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A Random Walk on Upper-Triangular Matrices

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1 Introduction

A random walk is a particular type of stochastic processes. The idea of a random walk generally is to study the path of something that is random. To study a random walk, we need to define the states and the probability transition. The states are all the possible values that the walk can take. The probability transition shows the probability of the event that the path is taken from one state to another or possibly itself. Now consider a group of upper-triangular matrices under a multiplication, which will be precisely defined in the next section. If we want to study a random walk on this group, we need to define all the states and the probabilities. There are many ways to choose the state space. We could pick each element to be a state itself. However, the choice of picking superclasses to be states is interesting.

2 Preliminaries

Let us now consider a particular family of groups.

Definition 2.1. *For two positive integers m and n. consider the following group*

$$UT_{m\times n} = \left\{ \begin{bmatrix} Id_m & X \\ 0 & Id_n \end{bmatrix} : X \in M_{m\times n}(\mathbb{F}_q) \right\}, \quad \textit{under multiplication},$$

where $\mathbb{F}_q$ is a finite field of size q. We call this group an ***upper-triangular group****.*

It is obvious to see that Id_{m+n} is the identity of this group. In particular, for any $\begin{bmatrix} Id_m & X \\ 0 & Id_n \end{bmatrix}$ and $\begin{bmatrix} Id_m & Y \\ 0 & Id_n \end{bmatrix} \in UT_{m\times n}$,

$$\begin{bmatrix} Id_m & X \\ 0 & Id_n \end{bmatrix} \cdot \begin{bmatrix} Id_m & Y \\ 0 & Id_n \end{bmatrix} = \begin{bmatrix} Id_m & X+Y \\ 0 & Id_n \end{bmatrix}.$$

Thus, The group is abelian and $\begin{bmatrix} Id_m & X \\ 0 & Id_n \end{bmatrix}^{-1} = \begin{bmatrix} Id_m & -X \\ 0 & Id_n \end{bmatrix}$.

To study a random walk on a group, we need to consider two main components :

(1) state space, and

(2) probability transition.

2.1 State Space

Before we define the particular state space on this group, it will be easier to consider the following set first.

Definition 2.2. *For any positive integer m, we define the set*

$$\mathcal{B}_m = \left\{ X \in M_{m\times m}(\mathbb{F}_q) : X_{ij} = \begin{cases} c_i & \textit{if } i = j, \\ 0 & \textit{if } i < j \end{cases}, \ \textit{where } c_i \neq 0, \ \textit{for } i = 1, 2, 3, \ldots, m \right\}.$$

Note that for a matrix $X \in M_{m\times n}(\mathbb{F}_q)$ and $A \in UT_m$, the product AX is a matrix obtained by applying a particular row operation; adding the multiple of one row to a higher one or scaling a row.

Example.

$$\begin{bmatrix}1 & a & 0\\0 & 1 & b\\0 & 0 & c\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\1 & 1 & 2\end{bmatrix}=\begin{bmatrix}1+a\cdot 0 & 0+a\cdot 1 & 2+a\cdot 1\\0+b\cdot 1 & 1+b\cdot 1 & 1+b\cdot 2\\c\cdot 1 & c\cdot 1 & c\cdot 2\end{bmatrix}$$

Similarly, for matrix $X \in M_{m\times n}(\mathbb{F}_q)$ and $B \in UT_n$, the product XB is a matrix obtained by applying a particular column operation; adding the multiple of one column to a higher one or scaling a column.

Example.

$$\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\1 & 1 & 2\end{bmatrix}\cdot\begin{bmatrix}1 & a & 0\\0 & 1 & b\\0 & 0 & c\end{bmatrix}=\begin{bmatrix}1 & 0+a\cdot 1 & c\cdot 2+b\cdot 0\\0 & 1+a\cdot 0 & c\cdot 1+b\cdot 1\\1 & 1+a\cdot 1 & c\cdot 2+b\cdot 1\end{bmatrix}$$

We pick the set of **superclasses** to be the state space of this walk. **Superclasses** are the equivalence classes of a particular equivalence relation obtained from *representation theory*. To be precise, the equivalence relation is defined as follow.

Definition 2.3. *Let* $\begin{bmatrix}Id_m & X\\0 & Id_n\end{bmatrix}$, *and* $\begin{bmatrix}Id_m & Y\\0 & Id_n\end{bmatrix} \in UT_{m\times n}$. *We say that they are in the same* ***superclass*** *if and only if there exist* $A = \begin{bmatrix}A_{11} & A_{12}\\0 & A_{22}\end{bmatrix}$ *and* $B = \begin{bmatrix}B_{11} & B_{12}\\0 & B_{22}\end{bmatrix}$, *where* $A_{11}, B_{11} \in \mathcal{B}_m$ *and* $A_{22}, B_{22} \in B_n$ *such that*

$$\begin{bmatrix}0 & Y\\0 & 0\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\0 & A_{22}\end{bmatrix}\cdot\begin{bmatrix}0 & X\\0 & 0\end{bmatrix}\cdot\begin{bmatrix}B_{11} & B_{12}\\0 & B_{22}\end{bmatrix}.$$

According to the definition 2.3, we can see that $\begin{bmatrix}Id_m & X\\0 & Id_n\end{bmatrix}$ and $\begin{bmatrix}Id_m & Y\\0 & Id_n\end{bmatrix}$ are in the same superclass if and only if there exist $A_{11} \in \mathcal{B}_m$ and $B_{22} \in \mathcal{B}_n$ such that

$$Y = A_{11}XB_{22}.$$

In particular, the matrix Y can be obtained by applying the particular row or column operation to the matrix X. This suggests that we should reindex the rows the other way. Thus, from now on we will index any matrix as the following definitions

Definition 2.4. *Let* $X \in M_{m\times n}(\mathbb{F}_q)$. *When we say* X_{ij}, *we actually mean* $X_{m+1-i,j}$ *for* $(i,j) \in \{1,2,3,\dots,m\}\times\{1,2,3,\dots,n\}$.

Definition 2.5. *Let* $X \in M_{m\times n}(\mathbb{F}_q)$. *The* ***transpose*** *of matrix* X *denoted by* X^T, *is defined by*

$$(X^T)_{ij} = X_{ji} \quad \text{for } (i,j) \in \{1,2,3,\dots,m\}\times\{1,2,3,\dots,n\}.$$

Example. Consider the matrix $X = \begin{bmatrix}1 & 2 & 3\\4 & 1 & 1\\2 & 1 & 2\end{bmatrix}$. By the definition 2.4, we have $X_{11} = 2, X_{21} = 4, X_{31} = 1$. Also, by definition 2.5, we have $X^T = \begin{bmatrix}2 & 1 & 3\\1 & 1 & 2\\2 & 4 & 1\end{bmatrix}$.

Notice that all the possible changes happen just in the top-left-$m \times n$ box of the matrix. Thus, we can possibly consider the top-left box of each element in the group. Precisely, consider the following group isomorphism.

$$\begin{array}{cccc} \phi: & UT_{m\times n} & \to & M_{m\times n}(\mathbb{F}_q) \\ & \begin{bmatrix} Id_m & X \\ 0 & Id_n \end{bmatrix} & \mapsto & X \end{array},$$

where $M_{m\times n}(\mathbb{F}_q)$ is considered as a group under addition. Clearly, ϕ is a bijection. Also, for any $\begin{bmatrix} Id_m & X \\ 0 & Id_n \end{bmatrix}$ and $\begin{bmatrix} Id_m & Y \\ 0 & Id_n \end{bmatrix} \in UT_{m\times n}$,

$$\phi\left(\begin{bmatrix} Id_m & X \\ 0 & Id_n \end{bmatrix}\right) + \phi\left(\begin{bmatrix} Id_m & Y \\ 0 & Id_n \end{bmatrix}\right) = X + Y = \phi\left(\begin{bmatrix} Id_m & X+Y \\ 0 & Id_n \end{bmatrix}\right) = \phi\left(\begin{bmatrix} Id_m & X \\ 0 & Id_n \end{bmatrix} \cdot \begin{bmatrix} Id_m & Y \\ 0 & Id_n \end{bmatrix}\right).$$

According to Definition 2.3, we say that $X, Y \in M_{m\times n}(\mathbb{F}_q)$ are in the same **superclass** if and only if there exist $A_{11} \in \mathcal{B}_m$ and $B_{22} \in \mathcal{B}_n$ such that

$$Y = A_{11} X B_{22}.$$

Since we have group elements into a superclass, we can choose a representative element for each superclass. We will discuss later why we want to pick a representative for each superclass. To choose a representative element, consider the following set

$$\mathcal{S}_{m\times n} = \left\{ X \in M_{m\times n}(\{0,1\}) \;\middle|\; \begin{array}{ll} \sum_{i=1}^{m} X_{ij} \le 1 & \text{for } j = 1, 2, \ldots, n, \\ \sum_{j=1}^{n} X_{ij} \le 1 & \text{for } i = 1, 2, \ldots, m. \end{array} \right\}.$$

To see how this set represents the set of the representative elements, we consider the following mapping

$$\begin{array}{cccc} \lambda: & M_{m\times n}(\mathbb{F}_q) & \to & \mathcal{S}_{m\times n} \\ & X & \mapsto & \lambda_X \end{array}, \text{ where } \lambda_X \text{ is in the same superclass as } X.$$

Note that $\mathcal{S}_{m\times n} \subseteq M_{m\times n}(\{0,1\})$. Also, for any $X \in \mathcal{S}_{m\times n}$, X is already an echelon form. In particular, for $X \in \mathcal{S}_{m\times n}, X = \lambda_X$. Hence, λ is surjective. We have defined and discussed about the state space for this random walk. In the next subsection, we will discuss the other component.

2.2 Probabilities

As mentioned earlier in this section, we need to consider the probability transition for this walk. In this particular walk, we allow either the zero matrix or **rank 1 matrices** to be chosen in each iteration. Before going further, we should understand what a **rank 1 matrix** looks like.

Definition 2.6. *For $C \in M_{m\times n}(\mathbb{F}_q)$, we say that C is a **rank 1 matrix** if and only if $\sum_{i=1}^{m}\sum_{j=1}^{n}(\lambda_C)_{ij} = 1$.*

Example. $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 4 \\ 1 & 2 & 2 \end{bmatrix}$ is a rank 1 matrix becuase $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 4 \\ 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. However, $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ is not a rank 1 matrix because $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -4 \\ 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

Our goal is to compute the total probability of going from one superclass to another superclass. According to the way we partition the group into this specific set of superclasses, we have the following fact for any $X, P, P' \in M_{m\times n}(\mathbb{F}_q)$,

$$|\{X | \lambda_X = \mu, \text{ and } \lambda_{X+P} = \nu\}| = |\{X | \lambda_X = \mu, \text{ and } \lambda_{X+P'} = \nu\}|.$$

This suggests that we can either fix an element in each superclass or fix a element in each rank 1 matrix. Hence, the idea of picking the representative element for each superclass makes sense now. In the next section, we will do all the essential computations for potentially obtaining the total probabilities.

3 Main Results

In this section, we will discuss about all the possibilities of λ_{X+P}, where $X \in \mathcal{S}_{m\times n}$ is given and P is an arbitrary rank 1 matrix. Before doing so, some notations and lemmas need to be stated.

3.1 Notation and Lemmas

Since from now on we will do a lot of row operations, so a nice notation will be helpful.

Definition 3.1. *For $X \in M_{m\times n}\mathbb{F}_q$,*

1. *"$c \times row_i + row_j$" means to multiple the i^{th} row of X by the constant c and add that to the j^{th} row,*
2. *"$c \times row_i$" means to multiple the i^{th} row of X by the constant c,*
3. *"$c\times col_i + col_j$" means to multiple the i^{th} column of X by the constant c and add that to the j^{th} column,*
4. *"$c \times col_i$" means to multiple the i^{th} column of X by the constant c.*

Definition 3.2. *Define*

$$\begin{aligned}
[a,b] &:= \{x \in \mathbb{Z} | a \le x \le b\},\\
[a,b) &:= \{x \in \mathbb{Z} | a \le x < b\},\\
(a,b] &:= \{x \in \mathbb{Z} | a < x \le b\},\\
(a,b) &:= \{x \in \mathbb{Z} | a < x < b\}.
\end{aligned}$$

This new notations of intervals will be useful in counting and keeping track of the entries.

Definition 3.3. *For $X \in M_{m\times n}(\mathbb{F}_q)$ and any subset $I \subseteq [1,m]\times[1,n]$, define $\cdot_I : M_{m\times n}(\mathbb{F}_q) \to M_{m\times n}(\mathbb{F}_q)$ such that*

$$X \to X_I\ , \qquad \text{where} \quad (X_I)_{ij} = \begin{cases} X_{ij} & (i,j) \in I \\ 0 & (i,j) \notin I. \end{cases}$$

Since row operations will be applied a number of times in the computations, this definition will be useful to represent a matrix when some of the rows or columns have been reduced. The following lemma is also helpful when we are doing row operations.

Lemma 3.4. *Let $X \in \mathcal{S}_{m\times n}$. For $i \in [1,m], j \in [1,n]$ such that*

(a) $X_{ij} \neq 0$,

(b) $X_{kj} = 0$ for all $k \in [1,i)$, and

(c) $X_{ik} = 0$ for all $k \in [1,n] - \{j\}$.

Then,

$$X \quad \sim \quad X - X_J + e_{ij}, \quad \text{where } J = [1,m]\times\{j\}.$$

Proof. For $k \in (i, m]$, we do the following row reductions

$$-X_{kj}X_{ij}^{-1} \times row_i + row_k.$$

Then, we do the following row reduction

$$X_{ij}^{-1} \times row_i.$$

Thus,

$$X \quad \sim \quad X - X_J + e_{ij}, \quad \text{where } J = [1, m] \times \{j\}.$$

□

Lemma 3.5. *Let $X \in \mathbb{F}_q^{m \times n}$. For $i \in [1, m], j \in [1, n]$ such that*

(a) $X_{ij} \neq 0$,

(b) $X_{kj} = 0$ *for all* $k \in [1, m] - \{i\}$, *and*

(c) $X_{ik} = 0$ *for all* $k \in [1, j)$.

Then,

$$X \quad \sim \quad X - X_I + e_{ij}, \quad \textit{where } I = \{i\} \times [1, n].$$

Proof. For $k \in (j, n]$, we do the following row (column) reductions

$$-X_{ik}X_{ij}^{-1} \times col_j + col_k.$$

Then, we do the following row reduction

$$X_{ij}^{-1} \times col_j.$$

Thus,

$$X \quad \sim \quad X - X_I + e_{ij}, \quad \text{where } I = \{i\} \times [1, n].$$

□

From now on, we will write a rank 1 matrix P as $p \cdot a^T \cdot b$, for some $p \in \mathbb{F}_q^*, a \in \mathbb{F}_q^m$, and $b \in \mathbb{F}_q^n$.
Before we move into the main results, we need to understand the following notations, which will turn out to be useful.

Definition 3.6. *A triple (i, j, ls) is a zig-zag path of length l if*

(a) i and j are sequences such that

  (i) $1 = i_0 < i_1 < \cdots < i_l \leq m$,

  (ii) $1 = j_0 < j_1 < \cdots < j_l \leq n$,

(b) $ls \in \{0, 1\}$.

Definition 3.7. *Let $X \in \mathcal{S}_{m \times n}$. A zig-zag path (i, j, ls) is downward in X if*

(a) $X_{i_k j_k} = 0$ *for all* $k = 0, 1, \ldots, l$,

(b) $X_{i_{k+1} j_k} = 1$ *for all* $k = 0, 1, \ldots, l - 1$.

Definition 3.8. *Let $X \in \mathcal{S}_{m\times n}$. A zig-zag path (i, j, ls) is backward in X if*

(a) $X_{i_k j_k} = 0$ *for all* $k = 0, 1, \ldots, l$,

(b) $X_{i_k j_{k+1}} = 1$ *for all* $k = 0, 1, \ldots, l-1$.

Lemma 3.9. *Let $X \in \mathcal{S}_{m\times n}$. A zig-zag path (i, j, ls) is downward in X if and only if a zig-zag path (j, i, ls) is backward in X^T.*

Proof. Let (i, j, ls) be a downward zig-zag path in X. Hence,

(i) $1 = i_0 < i_1 < \cdots < i_l \leq m$,

(ii) $1 = j_0 < j_1 < \cdots < j_l \leq n$,

(iii) $X_{i_k j_k} = 0$ for all $k = 0, 1, \ldots, l$,

(iv) $X_{i_{k+1} j_k} = 1$ for all $k = 0, 1, \ldots, l-1$, and

(v) $ls \in \{0, 1\}$.

Equivalently,

(i) $1 = j_0 < j_1 < \cdots < j_l \leq n$,

(ii) $1 = i_0 < i_1 < \cdots < i_l \leq n$,

(iii) $X^T_{j_k i_k} = 0$ for all $k = 0, 1, \ldots, l$,

(iv) $X^T_{j_k i_{k+1}} = 1$ for all $k = 0, 1, \ldots, l-1$, and

(v) $ls \in \{0, 1\}$.

Therefore, a zig-zag path (j, i, ls) is backward in X^T. □

Definition 3.10. *Let $X \in \mathcal{S}_{m\times n}$. A zig-zag path (i, j, ls) is upward in X if*

(a) $X_{i_k j_k} = 1$ *for all* $k = 0, 1, \ldots, l$,

(b) $X_{i_{k+1} j_k} = 0$ *for all* $k = 0, 1, \ldots, l-1$.

Definition 3.11. *Let $X \in \mathcal{S}_{m\times n}$. A zig-zag path (i, j, ls) is a forward in X if*

(a) $X_{i_k j_k} = 1$ *for all* $k = 0, 1, \ldots, l$,

(b) $X_{i_k j_{k=1}} = 0$ *for all* $k = 0, 1, \ldots, l-1$.

Lemma 3.12. *Let $X \in \mathcal{S}_{m\times n}$. A zig-zag path (i, j, ls) is upward in X if and only if a zig-zag path (j, i, ls) is forward in X^T.*

Proof. Let (i, j, ls) be a upwardward zig-zag path in X. Hence,

(i) $1 = i_0 < i_1 < \cdots < i_l \leq m$,

(ii) $1 = j_0 < j_1 < \cdots < j_l \leq n$,

(iii) $X_{i_k j_k} = 1$ for all $k = 0, 1, \ldots, l$,

(iv) $X_{i_{k+1} j_k} = 0$ for all $k = 0, 1, \ldots, l-1$, and

(v) $ls \in \{0, 1\}$.

Equivalently,

(i) $1 = j_0 < j_1 < \cdots < j_l \leq n$,

(ii) $1 = i_0 < i_1 < \cdots < i_l \leq n$,

(iii) $X^T_{j_k i_k} = 1$ for all $k = 0, 1, \ldots, l$,

(iv) $X^T_{j_k i_{k+1}} = 0$ for all $k = 0, 1, \ldots, l-1$, and

(v) $ls \in \{0, 1\}$.

Therefore, a zig-zag path (j, i, ls) is forward in X^T. □

3.2 Main Theorems

We will start with a summary theorem. The summary theorem describes all the possibilities of $X + P$, where $X \in \mathcal{S}_{m\times n}$ and P is a rank 1 matrix.

Theorem 3.13. *Given $X \in \mathcal{S}_{m\times n}$. For any rank 1 matrix $p \cdot a^T \cdot b, X + p \cdot a^T \cdot b$ is equivalent to one of the following:*

(i) $X + e_{11}$,

(ii) $X + \sum_{k=0}^{l-1} \left(e_{i_k j_k} - e_{i_{k+1} j_k}\right) + (ls) \cdot e_{i_l j_l}$ *for some downward zig-zag path (i, j, ls),*

(iii) $X + \sum_{k=0}^{l-1} \left(e_{i_k j_k} - e_{i_k j_{k+1}}\right) + (ls) \cdot e_{i_l j_l}$ *for some backward zig-zag path (i, j, ls),*

(iv) $X + p(1+p)^{-1} \cdot a^T_{(1,m]} \cdot b_{(1,n]}$,

(v) $X + e_{11} - \sum_{k=1}^{l} \left(e_{i_k j_{k-1}} - e_{i_k j_k}\right) - \sum_{k'=1}^{l'} \left(e_{i_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}\right) - e_{st} + a_s^{-1} b_t^{-1} \cdot a'^T_{(s,m]} \cdot b'_{(t,n]}$ *for some upward zig-zag path (i, j, ls) and forward zig-zag path (i', j', ls'),*

(vi) $X - e_{11} - \sum_{k=0}^{l-1} \left(e_{i_{k+1} j_k} - e_{i_k j_k}\right) - \sum_{k'=0}^{l'-1} \left(e_{i_{k'} j'_{k'+1}} - e_{i'_{k'} j'_{k'}}\right) + e_{st} + a_s^{-1} b_t^{-1} \cdot a'^T_{(s,m]} \cdot b'_{(t,n]}$ *for some downward zig-zag path (i, j, ls) and backward zig-zag path (i', j', ls'),*

(vii) $X + e_{11} - \sum_{k=1}^{l-1} \left(e_{i_k j_{k-1}} - e_{i_k j_k}\right) + (ls) \cdot e_{i_l j_l} - \sum_{k'=1}^{l'-1} \left(e_{i_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}\right) + (ls') \cdot e_{i_{l'} j_{l'}} + a_s^{-1} b_t^{-1} \cdot a'^T_{(s,m]} \cdot b'_{(t,n]}$ *for some upward zig-zag path (i, j, ls) and forward zig-zag path (i', j', ls'),*

Proof. *(i)* First, we choose $a_1 = b_1 = 1$. For $k \in (1, m]$, we do the following row operations

$$-a_k \times row_1 + row_k$$

Hence,

$$X + p \cdot a^T \cdot b \;\sim\; X + p \cdot a^T \cdot b - p \cdot a^T_{(1,m]} \cdot b \;=\; X + p \cdot e_1 \cdot b$$

Then, we apply Lemma 3.5 ($i = 1, j = 1$.) Thus,

$$X + p \cdot e_1 \cdot b \quad \sim \quad X + e_{11}.$$

Therefore,

$$X + p \cdot a^T \cdot b \quad \sim \quad X + e_{11}.$$

(ii) Let $p \in \mathbb{F}_q^*$, $a = [a_1\ a_2\ a_3 \cdots a_m]$ and $b = [b_1\ b_2\ b_3 \cdots b_n]$, where $a_1 = b_1 = 1$. Also, choose $b_k = 0$ for all $k \notin \{j_0, j_1, \ldots, j_l\}$ and $b_k \neq 0$ for all $\{j_0, j_1, \ldots, j_{l-1}\}$. Consider $X + p \cdot a^T \cdot b$. For $k \in (i_0, m] = (1, m]$, we do the following row reductions

$$-a_k \times row_1 + row_k.$$

Hence,

$$X + p \cdot a^T \cdot b \quad \sim \quad X + p \cdot e_1 \cdot b.$$

Then, we do the following row reduction

$$-p^{-1} \times row_1 + row_{i_1}.$$

Hence,

$$X + p \cdot e_1 \cdot b \quad \sim \quad X + p \cdot e_{i_0} \cdot b - e_{i_1} \cdot b.$$

Write $b = e_{j_0} + b_{(j_0,n]}$. So, $e_{i_1} \cdot (e_{j_0} + b_{(j_0,n]}) = e_{i_1 j_0} + e_{i_1} \cdot b_{(j_0,n]}$. Thus,

$$X + p \cdot e_{i_0} \cdot b - e_{i_1} \cdot b = X + p \cdot e_{i_0} \cdot b - e_{i_1 j_0} - e_{i_1} \cdot b_{(j_0,n]}.$$

Next, we apply Lemma 3.5 ($i = i_0, j = j_0$). Thus,

$$X + p \cdot e_{i_0} \cdot b - e_{i_1 j_0} - e_{i_1} \cdot b_{(j_0,n]} \quad \sim \quad X + e_{i_0 j_0} - e_{i_1 j_0} - e_{i_1} \cdot b_{(j_0,n]}.$$

Since $0 = b_{j_0+1} = \cdots = b_{j_1-1}$, so $b_{(j_0,n]} = b_{[j_1,n]}$. Thus,

$$X + e_{i_0 j_0} - e_{i_1 j_0} - e_{i_1} \cdot b_{(j_0,n]} = X + e_{i_0 j_0} - e_{i_1 j_0} - e_{i_1} \cdot b_{[j_1,n]}.$$

Hence,

$$X + p \cdot a^T \cdot b \quad \sim \quad X + e_{i_0 j_0} - e_{i_1 j_0} - e_{i_1} \cdot b_{[j_1,n]}.$$

Lemma 3.14. *For $t = 1, 2, \ldots, l-1$, we have*

$$X + \sum_{k=0}^{t-1} \left(e_{i_k j_k} - e_{i_{k+1} j_k}\right) - e_{i_t} \cdot b_{[j_t,n]} \quad \sim \quad X + \sum_{k=0}^{t} \left(e_{i_k j_k} - e_{i_{k+1} j_k}\right) - e_{i_{t+1}} \cdot b_{[j_{t+1},n]}.$$

Proof. First, we do the following row reduction

$$b_{j_t}^{-1} \times row_{i_t} + row_{i_{t+1}}.$$

Hence,

$$X + \sum_{k=0}^{t-1} \left(e_{i_k j_k} - e_{i_{k+1} j_k}\right) - e_{i_t} \cdot b_{[j_t,n]} \quad \sim \quad X + \sum_{k=0}^{t-1} \left(e_{i_k j_k} - e_{i_{k+1} j_k}\right) - e_{i_t} \cdot b_{[j_t,n]} - b_{j_t}^{-1} \cdot e_{i_{t+1}} \cdot b_{[j_t,n]}.$$

Next, we apply Lemma 3.5 ($i = i_t, j = j_t$). Thus,

$$X+\sum_{k=0}^{t-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)-e_{i_t}\cdot b_{[j_t,n]}-b_{j_t}^{-1}\cdot e_{i_{t+1}}\cdot b_{[j_t,n]} \quad\sim\quad X+\sum_{k=0}^{t-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)+e_{i_tj_t}-b_{j_t}^{-1}\cdot e_{i_{t+1}}\cdot b_{[j_t,n]}$$

Again, write $b_{[j_t,n]} = b_{j_t} \cdot e_{j_t} + b_{(j_t,n]}$. Hence,

$$b_{j_t}^{-1}\cdot e_{i_{t+1}}\cdot b_{[j_t,n]} = b_{j_t}^{-1}\cdot e_{i_{t+1}}\cdot\left(b_{j_t}\cdot e_{j_t}+b_{(j_t,n]}\right) = e_{i_{t+1}j_t}+b_{j_t}^{-1}\cdot e_{i_{t+1}}\cdot b_{(j_t,n]}.$$

Then,

$$X+\sum_{k=0}^{t-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)+e_{i_tj_t}-b_{j_t}^{-1}\cdot e_{i_{t+1}}\cdot b_{[j_t,n]} = X+\sum_{k=0}^{t-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)+e_{i_tj_t}-e_{i_{t+1}j_t}-b_{j_t}^{-1}\cdot e_{i_{t+1}}\cdot b_{(j_t,n]}.$$

Since $0 = b_{j_t+1} = \cdots = b_{j_{t+1}-1}$, so $b_{(j_t,n]} = b_{[j_{t+1},n]}$. Thus,

$$X+\sum_{k=0}^{t-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)-e_{i_t}\cdot b_{[j_t,n]} \quad\sim\quad X+\sum_{k=0}^{t}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)-b_{j_t}^{-1}\cdot e_{i_{t+1}}\cdot b_{[j_{t+1},n]}.$$

Then, we do the last row reduction and that is

$$b_{j_t}\times row_{i_{t+1}}.$$

Therefore,

$$X+\sum_{k=0}^{t-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)-e_{i_t}\cdot b_{[j_t,n]} \quad\sim\quad X+\sum_{k=0}^{t}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)-e_{i_{t+1}}\cdot b_{[j_{t+1},n]}.$$

□

Hence,

$$X+p\cdot a^T\cdot b \quad\sim\quad X+\sum_{k=0}^{l-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)-e_{i_l}\cdot b_{[j_l,n]}.$$

If $ls = 0$, then we choose $b_{j_l} = 0$. Therefore,

$$X+\sum_{k=0}^{l-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)-e_{i_l}\cdot b_{[j_l,n]} = X+\sum_{k=0}^{l-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right) = X+\sum_{k=0}^{l-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)+(ls)\cdot e_{i_lj_l}.$$

If $ls = 1$, then $X_{kj_l} = 0$ for all $k \in [1, m]$. So, We choose $b_{j_l} \neq 0$. By Lemma 3.5 ($i = i_l, j = j_l$), we have

$$X+\sum_{k=0}^{l-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)-e_{i_l}\cdot b_{[j_l,n]} \sim X+\sum_{k=0}^{l-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)+e_{i_lj_l} = X+\sum_{k=0}^{l-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)+(ls)\cdot e_{i_lj_l}.$$

Therefore,

$$X+p\cdot a^T\cdot b \quad\sim\quad X+\sum_{k=0}^{l-1}\left(e_{i_kj_k}-e_{i_{k+1}j_k}\right)+(ls)\cdot e_{i_lj_l}.$$

(iii) Let $X \in \mathcal{S}_{m\times n}$ and (i, j, ls) be a backward zig-zag path of length l in X. By Lemma 3.9, a zig-zag path (j, i, ls) is a downward zig-zag path of length l in X^T. Also, note that $(ls) \cdot X_{i_l k} = 0$ for all $k \in [1, n]$ implies that $(ls) \cdot X^T_{k i_l} = 0$ for all $k \in [1, n]$ Hence, by (ii), there exist $p \in \mathbb{F}_q^*$, $b \in \mathbb{F}_q^n$, and $a \in \mathbb{F}_q^m$ such that

$$X^T + p \cdot b^T \cdot a \quad \sim \quad X^T + \sum_{k=0}^{l-1} \left(e_{j_k i_k} - e_{j_{k+1} i_k}\right) + (ls) \cdot e_{j_l i_l}.$$

Equivalently,

$$X + p \cdot a^T \cdot b \quad \sim \quad X + \sum_{k=0}^{l-1} \left(e_{i_k j_k} - e_{i_k j_{k+1}}\right) + (ls) \cdot e_{i_l j_l}.$$

(iv) First, we choose p such that $1 + p \neq 0$ and $a_1 = b_1 = 1$. Also, we write

$$X + p \cdot a^T \cdot b = X - e_{11} + e_{11} + p \cdot a^T \cdot b.$$

Notice that

$$\begin{aligned} e_{11} + p \cdot a^T \cdot b &= e_{11} + p \cdot (e_1 + a^T_{(1,m]}) \cdot (e_1 + b_{(1,n]}) \\ &= e_{11} + p \cdot (e_{11} + e_1 \cdot b_{(1,n]} + a^T_{(1,m]} \cdot e_1 + a^T_{(1,m]} \cdot b_{(1,n]}) \\ &= e_{11} + p \cdot e_{11} + p \cdot e_1 \cdot b_{(1,n]} + p \cdot a^T_{(1,m]} \cdot e_1 + p \cdot a^T_{(1,m]} \cdot b_{(1,n]}) \\ &= e_1 \cdot ((1+p) \cdot e_1 + p \cdot b_{(1,n]}) + p \cdot a^T_{(1,m]} \cdot e_1 + p \cdot a^T_{(1,m]} \cdot b_{(1,n]}) \end{aligned}$$

Note that the first row of the matrix is $(1 + p) \cdot e_1 + p \cdot b_{(1,n]}$. For $k \in (1, m]$, we do the following row reductions

$$(1 + p)^{-1} p a_k \times row_1 + row_k.$$

Hence,

$$X + p \cdot a^T \cdot b \quad \sim \quad X - e_{11} + p \cdot a^T \cdot b - p(1 + p)^{-1} \cdot a^T_{(1,m]} \cdot ((1 + p) \cdot e_1 + p \cdot b_{(1,n]})$$

Notice that

$$\begin{aligned} p \cdot a^T \cdot b - p(1+p)^{-1} \cdot & a^T_{(1,m]} \cdot ((1+p) \cdot e_1 + p \cdot b_{(1,n]}) \\ &= e_1 \cdot ((1+p) \cdot e_1 + p \cdot b_{(1,n]}) + p \cdot a^T_{(1,m]} \cdot e_1 + p \cdot a^T_{(1,m]} \cdot b_{(1,n]}) \\ &\quad - p(1+p)^{-1} \cdot a^T_{(1,m]} \cdot ((1+p) \cdot e_1 + p \cdot b_{(1,n]}) \\ &= e_1 \cdot ((1+p) \cdot e_1 + p \cdot b_{(1,n]}) + p \cdot a^T_{(1,m]} \cdot e_1 + p \cdot a^T_{(1,m]} \cdot b_{(1,n]}) \\ &\quad - p \cdot a^T_{(1,m]} \cdot e_1 - p^2(1+p)^{-1} \cdot a^T_{(1,m]} \cdot b_{(1,n]} \\ &= e_1 \cdot ((1+p) \cdot e_1 + p \cdot b_{(1,n]}) + (p - p^2(1+p)^{-1}) \cdot a^T_{(1,m]} \cdot b_{(1,n]} \\ &= e_1 \cdot ((1+p) \cdot e_1 + p \cdot b_{(1,n]}) + p(1+p)^{-1} \cdot a^T_{(1,m]} \cdot b_{(1,n]} \end{aligned}$$

Hence,

$$X - e_{11} + p \cdot a^T \cdot b - p(1 + p)^{-1} \cdot a^T_{(1,m]} \cdot ((1 + p) \cdot e_1 + p \cdot b_{(1,n]})$$

$$=$$

$$X - e_{11} + e_1 \cdot ((1+p)\cdot e_1 + p \cdot b_{(1,n]}) + p(1+p)^{-1} \cdot a^T_{(1,m]} \cdot b_{(1,n]}.$$

Now, we apply Lemma 3.5($i=1, j=1$). Then,

$$X - e_{11} + e_1 \cdot ((1+p)\cdot e_1 + p \cdot b_{(1,n]}) + p(1+p)^{-1} \cdot a^T_{(1,m]} \cdot b_{(1,n]}$$

$$\sim$$

$$X - e_{11} + e_{11} + p(1+p)^{-1} \cdot a^T_{(1,m]} \cdot b_{(1,n]} = X + p(1+p)^{-1} \cdot a^T_{(1,m]} \cdot b_{(1,n]}.$$

There is another possibility of the matrix $p \cdot a^T \cdot b$. We could choose p to be -1. However, to take care of that case we need the following set of lemmas. The idea of employing the following lemmas is that we break the row & column operations into steps. Each step will have a pattern in one of the corresponding lemmas.

Lemma 3.15. *Let $X \in \mathcal{S}_{m\times n}$ such that $X_{i\hat{j}} = X_{\hat{i}'j'} = 1$. Let $a \in \mathbb{F}_q^m$ and $b \in \mathbb{F}_q^n$ such that $a_i \neq 0$ and $b_{j'} \neq 0$. Suppose $i' < \hat{i}'$ or $j < \hat{j}$. If $(i,\hat{j}) \neq (\hat{i}', j')$, then*

$$X + a^T_{[i,m]} \cdot e_j + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} - e_{i\hat{j}} + e_{i'j'} - e_{\hat{i}'j'} + a^T_{(i,m]} \cdot e_{\hat{j}} + e_{\hat{i}'} \cdot b_{(j',n]}.$$

Proof. First, we do the following row (column) reduction

$$-a_i^{-1} \times col_j + col_{\hat{j}}.$$

Hence,

$$X + a^T_{[i,m]} \cdot e_j + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + a^T_{[i,m]} \cdot e_j - a_i^{-1} \cdot a^T_{[i,m]} \cdot e_{\hat{j}} + e_{i'} \cdot b_{[j',n]}$$

Write $a_{[i,m]} = a_i \cdot e_i + a_{(i,m]}$. So, $a_i^{-1} \cdot a^T_{[i,m]} \cdot e_{\hat{j}} = e_{i\hat{j}} + a^{-1} \cdot a^T_{(i,m]} \cdot e_{\hat{j}}$. Thus,

$$X + a^T_{[i,m]} \cdot e_j - a_i^{-1} \cdot a^T_{[i,m]} \cdot e_{\hat{j}} + e_{i'} \cdot b_{[j',n]} = X + a^T_{[i,m]} \cdot e_j - e_{i\hat{j}} - a^{-1} \cdot a^T_{(i,m]} \cdot e_{\hat{j}} + e_{i'} \cdot b_{[j',n]}.$$

Then, we apply Lemma 3.4 ($i=i, j=j$). Thus,

$$X + a^T_{[i,m]} \cdot e_j - e_{i\hat{j}} - a^{-1} \cdot a^T_{(i,m]} \cdot e_{\hat{j}} + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} - e_{i\hat{j}} - a^{-1} \cdot a^T_{(i,m]} \cdot e_{\hat{j}} + e_{i'} \cdot b_{[j',n]}.$$

Then, we do the following row (column) reduction

$$-a_i^{-1} \times col_{\hat{j}}.$$

Therefore,

$$X + e_{ij} - e_{i\hat{j}} - a^{-1} \cdot a^T_{(i,m]} \cdot e_{\hat{j}} + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} - e_{i\hat{j}} + a^T_{(i,m]} \cdot e_{\hat{j}} + e_{i'} \cdot b_{[j',n]}$$

Similarly, we do the following row reduction

$$-b_{j'}^{-1} \times row_{i'} + row_{\hat{i}'}.$$

Hence,

$$X + e_{ij} - e_{i\hat{j}} + a^T_{(i,m]} \cdot e_{\hat{j}} + e_{i'} \cdot b_{[j',n]} \quad \sim \quad Y + e_{i'} \cdot b_{[j',n]} - b_{j'}^{-1} \cdot e_{\hat{i}'} \cdot b_{[j',n]},$$

where $Y = X + e_{i\hat{j}} - e_{i\hat{j}} + a_{(i,m]}^T \cdot e_{\hat{j}}$. Write $b_{[j',n]} = b_{j'} \cdot e_{j'} + b_{(j',n]}$. Hence, $b_{j'}^{-1} \cdot e_{\hat{i}'} \cdot b_{[j',n]} = e_{\hat{i}'\hat{j}'} + b_{j'}^{-1} \cdot e_{\hat{i}'} \cdot b_{(j',n]}$. Thus,

$$Y + e_{i'} \cdot b_{[j',n]} - b_{j'}^{-1} \cdot e_{\hat{i}'} \cdot b_{[j',n]} = Y + e_{i'} \cdot b_{[j',n]} - e_{\hat{i}'\hat{j}'} - b_{j'}^{-1} \cdot e_{\hat{i}'} \cdot b_{(j',n]}.$$

Then, we apply Lemma 3.5 ($i = i', j = j'$). Thus,

$$Y + e_{i'} \cdot b_{[j',n]} - e_{\hat{i}'\hat{j}'} - b_{j'}^{-1} \cdot e_{\hat{i}'} \cdot b_{(j',n]} \quad \sim \quad Y + e_{i'j'} - e_{\hat{i}'\hat{j}'} - b_{j'}^{-1} \cdot e_{\hat{i}'} \cdot b_{(j',n]}.$$

Then, we do the following row reduction

$$-b_{j'}^{-1} \times row_{\hat{i}'}.$$

Therefore,

$$Y + e_{i'j'} - e_{\hat{i}'\hat{j}'} - b_{j'}^{-1} \cdot e_{\hat{i}'} \cdot b_{(j',n]} \sim Y + e_{i'j'} - e_{\hat{i}'\hat{j}'} + e_{\hat{i}'} \cdot b_{(j',n]}.$$

Hence,

$$X + a_{[i,m]}^T \cdot e_j + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} - e_{i\hat{j}} + e_{i'j'} - e_{\hat{i}'j'} + a_{(i,m]}^T \cdot e_{\hat{j}} + e_{\hat{i}'} \cdot b_{(j',n]}.$$

□

Lemma 3.16. *Let $X \in \mathcal{S}^{m\times n}$ such that $X_{i\hat{j}} = X_{\hat{i}'j'} = 1$. Let $a \in \mathbb{F}_q^m$ and $b \in \mathbb{F}_q^n$ such that $a_i \neq 0$ and $b_{j'} \neq 0$. Suppose $i' < \hat{i}'$ or $j < \hat{j}$. If $(i, \hat{j}) = (\hat{i}', j')$, then*

$$X + a_{[i,m]}^T \cdot e_j + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} + e_{i'j'} - e_{ij'} + a_i^{-1} b_{j'}^{-1} \cdot a_{(i,m]}^T \cdot b_{(j',n]}.$$

Proof. First, we do the following row (column) reduction

$$-a_i^{-1} \times col_j + col_{j'}.$$

Hence,

$$X + a_{[i,m]}^T \cdot e_j + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + a_{[i,m]}^T \cdot e_j - a_i^{-1} \cdot a_{[i,m]}^T \cdot e_{j'} + e_{i'} \cdot b_{[j',n]}$$

Write $a_{[i,m]} = a_i \cdot e_i + a_{(i,m]}$. So, $a_i^{-1} \cdot a_{[i,m]}^T \cdot e_{j'} = e_{ij'} + a_i^{-1} \cdot a_{(i,m]} \cdot e_{j'}$.

$$X + a_{[i,m]}^T \cdot e_j - a_i^{-1} \cdot a_{[i,m]}^T \cdot e_{j'} + e_{i'} \cdot b_{[j',n]} = X + a_{[i,m]}^T \cdot e_j - e_{ij'} - a_i^{-1} \cdot a_{(i,m]}^T \cdot e_{j'} + e_{i'} \cdot b_{[j',n]}.$$

Then, we apply Lemma 3.4 ($i = i, j = j$). So,

$$X + a_{[i,m]}^T \cdot e_j - e_{ij'} - a_i^{-1} \cdot a_{(i,m]}^T \cdot e_{j'} + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} - e_{ij'} - a_i^{-1} \cdot a_{(i,m]}^T \cdot e_{j'} + e_{i'} \cdot b_{[j',n]}.$$

Next, for $k \in (i, m]$, we do the following row reductions

$$a_i^{-1} a_k b_{j'}^{-1} \times row_{i'} + row_k$$

Hence,

$$X + e_{ij} - e_{ij'} - a_i^{-1} \cdot a_{(i,m]}^T \cdot e_{j'} + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} - e_{ij'} - a_i^{-1} \cdot a_{(i,m]}^T \cdot e_{j'} + a_i^{-1} b_{j'}^{-1} \cdot a_{(i,m]}^T \cdot b_{[j',n]} + e_{i'} \cdot b_{[j',n]}$$

Note that

$$-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{j'}+a_i^{-1}b_{j'}^{-1}\cdot a_{(i,m]}^T\cdot b_{[j',n]}=a_i^{-1}b_{j'}^{-1}\cdot a_{(i,m]}^T\left(-b_{j'}\cdot e_{j'}+b_{[j',n]}\right)=a_i^{-1}b_{j'}^{-1}\cdot a_{(i,m]}^T\cdot b_{(j',n]}.$$

Hence,

$$X+e_{ij}-e_{ij'}-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{j'}+a_i^{-1}b_{j'}^{-1}\cdot a_{(i,m]}^T\cdot b_{[j',n]}+e_{i'}\cdot b_{[j',n]}=X+e_{ij}-e_{ij'}+a_i^{-1}b_{j'}^{-1}\cdot a_{(i,m]}^T\cdot b_{(j',n]}+e_{i'}\cdot b_{[j',n]}.$$

Then, we apply Lemma 3.5 ($i=i', j=j'$). So,

$$X+e_{ij}-e_{ij'}+a_i^{-1}b_{j'}^{-1}\cdot a_{(i,m]}^T\cdot b_{(j',n]}+e_{i'}\cdot b_{[j',n]}\quad\sim\quad X+e_{ij}+e_{i'j'}-e_{ij'}+a_i^{-1}b_{j'}^{-1}\cdot a_{(i,m]}^T\cdot b_{(j',n]}.$$

□

Lemma 3.17. *Let $X\in\mathcal{S}_{m\times n}$ such that $X_{kj'}=0$ for all $k=1,2,\ldots,m$ and $X_{i\hat{j}}=1$. Let $a\in\mathbb{F}_q^m$ and $b\in\mathbb{F}_q^n$ such that $a_i\neq 0$ and $b_{j'}\neq 0$. Suppose $i<i'$ and $j=j'$. Then,*

$$X+a_{[i,m]}^T\cdot e_j+e_{i'}\cdot b_{[j',n]}\quad\sim\quad X+e_{ij}-e_{i\hat{j}}-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{\hat{j}}+e_{i'}\cdot b_{(j,n]}.$$

Proof. First, note that $j'=j$. Hence,

$$X+a_{[i,m]}^T\cdot e_j+e_{i'}\cdot b_{[j',n]}=X+a_{[i,m]}^T\cdot e_j+e_{i'}\cdot b_{[j,n]}$$

Then, we do the following row (column) reduction

$$-a_i^{-1}\times col_j+col_{\hat{j}}.$$

Then,

$$X+a_{[i,m]}^T\cdot e_j+e_{i'}\cdot b_{[j,n]}\quad\sim\quad X+a_{[i,m]}^T\cdot e_j-a_i^{-1}\cdot a_{[i,m]}^T\cdot e_{\hat{j}}+e_{i'}\cdot b_{[j,n]}$$

We write $a_{[i,m]}=a_i\cdot e_i+a_{(i,m]}$. So, $a_i^{-1}\cdot a_{[i,m]}^T\cdot e_{\hat{j}}=e_{i\hat{j}}+a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{\hat{j}}$. Thus,

$$X+a_{[i,m]}^T\cdot e_j-a_i^{-1}\cdot a_{[i,m]}^T\cdot e_{\hat{j}}+e_{i'}\cdot b_{[j,n]}=X+a_{[i,m]}^T\cdot e_j-e_{i\hat{j}}-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{\hat{j}}+e_{i'}\cdot b_{[j,n]}$$

Next, we write $b_{[j,n]}=b_j\cdot e_j+b_{(j,n]}$. So,

$$a_{[i,m]}^T\cdot e_j+e_{i'}\cdot b_{[j,n]}=a_{[i,m]}^T\cdot e_j+e_{i'}\cdot\left(b_j\cdot e_j+b_{(j,n]}\right)=\left(a_{[i,m]}^T+b_j\cdot e_{i'}\right)\cdot e_j+e_{i'}\cdot b_{(j,n]}.$$

Thus,

$$X+a_{[i,m]}^T\cdot e_j-e_{i\hat{j}}-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{\hat{j}}+e_{i'}\cdot b_{[j,n]}=X+\left(a_{[i,m]}^T+b_j\cdot e_{i'}\right)\cdot e_j+e_{i'}\cdot b_{(j,n]}-e_{i\hat{j}}-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{\hat{j}}$$

Now, we apply Lemma 3.4 ($i=i, j=j$). Hence,

$$X+\left(a_{[i,m]}^T+b_j\cdot e_{i'}\right)\cdot e_j+e_{i'}\cdot b_{(j,n]}-e_{i\hat{j}}-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{\hat{j}}\quad\sim\quad X+e_{ij}-e_{i\hat{j}}-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{\hat{j}}+e_{i'}\cdot b_{(j,n]}.$$

Therefore,

$$X+a_{[i,m]}^T\cdot e_j+e_{i'}\cdot b_{[j',n]}\quad\sim\quad X+e_{ij}-e_{i\hat{j}}-a_i^{-1}\cdot a_{(i,m]}^T\cdot e_{\hat{j}}+e_{i'}\cdot b_{(j,n]}.$$

□

Lemma 3.18. *Let $X \in \mathcal{S}_{m\times n}$ such that $X_{\hat{i}j'} = 1$ and $X_{ik} = 0$ for all $k = 1, 2, \ldots, n$. Let $a \in \mathbb{F}_q^m$ and $b \in \mathbb{F}_q^n$ such that $a_i \neq 0$ and $b_{j'} \neq 0$. Suppose $i = i'$ and $j < j'$. Then,*

$$X + a^T_{[i,m]} \cdot e_j + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} - e_{\hat{i}j} + a^T_{(i,m]} \cdot e_j - b_j^{-1} \cdot e_{\hat{i}} \cdot b_{(j,n]}.$$

Proof. Note that $X^T \in \mathcal{S}_{n\times m}$ such that $X^T_{j'\hat{i}} = 1$ and $X^T_{ki} = 0$ for all $k = 1, 2, \ldots, n$. By Lemma 3.17, we have that

$$X^T + e_j \cdot a_{[i,m]} + b^T_{[j',n]} \cdot e_{i'} \quad \sim \quad X^T + e_{ji} - e_{\hat{j}i} - a_i^{-1} \cdot e_{\hat{j}} \cdot a_{(i,m]} + b^T_{(j,n]} \cdot e_{i'}.$$

Hence,

$$X + a^T_{[i,m]} \cdot e_j + e_{i'} \cdot b_{[j',n]} \quad \sim \quad X + e_{ij} - e_{\hat{i}j} + a^T_{(i,m]} \cdot e_j - b_j^{-1} \cdot e_{\hat{i}} \cdot b_{(j,n]}.$$

□

Lemma 3.19. *Let $X \in \mathcal{S}^{m\times n}$ such that $X_{kj} = 0$ for all $k = 1, 2, \ldots, m$ and $X_{ik} = 0$ for all $k = 1, 2, \ldots, n$. Let $a \in \mathbb{F}_q^m$ and $b \in \mathbb{F}_q^n$ such that $a_i \neq 0$ and $b_j \neq 0$. If $a_i + b_j \neq 0$, then*

$$X + a^T_{[i,m]} \cdot e_j + e_i \cdot b_{[j,n]} \quad \sim \quad X + e_{ij} - (a_i + b_j)^{-1} \cdot a^T_{(i,m]} \cdot b_{(j,n]}.$$

Proof. For $k \in (i, m]$, we do the following row reductions

$$-(a_i + b_j)^{-1} a_k \times row_i + row_k.$$

Then,

$$X + a^T_{[i,m]} \cdot e_j + e_i \cdot b_{[j,n]} \quad \sim \quad X + a^T_{[i,m]} \cdot e_j + e_i \cdot b_{[j,n]} - (a_i + b_j)^{-1} \cdot a^T_{(i,m]} \cdot \big((a_i + b_j) \cdot e_j + b_{(j,n]}\big),$$

since $b_{[j,n]} = b_j \cdot e_j + b_{(j,n]}$. Note that

$$a^T_{[i,m]} \cdot e_j - (a_i + b_j)^{-1} \cdot a^T_{(i,m]} \cdot \big((a_i + b_j) \cdot e_j + b_{(j,n]}\big) = a_i \cdot e_j - (a_i + b_j)^{-1} \cdot a^T_{(i,m]} \cdot b_{(j,n]}$$

Hence,

$$X + a^T_{[i,m]} \cdot e_j + e_i \cdot b_{[j,n]} - (a_i + b_j)^{-1} \cdot a^T_{(i,m]} \cdot \big((a_i + b_j) \cdot e_j + b_{(j,n]}\big) = X + e_i \cdot b_{[j,n]} + a_i \cdot e_j - (a_i + b_j)^{-1} \cdot a^T_{(i,m]} \cdot b_{(j,n]}.$$

Next, we apply Lemma 3.5 ($i = i, j = j$). Then,

$$X + e_i \cdot b_{[j,n]} + a_i \cdot e_j - (a_i + b_j)^{-1} \cdot a^T_{(i,m]} \cdot b_{(j,n]} \quad \sim \quad X + e_{ij} - (a_i + b_j)^{-1} \cdot a^T_{(i,m]} \cdot b_{(j,n]}.$$

□

Remark. If $a_i + b_j = 0$, then

$$X + a^T_{[i,m]} \cdot e_j + e_i \cdot b_{[j,n]} \quad = \quad X + a^T_{(i,m]} \cdot e_j + e_i \cdot b_{(j,n]}.$$

(v) We choose $a_1 = b_1 = 1$. Hence,

$$a^T \cdot b = (e_1 + a^T_{(1,m]}) \cdot (e_1 + b_{(1,n]}) = e_{11} + e_1 \cdot b_{(1,n]} + a^T_{(1,m]} \cdot e_1 + a^T_{(1,m]} \cdot b_{(1,n]}.$$

For $k \in (1, m]$, we do the following row reductions

$$-a_k \times row_1 + row_k.$$

Thus,

$$X - a^T \cdot b \quad \sim \quad X - a^T \cdot b + a^T_{(1,m]} \cdot b_{(1,n]}$$

Notice that

$$\begin{aligned} X - a^T \cdot b + a^T_{(1,m]} \cdot b_{(1,n]} &= X - \Big(e_{11} + e_1 \cdot b_{(1,n]} + a^T_{(1,m]} \cdot e_1 + a^T_{(1,m]} \cdot b_{(1,n]}\Big) + a^T_{(1,m]} \cdot b_{(1,n]} \\ &= X - e_{11} - e_1 \cdot b_{(1,n]} - a^T_{(1,m]} \cdot e_1 \\ &= (X - e_{11}) + (-a^T_{(1,m]}) \cdot e_1 + e_1 \cdot (-b_{(1,n]}) \\ &= (X - e_{11}) + (-a^T_{(i_0,m]}) \cdot e_{j_0} + e_{i'_0} \cdot (-b_{(j'_0,n]}) \end{aligned}$$

Next, we choose a and b such that $a_{(i_0,m]} = a_{[i_1,m]}$ and $b_{(j'_0,m]} = b_{[j'_1,m]}$. Hence,

$$X - a^T \cdot b \quad \sim \quad (X - e_{11}) + (-a^T_{[i_1,m]}) \cdot e_{j_0} + e_{i'_0} \cdot (-b_{[j'_1,n]})$$

In particular, we write $a'_{[i_1,m]} = -a_{[i_1,m]}$ and $b'_{[j'_1,n]} = -b_{[j'_1,n]}$. Therefore,

$$X - a^T \cdot b \quad \sim \quad (X - e_{11}) + a'_{[i_1,m]} \cdot e_{j_0} + e_{i'_0} \cdot b'_{[j'_1,n]}$$

Note that $X_{i1j_1} = 1$ and $X_{i'_1 j'_1} = 1$. Apply one of the Lemma 3.15, where $(i, j, \hat{j}) = (i_1, j_0, j_1)$, and $(i', j', \hat{i}') = (i'_0, j'_1, i'_1)$. Thus,

$$\begin{aligned} (X - e_{11}) + &a'_{[i_1,m]} \cdot e_{j_0} + e_{i'_0} \cdot b'_{[j'_1,n]} \\ &\sim (X - e_{11}) + e_{i_1 j_0} - e_{i_1 j_1} + e_{i'_0 j'_1} - e_{i'_1 j'_1} + a'^T_{(i_1,m]} \cdot e_{j_1} + e_{i'_1} \cdot b'_{(j'_1,n]} \\ &= X - e_{11} + (e_{i_1 j_0} - e_{i_1 j_1}) + (e_{i'_0 j'_1} - e_{i'_1 j'_1}) + a'^T_{(i_1,m]} \cdot e_{j_1} + e_{i'_1} \cdot b'_{(j'_1,n]} \\ &= X - e_{11} + \sum_{k=1}^{1} (e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{1} (e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + a'^T_{(i_1,m]} \cdot e_{j_1} + e_{i'_1} \cdot b'_{(j'_1,n]} \end{aligned}$$

Note that we chose a' and b' so that $a'^T_{(i_1,m]} = a'^T_{[i_2,m]}$ and $b'_{(j'_1,n]} = b'_{[j'_2,n]}$. Hence,

$$\begin{aligned} (X - e_{11}) + &a'_{[i_1,m]} \cdot e_{j_0} + e_{i'_0} \cdot b'_{[j'_1,n]} \\ &\sim X - e_{11} + \sum_{k=1}^{1} (e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{1} (e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + a'^T_{[i_2,m]} \cdot e_{j_1} + e_{i'_1} \cdot b'_{[j'_2,n]} \end{aligned}$$

So, we can iterate the Lemma 3.15 until we have

$$\begin{aligned} (X - e_{11}) + &a'_{[i_1,m]} \cdot e_{j_0} + e_{i'_0} \cdot b'_{[j'_1,n]} \\ &\sim X - e_{11} + \sum_{k=1}^{l-1} (e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'-1} (e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + a'^T_{[i_l,m]} \cdot e_{j_{l-1}} + e_{i'_{l-1}} \cdot b'_{[j'_l,n]} \end{aligned}$$

We cannot apply the same Lemma for the last step, since $(i_l, j_l) = (i'_l, j'_l)$. That, however, suggests us to use the Lemma 3.16. Apply Lemma 3.16, where $(i, j, \hat{j}) = (i_l, j_{l-1}, j'_l)$ and $(i', j', \hat{i}') = (i'_{l-1}, j'_l, i_l)$. Then,

$$\begin{aligned}
X - e_{11} &+ \sum_{k=1}^{l-1}(e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'-1}(e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + a'^T_{[i_l,m]} \cdot e_{j_{l-1}} + e_{i'_{l-1}} \cdot b'_{[j'_l,n]} \\
&\sim X - e_{11} + \sum_{k=1}^{l-1}(e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'-1}(e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + e_{i_l j_{l-1}} + e_{i'_{l-1} j'_l} - e_{i_l j'_l} + a_{i_l}^{-1} b_{j'_l}^{-1} \cdot a'^T_{(i_l,m]} \cdot b'_{(j'_l,n]} \\
&= X - e_{11} + \sum_{k=1}^{l-1}(e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'-1}(e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + e_{i_l j_{l-1}} + e_{i'_{l-1} j'_l} - e_{st} + a_{i_l}^{-1} b_{j'_l}^{-1} \cdot a'^T_{(i_l,m]} \cdot b'_{(j'_l,n]} \\
&= X - e_{11} + \sum_{k=1}^{l}(e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'}(e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + e_{st} + a_s^{-1} b_t^{-1} \cdot a'^T_{(s,m]} \cdot b'_{(t,n]}
\end{aligned}$$

(vi) We choose $a_1 = b_1 = 1$. Note that $(i_0, j_0) = (i'_0, j'_0) = (1, 1)$. Hence, we can write

$$X + p \cdot a^T \cdot b = (X - e_{i_1 j_0} - e_{i_0 j_1}) + e_{i_1 j_0} + e_{i_0 j_1} + p \cdot a^T \cdot b.$$

Also, write $a = e_{i_0} + a_{(1,m]}$. Hence,

$$\begin{aligned}
(X - e_{i_1 j_0} - e_{i_0 j_1}) &+ e_{i_1 j_0} + e_{i_0 j_1} + p \cdot a^T \cdot b \\
&= (X - e_{i_1 j_0} - e_{i_0 j_1}) + e_{i_1 j_0} + e_{i_0 j_1} + p \cdot (e_{i_0} + a^T_{(1,m]}) \cdot b \\
&= (X - e_{i_1 j_0} - e_{i_0 j_1}) + e_{i_1 j_0} + e_{i_0 j_1} + p \cdot e_{i_0} \cdot b + p \cdot a^T_{(1,m]} \cdot b \\
&= (X - e_{i_1 j_0} - e_{i_0 j_1}) + e_{i_1 j_0} + e_{i_0} \cdot (p \cdot b + e_{j_1}) + p \cdot a^T_{(1,m]} \cdot b
\end{aligned}$$

For $k \in (1, m]$, we do the following row reductions

$$-a_k \times row_1 + row_k.$$

Hence,

$$\begin{aligned}
X + p{\cdot}a^T \cdot b& \\
&\sim (X - e_{i_1 j_0} - e_{i_0 j_1}) + e_{i_1 j_0} + e_{i_0} \cdot (p \cdot b + e_{j_1}) + p \cdot a^T_{(1,m]} \cdot b - a^T_{(1,m]} \cdot (p \cdot b + e_{j_1}) \\
&= (X - e_{i_1 j_0} - e_{i_0 j_1}) + e_{i_1 j_0} + e_{i_0} \cdot (p \cdot b + e_{j_1}) - a^T_{(1,m]} \cdot e_{j_1}
\end{aligned}$$

Next we do another row reduction

$$-p^{-1} \times row_1 + row_{i_1}.$$

$$\begin{aligned}
(X - e_{i_1 j_0} - e_{i_0 j_1}) &+ e_{i_1 j_0} + e_{i_0} \cdot (p \cdot b + e_{j_1}) - a^T_{(1,m]} \cdot e_{j_1} \\
&\sim (X - e_{i_1 j_0} - e_{i_0 j_1}) + e_{i_1 j_0} + e_{i_0} \cdot (p \cdot b + e_{j_1}) - a^T_{(1,m]} \cdot e_{j_1} - p^{-1} \cdot e_{i_1} \cdot (p \cdot b + e_{j_1}) \\
&= (X - e_{i_1 j_0} - e_{i_0 j_1}) + e_{i_1 j_0} + e_{i_0} \cdot (p \cdot b + e_{j_1}) - a^T_{(1,m]} \cdot e_{j_1} - e_{i_1} \cdot b - p^{-1} \cdot e_{i_1 j_1} \\
&= (X - e_{i_1 j_0} - e_{i_0 j_1}) - e_{i_1} \cdot (b - e_{j_0}) + e_{i_0} \cdot (p \cdot b + e_{j_1}) - a^T_{(1,m]} \cdot e_{j_1} - p^{-1} \cdot e_{i_1 j_1} \\
&= (X - e_{i_1 j_0} - e_{i_0 j_1}) - e_{i_1} \cdot b_{(1,n]} + e_{i_0} \cdot (p \cdot b + e_{j_1}) - a^T_{(1,m]} \cdot e_{j_1} - p^{-1} \cdot e_{i_1 j_1}
\end{aligned}$$

Next, we use lemma 3.5 ($i = 1, j = 1$). Thus,

$$\begin{aligned}(X - e_{i_1j_0} - e_{i_0j_1}) - e_{i_1} \cdot b_{(1,n]} + e_{i_0} \cdot (p \cdot b + e_{j_1}) - a^T_{(1,m]} \cdot e_{j_1} - p^{-1} \cdot e_{i_1j_1} \\ \sim (X - e_{i_1j_0} - e_{i_0j_1}) - e_{i_1} \cdot b_{(1,n]} + e_{i_0j_0} - a^T_{(1,m]} \cdot e_{j_1} - p^{-1} \cdot e_{i_1j_1} \\ = (X - e_{i_1j_0} - e_{i_0j_1}) + e_{i_0j_0} - e_{i_1} \cdot b_{(1,n]} - a^T_{(1,m]} \cdot e_{j_1} - p^{-1} \cdot e_{i_1j_1}\end{aligned}$$

Now, we write $b'_{(1,n]} = -(b_{(1,n]} + p^{-1} \cdot e_{j_1})$ and $a'_{(i_0,m]} = -a_{(i_0,m]}$. Hence,

$$\begin{aligned}(X - e_{i_1j_0} - e_{i_0j_1}) + e_{i_0j_0} + e_{i'_1} \cdot b_{(1,n]} + a'^T_{(1,m]} \cdot e_{j_1} - p^{-1} \cdot e_{i_1j_1} \\ = (X - e_{i_1j_0} - e_{i'_0j'_1}) + e_{11} + e_{i'_1} \cdot b'_{(1,n]} + a'^T_{(1,m]} \cdot e_{j_1} \\ = (X - e_{i_1j_0} - e_{i'_0j'_1}) + e_{11} + e_{i'_1} \cdot b'_{(j'_0,n]} + a'^T_{(i_0,m]} \cdot e_{j_1}\end{aligned}$$

Note that we choose a' and b' such that $b'_{(j'_0,n]} = b'_{[j'_1,n]}$ and $a'_{(i_0,m]} = a'_{[i_1,m]}$. Thus,

$$\begin{aligned}(X - e_{i_1j_0} - e_{i'_0j'_1}) + e_{11} + e_{i'_1} \cdot b'_{(j'_0,n]} + a'^T_{(i_0,m]} \cdot e_{j_1} \\ = (X - e_{i_1j_0} - e_{i'_0j'_1}) + e_{11} + e_{i'_1} \cdot b'_{[j'_1,n]} + a'^T_{[i_1,m]} \cdot e_{j_1} \\ = X - e_{11} - (e_{i_1j_0} - e_{i_0j_0}) - (e_{i'_0j'_1} - e_{i'_0j'_0}) + a'^T_{[i_1,m]} \cdot e_{j_1} + e_{i'_1} \cdot b'_{[j'_1,n]} \\ = X - e_{11} - \sum_{k=0}^{1-1}(e_{i_1j_0} - e_{i_0j_0}) - \sum_{k=0}^{1-1}(e_{i'_0j'_1} - e_{i'_0j'_0}) + a'^T_{[i_1,m]} \cdot e_{j_1} + e_{i'_1} \cdot b'_{[j'_1,n]}\end{aligned}$$

Similarly, as last theorem, we can iterate the Lemma 3.15. Thus,

$$\begin{aligned}X + p \cdot a^T \cdot b \\ \sim X - e_{11} - \sum_{k=0}^{l-2}(e_{i_1j_0} - e_{i_0j_0}) - \sum_{k=0}^{l-2}(e_{i'_0j'_1} - e_{i'_0j'_0}) + a'^T_{[i_{l-1},m]} \cdot e_{j_{l'-1}} + e_{i'_{l'-1}} \cdot b'_{[j'_{l'-1},n]}.\end{aligned}$$

Lastly, we apply the Lemma 3.16 since the paths meet at the end. Thus,

$$\begin{aligned}X + p \cdot a^T \cdot b \\ \sim X - e_{11} - \sum_{k=0}^{l-1}\left(e_{i_{k+1}j_k} - e_{i_kj_k}\right) - \sum_{k'=0}^{l'-1}\left(e_{i_{k'}j'_{k'+1}} - e_{i'_{k'}j'_{k'}}\right) + e_{st} + a_s^{-1}b_t^{-1} \cdot a'^T_{(s,m]} \cdot b'_{(t,n]}.\end{aligned}$$

(vii) This is similar to *(v)* except for the last step. Since we do not require the two zig-zag paths to meet,

so we can make some changes

$$
\begin{aligned}
X - e_{11} &+ \sum_{k=1}^{l-1}(e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'-1}(e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + a'^{T}_{[i_l,m]} \cdot e_{j_{l-1}} + e_{i'_{l-1}} \cdot b'_{[j'_l,n]} \\
&\sim X - e_{11} + \sum_{k=1}^{l-1}(e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'-1}(e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + e_{i_l j_{l-1}} + e_{i'_{l-1} j'_l} - e_{i_l j'_l} + a_{i_l}^{-1} b_{j'_l}^{-1} \cdot a'^{T}_{(i_l,m]} \cdot b'_{(j'_l,n]} \\
&= X - e_{11} + \sum_{k=1}^{l-1}(e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'-1}(e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}}) + e_{i_l j_{l-1}} + e_{i'_{l-1} j'_l} - e_{st} + a_{i_l}^{-1} b_{j'_l}^{-1} \cdot a'^{T}_{(i_l,m]} \cdot b'_{(j'_l,n]} \\
&= X - e_{11} + \sum_{k=1}^{l}(e_{i_k j_{k-1}} - e_{i_k j_k}) + \sum_{k'=1}^{l'}(e_{i'_{k'-1} j'_{k'}} - e_{i'_{k'} j'_{k'}})
\end{aligned}
$$

According tot the computation in each proof, we can see that those are the only possibilities since they are forced by the algebra. □

4 Conclusion

So far we have considered all the possibilities of going from one state (superclass) to another state (superclass.) Next we possibly can actually count the number of elements that can take one from one state to another state. Hence, we can employ the idea in probability to compute the probability of each edge in the chain. Hence, we can apply the concepts in Markov chain.