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A Galois Correspondence with Generalized Covering Spaces

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A Galois Correspondence with Generalized Covering Spaces

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1. INTRODUCTION

The fundamental idea of algebraic topology is to convert problems about topological spaces into problems about associated algebraic objects. This is typically accomplished by assigning to each topological space X an algebraic object $F(X)$, in many examples $F(X)$ is a group. More importantly however, for each continuous map $f: X \rightarrow Y$ of topological spaces, we also associate a homomorphism $F(X) \rightarrow F(Y)$. It is also important that we make these assignments in a way that respects function composition.

The particular example we are interested in is the fundamental group, denoted π_1 . For any space X and point $x \in X$, there is an associated group $\pi_1(X, x)$. Informally, π_1 records informations about holes in X . We will discuss the specific construction later, but for now we show how it can be used to solve a topological problem.

Consider the topological space $D^2 \subseteq \mathbf{R}^2$ consisting of all points in \mathbf{R}^2 with distance less than or equal to 1 from the origin, and the subspace $W = D^2 - \{(0, 0)\}$.

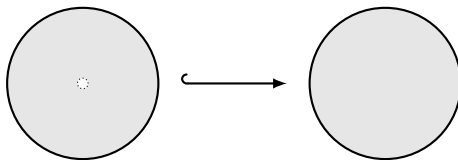


FIGURE 1. The spaces W and D^2 with the natural inclusion.

A natural question to ask is whether the spaces D^2 and W are homeomorphic, that is, if there is a continuous bijection $W \rightarrow D^2$ with continuous inverse. Our intuition says that they should not be, since W has a hole in it but D^2 does not. However, the topological problem of showing outright that there are no homeomorphisms $W \rightarrow D^2$ is difficult. Instead we use π_1 to translate this problem into one about groups. In particular, whenever $f: X \rightarrow Y$ is a homeomorphism and $x \in X$, then the associated map of groups $\pi_1(X, x) \rightarrow \pi_1(Y, f(x))$ is an isomorphism. In the case above, it can be shown that if $d \in D^2$ and $w \in W$ then $\pi_1(D^2, d)$ is the trivial group and $\pi_1(W, w)$ is isomorphic to the

additive group of integers. Since these groups are not isomorphic, it tells us that D^2 and W are not homeomorphic.

There are a few things that must be discussed before we can define π_1 . Throughout the following, we let $I = [0, 1]$ be the set of real numbers between (and including) 0 and 1. Then for a space X a path γ in X is a continuous map $\gamma: I \rightarrow X$. There is a fundamental equivalence relation on paths called homotopy. If γ, β are paths in X , we say that γ and β are homotopic if there exists a continuous map $h: I \times I \rightarrow X$ such that $h(0, t) = \gamma(t)$ and $h(1, t) = \beta(t)$ for all $t \in I$. A homotopy is really just a way of continuously deforming γ into β . Given a point $x \in X$, we define the set of loops in X based at x as

$$\Omega(X, x) = \{\gamma: \gamma \text{ is a path in } X \text{ and } \gamma(0) = \gamma(1) = x\}.$$

We form an equivalence relation \sim on $\Omega(X, x)$ by saying $\alpha \sim \beta$ if there is an endpoint preserving homotopy¹ between α and β . Now if X is a space with a distinguished point $x \in X$ then as a set $\pi_1(X, x) = \Omega(X, x) / \sim$, so a typical element in $\pi_1(X, x)$ is an equivalence class

$$[\gamma] = \{\alpha \in \Omega(X, x): \alpha \text{ and } \gamma \text{ are homotopic}\}.$$

In order to be a group, there needs to be multiplication and inversion maps. If $\alpha \in \Omega(X, x)$ we obtain a new loop $\bar{\alpha}$ defined by $t \mapsto \alpha(1 - t)$, and if β is another loop in $\Omega(X, x)$ we define the loop $\alpha * \beta$ by

$$\alpha * \beta(t) = \begin{cases} \beta(2t), & \text{if } t \in [0, \frac{1}{2}] \\ \alpha(2t - 1), & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

We now define maps $\mu: \pi_1(X, x) \times \pi_1(X, x) \rightarrow \pi_1(X, x)$ and $\iota: \pi_1(X, x) \rightarrow \pi_1(X, x)$ by $\mu([\alpha], [\beta]) = [\alpha * \beta]$ and $\iota([\alpha]) = [\bar{\alpha}]$. Of course the way these maps are defined it is not obvious they are well defined. For example, if α, β are representatives of the same equivalence class of loops (so that $[\alpha] = [\beta]$) it is necessary to check that $\bar{\alpha}$ and $\bar{\beta}$ represent the same class of loops in order to ensure ι is well defined. The maps μ, ι are well defined and make $\pi_1(X, x)$ into a group, but we do not prove this here. The identity element is the equivalence class $[c_x]$, where c_x is the constant loop at x defined by $c_x(t) = x$ for all $t \in I$. A loop in the same equivalence class of c_x is called nullhomotopic.

All of the above construction requires a distinguished choice of base point. We say that a pair (X, x) where X is a space and $x \in X$ is a based space. If (Y, y) is another based space, then a map of based spaces $f: (X, x) \rightarrow (Y, y)$ is a continuous map $f: X \rightarrow Y$ such that $f(x) = y$. In the above paragraph, we associated a group $\pi_1(X, x)$ to each based space, but we still need a homomorphism of groups $f_*: \pi_1(X, x) \rightarrow \pi_1(Y, y)$ for each map of based spaces $f: (X, x) \rightarrow (Y, y)$. Note that if γ is a loop in X based at x , then the composition

$$I \xrightarrow{\gamma} X \xrightarrow{f} Y$$

is continuous because γ and f are. It is a loop based at y because $\gamma(1) = \gamma(0) = x$ and f sends x to y . This suggests that we should define $f_*([\gamma]) = [f \circ \gamma]$. Again it should be checked that this is well defined and that it is a group homomorphism, which it is.

¹A homotopy $h: I \times I \rightarrow X$ between two loops $\alpha, \beta \in \Omega(X, x)$ is called endpoint preserving if $h(t, 0) = h(t, 1) = x$ for all $t \in I$. This will be a necessary condition for our discussion of covering spaces.

The last thing we need to know in order to translate from topology to algebra is that the assignment of maps respects composition. Suppose we have maps of based spaces $f: (X, x) \rightarrow (Y, y)$ and $g: (Y, y) \rightarrow (Z, z)$. The composition $g \circ f$ is a map $(X, x) \rightarrow (Z, z)$. This situation can be represented in the following commutative diagram.

$$\begin{array}{ccc} (X, x) & \xrightarrow{f} & (Y, y) \\ & \searrow^{g \circ f} & \swarrow_g \\ & & (Z, z) \end{array}$$

Now a priori there are two maps $\pi_1(X, x) \rightarrow \pi_1(Z, z)$. The first is $(g \circ f)_*$ and the second is the composition $g_* \circ f_*$. In order to respect composition, it should hold that $g_* \circ f_* = (g \circ f)_*$. But from the definition of f_*, g_*

$$g_* \circ f_*([\gamma]) = g_*([f \circ \gamma]) = [g \circ f \circ \gamma] = (g \circ f)_*([\gamma]).$$

So π_1 respects composition.

We have sorted out almost everything we need to show that D^2 and W are not homeomorphic, except for the actual computation of the fundamental groups. Let $d = (1, 0) \in D^2$ and let γ be a loop based at d . Define $h: I \times I \rightarrow D^2$ by $h(s, t) = s\gamma(t) + (1 - s)d$. Then h is a homotopy between γ and c_d , the constant loop at d . So every loop in D^2 is nullhomotopic and consequently $\pi_1(D^2, d) = \{[c_d]\}$, the trivial group. Let $w = (1, 0) \in W$ so that the inclusion $(W, w) \rightarrow (D^2, d)$ is a map of based spaces. All that is left now is to show that $\pi_1(W, w)$ is isomorphic to \mathbf{Z} , the additive group of integers. Unfortunately this is a difficult computation using only the above definitions. For example, let $\gamma: I \rightarrow W$ be the loop defined by $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. Pictorially, γ traverses the border of W once counter clockwise and intuitively γ should not be a nullhomotopic loop, since any continuous deformation of γ into c_w would have to cross the point $(0, 0)$ which is not in W . However it is difficult to show that there exist no possible homotopy between γ and c_w .

In order to complete this computation we need to develop some more machinery. One possible way to study groups is to look at their actions on sets. A (left) action of a group G on a set S is a map $\mu: G \times S \rightarrow S$ that respects the group structure. If we write $g \cdot s$ for $\mu(g, s)$ then by respecting the group structure, we mean that for all $s \in S$ and $g, h \in G$

$$e \cdot s = s \quad \text{and} \quad g \cdot (h \cdot s) = (gh) \cdot s,$$

where $e \in G$ is the identity element. A G -set is a set S with an action of G on it. Elements $g, h \in G$ can be distinguished by producing a G -set S and an element $s \in S$ such that $g \cdot s \neq h \cdot s$. This is how we will be able to tell the loops $\gamma, c_w \in \pi_1(W, w)$ apart.

Now the trick is to come up with non trivial $\pi_1(W, w)$ -sets, and this is where covering spaces come in. Informally, a covering space of a space X should be thought of as a space which unwinds loops in X .

Definition 1. (*Covering spaces*) Let X be a space. A covering space of X is a pair (Z, p) where Z is a topological space and $p: Z \rightarrow X$ is a continuous map that is locally trivial with discrete fiber.

The last condition means that for every $x \in X$ there is a neighborhood U of x so that $p^{-1}(U)$ is isomorphic to $U \times F$ for some discrete set F . Such a set U is often called an evenly covered neighborhood, and F is called the fiber. Pictorially, a covering space looks like this:

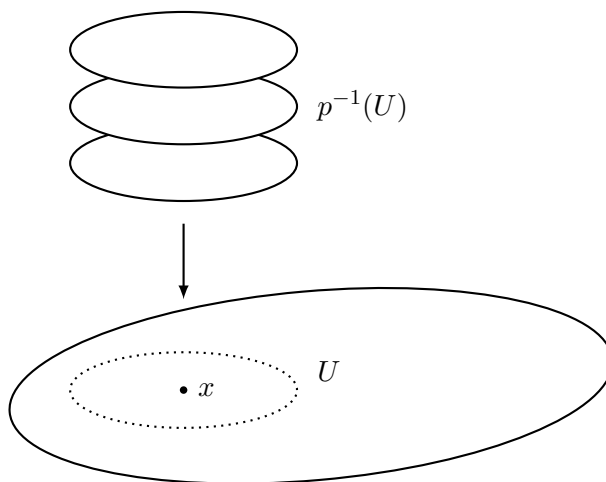


FIGURE 2. Covering spaces are like stacks of pancakes.

If $p: Z \rightarrow X$ is a covering space there are two particularly useful properties that p satisfies. The first is that p is a local homeomorphism.

Definition 2. (*Local homeomorphism*) A continuous map $p: Z \rightarrow X$ is a local homeomorphism if for all $z \in Z$ there is a neighborhood V of z such that $p|_V: V \rightarrow p(V)$ is a homeomorphism. Local homeomorphisms are also sometimes called étale maps.

A covering space $p: Z \rightarrow X$ is a local homeomorphism since for each $z \in Z$ we can find an evenly covered neighborhood U of $p(z)$ and the component of $p^{-1}(U) \simeq U \times F$ containing z projects isomorphically onto U . The second useful thing is that p has unique lifting of paths and homotopies of paths.

Definition 3. (*Unique lifting of paths and homotopies*) A map $p: Z \rightarrow X$ is said to have unique lifting of paths if for every path $\gamma: I \rightarrow X$ with $\gamma(0) = x$ and every point $z \in p^{-1}(x)$ there exists a unique path $\tilde{\gamma}: I \rightarrow Z$ such that $\tilde{\gamma}(0) = z$ and $p \circ \tilde{\gamma} = \gamma$. The map p is said to have unique lifting of homotopies of paths if for every homotopy $h: I \times I \rightarrow X$ and every continuous map $\tilde{h}_0: \{0\} \times I \rightarrow Z$ such that $p \circ \tilde{h}_0 = h|_{\{0\} \times I}$ there is a unique continuous map $\tilde{h}: I \times I \rightarrow Z$ such that $\tilde{h}|_{\{0\} \times I} = \tilde{h}_0$ and $p \circ \tilde{h} = h$.

This path lifting property can be summarized by saying whenever there is a commutative diagram of solid lines as below, there is a unique dashed line ($\tilde{\gamma}$) making the diagram commute.

$$\begin{array}{ccc}
\{0\} & \xrightarrow{\quad} & Z \\
\downarrow & \nearrow \exists! & \downarrow p \\
I & \xrightarrow{\quad \gamma \quad} & X
\end{array}$$

Unique lifting of homotopies of paths is essentially the same, except we start with a homotopy $h: I \times I \rightarrow X$ and a lift of $h|_{\{0\} \times I}$ and produce a homotopy $\tilde{h}: I \times I \rightarrow Z$ with $p \circ \tilde{h} = h$. To represent it diagrammatically we use the same diagram as above, replacing $\{0\}$ with $\{0\} \times I$ and I with $I \times I$.

All of this allows us to prove the following lemma, which we can use to produce non trivial $\pi_1(X, x)$ -sets. First though, if $p: Z \rightarrow X$ is a covering space, then for any class $[\alpha] \in \pi_1(X, x)$ and $s \in p^{-1}(x)$, we may produce a lift $\tilde{\alpha}$ of α with $\tilde{\alpha}(0) = s$ by unique lifting of paths. Note that $\tilde{\alpha}(1) \in p^{-1}(x)$ since $p \circ \tilde{\alpha}(1) = \alpha(1) = x$.

Lemma 1.1. (*Monodromy action*) *Let $p: Z \rightarrow X$ be a covering space and let $S = p^{-1}(x)$. Then the map $\mu: \pi_1(X, x) \times S \rightarrow S$ defined by $([\alpha], s) \mapsto \tilde{\alpha}(1)$ is well defined and a group action of $\pi_1(X, x)$ on S .*

Proof. We must check that this action is well defined. Suppose $\beta \in [\alpha]$ is another representative with lift $\tilde{\beta}$ starting at s . We must show $\tilde{\alpha}(1) = \tilde{\beta}(1)$. Let $h: I \times I \rightarrow X$ be an endpoint preserving homotopy between the two. Now $\tilde{\alpha}$ is a lift of $h|_{\{0\} \times I}$ and by unique lifting of homotopies of paths, there is a lift \tilde{h} such that $p \circ \tilde{h} = h$ and $h|_{\{0\} \times I} = \tilde{\alpha}$. Let $\gamma: I \rightarrow Z$ be the path given by $\tilde{h}|_{\{1\} \times I}$ so that $p \circ \gamma = \beta$. Since h is endpoint preserving, $\tilde{h}|_{I \times \{0\}}$ is a continuous map from a connected space into $p^{-1}(x)$, a discrete space, and consequently is constant. Thus

$$\gamma(0) = \tilde{h}(1, 0) = \tilde{h}(0, 0) = \tilde{\alpha}(0) = s.$$

This shows that γ is a lift of β starting at s and by uniqueness of lifts, $\gamma = \tilde{\beta}$. All that is left to show is that $\gamma(1) = \tilde{\alpha}(1)$. Again since h is endpoint preserving, $\tilde{h}|_{I \times \{1\}}$ is a continuous map from a connected space into the discrete space $p^{-1}(x)$, so it is constant. Thus

$$\gamma(1) = \tilde{h}(1, 1) = \tilde{h}(0, 1) = \tilde{\alpha}(1).$$

All this shows that $\mu: \pi_1(X, x) \times S \rightarrow S$ is well defined. It is fairly easy to check that this is a group action as well, so we omit the details. \square

The above action is called the monodromy action, and we write $[\alpha] \cdot s$ for $\mu([\alpha], s)$. Getting back to our original example, we would like to compute $\pi_1(W, w)$. Consider the map $f: \mathbf{R} \times (0, 1] \rightarrow W$ defined by $(t, s) \mapsto (s \cos(2\pi t), s \sin(2\pi t))$. This is continuous and is in fact a covering space.

This is an example of a non trivial covering space. With $w = (1, 0)$ we see that the fiber $f^{-1}(w)$ is $\mathbf{Z} \times \{1\}$. Recall that we wanted to show that the class $[c_w]$ of the constant path at w and the class $[\gamma]$ of the loop $\gamma: I \rightarrow W$ given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$ are different. Consider the monodromy action of $[c_w]$ and $[\gamma]$ on $0 \in f^{-1}(w)$. We must lift c_w and γ to paths in $\mathbf{R} \times (0, 1]$ starting at $(0, 1)$. It is easy to see that \tilde{c}_w is the constant

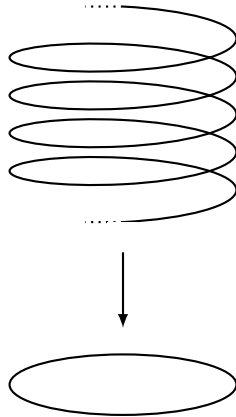


FIGURE 3. The portion of the map $f: \mathbf{R} \times (0, 1] \rightarrow W$ that is above S^1 .

loop at 0 hence $[c_w] \cdot 0 = 0$. Similarly, we see that $\tilde{\gamma}: I \rightarrow \mathbf{R} \times (0, 1]$ is just the inclusion $I \rightarrow I \times \{1\} \subseteq \mathbf{R} \times (0, 1]$ and consequently $[\gamma] \cdot 0 = \tilde{\gamma}(1) = (1, 1)$. Since $(1, 1) \neq (0, 1)$, we finally know that $[\gamma] \neq [c_w]$, which means $\pi_1(W, w)$ is not trivial! We can now confidently say the punctured disk is not the same (topologically) as the disk, since $\pi_1(W, w)$ is not trivial but $\pi_1(D^2, d)$ is.

Earlier, we claimed that $\pi_1(W, w) \simeq \mathbf{Z}$. It is not hard to see that there is a subgroup of $\pi_1(W, w)$ that is isomorphic to \mathbf{Z} by looking at the loops γ_n for $n \in \mathbf{Z}$ defined by $\gamma_n(t) = (\cos(2\pi nt), \sin(2\pi nt))$ and how they act on the fiber of f . In effect, this gives an injective group homomorphism $g: \mathbf{Z} \rightarrow \pi_1(W, w)$. To finish our computation, we show that g is bijective. This will be accomplished by showing every loop is homotopic to some γ_n . Let $[\alpha] \in \pi_1(W, w)$ and let $\tilde{\alpha}$ be a path lift of α in $\mathbf{R} \times (0, 1]$ starting at $(0, 1)$. Since it is a lift, $\tilde{\alpha}(1) = (n, 1)$ for some $n \in \mathbf{Z}$. Let $\tilde{\gamma}_n$ be a lift of γ_n to $\mathbf{R} \times (0, 1]$ starting at $(0, 1)$. Then $\tilde{\gamma}_n(1) = (n, 1)$ and hence $\tilde{\gamma}_n * \tilde{\alpha}$ is a loop in $\mathbf{R} \times (0, 1]$. However, any loop β in $\mathbf{R} \times (0, 1]$ is homotopic to the constant loop. We omit the proof of this as it is almost the same as the proof of analogous statement for D^2 . Consequently, there is a homotopy between $h: I \times I \rightarrow W$ between $\tilde{\gamma}_n * \tilde{\alpha}$ and the constant loop at $(0, 1)$. It follows that $f \circ h$ is a homotopy between $\gamma_n * \alpha$ and the constant loop c_w , i.e. $[c_w] = [\gamma_n * \alpha] = [\gamma_n] * [\alpha]$. Consequently $[\gamma_n] = [\alpha]$ as required.

All of this shows that $g: \mathbf{Z} \rightarrow \pi_1(W, w)$ is an isomorphism, which finishes the computation. The key fact in the above proof is that every loop in our covering space $\mathbf{R} \times (0, 1]$ is homotopic to the constant loop, i.e. that $\mathbf{R} \times (0, 1]$ is simply connected. We recall the definition.

Definition 4. (*Simply connected and universal covers*) A path connected space X is said to be simply connected if $\pi_1(X, x)$ is trivial for $x \in X$. A covering space $Z \rightarrow X$ is said to be universal if Z is simply connected.

As we can see from the computation of $\pi_1(W, w)$, the existence of a universal cover is extremely useful. There is a close connection between covering spaces and the fundamental group for spaces with universal covers, which we see in the following theorem. We will not

show this, but the proof that $\pi_1(W, w) = \mathbf{Z}$ can be modified to prove the difficult part of this theorem.

Theorem 1.1. (*Correspondence theorem*) *If X is connected, locally path connected and has a universal cover, then for any base point $x \in X$, the following sets are equivalent.*

- (1) $\{\text{Connected covering spaces of } X\}/\text{Isomorphism over } X$,
- (2) $\{\text{Transitive } \pi_1(X, x)\text{-sets}\}/\text{Isomorphism}$,
- (3) $\{\text{Subgroups of } \pi_1(X, x)\}/\text{Conjugation}$.

Moreover, this identification respects maps between the objects.

We have not introduced technical definitions to make the above theorem precise. If we use more abstract language, there is a clean and precise way of stating the previous theorem.

Theorem 1.2. (*Correspondence theorem, version 2*) *If X is connected, locally path connected and has a universal cover, then for any base point $x \in X$, there is an equivalence of categories*

$$\pi_1(X, x)\text{-Sets} \simeq \mathbf{Cov}(X),$$

where $\mathbf{Cov}(X)$ is the category of covering spaces over X .

Note that theorem 1.1 gives a way to compute the fundamental group. The theorem ensures there is a covering space $\tilde{X} \rightarrow X$ corresponding to the $\pi_1(X, x)$ -set $\pi_1(X, x)$. By the correspondence theorem, the group $\text{Aut}(\pi_1(X, x))$ of automorphisms of $\pi_1(X, x)$ as a left $\pi_1(X, x)$ -set is isomorphic to $\text{Aut}(\tilde{X})$, the group of automorphisms of \tilde{X} over X . The following lemma shows that $\text{Aut}(\pi_1(X, x))$ is isomorphic to $\pi_1(X, x)$.

Lemma 1.2. *Let G be a group. Then the group of automorphisms $\text{Aut}(G)$ of G as a left G -set is isomorphic to G .*

Proof. For any $g \in G$, define an automorphism $\varphi_g: G \rightarrow G$ given by $\varphi_g(h) = hg$. This gives an injective group homomorphism $G \rightarrow \text{Aut}(G)$, which we must show is surjective. Let $\psi: G \rightarrow G$ be an automorphism, and let $g = \psi(e)$, where $e \in G$ is the identity. Then for any $h \in G$ we have $\psi(h) = h\varphi(e) = hg$. Hence $\psi = \varphi_g$. \square

The lemma along with the correspondence theorem give the following isomorphisms

$$\pi_1(X, x) \simeq \text{Aut}(\pi_1(X, x)) \simeq \text{Aut}(\tilde{X}).$$

This is very useful because $\text{Aut}(\tilde{X})$ is simpler to compute. All of these results rely on X having a universal cover. In fact for a space X that is connected and locally path connected, the correspondence theorem holds if and only if X has a universal cover. The natural question to ask is when does a space X have a universal cover? This leads to the topological notion of semilocally simply connectedness.

Definition 5. (*Semilocally simply connected*) A space X is said to be semilocally simply connected if for every point $x \in X$, there is a neighborhood U of x such that if $\alpha: I \rightarrow X$ is a loop based at x with image contained in U , then α is homotopic to the constant loop at x .

Roughly this means there are not arbitrarily small holes. In the case that X is connected and locally path connected, X has a universal cover if and only if X is semilocally simply connected. It follows that for a space X , the correspondence theorem is true if and only if X is connected, locally path connected and semilocally simply connected.

Our paper seeks to extend the above correspondence theorem to spaces which are connected and locally path connected, but not necessarily semilocally simply connected. There are a few problems in trying to do this. In the non semilocally simply connected case, covers are not well behaved. For example, if X is not semilocally simply connected, it is possible to have covering spaces $f: Z \rightarrow Y$ and $g: Y \rightarrow X$ such that $g \circ f$ is not a covering space. In order to fix this, it is necessary to introduce a generalized notion of covering spaces, called *semicovers*. In [3], Brazas defines *semicovers*. We also introduce a notion of *semicovers* (which are defined differently than by Brazas) in section 2, and corollary 2.1 shows that the two notions are equivalent.

To see further obstructions to a generalized version of the correspondence theorem, we should look at how we get the subgroup associated to a covering space. If $p: Z \rightarrow X$ is a covering space (or more generally a *semicover*) and $x \in X$ is a basepoint, then for any $z \in p^{-1}(x)$ the induced map $p_*: \pi_1(Z, z) \rightarrow \pi_1(X, x)$ is injective. Thus the corresponding subgroup of $\pi_1(X, x)$ is $\text{im}(p_*)$. For the correspondence theorem to hold for X , we need some cover $p: \tilde{X} \rightarrow X$ whose corresponding subgroup is the trivial subgroup. This would imply that \tilde{X} is simply connected, i.e. that \tilde{X} is a universal cover. But in the non semilocally simply connected case, this cannot happen.

Therefore to generalize the results of the correspondence theorem, we need to change something about $\pi_1(X, x)$. One way of doing this is to add a topology to $\pi_1(X, x)$ so that *semicovers* correspond to open subgroups of $\pi_1(X, x)$. This is very similar to Galois theory for finite versus infinite extensions. In the case of finite extensions, intermediate extensions correspond to subgroups of the Galois group. For infinite extensions, it is necessary to put a topology on the Galois group, and then intermediate extensions correspond to closed subgroups of the Galois group. This approach is taken by Brazas in [3] where he introduces a topological group $\pi_1^{\text{Gal}}(X, x)$ whose underlying group is $\pi_1(X, x)$. The generalized correspondence theorem he proves is Theorem 7.19 in [3].

The other approach, which we take, is to change $\pi_1(X, x)$. We define a new group $\pi_1^{\text{Gal}}(X, x)$ called the Galois fundamental group for any connected and locally path connected based space (X, x) . Instead of looking at loops, $\pi_1^{\text{Gal}}(X, x)$ is defined in terms of automorphisms of generalized covers. The group naturally carries a topology making it into a topological group. The first main result is theorem 4.1, which states the following.

Theorem. *If X is a space which is connected and locally path connected, then there is an equivalence of categories*

$$\pi_1^{\text{Gal}}(X, x)\text{-Sets} \simeq \mathbf{SCov}(X),$$

where $\mathbf{SCov}(X)$ is the category of *semicovers* over X and $\pi_1^{\text{Gal}}(X, x)\text{-Sets}$ is the category of discrete sets with a continuous left action of $\pi_1^{\text{Gal}}(X, x)$.

The other main result we prove relates the two groups $\pi_1^{\text{Gal}}(X, x)$ and $\pi_1(X, x)$. We produce a topological group² $\pi_1^\sigma(X, x)$ whose underlying set is $\pi_1(X, x)$, and theorem 7.1 relates π_1^σ and π_1^{Gal} in the following way.

Theorem. *The completion $\pi_1^\sigma(X, x)^*$ of $\pi_1^\sigma(X, x)$ with respect to the two sided uniformity is isomorphic to $\pi_1^{\text{Gal}}(X, x)$.*

We will review basics of uniform spaces and completions of uniform spaces necessary for this theorem in section 3.

This paper can roughly be divided into two halves. The first half consists of sections 2-5 where we construct the Galois fundamental group and show it is functorial. Particularly, in section 2 we introduce semicovers and some basic facts about them. Section 3 is a review of uniform spaces, where we include the results used later in the paper. The Galois fundamental group is then defined in section 4 in terms of infinite Galois theories, which Bhatt and Scholze introduce in [1, Definition 7.2.1]. In this section we recall the definition of an infinite Galois theory, which consists of a category and a functor to the category of sets that satisfies certain axioms. We associate to each based space (X, x) the category $\mathbf{SCov}(X)$ of semicovers over X and the functor $i^*: \mathbf{SCov}(X) \rightarrow \mathbf{Sets}$ which takes a semicover $p: Z \rightarrow X$ to the fiber $p^{-1}(x)$. The main result of this section is that $(\mathbf{SCov}(X), i^*)$ is a tame infinite Galois theory if X is connected and locally path connected. This allows us to define $\pi_1^{\text{Gal}}(X, x)$ as the automorphisms of i^* , and theorem 4.1 is then a consequence of theorem 7.25 of [1]. In section 5 we show π_1^{Gal} is a functor from based spaces to uniform groups.

In the second half of the paper, we try and relate the Galois fundamental group and the usual fundamental group. Section 6 shows when X has a universal cover that $\pi_1^{\text{Gal}}(X, x)$ and $\pi_1(X, x)$ are isomorphic groups. The more general relationship is based on π_1^τ as defined by Brazas in [3]. We recall the construction of π_1^τ in section 7 and then prove that the Galois fundamental group is the completion of the usual fundamental group (theorem (7.1)). Sections 8 - 10 look at two specific examples of spaces which are not semilocally simply connected and examine the fundamental group and Galois fundamental group of each. The harmonic archipelago is introduced in 9 and we show that the Galois fundamental group is trivial, showing that in general the Galois fundamental group and the usual fundamental group are not isomorphic.

I would like to thank my adviser Jonathan Wise, who came up with this project. This thesis would certainly not have been possible without his guidance and many helpful conversations.

2. GENERALIZED COVERING SPACES

In the remainder of the paper, all spaces will be assumed to be locally path connected and connected.

To witness some bad behavior of covering spaces, it is necessary to consider spaces which are not semilocally simply connected. The simplest example of such a space is the Hawaiian earring, denoted \mathbf{E} . It is constructed as a subset of \mathbf{R}^2 by taking the union of the circles C_n , where C_n is the circle of radius $1/n$ centered at $(0, 1/n)$.

²The topology is obtained from open subgroups of $\pi_1^\tau(X, x)$, the group introduced by Brazas.

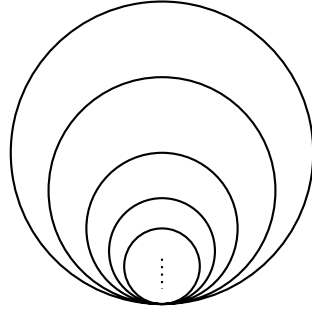


FIGURE 4. The Hawaiian Earring

The Hawaiian earring is not simply connected because every neighborhood of the origin $(0,0)$ contains infinitely many of the C_n and hence has many loops which are not nullhomotopic. Example 3.8 of [3] gives an example of a covering space $Y \rightarrow \mathbf{E}$ and a covering space $Z \rightarrow Y$ so that the composition $Z \rightarrow \mathbf{E}$ is not a covering space. Intuitively, it seems a cover of a cover should still be a cover, which suggests that the definition of a covering space is not always the best. One reason covering spaces are so useful is they provide geometric objects for the fundamental group to act on. We would like the results of lemma 1.1 to hold for generalized covers, and part of this action required being able to lift paths and homotopies of paths.

Definition 6. (*Unique homotopy lifting property*) A map $p: Z \rightarrow X$ satisfies the unique homotopy lifting property with respect to a class of spaces \mathcal{T} if given a map $f: Y \times I \rightarrow X$ where $Y \in \mathcal{T}$ and a lift $\tilde{f}_0: Y \times \{0\} \rightarrow Z$ such that $p \circ \tilde{f}_0 = f|_{Y \times \{0\}}$, there is a unique map $\tilde{f}: Y \times I \rightarrow Z$ such that $\tilde{f}|_{Y \times \{0\}} = \tilde{f}_0$ and $p \circ \tilde{f} = f$.

Again, the picture is the following, where Y is in \mathcal{T} .

$$\begin{array}{ccc}
 Y \times \{0\} & \xrightarrow{\quad} & Z \\
 \downarrow & \nearrow \exists! & \downarrow p \\
 Y \times I & \xrightarrow{\quad} & X
 \end{array}$$

Unique lifting of paths and homotopies of paths corresponds to the unique homotopy lifting property with respect to the class $\mathcal{T} = \{I^0, I\}$, where $I^0 = \{0\}$. The other fact we used was that covering spaces are also local homeomorphisms, hence have discrete fibers. It is evident from the proof of lemma 1.1 that for any space satisfying these two properties, the monodromy action will be well defined. It turns out that for local homeomorphisms, it is enough to satisfy the unique homotopy lifting property with respect to the class $\mathcal{T} = \{I^0\}$. This leads to the following definition.

Definition 7. (*Semicovers*) A map $p: Z \rightarrow X$ is called a semicovering map if it is a local homeomorphism and it satisfies the unique homotopy lifting property with respect to $\mathcal{T} = \{I^0\}$, where again I^0 is a single point space. A semicover of a space X is a pair

(Z, p) where $p: Z \rightarrow X$ is a semicovering map. A morphism $(Z, p) \rightarrow (Y, q)$ of semicovers over X consists of a continuous map $f: Z \rightarrow Y$ such that $p = q \circ f$, i.e. that makes the following diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & & X \end{array}$$

Semicovers over X and morphisms of semicovers over X form a category, denoted $\mathbf{SCov}(X)$.

By abuse of notation, we say that Z is a semicover when p is understood. Here is the more general monodromy lemma, whose proof is exactly the same as the previous monodromy lemma.

Proposition 2.1. (*Monodromy*) *Let $p: Z \rightarrow X$ be local homeomorphism that satisfies the unique homotopy lifting property with respect to $\mathcal{T} = \{I^0, I\}$, where I^0 is a single point space and $I = [0, 1]$ is the unit interval. Pick $x \in X$ and let $S = p^{-1}(x)$. Then there is a well defined action of $\pi_1(X, x)$ on S given by*

$$[\gamma].s = \tilde{\gamma}(1),$$

where $\tilde{\gamma}$ is a lift of γ starting at s .

In [3, Definition 3.1], Brazas defines semicovers as well. We will show definitions are equivalent, after we recall some necessary definitions.

Definition 8. If X, Y are topological spaces the compact open topology on $\text{Hom}_{\mathbf{Top}}(Y, X)$ is the topology with a sub-basis consisting of sets of the form

$$\langle K, U \rangle = \{f \in \text{Hom}(Y, X) : f(K) \subseteq U\},$$

where $K \subseteq Y$ is compact and $U \subseteq X$ is open.

Definition 9. (*Continuous lifting of paths*) For a space X , let $\mathcal{P}X$ be the space of paths in X (with the compact open topology). For any $x \in X$ define $\mathcal{P}X_x$ to be the subspace of paths starting at x . Given a continuous map $f: Y \rightarrow X$, there is a continuous map $\mathcal{P}f_y: \mathcal{P}Y_y \rightarrow \mathcal{P}X_{f(y)}$ obtained by composition. We say that f has *continuous lifting of paths* if $\mathcal{P}f_y$ is a homeomorphism.

Definition 10. (*Continuous lifting of homotopies*) For a space X , let \mathcal{H} denote the space of homotopies in X , i.e. the set of continuous maps $I \times I \rightarrow X$, again with the compact open topology. For any x in X we let $\mathcal{H}X_x$ be the subspace of homotopies beginning at x , i.e. maps $h: I \times I \rightarrow X$ such that $h|_{\{0\} \times I}$ is the constant map to $\{x\}$. If $f: Y \rightarrow X$ is continuous, we get a continuous map $\mathcal{H}f_y: \mathcal{H}Y_y \rightarrow \mathcal{H}X_{f(y)}$ by composition. We say that f has *continuous lifting of homotopies* if $\mathcal{H}f_y$ is a homeomorphism.

If $f: Y \rightarrow X$ is a semicovering map and $y \in Y$, then surjectivity of $\mathcal{P}f_y: \mathcal{P}Y_y \rightarrow \mathcal{P}X_{f(y)}$ follows from the path lifting property, and injectivity follows from uniqueness of path lifts. Consequently $\mathcal{P}f_y$ is bijective, and the same reasoning shows $\mathcal{H}f_y$ is bijective.

Definition 11. For any space X , we define the category $\mathbf{SCov}_{Br}(X)$ with objects being local homeomorphisms $p: Z \rightarrow X$ that have continuous lifting of paths and homotopies and morphisms being the obvious commuting triangles.

If $p: Y \rightarrow X$ is a map of space, then p satisfies the unique homotopy lifting property with respect to a one point space if and only for any $x \in X$ and $y \in p^{-1}(x)$ the map $\mathcal{P}p_y$ is bijective. In particular, this means $\mathbf{SCov}_{Br}(X) \subseteq \mathbf{SCov}(X)$.

Brazas shows ([3, Theorem 7.19]) a categorical equivalence between $\mathbf{SCov}_{Br}(X)$ and $\pi_1^\tau(X, x)$ -**Sets**, where $\pi_1^\tau(X, x)$ is the topologized fundamental group introduced in [4, 3.11], and $\pi_1^\tau(X, x)$ -**Sets** are the discrete sets with a continuous action. We will explore this later, but for now we show that any semicover has continuous lifting of paths and homotopies, i.e. that our notion of semicovers agrees with that of Brazas. Proposition 3.7 of [3] shows that any covering space is a semicover. However, the proof only uses that covering spaces are local homeomorphisms that satisfy the unique homotopy lifting property, so it extends to semicovers. We recall the proof here.

For any space X with basis \mathcal{B} , a convenient sub-basis for the compact open topology on $\mathcal{P}X$ are sets of the form

$$\langle K, U \rangle = \{\gamma \in \mathcal{P}X : \gamma(K) \subseteq U\},$$

where $K \subseteq I$ is compact and $U \in \mathcal{B}$. We can then form a basis for the topology on $\mathcal{P}X$ by taking sets of the form

$$\bigcap_{j=1}^n \langle K_n^j, U_j \rangle$$

where $K_n^j = [\frac{j-1}{n}, \frac{j}{n}]$ and $U_j \in \mathcal{B}$. First we need the following lemma, which shows that it is possible to lift homotopies of paths in semicovers.

Lemma 2.1. *Suppose $p: Y \rightarrow X$ is a local homeomorphism and satisfies the unique homotopy lifting property with respect to the class of single point spaces. Then p satisfies the unique homotopy lifting property with respect to I , the unit interval.*

Proof. Suppose $f: I \times I \rightarrow X$ is a continuous map and $\tilde{f}_0: I \times \{0\} \rightarrow Y$ is a continuous lift of $f|_{I \times \{0\}}$. For each $t \in I$, we may lift $f|_{\{t\} \times I}$ to a unique path h_t in Y starting at $\tilde{f}_0(t)$. Let $\tilde{f}: I \times I \rightarrow Y$ be defined by $\tilde{f}(t, s) = h_t(s)$. If this is continuous, it will be the unique lift of f .

Suppose $U \subseteq Y$ is an open set for which $p|_U: U \rightarrow p(U)$ is a homeomorphism. Given $w \in \tilde{f}^{-1}(U)$, we may find positive integers i, j, n so that $i, j \leq n$, $w \in K_n^{i,j}$ and $f(K_n^{i,j}) \subseteq p(U)$, where $K_n^{i,j} = K_n^i \times K_n^j$. For any $t \in K_n^i$ the path $h_t|_{K_n^j}$ is the unique lift the path $\gamma = f|_{\{t\} \times K_n^j}$. However, we can also lift γ by composing with $p|_{U^{-1}}$, hence the two must coincide. This means that $\text{im}(h_t|_{K_n^j}) \subseteq U$. This holds for each $t \in K_n^i$ and consequently $\tilde{f}(K_n^{i,j}) \subseteq U$. Since the U for which $U \rightarrow p(U)$ is a homeomorphism form a basis of Y , we have shown that \tilde{f} is continuous. \square

A consequence of the above lemma and proposition 2.1 is that for any semicovering $p: Y \rightarrow X$ and any $x \in X$, there is a well defined action of $\pi_1(X, x)$ on $p^{-1}(x)$.

Proposition 2.2. *(Brazas) If $p: Z \rightarrow X$ is a semicover, then p has continuous lifting of paths and homotopies.*

Proof. Suppose $x \in X$ and $z \in p^{-1}(X)$. The unique homotopy lifting property with respect to $\{I^0, I\}$ is equivalent to $\mathcal{P}_z, \mathcal{H}_z$ being bijective. We know these maps are continuous, so we only need to check they are open. Let

$$\mathcal{B}_p = \{U \subseteq Z: p|_U: U \rightarrow p(U) \text{ is a homeomorphism}\}.$$

Since p is a local homeomorphism, this is a basis of Z . A basic open set in $\mathcal{P}Z_z$ is of the form

$$\mathcal{U} = \bigcap_{j=1}^n \langle K_n^j, U_j \rangle \cap \mathcal{P}Z_z,$$

where $U_j \in \mathcal{B}_p$. Let

$$\mathcal{V} = \bigcap_{j=1}^n \langle K_n^j, p(U_j) \rangle \cap \mathcal{P}X_{p(z)}.$$

Since p is an open map, $p(U_j)$ is open for all j . It is clear that $\mathcal{P}p(\mathcal{U}) \subseteq \mathcal{V}$, and if we can show equality, it will follow that $\mathcal{P}p$ is a homeomorphism. If $\gamma \in \mathcal{V}$, since p satisfies the homotopy lifting property, we can find a lift $\tilde{\gamma} \in \mathcal{P}Z_z$. Suppose $t \in K_n^j$. Since $p|_{U_j}$ is a homeomorphism and $p|_{U_j}(\tilde{\gamma}(t)) = \gamma(t)$, it follows that $\tilde{\gamma}(t) \in U_j$. This shows that $\tilde{\gamma} \in \mathcal{U}$ and $\gamma = \mathcal{P}p(\tilde{\gamma})$, hence $\mathcal{P}p(\mathcal{U}) = \mathcal{V}$.

We now show that p has continuous lifting of homotopies. Suppose $\mathcal{U} \subseteq \mathcal{H}Z_z$ is a basic open set of the form

$$\mathcal{U} = \bigcap_{0 < i, j \leq n} \langle K_n^{i,j}, U_{i,j} \rangle,$$

where $K_n^{i,j} = K_n^i \times K_n^j$ and $U_{i,j} \in \mathcal{B}_p$. Let

$$\mathcal{V} = \bigcap_{0 < i, j \leq n} \langle K_n^{i,j}, p(U_{i,j}) \rangle.$$

It is clear that $\mathcal{H}p(\mathcal{U}) \subseteq \mathcal{V}$. If $h \in \mathcal{H}X_{p(z)}$ then we can lift h to some $\tilde{h} \in \mathcal{H}Z_z$. If $t \in K_n^{i,j}$ then using the homeomorphism $p|_{U_{i,j}}$ and identity $p|_{U_{i,j}} \circ \tilde{h}(t) = h(t)$ we see that $\tilde{h}(t) \in U_{i,j}$. As this holds for any t in any $K_n^{i,j}$ it follows that $\mathcal{H}p(\mathcal{U}) = \mathcal{V}$, so \mathcal{H} is an open map. \square

Therefore, we have proven the following.

Corollary 2.1. For a topological space X , there is an equivalence of categories between $\mathbf{SCov}_{Br}(X)$ and $\mathbf{SCov}(X)$.

Proof. The proposition above shows that every semicovering in the sense of definition 7 is a semicovering in the sense of Brazas. For the converse, continuous lifting of paths guarantees the unique homotopy lifting property with respect to single point space. Thus the two definitions are equivalent, and since the morphisms in both categories are just continuous maps that commute over the base, the categories are equivalent. \square

We now discuss a few features of semicovering maps. One nice feature that fails for regular covering spaces is the ‘two out of three’ property, as illustrated by example 3.8 in [3].

Proposition 2.3. *(Two out of three property) For spaces X, Y, Z and maps $f: X \rightarrow Y, g: Y \rightarrow Z$ with f surjective, if two of $f, g, h = g \circ f$ are semicovers so is the third.*

Proof. It is easy to show that if two of the three maps are local homeomorphisms, then the third is also. If either both g, h or f, g have continuous lifting of paths, then for any $x \in X$ two of the three maps $\mathcal{P}f_x, \mathcal{P}g_{f(x)}, \mathcal{P}h_x$ are homeomorphisms so the third must be. On the other hand suppose f, h both have continuous lifting of paths. For any $y \in Y$ we can write $y = f(x)$ for some $x \in X$. In this case $\mathcal{P}f_x, \mathcal{P}h_x$ are both homeomorphisms, hence $\mathcal{P}g_y$ is a homeomorphism. A similar argument will work for continuous lifting of homotopies. Then corollary 2.1 proves the two of three property for semicovers. \square

Another property worth mentioning is that semicovers are stable under pullback.

Proposition 2.4. *Let $f: Y \rightarrow X$ be a continuous map of spaces. Then there is a functor*

$$f^*: \mathbf{SCov}(X) \rightarrow \mathbf{SCov}(Y)$$

defined by pullback.

The proof follows from the universal property of the fiber product and the well know fact that the pullback of a local homeomorphism is again a local homeomorphism. Alternatively, this is proven by Brazas [3, Proposition 3.9].

3. UNIFORM SPACES AND TOPOLOGICAL GROUPS

In this section, we go over some of the basic facts about uniform structures that will be used later on the paper. Uniform structures can be considered as a generalization of metric spaces. Most of the proofs will be skipped, all proofs can found in Bourbaki [2]. We begin with some definitions.

Definition 12. *(Filters) Let X be a set. A filter \mathcal{F} on X is a set of subsets of X that satisfy the following axioms:*

- F1 If $U \in \mathcal{F}$ and $U \subseteq V$ then $V \in \mathcal{F}$.
- F2 If $U, V \in \mathcal{F}$ then $U \cap V \in \mathcal{F}$.
- F3 $\emptyset \notin \mathcal{F}$.

Definition 13. *(Filter Bases) If \mathcal{B} is a non empty set of subsets of a set X then the set of $A \subseteq X$ that contain some element of \mathcal{B} forms a filter if and only if the following hold:*

- B1 If $A, B \in \mathcal{B}$, there is some $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.
- B2 $\emptyset \notin \mathcal{B}$.

The set \mathcal{B} is said to be a filter base of the filter it generates.

Proof. Let \mathcal{F} be the set of supersets of an element of \mathcal{B} , and suppose B1 and B2 hold. Clearly F1 holds and F3 holds because $\emptyset \notin \mathcal{B}$. If $U, V \in \mathcal{F}$ there are $A, B \in \mathcal{B}$ such that $A \subseteq U$ and $B \subseteq V$ so by B1 there is some $C \in \mathcal{B}$ with $C \subseteq A \cap B \subseteq U \cap V$ hence $U \cap V \in \mathcal{F}$, showing \mathcal{F} satisfies F2.

Now suppose \mathcal{F} is a filter. By F3, $\emptyset \notin \mathcal{B}$ so B2 must hold. If $A, B \in \mathcal{B}$ then by F2, $A \cap B \in \mathcal{F}$ so there is some $C \in \mathcal{B}$ such that $C \subseteq A \cap B$, i.e. B1 holds. \square

Example 3.1. For any topological space X and $x \in X$, the set of (not necessarily open) neighborhoods of x form a filter $\mathcal{N}(x)$. This is called the *neighborhood filter* of x .

Recall that for any subset $U \subseteq X \times X$, the inverse of U is

$$U^{-1} = \{(x, y) : (y, x) \in U\}.$$

If $V \subseteq X \times X$ also, then the composition of U with V is

$$U \circ V = \{(x, z) : \text{there exists } y \in X \text{ such that } (x, y) \in U \text{ and } (y, z) \in V\}.$$

We may now give the definition of a uniform space.

Definition 14. (*Uniform structures, spaces, and continuity. Fundamental system of entourages*) Let X be a set. A uniform structure Φ on X is a filter on $X \times X$ that satisfies the following axioms.

U1 For all $U \in \Phi$, we have $\Delta X \subseteq U$.

U2 If $U \in \Phi$ then $U^{-1} \in \Phi$.

U3 If $U \in \Phi$ then there exists $V \in \Phi$ such that $V \circ V \subseteq U$.

The pair (X, Φ) is called a uniform space, and the elements of Φ are called entourages. If (Y, Ψ) is another uniform space a map, a uniformly continuous map $f: (X, \Phi) \rightarrow (Y, \Psi)$ is a function $f: X \rightarrow Y$ so that if $U \in \Psi$ then $(f \times f)^{-1}(U) \in \Phi$. A set of entourages $\{V_i\}_{i \in I}$ of a uniform space is said to be a fundamental system of entourages if every entourage contains some V_i .

Let (X, Φ) be a uniform space. For any $V \in \Phi$ and $x \in X$, let

$$V(x) = \{y : (x, y) \in V\}.$$

The associated topology of a uniform structure is defined so that a set $U \subseteq X$ is open if and only if for each $x \in U$, there is an entourage $V \in \Phi$ such that $V(x) \subseteq U$. Note that this gives a topology on X , where the open sets are the sets $V(x)$ for $V \in \Phi$ and $x \in X$. This is called the topology induced by the uniform structure Φ .

Example 3.2. Let X be a metric space with metric d . For any $\varepsilon > 0$, let

$$V_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}.$$

These form a fundamental system of entourages of the uniform structure associated to (X, d) .

Uniform spaces form a category once we make our choice of maps.

Definition 15. (*Uniform continuity*) Let X, Y be uniform spaces. A map $f: X \rightarrow Y$ is said to be uniformly continuous if for every entourage V of Y , the inverse image $(f \times f)^{-1}(V)$ is an entourage of X .

We come to some important examples of uniform structures that are associated to topological group.

Definition 16. (*The left, right and two sided uniformities*) Let G be a topological group. The right (resp. left) uniformity has a fundamental system of entourages given by sets of pairs

$$\{(g, h) \in G \times G : \text{such that } hg^{-1} \in U \text{ (resp. } g^{-1}h \in U)\}$$

as U runs over neighborhood of the identity $e \in G$. Denote the uniform structure of the left and right uniformity as \mathcal{L}, \mathcal{R} respectively.

The two sided uniformity \mathcal{I} is the coarsest uniformity on G so that the identity maps $(G, \mathcal{I}) \rightarrow (G, \mathcal{L})$ and $(G, \mathcal{I}) \rightarrow (G, \mathcal{R})$ are both uniformly continuous.

Definition 17. (*Cauchy filters, limits, separated and complete uniform spaces*) Let X be a uniform space, and \mathcal{F} a filter on X . We say that \mathcal{F} is a Cauchy filter if for any entourage V there is some $A \in \mathcal{F}$ such that $A \times A \subseteq V$. We say that $x \in X$ is the limit of \mathcal{F} and that \mathcal{F} converges (to x) if for any entourage V there is some $A \in \mathcal{F}$ such that $A \subseteq V(x)$. A filter base for a Cauchy filter is called a Cauchy filter base. A uniform space is complete if every Cauchy filter converges. A uniform space X is separated if the intersection of all entourages is the diagonal ΔX .

All of these ideas are generalizations of ideas for metric spaces. Cauchy filters in a uniform space generalize Cauchy sequences in a metric space, limits of Cauchy filters generalize limits of Cauchy sequences and complete uniform spaces generalize complete metric spaces. Separatedness implies that the topology is hausdorff and also implies there is at most one limit for a Cauchy filter. It is necessary to use filters for these generalized definitions unless some countability axioms on the uniform space are satisfied.

Definition 18. (*Complete topological groups*) A topological group G is said to be complete if G with the two sided uniformity is a complete uniform space.

Proposition 3.1. *A Hausdorff topological group G is isomorphic to a dense subgroup of a complete group \hat{G} if and only if the image of a Cauchy filter base under the transformation $g \mapsto g^{-1}$ is again a Cauchy filter base. In this case \hat{G} is unique up to isomorphism.*

Proof. Bourbaki Chapter 3 §3.4 theorem 1 [2]. □

We now come to some examples which will be useful later. First, if Φ is of a collection of equivalence relations on X , then the filter Φ generates is a uniform structure. Indeed if $V \in \Phi$ then $\Delta X \subseteq V$ by reflexivity, $V^{-1} = V$ by the symmetric property and $V \circ V = V$ by transitivity.

Suppose S is a set. For any set T we can form the following set

$$(1) \quad V_T = \{(\sigma, \tau) : \sigma^{-1}\tau|_T = \sigma\tau^{-1}|_T = \text{id}_T\} \subseteq \text{Aut}(S) \times \text{Aut}(S)$$

This is an equivalence relation, and one easily sees that for finite sets $T_1, T_2 \subseteq S$, that $V_{T_1} \cap V_{T_2} = V_{T_1 \cup T_2}$.

Definition 19. (*Compact open uniformity*) If S is a set then the equivalence relations $V_T \subseteq S \times S$ (where T is finite) form a fundamental system of entourages a uniform structure. We call this the compact open uniformity.

The topology we get from this uniformity is in fact the same as the compact open topology for $\text{Aut}(S)$ where S has been given the discrete topology hence the name.

In the next section, we will be interested in putting a uniformity on the set of natural automorphisms of a functor. Given any functor $F: \mathcal{C} \rightarrow \mathbf{Sets}$, we can put a uniformity on $\text{Aut}(F)$ (provided $\text{Aut}(F)$ is actually a set). We do this by putting the compact open

uniformity on $\text{Aut}(F(C))$ for any $C \in \mathcal{C}$ and then putting the coarsest uniformity on $\text{Aut}(F)$ that makes all the maps $\text{Aut}(F) \rightarrow \text{Aut}(F(C))$ uniformly continuous.

Lemma 3.1. *With the topology from the compact open uniformity, $G = \text{Aut}(S)$ is a topological group, and the compact open uniformity on G is the same as the two sided uniformity of G .*

Proof. If $\sigma \in G$ and $T \subseteq S$ is finite, then the image of the neighborhood $V_T(\sigma)$ under inversion is clearly just $V_T(\sigma^{-1})$. Thus inversion is continuous.

For multiplication, fix $(\sigma, \tau) \in G^2$ and let $V_T(\sigma\tau)$ be a neighborhood of their product. Consider the neighborhood

$$W = V_{T \cup \tau(T)}(\sigma) \times V_{T \cup \sigma^{-1}(T)}(\tau).$$

One can easily check that if $(\alpha, \beta) \in W$, then $\alpha\beta \in V_T(\sigma\tau)$. This shows continuity of multiplication.

The final statement is easy to verify and is left to the reader. \square

Lemma 3.2. *For a set S , the group $G = \text{Aut}(S)$ with the compact open uniformity is a complete, separated uniform space.*

Proof. If (σ, τ) are in every entourage, then they agree on every point, and since functions are determined by their values on points, they are equal. Thus G is separated.

Now let \mathcal{F} be a Cauchy filter on G . We define $\varphi, \psi \in G$ in the following way: For any $t \in S$ we can find some $A \in \mathcal{F}$ such that $A \times A \subseteq V_{\{t\}}$. Let $\varphi(t) = \sigma(t)$, and let $\psi(t) = \sigma^{-1}(t)$, where $\sigma \in A$. Since every thing in A agrees on $\{t\}$, this is independent of the choice of $\sigma \in A$. It is also independent of the choice of A , since if $B \in \mathcal{F}$ satisfies $B \times B \subseteq V_{\{t\}}$, then $A \cap B$ is non empty since \mathcal{F} is a filter, hence for $\sigma \in A$ and $\tau \in B$, we have $\sigma(t) = \tau(t)$ and $\sigma^{-1}(t) = \tau^{-1}(t)$. Thus φ and ψ are well defined functions, who are mutually inverse, and hence bijections.

We verify that \mathcal{F} converges to φ . A neighborhood of φ is of the form

$$V_T(\varphi) = \{\sigma: \sigma|_T = \varphi|_T, \sigma^{-1}|_T = \varphi^{-1}|_T\}.$$

For each $t \in T$ we can find $A_t \in \mathcal{F}$ so that $A_t \times A_t \subseteq V_{\{t\}}$. Let A denote the intersection of the sets A_t indexed over T . Since this is a finite intersection, $A \in \mathcal{F}$. We see by the definition of φ that $A \subseteq V_T(\varphi)$. Thus φ is the limit of \mathcal{F} . \square

The main result is the following.

Proposition 3.2. *Let \mathcal{C} be a category and $F: \mathcal{C} \rightarrow \mathbf{Sets}$ a functor such that the natural automorphisms $\text{Aut}(F)$ form a set. Give $G = \text{Aut}(F)$ be the coarsest uniformity making the maps $\text{Aut}(F) \rightarrow \text{Aut}(F(X))$ uniformly continuous for each $X \in \mathcal{C}$, where $\text{Aut}(F(X))$ is given the compact open uniformity. Then the topology obtained by this uniformity makes G a separated complete topological group.*

Proof. For each $X \in \mathcal{C}$ the maps $G \rightarrow \text{Aut}(F(X)) \xrightarrow{\text{inv}} \text{Aut}(F(X))$ and $G \times G \rightarrow \text{Aut}(F(X)) \times \text{Aut}(F(X)) \xrightarrow{m} \text{Aut}(F(X))$ are continuous by definition of the uniformity put on G and lemma 3.1. It is clear the the two compositions are the same as the compositions $G \xrightarrow{\text{inv}} G \rightarrow \text{Aut}(F(X))$ and $G \times G \xrightarrow{m} G \rightarrow \text{Aut}(F(X))$ respectively. Since these maps are continuous, by definition of the uniformity on G a map from a space $Y \rightarrow G$ is

continuous if and only if the maps $Y \rightarrow \text{Aut}(F(X))$ are continuous for each $X \in \mathcal{C}$. Thus multiplication and inversion are continuous in G showing that it is a topological group.

We now show G is separated and complete.

If $(\eta, \xi) \in G \times G$ is in the intersection of all entourages of G , then by lemma 3.2, $\eta_{F(X)} = \xi_{F(X)}$ for all $X \in \mathcal{C}$. Thus $\eta = \xi$ and G is separated.

Let \mathcal{F} be a Cauchy filter on G . This clearly induces a Cauchy filter \mathcal{F}_X on $F(X)$ for all $X \in \mathcal{C}$. By lemma 3.2, these have a limit, which we call $\eta_{F(X)}$. We need to show these maps together form a natural automorphism. Suppose that $f: X \rightarrow Y$ is an arrow in \mathcal{C} . Let $x \in F(X)$. Since \mathcal{F} is Cauchy, we can find some $\xi \in G$ such that $\eta_{F(Y)} \circ Ff(x) = \xi_{F(Y)} \circ Ff(x)$ and $\eta_{F(X)}(x) = \xi_{F(X)}(x)$. As ξ is a natural automorphism, we have $\xi_{F(Y)} \circ Ff(x) = Ff \circ \xi_{F(X)}(x)$. Thus

$$\eta_{F(Y)} \circ Ff(x) = \xi_{F(Y)} \circ Ff(x) = Ff \circ \xi_{F(X)}(x) = Ff \circ \eta_{F(X)}(x),$$

which shows that the maps are natural, and hence give a natural transformation η . Since all of the maps $\eta_{F(X)}$ are bijections, it is indeed a natural automorphism, hence it is in G .

Finally we show that η is the limit of \mathcal{F} . A neighborhood of η in G is of the form

$$V_{T_1, \dots, T_n}(\eta) = \{\xi \in G: \xi_{F(X_i)} \circ \eta_{F(X_i)}^{-1}|_{T_i} = \xi_{F(X_i)}^{-1} \circ \eta_{F(X_i)}|_{T_i} = \text{id}_{T_i} \text{ for } i = 1, \dots, n\}.$$

where X_1, \dots, X_n are objects of \mathcal{C} and $T_i \subseteq F(X_i)$. By definition of being Cauchy, there is some $A \in \mathcal{F}$ such that $A \times A \subseteq V_{T_1, \dots, T_n}$. Here V_{T_1, \dots, T_n} is the entourage of G given by pairs (η, ξ) such that $\xi \in V_{T_1, \dots, T_n}(\eta)$. From the construction of η it is clear that $A \subseteq V_{T_1, \dots, T_n}(\eta)$ because $\eta_{F(X_i)}$ is the limit of \mathcal{F}_{X_i} , so η is the limit of \mathcal{F} and G is complete. \square

4. GALOIS THEORY OF SEMICOVERS

In this section, we introduce categorical Galois theory in the framework of objects called infinite Galois theories as defined in [1, Definition 7.2.1]. Categorical Galois theory encompasses the usual Galois theory of fields, and, as we shall see, the Galois theory of covering spaces.

Choice of a base point x of a space X can be seen as a map $i: \{x\} \rightarrow X$. We get a functor $i^*: \mathbf{SCov}(X) \rightarrow \mathbf{Sets}$ that assigns to a semicover (Z, p) the set $i^*Z = p^{-1}(x)$. If $f: (Z, p) \rightarrow (Y, q)$ is a map of semicovers, then the associated map of sets $i^*f: i^*Z \rightarrow i^*Y$ is the restriction of f to $p^{-1}(x)$. In this section, we aim to show that the pair $(\mathbf{SCov}(X), i^*)$ constitute a tame infinite Galois theory. We recall definitions.

Definition 20. (*Faithful and conservative functors*) If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor between categories and let X, Y be objects of \mathcal{C} . Then F is said to be faithful if the map

$$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is injective. The functor F is said to be conservative if whenever $p: X \rightarrow Y$ is a morphism and $F(p): F(X) \rightarrow F(Y)$ is an isomorphism, then p is an isomorphism.

Definition 21. (*GT connected*) Let \mathcal{C} is a category and X an object. We say X is GT connected if whenever $f: Y \rightarrow X$ is a monomorphism either Y is initial or f is an isomorphism.

Definition 22. (*Infinite Galois theory*) Let \mathcal{C} be a category and F a functor from \mathcal{C} to \mathbf{Sets} . The pair (\mathcal{C}, F) constitutes an infinite Galois theory if it satisfies the following axioms.

- IGT1 \mathcal{C} has small colimits and finite limits.
- IGT2 Each object in \mathcal{C} can be written as a coproduct of GT connected objects.
- IGT3 \mathcal{C} is generated under colimits by a set of GT connected objects.
- IGT4 F is faithful, conservative and commutes with colimits and finite limits.

Remark: The category \mathcal{C} is necessarily a topos.

Definition 23. (*Tame infinite Galois theory*) Let (\mathcal{C}, F) be an infinite Galois theory. We say that (\mathcal{C}, F) is tame if it satisfies the following axiom as well.

- IGT5 For any connected object $X \in \mathcal{C}$, the action of $\pi_1(\mathcal{C}, F)$ on $F(X)$ is transitive.

In showing that (\mathbf{SCov}, i^*) is an infinite Galois theory, the hardest part is showing that colimits exist, and we would also like to know that GT connectedness agrees with our intuition. We begin with a few lemmas.

Lemma 4.1. *Let X be a connected, locally path connected space. For any $(Z, p) \in \mathbf{SCov}(X)$ the following are equivalent:*

- (1) Z is a connected topological space.
- (2) Z is path connected.
- (3) (Z, p) is GT connected.

Proof.

(1 \implies 2) Note Z is locally path connected since X is and because connected locally path connected spaces are path connected, Z is path connected.

(2 \implies 3) Any map $f: Y \rightarrow Z$ must be a semicover by proposition 2.3. Suppose f is a monomorphism and Y is not empty. Let $y \in Y$ and $z \in Z$. Since Z is path connected, there is a path γ from $f(y)$ to z . If $\tilde{\gamma}$ is a lift of this in Y , then $\tilde{\gamma}(1)$ maps to z . Thus f is surjective, meaning that f is bijective. Since bijective maps of local homeomorphisms are homeomorphisms, Z is GT connected.

(3 \implies 1) Let $Z' \subseteq Z$ be a connected component. The inclusion $Z' \hookrightarrow Z$ is a monomorphism so either $Z' = \emptyset$ or $Z' = Z$. Either way Z is connected. \square

In the following we use étale to mean local homeomorphism. Let $\mathbf{\acute{E}t}(X)$ be the category of étale covers of X .

Lemma 4.2. *Small colimits exist in $\mathbf{\acute{E}t}(X)$.*

Proof. Suppose I is a small category and $D: I \rightarrow \mathbf{\acute{E}t}(X)$ is a diagram. There is a forgetful functor $F: \mathbf{\acute{E}t}(X) \rightarrow \mathbf{Top}$ that forgets about the map to X . Since \mathbf{Top} has small colimits, let Z be the the colimit of $F \circ D$. For any $i \in I$ denote $D(i)$ by $p_i: Z_i \rightarrow X$, and let $f_i: Z_i \rightarrow Z$ be the map to the colimit in \mathbf{Top} . Then by the universal property of the colimit, we get a map $p: Z \rightarrow X$ so that $p \circ f_i = p_i$ for all $i \in I$.

Given $z \in Z$, we have that $z = f_i(z_i)$ for some i and some $z_i \in Z_i$. Since p_i is a local homeomorphism, there are open subsets U_i of z_i and an open set $V \subseteq X$ so that each p_i restricts to give a homeomorphism $U_i \rightarrow V$. Let $U = p^{-1}(V) \cap f_i(U_i) \subseteq Z$. Then there is a commuting diagram

$$\begin{array}{ccc}
U_i & \xrightarrow{f_i|_{U_i}} & U \\
& \searrow p_i|_{U_i} & \swarrow p|_U \\
& & V
\end{array}$$

Since $p_i = p \circ f_i$, we have $p(f(U_i)) = p_i(U_i) = V$. This means that $p|_U$ and $f_i|_{U_i}$ are surjective. From this, it follows that $p|_U, f_i|_{U_i}$ are also injective, hence are bijective. Thus to show that $p|_U$ is a homeomorphism onto V , we only need to know that U is open. We can check that U is open by checking if $f_j^{-1}(U) \subseteq Z_j$ is open for all j . If $f_j^{-1}(U)$ is empty it is open, otherwise assume there is some $y \in f_j^{-1}(U)$. Let $x = p_j(y) \in V$. Since p_j is a local homeomorphism, we can find an open neighborhood W of y such that p_j induces a homeomorphism of W onto its image. By shrinking W if necessary, we may also assume that $p_j(W) \subseteq V$. Thus we get a commutative diagram

$$\begin{array}{ccc}
W & \xrightarrow{f_j|_W} & Z \\
& \searrow p_j|_W & \swarrow p \\
& & p_j(W)
\end{array}$$

Note that $p_j(W) \subseteq V = p(U)$. But $p|_U$ is bijective, so $f_j(W) = (p|_U)^{-1}(p_j(W)) \subseteq U$. This means that $p_j^{-1}(U)$ is open in Z_j , and hence U is open. Note that this implies that $f_j|_W$ is an isomorphism. This result holds for arbitrary $y \in Z_j$ (since each $y \in f_j^{-1}(U)$ for some open set $U \subseteq Z$). \square

Remark: The proof above shows that $f_j: Z_j \rightarrow Z$ is a local homeomorphism. This will be important to show the homotopy lifting property. The above lemma is equivalent to the fact that small colimits of sheaves on a space X exist.

Lemma 4.3. *Let $(W, r), (Y, q)$ be in $\mathbf{SCov}(X)$ and let $g, h \in \text{Hom}_{\mathbf{SCov}(X)}(W, Y)$. Let (Z, p) be the coequalizer in \mathbf{Top} . Then the map $Y \rightarrow Z$ is a semicover.*

Proof. Let f denote the map $Y \rightarrow Z$. From the remark above, we see that f is a local homeomorphism.

The maps g, h induce a map $W \rightarrow Y \times_X Y$. This gives a relation which generates an equivalence relation, and it is easy to see that Z is the quotient of Y by the corresponding equivalence relation. To give an explicit description of the equivalence relation, $a, b \in Y$ are related if there is a chain w_1, \dots, w_n so that $g(w_1) = a, h(w_n) = b$ and $g(w_{i+1}) = h(w_i)$ for $1 \leq i < n$.

We first show that f has unique path lifting, and for this sake let $\gamma: I \rightarrow Z$ be any path. To show uniqueness, we suppose α, β are liftings of γ . Then α, β are both lifts of $p \circ \gamma$ and since q is a semicovering map, we must have $\alpha = \beta$.

Now we show existence, with γ as above. Since p_2 is a semicover, for each $y \in f^{-1}(\gamma(0))$ we can lift $p \circ \gamma$ (a path in X) to some loop $\tilde{\gamma}_y$ in Y starting at y . For any fixed

$y_0 \in f^{-1}(\gamma(0))$, we must show that $f \circ \tilde{\gamma}_{y_0} = \gamma$. Since $\gamma(I)$ is compact and f is a local homeomorphism, we may find open sets $V_1, \dots, V_n \subseteq Y$ so that $f|_{V_i}$ is a homeomorphism onto its image, and the sets $f(V_i)$ cover $\gamma(I)$. By shrinking the V_i if necessary and reordering them, we may assume there exist $a_1, \dots, a_{n+1} \in [0, 1]$ with $0 = a_1 < a_2 < \dots < a_{n+1} = 1$ and $\gamma([a_i, a_{i+1}]) \subseteq f(V_i)$. We may lift $\gamma|_{[a_0, a_1]}$ to some path β via the homeomorphism $f|_{V_1}$. Now $f(\beta(a_0)) = f(y_0)$, so there is some chain w_1, \dots, w_m such that $g(w_1) = \beta(a_0)$, $h(w_m) = y_0$ and $g(w_{i+1}) = h(w_i)$. Since the maps g, h are semicovers, we can lift α via g to a path β_1 starting at w_1 . Inductively, we let β_i be the lift of $h \circ \beta_{i-1}$ via g . We must have that $h \circ \beta_m(a_0) = y_0$. By the property of the coequalizer, the compositions of the paths $g \circ \beta_1, h \circ \beta_2, \dots, h \circ \beta_m$ with f and with q all agree. Now $h \circ \beta_m$ is a lift of $p \circ \gamma|_{[a_0, a_1]}$ starting at y_0 so by uniqueness of path lifting, $h \circ \beta_m = \tilde{\gamma}_{y_0}|_{[a_0, a_1]}$. We also know that $f \circ g \circ \beta_1$ is a lift of $\gamma|_{[a_0, a_1]}$ and $g \circ \beta_1$ agrees with $h \circ \beta_m = \tilde{\gamma}_{y_0}|_{[a_0, a_1]}$. Thus $f \circ \tilde{\gamma}(t) = \gamma(t)$ for $t \in [a_0, a_1]$. We may inductively repeat this procedure for $[a_i, a_{i+1}]$ to show that $f \circ \tilde{\gamma}(t) = \gamma(t)$ for $t \in [a_i, a_{i+1}]$. This shows that $\tilde{\gamma}_{y_0}$ is a lift of γ , as required. \square

Proposition 4.1. *Let X be a connected, locally path connected space, and let $i: \{x\} \rightarrow X$ be a base point. Then $(\mathbf{SCov}(X), i^*)$ is an infinite Galois theory.*

Proof. We must show that $(\mathbf{SCov}(X), i^*)$ satisfy IGT1–IGT4.

(IGT1) We show that small colimits exist by showing small coproducts and coequalizers exist. To show existence of finite limits, we need only to show finite products and equalizers exist. Since $\text{id}: X \rightarrow X$ is terminal in $\mathbf{SCov}(X)$, products are already fiber products. Since any category with products and fiber products has equalizers, we need only to show the existence of finite fiber products.

Suppose that $D: I \rightarrow \mathbf{SCov}(X)$ is a small diagram of semicovers with $D(i) = (Z_i, p_i: Z_i \rightarrow X)$. Let Z be the colimit in \mathbf{Top} , as described in lemma 4.2. By the same lemma, we get that $p: Z \rightarrow X$ is a local homeomorphism. If p satisfies the unique homotopy lifting property, then (Z, p) will satisfy the correct universal property to be the colimit in $\mathbf{SCov}(X)$.

If Z is the disjoint union of the Z_i , then it is clear that it satisfies the unique homotopy lifting property. For coequalizers, lemma 4.3 tells us that the maps from the diagram to the coequalizer are semicovering. Since the map to the coequalizer is always surjective, the two out of three property shows that the coequalizer is a semicover.

It is easy to see that the unique homotopy lifting property is preserved by limits. To show this, suppose we are given a diagram (Z_i, p_i) in $\mathbf{SCov}(X)$ and Z is the limit in \mathbf{Top} . Note that the following diagram commutes for each i .

$$\begin{array}{ccccc} Y & \longrightarrow & Z & \longrightarrow & Z_i \\ \downarrow & & \downarrow & & \downarrow \\ Y \times I & \longrightarrow & X & \xrightarrow{=} & X \end{array}$$

So if we are given that the inner square commutes, we get that the outer square commutes and by the homotopy lifting property, there are unique maps $Y \times I \rightarrow Z_i$ for each I

making the diagram commute. By the universal property of limits, this means that there is a unique map $Y \times I \rightarrow Z$ making the diagram commute. This is exactly the unique homotopy lifting property.

We move on to finite fiber products. Given maps of semicovers

$$f_1: (Z_1, p_1) \rightarrow (W, q) \quad f_2: (Z_2, p_2) \rightarrow (W, q),$$

let $Z = Z_1 \times_W Z_2$ be the fiber product in **Top**. We get a map $s: Z \rightarrow W$ and define $p: Z \rightarrow X$ by $p = q \circ s$. If (Z, p) is a semicover, then it will be the fiber product in **SCov**(X) of $(Z_1, p_1), (Z_2, p_2)$ over (W, q) . If one of Z_1 or Z_2 is empty, then Z is empty and is trivially a semicover. Otherwise let $(z_1, z_2) \in Z$. We may find neighborhoods U_1, U_2 of z_1, z_2 respectively and an open set $V \subseteq W$ such that $f_i|_{U_i}: U_i \rightarrow V$ for $i = 1, 2$. One can easily see that for any subspace $T \subseteq Y$, the fiber product $T \times_Y T$ is homeomorphic to T . From this, we deduce that $(U_1 \times_W U_2) \cap Z$ is open in Z and isomorphic to V . This shows that s is a local homeomorphism. Since q is also a local homeomorphism, the composition $p = q \circ s$ is a local homeomorphism, as required.

(IGT2) We know that space Z splits up into path components, which are open since X (hence Z) is locally path connected and are GT connected by lemma 4.1.

(IGT3) In light of (2), we need to only show that the class of path connected semicovers is essentially small. Fixing $x \in X$, the cardinality of a semicover Z is bounded by the set of paths starting at x and the set of points of X by the path lifting property. For any given set Z , there are only a set's worth of topologies on Z , and only a set's worth of maps $Z \rightarrow X$. Thus up to isomorphism, there is only a sets worth of connected semicovers, bounded in terms of the cardinality of $\Omega(X, x)$.

(IGT4) Suppose $(Z, p), (Y, q)$ are semicovers and f, g are maps $(Z, p) \rightarrow (Y, q)$. Suppose $i^*f = i^*g$ and let $z_0 \in Z$. Fix some path γ such that $\gamma(0) = z_0$ and $z = \gamma(1) \in i^*Z$. Since $f(z) = i^*f(z) = i^*g(z) = g(z)$ the loop $\alpha = \overline{g\gamma} * f\gamma$ (where $\overline{g\gamma}$ is the reverse of $g\gamma$ and $*$ represents path concatenation) is a lift of the nullhomotopic loop $\overline{p\gamma} * p\gamma$, so $\alpha(0) = \alpha(1)$. Consequently $f(z_0) = \alpha(0) = \alpha(1) = g(z_0)$. Hence $f = g$ and i^* is faithful.

To show that i^* is conservative, suppose i^*f is a bijective map. Using the same technique as above, it is clear that f is an injection since i^*f is an injection. Suppose that i^*f is a surjection. Given any $y \in Y$ we may find a path γ in Y so that $\gamma(0) \in i^*Y$ and $\gamma(1) = y$. Since there is some $z_0 \in i^*Z$ that maps to $\gamma(0)$ we lift γ to a path $\tilde{\gamma}$ in Z starting at z_0 . This path has the property that $f(\tilde{\gamma}(1)) = y$, hence f is surjective. Thus f is homeomorphism, as it is a bijective local homeomorphism.

From the construction given for the colimits and finite limits, it is easy to see that i^* preserves colimits and finite limits. \square

Remark: We have shown that i^* preserves injections and surjections, which is stronger than being conservative. It is now possible to define the Galois fundamental group.

Definition 24. (*Fundamental group of an infinite Galois theory*) Let (\mathcal{C}, F) be an infinite Galois theory. Then the fundamental group of (\mathcal{C}, F) is defined as

$$\pi_1(\mathcal{C}, F) := \text{Aut}(F),$$

where $\text{Aut}(F)$ is the set of natural automorphisms of F . We give the uniform structure of proposition 3.2 to $\pi_1(\mathcal{C}, F)$.

Remark: We have to justify that $\text{Aut}(F)$ forms a set. This follows from the fact that a natural transformation $\eta: F \rightarrow F$ is determined by its action on connected objects, since every object is a disjoint union of connected objects by assumption, and up to isomorphism these form a set.

Proposition 3.2 shows that $\pi_1^{\text{Gal}}(X, x)$ is a complete and separated topological group.

Definition 25. (*Galois fundamental group*) Let X be a connected, locally path connected space, and let $i: \{x\} \rightarrow X$ be a base point. We define the Galois fundamental group of (X, x) as $\pi_1^{\text{Gal}}(X, x) := \pi_1(\mathbf{SCov}(X), i^*)$.

As we mentioned earlier, we aim to show that $(\mathbf{SCov}(X), i^*)$ is a tame infinite Galois theory. This gives an equivalence of categories between $\mathbf{SCov}(X)$ and $\pi_1^{\text{Gal}}(X, x)\text{-Sets}$, the category of discrete sets with a continuous action of $\pi_1^{\text{Gal}}(X, x)$ on them. In this next portion, we present the idea of a tame infinite Galois theory.

If (\mathcal{C}, F) is an infinite Galois theory and X is an object of \mathcal{C} , there is an action of $\pi_1(\mathcal{C}, F)$ on $F(X)$. If $\eta \in \pi_1(\mathcal{C}, F)$, then the action on $x \in F(X)$ is given by $\eta \cdot x = \eta_{F(X)}(x)$.

Proposition 4.2. *Let X be a connected, locally path connected space, with base point $i: \{x\} \rightarrow X$. Then there is a group homomorphism $\nu: \pi_1(X, x) \rightarrow \pi_1^{\text{Gal}}(X, x)$, where $\nu(\gamma)$ is the natural transformation defined by the monodromy action of γ , i.e.*

$$\nu(\gamma)_{i^*Z}(z) = \gamma \cdot z.$$

Proof. Given $\gamma \in \pi_1(X, x)$ we must show that $\nu(\gamma)$ is a natural transformation. Suppose $f: (Z, p) \rightarrow (Y, q)$ is a map of semicovers over X . In other words, the square

$$\begin{array}{ccc} i^*Z & \xrightarrow{i^*f} & i^*Y \\ \nu(\gamma)_{i^*Z} \downarrow & & \downarrow \nu(\gamma)_{i^*Y} \\ i^*Z & \xrightarrow{i^*f} & i^*Y \end{array}$$

commutes. We also can clearly see that if e is the constant loop at x , then $\nu(e)$ is the natural automorphism that is the identity on all i^*Z .

To show that ν is homomorphism, let $\gamma, \alpha \in \pi_1(X, x)$. Then

$$\nu(\gamma * \alpha)_{i^*Z}(z) = (\gamma * \alpha) \cdot z = \gamma \cdot (\alpha \cdot z) = \gamma \cdot \nu(\alpha)_{i^*Z}(z) = \nu(\gamma)_{i^*Z} \circ \nu(\alpha)_{i^*Z}(z).$$

Thus $\nu(\gamma * \alpha) = \nu(\gamma) \circ \nu(\alpha)$ so ν is a group homomorphism. \square

Corollary 4.1. With X, i as above, $(\mathbf{SCov}(X), i^*)$ is a tame infinite Galois theory.

Proof. If $p: Z \rightarrow X$ is a path connected semicover, then the action of $\pi_1(X, x)$ is transitive on i^*Z . Indeed if $z, z' \in i^*Z$ we construct a path γ from z to z' . Then $p \circ \gamma$ is a loop in X and $[\gamma].z = z'$. From the construction of ν , we see that $\pi_1^{\text{Gal}}(X, x)$ will also act transitively on i^*Z . \square

From now on, if G is a topological group, then G -**Sets** refers to the category of discrete sets with a continuous action of G . A basic fact about infinite Galois theories is that their fundamental groups are Noohi groups. We recall the definition of a Noohi groups from [1, Definition 7.1.1].

Definition 26. (*Noohi groups*) For a topological group G , let $F_G: G\text{-}\mathbf{Sets} \rightarrow \mathbf{Sets}$ be the forgetful functor. Then G is said to be Noohi if the natural map $G \rightarrow \text{Aut}(F_G)$ is an isomorphism of topological groups, where $\text{Aut}(F_G)$ is given the topology of proposition 3.2.

Proposition 4.3. *If (\mathcal{C}, F) is an infinite Galois theory, then $\pi_1(\mathcal{C}, F)$ is a Noohi group.*

Proof. Bhatt and Scholze [1, Theorem 7.2.5] □

Theorem 7.2.5 in [1] also states that if (\mathcal{C}, F) is a tame infinite Galois theory, then the category \mathcal{C} is equivalent to $\pi_1(\mathcal{C}, F)\text{-}\mathbf{Sets}$, where $\pi_1(\mathcal{C}, F)$ have the topology from proposition 3.2. In our case, this gives the following.

Theorem 4.1. *If X is a connected locally path connected space with base point x , then the categories $\mathbf{SCov}(X)$ and $\pi_1^{\text{Gal}}(X, x)\text{-}\mathbf{Sets}$ are equivalent.*

The functor $\mathbf{SCov}(X) \rightarrow \pi_1^{\text{Gal}}(X, x)\text{-}\mathbf{Sets}$ is just i^* . For a semicover Z the action of $\eta \in \pi_1^{\text{Gal}}(X, x)$ on $z \in i^*Z$ is $\eta \cdot z = \eta_{i^*Z}(z)$. If $f: Z \rightarrow Y$ is a map of semicovers then $i^*f: i^*Z \rightarrow i^*Y$ is a map of $\pi_1^{\text{Gal}}(X, x)$ -sets, since $\pi_1^{\text{Gal}}(X, x)$ acts by natural transformations.

5. FUNCTORIALITY OF THE GALOIS FUNDAMENTAL GROUP

We have introduced $\pi_1^{\text{Gal}}(X, x)$ for based space (X, x) that is connected and locally path connected and shown that this is a complete separated topological group. In this section, we will show that π_1^{Gal} is actually a functor from the category of based connected, locally path connected spaces to the category of complete topological groups. This requires that for any map $f: (X, x) \rightarrow (Y, y)$, we get a map $f_*: \pi_1^{\text{Gal}}(X, x) \rightarrow \pi_1^{\text{Gal}}(Y, y)$, satisfying the composition laws and preserving identities. Recall from proposition 2.4 that pullback along f defines a functor $f^*: \mathbf{SCov}(Y) \rightarrow \mathbf{SCov}(X)$. Let $i: \{x\} \rightarrow X, j: \{y\} \rightarrow Y$ be the choice of base point. Then $f \circ i = j$ hence $i^* \circ f^* = j^*$. That is, the diagram

$$\begin{array}{ccc} \mathbf{SCov}(Y) & \xrightarrow{f^*} & \mathbf{SCov}(X) \\ & \searrow j^* & \swarrow i^* \\ & \mathbf{Sets} & \end{array}$$

commutes. Thus if we are given a natural automorphism of i^* , we obtain a natural automorphism of j^* by precomposition with f^* . More explicitly, given $\eta \in \pi_1^{\text{Gal}}(X, x)$ we define $f_*(\eta)$ by

$$f_*(\eta)_Z = \eta_{f^*Z}$$

for any $Z \in \mathbf{SCov}(Y)$. One can check under these assumptions that f_* is a homomorphism.

Recall that for a space X and base point $i: \{x\} \rightarrow X$, there are maps $\pi_1^{\text{Gal}}(X, x) \rightarrow \text{Aut}(i^*Z)$ for all $Z \in \mathbf{SCov}(X)$. The uniform structure on $\pi_1^{\text{Gal}}(X, x)$ is the one that is

pulled back from the uniform structure on $\text{Aut}(i^*Z)$ for all $Z \in \mathbf{SCov}(X)$. The uniformity on i^*Z is generated by entourages of the form

$$V_{Z,T} = \{(\eta, \xi): \eta^{-1}\xi \cdot t = \eta\xi^{-1} \cdot t = t \text{ for all } t \in T\},$$

where $Z \in \mathbf{SCov}(X)$ and $T \subseteq i^*Z$ is finite. A fundamental system of entourages is given by taking finite intersections of such sets. In fact, this can be simplified.

Proposition 5.1. *Entourages of the form $V_{Z,\{t\}}$ where Z is connected form a fundamental system of entourages.*

Proof. To begin, we can assume that T consists of only a single element in the above entourage. To see this, we replace Z with $Z' = Z \times_X \cdots \times_X Z$, where there are $|T|$ many copies (we will show later that $\mathbf{SCov}(X)$ is closed under finite limits so that the fiber product is indeed a semicovering). Let $t \in i^*Z'$ be any tuple that contains all members of T . The condition for $\eta, \xi \in \pi_1^{\text{Gal}}(X, x)$ to agree on T is the same as requiring that they agree on t .

Now suppose that we are given Z_1, \dots, Z_n and $t_i \in i^*Z_i$. We can assume that the Z_i are connected. In this case, we let $Z = Z_1 \times_X \cdots \times_X Z_n$ and $t = (t_1, \dots, t_n)$. The intersection of the entourages $V_{Z_i, \{t_i\}}$ is then clearly the same as $V_{Z, \{t\}}$. If Z is not connected, we take Z' to be the connected component of Z containing t . Any natural transformation must map i^*Z' into i^*Z' , so $V_{Z, \{t\}} = V_{Z', \{t\}}$. \square

Remark: If X has a universal cover, then this uniformity induces the discrete topology on $\pi_1(X, x)$. This is because for any finite subset $T \subseteq i^*\tilde{X}$, we have $V_T = \Delta X$.

We will write $V_{Z,t}$ instead of $V_{Z, \{t\}}$ from now on.

Proposition 5.2. *Suppose $f: (X, x) \rightarrow (Y, y)$ is a continuous map of based spaces. Let $G = \pi_1^{\text{Gal}}(X, x), H = \pi_1^{\text{Gal}}(Y, y)$. Then $f_*: G \rightarrow H$ is a continuous homomorphism of topological groups. Further, if $g: (Y, y) \rightarrow (Z, z)$ is another map of based spaces, then $(g \circ f)_* = g_* \circ f_*$.*

Proof. Let $i: \{x\} \rightarrow X$ be the inclusion of x and similarly for $j: \{y\} \rightarrow Y$. Suppose $V_{Z,t}(h)$ is a neighborhood of h , with Z connected. Now f^*Z is in $\mathbf{SCov}(X)$ and since $i^*f^*Z = j^*Z$, we have $t \in i^*f^*Z$. So it at least makes sense to talk about the neighborhood $V_{f^*Z,t}(g)$. If $g' \in V_{f^*Z,t}(g)$, then $g' \cdot t = g \cdot t$ and $(g')^{-1} \cdot t = g^{-1} \cdot t$. Applying f_* shows that $f_*(g') \cdot t = f_*(g) \cdot t = h \cdot t$ and likewise $f_*(g')^{-1} \cdot t = h^{-1} \cdot t$. In other words, $V_{f^*Z,t}(g)$ is a neighborhood of g in G , and it maps into $V_{Z,t}(h)$, thus f_* is continuous. \square

This leads to the following corollary.

Corollary 5.1. π_1^{Gal} is a functor from based topological spaces into topological groups.

6. UNIVERSAL COVERS

We can compare the previous construction of the fundamental group to the construction of based loops up to homotopy, which we denote π_1^{top} .

Proposition 6.1. *If X is a space with a universal cover \tilde{X} , and $x \in X$ is a base point, then $\pi_1^{\text{Gal}}(X, x)$ and $\pi_1^{\text{top}}(X, x)$ are isomorphic as topological groups if $\pi_1^{\text{top}}(X, x)$ is given the discrete topology.*

Proof. Suppose $p: \tilde{X} \rightarrow X$ is a universal cover. It is well known that if X has a universal cover, then $\pi_1^{\text{top}}(X, x) \simeq \text{Aut}(\tilde{X})$. By Yoneda's lemma and the fact that $\pi_1^{\text{top}}(X, x)$ is isomorphic to the automorphisms of \tilde{X} over X , showing that \tilde{X} represents i^* is sufficient to complete the proof. Let $p: \tilde{X} \rightarrow X$ be the universal cover of X . Fix a point $s \in i^*\tilde{X}$ and let $(Z, q) \in \mathbf{SCov}(X)$. For any $z \in i^*Z$, construct a map $f_z: \tilde{X} \rightarrow Z$ in the following way: For $t \in \tilde{X}$, choose a path γ from s to t . This projects down to a path in X , which we lift up to a path $f_z\gamma$ in Z starting at z . Let $f(t)$ be the endpoint of this path. We must show that this is well defined, so suppose γ' is another path from s to t . As \tilde{X} is simply connected, these paths are homotopic, so they map to homotopic paths in X . By the homotopy lifting property and local homeomorphism, we see that the lifts $f_z\gamma, f_z\gamma'$ have the same endpoint. Thus f_z is well defined.

To show continuity, suppose $x \in \tilde{X}$ and $U \subseteq Z$ is an open neighborhood of $f_z(x)$. We will show that there is a neighborhood of x that maps into U . Assume $q|_U: U \rightarrow X$ is an open embedding and U is path connected. Let $V \subseteq \tilde{X}$ be the set of points $y \in \tilde{X}$ so that there exists a path γ from x to y where $p\gamma$ is a path in $q(U)$. For all $y \in V$ there is an open path connected neighborhood W of y . Then $y \in p^{-1}(q(U)) \cap W \subseteq V$ which is open since $q(U)$ is open, so V is open. Finally for any $y \in V$, let γ be a path from x to y so that $p\gamma$ is a path in $q(U)$. Lift $p\gamma$ to a path $\tilde{p}\gamma$ starting at $f(x)$. This will be a path in U , hence $\tilde{p}\gamma(1) \in U$. All that is left is to show $f(y) = \tilde{p}\gamma(1)$ to prove continuity. Let α is a path in \tilde{X} from s to x , and $\tilde{p}\alpha$ a lift in Z of $p\alpha$. Then $\alpha * \gamma$ is a path from s to y and $\tilde{p}\alpha * \tilde{p}\gamma$ is a lift of $p\alpha * p\gamma$, so $f(y) = \tilde{p}\alpha * \tilde{p}\gamma(1) = \tilde{p}\gamma(1)$. Consequently f is continuous.

This defines a map $\eta_Z: i^*Z \rightarrow \text{Hom}(\tilde{X}, Z), z \mapsto f_z$. It is injective since $f_z(s) = z$. We show that this map is surjective. Suppose $f: \tilde{X} \rightarrow Z$. In order to commute over X , for any $t \in \tilde{X}$ the element $f(t)$ must be the same as the element we get by constructing a path from s to t , projecting it into X , lifting it to a path in Z starting at $f(s)$ and looking at the endpoint.

Finally all that is left is to show that the maps η_Z are natural. Let $g: (Z, q) \rightarrow (Y, r)$ be a map of semicovers and let $z \in i^*Z$. For any $t \in \tilde{X}$, we need to show $g(f_z(t)) = f_{g(z)}(t)$. Suppose α is a path from s to t in \tilde{X} and let $\tilde{p}\alpha$ be the lift of $p\alpha$ in Z starting at z , so that $f_z(t) = \tilde{p}\alpha(1)$. Note $g(\tilde{p}\alpha(t)) = g(f_z(t))$. However, $g\tilde{p}\alpha$ is a lift of $p\alpha$ in Y starting at $g(z)$, so $g(\tilde{p}\alpha(1)) = f_{g(z)}(t)$. This means $gf_z = f_{g(z)}$, showing the η_Z give a natural transformation $\eta: i^* \rightarrow \text{Hom}(\tilde{X}, -)$. Since each η_Z is a bijection η is a natural isomorphism. Consequently, $\pi_1^{\text{Gal}}(X, x)$ and $\pi_1^{\text{top}}(X, x)$ are isomorphic as groups.

The remark after proposition 5.1 shows that $\pi_1^{\text{Gal}}(X, x)$ has the discrete topology, which shows the two groups are isomorphic as topological groups. \square

Thus for a space X with a universal cover, we may write $\pi_1(X, x)$ without confusion.

7. THE TOPOLOGIZED FUNDAMENTAL GROUP

The contents of this section are a summary of a few of the ideas from section 4 of [3].

The fundamental group $\pi_1(X, x)$ for a space X is a quotient of the loop space $\Omega(X, x) = \mathcal{P}_x \cap \mathcal{P}^x$. Since $\Omega(X, x)$ is a topological space under the compact open topology, $\pi_1(X, x)$ inherits a topology via the quotient map. While in general this does not make $\pi_1(X, x)$ into a topological group as seen in [7], it does make it into a quasitopological group in

the sense that the inversion map is continuous and multiplication is continuous in each variable.

The category **TopGrp** of topological groups is a full sub category of **qTopGrp**, the category of quasitopological groups, so there is a forgetful functor $F: \mathbf{TopGrp} \rightarrow \mathbf{qTopGrp}$. We will show that this functor has a left adjoint using Freyd's adjoint functor theorem. In order to use this theorem the following lemma is necessary.

Lemma 7.1. *The categories **TopGrp** and **qTopGrp** are complete and F preserves all small limits.*

Proof. Suppose there is a diagram G_i of quasitopological groups. Since the category of groups is complete, let G be the limit of the diagram G_i in this category. We put the coarsest topology on G that make the homomorphisms $f_i: G \rightarrow G_i$ continuous for all i . If H is a quasitopological group that maps in a compatible way into the diagram G_i , then there is a unique group homomorphism $H \rightarrow G$ by the universal property of limits of groups. The open sets of G are finite intersections of open sets pulled back from the G_i , and since $H \rightarrow G_i$ is continuous for each i , we must have $H \rightarrow G$ continuous. Thus G satisfies the correct universal property. We need now to show that G is in fact a quasitopological group.

Now we must show that inversion is continuous and multiplication is continuous in each variable. For each group G_i in the diagram, let inv_i be the inversion map and $\mu_i: G_i \times G_i \rightarrow G_i$ the multiplication map for G_i . We also let $\text{inv}: G \rightarrow G$ be the inversion map and $\mu: G \times G \rightarrow G$ the multiplication map. Now for each i , the following diagram of topological spaces commutes

$$\begin{array}{ccc} G & \xrightarrow{\text{inv}} & G \\ f_i \downarrow & & \downarrow f_i \\ G_i & \xrightarrow{\text{inv}_i} & G_i \end{array}$$

It follows that if $U \subseteq G_i$ is open, then $(f_i \circ \text{inv})^{-1}(U) = (\text{inv}_i \circ f_i)^{-1}(U)$ is open. This means that inv is continuous since the sets $f_i^{-1}(U)$, where U is an open subset of some G_i , form a sub basis for the topology on G .

Now to see that multiplication is continuous in each variable, we fix some $g \in G$. For each G_i , the following diagram of topological spaces commutes

$$\begin{array}{ccc} G \times \{g\} & \xrightarrow{\mu} & G \\ f_i \times f_i \downarrow & & \downarrow f_i \\ G_i \times \{f_i(g)\} & \xrightarrow{\mu_i} & G_i \end{array}$$

The map $f_i \times f_i|_{\{g\}}$ is continuous since the products of continuous maps are continuous. For similar reasons as inversion, this implies that the top map is continuous. An analogous

argument shows that fixing an element on the left will give a continuous map $\{g\} \times G \rightarrow G$. Thus G is a quasitopological group.

Now suppose that all of the G_i are topological groups, and let G denote the same construction. Then G will satisfy the correct universal property, and we also have that the following diagram commutes for each G_i .

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ f_i \times f_i \downarrow & & \downarrow f_i \\ G_i \times G_i & \xrightarrow{\mu_i} & G_i \end{array}$$

For analogous reasons, the map μ must be continuous. This shows that G is intact a topological group.

All of this shows that the limit of a diagram of topological groups is the same whether this limit is taken in the category of quasitopological or the category of topological groups. In other words, the forgetful functor F preserves limits. \square

Proposition 7.1. *The forgetful functor $F: \mathbf{TopGrp} \rightarrow \mathbf{qTopGrp}$ has a left adjoint $\tau: \mathbf{qTopGrp} \rightarrow \mathbf{TopGrp}$.*

Proof. In light of the previous lemma, all we need to show is that for any quasitopological group G , there is a small collection of topological groups G_i indexed by I and arrows $f_i: G \rightarrow G_i$ such that any continuous morphism $h: G \rightarrow H$ whose target is a topological group can be written as $h = t \circ f_i$ for some $t: G_i \rightarrow H$. First, if G is a topological group and K is a normal subgroup, a topology \mathcal{T} on G/K is said to be a G -compatible topology for K if G/K is a topological group and $G \rightarrow G/K$ is continuous. Now let

$$I = \{(K, \mathcal{T}): K \subseteq G \text{ is a normal subgroup, } \mathcal{T} \text{ is a } G\text{-compatible topology for } K\}$$

Then if $i = (K, \mathcal{T})$, let $G_i = G/K$ with the topology \mathcal{T} and $f_i: G \rightarrow G/K = G_i$ be the usual quotient map for groups. Any map from $h: G \rightarrow H$ where H is a topological group can be factored as $G \rightarrow \text{im}(h) \hookrightarrow H$. The underlying group of $\text{im}(h)$ is isomorphic to $G/\ker(h)$ which is isomorphic to G_i for some $i \in I$. The inclusion $\text{im}(h) \hookrightarrow H$ is a morphism of topological groups. Hence we apply Freyd's adjoint functor theorem (Theorem 2 §6 Chapter 5 [9]) which shows that τ exists. \square

Note that if G is a quasitopological group, then $\tau(G)$ has the same underlying group as G . Indeed, if we let $\text{in}(G)$ be the same underlying group as G with the indiscrete topology then the identity $G \rightarrow \text{in}(G)$ is continuous. We get a map $\tau(G) \rightarrow \text{in}(G)$ that makes the following diagram commute

$$\begin{array}{ccc}
G & \longrightarrow & \tau(G) \\
& \searrow \text{id} & \downarrow \\
& & \text{in}(G)
\end{array}$$

This means that $G \rightarrow \tau(G)$ is injective. It is clear to see that the the image of G in $\tau(G)$ satisfies the correct universal property to be the adjoint, hence by uniqueness of the adjoint, the image of G must be $\tau(G)$.

Definition 27. (*The topological fundamental group*) Given any based space (X, x) we define $\pi_1^\tau(X, x)$ to be $\tau(\pi_1(X, x))$.

It is a result of [3] that for a connected, locally path connected based space (X, x) there is an equivalence between the category $\pi_1^\tau(X, x)\text{-Sets}$ (where the sets are considered as discrete spaces and the action is continuous) and the category $\mathbf{SCov}_{Br}(X)$. In fact, the conditions of being connected and locally path connected may be weakened, but for our purposes it is enough. From 2.2, we know there is a categorical equivalence between $\mathbf{SCov}_{Br}(X)$ and $\mathbf{SCov}(X)$. These remarks prove the following proposition.

Proposition 7.2. *If (X, x) is a based topological space that is connected and locally path connected, there is a categorical equivalence between $\pi_1^\tau(X, x)\text{-Sets}$ and $\mathbf{SCov}(X)$.*

If $\pi_1^\tau(X, x)$ had a basis of open subgroups of the neighborhood filter of the identity, the proposition above combined with proposition 7.1.5 of [1] would show that $\pi_1^{\text{Gal}}(X, x)$ is the completion of $\pi_1^\tau(X, x)$ with respect to the two sided uniformity. Unfortunately, we don't know whether or not this is true, so instead we introduce a new topology on $\pi_1(X, x)$ that lets us use proposition 7.1.5 of [1].

Definition 28. (*The σ -topology*) Let G be a quasitopological group. The σ -topology on G is the unique topology where the neighborhoods of the identity are sets that contain an open subgroup of $\tau(G)$. Let $\pi_1^\sigma(X, x)$ denote $\pi_1(X, x)$ with the σ -topology.

Proposition 7.3. *For any based topological space, $\pi_1^\sigma(X, x)$ is a topological group.*

Proof. The neighborhood filter of the identity is generated by certain subgroups of π_1 so this filter satisfies GV1-GV2 as defined in Chapter 3, §2 of [2]. If $N \subseteq \pi_1^\tau(X, x)$ is an open subgroup and $g \in \pi_1^\tau(X, x)$, then gNg^{-1} is an open subgroup because conjugation is a homeomorphism in a topological group. Thus the neighborhood satisfies GV3. Therefore by proposition 1 of Chapter 3, §2 in [2], $\pi_1^\sigma(X, x)$ is a topological group. \square

The σ -topology is coarser than the τ topology, so the identity $\pi_1^\tau(X, x) \rightarrow \pi_1^\sigma(X, x)$ is continuous. We will now show that $\pi_1^{\text{Gal}}(X, x)$ is the completion of $\pi_1^\sigma(X, x)$, but first we need a lemma. For the rest of the paper, whenever G is a topological group, $G\text{-Sets}$ denotes the category of discrete sets with a continuous action of G .

Lemma 7.2. *Suppose that G, H are topological groups so that there is a bijective continuous map $\varphi: G \rightarrow H$. If G and H have the same open subgroups, then there is a categorical equivalence between $G\text{-Sets}$ and $H\text{-Sets}$.*

Proof. We need to show that if S is a set with an action of the underlying group of G, H on it, the action of G on S is continuous if and only if the action of H is continuous. In this case, the equivalence will be given by the functor $G\text{-Sets} \rightarrow H\text{-Sets}$ which doesn't change the actions, sets or morphisms.

Without loss of generality we assume that S is transitive, since every G -set decomposes as a disjoint union of transitive G -sets (the orbits). Recall that for any topological group M , a transitive action of M on a discrete set T is continuous if and only if for any $t \in T$, the stabilizer of t is an open subgroup of M . Since G, H have the same open subgroups, it follows that if the action of G on S is continuous if and only if the action of H on S is continuous. \square

Corollary 7.1. The categories $\pi_1^\tau(X, x)\text{-Sets}$ and $\pi_1^\sigma\text{-Sets}$ are equivalent.

Proof. By construction, $\pi_1^\tau(X, x)$ and $\pi_1^\sigma(X, x)$ have the same open subgroups, hence by lemma 7.2, the categories are equivalent. \square

For a topological group G , denote the completion of G with respect to the two sided uniformity by G^* .

Theorem 7.1. The completion $\pi_1^\sigma(X, x)^*$ of $\pi_1^\sigma(X, x)$ with respect to the two sided uniformity is $\pi_1^{\text{Gal}}(X, x)$.

Proof. Let $F: \pi_1^\sigma(X, x)\text{-Sets} \rightarrow \text{Sets}$ be the forgetful functor. Then $\pi_1^\sigma(X, x)^* \simeq \text{Aut}(F)$ by proposition 7.1.5 of [1]. On the other hand, we have equivalences

$$\pi_1^{\text{Gal}}(X, x)\text{-Sets} \simeq \mathbf{SCov}(X) \simeq \mathbf{SCov}_{Br}(X) \simeq \pi_1^\tau(X, x)\text{-Sets} \simeq \pi_1^\sigma(X, x)\text{-Sets}.$$

The first equivalence is theorem 4.1, the second is corollary 2.1, the third is theorem 7.19 in [3] and the last follows from lemma 7.2. Since $\pi_1^{\text{Gal}}(X, x)$ is Noohi (proposition 4.3) and $\pi_1^{\text{Gal}}(X, x)\text{-Sets} \simeq \pi_1^\sigma(X, x)\text{-Sets}$ we have $\pi_1^{\text{Gal}}(X, x) \simeq \text{Aut}(F) \simeq \pi_1^\sigma(X, x)^*$. \square

8. COVERS OF THE EARRING

Let C_n (for $n > 0$) be the circle of radius $1/n$ centered at $(0, 1/n)$ in the plane. Then the hawaiian earring is $\mathbf{E} = \bigcup_{n=1}^{\infty} C_n$ and is given the subspace topology. Let \mathbf{Y} be the countably infinite wedge of circles. There is an obvious bijection $f: \mathbf{Y} \rightarrow \mathbf{E}$ which is continuous. Pullback by f defines a functor $\mathbf{SCov}(\mathbf{E}) \rightarrow \mathbf{SCov}(\mathbf{Y})$. For any $Z \in \mathbf{SCov}(\mathbf{E})$ the corresponding semicovering space f^*Z is the same set as Z (since f is a bijection), it just has a different topology which we call the \mathbf{Y} topology.

Proposition 8.1. The functor $f^*: \mathbf{SCov}(\mathbf{E}) \rightarrow \mathbf{SCov}(\mathbf{Y})$ is fully faithful.

Proof. It is faithful since f^* doesn't change the points and maps are determined by what they do on points. Now suppose $g: (W, q) \rightarrow (Z, p)$ is continuous in the \mathbf{Y} topology and $U \subseteq Z$ is open in the \mathbf{E} topology. If there is no point in U that maps onto the origin, then $g^{-1}(U)$ is open in the \mathbf{E} topology, as \mathbf{E} and \mathbf{Y} are homeomorphic away from the origin, so we assume that there is some $z \in U$ that maps to the origin in \mathbf{E} . By choosing a smaller neighborhood if necessary, we can assume that U is homeomorphic to a connected open neighborhood of the origin in \mathbf{E} . For any point $w \in g^{-1}(z)$ we can find an open neighborhood V of w that is homeomorphic to a connected open neighborhood of the origin. Now $g^{-1}(U) \cap V$ is open in the \mathbf{Y} topology. One sees that $g^{-1}(U) \cap V$

will be open in the \mathbf{E} topology if $q(g^{-1}(U) \cap V)$ contains all but finitely many of the circles. As $p(U) \cap q(V)$ is open, it contains all but finitely many of the circles, and since $p(U) \cap q(V) \subseteq q(g^{-1}(U) \cap V)$, we see that $g^{-1}(U) \cap V \subseteq g^{-1}(U)$ is open in the \mathbf{E} topology. From this it is clear that g is continuous in the \mathbf{E} topology. Thus f^* is full. \square

Let $e \in \mathbf{E}$ denote the origin of the hawaiian earring, $y \in \mathbf{Y}$ be the point that maps to e , and $G = \pi_1(\mathbf{Y}, \mathbf{y})$. Using the equivalence of G -Sets and $\mathbf{SCov}(\mathbf{Y})$, we can classify the essential image of $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$ in terms of G -Sets. Now G is isomorphic to a free group on countably a infinite set of generators, so a giving a G -set is the same as giving a set S and a bijection $\varphi_i: S \rightarrow S$ for each integer i such that $\varphi_{-i} = \varphi_i^{-1}$ and φ_0 is the identity.

Proposition 8.2. *A G -set S is in the essential image of $\mathbf{SCov}(\mathbf{E}) \rightarrow \mathbf{G}\text{-Sets}$ if and only if the following two conditions hold:*

- (1) *For any $s \in S$, all but finitely many of the φ_i fix s .*
- (2) *If there is a sequence $\varphi_{i_1}, \varphi_{i_2}, \dots$ where any φ_k appears only finitely many times, then for any $s \in S$, the sequence $\varphi_{i_1}(s), \varphi_{i_2}\varphi_{i_1}(s), \dots$ eventually stabilizes.*

Proof. In the equivalence of $\mathbf{SCov}(\mathbf{Y})$ and G -Sets the automorphism φ_k is the action of the fiber that we get from the loop c_k that goes around the $|k|$ -th circle clockwise if k is positive or counterclockwise otherwise (and is constant if $k = 0$). Thus if $Z \in \mathbf{SCov}(\mathbf{E})$ then for each $z \in i^*Z$, there is a neighborhood U of z that is homeomorphic to its image in E , and hence all but finitely many of the circles are unwound. This means that all but finitely many of the φ_k act trivially on z . Now a sequence $\varphi_{i_1}, \varphi_{i_2}, \dots$ where φ_k only appears finitely many times corresponds to an element of $\pi_1^{\text{top}}(\mathbf{E}, \mathbf{e})$ as seen in [6]. Let γ be a representative of this loop. The set S corresponds to i^*Z for some $Z \in \mathbf{SCov}(\mathbf{E})$. Since we produce a path $\tilde{\gamma}$ that is a lift of γ to Z , we see that the sequence stabilizes to $\tilde{\gamma}(1)$.

Conversely, suppose we are given a set S and automorphisms φ_k that satisfy the two conditions. Let Z' be the corresponding cover of \mathbf{Y} . Let $p': Z' \rightarrow \mathbf{E}$ be the composition $Z' \rightarrow \mathbf{Y} \rightarrow \mathbf{E}$. The first condition ensures that all but finitely many of the loops at any point in the fiber are unwound. We make a new cover $p: Z \rightarrow \mathbf{E}$ so that $Z = Z'$ as a set and $p = p'$. A subset $U \subseteq Z$ is open if and only if U is open in Z' and for all $s \in S \cap U$, all but finitely many of the wound loops based at s are contained in U . Already $p: Z \rightarrow \mathbf{E}$ is locally bijective, and the topology is defined to make it a local homeomorphism, thus we only need to show the unique homotopy lifting property.

Let $\gamma: I \rightarrow \mathbf{E}$ be a path, $z \in Z$ and $F = \gamma^{-1}(e)$. We assume without loss of generality that F contains no closed intervals. Let $0 = x_1 = \inf F$ and $x_n = \inf F - \{x_1, \dots, x_{n-1}\}$. If there are only finitely many x_i then γ is continuous when considered as a loop $I \rightarrow \mathbf{Y}$, so we may lift it to a loop in Z' starting at z and compose the lift with $Z \rightarrow Z'$ to get a lift starting at z . Otherwise, there is an $x \in F$ that is a limit of the sequence. Let $\gamma_n = \gamma|_{[x_n, x_{n+1}]}$. By definition of the x_i , γ_n must be a loop that traverses some circle $C_{|k(n)|}$ once, either counter clockwise or clockwise. This means it is homotopic to $c_{k(n)}$, where $k(n)$ is signed according to the direction γ_n traverses $C_{|k(n)|}$. We have shown we can lift γ_n . Let $\tilde{\gamma}_1$ be a lift of γ_1 starting at z and $\tilde{\gamma}_n$ be a lift of γ_n starting at $\tilde{\gamma}_{n-1}(1)$. Note that $\gamma_n(1) = \varphi_{k(1)} \cdots \varphi_{k(n)}(z)$. The sequence $\varphi_{k(1)}(z), \varphi_{k(2)}\varphi_{k(1)}(z), \dots$ must have any fixed k appear finitely many times, otherwise γ traverses $C_{|k|}$ infinitely many times violating the

fact that γ has compact image. Hence the sequence $\gamma_1(1), \gamma_2(1), \dots$ eventually stabilizes by the second condition. That means we may find some N and some $U \subseteq Z$ so that $\gamma_n(1) = \gamma_m(1)$ for all $m > n > N$, $\gamma|_{[x_n, x_m]} \subseteq U$ and $p|_U$ is an open embedding. We have shown $\gamma|_{[0, x_N]}$ can be lifted, and since the image of $\gamma|_{[x_N, x]}$ is contained in $p(U)$, this portion can be lifted using the homeomorphism $p|_U: U \rightarrow p(U)$. \square

9. THE GALOIS FUNDAMENTAL GROUP OF THE HARMONIC ARCHIPELAGO

The harmonic archipelago \mathbf{A} is the space that is obtained by filling in the gaps between the loops of the hawaiian earring and adding a bump of height one between loops c_n, c_{n+1} . A more explicit description can be found in [8]. In particular, there is a continuous inclusion $f: \mathbf{E} \rightarrow \mathbf{A}$. We view \mathbf{E} as a subset of \mathbf{A} and let e denote the shared origin of the harmonic archipelago and the hawaiian earring.

Proposition 9.1. $\pi_1^{\text{Gal}}(\mathbf{A}, \mathbf{e})$ is trivial.

Proof. We prove this by showing that \mathbf{A} has no non trivial connected semicovers.

Suppose $p: Z \rightarrow \mathbf{A}$ is a non trivial connected semicover. Then $W = p^{-1}(E)$ is a non trivial connected semicover of \mathbf{E} . Since W is non trivial, there is some $z \in i^*W$ and some n so that $c_n \cdot z \neq z$. Since for all k , c_k is homotopic to c_n in \mathbf{A} we have $c_k \cdot z = c_n \cdot z$. This contradicts proposition 8.2, thus there are no non trivial connected semicovers of \mathbf{A} . \square

In [8] it is shown that $\pi_1(\mathbf{A}, \mathbf{e}) \neq \mathbf{0}$, so in general the groups $\pi_1(X, x)$ and $\pi_1^{\text{Gal}}(X, x)$ are not isomorphic.

10. THE GALOIS FUNDAMENTAL GROUP OF THE EARRING

Let $i: \{e\} \rightarrow \mathbf{E}$ be the inclusion of the origin into the hawaiian earring, \mathbf{Y} the infinite wedge of circles and $j: \{y\} \rightarrow \mathbf{Y}$ be the inclusion of the point at which all of the circles meet in the infinite wedge of circles. The bijection $f: \mathbf{Y} \rightarrow \mathbf{E}$ takes e to y , so there is a group morphism $\pi_1(\mathbf{Y}, \mathbf{y}) \rightarrow \pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$. To show that this map is injective, we use the covers \mathbf{E}_n , where the first n circles of the hawaiian earring are unwound completely and the rest are still wound. Now $\alpha, \beta \in \pi_1(\mathbf{Y}, \mathbf{y})$ correspond to words in an alphabet that is indexed by \mathbf{N} . Since words are necessarily finite, we can choose some n such that no letters corresponding to $k > n$ appear in the words α, β . Then α and β will act differently on \mathbf{E}_n , which means that they correspond to different natural transformations. Thus $\pi_1(\mathbf{Y}, \mathbf{y})$ is naturally a subgroup of $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$.

We want to show that $\pi_1(\mathbf{Y}, \mathbf{y})$ is dense in $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$, and for this we will use the following lemma.

Lemma 10.1. Let $Z \in \mathbf{SCov}(\mathbf{E})$ be a connected semicover. Then for any $s, t \in i^*Z$ there exists some $\alpha \in \pi_1(\mathbf{Y}, \mathbf{y})$ such that $\alpha \cdot s = t$.

Proof. Since Z is connected, there is a path $\tilde{\gamma}$ that starts s and ends at t . If we project this down, we get a loop γ in E , which gives an element of the fundamental group. Let $F = \tilde{\gamma}^{-1}(i^*Z)$. We can assume that F does not contain any closed intervals, since if it did we could adjust $\tilde{\gamma}$ by gluing the endpoints together and still have a continuous loop that is homotopic to $\tilde{\gamma}$. It is clear that if F is finite, then $\gamma \in \pi_1(\mathbf{Y}, \mathbf{y})$, and we are

done. Otherwise F is infinite and we consider the set of limit points $L(F)$ of the subspace $F \subseteq [0, 1]$. Since F is infinite and bounded, $L(F)$ is not empty. Let $x \in L(F)$. Thus there is a sequence $x_1, x_2, \dots \in F$ that converges to x . Now $\tilde{\gamma}$ induces a continuous map $F \rightarrow i^*Z$ with their respective subspace topologies, and thus the sequence $\tilde{\gamma}(x_1), \tilde{\gamma}(x_2), \dots$ converges. Since i^*Z is discrete, this means there is some N such that $\tilde{\gamma}(x_n) = \tilde{\gamma}(x_m)$ for all $n, m \geq N$. This means that we can glue x_N to x to get a new loop that acts the same way as $\tilde{\gamma}$ on the fiber. This gets rid of the limit point x , and as all limit points can be removed this way, we can find a loop α that acts the same way on the fiber, but such that $\alpha^{-1}(i^*Z)$ is finite. Thus α projects down to a loop in $\pi_1(\mathbf{Y}, \mathbf{y})$. \square

Proposition 10.1. $\pi_1(\mathbf{Y}, \mathbf{y})$ is dense in $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$.

Proof. Let $\eta \in \pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$, and consider the neighborhood $V = V_{Z,t}(\eta)$ (where $z \in i^*Z$) of η . By Proposition 5.1 we may assume Z is connected. Let $Z' = Z \times_{\mathbf{E}} Z$. The elements $(\eta_Z^{-1}(t), t), (t, \eta_Z(t))$ are in the same connected component of Z' . Thus by Lemma 10.1, there is some $\alpha \in \pi_1(Y)$ such that $\alpha \cdot (\eta_Z^{-1}(t), t) = (t, \eta_Z(t))$. It follows that $\alpha \in V_{Z,t}$. \square

Corollary 10.1. $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$ is the completion of $\pi_1(\mathbf{Y}, \mathbf{y})$.

Proof. Since $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$ is complete, and $\pi_1(\mathbf{Y}, \mathbf{y})$ is dense in $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$, this result follows from the uniqueness of the completion of a topological group. \square

Recall that the completion of a topological group can be described as the set of minimal Cauchy filters on the group. We now describe how to go from a Cauchy filter \mathcal{F} on $\pi_1(\mathbf{Y}, \mathbf{y})$ to an element of $\pi_1^{\text{Gal}}(\mathbf{E}, \mathbf{e})$. Let $Z \in \mathbf{SCov}(\mathbf{E})$ and define $\varphi_Z(t) = \alpha \cdot t$, where $\alpha \in A$ and $A \in \mathcal{F}$ is $V_{Z,\{t\}}$ close. As in Lemma 3.2, we see that this is a bijection. The argument used in proposition 3.2 shows that the φ_Z give a natural transformation. A topological group G has a completion if and only if the image of a Cauchy filter base under the inversion map is a Cauchy filter, base, hence the image of \mathcal{F} under inversion generates a Cauchy filter, and the natural transformation associated to this is clearly the inverse the the natural transformation associated to \mathcal{F} . Thus we get a natural automorphism.

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