16) and (3.23)

$$\tilde{\alpha} > \frac{(1-\eta)b_2}{b_1 b_3}. \tag{3.24}$$

Thus from (3.20) we conclude that the steplengths α_j are bounded away from zero for all $j \in J$, and (3.17) holds with $\gamma'_\mu = \eta b_2 \min\{1, (1-\eta)b_2/(b_1 b_3)\}$.

□

Now that we know from (3.15) that the line search can guarantee decrease in ϕ at every iteration, and from (3.17) that ϕ decreases significantly at the good iterates, we can prove a global convergence result for the ℓ_1 merit function. (Actually (3.17) is stronger than we need for global convergence but we will make full use of it in Section 4 to prove local R-linear convergence).

Theorem 3.4 *Let the sequence $\{x_k\}$ be generated by Algorithm 2.1 using the ℓ_1 merit function with weights $\{\mu_k\}$ chosen by (2.27). Suppose that Assumptions 3.1 are satisfied. Then the weights $\{\mu_k\}$ are constant for all sufficiently large k and* $\liminf_{k\to\infty}(\|Z_k^T g_k\| + \|c_k\|) = 0$.

Proof: First note that by Assumptions 3.1 and (2.21) $\{\|\hat{\lambda}_k\|\}$ is bounded. Therefore, since the procedure (2.27) increases μ_k by at least ρ whenever it changes the weight, it follows that there is an index k_0 and a value μ such that for all $k > k_0$, $\mu_k = \mu \ge \|\hat{\lambda}_k\| + \rho$. Now by Assumption 3.1-3 there is a set J of good iterates, and by Lemma 3.3 and the fact that $\phi_\mu(x_k)$ decreases at each iterate, we have that for $k > k_0$,

$$\begin{aligned}
\phi_\mu(x_{k_0}) - \phi_\mu(x_{k+1}) &= \sum_{j=k_0}^{k} (\phi_\mu(x_j) - \phi_\mu(x_{j+1})) \\
&\ge \sum_{j\in J\cap[k_0,k]} (\phi_\mu(x_j) - \phi_\mu(x_{j+1})) \\
&\ge \gamma'_\mu \sum_{j\in J\cap[k_0,k]} [\|Z_j^T g_j\|^2 + \|c_j\|_1].
\end{aligned}$$

By Assumption 3.1-1 $\phi_\mu(x)$ is bounded below for all $x \in D$, so the sum is finite, and thus the term inside the square brackets converges to zero. Therefore

$$\lim_{j\in J, j\to\infty} (\|Z_j^T g_j\| + \|c_j\|_1) = 0. \tag{3.25}$$

and since, by Theorem 3.1, J is infinite the theorem follows.

□

Actually this result could have been proved with the boundedness of $|f|$ and $||c||$ in Assumption 3.1 replaced with the assumption that ϕ_{μ_k} is bounded below over D for some k, but the analysis would have been somewhat more complicated.

3.2 Fletcher's Merit Function

Now we consider Algorithm 2.1 using Fletcher's merit function (2.20). Even though the analysis is similar to that with the ℓ_1 merit function, we will be forced to make some additional optimistic assumptions in order to establish convergence.

Recall the directional derivative (3.3),

$$D\Phi_{\nu_k}(x_k; d_k) = -||Z_k^T g_k|| ||h_k|| \cos\theta_k + c_k^T \hat{\lambda}_k' d_k - \nu_k ||c_k||^2. \tag{3.26}$$

In this case the weight ν_k appears to be playing the same role as the difference $(\mu - ||\hat{\lambda}_k||_\infty)$ does in (3.2). However, since the term involving the derivative of $\hat{\lambda}$ appears to be of unpredictable sign, ν_k may have to be increased to ensure that the descent condition holds. Considering (3.26) we see that d_k is a descent direction if and only if

$$\nu_k > \frac{c_k^T \hat{\lambda}_k' d_k - ||Z_k^T g_k|| ||h_k|| \cos\theta_k}{||c_k||^2}. \tag{3.27}$$

(If $||c_k|| = 0$ we obtain a strong direction of descent for any choice of ν_k, and the analysis that follows becomes very simple. We therefore assume that $||c_k|| \neq 0$.) Condition (3.27) certainly appears more complex than the corresponding condition (3.14) for the ℓ_1 function. Setting that issue aside for the moment, we now show that if we choose ν_k to satisfy a slightly stronger condition than (3.27) we can prove a result analogous to Lemma 3.3.

Lemma 3.5 *Let the iterates $\{x_k\}$ be generated by Algorithm 3.1 using Fletcher's merit function (2.20) where, for all $k \geq 1$, the weights are chosen so that*

$$\nu_k > \left[\frac{c_k^T \hat{\lambda}_k' d_k + \frac{1}{2} g_k^T h_k}{||c_k||^2}\right] + \rho \equiv \overline{\nu}_k + \rho, \tag{3.28}$$

for some positive constant ρ. Suppose that Assumptions 3.1 are satisfied. Then for all $k \geq 1$ we have that

$$D\Phi_{\nu_k}(x_k; d_k) \leq -\tfrac{1}{2} ||Z_k^T g_k|| ||h_k|| \cos\theta_k - \rho ||c_k||^2, \tag{3.29}$$

and there exists a positive constant b_4 such that, for all $j \in J$,

$$D\Phi_{\nu_j}(x_j; d_j) \leq -b_4 \left[||Z_j^T g_j||^2 + ||c_j||^2\right]. \tag{3.30}$$

Moreover for any value ν there is a constant γ_ν' such that, if $j \in J$ and $\nu_j = \nu$,

$$\Phi_{\nu_j}(x_j) - \Phi_{\nu_j}(x_{j+1}) \geq \gamma_\nu' \left[||Z_j^T g_j||^2 + ||c_j||^2\right]. \tag{3.31}$$

Proof: From (2.29) and the definition of $\overline{\nu}_k$

$$D\Phi_{\nu_k}(x_k; d_k) = \tfrac{1}{2} g_k^T h_k + (\overline{\nu}_k - \nu_k)\|c_k\|^2, \tag{3.32}$$

and using (3.28) and (3.1), equation (3.29) follows. Next, note that, for $j \in J$, equation (3.30) follows from (3.29) using (3.11), and (3.6).

The rest of the proof is analogous to the proof of Lemma 3.3. Since the line search enforces the condition (2.2), it is clear from (3.30) that (3.31) holds, provided the α_j can be bounded from below. As in the proof of Lemma 3.3 we see that if $\alpha_j < 1$, we have (3.19) and (3.20) for the Fletcher function. Using Taylor's theorem we see that (3.21) also holds in this case, except that b_1 now stands for a constant different form the one defined before (2.22). We therefore obtain (3.22). From (3.12) we have

$$\|d_j\|^2 \le b_5 \left[\|Z_j^T g_j\|^2 + \|c_j\|^2 \right], \tag{3.33}$$

for some positive constant b_5. We see, from (3.22), (3.30), and (3.33), that

$$\tilde{\alpha} > \frac{(1-\eta) b_4}{b_1 b_5}. \tag{3.34}$$

Thus from (3.20) we conclude that the steplengths α_j are bounded away from zero for all $j \in J$.

□

Note that (3.28) gives a computable value, and ν_k could be increased if necessary, at each iteration, to satisfy (3.28). In order to use Lemma 3.5 to prove any convergence result we must know that eventually ν_k becomes fixed while still satisfying (3.28). Therefore, by analogy with (2.27), we suggest choosing ν_k at each iteration by

$$\nu_k = \begin{cases} \nu_{k-1} & \text{if } \nu_{k-1} \ge \overline{\nu}_k + \rho \\ \overline{\nu}_k + 2\rho & \text{otherwise,} \end{cases} \tag{3.35}$$

where ρ is some positive constant.

Note that the sequence $\{\nu_k\}$ will diverge if $\{\overline{\nu}_k\}$ is unbounded, and in that case Lemma 3.5 cannot be used to prove convergence. Thus it is essential that the sequence $\overline{\nu}_k$ be bounded. However, in contrast to $\|\hat{\lambda}_k\|$, the quantity $\overline{\nu}_k$ depends on d_k, and thus B_k, as well as on x_k, making its boundedness a difficult question. The most we are able to say about the boundedness of $\overline{\nu}_k$ is contained in the following result.

Lemma 3.6 *Suppose that the iterates $\{x_k\}$ are generated by Algorithm 2.1 using Fletcher's merit function (2.20) and that Assumptions 3.1 are satisfied. Then, there is a constant b_6 such that for any k,*

$$\overline{\nu}_k \le b_6 \left(\frac{s_k^T s_k}{s_k^T B_k s_k} + 1 \right), \tag{3.36}$$

and the sequence $\{\overline{\nu}_j\}$ is thus uniformly bounded above for all $j \in J$.

Proof: By the geometric/arithmetic mean inequality,

$$\begin{aligned}
c_k^T \hat{\lambda}_k' h_k &= \left[|g_k^T h_k| \frac{(c_k^T \hat{\lambda}_k' h_k)^2}{|g_k^T h_k|} \right]^{\frac{1}{2}} \\
&\leq \frac{1}{2} |g_k^T h_k| + \frac{1}{2} \frac{(c_k^T \hat{\lambda}_k' h_k)^2}{|g_k^T h_k|},
\end{aligned}$$

since $g_k^T h_k < 0$. Therefore by (3.28), (2.8)-(2.10), and (3.8)

$$\begin{aligned}
\overline{\nu}_k &\leq \left[\frac{(c_k^T \hat{\lambda}_k' h_k)^2}{2|g_k^T h_k|} + c_k^T \hat{\lambda}_k' v_k \right] \frac{1}{\|c_k\|^2} \\
&\leq \left[\frac{(\|c_k\| \, \|\hat{\lambda}_k'\| \, \|s_k\|)^2}{2 s_k^T B_k s_k} + \|c_k\| \, \|\hat{\lambda}_k'\| \, \|v_k\| \right] \frac{1}{\|c_k\|^2} \\
&\leq \frac{\|\hat{\lambda}_k'\|^2}{2} \frac{s_k^T s_k}{s_k^T B_k s_k} + \gamma_0 \|\hat{\lambda}_k'\|.
\end{aligned}$$

Referring to (2.21) we note that by Assumptions 3.1-1 and 3.1-2, $\|\hat{\lambda}_k'\|$ is uniformly bounded for all x_k. By (3.6) and (3.7) it follows that $\{\overline{\nu}_j\}$ is bounded for all $j \in J$. □

This result is not as strong as one might hope for, since we are not able to bound the Rayleigh quotient $s_k^T B_k s_k / s_k^T s_k$ away form zero for all k. Therefore we cannot rule out the possibility that a subsequence of these Rayleigh quotients goes to zero in such a way that $\{\nu_k\}$ must diverge to yield a descent direction at each iteration. It is not clear whether this is likely to be a problem in practice or not. It is interesting to note that Powell and Yuan (1986) avoid these difficulties, when analyzing the Fletcher function, by assuming *a priori* that $\|B_k\|$ and $\|B_k^{-1}\|$ are bounded. Under these conditions they show that, if ν_k is chosen by a procedure analogous to (3.28), it will be bounded.

Therefore, to prove a global convergence theorem analogous to Theorem 3.4 we will simply make the optimistic assumption that the sequence $\{\overline{\nu}_k\}$ is bounded.

Theorem 3.7 *Let the sequence $\{x_k\}$ be generated by Algorithm 2.1 using the Fletcher merit function with the weights ν_k chosen by (3.35). Suppose that Assumptions 3.1 are satisfied and that the sequence $\{\overline{\nu}_k\}$ defined by (3.28) is bounded above for all k. Then ν_k is eventually constant and* $\liminf_{k \to \infty}(\|Z_k^T g_k\| + \|c_k\|) = 0$.

Proof: Since the sequence $\{\overline{\nu}_k\}$ is bounded, the procedure (3.35) guarantees that ν_k will eventually be constant. By Assumptions 3.1, Φ_ν is bounded below for all $x \in D$. Then, using Lemma 3.5, the result follows by the same argument as in the proof of Theorem 3.4.

4. Local Convergence

Now we consider a local minimizer x_* that satisfies the second order sufficiency conditions, and show that the algorithm is locally and R-linearly convergent to it. We will make the following assumptions in a neighborhood of x_*, and for the rest of the paper, these replace Assumptions 3.1.

Assumptions 4.1 The point x_* is a local minimizer for problem (1.1) at which that the following conditions hold.

(1) The functions $f : \mathbf{R}^n \to \mathbf{R}$, and $c : \mathbf{R}^n \to \mathbf{R}^t$ are three times continuously differentiable in a neighborhood of x_*.

(2) The matrix $A(x_*)$ has full column rank. This implies that x_* is a Karush-Kuhn-Tucker point of (1.1), i.e. there exists a vector $\lambda_* \in \mathbf{R}^t$ such that

$$\nabla_x L(x_*, \lambda_*) = g(x_*) + A(x_*)\lambda_* = 0.$$

(3) For all $w \in \mathbf{R}^{n-t}$, $w \neq 0$, we have $w^T G_* w > 0$.

Note that (1) and (2) imply that there are constants γ_0, γ_L such that, for all x near x_*,

$$\|A(x)[A(x)^T A(x)]^{-1}\| \leq \gamma_0, \tag{4.1}$$

and for all x and z near x_*,

$$\|\hat{\lambda}(x) - \hat{\lambda}(z)\| \leq \gamma_L \|x - z\|, \tag{4.2}$$

where $\hat{\lambda}(x)$ is given by (2.21). Also, (1) and (3) imply that for all (x, λ) sufficiently near (x_*, λ_*), and for all $w \in \mathbf{R}^{n-t}$,

$$m\|w\|^2 \leq w^T G(x, \lambda) w \leq M\|w\|^2, \tag{4.3}$$

for some positive constants m, M. The condition $f, c \in C^3$ is only needed for Fletcher's function; for the ℓ_1 merit function it suffices to assume that $f, c \in C^2$ and that their Hessians are Lipschitz continuous near x_*.

We need to establish some results about such a local minimizer and its relationship to the merit functions. First we note that, near x_*, the quantities $c(x)$ and $Z(x)^T g(x)$ may be regarded as a measure of the error at x. This result is not new (see e.g. Powell (1978)), but we give a proof for the sake of completeness. We recall that $Z(x)$ stands for any orthogonal matrix with the property $A(x)^T Z(x) = 0$.

Lemma 4.1 *If Assumptions 4.1 hold, then for all x sufficiently near x_**

$$\gamma_1 \|x - x_*\| \leq \|c(x)\| + \|Z(x)^T g(x)\| \leq \gamma_2 \|x - x_*\|, \tag{4.4}$$

for some positive constants γ_1, γ_2.

Proof: Define the function $H : \mathbf{R}^{n+t} \to \mathbf{R}^{n+t}$ by

$$H(x, \lambda) = \begin{bmatrix} \nabla_x L(x, \lambda) \\ c(x) \end{bmatrix}.$$

Then $H(x_*, \lambda_*) = 0$, and

$$H'(x_*, \lambda_*) = \begin{bmatrix} \nabla^2_{xx} L(x_*, \lambda_*) & A(x_*) \\ A(x_*)^T & 0 \end{bmatrix}.$$

We note that $H'(x_*, \lambda_*)$ is nonsingular, for if $H'(x_*, \lambda_*)(u^T, v^T)^T = 0$ for some $u \in \mathbf{R}^n$ and some $v \in \mathbf{R}^t$, then

$$\nabla^2_{xx} L(x_*, \lambda_*)u + A(x_*)v = 0 \tag{4.5}$$

$$A(x_*)^T u = 0. \tag{4.6}$$

Thus $u^T \nabla^2_{xx} L(x_*, \lambda_*)u = 0$, and by (4.6) and Assumption 4.1-3 this implies that $u = 0$. Then, since $A(x_*)$ has full rank, (4.5) implies $v = 0$. Therefore $H'(x_*, \lambda_*)$ is nonsingular.

Let $\|\cdot\|_e$ denote the norm defined by $\|(u^T, v^T)^T\|_e = \|u\| + \|v\|$, for vectors in $\mathbf{R}^{n+t}$, and by the corresponding induced matrix norm, for $(n+t) \times (n+t)$ matrices. The differentiability of H at (x_*, λ_*) implies that for any $\epsilon > 0$,

$$\left\| H(x, \lambda) - H'(x_*, \lambda_*) \begin{bmatrix} x - x_* \\ \lambda - \lambda_* \end{bmatrix} \right\|_e \leq \epsilon(\|x - x_*\| + \|\lambda - \lambda_*\|),$$

for all (x, λ) sufficiently close to (x_*, λ_*). Since $H'(x_*, \lambda_*)$ is nonsingular, if ϵ is taken sufficiently small it follows that

$$\gamma_1(\|x - x_*\| + \|\lambda - \lambda_*\|) \leq \|H(x, \lambda)\|_e \leq \gamma_2'(\|x - x_*\| + \|\lambda - \lambda_*\|), \tag{4.7}$$

where $\gamma_2' = \|H'(x_*, \lambda_*)\|_e + \epsilon$ and $\gamma_1 = 1/\|H'(x_*, \lambda_*)^{-1}\|_e - \epsilon$. If we set $\lambda = \hat{\lambda}(x)$, the least squares multiplier, in (4.7) then since $\nabla_x L(x, \hat{\lambda}(x)) = Z(x)Z(x)^T g(x)$, the left inequality in (4.4) follows immediately, and the right inequality follows from (4.2) if we let $\gamma_2 = \gamma_2'(1 + \gamma_L)$.

□

Now we show that, for a fixed weight, either merit function may also be regarded as a measure of the error.

Lemma 4.2 *Suppose that Assumptions 4.1 hold at x_*. Then for any $\mu > \|\lambda_*\|_\infty$ there exist constants γ_3 and γ_4, such that for all $\|x - x_*\|$ sufficiently small*

$$\gamma_3\|x - x_*\|^2 \leq \phi_\mu(x) - \phi_\mu(x_*) \leq \gamma_4 \left[\|Z(x)^T g(x)\|^2 + \|c(x)\|_1\right]. \tag{4.8}$$

Furthermore, for any ν sufficiently large there are constants γ_5 and γ_6 such that for all $\|x - x_*\|$ sufficiently small

$$\gamma_5\|x - x_*\|^2 \leq \Phi_\nu(x) - \Phi_\nu(x_*) \leq \gamma_6 \left[\|Z(x)^T g(x)\|^2 + \|c(x)\|^2\right]. \tag{4.9}$$

Proof: First we consider the Fletcher merit function, which by Assumptions 4.1 is at least twice continuously differentiable near x_*. We have

$$\nabla\Phi_\nu(x) = g(x) + A(x)[\hat{\lambda}(x) + \nu c(x)] + \hat{\lambda}'(x)^T c(x)$$
$$\nabla^2\Phi_\nu(x_*) = \nabla^2_{xx}L(x_*, \lambda_*) + A_*\hat{\lambda}'(x_*) + \hat{\lambda}'(x_*)^T A_*^T + \nu A_* A_*^T.$$

By Lemma 4.1, and since $\nabla\Phi_\nu(x_*) = 0$, we have that for any $\epsilon > 0$ there is a constant γ_6 such that

$$\begin{aligned}\Phi_\nu(x) - \Phi_\nu(x_*) &\leq \frac{1}{2}\left(\|\nabla^2\Phi_\nu(x_*)\| + \epsilon\right)\|x - x_*\|^2 \\ &\leq \gamma_6\left[\|Z(x)^T g(x)\|^2 + \|c(x)\|^2\right],\end{aligned}$$

for all x sufficiently near x_*.

To establish the left inequality we define

$$\hat{G} = \nabla^2_{xx}L(x_*, \lambda_*) + A_*\hat{\lambda}'(x_*) + \hat{\lambda}'(x_*)^T A_*^T,$$

so that $\hat{G} + \nu A_* A_*^T = \nabla^2\Phi_\nu(x_*)$. Note that $Z_*^T\hat{G}Z_*$ is positive definite. We now show that $\hat{G} + \nu A_* A_*^T$ is positive definite for ν sufficiently large.

Let K be an $n \times t$ matrix with full column rank such that $Z_*^T\hat{G}K = 0$. The span of K could be considered as a subspace that is $\hat{G}$ conjugate to the span of Z_*. Note that the $t \times t$ matrix $A_*^T K$ is nonsingular, since if $A_*^T K v = 0$ for some $v \in \mathbf{R}^t$ then $Kv = Z_* w$ for some $w \in \mathbf{R}^{n-t}$. But then $Z_*^T\hat{G}Z_* w = Z_*\hat{G}Kv = 0$, which implies that $w = 0$, and so $v = 0$.

Now consider the $n \times n$ matrix

$$\begin{bmatrix} Z_*^T \\ K^T \end{bmatrix}\left[\hat{G} + \nu A_* A_*^T\right][Z_* \; K] = \begin{bmatrix} Z_*^T\hat{G}Z_* & 0 \\ 0 & K^T\hat{G}K + \nu K^T A_* A_*^T K \end{bmatrix}. \tag{4.10}$$

The matrix on the right hand side is positive definite if ν is greater than the smallest eigenvalue of $(K^T A_*)^{-1}K^T\hat{G}K(A_*^T K)^{-1}$. In this case, since the product of the three matrices on the left side is nonsingular, the matrix $[Z_* \, K]$ must be nonsingular, and thus $\hat{G} + \nu A_* A_*^T = \nabla^2\Phi_\nu(x_*)$ is positive definite for such ν.

Since $\nabla^2\Phi_\nu(x)$ is continuous, there is a constant $\gamma_5 > 0$ such that for all x in some neighborhood of x_*, all eigenvalues of $\nabla^2\Phi_\nu(x)$ are greater than $2\gamma_5$. Therefore, since $\nabla\Phi_\nu(x_*) = 0$,

$$\Phi_\nu(x) - \Phi_\nu(x_*) \geq \gamma_5\|x - x_*\|^2.$$

We now treat the ℓ_1 merit function with some fixed value of $\mu > \|\lambda_*\|_\infty$. Consider a neighborhood N of x_* over which (4.9) holds for some ν, and such that $\mu - \|\hat{\lambda}(x)\|_\infty > \frac{1}{2}[\mu - \|\lambda_*\|_\infty]$, and $\|c(x)\| \leq \frac{1}{2\nu}[\mu - \|\lambda_*\|_\infty]$ for all $x \in N$. Then we have that for $x \in N$

$$\begin{aligned}\phi_\mu(x) &= \Phi_\nu(x) - \hat{\lambda}(x)^T c(x) - \frac{1}{2}\nu\|c(x)\|^2 + \mu\|c(x)\|_1 \\ &\geq \Phi_\nu(x) + \left[\mu - \|\hat{\lambda}(x)\|_\infty - \frac{1}{2}\nu\|c(x)\|\right]\|c(x)\|_1 \\ &\geq \Phi_\nu(x) + \frac{1}{4}[\mu - \|\lambda_*\|_\infty]\|c(x)\|_1.\end{aligned}$$

Since $\phi_\mu(x_*) = \Phi_\nu(x_*)$ the left inequality of (4.8) follows from (4.9) with $\gamma_5 = \gamma_3$. Now

$$\begin{aligned}\phi_\mu(x) &\le L(x,\lambda_*) + (\mu + \|\lambda_*\|_\infty)\|c(x)\|_1 \\ &\le L(x_*,\lambda_*) + \|\nabla^2_{xx}L(x_*,\lambda_*)\|\|x - x_*\|^2 + (\mu + \|\lambda_*\|_\infty)\|c(x)\|_1.\end{aligned}$$

Since $L(x_*,\lambda_*) = \phi_\mu(x_*)$, the right inequality follows from (4.4), and from the boundedness of $\|c(x)\|$ near x_*.

□

A consequence of this lemma is that, for a sufficiently large value of the weight, either merit function will have a strong local minimizer at x_*. We would like to use the descent property of Algorithm 2.1 to show that x_* is a point of attraction of the algorithm. To do this we make the following assumption on the line search.

Assumption 4.2 The line search has the property that, for x_k sufficiently close to x_*, $\varphi((1-\theta)x_k + \theta x_{k+1}) \le \varphi(x_k)$ for all $\theta \in [0,1]$.

This assumption is rather similar to, but weaker than, the Curry-Altman condition, and similarly, there is no practical line search algorithm which can guarantee it absolutely. However, it seems unlikely that it is violated close to x_*. We should note that an assumption of this type is needed also in the context of unconstrained optimization; see for example §7 of Byrd, Nocedal and Yuan (1987).

Now we consider Algorithm 2.1 using the ℓ_1 merit function and show that if an iterate x_k gets close enough to x_*, with k large enough, the sequence will stay close to x_* and converge to x_* R-linearly.

Theorem 4.3 *Let $\{x_k\}$ be generated by Algorithm 2.1 using the ℓ_1 merit function (2.19), with μ_k chosen by (2.27). Suppose that x_* satisfies Assumptions 4.1, that Assumption 4.2 holds, and that $\{\|\hat\lambda(x_k)\|\}$ is bounded. Then the weight has a fixed value μ for all sufficiently large k, and there is a neighborhood of x_* such that if any iterate x_{k_0} falls in that neighborhood, with $\mu_{k_0} = \mu$, then $\{x_k\} \to x_*$. Furthermore*

$$\phi_\mu(x_{k+1}) - \phi_\mu(x_*) \le r^{k-k_0}[\phi_\mu(x_{k_0}) - \phi_\mu(x_*)], \qquad k \ge k_0 \tag{4.11}$$

for some constant $r < 1$, and

$$\sum_{k=1}^{\infty} \|x_k - x_*\| < \infty. \tag{4.12}$$

Proof: By Assumptions 4.1 there exists $\delta_1 > 0$ such that, for all x in the neighborhood $N_1 = \{x : \|x - x_*\| < \delta_1\}$, Assumptions 3.1-1 and 3.1-2 are satisfied, and

$$\|\hat\lambda(x)\|_\infty + \rho > \|\lambda_*\|_\infty. \tag{4.13}$$

Also, by choosing δ_1 small enough we can guarantee (as in Lemma 2.1) that, if x_k and x_{k+1} are in N_1 and λ_k satisfies (2.12), then Assumption 3.1-3 is satisfied.

Now, since $\{\|\hat{\lambda}(x_k)\|_\infty\}$ is bounded, the procedure (2.27) implies that for all k greater than some value $\bar{k}$, μ_k is fixed at some value μ. By (4.13) and (2.27), if an iterate x_k, with $k > \bar{k}$, occurs in N_1 then it must be that $\mu > \|\lambda_*\|_\infty$. For such μ it follows from Lemma 4.2 that the function ϕ_μ has a strict local minimizer at x_*. Therefore, there exists $\delta_2 \in (0, \delta_1]$ such that if $\|x - x_*\| < \delta_2$, the connected component of the level set $\{z : \phi_\mu(z) < \phi_\mu(x)\}$ containing x_* is a subset of N_1 over which equation (4.8) holds.

Now Assumption 4.2 implies that if for some $k_0 > \bar{k}$, $\|x_{k_0} - x_*\| < \delta_2$, then $x_k \in N_1$ for all $k > k_0$, since ϕ_μ is decreased at each step.

Thus we have that Assumptions 3.1 hold on N_1 for $k > k_0$, and we may identify N_1 with the set D of those assumptions, so that all of the results in Subsection 3.1 for the ℓ_1 merit function hold for $k > k_0$. Therefore, if B_{k_0} is positive definite B_k remains positive definite for all subsequent iterates, and by Theorem 3.1 there is a set of good iterates J. From Lemma 3.3 and Lemma 4.2 we have, for all $j \in J$, $j > k_0$,

$$\phi_\mu(x_j) - \phi_\mu(x_{j+1}) \geq \frac{\gamma'_\mu}{\gamma_4}\left[\phi_\mu(x_j) - \phi_\mu(x_*)\right], \tag{4.14}$$

and so

$$\phi_\mu(x_{j+1}) - \phi_\mu(x_*) \leq r^{\frac{6}{5}}\left[\phi_\mu(x_j) - \phi_\mu(x_*)\right],$$

where $r^{\frac{6}{5}} \equiv 1 - \frac{\gamma'_\mu}{\gamma_4} < 1$. From Lemma 3.3 we see that $\phi_\mu(x_{k+1}) < \phi_\mu(x_k)$ for all k, and since $J \cap [k_0, k]$ has at least $\lceil 5(k - k_0)/6 \rceil$ elements, we have for all $k \geq k_0$

$$\phi_\mu(x_{k+1}) - \phi_\mu(x_*) \leq r^{k-k_0}\left[\phi_\mu(x_{k_0}) - \phi_\mu(x_*)\right].$$

From this relation and (4.8) we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \|x_k - x_*\| &\leq \sum_{k=1}^{k_0} \|x_k - x_*\| + (\gamma_3)^{-1/2} \sum_{k=k_0}^{\infty} \left[\phi_\mu(x_{k+1}) - \phi_\mu(x_*)\right]^{1/2} \\
&\leq \sum_{k=1}^{k_0} \|x_k - x_*\| + \left[\frac{1}{\gamma_3 r^{k_0}}(\phi_\mu(x_{k_0+1}) - \phi_\mu(x_*))\right]^{1/2} \sum_{k=k_0}^{\infty} (r^{1/2})^k \\
&< \infty.
\end{aligned}$$

□

It is possible to strengthen this result and show that there is a neighborhood of x_* such that if *any* iterate lands in the neighborhood, the sequence converges to x_* R-linearly. However the analysis of this result is much more complex.

Note that the local result of Theorem 4.3 fits together well with the global analysis of Section 3.1. If Assumptions 3.1 hold for a set D which is in addition compact then by Theorem 3.4 the sequence $\{x_k\}$ will have a cluster point that is a stationary point. If this stationary point satisfies Assumptions 4.1 then Theorem 4.3 implies that the sequence will converge to it R-linearly.

For Fletcher's merit function one cannot show such a strong result since, as was discussed in Section 3.2, there appear to be no assumptions on the problem that will guarantee $\{\nu_k\}$ is bounded. However, if we make the optimistic assumption that the sequence $\{\overline{\nu}_k\}$ defined by (3.28) is uniformly bounded, we may prove an R-linear convergence result.

Theorem 4.4 *Let $\{x_k\}$ be generated by Algorithm 2.1 using the Fletcher merit function (2.20), with ν_k chosen by (3.35). Suppose that x_* satisfies Assumptions 4.1, that Assumption 4.2 holds, that the sequence $\{\overline{\nu}_k\}$ defined by (3.28) is bounded, and that ν_k is eventually large enough to satisfy the conditions of Lemma 4.2. Then the weight has a fixed value ν for all sufficiently large k, and there is a neighborhood of x_* such that if any iterate x_{k_0} falls in that neighborhood, with $\nu_{k_0} = \nu$, then $\{x_k\} \to x_*$. Furthermore*

$$\Phi_\nu(x_{k+1}) - \Phi_\nu(x_*) \leq r^{k-k_0}[\Phi_\nu(x_{k_0}) - \Phi_\nu(x_*)], \qquad k \geq k_0 \tag{4.15}$$

for some constant $r < 1$, and

$$\sum_{k=1}^{\infty} \|x_k - x_*\| < \infty. \tag{4.16}$$

Proof: By the assumed boundedness of $\{\overline{\nu}_k\}$, the procedure (3.35) guarantees that the weight ν_k is equal to some fixed value ν for all k sufficiently large. Since we also assume that eventually ν_k becomes large enough that (4.9) holds for some constants γ_5 and γ_6, then Assumption 4.2 implies the sequence eventually stays in a neighborhood in which Assumptions 3.1 hold. At this point Lemma 3.5 and Lemma 4.2 imply that

$$\Phi_\nu(x_k) - \Phi_\nu(x_{k+1}) \geq \frac{\gamma_\nu'}{\gamma_6}[\Phi_\nu(x_k) - \Phi_\nu(x_*)]. \tag{4.17}$$

This expression has the same form as equation (4.14) in the proof of Theorem 4.3, and the result follows by the same argument, using equation (4.9) in place of (4.8).

□

It is interesting to note that, once R-linear convergence has been established, it follows that $\|B_k\|$ and $\|B_k^{-1}\|$ are uniformly bounded (we prove this later in Theorem 5.1). Then, by Lemma 3.6 we have that $\overline{\nu}_k$ is bounded. However, we know of no way to establish the boundedness of $\overline{\nu}_k$ *a priori*, and thus give a proof of R-linear convergence of the algorithm using the Fletcher function without making such optimistic assumptions.

5. Superlinear Convergence

We have shown in §4 that Algorithm 2.1 is R-linearly convergent. We now investigate whether superlinear convergence occurs, under the assumptions of §4. In §5.1 we discuss the relevant properties of the null space basis and give an attainable condition which, as we show in §5.2, implies a consistency property of B_k yielding two-step superlinear convergence, if steplengths of one are eventually taken at every iteration. For the Fletcher function this implies superlinear convergence of Algorithm 2.1, as we show in

§5.3. However with the ℓ_1 function steplengths of one may be impossible even very close to the solution. In §5.4-5 we consider two modified versions of Algorithm 2.1 and show that they both overcome this difficulty and yield two-step superlinear convergence.

5.1 Choice of null space basis.

The results of §4 only require of the matrix Z_k that its columns form an orthonormal basis for the null space of A_k^T, i.e. that $A_k^T Z_k = 0$, and $Z_k^T Z_k = I$. However, this does not completely specify Z_k, and if the choice of null space basis changes too much from one iterate to the next, superlinear convergence can be impeded. Byrd and Schnabel (1986) point out that any algorithm that chooses Z_k as a function of $A(x_k)$ alone will have discontinuities at some points. Coleman and Sorensen (1984) and Gill, Murray, Saunders, Stewart and Wright (1985) consider this issue and suggest several procedures for computing Z_k, based in part on information at previous iterates, which guarantee that Z varies smoothly.

The approach of Coleman and Sorensen is to obtain Z_k by computing a QR factorization of A_k, in which the inherent arbitrary sign choices in the factorization algorithm are made, if A_k is sufficiently close to A_{k-1}, the same way as they were done in computing Z_{k-1} from A_{k-1}. If $\{x_k\} \to x_*$, then for k sufficiently large all the matrices A_k will be close enough together that the same sign choices will be made at each step. Therefore, for the rest of the sequence we have $Z_k = z(A_k)$ where z is a smooth function of $n \times (n-t)$ matrices in a neighborhood of $A(x_*)$. This implies that there a constant a_* such that $\|Z_k - z(x_*)\| \leq a_* \|x_k - x_*\|$.

Gill, Murray, Saunders, Stewart and Wright (1985) propose applying the orthogonal factor of the QR factorization of A_{k-1} to A_k, and then computing the QR factorization of $Q_{k-1}^T A_k$ to get Q_k and thus Z_k. They show that with this method

$$\|Z_{k+1} - Z_k\| \leq \bar{a} \|x_{k+1} - x_k\|,$$

for some constant $\bar{a}$. If we consider the null space bases at two iterates x_k and x_j, with $j < k$, we have

$$\begin{aligned} \|Z_k - Z_j\| &\leq \sum_{i=j}^{k-1} \|Z_{i+1} - Z_i\| \\ &\leq \bar{a} \sum_{i=j}^{k-1} \|x_{i+1} - x_i\|. \end{aligned}$$

If the sequence $\{x_k\}$ converges R-linearly, then the sum $\sum_{i=1}^{\infty} \|x_{i+1} - x_i\|$ is finite. Therefore, we must have that $\|Z_k - Z_j\| \to 0$ as j and k go to infinity. This means that $\{Z_k\}$ is a Cauchy sequence, and must thus converge to some matrix Z_*, which by continuity satisfies $A(x_*)^T Z_* = 0$. Therefore for the Gill, Murray, Saunders, Stewart and Wright procedure, as well as for the Coleman and Sorensen procedure, there is a constant a_* such that for all k

$$\|Z_k - Z_*\| \leq a_* \|x_k - x_*\|, \tag{5.1}$$

where Z_* is a particular null space basis for $A(x_*)$. As we shall show the condition (5.1) is all that is required of the null space basis to give superlinear convergence of the reduced Hessian algorithm.

5.2 Consistency of the Matrix Approximation

Since Algorithm 2.1 approximates only the reduced Hessian G_k, one cannot expect it to be 1-step Q-superlinearly convergent. (See the examples of Byrd (1985) and Yuan (1985)). However, results of Powell (1978) show that if $\{x_k\} \to x_*$, if $\alpha_k = 1$ at each step, and if the matrices B_k satisfy

$$\omega_k \equiv \frac{\|(B_k - G_*)s_k\|}{\|x_{k+1} - x_k\|} \to 0, \tag{5.2}$$

then Algorithm 2.1 is 2-step superlinearly convergent, i.e.

$$\frac{\|x_{k+2} - x_*\|}{\|x_k - x_*\|} \to 0. \tag{5.3}$$

In fact, Coleman and Conn (1984) prove that Algorithm 2.1, using the DFP update, satisfies (5.2). Their arguments are based on the theory of Dennis and Moré (1977) and, with some changes, apply to the BFGS method also. However, it is also possible to obtain (5.2) using the techniques of Byrd and Nocedal (1987), as we now show.

Theorem 5.1 *Suppose that Assumptions 4.1 hold at x_*, and that the iterates $\{x_k\}$ generated by Algorithm 2.1, using any merit function, are contained in a neighborhood of x_* in which (4.1) - (4.3) hold. Furthermore assume that $\{x_k\}$ converges to x_* R-linearly, and that the matrices Z_k satisfy (5.1). Then*

$$\lim_{k \to \infty} \omega_k = 0,$$

and $\{\|B_k\|\}$ and $\{\|B_k^{-1}\|\}$ are bounded.

Proof: If $s_k = 0$ then $\omega_k = 0$. If $s_k \neq 0$, then we have from (4.3) and (2.18) that $y_k^T s_k > 0$. Since $h_k \leq \|x_{k+1} - x_k\|$, and since $\alpha_k \leq 1$, we have for any $\tau \in [0, 1]$ that

$$\|(x_k + \tau \alpha_k h_k) - x_*\| \leq \|e_k\| + \|x_{k+1} - x_k\| \leq 2\|e_k\| + \|e_{k+1}\|,$$

where $e_k = x_k - x_*$. Using this, (2.18), (4.2) and (5.1) we have

$$\begin{aligned} \frac{\|y_k - G_* s_k\|}{\|s_k\|} &= \frac{\|(\overline{G}_k - G_*)s_k\|}{\|s_k\|} \\ &\leq \hat{a} \max(\|e_{k+1}\|, \|e_k\|), \end{aligned}$$

for some constant $\hat{a}$. Due to the R-linear convergence, $\sum_{k=1}^{\infty} ||e_k|| < \infty$. We can therefore apply Theorem 3.2 of Byrd and Nocedal (1987) to obtain (5.2), since $||x_{k+1} - x_k|| \geq ||s_k||$, and to conclude that $\{||B_k||\}$ and $\{||B_k^{-1}||\}$ are bounded.

□

This theorem implies that, if $\alpha_k = 1$ at each step, then the sequence $\{x_k\}$ converges 2-step superlinearly to x_*. However, it turns out that with the ℓ_1 merit function (2.19) even very close to x_*, a steplength of 1 may not satisfy the steplength condition (2.2) in Algorithm 2.1. As pointed out in Chamberlain et. al. (1982) this "Maratos effect" can slow the convergence rate. To ensure that eventually $\alpha_k = 1$ is used at each step some slight modifications of Algorithm 2.1 must be made, when using the ℓ_1 merit function . We discuss two of them, the correction step, and the watchdog technique in §5.4 and §5.5. Before doing so we will show that these difficulties do not arise with Fletcher's merit function.

5.3 Fletcher's Merit Function

Since this merit function is differentiable with a strong local minimizer at x_*, one can show that for all sufficiently large k the algorithm accepts steplengths of 1, provided the weight ν is large enough. To show this and to establish the results of the next sections it is useful to first prove the following technical lemma about the decrease in the Lagrangian function produced by a single step of the algorithm.

Lemma 5.2 *Suppose that Assumptions 4.1 hold at x_* and that the matrices Z_k satisfy (5.1). If x_k is sufficiently close to x_*, and if ω_k defined by (5.2) is sufficiently small, then*

$$\frac{m}{2}||h_k|| - ||v_k|| \leq ||Z_k^T g_k|| \leq 2M||h_k|| + ||v_k||, \tag{5.4}$$

and therefore

$$||d_k|| = O(||e_k||). \tag{5.5}$$

Moreover, for any $\eta < \frac{1}{2}$ there exist constants $\hat{\gamma}$ and $\bar{\gamma}$ such that for e_k and ω_k sufficiently small,

$$L(x_k + d_k, \hat{\lambda}_k) \leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k - \bar{\gamma}||Z_k^T g_k||^2 + \hat{\gamma}||c_k||^2. \tag{5.6}$$

Proof: Since $s_k = \alpha_k Z_k^T h_k$ and $B_k s_k = -\alpha_k Z_k^T g_k$, we have from the definition of ω_k

$$\frac{||h_k||}{||G_*^{-1}||} - \omega_k(||h_k|| + ||v_k||) \leq ||Z_k^T g_k|| \leq ||G_*|| ||h_k|| + \omega_k(||h_k|| + ||v_k||).$$

If ω_k is small enough, and using (4.3), we obtain (5.4). The left inequality in (5.4) together with (2.9) and (4.1) give (5.5).

By Taylor's theorem

$$L(x_k + d_k, \hat{\lambda}_k) = L(x_k, \hat{\lambda}_k) + \nabla_x L(x_k, \hat{\lambda}_k)^T d_k + \frac{1}{2} d_k^T \nabla_{xx}^2 L(z, \hat{\lambda}_k) d_k,$$

where $z = x_k + \tau d_k$ for some $\tau \in (0,1)$. From (2.9) and (2.21) we have that $\nabla_x L(x_k, \hat{\lambda}_k)^T v_k = 0$. Therefore, since the second derivatives of f and c are bounded near x_*, we have by (2.7) and (2.8)

$$\begin{aligned} L(x_k + d_k, \hat{\lambda}_k) &\leq L(x_k, \hat{\lambda}_k) + g_k^T h_k + \frac{1}{2} h_k^T \nabla_{xx}^2 L(z, \hat{\lambda}_k) h_k + a_1 \|v_k\|(2\|h_k\| + \|v_k\|) \\ &\leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k \\ &\quad -(1-\eta)[h_k^T Z_k (B_k - G_*) Z_k^T h_k + h_k^T Z_k G_* Z_k^T h_k] \\ &\quad + \frac{1}{2} h_k^T \nabla_{xx}^2 L(z, \hat{\lambda}_k) h_k + a_1 \|v_k\|(2\|h_k\| + \|v_k\|), \end{aligned}$$

for some constant a_1. From the definition of ω_k

$$\|h_k^T Z_k (B_k - G_*) Z_k^T h_k\| \leq \|h_k\|(\|h_k\| + \|v_k\|)\omega_k,$$

and therefore

$$\begin{aligned} L(x_k + d_k, \hat{\lambda}_k) &\leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k + (1-\eta)\|h_k\|(\|h_k\| + \|v_k\|)\omega_k \\ &\quad + (1-\eta) h_k^T Z_k [Z_k^T \nabla_{xx}^2 L(z, \hat{\lambda}_k) Z_k - G_*] Z_k^T h_k \\ &\quad - (\frac{1}{2} - \eta) h_k^T \nabla_{xx}^2 L(z, \hat{\lambda}_k) h_k + a_1 \|v_k\|(2\|h_k\| + \|v_k\|). \end{aligned} \tag{5.7}$$

Using (4.3), (5.5) and (5.1) we have

$$\begin{aligned} L(x_k + d_k, \hat{\lambda}_k) &\leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k + (1-\eta)\|h_k\|(\|h_k\| + \|v_k\|)\omega_k \\ &\quad + a_2 \|h_k\|^2 \|e_k\| - (\frac{1}{2} - \eta)\|h_k\|^2 m \\ &\quad + a_1 \|v_k\|(2\|h_k\| + \|v_k\|), \end{aligned}$$

for some constant a_2. Thus if $\|e_k\|$ and ω_k are sufficiently small

$$L(x_k + d_k, \hat{\lambda}_k) \leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k - \frac{1}{2}(\frac{1}{2} - \eta)\|h_k\|^2 m + a_1 \|v_k\|(3\|h_k\| + \|v_k\|). \tag{5.8}$$

By the geometric/arithmetic mean inequality,

$$\begin{aligned} \|h_k\|\|v_k\| &= \left[\frac{(\frac{1}{2} - \eta)m}{6a_1} \|h_k\|^2 \frac{6a_1}{(\frac{1}{2} - \eta)m} \|v_k\|^2 \right]^{\frac{1}{2}} \\ &\leq \frac{(\frac{1}{2} - \eta)m}{12a_1} \|h_k\|^2 + \frac{3a_1}{(\frac{1}{2} - \eta)m} \|v_k\|^2. \end{aligned}$$

Substituting this into (5.8) we obtain

$$L(x_k + d_k, \hat{\lambda}_k) \leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k - (\frac{1}{2} - \eta)\frac{m}{4}\|h_k\|^2 + \left(a_1 + \frac{9a_1^2}{(\frac{1}{2} - \eta)m} \right) \|v_k\|^2.$$

From (5.4) we have that $\|h_k\|^2 \geq \frac{1}{4M^2}(\|Z_k^T g_k\| - \|v_k\|)^2 \geq \frac{1}{4M^2}(\frac{1}{2}\|Z_k^T g_k\|^2 - \|v_k\|^2)$. Using this (2.9) and (4.3) we have

$$L(x_k + d_k, \hat{\lambda}_k) \leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k - \bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2,$$

for some constants $\bar{\gamma}$ and $\hat{\gamma}$.

□

It is interesting to note from this result that the Lagrangian is decreased, unless the term $\hat{\gamma}\|c_k\|^2$ is large. This term occurs because the point x_* is not in general a local minimizer of $L(x, \lambda_*)$ but may be a saddle point; thus the v_k component of the step which decreases $\|c(x)\|$ may actually increase the Lagrangian. This fact prevents the Lagrangian from serving as a good merit function. It appears that a good merit function must have a term which gives sufficient weight to decreases in the value of $c(x)$, and it can be seen that both merit functions considered here are equal to the Lagrangian plus a term dependent on $\|c\|$.

Looking at the Fletcher merit function in this way and using Lemma 5.2 we can prove superlinear convergence.

Theorem 5.3 *Suppose that Assumptions 4.1 hold at x_*, and that Algorithm 2.1, using Fletcher's merit function, generates a sequence $\{x_k\}$ which converges R-linearly to x_*. Assume also that the matrices Z_k satisfy (5.1). Then, if for all sufficiently large k the weight has a fixed value ν, which is large enough, the rate of convergence is two-step Q-superlinear.*

Proof: We only need to show that for all sufficiently large k the point $x_{k+1} \equiv x_k + d_k$ satisfies the line search condition (2.2), for Theorem 5.1 and the results of Powell (1978) then imply (5.3).

By (5.5) we have that

$$\|c_{k+1}\| \leq \|c_k + A_k^T d_k\| + O(\|d_k\|^2) \leq a_3\|e_k\|^2, \tag{5.9}$$

for some constant a_3. Using this, (2.20), (5.6), (5.5) and (2.29) we obtain

$$\begin{aligned}
\Phi_\nu(x_{k+1}) &= L(x_{k+1}, \hat{\lambda}_k) + \left[L(x_{k+1}, \hat{\lambda}_{k+1}) - L(x_{k+1}, \hat{\lambda}_k)\right] + \tfrac{1}{2}\nu\|c_{k+1}\|^2 \\
&\leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k - \bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2 + \\
&\quad \|\hat{\lambda}_{k+1} - \hat{\lambda}_k\|\|c_{k+1}\| + \tfrac{1}{2}\nu\|c_{k+1}\|^2 \\
&\leq \Phi_\nu(x_k) - \tfrac{1}{2}\nu\|c_k\|^2 + \eta\left[g_k^T h_k + d_k^T {\hat{\lambda}_k'}^T c_k - \nu\|c_k\|^2\right] - \eta d_k^T {\hat{\lambda}_k'}^T c_k + \\
&\quad \eta\nu\|c_k\|^2 - \bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2 + a_4\|e_k\|^3. \\
&\leq \Phi_\nu(x_k) + \eta D\Phi_\nu(x_k; d_k) - \left\{[(\tfrac{1}{2} - \eta)\nu - \hat{\gamma}]\|c_k\|^2 + \bar{\gamma}\|Z_k^T g_k\|^2\right\} + \\
&\quad a_5\eta\|e_k\|\|c_k\| + a_4\|e_k\|^3,
\end{aligned} \tag{5.10}$$

for some constants a_4, a_5. Using Lemma 4.1 and the geometric/arithmetic mean inequality (as in the proof of Lemma 5.2), we see that there is a constant a_6 such that

$$a_5 \eta \|e_k\| \|c_k\| \leq a_6 \|c_k\|^2 + \tfrac{1}{2}\bar{\gamma} \|Z_k^T g_k\|^2,$$

from which one can show that, if ν is sufficiently large, $a_5 \eta \|e_k\| \|c_k\|$ is less than half the term inside the curly brackets. Also, if $\|e_k\|$ is sufficiently small, we have from Lemma 4.1 that the last term in (5.10) is less than half the term inside the curly brackets. Therefore

$$\Phi_\nu(x_{k+1}) \leq \Phi_\nu(x_k) + \eta D\Phi_\nu(x_k; d_k),$$

and the unit steplength is accepted by the algorithm.

□

5.4 The Second Order Correction Technique

Since the difficulty with the ℓ_1 merit function is caused by the nondifferentiability of the term $\|c(x)\|_1$, a very simple measure is to add to the step a correction of the form

$$w_k = -A_k (A_k^T A_k)^{-1} c(x_k + d_k).$$

This is very similar to strategies proposed by Coleman and Conn (1982), Fletcher (1982), Gabay(1982) and Mayne and Polak (1982) to deal with this problem. The effect of this correction step, which is normal to the constraints, is to decrease the quantity $\|c(x)\|$ so that it is of the order of $\|e_k\|^3$. This means that the merit function will then be decreased at the point $x_k + d_k + w_k$, as we will show.

We therefore consider the following variation of Algorithm 2.1.

Algorithm 5.1

The constants $\eta \in (0, \frac{1}{2})$ and τ, τ' with $0 < \tau < \tau' < 1$ are given.

(1) Set $k = 1$ and choose a starting point x_1 and a symmetric and positive definite starting matrix B_1.

(2) Compute d_k as the solution of the quadratic program (2.1)

(3) Set $\alpha_k = 1$.

(4) If

$$\phi_\mu(x_k + \alpha_k d_k) \leq \phi_\mu(x_k) + \eta \alpha_k D\phi_\mu(x_k; d_k), \tag{5.11}$$

set $x_{k+1} = x_k + \alpha_k d_k$ and go to (8).

(5) If (5.11) does not hold and if $\alpha_k < 1$ go to 7.

(6) Compute

$$w_k = -A_k(A_k^T A_k)^{-1} c(x_k + d_k). \tag{5.12}$$

If

$$\phi_\mu(x_k + d_k + w_k) \le \phi_\mu(x_k) + \eta D\phi(x_k; d_k) \tag{5.13}$$

holds, set $x_{k+1} = x_k + d_k + w_k$ and go to (8); otherwise go to (7).

(7) Choose a new α_k in $[\tau\alpha_k, \tau'\alpha_k]$ and go to (4).

(8) Update B_k using the BFGS formula (2.6).

(9) Set $k := k + 1$, and go to (2).

We will show that after a finite number of iterations backtracking is never needed, i.e. the step taken by this algorithm is either $x_{k+1} = x_k + d_k$ or $x_{k+1} = x_k + d_k + w_k$, which will imply superlinear convergence.

First we need to verify that Algorithm 5.1 is locally R-linearly convergent. This is easy to do, because Algorithm 5.1 differs from Algorithm 2.1 only if the step is accepted by (5.13), and this test enforces a sufficient reduction in the merit function. To show that Theorem 4.3 applies we only need to consider an iteration such that $j \in J$ and $x_{j+1} = x_j + d_j + w_j$. From (5.13) and (3.16) we see that (3.17) holds, and the proof of Theorem 4.3 applies without change. Therefore Algorithm 5.1 is R-linearly convergent.

Now we argue that Theorem 5.1 also holds for Algorithm 5.1. We consider an iteration for which the second order correction is used: $x_{k+1} = x_k + d_k + w_k$. Then

$$\|w_k\| \le \|e_{k+1}\| + \|e_k\|, \tag{5.14}$$

due to the orthogonality of w_k and d_k. Proceeding as in the proof of Theorem 5.1 (except that $\alpha_k = 1$) we have $x_k + h_k = x_{k+1} - v_k - w_k$, and therefore using (5.14) and (4.1)

$$\|x_k + h_k - x_*\| \le \|e_{k+1}\| + \hat{\gamma}_0\|e_k\| + \|e_{k+1}\| + \|e_k\|.$$

The rest of the proof is identical to that of theorem 5.1. Therefore we know that for Algorithm 5.1 condition (5.2) holds and that the matrices B_k and their inverses are bounded.

We now show that after a finite number of iterations backtracking is never needed.

Theorem 5.4 *Let Assumptions 4.1 hold at x_*. If x_k is sufficiently close to x_* and ω_k, defined by (5.2), is sufficiently small*

$$\phi_\mu(x_k + d_k + w_k) \le \phi_\mu(x_k) + \eta D\phi(x_k; d_k)$$

Proof: From (5.12), (4.1) and (5.9) we have

$$\|w_k\| = O(\|e_k\|^2). \tag{5.16}$$

Since $\nabla_x L(x_k, \hat{\lambda}_k)^T w_k = 0$, and using (5.5), we have

$$\begin{aligned} L(x_k + d_k + w_k, \hat{\lambda}_k) - L(x_k + d_k, \hat{\lambda}_k) &= \nabla_x L(x_k + d_k + \tau w_k, \hat{\lambda}_k)^T w_k \\ &= \nabla_x L(x_k, \hat{\lambda}_k)^T w_k + \\ &\quad O(\|d_k + \tau w_k\| \|w_k\|), \\ &= O(\|e_k\|^3), \end{aligned} \tag{5.17}$$

for some $\tau \in (0, 1)$. Similarly

$$c(x_k + d_k + w_k) = c(x_k + d_k) + A_k^T w_k + \int_0^1 [A(x_k + d_k + \tau w_k) - A_k]\, w_k d\tau.$$

Since the first two terms on the right hand side cancel, we have from (5.16) and (5.5)

$$\|c(x_k + d_k + w_k)\| = O(\|e_k\|^3). \tag{5.18}$$

Now

$$\begin{aligned} \phi_\mu(x_k + d_k + w_k) &= f(x_k + d_k + w_k) + \hat{\lambda}_k^T c(x_k + d_k + w_k) + \mu\|c(x_k + d_k + w_k)\|_1 \\ &\quad -\hat{\lambda}_k^T c(x_k + d_k + w_k) \\ &\le L(x_k + d_k + w_k, \hat{\lambda}_k) + (\mu + \|\hat{\lambda}_k\|_\infty)\|c(x_k) + d_k + w_k)\|_1 \\ &\le L(x_k + d_k + w_k, \hat{\lambda}_k) + O(\|e_k\|^3) \end{aligned} \tag{5.19}$$

Using (5.17), (5.6) and (2.26)

$$\begin{aligned} \phi_\mu(x_k + d_k + w_k) &\le L(x_k + d_k, \hat{\lambda}_k) + O(\|e_k\|^3) \\ &\le L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k - \bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2 + O(\|e_k\|^3) \\ &= f_k + \mu\|c_k\|_1 + \hat{\lambda}_k^T c_k - \mu\|c_k\|_1 + \eta g_k^T h_k - \bar{\gamma}\|Z_k^T g_k\|^2 \\ &\quad +\hat{\gamma}\|c_k\|^2 + O(\|e_k\|^3) \\ &= \phi_\mu(x_k) + \eta\left[g_k^T h_k + \hat{\lambda}_k^T c_k - \mu\|c_k\|_1\right] + (1-\eta)\hat{\lambda}_k^T c_k \\ &\quad -(1-\eta)\mu\|c_k\|_1 - \bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2 + O(\|e_k\|^3) \\ &= \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k) - (1-\eta)\rho\|c_k\|_1 - \bar{\gamma}\|Z_k^T g_k\|^2 + \\ &\quad \hat{\gamma}\|c_k\|^2 + O(\|e_k\|^3). \end{aligned}$$

Assuming that $\|c_k\| \le (1-\eta)\rho/(2\hat{\gamma})$, we have

$$\phi_\mu(x_k + d_k + w_k) \le \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k) - \{\tfrac{1}{2}(1-\eta)\rho\|c_k\|_1 + \bar{\gamma}\|Z_k^T g_k\|^2\} + O(\|e_k\|^3).$$

By (4.4), if $\|e_k\|$ is sufficiently small, the last term is smaller in magnitude than the term inside the curly brackets.

$\square$

Now we need to show that Powell's condition (5.2) implies 2-step Q-superlinear convergence also for Algorithm 5.1, if for all large k backtracking is not used.

Theorem 5.5 *Suppose that Assumptions 4.1 hold at x_*, and that Algorithm 5.1, generates a sequence $\{x_k\}$ which converges R-linearly to x_*. Assume also that the matrices Z_k satisfy (5.1). Then the rate of convergence is two-step Q-superlinear.*

Proof: Since we have shown that the matrices B_k and their inverses are bounded, Theorem 4.1 of Nocedal and Overton (1985) gives

$$\|x_{k-1} + d_{k-1} - x_*\| \leq C_1 \|e_{k-1}\| \tag{5.20}$$

for some constant C_1. Note also that by (5.9)

$$\|c(x_{k-1} + d_{k-1})\| \leq a_3 \|e_{k-1}\|^2. \tag{5.21}$$

Now, if the second order correction is used at step $k-1$, by (5.16) it satisfies $\|w_{k-1}\| = O(\|e_{k-1}\|^2)$. Therefore regardless of whether the correction step was used we have from (5.20) and (5.21) that

$$\|e_k\| \leq O(\|e_{k-1}\|) \tag{5.22}$$

and

$$\|c_k\| \leq O(\|e_{k-1}\|^2). \tag{5.23}$$

Now Lemma 6 of Powell (1978) implies that for any step on a quadratic program of the form (2.1) at x_k, under Assumptions 4.1, we have

$$\begin{aligned} \|x_k + d_k - x_*\| &\leq O(\|c_k\| + O(\|d_k\|^2) + \\ &\quad O(\|Z_k[G_k - B_k]Z_k^T d_k) \\ &\leq O(\|c_k\|) + O(\|e_k\|^2) + O(\omega_k \|d_k\|) \\ &\leq O(\|e_{k-1}\|^2) + O(\omega_k \|e_{k-1}\|), \end{aligned}$$

by (5.5), (5.22) and (5.23). If the second order correction is used at x_k then by (5.16) $\|w_k\| = O(\|e_k\|^2) = O(\|e_{k-1}\|^2)$, so that whether a correction step is taken or not,

$$\|e_{k+1}\| \leq O(\|e_{k-1}\|^2) + O(\omega_k \|e_{k-1}\|). \tag{5.24}$$

Since we have shown that $\omega_k \to 0$, we conclude from (5.24) that

$$\|e_{k+1}\| / \|e_{k-1}\| \to 0.$$

□

It is interesting to note that, if the correction step is tried at every iteration, the result of Byrd (1984) applies, giving a better convergence rate for the sequence $\{x_k + d_k\}$.

Theorem 5.6 *Consider a modification to Algorithm 5.1 such that, at every iteration, w_k is computed and if (5.13) holds then $x_{k+1} = x_k + d_k + w_k$. For this iteration, under the conditions of Theorem 5.5, the sequence $\{x_k + d_k\}$ converges to x_* one-step Q-superlinearly, that is*

$$\frac{\|x_{k+1} + d_{k+1} - x_*\|}{\|x_k + d_k - x_*\|} \to 0. \tag{5.25}$$

Proof: By Theorem 5.4, for k sufficiently large a full corrected step is taken so that $x_{k+1} = x_k + d_k + w_k$. The iteration is then equivalent to Algorithm 3 discussed by Byrd (1984) with the full Hessian approximation of that algorithm given by $Z_k B_k Z_k^T$. By Theorem 3.5 of that paper, since R-linear convergence implies boundedness of the Hessian approximations, (5.25) holds .

5.5 The Watchdog Technique

To avoid the inefficiencies caused by the Maratos effect, Chamberlain et al (1982) propose to sometimes accept the unit steplength even if this results in an increase in the ℓ_1 merit function. They call this a "relaxed step". However if after $\hat{t}$ steps a sufficient reduction has not been obtained, they go back to the iterate where the relaxed step was performed. We now describe a special case of this watchdog algorithm in which $\hat{t} = 1$. For simplicity we will assume that the matrix is updated at each iterate along the direction moved to reach that iterate, even though in practice it may be preferable not to do so at certain iterates that will be rejected. We note that an update at x_{k+1} is always done using information from the immediately preceeding step $x_{k+1} - x_k$. The algorithm uses the ℓ_1 merit function with the weight μ_k adjusted by (2.27); however in the description that follows we omit the subscripts of μ, for simplicity.

Watchdog Algorithm
The constant $\eta \in (0, \frac{1}{2})$ is given.

(0) Choose a starting point x_1 and a symmetric and positive definite starting matrix B_1. Set $k := 1$ and let $S = \{1\}$.

(1) Compute $x_{k+1} = x_k + d_k$, where d_k is the solution of (2.1). Update B_k by means of (2.6) to obtain B_{k+1}.

(2) Test the condition

$$\phi_\mu(x_{k+1}) \leq \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k). \tag{5.26}$$

If (5.26) holds, set $k := k + 1$, $S = S \cup \{k\}$, and go to (1).

(3) Compute $x_{k+2} = x_{k+1} + \alpha_{k+1} d_{k+1}$, where d_{k+1} solves (2.1) and α_{k+1} is such that

$$\phi_\mu(x_{k+2}) \leq \phi_\mu(x_{k+1}) + \eta\alpha_{k+1} D\phi_\mu(x_{k+1}; d_{k+1}). \tag{5.27}$$

Update B_{k+1} to get B_{k+2}.

(4) If

$$\phi(x_{k+1}) \leq \phi(x_k) \tag{5.28}$$

or

$$\phi_\mu(x_{k+2}) \leq \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k). \tag{5.29}$$

set $k := k + 2$, $S = S \cup \{k\}$, and go to 1.

(5) If $\phi_\mu(x_{k+2}) > \phi_\mu(x_k)$ compute $x_{k+3} = x_k + \alpha_k d_k$, where α_k is such that

$$\phi_\mu(x_{k+3}) \leq \phi_\mu(x_k) + \eta\alpha_k D\phi_\mu(x_k; d_k). \tag{5.30}$$

If $\phi_\mu(x_{k+2}) \leq \phi_\mu(x_k)$, compute d_{k+2} by solving (2.1), let $x_{k+3} = x_{k+2} + \alpha_{k+2}d_{k+2}$, where α_{k+2} is such that

$$\phi_\mu(x_{k+3}) \leq \phi_\mu(x_{k+2}) + \eta\alpha_{k+2} D\phi_\mu(x_{k+2}; d_{k+2}). \tag{5.31}$$

Update B_{k+2} to get B_{k+3}, set $k := k+3$, $S = S \cup \{k\}$, and go to 1.

The set S is not required by the algorithm and is introduced only to facilitate the analysis. It identifies the iterates for which a sufficient merit function reduction was obtained. Note that at least one third of the iterates have their indices in S.

For this algorithm it is possible to establish the R-linear convergence of the iterates in S, that is the set of iterates that satisfy a sufficient decrease condition. However the Watchdog Algorithm updates B_k at every iteration, and in order to conclude that $\omega_k \to 0$ we must have that

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_*\| < \infty,$$

where the sum is taken over all the iterates. It appears to be possible that when B_k is updated in step (1) at a point x_{k+1} that fails the test (5.26), x_{k+1} may be much farther from the solution than x_k, so that updating along d_k will move B_{k+1} away from the true Hessian. To avoid this difficulty and ensure R-linear convergence of all the iterates we now change the algorithm so that a point x_{k+1} that fails to satisfy (5.26) is accepted only if it satisfies

$$\|Z_{k+1}^T g_{k+1}\| + \|c_{k+1}\| \leq 2(\|Z_k^T g_k\| + \|c_k\|), \tag{5.32}$$

where the factor 2 is an arbitrary parameter. Otherwise, we do a line search and revoke the update of step (2). In the Watchdog Algorithm this amounts to adding the following after step (2).

(2a) If (5.26) does not hold, and (5.32) is not satisfied then compute α such that

$$\phi_\mu(x_k + \alpha d_k) \leq \phi_\mu(x_k) + \eta\alpha D\phi_\mu(x_k; d_k), \tag{5.33}$$

update B_k to get B_{k+1}, set $x_{k+1} = x_k + \alpha d_k, k := k+1, S = S \cup \{k\}$, and go to 1.

For this modified algorithm we are able to prove R-linear convergence of the entire sequence.

Lemma 5.7 *Let $\{x_k\}$ be generated by the Watchdog Algorithm using the additional step (2a). Suppose that x_* satisfies Assumptions 4.1, and that for all k greater than some index k_0, the weight μ_k has constant value μ and the iterates x_k are contained in a* *neighborhood of x_* for which Lemma 4.2, and (4.1)-(4.3) hold. Then $\{x_k\} \to x_*$, and there exists $r < 1$ and a_8 such that for any $k > k_0$*

$$\phi_\mu(x_k) - \phi_\mu(x_*) \le a_8 r^{k-k_0} \tag{5.34}$$

Therefore

$$\sum_{k=0}^{\infty} \|x_{k+1} - x_*\| < \infty, \tag{5.35}$$

and $\omega_k \to 0$.

Proof: Let $S = \{l_1, l_2, ...\}$. From (5.26), (5.29), (5.30) and (5.31) we see that for any $l_i > 0$ there is an integer j_i such that $1 \le j_i \le l_i - l_{i-1} \le 3$, and such that

$$\phi_\mu(x_{l_i}) \le \phi_\mu(x_{l_i - j_i}) + \eta\alpha D\phi_\mu(x_{l_i-j_i}; d_{l_i-j_i}), \tag{5.36}$$

where α is a steplength computed by the algorithm. We also see that the inequality

$$\phi_\mu(x_{l_i-j_i}) \le \phi_\mu(x_{l_{i-1}}) \tag{5.37}$$

holds for j_i.

Now suppose $l_i - j_i \in J$ so that (3.16) holds. Either $\alpha = 1$ or a backtracking linesearch was done along $d_{l_i-j_i}$ to determine α, and in either case the arguments in the proof of Lemma 3.3 together with (5.36) imply that

$$\phi_\mu(x_{l_i}) \le \phi_\mu(x_{l_i-j_i}) - \gamma'[\|Z_{l_i-j_i}^T g_{l_i-j_i}\|^2 + \|c_{l_i-j_i}\|_1]. \tag{5.38}$$

for some constant γ'. Now (5.38) together with (4.8) and then (5.37) imply

$$\begin{aligned}\phi_\mu(x_{l_i}) - \phi_\mu(x_*) &\le r_0^2[\phi_\mu(x_{l_i-j_i}) - \phi_\mu(x_*)] \\ &\le r_0^2[\phi_\mu(x_{l_{i-1}}) - \phi_\mu(x_*)] \end{aligned} \tag{5.39}$$

where $r_0^2 \equiv 1 - \frac{\gamma'}{\gamma_4} < 1$. Theorem 3.1 implies that $J \cap [1,k]$ contains at least $\frac{5}{6}k$ iterates, that is $[1,k]$ contains at most $\frac{k}{6}$ elements not in J. Therefore $|S \cap J \cap [1,k]| \ge |S \cap [1,k]| - \frac{k}{6}$. The structure of the watchdog procedure implies that $\frac{k}{3} \le |S \cap [1,k]|$ so that

$$|S \cap J \cap [1,k]\,| \ge \frac{1}{2}|S \cap [1,k]|.$$

Therefore (5.39) holds for at least half of the elements in S, and since $\{\phi_\mu(x_{l_i})\}$ is a decreasing sequence, we have that

$$\phi_\mu(x_k) - \phi_\mu(x_*) \le r_0^{k-1}[\phi_\mu(x_1) - \phi_\mu(x_*)] \tag{5.40}$$

holds for all $k \in S$.

Now we will show that step (2a) ensures that (5.34) holds for all the iterates. To show this we divide the iterates into three groups: (i) S; (ii) $S_1 = \{k \notin S \colon k-1 \in S\}$; (iii) S_2, the set of indices of the remaining iterates; (note that if $k \in S_2$ then $k-1 \in S_1$). Now if $k \in S$, we have from (5.40) that (5.34) holds. If $k \in S_1$ is large enough, we have from (4.8), (5.32), (4.4), again (4.8) and (5.40)

$$\begin{aligned} \phi_\mu(x_k) - \phi_\mu(x_*) &\leq \gamma_4[\|Z_k^T g_k\|^2 + \|c_k\|_1] \\ &\leq 2\gamma_4[\|Z_{k-1}^T g_{k-1}\| + \|c_{k-1}\|_1] \\ &\leq 2\gamma_4\gamma_2\|e_{k-1}\| \\ &\leq \frac{2\gamma_4\gamma_2}{\sqrt{\gamma_3}}(\phi_\mu(x_{k-1}) - \phi_\mu(x_*))^{\frac{1}{2}} \\ &\leq \frac{2\gamma_4\gamma_2}{\sqrt{\gamma_3}} r_0^{\frac{k-2}{2}}(\phi_\mu(x_1) - \phi_\mu(x_*))^{\frac{1}{2}} \end{aligned} \tag{5.41}$$

so that (5.34) is satisfied for $r \geq \sqrt{r_0}$ and $a_8 \geq \frac{2\gamma_4\gamma_2}{\sqrt{\gamma_3}} r_0^{-1}(\phi_\mu(x_1) - \phi_\mu(x_*))^{\frac{1}{2}}$. If $k \in S_2$ then $\phi_\mu(x_k) < \phi_\mu(x_{k-1})$ and $x_{k-1} \in S_1$, which gives (5.34) for some r less than 1.

We obtain $\sum_{k=0}^{\infty} \|x_{k+1} - x_*\| < \infty$ as in the proof of Theorem 4.3. The condition $\omega_k \to 0$ is proved as in Theorem 5.1.

□

Theorem 5.8 *Let Assumptions 4.1 hold at x_* and assume that the sequence $\{x_k\}$ generated the Watchdog Algorithm converges R-linearly to x_*. Then for all sufficiently large k the steplength is $\alpha_k = 1$, and the rate of convergence is 2-step Q-superlinear.*

Proof: Consider an iterate x_k at step (1) of the Watchdog Algorithm. The algorithm then sets $x_{k+1} = x_k + d_k$, and if x_{k+1} satisfies the sufficient decrease condition in step (2), then it is accepted and the algorithm goes back to step (1). Thus in this case the algorithm loops using $\alpha_k = 1$.

Let us now assume that the sufficient decrease condition is not satisfied at x_{k+1}. We will show that, if e_k and ω_k are sufficiently small, then x_{k+1} will satisfy the test (5.32). We then show that the line search, which will be made in step (3), will set $\alpha = 1$, and then in step (4) either (5.28) or (5.29) will be satisfied. Thus x_{k+1} and x_{k+2} will both be accepted with steplengths of 1.

To do this we first note that, since $\{\hat\lambda_k\}$ is bounded, there is a constant $\tilde\gamma$ such that $\mu + \|\hat\lambda_k\|_\infty < \tilde\gamma$. Also, since d_k is generated by (2.1), we apply Lemma 5.2 to obtain

$$\begin{aligned} \phi_\mu(x_{k+1}) &= L(x_{k+1}, \hat\lambda_k) + \mu\|c_{k+1}\|_1 - \hat\lambda_k^T c_{k+1} \\ &\leq L(x_k, \hat\lambda_k) + \eta g_k^T h_k - \bar\gamma\|Z_k^T g_k\|^2 + \hat\gamma\|c_k\|^2 + \tilde\gamma\|c_{k+1}\|_1 \\ &= f_k + \hat\lambda_k^T c_k + \eta\left[g_k^T h_k + \hat\lambda_k^T c_k - \mu\|c_k\|_1\right] - \bar\gamma\|Z_k^T g_k\|^2 + \hat\gamma\|c_k\|^2 \\ &\quad + \eta\mu\|c_k\|_1 - \eta\hat\lambda_k^T c_k + \tilde\gamma\|c_{k+1}\|_1 \\ &\leq \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k) - \bar\gamma\|Z_k^T g_k\|^2 + \hat\gamma\|c_k\|^2 \end{aligned}$$

$$
\begin{aligned}
&+(1-\eta)\left[\hat{\lambda}_k^T c_k - \mu\|c_k\|_1\right] + \tilde{\gamma}\|c_{k+1}\|_1 \\
\leq\ & \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k) - \bar{\gamma}\|Z_k^T g_k\|^2 + [\hat{\gamma}\|c_k\| - \rho(1-\eta)]\,\|c_k\|_1 \\
&+\tilde{\gamma}\|c_{k+1}\|_1.
\end{aligned}
$$

Thus for k large enough we have

$$
\phi_\mu(x_{k+1}) \leq \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k) - \bar{\gamma}\|Z_k^T g_k\|^2 - \tfrac{1}{2}\rho(1-\eta)\|c_k\|_1 + \tilde{\gamma}\|c_{k+1}\|. \tag{5.42}
$$

Since we assume that the sufficient decrease condition failed from x_k to x_{k+1},

$$
\phi_\mu(x_{k+1}) > \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k),
$$

which together with (5.42) implies

$$
-\bar{\gamma}\|Z_k^T g_k\|^2 - \tfrac{1}{2}\rho(1-\eta)\|c_k\|_1 + \tilde{\gamma}\|c_{k+1}\|_1 > 0. \tag{5.43}
$$

Using (5.9) this implies there exists a constant γ_5 such that

$$
\|c_k\| \leq \gamma_5\|e_k\|^2 \tag{5.44}
$$

whenever x_{k+1} does not satisfy (5.26). Now Lemma 6 of Powell (1978) implies that for any step on a quadratic program of the form (2.1), under Assumptions 4.1, we have

$$
\|x_k + d_k - x_*\| \leq O(\|c_k\|) + O(\|d_k\|^2) + O(\omega_k\|d_k\|), \tag{5.45}
$$

which together with (5.5) and (5.44) implies that

$$
\|e_{k+1}\| \leq O(\|e_k\|^2) + O(\omega_k\|e_k\|), \tag{5.46}
$$

when (5.26) is not satisfied. Since, by Lemma 4.1 $\|e_k\|$ and $\|Z_k^T g_k\| + \|c_k\|$ are of the same order, this relation implies that (5.32) will be satisfied for sufficiently large k, since $\omega_k \to 0$.

Now we must show that the step length in the direction d_{k+1} will be one, which happens if

$$
\phi_\mu(x_{k+1} + d_{k+1}) \leq \phi_\mu(x_{k+1}) + \eta D\phi(x_{k+1}; d_{k+1}). \tag{5.47}
$$

To do this apply (5.42) to the step from x_{k+1} to $x_{k+1} + d_{k+1}$:

$$
\begin{aligned}
\phi_\mu(x_{k+1} + d_{k+1}) \ \leq\ & \phi_\mu(x_{k+1}) + \eta D\phi_\mu(x_{k+1}; d_{k+1}) - \bar{\gamma}\|Z_{k+1}^T g_{k+1}\|^2 \\
& -\tfrac{1}{2}\rho(1-\eta)\|c_{k+1}\|_1 + \tilde{\gamma}\|c(x_{k+1} + d_{k+1})\|_1.
\end{aligned} \tag{5.48}
$$

Now note that by (5.9) and (5.46)

$$
\|c(x_{k+1} + d_{k+1})\| \leq O(\|e_{k+1}\|^2) \leq O(\|e_k\|^2(\|e_k\| + \omega_k)^2). \tag{5.49}
$$

Note also that by (5.43) and Lemma 4.1

$$\|c_{k+1}\|_1 > \frac{1}{\tilde{\gamma}}\left[\bar{\gamma}\|Z_k^T g_k\|^2 + \tfrac{1}{2}\rho(1-\eta)\|c_k\|\right] \geq a_9\|e_k\|^2, \tag{5.50}$$

for some constant a_9. Together, (5.49) and (5.50) imply that the sum of the last three terms in (5.48) is negative, and (5.47) follows.

Now we consider step (4) of the algorithm. If $\phi(x_{k+1}) \leq \phi(x_k)$ then x_{k+2} is accepted and we are finished. Otherwise, we need to show that

$$\phi_\mu(x_{k+1} + d_{k+1}) \leq \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k). \tag{5.51}$$

Using Lemma 5.2

$$\begin{aligned}
\phi_\mu(x_{k+1} + d_{k+1}) &= f(x_{k+1} + d_{k+1}) + \hat{\lambda}_k^T c(x_{k+1} + d_{k+1}) + \mu\|c(x_{k+1} + d_{k+1})\|_1 \\
&\quad -\hat{\lambda}_k^T c(x_{k+1} + d_{k+1}) \\
&\leq L(x_{k+1} + d_{k+1}, \hat{\lambda}_k) + \tilde{\gamma}\|c(x_{k+1} + d_{k+1})\|_1 \\
&= L(x_k, \hat{\lambda}_k) + \left[L(x_{k+1}, \hat{\lambda}_k) - L(x_k, \hat{\lambda}_k)\right] \\
&\quad + \left[L(x_{k+1} + d_{k+1}, \hat{\lambda}_k) - L(x_{k+1}, \hat{\lambda}_k)\right] \\
&\quad + \tilde{\gamma}\|c(x_{k+1} + d_{k+1})\|_1 \\
&\leq L(x_k, \hat{\lambda}_k) + \eta g_k^T h_k - \bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2 \\
&\quad + \left[L(x_{k+1} + d_{k+1}, \hat{\lambda}_{k+1}) - L(x_{k+1}, \hat{\lambda}_{k+1})\right] \\
&\quad + \left[L(x_{k+1} + d_{k+1}, \hat{\lambda}_k) - L(x_{k+1} + d_{k+1}, \hat{\lambda}_{k+1})\right] \\
&\quad - \left[L(x_{k+1}, \hat{\lambda}_k) - L(x_{k+1}, \hat{\lambda}_{k+1})\right] + \tilde{\gamma}\|c(x_{k+1} + d_{k+1})\|_1.
\end{aligned}$$

Applying Lemma 5.2 once more

$$\begin{aligned}
\phi_\mu(x_{k+1} + d_{k+1}) &\leq \phi_\mu(x_k) + \hat{\lambda}_k^T c_k - \mu\|c_k\|_1 + \eta\left[g_k^T h_k + \hat{\lambda}_k^T c_k - \mu\|c_k\|_1\right] \\
&\quad -\bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2 - \eta(\hat{\lambda}_k^T c_k - \mu\|c_k\|_1) \\
&\quad + \left\{\eta g_{k+1}^T h_{k+1} - \bar{\gamma}\|Z_{k+1}^T g_{k+1}\|^2\right\} + \hat{\gamma}\|c_{k+1}\|^2 \\
&\quad + \|\hat{\lambda}_{k+1} - \hat{\lambda}_k\|_\infty(\|c(x_{k+1} + d_{k+1})\|_1 + \|c_{k+1}\|_1) + \tilde{\gamma}\|c(x_{k+1} + d_{k+1})\|_1 \\
&\leq \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k) - (1-\eta)(\mu\|c_k\|_1 - \hat{\lambda}_k^T c_k) \\
&\quad -\bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2 + \hat{\gamma}\|c_{k+1}\|^2 \\
&\quad + \|\hat{\lambda}_{k+1} - \hat{\lambda}_k\|_\infty(\|c(x_{k+1} + d_{k+1})\|_1 + \|c_{k+1}\|_1) + \tilde{\gamma}\|c(x_{k+1} + d_{k+1})\|_1,
\end{aligned}$$

since both terms inside the curly brackets are non-positive. By (5.9) $\|c_{k+1}\| = O(\|e_k\|)^2$, and by (5.49) $(\|c(x_{k+1} + d_{k+1})\|_1) = o(\|e_k\|^2)$. Therefore

$$\begin{aligned}
\phi_\mu(x_{k+1} + d_{k+1}) &\leq \phi_\mu(x_k) + \eta D\phi_\mu(x_k; d_k) - \rho(1-\eta)\|c_k\|_1 - \bar{\gamma}\|Z_k^T g_k\|^2 + \hat{\gamma}\|c_k\|^2 \\
&\quad + o(\|e_k\|^2)
\end{aligned} \tag{5.52}$$

For k sufficiently large, $-\rho(1-\eta)||c_k||_1 + \hat{\gamma}||c_k||^2 \leq -\frac{1}{2}\rho(1-\eta)||c_k||_1$. Therefore the sum of the last three terms in (5.52) is negative, since by Lemma 4.1, $-\bar{\gamma}||Z_k^T g_k||^2 + -\frac{1}{2}\rho(1-\eta)||c_k||_1$ is of magnitude $||e_k||^2$. This establishes (5.51).

□

6. Summary and Conclusions.

We have studied the convergence properties of reduced Hessian successive quadratic programming, using the updating procedure of Coleman and Conn, and a backtracking line search. We have considered the effect of two merit functions: the ℓ_1 and the Fletcher functions. Our work differs from previous studies of these methods in that we have made no assumptions about the quasi-Newton matrices other than that the initial matrix is positive definite.

We now summarize, in general terms, the main results of this paper, considering the ℓ_1 merit function first. In section 3 it is shown that if the iterates are contained in a convex set in which the problem satisfies some smoothness and regularity conditions, and in which s_k and y_k satisfy (2.16) and (2.17) then $\liminf_{k\to\infty}(||Z_k^T g_k|| + ||c_k||) = 0$.

The local results proved in section 4 are somewhat stronger. If a local minimizer is a regular point satisfying the second order sufficiency conditions and if $\{||\hat{\lambda}(x_k)||\}$ is bounded, then there is a neighborhood of the minimizer such that if an iterate x_k lands in that neighborhood with k sufficiently large, the sequence converges to that minimizer R-linearly. The assumption that $\{||\hat{\lambda}(x_k)||\}$ is bounded is stronger than we would like, but follows from a regularity assumption on the constraints and thus meshes well with the global theory.

To obtain a superlinear rate of convergence we first impose some conditions on the choice of the null space basis Z_k, which are fairly easy to enforce in practice. Then, due to the difficulties associated with the Maratos effect, we are forced to make some modifications to the algorithm in section 5. Use of either modification ensures that steplengths of one are taken near the solution, but requires some extra cost in terms of function evaluations. One is to add a second order correction step to the iteration and the other is a variant of the watchdog technique. We show that both modifications retain the original local and global convergence properties and guarantee two-step Q-superlinear convergence. In addition we show that if the second order correction is in effect at every step, the sequence $x_k + d_k$ converges one-step Q-superlinearly.

For reduced Hessian methods using the Fletcher merit function similar global and local properties are proved in Sections 3 and 4, but only by making additional assumptions on the boundedness of B_k^{-1}. These *a priori* assumptions on the behavior of the algorithm are needed to guarantee the boundedness of the merit function weights, and the need for them makes the convergence theory in sections 3 and 4 significantly weaker for this merit function than for the ℓ_1 function. However, in section 5 we show that when the Fletcher function is used, no modifications are necessary to ensure steplengths of one. It is then easy to show, under the same conditions on the null space basis, that the rate of convergence is two-step superlinear.

We believe that this paper, at least in the local and superlinear sections, provides a realistic and informative analysis of the behavior of reduced Hessian successive quadratic programming in a practical implementation. We think that similar analysis should be possible when the update studied by Nocedal and Overton is used, and we hope that it will prove possible to analyze full Hessian SQP in a similar fashion.

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