

Normal Forms and Near-Axis Expansions for Beltrami Magnetic Fields

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A formal series transformation to Birkhoff-Gustavson normal form is obtained for toroidal magnetic field configurations in the neighborhood of a magnetic axis. Bishop's rotation-minimizing coordinates are used to obtain a local orthogonal frame near the axis in which the metric is diagonal, even if the curvature has zeros. We treat the cases of vacuum and force-free (Beltrami) fields in a unified way, noting that the vector potential is essentially the Poincaré-Liouville one-form of Hamiltonian dynamics, and the resulting magnetic field corresponds to the canonical two-form of a nonautonomous one-degree-of-freedom system. Canonical coordinates are obtained and Floquet theory is used to transform to a frame in which the lowest-order Hamiltonian is autonomous. The resulting magnetic axis can be elliptic or hyperbolic, and resonant elliptic cases are treated. The resulting expansion for the field is shown to be well-defined to all orders, and is explicitly computed to degree four. An example is given for an axis with constant torsion near a $1 : 3$ resonance.

I. INTRODUCTION

The utility of a device for confining plasma by a magnetic field depends crucially on the geometry of the field, especially for the case of toroidal confinement, like that in tokamaks or stellarators. Any such configuration should ensure that plasma pressure and electromagnetic forces balance to obtain an MHD equilibrium. Additional considerations, such as omnigenity or quasisymmetry, are necessary to ensure good confinement of gyrating particles (see the review Ref. [22]). For any of these desired properties, two crucial questions must be answered. Firstly, do magnetic fields with the desired property exist? If so, what is the topology, geometry, and dynamics of these fields? Many theoretical tools have been constructed to gain insight into these questions. One classical tool, which is enjoying a recent resurgence, is the near-axis expansion.

In essence, a near-axis expansion is a method for computing the magnetic field B as a power series in distance from a *magnetic axis*; a closed field line $r_0 : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ of B that is an isolated fixed point of its Poincaré first-return map. In the axisymmetric case, such an axis is a circle at the center of a nested family of toroidal magnetic surfaces, but more generally there may not be a smooth family of such surfaces. Two techniques have been developed for near-axis expansions: the direct and the inverse method.

The direct method was pioneered by Mercier³⁵ and Solov'ev and Shafranov³⁹ for studying solutions to the force-balance equations

$$J \times B = \nabla p, \quad \nabla \cdot B = 0, \quad (1.1)$$

where $J = \nabla \times B$ is the current vector and p is the (scalar) plasma pressure. The core idea is to use a Frenet-Serret frame based on r_0 to obtain what are now called Mercier coordinates $(\rho, \theta, s) \in \mathbb{R}^+ \times \mathbb{T}^2$. In these coordinates the axis $r_0(s)$ is simply $\rho = 0$. All physical quantities are then expanded as formal power series in ρ and (1.1) is solved order by order. Key goals of the direct method are to establish formal solutions to (1.1) (or, perhaps equally as interestingly, imply their non-existence), and to compute an integral of the system, e.g., the toroidal magnetic flux ψ , in terms of the Mercier coordinates. The direct

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method is beneficial when no assumptions are made about the possible topology, geometry, or dynamics of B . Since the pioneering work, the direct method has been implemented by many authors,^{3,4,25,26,29–32,38,40}

In contrast, the inverse method, as used most prominently by Garren and Boozer,^{17,18} but appearing earlier in Ref. [31], assumes the existence of special magnetic coordinates such as Boozer or Hamada coordinates.²² These consist of an integral ψ and a pair of angles θ, ϕ , so that the contravariant components of B , for example, depend only on ψ . The core aim of the inverse method is to determine the Euclidean coordinates (x, y, z) as a series expansion in the magnetic coordinates. The benefit of this method is that it can efficiently provide expressions for physical quantities in terms of magnetic coordinates, and these, in turn are useful for further theoretical exploration. However, since the inverse method necessarily assumes the existence of magnetic coordinates, it implicitly assumes that the field line flow is integrable (see, for instance, Ref. [9]). Conversely, if a nonvanishing magnetic field has toroidal flux surfaces, then, in the neighborhood of any flux surface, there exist magnetic coordinates;²⁸ this construction can be extended to a neighborhood of an axis as well.⁹ Nevertheless, given a magnetic axis r_0 , there may not exist a local, integrable field B . Indeed, an outstanding conjecture of Grad is that smooth solutions to (1.1) do not exist for a general toroidal domain.¹⁹

In this paper we study the direct method for near-axis expansion using a Hamiltonian perspective. Any nonzero, divergence-free vector field can be locally written as a non-autonomous $1\frac{1}{2}$ degree Hamiltonian system.²¹ The true power of this Hamiltonian reformulation is that all information about the vector field B is stored in a single function, the Hamiltonian H . Consequently, the dynamics of the field lines of B can be understood through this single function. Moreover, the perspective lends itself to the many ideas and tools of Hamiltonian mechanics and more generally, of symplectic geometry. In this paper, we demonstrate the benefits of this view through novel applications of classical ideas of Hamiltonian mechanics to near-axis expansions.

A similar perspective was adopted by Bernardin, Moses, and Tataronis^{3,4} to investigate magnetic fields satisfying (1.1) assuming that $\nabla p \neq 0$. In contrast to these papers, we treat (1.1) under the assumption that $\nabla p = 0$ in a neighborhood of the axis. In this case, the current must be parallel to the field,

$$\nabla \times B = \lambda B. \quad (1.2)$$

When $\lambda \neq 0$ such a field is called *Beltrami* (or *force-free*); the *vacuum* case corresponds to $\lambda = 0$. In these cases, the field lines are generically chaotic, as was first emphasized by Arnold in the fluid context¹ (see also Ref. [10, 14, and 15]). Since flux surfaces do not generically exist, the inverse method cannot be used. The vacuum field case has been treated previously in Ref. [26]. There, the authors assume the existence of a magnetic potential ϕ such that $B = \nabla\phi$, which of course implies that $J = 0$. Instead, we consider the vector potential A such that $B = \nabla \times A$, allowing, for the first time, a unified expansion for both Beltrami and vacuum fields. The expansion is given explicitly, to all orders, in Proposition V.2. All techniques from Hamiltonian mechanics applied in this paper to the Beltrami and vacuum fields naturally carry over to $\nabla p \neq 0$; however for brevity, we leave this to future work.

Our work further differs from Ref. [3] by recasting the expansion in terms of differential forms. A tutorial on differential forms specifically for plasma physics is given in Ref. [33]. By translating the theory into the language of differential forms we reveal the intrinsic geometry of vacuum and Beltrami fields: they give \mathbb{R}^3 , or more generally any 3-manifold M on which the equations are considered, the structure of a cosymplectic and contact manifold, respectively. For Beltrami fields, the utility of a contact structure was first understood by Etnyre and Ghrist¹⁵ and subsequently has been the source of many interesting results, most recently Ref. [10]. As far as we are aware, the result that vacuum fields are cosymplectic is novel. While the work here does not crucially depend on the understanding of these geometries, we believe that there can be further synergies between symplectic topology and plasma physics.

We will apply two useful tools from Hamiltonian mechanics: Floquet and normal form theory. Floquet theory¹⁶ is the study of time-periodic, linear differential equations and was specialized to the Hamiltonian case by Moser.³⁷ It provides a canonical coordinate system in which the linear dynamics becomes autonomous, thus giving an efficient way to compute its stability. In our context, the leading order terms in the near-axis expansion become independent of the toroidal angle, and the axis is revealed to be hyperbolic or elliptic. The Floquet transformation was implicitly computed in Ref. [4, 26, 35, and 39] as a sequence of transformations based on geometric assumptions about the flux surfaces near the axis. As we will demonstrate, the composition of these transformations is indeed the Floquet transformation.

An important result of Floquet theory is that when the axis is elliptic, its rotational transform, τ_0 , is related to the torsion, τ , of the curve r_0 ; we will show this holds for the Beltrami case as well. Moreover, our results also hold for hyperbolic axes which have stable and unstable manifolds with “expansivity” ν_0 . Such configurations are of importance in the design of divertors.^{7,8}

Normal form theory for Hamiltonian systems was pioneered by Birkhoff.⁵ The theory gives a way to compute “simple” coordinates in the neighborhood of a periodic orbit; a nice exposition is given in Ref. [36]. We will apply this technique to near-axis expansions. Essentially, normal form theory provides an iterative procedure to remove as many terms in a power series expansion of the Hamiltonian as possible. If the axis is elliptic and non-resonant, that is if $\tau_0 \notin \mathbb{Q}$, or if the axis is hyperbolic, then normal form theory gives coordinates $(x, y, s) \in D^2 \times \mathbb{S}^1$ so that H is (formally) of the form $H(x^2 + y^2)$ or $H(xy)$, respectively. If the axis is resonant with $\tau_0 = p/q$ then Gustavson’s normal form theory gives coordinates $(\rho, \theta, \phi) \in \mathbb{R}^+ \times \mathbb{T}^2$ so that $H = \frac{1}{2} \frac{p}{q} \rho^2 + K(\rho, q\theta + p\phi)$.²⁰ In each case, H is formally integrable: normal form theory provides both simple coordinates and an approximate integral. If the normal form series were to converge, these coordinates would give a true integral, defining flux surfaces, even in the resonant case.

Our normal form results should be directly compared to previous work for the nonresonant elliptic case;⁴ these authors compute an adiabatic invariant near the axis. Their method uses generating functions to implicitly give the coordinate transformation. A similar procedure was used in Ref. [26] to compute flux surfaces for a vacuum field; their flux coordinate ψ is, in essence, the adiabatic invariant of Bernardin et al. As we will show, normal form theory applies to this case, but also applies to hyperbolic and resonant elliptic axes. Moreover, we will use a near-resonant normal form³⁶ to give approximate flux surfaces when the on-axis rotational transform τ_0 is near a low order rational. A key difference from the generating function method is our use of Lie series to compute the normalizing transformation, in line with Ref. [13]. As they argued, the crucial benefit Lie series provide over the generating functions is efficiency as well as the ease of computing the inverse.

The paper is outlined as follows. In Section II, Beltrami and vacuum fields are introduced through the lens of differential forms. This translation from vector calculus notation establishes the intrinsic geometry of vacuum and Beltrami fields. In Section III the magnetic axis is defined and Bishop’s coordinates⁶ are introduced. These give Mercier coordinates without the assumption of non-vanishing curvature. A further advantage of these coordinates is that the metric is diagonal. In Section IV the Hamiltonian formulation is given and the classical theory of Floquet and of normal forms, including the near-resonant case, is recalled. Section V contains the application to near-axis expansions and the formal expansion for the Hamiltonian for Beltrami fields to all orders is found in Proposition V.2. Lastly, we apply the Floquet transformation and deconstruct it into the geometric transformations of previous work. Finally, in Section VI we give two examples of the normal form computation. Our examples use discrete symmetry to obtain closed curves. We apply this method to obtain a family of curves with constant torsion. These examples are chosen so that the axis is elliptic and the on-axis rotational transform τ_0 is arbitrarily close to a 1 : 3 resonance. The first example uses a regular normal form, while the second uses the near-resonant normal form. The calculated approximate integrals are then compared to the true field line dynamics. Future directions and concluding remarks are given in Section VII.

II. GEOMETRY OF VACUUM AND BELTRAMI FIELDS

In this paper we will consider a solid torus $D^2 \times \mathbb{S}^1$ in \mathbb{R}^3 , with the Euclidean metric and the standard volume form. However, the equations defining a vacuum or Beltrami field can be given for any Riemannian three-manifold M . In this section we give this general description through the use of differential forms. A summary of the translation is given in Table I and further exposition is given by MacKay.³³

Suppose that M is an orientable three-dimensional manifold with metric g and Riemannian volume form Ω . Associated with any non-vanishing magnetic field B on M is the so-called *flux form*; a two-form β defined by taking the interior product of B with the volume form

$$\beta := \iota_B \Omega. \quad (2.1)$$

The name follows from the fact that, given any two-dimensional surface S in M , the magnetic flux

through S is given by $\int_S \beta$.

The requirement that the magnetic field is divergence free, $\nabla \cdot B = 0$, can be restated in terms of the flux form β as $d\beta = 0$, that is, that β is closed. If B is non-vanishing, it also follows that β has maximal rank. Any two-form that is both closed and of maximal rank is called *presymplectic*.⁹ Conversely, as shown in Ref. [9], given any presymplectic form β , there exists a unique, non-vanishing, divergence-free vector field B such that $\iota_B \Omega = \beta$. Hence, the magnetic field B and flux-form β are dual views of the same mathematical object.

With the metric g in hand, there is a third view of a magnetic field. This is provided through the musical isomorphisms relating one-forms to vector fields, namely,

$$B^\flat := \iota_B g = g(B, \cdot).$$

One can think of B^\flat as the covariant representation of B . Indeed, in an arbitrary coordinate system $(x^1, x^2, x^3) \in M$, B^\flat is the covariant representation of a magnetic field as a one-form:

$$B^\flat = B_i dx^i. \quad (2.2)$$

The relationship between B^\flat and β is given through the Hodge star operator \star , which provides an isomorphism between k -forms and $(3-k)$ -forms. In local coordinates, the operator is defined on two-forms β as

$$\beta = \frac{1}{2} \epsilon_{ijk} \beta^i dx^j \wedge dx^k \mapsto \star \beta = \frac{1}{\rho} g_{ij} \beta^j dx^i, \quad (2.3)$$

and on one-forms as

$$\alpha = \alpha_i dx^i \mapsto \star \alpha = \frac{1}{2} \epsilon_{ijk} \rho g^{il} \alpha_l dx^j \wedge dx^k, \quad (2.4)$$

where $\rho = \sqrt{|\det g_{ij}|}$. The correspondence between B^\flat and β is then

$$\beta = \star B^\flat. \quad (2.5)$$

It is well known for $M = \mathbb{R}^3$ that any divergence-free vector field has a vector potential A : $B = \nabla \times A$. Equivalently, since β is closed, and all closed two-forms on \mathbb{R}^3 are exact, there is a primitive one-form $A^\flat =: \alpha$ for β :

$$\beta = d\alpha. \quad (2.6)$$

Using (2.5) this is also written

$$B^\flat = \star d\alpha. \quad (2.7)$$

More generally, the vector potential exists for any manifold M on which closed two-forms are exact.²

Given some additional structure on B , e.g., if it obeys magneto-hydrostatics (MHS), is Beltrami, or is a vacuum field, then there is a corresponding geometric interpretation. To see this, firstly note that the current J , defined by $J = \nabla \times B$, becomes $\iota_J \Omega = dB^\flat$. If B satisfies MHS, then there must exist p satisfying (1.1), or equivalently

$$\iota_J \beta = -dp.$$

In open regions where $dp \neq 0$, (B, J, p) is an *integrable presymplectic system*, see Ref. [9] for details.

Alternatively, if B is a vacuum field then $J = 0$, so $dB^\flat = 0$. Thus the one-form B^\flat is closed and $\beta \wedge B^\flat$ is a volume form on M . A manifold M together with a presymplectic form β and a closed one-form η such that $\beta \wedge \eta$ is a volume form, is called a *cosymplectic manifold*. It follows that B is a vacuum magnetic field if (M, β, B^\flat) is a *cosymplectic manifold*. This places vacuum fields in the realm of cosymplectic geometry.

Lastly, for a Beltrami (force-free) field $J = \lambda B$. Translating to differential forms gives

$$dB^\flat = \iota_J \Omega = \lambda \iota_B \Omega = \lambda \beta. \quad (2.8)$$

In this paper we will assume that λ is constant. A manifold M together with an exact, presymplectic form $\beta = d\eta$, so that $\beta \wedge \eta$ is a volume form, is called a *contact manifold*. Since (2.8) implies that $\eta = \lambda^{-1}B^\flat$, is indeed a primitive, $d\eta = d(\lambda^{-1}B^\flat) = \beta$, it follows that when B is Beltrami, $(M, \beta, \lambda^{-1}B^\flat)$ is a contact manifold. This places Beltrami fields in the realm of contact geometry.

We summarize these ideas as:

Lemma II.1. *Suppose B is a non-vanishing magnetic field on an orientable three-manifold M with volume form Ω . Let $\beta = \iota_B\Omega$. Then:*

1. B is MHS if (M, β) is a presymplectic manifold, $\iota_J\Omega = dB^\flat$ and $\iota_J\beta = -dp$;
2. B is a vacuum field if (M, β, B^\flat) is a cosymplectic manifold; and
3. B is a Beltrami field if there exists $\lambda \neq 0$ such that $(M, \beta, \lambda^{-1}B^\flat)$ is a contact manifold.

By viewing field line dynamics in this way, one can not only instantly see the differing geometries of these three cases, but also the relationship between magnetic fields and Hamiltonian mechanics. This relationship will be used heavily below. This geometric view of magnetic fields is not new and many interesting properties of magnetic fields have already been uncovered through this perspective. See, for instance, Ref. [10, 14, and 15].

The Beltrami condition (2.8) can be reformulated as a partial differential equation (PDE) for the coefficients of the vector potential α . Indeed, this will be used to compute the Hamiltonian in Section V, below. Using (2.6) with (2.8) requires that

$$\lambda d\alpha = dB^\flat, \quad \iota_B\Omega = d\alpha.$$

Then from (2.5), $B^\flat = \star d\alpha$, which implies

$$\lambda B^\flat = \star d(\lambda\alpha) = (\star d)^2\alpha. \quad (2.9)$$

Equivalently, $(\star d - \lambda)\alpha = \vartheta$ where ϑ is some closed one-form. However, note that it is not α but $d\alpha$ that defines the field-line dynamics: gauge freedom implies any closed one-form can be added to α without changing B . Thus without loss of generality, we could set $\vartheta = 0$. Nevertheless, we will retain (2.9) because, as will be seen, it is more useful to use the gauge freedom to select a desired form for α .

From Lemma II.1, the vacuum field case implies

$$dB^\flat = 0 \quad \implies \quad (\star d)^2\alpha = 0. \quad (2.10)$$

Of course, this is exactly the Beltrami equation (2.9) with $\lambda = 0$. This enables the simultaneous treatment of vacuum and Beltrami fields; simply treat the Beltrami case and then let $\lambda \rightarrow 0$.

Thus, the fundamental system to solve is (2.9). We will expand this PDE in the neighborhood of a magnetic axis, order-by-order in the radius to obtain an explicit construction of a normal form and a relation to Hamiltonian dynamics in Section V.

III. COORDINATES NEAR A MAGNETIC AXES

A. Magnetic axes

Magnetic axes are unavoidable in the study of plasma confinement since most containment designs are based on toroidal geometry. Such a device must have an axis that is a closed field line. In the simplest case this is the “center” of a family of nested toroidal surfaces. However, a general definition should not assume integrability and should also exclude closed field lines on rational tori.

Generally, suppose that $r_0 : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is a closed field line of a nonzero, smooth magnetic field B . Let $U = D^2 \times \mathbb{S}^1$ be a tubular neighborhood of the axis and Σ be some local section transverse to B containing a point $z \in r_0$. The flow of B produces a well-defined Poincaré first-return map $\pi_\Sigma : \Sigma \rightarrow \Sigma$

	Vector Calculus	Differential Forms
Metric	$g^{ij} = \nabla x^i \cdot \nabla x^j$	$ds^2 = g_{ij} dx^i dx^j$
Volume	$\rho = \sqrt{\det g_{ij}}$	$\Omega = \rho dx^1 \wedge dx^2 \wedge dx^3 = \star 1$
Covariant	$\vec{B} = B_j \nabla x^j$	$B^b = B_j dx^j$
Contravariant	$B^i = \vec{B} \cdot \nabla x^i$	$B = B^i \partial_{x^i}$
	$\vec{B} = \frac{1}{2} \epsilon_{ijk} \rho B^i \nabla x^j \times \nabla x^k$	$\beta = \frac{1}{2} \epsilon_{ijk} \rho B^i dx^j \wedge dx^k$
Hodge Star	$B_i = g_{ij} B^j$	$B^b = \star \beta$
Divergence	$\nabla \cdot \vec{B}$	$d \star B^b = d \iota_B \Omega$
Flux	$\vec{B} \cdot d^2 S$	$\beta = \iota_B \Omega = \star B^b$
Current	$\vec{J} = \nabla \times \vec{B}$	$J^b = \star dB^b$
Vector Potential	$\vec{B} = \nabla \times \vec{A}$	$B^b = \star d\alpha \ (A^b \equiv \alpha)$

TABLE I: Translations between vector calculus and differential forms. We use the summation convention. Note that the matrix (g_{ij}) is the inverse of (g^{ij})

with a fixed point z . The local dynamics of the closed field line r_0 can be characterized by the dynamics of the map π_Σ .

Using the Poincaré map we can exclude closed orbits on rational surfaces from our definition of a magnetic axis as follows.

Definition 1. A closed field line $r_0 : \mathbb{S}^1 \rightarrow \mathbb{R}^3$ is a *magnetic axis* if each point $z \in r_0$ is an isolated fixed point of its Poincaré first-return map, π_Σ .

This condition is coordinate independent and does not depend on the choice of section Σ . Indeed the flow of B provides a conjugacy between the first-return maps on any pair of sections.³⁴

However, as is sketched in Fig. 1, there could be several such axes, perhaps of differing local topology. In Section IV B, the notion of a *degenerate* and *nondegenerate* magnetic axes is defined. As will be seen, a nondegenerate axis must be elliptic or hyperbolic.

B. Framing a magnetic axis

In order to understand the possible field line behavior near a magnetic axis $r_0(s)$, it is useful to have good coordinates defined in its neighborhood. As first demonstrated by Mercier,³⁵ when $r_0 \in C^3([0, T], \mathbb{R}^3)$ and the curvature of r_0 is non-vanishing, these can be provided through the Frenet-Serret *moving frame* (see, for instance, Ref. [6]). Specifically, when s is the arc length, define the unit tangent, $\hat{t}(s) = r'_0$, normal $\hat{n}(s)$, and binormal $\hat{b}(s)$ vectors. Taking these to be row vectors, they satisfy the matrix ordinary differential equation

$$\frac{d}{ds} \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix}. \quad (3.1)$$

Here $\kappa(s)$ and $\tau(s)$ are the curvature and torsion of r_0 , respectively. Under the assumption that $\kappa(s)$ is non-vanishing, they are given explicitly by

$$\kappa = |r_0''|, \quad \tau = \frac{(r_0' \times r_0'') \cdot r_0'''}{\kappa^2}.$$

The Frenet-Serret frame defines a local embedding

$$\pi_{FS} : D^2 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3, \quad (x, y, s) \mapsto r_0(s) + x\hat{n}(s) + y\hat{b}(s). \quad (3.2)$$

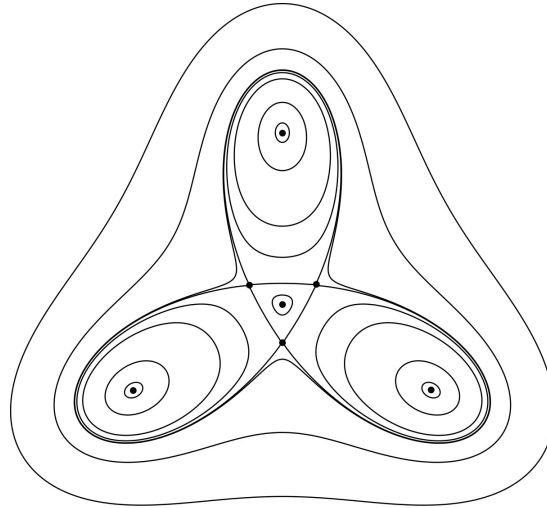


FIG. 1: A global Poincaré section of an example magnetic field. There are seven “magnetic axes” by Definition 1, four are elliptic and three, hyperbolic. Perhaps only one might be called the “central axis.”

In other words, π_{FS} is an embedding of the trivial disk bundle $D^2 \times S^1$ into a tubular neighborhood of $r_0(s)$ in \mathbb{R}^3 . In the plasma physics literature, these coordinates are often referred to as Mercier coordinates.²⁶

While the Frenet-Serret frame constructs coordinates in terms of the geometrically significant quantities κ and τ , in some practical cases this frame does not exist. This occurs, for example, if r_0 is not C^3 , or—more crucially—if r_0 has any inflection points or straight segments, i.e., points with $\kappa = 0$.

There are other choices for an orthonormal frame based on the curve $r_0(s)$. Moreover, such a frame can have an induced metric that is diagonal (in contrast to the Frenet-Serret case); in this case it is called *rotation minimizing*.⁶

Note that Mercier³⁵ established a rotation minimizing frame starting with the Frenet-Serret frame. However, the former can be constructed independently of the existence of the latter. There are at least two ways to do this. The first is to use a three-dimensional version of Fermi-Walker transport.¹² The second is Bishop’s *relatively parallel adapted frame*.⁶ A *relatively parallel* vector field, $v(s)$, is one that is *normal* to the curve, that is $v(s) \cdot \hat{t}(s) = 0$, but such that $v'(s)$ is *parallel* to $\hat{t}(s)$. Provided that r_0 is at least C^2 , there exists a unique relatively parallel vector field $v(s)$ such that $v(0) = v_0$ for every initial normal vector v_0 , see Thm. 1 of Ref. [6]. These vector fields can be constructed from a Frenet-Serret frame; however, they may also be constructed from any orthonormal frame. Crucially, this means that the curvature need not be nonzero, and the curve need not be C^3 .

As a consequence, for each initially orthonormal basis $(\hat{t}(0), \hat{n}_1(0), \hat{n}_2(0))$ we can compute a unique, relatively parallel adapted orthonormal frame $(\hat{t}, \hat{n}_1, \hat{n}_2)$ along the curve r_0 . Being relatively parallel, $\hat{n}'_1 = -\kappa_1 \hat{t}$, $\hat{n}'_2 = -\kappa_2 \hat{t}$ for some functions κ_1, κ_2 . Thus

$$\begin{pmatrix} \hat{t}' \\ \hat{n}'_1 \\ \hat{n}'_2 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & 0 \\ -\kappa_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{t} \\ \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}. \quad (3.3)$$

The functions (κ_1, κ_2) define the so-called *normal development* of the curve r_0 . If the Frenet-Serret frame exists, then

$$\kappa_1(s) = \kappa(s) \cos(\gamma(s) - \gamma^*), \quad \kappa_2(s) = \kappa(s) \sin(\gamma(s) - \gamma^*), \quad (3.4)$$

where γ^* is the angle between $\hat{n}_1(0)$ and $\hat{n}(0)$, and $\gamma(s)$ the *integral torsion*

$$\gamma(s) = \int_0^s \tau(s) ds. \quad (3.5)$$

As we mentioned above, rotation minimizing coordinates have another prominent advantage: the induced metric is diagonal, unlike that of the Frenet-Serret frame. Indeed, if g_e is the Euclidean metric on \mathbb{R}^3 and π_{FS} is the embedding (3.2), then the induced metric $g = \pi_{FS}^* g_e$ for the Frenet-Serret frame is³⁵

$$g = (\tilde{h}_s^2 + \tau^2(x^2 + y^2)) ds^2 + 2\tau(xdsdy - ydsdx) + dx^2 + dy^2, \quad (3.6)$$

$$h_s \equiv 1 - \kappa x.$$

This metric has non-diagonal terms due to the torsion. By contrast, for the rotation minimizing frame and the embedding, $\pi_B : D^2 \times \mathbb{R} \rightarrow \mathbb{R}^3$

$$\pi_B(\tilde{x}, \tilde{y}, s) = r_0(s) + \tilde{x}\hat{n}_1 + \tilde{y}\hat{n}_2, \quad (3.7)$$

the induced metric $\tilde{g} = \pi_B^* g_e$ becomes

$$\tilde{g} = \tilde{h}_s^2 ds^2 + d\tilde{x}^2 + d\tilde{y}^2, \quad (3.8)$$

$$\tilde{h}_s \equiv 1 - \kappa_1 \tilde{x} - \kappa_2 \tilde{y},$$

which is now diagonal.

Rotation minimizing coordinates are not without their drawbacks. The frame is not necessarily periodic in s , even for a periodic r_0 . Hence, it must be ensured that functions, forms or vectors defined on $\tilde{M} := D^2 \times \mathbb{R}$ are periodic when pulled back to M . If γ_0 is defined as the positively oriented angle between $\hat{n}_1(0)$ and $\hat{n}_1(T)$, and $\gamma(s)$ is *any* function satisfying

$$\gamma(T + s) - \gamma(s) = \gamma_0, \quad (3.9)$$

then this periodicity condition is equivalent to ensuring any object is well-defined under the push-forward by $\pi_S \circ R_\gamma$ where $\pi_S(x, y, s) = (x, y, s \bmod T)$ is the natural projection from the cover \tilde{M} to $D^2 \times \mathbb{S}^1$, and R_γ is a positive rotation in the plane normal to $\hat{t}_0(s)$ by $\gamma(s)$ for each s . Note that if γ is specifically the integral torsion (3.5) then we will push-forward to the Frenet-Serret frame; however, γ can be any function satisfying (3.9) and we will push-forward to some orthonormal periodic frame of r_0 .

One other drawback of the rotation minimizing coordinates is that, unlike κ and τ , the quantities κ_1 and κ_2 do not uniquely define the curve r_0 . However, it is clear from (3.4) that the normal development of a Frenet-Serret curve is unique up to rotation (essentially up to the constant γ^* in (3.9)).

Another trick that we will find useful is to think of $D^2 \subset \mathbb{C}$ and use the complex coordinate $z = x + iy$, so that the metric g (3.6) becomes

$$g = (h_s^2 + \tau^2 z \bar{z}) ds^2 + i\tau(zd\bar{z}ds - \bar{z}dzds) + dzd\bar{z},$$

$$h_s = 1 - \frac{1}{2}\kappa(z + \bar{z})$$

Setting the initial phase γ^* (3.4) to zero, the rotation minimizing coordinates (\tilde{x}, \tilde{y}) then become

$$u \equiv \tilde{x} + i\tilde{y} = e^{i\gamma} z, \quad (3.10)$$

so that \tilde{g} (3.8) is now

$$\tilde{g} = \tilde{h}_s^2 ds^2 + du d\bar{u}, \quad (3.11)$$

$$\tilde{h}_s = 1 - \frac{1}{2}(\bar{\kappa}_u u + \kappa_u \bar{u}), \quad \kappa_u = \kappa_1 + i\kappa_2.$$

Note, that even though we use this complex notation, all physical functions will be taken to be real-valued.

Under the transformation to complex coordinates $(\tilde{x}, \tilde{y}) \mapsto (\tilde{x} + i\tilde{y}, \tilde{x} - i\tilde{y}) = (u, \bar{u})$ on $T\tilde{M}$, the basis vectors $\partial_{\tilde{x}}, \partial_{\tilde{y}}$ of the tangent bundle and $d\tilde{x}, d\tilde{y}$ of the cotangent bundle push forward to

$$\begin{aligned} \partial_{\tilde{x}} &= \partial_u + \partial_{\bar{u}}, & \partial_{\tilde{y}} &= i\partial_u - i\partial_{\bar{u}}, \\ d\tilde{x} &= \frac{1}{2}(du + d\bar{u}), & d\tilde{y} &= \frac{1}{2i}(du - d\bar{u}). \end{aligned} \quad (3.12)$$

It follows that an arbitrary vector field becomes

$$B^s \partial_s + B^{\tilde{x}} \partial_{\tilde{x}} + B^{\tilde{y}} \partial_{\tilde{y}} = B^s \partial_s + B^u \partial_u + \bar{B}^{\bar{u}} \partial_{\bar{u}},$$

where $B^u = B^{\tilde{x}} + iB^{\tilde{y}}$. Similarly, an arbitrary one-form becomes

$$a = a_s ds + a_{\tilde{x}} d\tilde{x} + a_{\tilde{y}} d\tilde{y} = a_s ds + a_u du + \bar{a}_{\bar{u}} d\bar{u}, \quad (3.13)$$

with $a_u = \frac{1}{2}(a_{\tilde{x}} - ia_{\tilde{y}})$. For the case of the vector potential, α , (2.7) gives the covariant representation

$$\star d\alpha = B^s ds + B_u du + \bar{B}_{\bar{u}} d\bar{u}. \quad (3.14)$$

Note that for the metric (3.11), these components are related to the contravariant ones by

$$B_s = \bar{h}_s^2 B^s, \quad B_u = \frac{1}{2} \bar{B}^u, \quad \bar{B}_{\bar{u}} = \frac{1}{2} B^u.$$

IV. NEAR-AXIS HAMILTONIANS, FLOQUET THEORY, AND NORMAL FORMS

In this section we establish the Hamiltonian nature of magnetic fields near an axis, opening the study of magnetic fields to the tools of Hamiltonian mechanics. We then describe two such useful tools: Floquet theory and normal form theory. Both of these are useful in finding simple coordinates in the neighborhood of a magnetic axis, and we will use them in Section V to construct the “simplest” coordinates near a magnetic axis.

A. Hamiltonian near a magnetic axis

As is well known, the dynamics of the field lines in a neighborhood of r_0 can be described by a non-autonomous Hamiltonian system, see Ch. 9 of Ref. [21].

Theorem IV.1. *There is a tubular neighborhood $U \cong D^2 \times \mathbb{S}^1$ of r_0 with coordinates $(x, y, s) \in U$ such that the closed orbit becomes $r_0(s) = (0, 0, s)$ and there is a Hamiltonian $H : U \rightarrow \mathbb{R}$ such that*

$$\begin{aligned} \alpha &= ydx - H(x, y, s)ds \\ \beta &= dy \wedge dx - dH \wedge ds. \end{aligned}$$

That is, the one-form α is the Liouville one-form of a non-autonomous Hamiltonian function H . Moreover at the magnetic axis, $d_{\perp} H|_{r_0} = 0$, where d_{\perp} is the derivative perpendicular to r_0 .

Proof. Take some orthonormal frame at each point on r_0 to define coordinates in a tubular neighborhood $(\tilde{x}, \tilde{y}, s) \in U \cong D^2 \times \mathbb{S}^1$ of r_0 such that $r_0(s) = (0, 0, s)$. In such a neighborhood, the fact that β is closed implies that it is exact, that is, there exists α such that $\beta = d\alpha$. Using the gauge freedom of α we can assume

$$\alpha = \alpha_{\tilde{x}}(\tilde{x}, \tilde{y}, s)d\tilde{x} + \alpha_s(\tilde{x}, \tilde{y}, s)ds, \quad \partial_s \alpha_{\tilde{x}}(0, 0, s) = 0,$$

so that

$$\beta = d\alpha = \partial_{\tilde{y}} \alpha_{\tilde{x}} d\tilde{y} \wedge d\tilde{x} + (\partial_s \alpha_{\tilde{x}} - \partial_{\tilde{x}} \alpha_s) ds \wedge d\tilde{x} + \partial_{\tilde{y}} \alpha_s d\tilde{y} \wedge ds.$$

The magnetic field is tangent to the axis, so $B|_{r_0} = B_0(s)\partial_s$, where $B_0(s) \neq 0$ by assumption. Moreover, the volume form in $(\tilde{x}, \tilde{y}, s)$ has the form $\Omega = \rho(\tilde{x}, \tilde{y}, s)d\tilde{x} \wedge d\tilde{y} \wedge ds$ for some nonzero density ρ . Therefore, since $\beta = \iota_B \Omega$, we know $\beta|_{r_0} = -\rho(0, 0, s)B_0(s)d\tilde{y} \wedge d\tilde{x}$ and it follows that $\partial_{\tilde{y}}\alpha_s|_{r_0} = -\rho(0, 0, s)B_0 \neq 0$, $-\partial_{\tilde{x}}\alpha_s(0, 0, s) = 0$ and $-\partial_{\tilde{y}}\alpha_s(0, 0, s) = 0$.

Choose new coordinates $(x, y, s) = (\tilde{x}, \alpha_{\tilde{x}}, s)$. This is a diffeomorphism, locally in (\tilde{x}, \tilde{y}) , for all s by the inverse function theorem. In these new coordinates define $H = -\alpha_s$, and then $\alpha = ydx - Hds$ and $\beta = dy \wedge dx - dH \wedge ds$ as desired.

Note that $d_{\perp}H|_{r_0} = -\partial_x\alpha_s(0, 0, s)dx - \partial_y\alpha_s(0, 0, s)dy = 0$ by the assumed form of β on r_0 . \square

In the language of vector calculus, Theorem IV.1 is equivalent to showing that there are coordinates such that the contravariant representation of B is

$$B = \nabla y \times \nabla x - \nabla H(x, y, s) \times \nabla s.$$

B. Normal Forms: Set-up

Birkhoff's normal form theory seeks a choice of canonical coordinates near a periodic orbit, or fixed point, for which the Hamiltonian takes its "simplest" form. The definition of "simplest" is perhaps a matter of taste; for the Birkhoff normal form, the goal is to have as few terms as possible in the series expansion of H . The normal form will be the result of an iterative construction of a new coordinate system.

A review of normal form theory is given in Appendix A. Here, we will outline the core details for the normal form near a periodic orbit or magnetic axis, r_0 , such that $d_{\perp}H|_{r_0} = 0$. It is convenient to introduce the angle

$$\phi = 2\pi \frac{s}{T},$$

so that the axis can be thought of as a periodic orbit with period 2π .

Assume that $H = H(x, y, \phi)$ is a non-autonomous Hamiltonian on $D^2 \times \mathbb{S}^1$ with canonical variables (x, y) and such that $x = y = 0$ corresponds to an isolated, 2π -periodic orbit. We begin by expanding H in a Taylor expansion in x, y :

$$H \sim H_0 + H_1 + \dots \quad (4.1)$$

Here we denote the *lowest-degree* terms by H_0 , i.e., we assume that there is a $k \in \mathbb{N}$ such that H_0 is a degree k polynomial in x, y . Similarly, H_i denotes a degree $k + i$ polynomial in (x, y) . All of these coefficients are 2π -periodic in ϕ .

Generally, H begins with quadratic terms so that $k = 2$. If it does not, then the orbit r_0 is said to be *degenerate*. Such cases can still be treated by normal form theory, however, it is much more difficult to deduce the final normal form of H (see Appendix A for further details). Henceforth, assume that $k = 2$.

C. Floquet Theory

We will first ignore the higher order terms and treat the dynamics of the quadratic Hamiltonian $H_0(x, y, \phi)$ using Floquet theory. The resulting linear system is

$$\frac{d}{d\phi} \begin{pmatrix} x \\ y \end{pmatrix} = J\nabla H_0 = A(\phi) \begin{pmatrix} x \\ y \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4.2)$$

where the matrix $A(\phi)$ is a 2π -periodic Hamiltonian matrix, i.e., $JA = (JA)^T$.

Since this is a linear, time-periodic system, the core result of Floquet theory³⁴ applies:

Theorem IV.2 (Floquet-Lyapunov). *The fundamental matrix solution $X(\phi)$ of*

$$\dot{X} = A(\phi)X, \quad X(0) = I, \quad (4.3)$$

is of the form

$$X(\phi) = P(\phi)e^{C\phi},$$

where the matrix $P(\phi)$ is symplectic and 2π -periodic and C is a constant Hamiltonian matrix. Moreover, $P(\phi)$ and C can be assumed to be real by letting $P(\phi)$ be 4π -periodic if necessary.

As noted, one can take $P(\phi)$ to be a symplectic matrix whenever (4.3) is Hamiltonian (Thm. 3.4.2 of Ref. [36]), i.e., $P^T J P = J$. In this case, C must be a Hamiltonian matrix.

The eigenvalues of C are called the *Floquet exponents*. Taking coordinates $w \in \mathbb{R}^2$ via $(x, y)^T = P(\phi)w$ transforms (4.3) to the autonomous system $\dot{w} = Cw$. Consequently, in the new coordinates $H_0 = \frac{1}{2}w^T C w$ is autonomous.

For a one and a half degree-of-freedom Hamiltonian system, there are two Floquet exponents, ω_1, ω_2 , which must satisfy $\omega_1 = -\omega_2$. Thus they are either purely imaginary, purely real or both zero.

When the exponents are purely imaginary, say $\pm i\tau_0$, with *rotational transform* $\tau_0 \in \mathbb{R} \setminus \{0\}$, then the linear system (4.3) is stable. More precisely, solutions to (4.3) lie on invariant tori with elliptical cross sections on $\phi = \text{const}$ surfaces. It is always possible in this case to assume that $P(\phi)$ is 2π -periodic.

In contrast, when the exponents are purely real, say $\pm\nu_0$ with *expansivity* $\nu_0 \in \mathbb{R} \setminus \{0\}$, equation (4.3) is hyperbolic and the periodic orbit has invariant stable and unstable manifolds. For so-called *reflection hyperbolic* orbits, the matrix $P(\phi)$ must be taken 4π -periodic. Geometrically, these orbits have stable manifolds that make a $(2j+1)\pi$ rotation as ϕ goes from 0 to 2π , for some $j \in \mathbb{Z}$. In contrast, $P(\phi)$ can be taken 2π periodic for *direct hyperbolic* orbits, which have stable manifolds that make a $2j\pi$ rotation. These invariant manifolds serve as separatrices for H_0 .

The full, nonlinear system may still have bounded solutions when the axis is hyperbolic. For example, in Fig. 1 the three points on the separatrix are hyperbolic orbits, while the remaining four are elliptic orbits, and the overall system still has bounded orbits.

Finally, the Floquet exponents may vanish, and then the axis is *degenerate*. More generally an elliptic case could be said to be degenerate, or *resonant*, when $\tau \in \mathbb{Q}$. Even though a resonant axis is linearly elliptic, higher order terms may destroy the tori of the quadratic part.

D. Normal Forms: Higher order

Returning to the normal form procedure, we will assume that the Floquet transformation has been made so that H_0 does not depend on ϕ and that the Floquet exponents are nonzero, so the axis is linearly elliptic or hyperbolic.

We seek coordinates in a neighborhood of r_0 that transform H to its “simplest” form, so that the core aspects of the dynamics can easily be understood. The most concise way to state the normal form theorem is to use the Poisson bracket; if $f, g \in C^\infty(M)$ and we have canonical coordinates x, y normal to r_0 , then the *Poisson bracket* is defined as

$$\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g). \quad (4.4)$$

To simplify the calculations for the elliptic case, we will use the complex conjugate variables (u, \bar{u}) , with $u = x + iy$. In these coordinates the Poisson bracket becomes

$$\{f, g\} = 2i[(\partial_{\bar{u}} f)(\partial_u g) - (\partial_u f)(\partial_{\bar{u}} g)]. \quad (4.5)$$

Note that even with the complex coordinates all physical functions are real-valued.

The following theorem gives the desired normal form for H .

Theorem IV.3. *Let H be a Hamiltonian system containing a linearly elliptic or hyperbolic periodic orbit r_0 of period 2π . There exists a formal, canonical, 2π -periodic (possibly 4π -periodic), near-identity, change of variables $w = \Phi(u, \phi)$ that transforms the Hamiltonian (4.1) to*

$$\tilde{H} \sim \sum \tilde{H}_j(w, \phi),$$

such that

$$\{\tilde{H}_j, H_0\} + \partial_\phi \tilde{H}_j = 0, \quad (4.6)$$

for all $j \geq 0$.

This theorem is due to Birkhoff,⁵ and most books on Hamiltonian mechanics contain a proof. A particularly thorough account is given in Ref. [36]. The proof is constructive, giving an iterative procedure to compute the normal form at each order j . Some of the details of the computation are given in Appendix A.

The terms in the normal form \tilde{H} depend on whether the axis is hyperbolic or elliptic and, in the latter case, resonant or not.

Corollary IV.4. *Let \tilde{H} be a non-autonomous Hamiltonian system that contains a periodic orbit r_0 . Then there are local coordinates (x, y, ϕ) such that:*

(i) *if r_0 is linearly elliptic with Floquet exponents $\pm it_0$ then, if $t_0 \notin \mathbb{Q}$ the formal normal form becomes*

$$\tilde{H}(x, y, \phi) \sim F(x^2 + y^2),$$

for some function $F : \mathbb{R} \rightarrow \mathbb{R}$; by contrast, in the resonant case, $t_0 = p/q \in \mathbb{Q}$,

$$\tilde{H} \sim \frac{1}{2} \frac{p}{q} \rho^2 + K(\rho, q\theta + p\phi), \quad (4.7)$$

for some function K where (θ, ρ) are defined by $x + iy = \rho e^{i\theta}$; and

(ii) *if r_0 is linearly hyperbolic then the formal normal form becomes*

$$\tilde{H}(x, y, \phi) \sim F(xy).$$

A remarkable fact about normal forms for $1\frac{1}{2}$ degree of freedom systems is that they are always formally integrable. This is most easily seen when the axis is non-resonant ($t \notin \mathbb{Q}$) elliptic or hyperbolic. In this case the normal form Hamiltonian of Corollary IV.4 is independent of ϕ , so that it is a formal integral of the system.

For the resonant elliptic case, the normal form (4.7) depends only on the single angle-like variable $q\theta + p\phi$. Thus one can do a time-dependent canonical transformation to a frame that rotates with this angle to obtain a new Hamiltonian that is autonomous.³⁶ In these new coordinates, the lowest-order term H_0 is removed, and the Hamiltonian begins with terms of degree q . Thus the elliptic orbit becomes a *degenerate* magnetic axis. Nevertheless, since the system is now autonomous, it is formally integrable. An example is shown in Fig. 2 for $p/q = 1/4$. Note that the lowest-order resonant terms in this case are quartic. The (nonlinear) stability of the axis $x = y = 0$ depends, in this case, on the size of the resonant terms.³⁶

It is, however, important to note that the integrability of the normal form is misleading since the normal form expansion is generally only formal. Indeed, the power series for the coordinate transformation Φ of Theorem IV.3 typically does not converge, even in a neighborhood of the magnetic axis. Of course, if one knows that Φ is smooth or analytic then immediately one obtains the integrability of the system. There is a partial converse; if it is known that the system is integrable and the integral is nondegenerate (in particular non-resonant), then Φ must be smooth or analytic. The proof is recalled in Ref. [9].

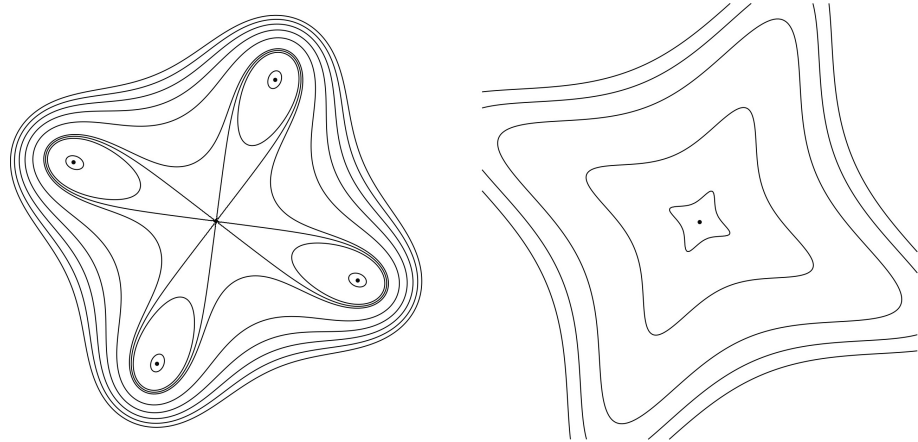


FIG. 2: Two examples of an integral corresponding to a 1:4 resonance. On the left the central magnetic axis is unstable, whilst on the right it is stable.

E. Normal forms: Near Resonance

A great benefit of understanding near-axis expansions through normal form theory is the ability to understand *near* resonant phenomena. Suppose that the on-axis rotational transform $\tau_0 = (p/q + \varepsilon)$ for a *resonance detuning* ε . The key idea is to treat ε as formally small and to find the normal form of H using the resonance p/q . In doing so, the normal form will be valid as ε crosses zero and may produce a better understanding of the phase space topology further away from the magnetic axis.

Of course, the rationals are dense in the reals, so there is always a p/q arbitrarily close to τ_0 . However, if q is large, any resonant terms that appear will not enter the normal form until the q^{th} degree terms in x, y . Although the following analysis will still work, it is only low-order resonances that are of primary concern near the axis.

Concretely, suppose the Hamiltonian is of the form

$$H = \frac{1}{2}(p/q + \varepsilon)(x^2 + y^2) + \dots$$

When ε is formally small, it can be neglected in H_0 and the resonant normal form becomes (4.7). At this stage, we can add back the ε term under the ordering assumption it is as small as the first resonant term, i.e., $\varepsilon \rho^2 \sim \rho^q$. The resulting Hamiltonian again depends only on the combination $q\theta + p\phi$, and so it is integrable—with an invariant that can be obtained by a time-dependent transformation as before.

The topology and bifurcations of the phase portraits for different values of q as ε passes through 0 are well understood (see, for example, Ref. [2]). The usual consequence is a stable region about the axis followed by a q -island chain at a distance $\mathcal{O}(\sqrt{\varepsilon})$ from the axis. However, the cases $q = 2, 3, 4$ are special, since the detuning term appears at an order comparable with the resonant normal form terms. A 1:3 near-resonant example was depicted in Fig. 1.

We will give an example use of this near resonance analysis in Section VI.

V. APPLICATION TO MAGNETIC AXES

In this section we apply the classical Floquet and normal form theory to magnetic axes.

A. Formal Hamiltonians for magnetic axes

Given the rotation minimizing coordinates of Section III, defined in a tubular neighborhood of the magnetic axis, we now present an iterative scheme to directly compute the Hamiltonian and normal form coordinates for a given nondegenerate magnetic axis.

1. Series Expansions

In this section we construct the canonical Hamiltonian H for Beltrami and vacuum magnetic fields in the neighborhood of a magnetic axis r_0 by solving (2.9) for the vector potential α . It will be convenient to solve for $\tilde{\alpha}$, the vector potential in the (s, u) rotation minimizing coordinates since the metric is diagonal on \tilde{M} . Once this is done, we will use (3.7) to impose the constraint that there exists $\alpha = \pi_B^* \tilde{\alpha}$, that is, that $\tilde{\alpha}$ pushes forward to a periodic one-form on M when rotated to a periodic frame through some $\gamma(s)$ satisfying (3.9).

For ease in computing canonical coordinates in Section V A 2, it will be convenient to use gauge freedom to choose a representation of $\tilde{\alpha}$ different from (3.13).

Lemma V.1. *Up to gauge freedom, any real-valued one-form on \tilde{M} can formally be written as*

$$\tilde{\alpha}(u, \bar{u}, s) = \tilde{\alpha}_s(u, \bar{u}, s)ds - \frac{1}{4i}\tilde{\alpha}_u(u, \bar{u}, s)(u\bar{u} - \bar{u}u), \quad (5.1)$$

where $\tilde{\alpha}_s, \tilde{\alpha}_u : \tilde{M} \rightarrow \mathbb{R}$. Furthermore if the original form is analytic at $u = 0$, then so is $\tilde{\alpha}$.

Proof. An arbitrary one-form a (3.13) is equivalent to (5.1) under a gauge transformation if there exists a function $F : \tilde{M} \rightarrow \mathbb{R}$ such that $a - \tilde{\alpha} = dF$. In this case, necessarily $da - d\tilde{\alpha} = 0$, and, since \tilde{M} is simply connected, this condition is also sufficient. Writing out each component of the condition yields,

$$\partial_u \tilde{\alpha}_s = \frac{1}{4i} \bar{u} \partial_s \tilde{\alpha}_u + \partial_u a_s - \partial_s a_u, \quad (5.2)$$

$$\partial_{\bar{u}}(\bar{u}\tilde{\alpha}_u) + \partial_u(u\tilde{\alpha}_u) = 4i(\partial_{\bar{u}}a_u - \partial_u\bar{a}_u), \quad (5.3)$$

since the third, $ds \wedge d\bar{u}$, component simply gives the complex conjugate of (5.2). First consider (5.3) as an equation determining a real-valued function $\tilde{\alpha}_u$ given an arbitrary complex-valued a_u . Indeed, this can be solved at least formally about $u = 0$. To see this, expand each function as a power series in u, \bar{u} . Note that the operator $f \mapsto \partial_{\bar{u}}(\bar{u}f) + \partial_u(uf)$ maps monomials $u^{k_1}\bar{u}^{k_2} \mapsto (2 + k_1 + k_2)u^{k_1}\bar{u}^{k_2}$. Hence, we can solve for each monomial by simply dividing by $(2 + k_1 + k_2)$, which is always nonzero.

Given such a solution to (5.3) there then exists, for each value of $s \in \mathbb{R}$, a function F such that

$$a_u - \frac{1}{4i}\bar{u}\tilde{\alpha}_u = \partial_u F, \quad \bar{a}_u + \frac{1}{4i}u\tilde{\alpha}_u = \partial_{\bar{u}} F. \quad (5.4)$$

Since $\tilde{\alpha}_u$ is smooth in s then so is F . Substituting this form into (5.2) yields

$$\partial_u \tilde{\alpha}_s = \partial_u a_s - \partial_s \partial_u F.$$

Hence, taking $\tilde{\alpha}_s = a_s - \partial_s F$ gives a solution to (5.2). \square

Now we use the form (5.1) to solve the Beltrami equation (2.9). In the metric (3.11), the covariant components (3.14) become

$$\begin{aligned} B_s &= \frac{1}{2}\tilde{h}_s(\partial_{\bar{u}}(\bar{u}\tilde{\alpha}_u) + \partial_u(u\tilde{\alpha}_u)), \\ B_u &= \frac{i}{\tilde{h}_s}(\partial_u \tilde{\alpha}_s - \frac{1}{4i}\bar{u}\partial_s \tilde{\alpha}_u), \\ \bar{B}_u &= -\frac{i}{\tilde{h}_s}(\partial_{\bar{u}} \tilde{\alpha}_s + \frac{1}{4i}u\partial_s \tilde{\alpha}_u). \end{aligned} \quad (5.5)$$

Applying the operator $\star d$ once more to obtain (2.9) gives

$$\begin{aligned} 2i\tilde{h}_s(\partial_{\bar{u}}B_u - \partial_u\bar{B}_u) &= \lambda B_s, \\ i(\partial_u B_s - \partial_s B_u) &= \lambda\tilde{h}_s B_u, \\ -i(\partial_{\bar{u}}B_s - \partial_s\bar{B}_u) &= \lambda\tilde{h}_s\bar{B}_u. \end{aligned} \quad (5.6)$$

This set, upon substitution for B in terms of $\tilde{\alpha}$ from (5.5), corresponds to three PDEs for the vector potential components $\tilde{\alpha}_s$ and $\tilde{\alpha}_u$.

To formally solve (5.6) for α we expand each component in a series in u and \bar{u} ,

$$\begin{aligned} \tilde{\alpha}_s &\sim \sum_{j=0} \tilde{\alpha}_s^j(s, u, \bar{u}), \\ \tilde{\alpha}_u &\sim \sum_{j=0} \tilde{\alpha}_u^j(s, u, \bar{u}), \end{aligned} \quad (5.7)$$

where each $\tilde{\alpha}_\times^j$ is a degree j , homogeneous polynomial in u, \bar{u} with complex coefficients that are functions of s . Substituting the series expansion (5.7) into the Beltrami condition (5.6) then gives

$$\partial_{u\bar{u}}\tilde{\alpha}_s^n = \frac{1}{4} \left\{ \text{im} \left(\partial_{\bar{u}}(\bar{u}\partial_s\tilde{\alpha}_u) + \frac{1}{2}\tilde{h}_s^{-1}\kappa_u\bar{u}\partial_s\tilde{\alpha}_u \right) - 2\text{re}(\kappa_u\partial_u\tilde{\alpha}_s) - \lambda B_s \right\}_{n-2}, \quad (5.8a)$$

$$L_u\tilde{\alpha}_u^n = \left\{ \frac{1}{2}\tilde{h}_s^{-2}\bar{\kappa}_u B_s + \tilde{h}_s^{-1}\partial_s B_u - i\lambda B_u \right\}_{n-1}, \quad (5.8b)$$

$$\bar{L}_u\tilde{\alpha}_u^n = \left\{ \frac{1}{2}\tilde{h}_s^{-2}\kappa_u B_s + \tilde{h}_s^{-1}\partial_s\bar{B}_u + i\lambda\bar{B}_u \right\}_{n-1}, \quad (5.8c)$$

where L_u is defined by

$$L_u\tilde{\alpha} =: \frac{1}{2}\partial_u(\partial_{\bar{u}}\tilde{\alpha}_u + \partial_u\tilde{\alpha}_u),$$

and \bar{L}_u is the equivalent under $u \leftrightarrow \bar{u}$. The braces $\{\cdot\}_j$ in (5.8) denote the j^{th} order term from the formal series (5.7). The right hand sides of (5.8) depend on the components of α to at most degree $n-1$. As a consequence, the equations can be solved iteratively. We formulate this as a proposition. In the proposition, we will see that it is convenient to think of the polynomial coefficients $\tilde{\alpha}_s$ and $\tilde{\alpha}_u$ as functions of the rotated variable $z = e^{-i\gamma(s)}u$. An advantage of this is that, if we can show the remaining dependence on s is periodic with period T , then the resulting fields project from the covering space \bar{M} to M .

Proposition V.2. *For any smooth $\gamma(s)$ satisfying (3.9) there is a formal solution to (5.8) of the form*

$$\begin{aligned} \tilde{\alpha}_s^n(u, \bar{u}, s) &= A_n(s)z^n + \bar{A}_n(s)\bar{z}^n + R^n(z, \bar{z}, s), \\ \tilde{\alpha}_u^n(u, \bar{u}, s) &= \alpha_z^n(z, \bar{z}, s), \end{aligned} \quad (5.9)$$

where

$$z = e^{-i\gamma}u, \quad (5.10)$$

and R^n and α_z^n are real, degree- n homogeneous polynomials in z, \bar{z} with coefficients periodic in s and dependent on $\tilde{\alpha}_s^k, \tilde{\alpha}_u^k$ for $k < n$, and each A_n is a free, complex-valued function. In particular, if each A_n is taken T -periodic in s then the formal series $\tilde{\alpha}_s, \tilde{\alpha}_u$ are T -periodic in s .

Moreover, by subjecting (2.9) to the additional constraint $\star d\alpha|_{r_0} = B_0 ds$, we have $\tilde{\alpha}_u^0 = \alpha_z^0 = B_0(s)$, and we can choose $\alpha_s^0 = \alpha_s^1 = 0$ without changing B .

Proof. We prove the proposition by induction on the degree in (5.9). As the right hand side of (5.8) vanishes for $n=0$, and for $n=1$ for (a), it follows that α_z^0, α_s^0 , and α_s^1 are free functions. Make the particular choice $\alpha_s^0 = \alpha_s^1 = 0$ and $\alpha_z^0 = B_0(s)$ where $B_0(s) = B_s(0, 0, s) = B^s(0, 0, s)$ is the magnetic field on axis.

Assume the result is true for all $k < n$ and consider order n . The right hand side of (5.8a) — evaluated at order $n - 2$ — depends on α_z^j for $j \leq n - 2$ and α_s^j for $j \leq n - 1$. For the second and third equations, we must know α_s^n ; therefore, the first equation should be solved first before solving for α_z^n .

The left hand side of (5.8a) can be thought of as a linear operator on a vector space, specifically, \mathcal{H}_n^γ , the vector space of homogeneous, degree- n , real polynomials in z , and \bar{z} with T -periodic coefficients in s . Then

$$\partial_{u\bar{u}} : \mathcal{H}_n^\gamma \rightarrow \mathcal{H}_{n-2}^\gamma.$$

To show that (5.8a) is solvable we need to prove the right hand side is in the image of $\partial_{u\bar{u}}$. Necessarily, the right hand side must be in \mathcal{H}_{n-2}^γ . With the assumption that the proposition is true for all $n \leq k - 1$, a calculation confirms this is indeed true. In order to see this, note that $\kappa_z = e^{-i\gamma} \kappa_u$ is periodic by (3.11); as a consequence h_s is also periodic. Note that $\dim(\mathcal{H}_n^\gamma) = n + 1$. The kernel of $\partial_{u\bar{u}}$ acting on \mathcal{H}_n^γ is spanned by $\text{re}(u^n)$, and $\text{im}(u^n)$. Hence, as the kernel is two-dimensional and $\dim(\mathcal{H}_{n-2}^\gamma) = n - 1$, then the rank-nullity theorem guarantees that $\partial_{u\bar{u}}$ is surjective. Hence, assuming the proposition for $n \leq k - 1$, there is a solution to (5.8a) for $n = k$.

Now, a necessary and sufficient condition for Eqs. (5.8b) and (5.8c) is for $\nabla \cdot B = 0$. Indeed, these equations correspond to the system (5.6). By knowing $\bar{\alpha}_s$ to degree n , and $\bar{\alpha}_z$ to degree $n - 1$, we can obtain B_u to order n . We want to show that the final two equations are consistent for B_s . In turn this will show that (5.8) is consistent. A quick rearrangement yields

$$\partial_u B_s = \partial_s B_u - i\bar{h}_s \lambda B_u, \quad \partial_{\bar{u}} B_s = \partial_s \bar{B}_u + i\bar{h}_s \lambda \bar{B}_u.$$

Since \bar{M} is simply connected, a necessary and sufficient condition for these equations to be solvable is for

$$\partial_{\bar{u}} \left(\partial_s B_u - i\bar{h}_s \lambda B_u \right) - \partial_u \left(\partial_s \bar{B}_u + i\bar{h}_s \lambda \bar{B}_u \right) = 0.$$

This condition is satisfied as,

$$\begin{aligned} \partial_{\bar{u}} \left(\partial_s B_u - i\bar{h}_s \lambda B_u \right) - \partial_u \left(\partial_s \bar{B}_u + i\bar{h}_s \lambda \bar{B}_u \right) &= \partial_s (\partial_{\bar{u}} B_u - \partial_u \bar{B}_u) - i\lambda \partial_{\bar{u}} (\bar{h}_s B_u) - i\lambda \partial_u (\bar{h}_s \bar{B}_u) \\ &= -i\lambda \partial_s (\tfrac{1}{2} \bar{h}_s^{-1} B_s) - i\lambda \partial_{\bar{u}} (\bar{h}_s B_u) - i\lambda \partial_u (\bar{h}_s \bar{B}_u) \\ &= -\frac{i}{2} \lambda \bar{h}_s \nabla \cdot B. \end{aligned}$$

It follows that the equations are consistent since $\nabla \cdot B = 0$ by assumption.

Since (5.8) is consistent, a solution at order $n = k$ will exist provided the all coefficients are known for $n \leq k - 1$. The result follows by induction. \square

From the preceding proposition we can obtain a formal solution of the form

$$\bar{\alpha} = \alpha_s(z, \bar{z}, s) ds - \frac{1}{4i} \alpha_z(z, \bar{z}, s) (ud\bar{u} - \bar{u}du). \quad (5.11)$$

Consequently, by choosing γ that satisfies (3.9) and then rotating through $R_{-\gamma}$ we obtain periodic coordinates and a primitive form α satisfying (2.9) so that

$$\alpha = \left(\frac{1}{2} \tau \alpha_z(z, \bar{z}, s) z \bar{z} + \alpha_s(z, \bar{z}, s) \right) ds - \frac{1}{4i} \alpha_z(z, \bar{z}, s) (z d\bar{z} - \bar{z} dz), \quad (5.12)$$

where $\tau(s) := \gamma'(s)$. Note that, if the Frenet-Serret frame exists and γ is taken to be the integral torsion (3.5), then τ is the torsion of r_0 .

The explicit solution to order $n = 4$ of equations (5.8) is given in Appendix B. The leading order terms will be needed in Section V B and are given by

$$\begin{aligned} \alpha_s^2 &= A_2 z^2 + \bar{A}_2 \bar{z}^2 - \frac{1}{4} \lambda B_0 z \bar{z}, \\ \alpha_z^0 &= B_0(s), \\ \alpha_z^1 &= \frac{1}{3} B_0 (\bar{\kappa}_2 z + \kappa_2 \bar{z}), \end{aligned}$$

where the A_2 is a complex-valued, T -periodic function of s , and $\kappa_z = e^{-i\gamma(s)} \kappa_u$. Note that, if γ is the integral torsion (3.5), then $\kappa_z = \bar{\kappa}_z = \kappa$ is the curvature, and τ is the torsion of r_0 .

2. Canonical coordinates

We have now computed in Proposition V.2 the vector potential α . By finding canonical coordinates, we will be able to get an expression for the Hamiltonian H . A method to compute canonical coordinates using series expansions could be obtained using the work of Cary and Littlejohn.¹¹ However, the canonical coordinates are easily obtained as a consequence of our choice of gauge from Lemma V.1.

In complex coordinates $Z = X + iY$, the canonical Liouville one-form is given, up to a closed one-form, by

$$\begin{aligned}\alpha &= \frac{1}{4i} (Z d\bar{Z} - \bar{Z} dZ) - H ds \\ &= \frac{1}{2} (Y dX - X dY) - H ds,\end{aligned}\quad (5.13)$$

where $H : M \rightarrow \mathbb{R}$ is the Hamiltonian. Recalling that α_z is real, (5.12) can be transformed into the form (5.13) using

$$Z = \bar{z} \sqrt{\alpha_z(s, z, \bar{z})}, \quad (5.14)$$

and H becomes

$$H = -\frac{1}{2}\tau(s)Z\bar{Z} - \alpha_s. \quad (5.15)$$

The first four orders of H are explicitly given in Appendix B. The leading order is

$$H_0 = -\bar{A}_2 B_0^{-1} Z^2 - A_2 B_0^{-1} \bar{Z}^2 + \left(\frac{1}{4}\lambda - \frac{1}{2}\tau\right) Z\bar{Z} \quad (5.16)$$

3. Recovering the magnetic field

We have found canonical coordinates (Z, \bar{Z}) which put the one-form α into the form (5.13). The beauty of this form is that almost all of the information about the magnetic vector field is stored in the single function H . In order to extract the magnetic field from H , we can first use the fact that $\iota_B \Omega = \beta = d\alpha$. Explicitly, we have that $\Omega = \rho dZ \wedge d\bar{Z} \wedge ds$ where $\rho = \sqrt{g_Z}$ is the density from the metric g_Z induced by the transformation to canonical coordinates. Then

$$\begin{aligned}\iota_B \Omega &= \rho B_Z d\bar{Z} \wedge ds - \rho B_{\bar{Z}} dZ \wedge ds + \rho B_s dZ \wedge d\bar{Z}, \\ d\alpha &= \frac{1}{2i} dZ \wedge d\bar{Z} - dH \wedge ds.\end{aligned}$$

It follows that

$$B_Z = -\rho^{-1} \partial_{\bar{Z}} H, \quad B_{\bar{Z}} = \rho^{-1} \partial_Z H, \quad B_s = \frac{1}{2i} \rho^{-1}.$$

Note that these are the Euler-Lagrange equations scaled by $\frac{1}{2i} \rho^{-1}$.

The density ρ can be found by directly computing $\sqrt{g_Z}$ or by transforming the Euclidean volume form in Frenet-Serret coordinates, $\Omega = -ih_s d\bar{z} \wedge dz \wedge ds$, by the transformation to canonical coordinates. Both methods involve inverting (5.14), which is a rather complicated expression. However, to compute the normal form, we will not be working directly with B , but instead with the Hamiltonian vector field for H ,

$$B_H = 2i\rho B = -2i\partial_{\bar{Z}} H \partial_Z + 2i\partial_Z H \partial_{\bar{Z}} + \partial_s.$$

Consequently, the complication of computing ρ is bypassed.

B. Normal form coordinates near magnetic axis

Normal form theory is useful in our analysis of near-axis expansions as it will allow for the elimination of as many terms as possible in the series expansion of $H(s, Z, \bar{Z})$. These removable terms are dependent on the quadratic component H_0 given explicitly in (5.16). Specifically, as was highlighted by Corollary IV.4, if the axis is elliptic we need to compute the on-axis rotational transform t_0 , or, if the axis is hyperbolic, we need the on-axis expansivity ν_0 .

It is possible to get an explicit formula for these constants in terms of geometric quantities, such as the total torsion and curvature of the magnetic axis. First, note the leading order dynamics from H_0 is given by

$$\dot{Z} = -2i\partial_Z H_0 = 4i\bar{C}_2(s)\bar{Z} - i\left(\frac{1}{2}\lambda - \tau(s)\right)Z, \quad (5.17)$$

with $C_2(s) := B_0(s)^{-1}\bar{A}_2(s)$. Theorem IV.2 guarantees a canonical transformation $w = F(s, Z, \bar{Z})$ that is linear in Z , and \bar{Z} bringing (5.17) into

$$\dot{w} = it_0 w, \quad (5.18)$$

in the elliptic case and

$$\dot{w} = \nu_0 \bar{w}. \quad (5.19)$$

in the hyperbolic case.

In order to obtain F explicitly, it is useful to separately consider the elliptic and hyperbolic cases. In the elliptic case, the transformation F takes invariant tori of the linearized dynamics that are in a tubular neighborhood U of the axis, into other invariant tori in $D^2 \times \mathbb{S}^1$ that are given simply by level sets of $w\bar{w}$ with $w \in D^2$. Similarly, in the hyperbolic case, F maps invariant surfaces of the linearized dynamics, that are in a tubular neighborhood U of the axis, to invariant surfaces that are level sets of $w^2 + \bar{w}^2$ in $D^2 \times \mathbb{S}^1$.

In either case, this transformation can be broken down into a rotation

$$Z = e^{i\delta(s)}V, \quad (5.20)$$

that aligns the principal axes of each ellipse or hyperbola with the coordinate axes. Here $\delta(s+T) - \delta(s) = 2\pi j$ in the elliptic or direct hyperbolic case, and $\delta(s+T) - \delta(s) = (2j+1)\pi$ in the reversed hyperbolic case, for some $j \in \mathbb{Z}$. This is followed by a canonical scaling

$$V = \cosh \eta(s)v - \sinh \eta(s)\bar{v}, \quad (5.21)$$

for T -periodic $\eta(s)$, in order to scale the major and minor axes of each ellipse or hyperbola to obtain symmetry.

Applying the transformations to (5.17) yields

$$\begin{aligned} \dot{v} = & -\frac{i}{2} \left(4(C_2 e^{2i\delta} + \bar{C}_2 e^{-2i\delta}) \sinh 2\eta + (\lambda + 2\delta' - 2\tau) \cosh 2\eta \right) v \\ & + i \left(4e^{-2i\delta} \bar{C}_2 \cosh^2 \eta + 4e^{2i\delta} C_2 \sinh^2 \eta + \left(\frac{1}{2}\lambda + \delta' - \tau\right) \sinh 2\eta - i\eta' \right) \bar{v}. \end{aligned} \quad (5.22)$$

In order for ellipses to be invariant we require $\frac{d}{ds}v\bar{v} = 0$. Computing this condition yields

$$C_2(s) \cosh 2\eta = -\frac{1}{4}e^{-2i\delta} \left(\left(\frac{1}{2}\lambda + \delta' - \tau\right) \sinh 2\eta + i\eta' \cosh 2\eta \right), \quad (5.23)$$

In order for hyperbolas to be invariant we require $\frac{d}{ds}(v^2 + \bar{v}^2) = 0$. Computing this condition yields

$$C_2(s) \sinh 2\eta = -\frac{1}{4}e^{-2i\delta} \left(\left(\frac{1}{2}\lambda + \delta' - \tau\right) \cosh 2\eta + i\eta' \sinh 2\eta \right). \quad (5.24)$$

Note that there is an additional constraint here; whenever $\eta = 0$ the left hand side of (5.24) vanishes, and this implies $\delta' = \tau - \frac{1}{2}\lambda$. In other words, to avoid singularities in $C_2(s)$, it must be that $\delta' = \tau - \frac{1}{2}\lambda + F(s) \sinh 2\eta$ for some smooth function $F(s)$.

Finding the functions δ, η from (5.23) or (5.24) given the function $C_2(s)$ can not, in general, be done analytically. Indeed this would amount to solving the general Floquet problem for (5.17). However, when designing useful magnetic fields, it is perhaps easier to work backwards by choosing δ, η and letting them determine the free function C_2 .

Applying the two transformations to the leading order Liouville one-form α^0 yields

$$\alpha^0 = \frac{1}{4i} (vd\bar{v} - \bar{v}dv) - H_0 ds, \quad (5.25)$$

where the Hamiltonian has transformed to either

$$H_0 = \frac{1}{2} \left(\frac{1}{2} \lambda + \delta' - \tau \right) \operatorname{sech}(2\eta) v \bar{v},$$

for the elliptic case or

$$H_0 = \frac{1}{4} \left(\frac{1}{2} \lambda + \delta' - \tau \right) \operatorname{csch}(2\eta) (v^2 + \bar{v}^2).$$

for the hyperbolic case.

In the elliptic case, the dynamics are now given by orbits contained in invariant tori $\frac{1}{2}v\bar{v} = \text{const}$ with frequency of poloidal rotation

$$\iota(s) = \int_0^s \left(\frac{1}{2} \lambda + \delta' - \tau \right) \operatorname{sech} 2\eta ds. \quad (5.26)$$

To complete the transformation F we require that the rotation rate is constant along the toroidal angle s . This can be done by averaging through the nonautonomous canonical transformation

$$v = e^{-i(\iota(s) - \tau_0 s)} w, \quad \tau_0 := \frac{1}{T} \iota(T), \quad (5.27)$$

to transform the Hamiltonian to

$$H_0 = \frac{1}{2} \tau_0 w \bar{w},$$

as desired.

In the hyperbolic case, the dynamics have been reduced to orbits contained in invariant level sets of $\frac{1}{4}(v^2 + \bar{v}^2) = \text{const}$ with a contraction rate of

$$\nu(s) = \int_0^s \left(\frac{1}{2} \lambda + \delta' - \tau \right) \operatorname{csch}(2\eta) ds. \quad (5.28)$$

To complete the transformation F we require that this contraction rate be constant along the toroidal angle s . This can be done through a canonical rescaling using the average contraction rate. Specifically, make the canonical transformation

$$v = \cosh(\nu(s) - \nu_0 s) w - i \sinh(\nu(s) - \nu_0 s) \bar{w}, \quad \nu_0 := \frac{1}{T} \nu(T). \quad (5.29)$$

This transforms the Hamiltonian to

$$H_0 = \frac{1}{4} \nu_0 (w^2 + \bar{w}^2) ds,$$

as desired.

We summarize the results of this subsection as the following lemma.

Lemma V.3. *The Liouville one-form can be transformed to*

$$\begin{aligned} \alpha &= \frac{1}{4i} (wd\bar{w} - \bar{w}dw) - H_0 ds - \sum_{n=1} H_n(s, w, \bar{w}) ds, \\ H_n &= C_n(s) w^{n+2} + \bar{C}_n(s) \bar{w}^{n+2} + P_n(s, w, \bar{w}), \end{aligned} \quad (5.30)$$

where $P_n(s, w, \bar{w})$ is a polynomial in w, \bar{w} of homogeneous degree $n+2$ with coefficients that are T periodic functions in s and dependent on $C_j, j \leq n$.

Specifically there are functions δ and η such that when the axis is elliptic the transformation, using (5.26), is

$$Z = e^{i\delta}(\cosh(\eta)v - \sinh(\eta)\bar{v}), \quad v = e^{-i(\nu(s) - \nu_0 s)}w,$$

and $H_0 = \frac{1}{2}\tau_0 w \bar{w}$

Similarly, when the axis is hyperbolic the transformation, using (5.28), is

$$Z = e^{i\delta}(\cosh(\eta)v - \sinh(\eta)\bar{v}), \quad v = \cosh(\nu(s) - \nu_0 s)w - i \sinh(\nu(s) - \nu_0 s)\bar{w},$$

and $H_0 = \frac{1}{4}\nu_0(w^2 + \bar{w}^2)$.

Once the leading order terms of the Hamiltonian H_0 are in the autonomous form given in Lemma V.3, we are in a position to determine the Birkhoff normal form for H using Theorem IV.3 from Section IV. The result will, of course, depend upon whether the axis is elliptic or hyperbolic, and in the elliptic case, on whether t_0 is rational or not. An example of a near-resonant normal form for the elliptic case is given in Section VI.

VI. EXAMPLE: CURVES WITH CONSTANT TORSION

In this section normal form theory is applied to an example magnetic axis curve. For simplicity both this curve and the free functions are chosen to ensure the Hamiltonian has only finitely many harmonics at each order. In Section VI A, a family of closed curves with simple curvature and torsion functions is described. In Section VI B a near-axis expansion for these curves is given, the corresponding Hamiltonian function computed, and the normal form analyzed.

A. Closed curves of constant torsion

To obtain a simple form for the magnetic axis, we follow Karcher²⁷ who obtained a simple, discrete symmetry condition under which a Frenet curve, $r(s)$, is a closed loop. More generally a necessary condition is that the Floquet matrix of the system (3.1) is the identity,²⁴ but this is a harder computation. To start, assume that the curvature and torsion functions are even, i.e., there is a point s_j such that

$$\kappa(s_j + s) = \kappa(s_j - s), \quad \tau(s_j + s) = \tau(s_j - s). \quad (6.1)$$

Such a symmetry implies a corresponding symmetry of the resulting space curve:

Lemma VI.1. *If $r(s)$ is a Frenet space curve (3.1) satisfying (6.1) for some $s_j \in \mathbb{R}$, then it is symmetric at s_j with respect to a rotation by π about the curvature normal $\hat{n}(s_j)$.*

Proof. Without loss of generality, suppose that $s_j = 0$. For any smooth functions $\kappa(s)$ and $\tau(s)$, there exists a unique curve $r(s)$, up to Euclidean transformations. Denote by $r_+(s)$ the curve with curvature $\kappa_+(s) = \kappa(s)$ and torsion $\tau_+(s) = \tau(s)$ and by $r_-(s)$ the curve with curvature $\kappa_-(s) = \kappa(-s)$ and torsion $\tau_-(s) = \tau(-s)$. Assuming that both satisfy the same initial conditions, $r_{\pm}(0) = 0$, $\hat{n}_{\pm}(0) = (1, 0, 0)$, then uniqueness implies that $r_-(s) = r_+(-s)$. It follows that

$$\hat{t}_-(0) = -\hat{t}_+(0), \quad \hat{n}_-(0) = \hat{n}_+(0), \quad \hat{b}_-(0) = -\hat{b}_+(0).$$

Thus, since $\kappa_+ = \kappa_-$ and $\tau_+ = \tau_-$, and $r_+(0) = r_-(0)$ the curves $r_{\pm}(s)$ must be identical up to a π -rotation about the shared normal $\hat{n}_{\pm}(0)$, denoted by R_{π} . It follows that $r_+(s) = R_{\pi} \cdot r_-(s) = R_{\pi} \cdot r_+(-s)$.

More generally when $s_j \neq 0$, this symmetry holds for $r(s) = r_+(s - s_j)$. \square

If we suppose that there are at least two of these even symmetry points, say $s_0 \neq s_1$, for (6.1), then there are infinitely many of them.

Lemma VI.2. *If a function has two even symmetry points $s_0 \neq s_1$ then it is periodic with period $2(s_1 - s_0)$.*

Proof. By a simple calculation,

$$\begin{aligned} f(s + 2(s_1 - s_0)) &= f(s_1 + (s_1 - 2s_0 + s)) = f(s_1 - s_1 + 2s_0 - s) \\ &= f(2s_0 - s) = f(s_0 + (s_0 - s)) = f(s_0 - s_0 + s) \\ &= f(s), \end{aligned}$$

implying that f is periodic with period $2(s_1 - s_0)$. \square

Thus if both κ and τ are even about two symmetry points, then they are both periodic. Consequently, if the curve $r(s)$ is not closed, then it has infinitely many even symmetry points. More generally, if the set of symmetry points is assumed discrete, then Lemma VI.2 immediately implies that the full set is countable.

Geometrically, the proof of Lemma VI.2 shows that the entire curve $r(s)$ can be constructed from a fundamental segment $\{r(s) \mid s \in [s_0, s_1]\}$. This is done by first rotating about $\hat{n}(s_1)$ to get a symmetry point $s_2 = 2(s_1 - s_0) + s_0$. A similar rotation about $\hat{n}(s_2)$ then gives s_3 , and so on. Likewise, rotation about $\hat{n}(s_0)$ gives s_{-1} and so on.

The discrete symmetry established above can be used to find closed curves. If we can find that there exists some $m \in \mathbb{N}$ such that $r(s_m) = r(s_0)$ then we are guaranteed that $r(s)$ is periodic. The following gives a sufficient condition for periodicity.

Proposition VI.3. *Suppose $r(s)$ has curvature and torsion satisfying (6.1) for at least two distinct values s_0 and s_1 , and $r(s_0) \neq r(s_1)$. Let ℓ_j be the normal line $\{r(s_j) + t\hat{n}(s_j) \mid t \in \mathbb{R}\}$. If ℓ_0 and ℓ_1 intersect and do so at an angle that is a rational multiple of π , then $r(s)$ is closed.*

Proof. From Lemma VI.1, the discrete maps R_0, R_1 corresponding to π -rotation about $\hat{n}(s_0)$, and $\hat{n}(s_1)$, respectively, are discrete symmetries of $r(s)$. From Lemma VI.2, the points $s_j = j(s_1 - s_0) + s_0$ for $j \in \mathbb{N}$ are also symmetry points with discrete maps R_j (that are π -rotations about $\hat{n}(s_j)$). We claim that if ℓ_0, ℓ_1 intersect at a point p then all the ℓ_j intersect at p . Indeed, the line ℓ_1 is the fixed set of the map R_1 . Hence, applying R_1 to ℓ_0 yields another line which will also intersect ℓ_1, ℓ_0 at p . But this line is precisely ℓ_2 . By induction, it must be true that all ℓ_j intersect at a point p . Moreover, since all the rotations are by π , all these lines lie on the same plane spanned by ℓ_0 and the vector $r(s_1) - r(s_0)$.

Finally, the angle of intersection between ℓ_0, ℓ_1 is the same as the angle of intersection between ℓ_j, ℓ_{j+1} . Hence, if this angle is a rational multiple of π , then the discrete mapping of ℓ_{j-1} into ℓ_{j+1} by R_j will be periodic. Hence, for some $m \in \mathbb{N}$, $r(s_m) = r(s_0)$, implying the curve is closed. \square

The sufficient conditions of this proposition can be reformulated as

$$\cos^{-1}(\hat{n}(s_0) \cdot \hat{n}(s_1)) = \frac{p}{q}\pi, \quad (r(s_0) - r(s_1)) \cdot (\hat{n}(s_0) \times \hat{n}(s_1)) = 0, \quad (6.2)$$

for $p, q \in \mathbb{N}$. Here the first condition guarantees that the angle is rational and the second that the two lines ℓ_0 and ℓ_1 both lie in a plane containing the points $r(s_0)$ and $r(s_1)$, so they necessarily intersect.

As a simple example with two even symmetry points, (6.1), consider curves with torsion that is constant and curvature that has a single harmonic:

$$\tau(s) = \tau_0, \quad \kappa(s) = a_0 + a_1 \sin(s). \quad (6.3)$$

Clearly, $s_0 = \pi/2$, and $s_1 = 3\pi/2$ are even symmetry points, and by Lemma VI.2 so are $s_j = (\frac{1}{2} + j)\pi$ for each $j \in \mathbb{Z}$. Approximate values of (τ_0, a_0, a_1) that give closed curves can be obtained by solving the Frenet-Serret equations numerically for $s \in [s_0, s_1]$, for a range of values of (τ_0, a_0, a_1) , and applying a Newton iteration to find solutions to (6.2). One such closed curve, corresponding to the parameter values

$$\frac{p}{q} = \frac{4}{5}, \quad (\tau_0, a_0, a_1) \approx (0.8, 0.549008, 0.48878), \quad (6.4)$$

is shown in Fig. 3. Note that this curve has length $T = 10\pi$.

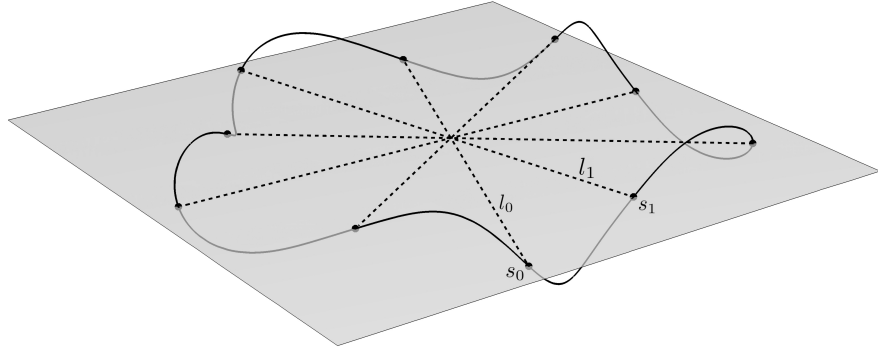


FIG. 3: A closed curve satisfying (6.2) with parameters (6.4). It has a 10-fold symmetry.

B. Normal form examples

In this section we give two examples of a normal form calculation to quartic degree in the variables w, \bar{w} of (5.30). We follow the iterative procedure outlined in Section IV. The available free functions to this order, as determined in Section V A, are summarized in Table II for an elliptic axis.

TABLE II: Available free functions at a given degree.

Degree	Free functions	Conditions
2	λ	real
2	$B_0(s)$	real, periodic
2	$\delta(s)$	periodic mod 2π
2	$\eta(s)$	real, periodic.
3	$A_3(s)$	complex, periodic.
4	$A_4(s)$	complex, periodic.

The free functions could be constrained based on desired properties of the magnetic field. For instance, if quasisymmetry or omnigenity were desired,²² then additional constraints on the free functions would need to be imposed. In the following examples we will be far less ambitious so that our choice of free functions will merely simplify the number of terms appearing in the Floquet-transformed Hamiltonian (Eq. (5.30)). We will then compute the normal form for this choice. Moreover, we will choose the Beltrami constant λ so that the on-axis rotation is close to a resonance, $\tau_0 = 1/3 + \varepsilon$. This allows for two comparative normal form examples; the first for a value of τ_0 with no resonance terms up to degree four, and the second using the near-resonant normal form as described in Section IV E.

We will use the family $r_0(s)$ (6.3) with the parameters (6.4). At degree zero, choose $B_0(s) = 1$. At degree two, take $\eta(s) = \frac{1}{2} \ln 2$ and $\delta(s) = s$, see Table III. This choice ensures that we can simply compute (5.26) as

$$\iota(s) = \int_0^s (\frac{1}{2}\lambda + \delta' - \tau) \operatorname{sech} 2\eta ds = \frac{4}{5}(\frac{1}{2}\lambda + 1 - \tau_0)s,$$

a linear function of s . As a consequence, the Floquet transformations (Eqs. (5.20), (5.21) and (5.27)) will produce coefficients of the Hamiltonian (5.30) that are periodic functions with only low order harmonics. Moreover, by choosing λ appropriately, we can ensure that $\tau_0 = \iota(T)/T = 1/3 + \varepsilon$ for some small value of ε .

For the degree 3 and 4 terms, there is a choice of periodic, complex-valued functions $A_3(s), A_4(s)$ as given in Proposition V.2. We want to ensure that there are some 1:3 resonance terms when the near-resonant normal form is computed in Section VIB 2. By Corollary IV.4, this amounts to requiring that the coefficient of w^3 is non-zero. Hence, we choose $A_3(s)$ so that, after Floquet transformation, the coefficients of $w\bar{w}^2$ and $w^2\bar{w}$ vanish, but the coefficient of w^3 and \bar{w}^3 do not. Similarly, we will choose $A_4(s)$ so that the coefficients of $w^3\bar{w}, w\bar{w}^3$ vanish. This guarantees as few coefficients at this order as possible.

A summary of the choice of free functions is given in Table III. Through appropriate substitution of these specific free functions and consequent transformation of (5.16) by the Floquet transformations, the resulting Hamiltonian (5.30) to degree-four is of the form,

$$\begin{aligned} H &= H_0 + H_1^0 + H_2^0, & H_0 &= \frac{1}{2}(\frac{1}{3} + \varepsilon)w\bar{w}, \\ H_1^0 &= \sum_{m=-2}^2 e^{ims} (b_{3,0,m}w^3 + b_{0,3,m}\bar{w}^3), \\ H_2^0 &= \sum_{m=-4}^4 e^{ims} (b_{4,0,m}w^4 + b_{2,2,m}w^2\bar{w}^2 + b_{0,4,m}\bar{w}^4). \end{aligned} \quad (6.5)$$

Note that $b_{j,k,m} = \bar{b}_{k,j,-m}$ since H is real.

TABLE III: Choice of free functions for examples.

Degree	Free functions	Choice
2	λ	$\frac{1}{6}(-7 + 15\varepsilon + 12\tau_0)$
2	$B_0(s)$	1
2	$\delta(s)$	s
2	$\eta(s)$	$\frac{1}{2} \ln 2$
3	$A_3(s)$	so that coefficients of $w\bar{w}^2, w^2\bar{w}$ vanish
4	$A_4(s)$	so that coefficients of $w\bar{w}^3, w^3\bar{w}$ vanish

1. Example 1: non-resonant normal form

Using the choice of functions given in Table III, we will compute the non-resonant normal form for the Hamiltonian (6.5). We follow Appendix A 2 in moving to extended phase space and using the extended bracket, (A4), $\{\cdot, \cdot\} + \partial_s$. The following calculation will hold provided $\tau_0 \neq \frac{p}{q}$ with $q \leq 4$.

At the d^{th} iteration in the normal form procedure, we need to solve the homological equation

$$\{F_d, H_0\} + \partial_s F_d = H_d^{d-1} - H_d^d,$$

where the first term represents the degree d terms after the previous $d - 1$ normal form transformations have been made, and the second contains the resonant (or irremovable) terms. That is, at each order the Hamiltonian will have the form

$$H_d^{d-1} = \sum_{|m| \leq d} \sum_{j+k=d} b_{j,k,m} e^{ims} w^j \bar{w}^k,$$

with $b_{j,k,m} = \bar{b}_{k,j,-m}$ since H is real. It is important to note that each coefficient $b_{j,k,m}$ must be computed by applying all the previous, lower order, transformations (see Appendix A).

Letting the degree- d terms have the power series representation

$$F_d = \sum_{|m| \leq d} \sum_{j+k=d} F_{j,k,m} e^{ims} w^j \bar{w}^k.$$

then, solving the homological equation, term-by-term gives

$$F_{j,k,m} = -i \frac{b_{j,k,m}}{(k-j)t_0 + m}, \quad (6.6)$$

providing there is no resonance: $(k-j)t_0 + m \neq 0$. Any “resonant” terms are not removed, and accumulated in the normal form Hamiltonian H_d^d .

With ε taken so that $t_0 \neq \frac{p}{q}$ with $q \leq 4$, the degree 4 Hamiltonian is given as,

$$H^4 = H_0^0 + H_1^1 + H_2^2 = \frac{1}{2}t_0 w \bar{w} + b_{2,2,0} w^2 \bar{w}^2. \quad (6.7)$$

Note that $b_{2,2,0} \in \mathbb{R}$ as H is real. As there are no resonant terms in the normal form (6.7), and consequently it does not depend on s , then \tilde{H} is an approximate integral for the system. By pulling back this approximate integral by the normal form transformation, we obtain an integral in the original Floquet coordinates w, \bar{w} . The resulting level sets of this function at a slice $s = 0$ is given in Fig. 4 for a value of $\varepsilon = 0.01$. Moreover, some trajectories of the degree four Hamiltonian Eq. (6.5) are plotted for comparison.

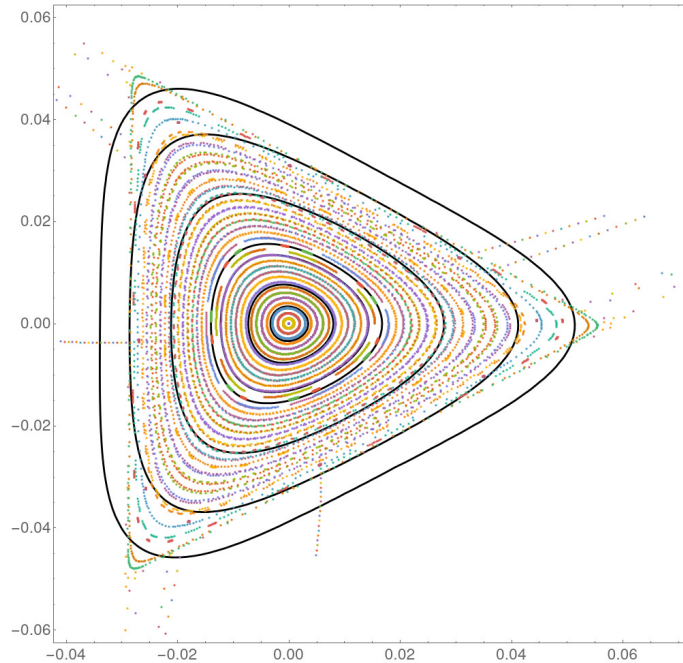


FIG. 4: In black are the contour lines of the approximate invariant \tilde{H} at a slice $s = 0$ and for $\varepsilon = 0.01$. Overlaid is a Poincaré plot of the orbits of the Hamiltonian system (6.5).

2. Example 2: near-resonant normal form

As in the previous example, we will make a choice of functions as given in Table III. In contrast however, we will compute the near-resonant normal form for the Hamiltonian (6.5). That is we will assume ε to be small and follow the theory outlined in Section IV E.

As before, the general solution for the normalizing functions F_d is given Eq. (6.6). As we are essentially taking $\nu_0 = 1/3$, at degree 3 there is a resonant term

$$H_1^1 = b_{3,-1}e^{-is}w^3 + \bar{b}_{3,-1}e^{is}\bar{w}^3.$$

The degree 4 terms of the normal form, H_2^2 , will only contain the irremovable term $\bar{b}_{2,2,0} \in \mathbb{R}$, similar to the previous example. Here $\bar{b}_{2,2,0}$ is the coefficient of $w\bar{w}$ after the transformation generated by F_1 has been applied to H^0 to give H^1 . The degree 4 normal form becomes

$$H^4 = H_0 + H_1^1 + H_2^2 = \frac{1}{2}\varepsilon_0 w\bar{w} + b_{3,0,-1}e^{-is}w^3 + \bar{b}_{3,0,-1}e^{is}\bar{w}^3 + \bar{b}_{2,2,0}w^2\bar{w}^2. \quad (6.8)$$

The resonant terms in the normal form (6.8) prevent the system from being autonomous. As a consequence, the normal form itself is not an integral. However, through a rotation, $\bar{w} = e^{-1/3is}w$ we obtain the approximate integral

$$J = \frac{1}{2}\varepsilon w\bar{w} + b_{3,-1}\bar{w}^3 + \bar{b}_{3,-1}w^3 + b_{2,2,0}w^2\bar{w}^2. \quad (6.9)$$

By pulling back this approximate integral by the rotation and the normal form transformation, we obtain an integral in the original Floquet coordinates w, \bar{w} . The resulting level sets of this function at a slice $s = 0$ is given in Fig. 5 for a value of $\varepsilon = 0.01$. Moreover, some trajectories of the degree 4 Hamiltonian Eq. (6.5) are plotted for comparison.

Note that the level sets appear to capture more of the structure of the actual field line flow in this case. This is apparently due to retaining the formally resonant terms—which can be quite large—even though this is not strictly necessary when the axis is non-resonant.

VII. CONCLUDING REMARKS

In this paper, we studied near-axis expansions for Beltrami and vacuum magnetic fields. These were introduced through the lens of differential forms, motivating a Hamiltonian perspective and facilitating the application of Floquet and normal form theory. Ultimately these techniques gave a way to iteratively compute simple coordinates and an approximate integral for fields near a magnetic axis. We gave two examples, the first analyzed using a regular normal form, and the second using a near-resonant normal form.

The language of differential forms reveals how Beltrami and vacuum fields give contact and cosymplectic structures to the manifold. The fact that Beltrami fields form contact structures has been used previously to establish topological properties of Beltrami fields (for example see Ref. [10, 14, and 15]). As far as we are aware, the cosymplectic structure of a vacuum field has yet to be explored. We hope that future work may further illuminate this subtle difference.

In this paper, several generalizations were made that permit the application of near-axis expansions to a wider variety of configurations. Firstly, we showed how to construct rotation minimizing coordinates without a Frenet-Serret frame, using the ideas of Bishop.⁶ Consequently, it is possible to study axes with points of vanishing curvature while still using the traditional framework of Mercier.³⁵ Secondly, we demonstrated how to implement a near-axis expansion for a hyperbolic axis, which may prove useful for the study of divertors.^{7,8} Finally, we were able to carry out these expansions without assuming the existence of flux surfaces or non-resonance. Nevertheless, we constructed approximate integrals using the normal form. This contrasts the common claim that an existence requirement for flux surfaces is non-resonance of the axis.²⁶

As we demonstrated, normal form theory gives a way to understand the possible topology of magnetic fields near an axis, in particular for those near or at resonance. The resonant normal form will allow

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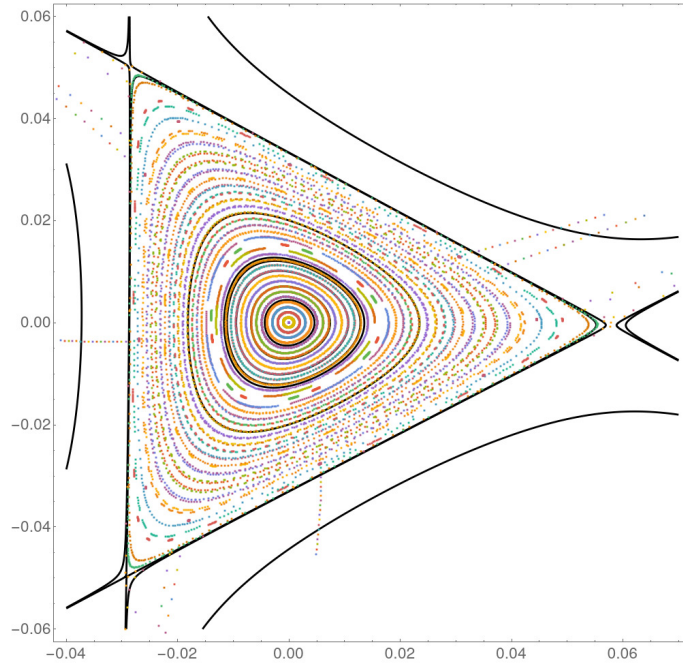


FIG. 5: In black are the contour lines of the approximate invariant J at a slice $s = 0$ and for $\varepsilon = 0.01$. Overlaid is a Poincaré plot of the orbits of the Hamiltonian system (6.5).

the investigation of properties, such as omnigenity or quasisymmetry, near island chains and hyperbolic axes. This may enlarge the category of configurations with these properties. Furthermore, application of normal form theory to magnetic fields with non-vanishing pressure gradients $\nabla p \neq 0$ was omitted from the current work. However, the theory naturally carries over to this case. This opens the potential to understand phenomena near or at resonance using normal form coordinates.

Moreover, as demonstrated by comparing Fig. 5 and Fig. 4, a near-resonant normal form can give a better approximation of flux surfaces as well as help locate any separatrices.

Finally, this paper applies the near-axis expansion, for the first time, to Beltrami fields. Such fields have recently become the central tool in computing stepped-pressure MHD equilibria, which have open regions with $\nabla p = 0$ where the field is Beltrami.²³ Although Beltrami fields are generically chaotic, the existence of even approximate flux surfaces would be advantageous. Our techniques will be useful to construct fields with such flux surfaces.

Appendix A: Normal Form Theory

In this section we will give a proof of Theorem IV.3. There are many proofs in the literature (see, for instance, Ref. [36]). However, because these proofs are constructive, and are used in the computations in Section IV and Section VIB, it is of use to be more explicit.

As outlined in Section IV, normal form theory seeks a choice of canonical coordinates near a periodic orbit for which the Hamiltonian takes its “simplest” form. We want a way to easily apply canonical

transformations to the Hamiltonian, and study how this transformation modifies the Hamiltonian at each degree in the Taylor series expansion of H about the periodic orbit. As pioneered by Ref. [13], one relatively easy way to do this is through the method of Lie series.

This method takes advantage of the fact that the set of diffeomorphisms $\text{Diff}(M)$ on a manifold M is a Lie group under composition. The corresponding Lie algebra $\chi(M)$ is the vector space of complete vector fields on M . This gives an efficient method for computing the action of a flow of a Hamiltonian vector field on a function. Explicitly if $\varphi_X^t, \epsilon \in \mathbb{R}$, is the time- t flow of a vector field $X \in \chi(M)$, then the action of φ_X^t on a function f is given by the *Lie series*

$$\varphi_X^t f = \exp(tX)f := \sum_{k=0}^{\infty} \frac{t^k}{k!} (X^k f), \quad (\text{A1})$$

where the vector field $X = X^i \partial_i$ can be thought of as the usual Lie derivative operator that acts on functions.

For the case of Hamiltonian vector fields, the flow will be a symplectic transformation. These can be generated using the Poisson bracket $\{\cdot, \cdot\}$ on the manifold M , recall (4.4). Given $H \in C^\infty(M, \mathbb{R})$, the Hamiltonian vector field is $X_H = \{\cdot, H\}$, and the Lie derivative operator becomes

$$L_H := \{\cdot, H\} = X_H.$$

Moreover, it can be shown that the Lie bracket for Hamiltonian vector fields is given by

$$[X_G, X_F] = X_{\{F, G\}}.$$

The point is that the map $F \mapsto \{\cdot, F\}$ is a Lie algebra homomorphism.

Thus, given a Hamiltonian function $F : M \rightarrow \mathbb{R}$, there is a Hamiltonian vector field $X_F = \{\cdot, F\}$, which in turn, can be used to generate a symplectic transformation $\varphi_{X_F}^t$, the time t flow of X_F . Consequently, the action on a function H by can be computed using Lie series through (A1), that is,

$$\varphi_{X_F}^t H = \exp(L_F)H.$$

Through this relation, we never have to deal directly with the flow φ_{X_F} , or even the vector field X_F , we can simply use Hamiltonian F and the Poisson Bracket operator L_F , to compute the transformation of H .

We will proceed by first recalling the normal form transformation for an autonomous Hamiltonian, before extending it to the non-autonomous case.

1. Time Independent

Assume that H is independent of time and has an equilibrium, without loss of generality, at the origin. Let $(x, y) \in T^*\mathbb{R}^n$ be local canonical coordinates and $\{\cdot, \cdot\}$ the canonical Poisson bracket (4.4).

We review here the iterative procedure to transform the Hamiltonian H to a normal form and produce the required transformation. To start this procedure, we expand H into a Taylor series about the origin

$$H = H_0 + \epsilon H_1^0 + \epsilon^2 H_2^0 + \dots,$$

denoting the degree of the homogeneous component H_j using a subscript. We will make the non-degeneracy assumption that H_0 is degree two, and so that a lower index j indicates a degree $2 + j$ polynomial in (x, y) . The upper index—on the higher order terms—will denote the step in the normal form procedure. We omit this for H_0 , as it will stay fixed. The ϵ is introduced purely for bookkeeping.

For the general normalization step, consider a $j+2$ Hamiltonian $\epsilon^j F_j$ with degree $j+1$ vector field $\epsilon^j X_{F_j}$. Since the ϵ^j just scales time, the corresponding “time-one” flow becomes the symplectic transformation $\varphi_{X_j}^{\epsilon^j}$. Using Eq. (A1) this transforms the Hamiltonian H to a new form \tilde{H} , given by,

$$\begin{aligned} \tilde{H} &= \exp(\epsilon^j L_{F_j})(H_0 + \epsilon H_1^0 + \dots) \\ &= H_0 + \epsilon H_1^0 + \dots + \epsilon^j (L_{F_j} H_0 + H_j^0) + O(\epsilon^{j+1}). \end{aligned}$$

Note that the lowest-order effect of this transformation is to transform the degree $j + 2$ homogeneous component of H , namely H_j^0 . The corresponding equation at this order is

$$H_j^1 = L_{F_j} H_0 + H_j^0 = \{H_0, F_j\} + H_j^0.$$

This equation, when thought of as an equation to determine the desired F_j :

$$L_{H_0} F_j = \{F_j, H_0\} = H_j^0 - H_j^1, \quad (\text{A2})$$

is referred to as the *homological equation*. The central object of study is now the linear operator $L_{H_0} = \{\cdot, H_0\}$. Letting \mathcal{H}_j be the vector space of degree- j homogeneous polynomials in (x, y) , then, since H_0 is quadratic,

$$L_{H_0} : \mathcal{H}_j \rightarrow \mathcal{H}_j.$$

Ideally F_j would be chosen so that $H_j^1 = 0$, and the resulting Hamiltonian would have no order- j terms. However, this is only possible if $H_j^0 \in \text{Im}(L_{H_0})$. However, if L_{H_0} is not onto, then there can be components of H_j^0 not in $\text{Im} L_{H_0}$. For the case that the linearized matrix $DJ\nabla H_0|_{q=p=0}$ is diagonalizable (recall (4.2)), then so is the operator L_{H_0} .³⁶ Under this assumption, it is always possible to write

$$\mathcal{H}_j = \text{Im}(L_{H_0}) \oplus \text{Ker}(L_{H_0}).$$

Then if we choose H_j^1 to be the projection of H_j^0 onto $\text{Ker}(L_{H_0})$, the homological equation (A2) can be solved for F_j . Of course, then the terms H_j^1 , the “resonant terms,” remain in the transformed Hamiltonian H^1 .

The normal form procedure thus begins by diagonalizing the matrix associated with the quadratic H_0 . To normalize the cubic terms, H_1^0 , we act on H by the flow generated by a degree-three Hamiltonian F_1 , or equivalently its degree-two vector field X_1 . The transformed Hamiltonian becomes

$$H^1 = \exp(\epsilon L_{F_1}) H = H_0 + \epsilon H_1^1 + \epsilon^2 H_2^1 + \dots$$

The next step in the iterative procedure is to normalize the order two terms, using a transformation generated by a Hamiltonian $\epsilon^2 F_2$. The result with be

$$H^2 = H_0 + \epsilon H_1^1 + \epsilon^2 H_2^2 + \dots,$$

where the order zero and order one terms remain unchanged. To do this, we must solve the homological equation

$$L_{H_0} F_2 = H_2^1 - H_2^2.$$

Again we choose H_2^2 so that $H_2^1 - H_2^2 \in \text{Im}(L_{H_0})$ to compute F_2 .

Continuing in this fashion, we can obtain the j^{th} order Hamiltonian by iteration of F_k for $k = 1, \dots, j$, namely

$$H^j = H_0 + \epsilon H_1^1 + \dots + \epsilon^j H_j^j + \dots$$

The transformation Φ_j bringing H to j^{th} order normal form is given by the Lie series

$$\Phi_j = \exp(\epsilon^j L_{F_j}) \exp(\epsilon^{j-1} L_{F_{j-1}}) \dots \exp(\epsilon L_{F_1}).$$

This can be computed as a series expansion. Note that the inverse transformation is easily computed as

$$\Phi_j^{-1} = \exp(-\epsilon L_{F_1}) \dots \exp(-\epsilon^{j-1} L_{F_{j-1}}) \exp(-\epsilon^j L_{F_j}).$$

2. Time Periodic

We will now present an outline of the time dependent normal form. Consider a Hamiltonian that depends periodically on time t so that $H : M \times S^1 \rightarrow \mathbb{R}$. As before, we assume that we are given canonical coordinates $(x, y) \in T^* \mathbb{R}^n$ near an orbit $r_0 : S^1 \rightarrow M$ that has period T . We assume that coordinates (e.g. using Floquet theory) have been chosen so that $r_0(t) = (0, 0)$.

Now, decompose H into its various homogeneous terms by expanding in a Taylor series in (x, y) :

$$H(x, y, t) = H_0(x, y, t) + \epsilon H_1(x, y, t) + \dots \quad (\text{A3})$$

As before, ϵ is introduced purely for bookkeeping.

We assume that the Hamiltonian is in Floquet coordinates so that $H_0 = H_0(x, y)$ is independent of time t (see Section IV) and is quadratic. We would now like to do the same as in the autonomous case, namely, simplify the system by a near identity, canonical coordinate transformation. Unfortunately, the time dependence prevents us from directly using the Lie derivatives L_{H_j} . However, this problem can be circumnavigated by moving to extended phase space.

A point in extended phase space $(x, y, t, E) \in M \times S^1 \times \mathbb{R}$ has the energy variable E as its fourth coordinate. The Hamiltonian is now

$$\tilde{H} = \tilde{H}_0 + \epsilon H_1 + \dots, \quad \tilde{H}_0 := H_0 + E,$$

and the Poisson bracket becomes

$$\{\widetilde{\cdot}, \widetilde{\cdot}\} = \{\cdot, \cdot\} + \{\cdot, \cdot\}_E, \quad \{F, G\}_E := \partial_t F \partial_E G - \partial_E F \partial_t G.$$

The normal form transformations do not need to change the time coordinate, so we consider vector fields $X = \{\widetilde{\cdot}, F\}$, that are zero in the time-direction. As a consequence, the corresponding Hamiltonian, F , in the extended phase space is independent of E , so that

$$\begin{aligned} \tilde{L}_{H_0} &:= \{\widetilde{\cdot}, \tilde{H}_0\} = \{F, H_0\} + \{F, E\}_E \\ &= \{F, H_0\} + \partial_t F. \end{aligned} \quad (\text{A4})$$

As before, the transformation at order j corresponds to a Hamiltonian $\epsilon^j F_j$ that is degree $j + 2$ in (x, y) , but now has T periodic coefficients. This gives a degree $j + 1$ vector field $\epsilon^j X_j$, and generates the transformation $\varphi_{X_j}^{\epsilon^j}(x, y, t, E)$.

Now, from the computation in Appendix A 1, application of the symplectic transformation φ produces the homological equation

$$\tilde{L}_{H_0} F_j = H_j^{j-1} - H_j^j.$$

It follows that, in the time dependent case, the appropriate linear operator is now \tilde{L}_{H_0} , and it is $\ker \tilde{L}_{H_0}$ that determines the resonant terms in the normal form.

The normal form procedure is carried out in Section IV, following the autonomous case. The only difference is the modification of the homological equation and that \tilde{L}_{H_0} must be used in computing the Lie series of Eq. (A1).

Appendix B: Explicit Expansions for the Vector Potential and Hamiltonian

The explicit solution to the recursive equations (5.8) to order $n = 4$ is given as

$$\begin{aligned}\alpha_s^2 &= A_2 z^2 + \bar{A}_2 \bar{z}^2 - \frac{1}{4} \lambda B_0 z \bar{z}, \\ \alpha_s^3 &= A_3 z^3 + \bar{A}_3 \bar{z}^3 + z \bar{z} (R_{2,1} z + \bar{R}_{2,1} \bar{z}), \\ \alpha_s^4 &= A_4 z^4 + \bar{A}_4 \bar{z}^4 + R_{2,2} z^2 \bar{z}^2 + z \bar{z} (R_{3,1} z^2 + \bar{R}_{3,1} \bar{z}^2), \\ \alpha_z^0 &= B_0(s), \\ \alpha_z^1 &= \frac{1}{3} B_0 (\bar{\kappa}_z z + \kappa_z \bar{z}), \\ \alpha_z^2 &= \frac{1}{4} \operatorname{re} \left((B_0 \kappa_z^2 + 4A_2(\lambda + 2\tau) + 4iA_2') z^2 \right) - \frac{1}{8} (B_0 (\lambda^2 - 2|\kappa_z|^2) + B_0'') z \bar{z}, \\ \alpha_z^3 &= Q_{3,0} z^3 + \bar{Q}_{3,0} \bar{z}^3 + z \bar{z} (Q_{2,1} z + \bar{Q}_{2,1} \bar{z}),\end{aligned}$$

where

$$\begin{aligned}R_{2,1} &= -\frac{1}{4} A_2 \kappa_z + \frac{1}{96} (5i\bar{\kappa}_z B_0' + B_0 (\bar{\kappa}_z (3\lambda + 2\tau) + 2i\bar{\kappa}_z')), \\ R_{2,2} &= -\frac{1}{16} \operatorname{re} (A_2 \kappa_z^2) + \frac{1}{128} (2\lambda B_0'' + B_0 (2\lambda^3 + |\kappa_z|^2 (\lambda - 2\tau) - 2\operatorname{im}(\bar{\kappa}_z \kappa_z'))), \\ R_{3,1} &= \frac{1}{48} A_2 (-3|\kappa_z|^2 - 4(\lambda^2 + \tau(\lambda - 2\tau) - i\tau')) - \frac{1}{4} A_3 \kappa_z - \frac{1}{24} A_2', \\ &\quad + \frac{1}{384} \bar{\kappa}_z (9i\bar{\kappa}_z B_0' + B_0 (\bar{\kappa}_z (3\lambda + 10\tau) + 10i\bar{\kappa}_z')) - \frac{1}{24} i(\lambda - 4\tau) \bar{A}_2', \\ Q_{3,0} &= \frac{1}{15} A_2 (3\bar{\kappa}_z (\lambda + 4\tau) + 2i\bar{\kappa}_z') + \frac{2}{5} A_3 (\lambda + 3\tau) + \frac{1}{20} B_0 \bar{\kappa}_z^3 + \frac{1}{3} i\bar{\kappa}_z A_2' + \frac{2}{5} iA_3', \\ Q_{2,1} &= \frac{1}{10} A_2 (\kappa + z(\lambda + 5\tau) + i\kappa_z') + \frac{3}{10} i\kappa_z A_2' + \frac{1}{80} (-9\bar{\kappa}_z B_0'' + i\bar{\kappa}_z B_0' (2\lambda + 7\tau) \\ &\quad - 7B_0' \bar{\kappa}_z' + B_0 (\bar{\kappa}_z (-3\lambda^2 + \lambda\tau + 2i\tau' + 2\tau^2) \\ &\quad + i\bar{\kappa}_z' (\lambda + 4\tau) - 2\bar{\kappa}_z'' + 12|\kappa|^2 \bar{\kappa}_z)).\end{aligned}$$

Here the A_j are T periodic functions of s , and $\kappa_z = e^{-i\gamma(s)} \kappa_u$. Note that, if γ can be taken as the integral torsion (3.5), we have $\kappa_z = \bar{\kappa}_z = \kappa$ the curvature and τ the torsion of r_0 .

To degree four, the Hamiltonian (5.15) is given explicitly by

$$\begin{aligned}H_0 &= -\bar{A}_2 B_0^{-1} Z^2 - A_2 B_0^{-1} \bar{Z}^2 + \left(\frac{1}{4}\lambda - \frac{1}{2}\tau\right) Z \bar{Z}, \\ H_1 &= H_{3,0} Z^3 + \bar{H}_{3,0} \bar{Z}^3 + Z \bar{Z} (H_{2,1} Z + \bar{H}_{2,1} \bar{Z}), \\ H_2 &= H_{4,0} Z^4 + \bar{H}_{4,0} \bar{Z}^4 + Z \bar{Z} (H_{3,1} Z^2 + \bar{H}_{3,1} \bar{Z}^2) + H_{2,2} Z^2 \bar{Z}^2,\end{aligned}$$

where

$$\begin{aligned}H_{3,0} &= -B_0^{-3/2} (\bar{A}_3 - \frac{1}{3} \bar{A}_2 \kappa_z), \\ H_{2,1} &= \frac{1}{96} B_0^{-3/2} (56\bar{\kappa}_z \bar{A}_2 + 5i\kappa_z B_0' - B_0 \kappa_z (11\lambda + 2\tau) + 2iB_0 \kappa_z'), \\ H_{4,0} &= -B_0^{-2} \left(\frac{1}{24} \kappa_z^2 \bar{A}_2 - \frac{1}{2} \kappa_z \bar{A}_3 + \bar{A}_4 - \frac{1}{2} B_0^{-1} \bar{A}_2 (\bar{A}_2 (\lambda + 2\tau) - i\bar{A}_2') \right), \\ H_{3,1} &= -\frac{1}{384} B_0^{-3} (\bar{A}_2 (48B_0'' + 8B_0 (8(\lambda^2 + \tau(\lambda + \tau)) - 4i\tau'(s) + 7|\kappa_z|^2))) \\ &\quad + \frac{1}{12} iB_0^{-2} \bar{A}_2' (\lambda + 2\tau) + \frac{3}{4} B_0^{-2} \bar{\kappa}_z \bar{A}_3 + \frac{1}{24} B_0^{-2} \bar{A}_2'', \\ &\quad + \frac{1}{384} B_0^{-2} \kappa_z (B_0 (\kappa_z (7\lambda - 6\tau) + 6i\kappa_z') - i\kappa_z B_0'), \\ H_{2,2} &= \frac{1}{64} B_0^{-1} \lambda^3 + \frac{1}{64} \lambda B_0^{-2} B_0'' + \frac{1}{384} B_0^{-1} |\kappa_z|^2 (17\lambda + 14\tau) + B_0^{-3} |A_2|^2 (\lambda + 2\tau) \\ &\quad - \frac{13}{48} B_0^{-2} \operatorname{re} (\kappa_z^2 A_2) + \frac{14}{384} B_0^{-1} \operatorname{im} (\bar{\kappa}_z \kappa_z') - B_0^{-3} \operatorname{im} (\bar{A}_2 A_2').\end{aligned}$$

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DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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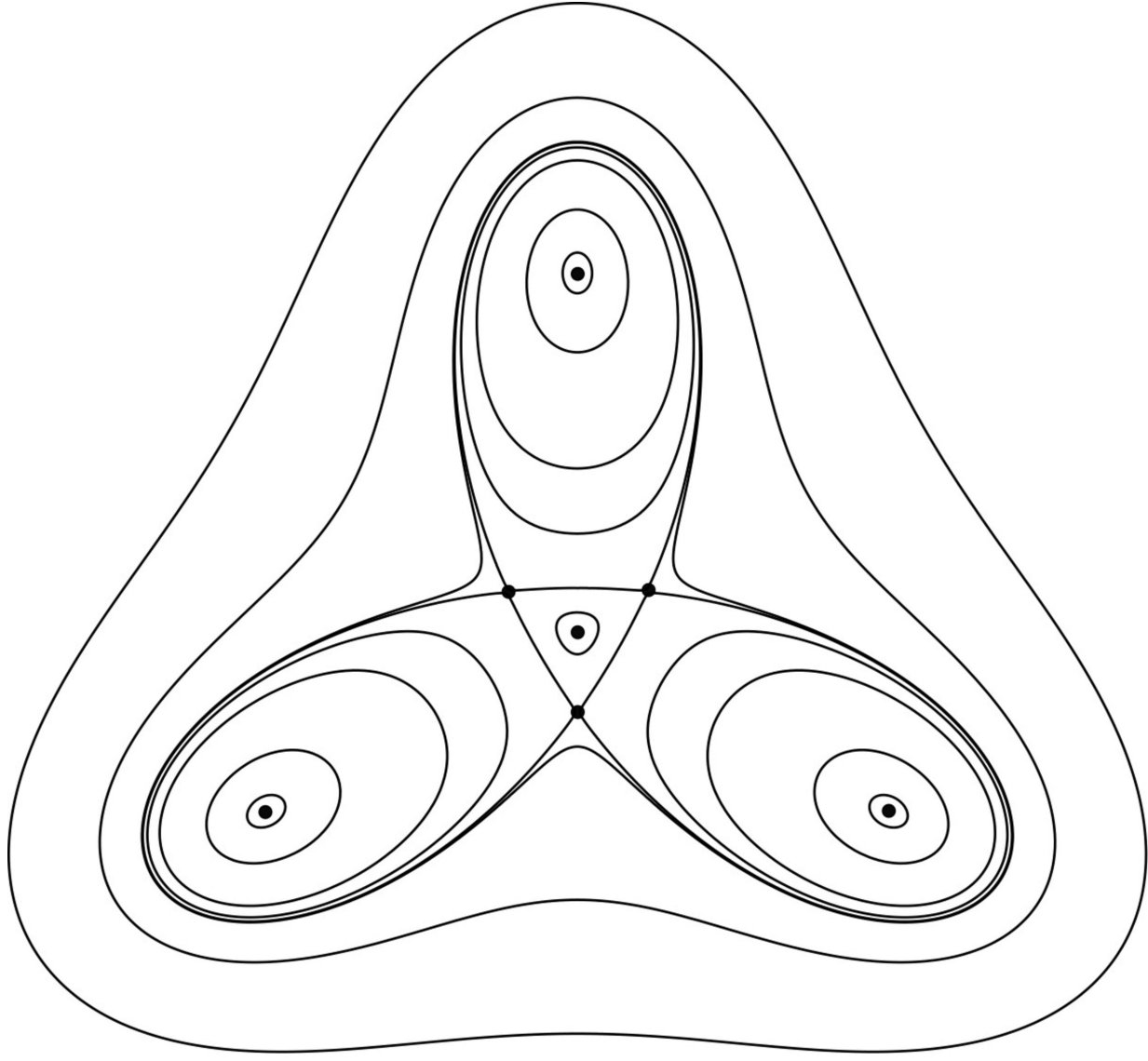
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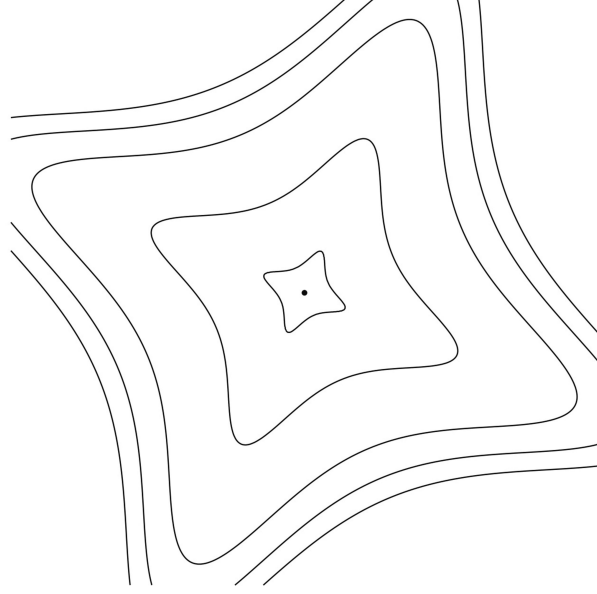
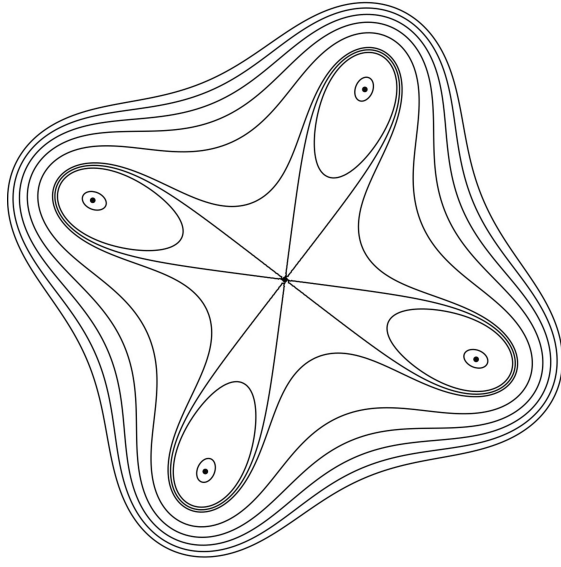
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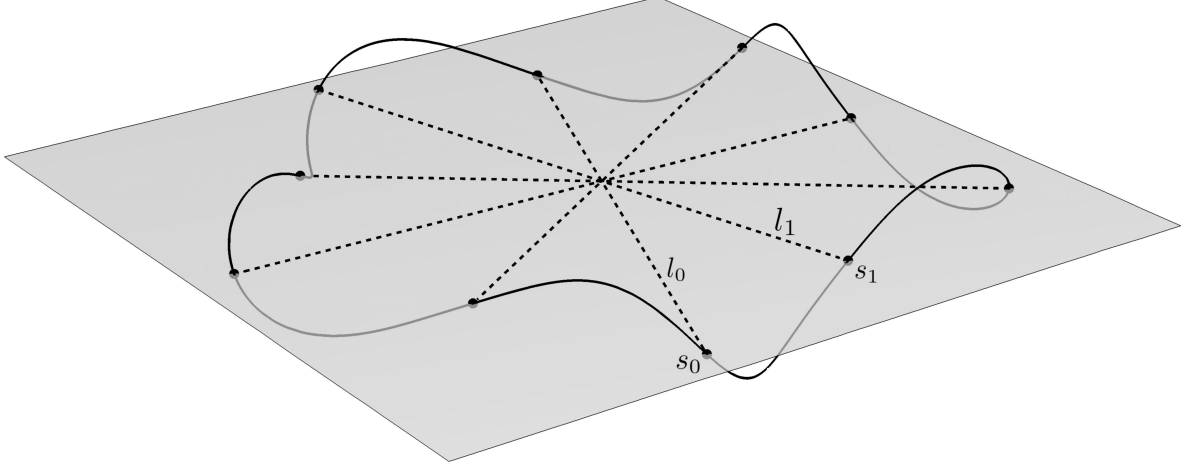
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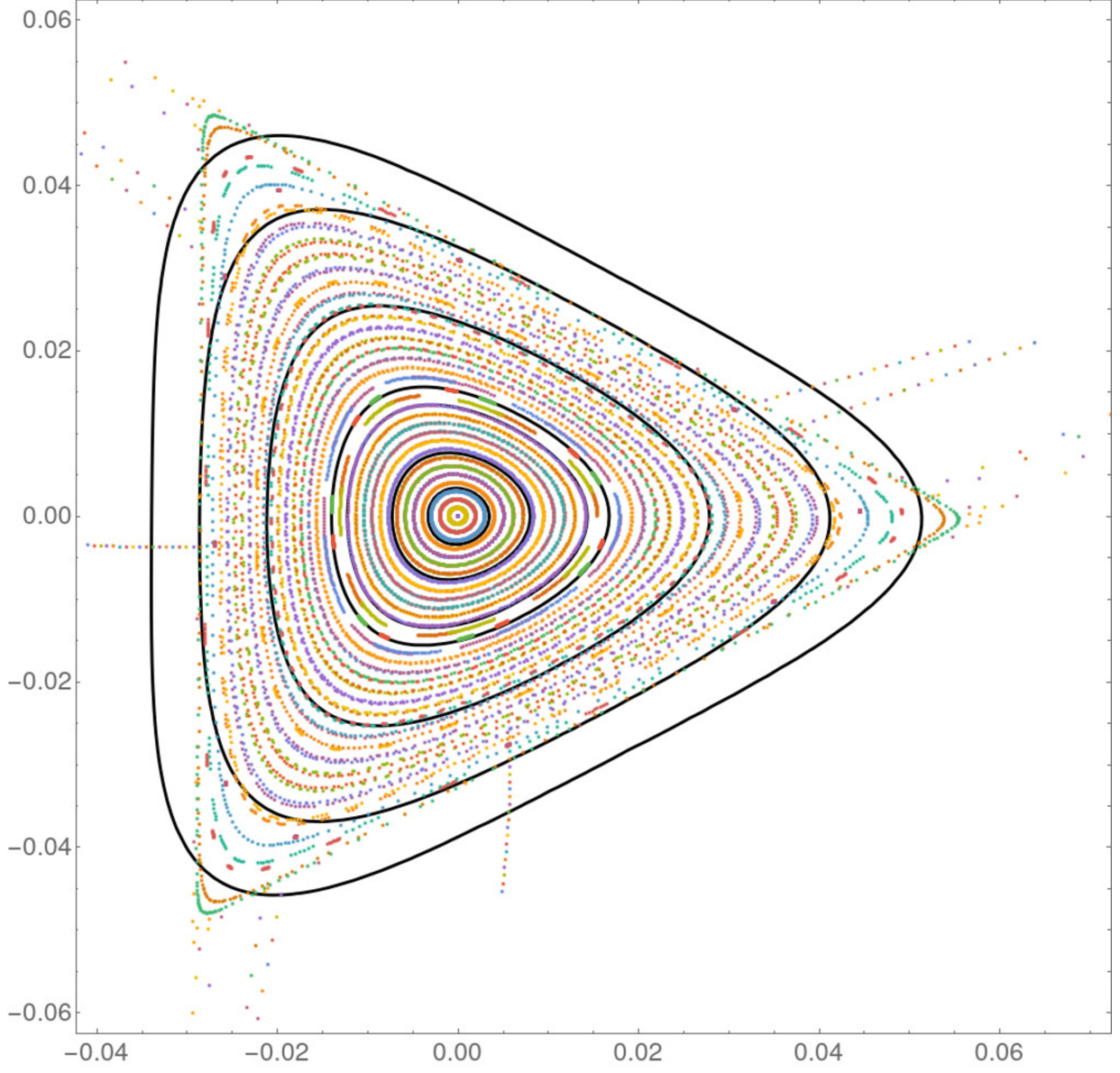
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