The Geometry of Null Surfaces in Minkowski Space

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The Geometry of Null Surfaces in Minkowski Space

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Thank you to my advisor Professor Jeanne Clelland who helped me wade through this experience, and without this project would not have come to fruition.
ABSTRACT

This paper investigates the geometric properties of null surfaces in the Minkowski space $\mathbb{M}^{2,1}$. Invariants of these surfaces are found using Cartan’s method of moving frames. This leads to a complete classification of null surfaces. Examples of these surfaces are then given.
At the end of the 19th century the study of geometry was split. The geometry of Euclidean spaces, and non-Euclidean spaces were studied separately. However, in 1872 Felix Klein published his *Erlangen Program*, which united these fields. The unifying principle he proposed was that geometry is actually the study of symmetries among groups of transformations.

Around this same time physics was in a state of uncertainty. At the end of the 19th century Maxwell had unified electricity and magnetism. The pervading feeling among physicists was that "...the only occupation which will then be left to men of science will be to carry these measurements to another place of decimals" (James Clerk Maxwell). There was one lingering problem, however. Nobody knew how light travelled through vacuum. Many believed there was a substance called the ether. Many experiments tried to measure the earth’s motion through this ether by measuring the speed of light in different directions relative to the earth. However, all these experiments showed that light travelled at the same speed, regardless of direction. This meant that there could be no ether which the earth moved through, and which light travelled through. Luckily, in 1905 Einstein released four articles dubbed the *Annus Mirabilis* papers, among these was "On the Electrodynamics of Moving Bodies", which addressed this issue by assuming the speed of light is constant relative to all inertial frames. Just five years before Poincaré began to study the "principle of relative motion" and noticed that by taking time to be the fourth component in $\mathbb{M}^{1,3}$ the Lorentz transformations are rotations of $\mathbb{R}^4$. Minkowski then used this observation to show that Maxwell’s equations remained invariant under these transformations, uniting space and time as one entity, spacetime. From this Minkowski was able to reformulate Einstein’s special theory of relativity.
2. PRELIMINARIES

We assume the reader has a basic knowledge of differential geometry. This should include the notions of differentiable manifolds, curves, and surfaces, tangent spaces, dual spaces, sections, fibers, bundles, and tensor operations (mainly the wedge product).

2.1 Differential Geometry Basics

Our main object of study will be regular surfaces in Minkowski space. We first establish some basic conventions, notation, and notions of differential geometry.

Convention 2.1.1 (Einstein summation). Throughout this paper we will be using the Einstein summation convention. This convention asserts that when an index variable appears in both an upper and lower index position it indicates a sum over that variable. For example,

\[ x_i y^i = \sum_i x_i y^i \]

Definition 2.1.2 (Lie group). A Lie group is a set \( G \) which is both a group and a differentiable manifold, for which the map \( \mu : G \times G \to G \) given by

\[ \mu(g, h) = gh^{-1} \]

is differentiable.

To every Lie group is associated a Lie algebra.

Definition 2.1.3 (Lie algebra). A Lie Algebra, \( \mathfrak{g} \), is the tangent space to the Lie group \( G \) at the identity element \( e \in G \).

There is a product structure on the Lie algebra called the Lie bracket. We will not need to use the Lie bracket, and will therefore leave it undefined.

Convention 2.1.4 (Left translation). Throughout this paper we will be using the left translation, or left multiplication map, \( L_h \), on Lie groups where

\[ L_h(g) = hg \]

for \( g \in G \) with \( G \) a Lie group.
The only Lie groups we will be working with are subgroups of $\text{GL}(n)$.

The last notion we need to cover is that of moving frames. The idea behind moving frames, is that to each point on a surface $p \in \Sigma$ in $\mathbb{R}^3$ we assign a basis for the tangent space $T_p\mathbb{R}^3$. By examining properties of the basis at different points of the surface we can discover the geometry of the surface. In $\mathbb{R}^3$ we will create a basis $\{e_0, e_1, e_2\}$ and use the projection map $\pi(x; e_0, e_1, e_2) = x$ for $x \in \mathbb{R}^3$. The ordered set of vectors $\{x; e_0, e_1, e_2\}$ is called a frame. From this definition of $\pi$ we see that the fiber of $\pi$ at $x$ consists of all frames for the tangent space $T_x\mathbb{R}^3$. We can group these frame spaces together creating the frame bundle, denoted by $F(\mathbb{R}^3)$, which consists of all frames at all points.

### 2.2 Differential Forms

**Definition 2.2.1** (Exterior derivative). If $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable then the exterior derivative of $f$ is the 1-form $df$ where

$$df_x(v) = v(f)$$

for $x \in \mathbb{R}^3$ and $v \in T_x\mathbb{R}^3$.

So $df_x$ is the directional derivative of $f$ at $x$ in the direction of $v$. The exterior derivative obeys Leibniz rule

$$d(fg) = gdf + fdg$$

and the chain rule

$$d(g(f)) = g'(f)df$$

where $f$ and $g$ are functions $f, g : \mathbb{R}^3 \to \mathbb{R}$.

**Theorem 2.2.2.** $d^2 \Phi = 0$, for any differentiable form $\Phi$ on $\mathbb{R}^3$

This is one of the most useful theorems we will be working with, and results from the fact that mixed partials commute.

**Definition 2.2.3** (1-form). A smooth 1-form, $\phi$, on the 3-dimensional vector space, $\mathbb{R}^3$, is a smooth section of the cotangent bundle $T^*\mathbb{R}^3$.

$$\phi : T\mathbb{R}^3 \to \mathbb{R}$$

The 1-forms $(dx^1, \ldots, dx^n)$ are defined by

$$dx^i(v) = v^i, \text{ for } v = (v^1, \ldots, v^n) \in T\mathbb{R}^3$$
We can multiply 1-forms together using the *wedge product*

**Definition 2.2.4** (Wedge product). The wedge product of two tensors, $v$ and $w$, denoted $v \wedge w$, is defined by

$$v \wedge w = v \otimes w - w \otimes v$$

where $\otimes$ is the standard tensor product.

From this definition we see that the wedge product is skew-symmetric, giving $v \wedge w = -w \wedge v$ and $v \wedge v = 0$. The 1-forms along with the wedge product form an algebra called the *algebra of differential forms*. Using the wedge product, if a form $\Phi$ is the product of $p$ 1-forms, then $\Phi$ is called a $p$-form.

The reason the method of moving frames is so powerful is that we can express the derivatives of the frame in terms of the frame itself. With this in mind let’s examine the derivatives of a frame $(x; e_1, \ldots, e_n)$. Let’s consider the exterior derivative of $x$ first. First, notice that $dx$ maps elements of the frame bundle to the tangent space $dx : F(\mathbb{R}^3) \to T\mathbb{R}^3$. We can now see how the method of moving frames comes into play. Because $dx$ maps into the tangent space and our frame forms a basis of the tangent space, we can express $dx$ in terms of the frame basis $\{e_1, \ldots, e_n\}$

$$dx = e_i \omega^i$$

where $\omega^i$ are scalar-valued 1-forms. Repeating this process for the frame basis gives us $de_i : T_F(\mathbb{R}^3) \to T_x \mathbb{R}^3$ where $f = (x; e_0, e_1, e_2) \in F \mathbb{R}^3$ is a frame of $\mathbb{R}^3$. So

$$de_i = e_j \omega^j_i$$

**Definition 2.2.5** (Dual forms and connection forms). The *dual forms*, $\omega^i$, are 1-forms defined by the equation

$$dx = e_i \omega^i$$

(2.2.1)

The *connection forms*, $\omega^j_i$, are 1-forms defined by the equation

$$de_i = e_j \omega^j_i$$

(2.2.2)

The *dual form* $\omega^i$ tells us how a surface changes in the $e_i$ direction. The *connection form* $\omega^j_i$ tells us how the frame vector $e_i$ changes in the $e_j$ direction. That is to say that for some vector $v \in \mathbb{R}^3$
• $\omega^i(v)$ tells us how the surface changes in the $e_i$ direction as we move in the $v$ direction along the surface.

• $\omega^j(v)$ tells us how $e_i$ changes in the $e_j$ direction as we move in the $v$ direction along the surface.

To see how the dual and connection forms behave let’s differentiate their defining equations. Differentiating (2.2.1) gives

\[
\begin{align*}
\frac{d}{dx}(dx) &= d(e_i \omega^i) \\
\frac{d^2}{d^2x}(x) &= de_i \wedge \omega^i + e_i \wedge d\omega^i \\
0 &= (e_j \omega^j_i) \wedge \omega^i + e_i \wedge d\omega^i \\
&= (e_i \omega^j_i) \wedge \omega^j + e_i \wedge d\omega^i \\
&= e_i(\omega^i_j \wedge \omega^j + d\omega^i)
\end{align*}
\]

differentiating (2.2.2) gives

\[
\begin{align*}
\frac{d}{dx}(d(e_i)) &= d(e_j \omega^j_i) \\
\frac{d^2}{d^2x}(e_i) &= de_j \wedge \omega^j_i + e_j \wedge d\omega^j_i \\
0 &= (e_k \omega^j_k) \wedge \omega^j_i + e_j \wedge d\omega^j_i \\
&= e_j(\omega^j_k \wedge \omega^k_i + d\omega^j_i) \\
&= e_j(\omega^j_k \wedge \omega^k_j + d\omega^j_i)
\end{align*}
\]

these equations lead us to the Cartan structure equations.

**Definition 2.2.6** (Cartan Structure Equations).

\[
\begin{align*}
d\omega^i &= -\omega^j \wedge \omega^i \\
d\omega^j_i &= -\omega^k \wedge \omega^j
\end{align*}
\]

We are able to unify the dual and connection forms by defining the Maurer-Cartan form, $\Omega$. Let

\[
g(f) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x & e_1 & \cdots & e_n \end{bmatrix}
\]

where $f = \{x; e_1, \ldots, e_n\} \in \mathbb{L}$. We define the Maurer-Cartan form $\Omega$ as

\[
\Omega = g^{-1}dg
\]
where

\[ dg = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ dx & de_1 & \cdots & e_n \end{bmatrix} \]

This means

\[ dg = g\hat{\Omega} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ dx & de_1 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ x & e_1 & \cdots & e_n \end{bmatrix} \hat{\Omega} \]

Recall from (2.2.1) and (2.2.2) that

\[ dx = e_i\omega^i \]
\[ de_i = e_j\omega^i_j \]

So

\[ \hat{\Omega} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \omega^1 & \omega^1_1 & \cdots & \omega^1_n \\ \vdots & \vdots & \ddots & \vdots \\ \omega^n & \omega^n_1 & \cdots & \omega^n_n \end{bmatrix} \]

We will denote the \( n \times n \) submatrix of connection forms as \( \Omega \) and the \( n \times 1 \) submatrix of dual forms as \( \omega \).

\[ \Omega = \begin{bmatrix} \omega^0_0 & \omega^0_1 & \omega^0_2 \\ \omega^1_0 & \omega^1_1 & \omega^1_2 \\ \omega^2_0 & \omega^2_1 & \omega^2_2 \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \end{bmatrix} \]

We end our discussion of forms with Cartan’s Lemma.

**Lemma 2.2.7** (Cartan’s Lemma). Suppose that \( \eta^1, \ldots, \eta^n \) are linearly independent 1-forms and that \( \phi_1, \ldots, \phi_n \) are 1-forms such that

\[ \phi_i \wedge \eta^i = 0 \]

Then there exist functions \( h_{ij} = h_{ji} \), symmetric in their lower indices, such that

\[ \phi_i = h_{ij}\eta^j \]
2.3 Minkowski space

Minkowski space is very similar to Euclidean space. The main difference is the metric imposed on the space.

**Definition 2.3.1** (Minkowski Space). The *Minkowski space* $\mathbb{M}^{(n,1)}$ is the vector space $\mathbb{R}^{n+1}$ with the metric

$$g_{\alpha,\beta} = \langle e_\alpha, e_\beta \rangle = \begin{cases} -1 & \alpha = \beta = 0 \\ 1 & \alpha = \beta = 1, \ldots, n \\ 0 & \alpha \neq \beta \end{cases}$$

Because the metric is not positive definite as it is in Euclidean space, we designate three different type of vectors

**Definition 2.3.2** (Vector types). A non-zero vector is called

*timelike* if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$

*spacelike* if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$

*lightlike* or *null* if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$

Throughout this paper we will be working in the 3 dimensional space $\mathbb{M}^{2,1}$. Let us find some of the basic properties of $\mathbb{M}^{2,1}$.

We want to find transformations, $A \in \text{GL}(3)$, which preserve the length of all vectors, $\mathbf{v} \in \mathbb{M}^{2,1}$.

$$\langle A\mathbf{v}, A\mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle$$

$$(A\mathbf{v})^t g (A\mathbf{v}) = \mathbf{v}^t g \mathbf{v}$$

$$\mathbf{v}^t A^t g A \mathbf{v} = \mathbf{v}^t g \mathbf{v}$$

Evidently we are looking for matrices, $A$, such that $A^t g A = g$, where

$$g = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the Minkowski metric. Matrices that satisfy this condition form a sub-group of $\text{GL}(3)$ called the *Lorentz group*, denoted by $O(2,1)$. The Lorentz group has four connected components. We will focus on *proper, orthochronous* Lorentz transformations, denoted by $SO^+(2,1)$. These transformations are in the connected component of $O(2,1)$ which contains the identity matrix.
The transformations in this group have a determinant equal to 1 and preserve the orientation of timelike vectors.

So far we have only focused on rotations; however the Minkowski inner product is also preserved under translations. Accounting for translations gives us the Poincaré group

$$M(2, 1) = \left\{ \begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix} : A \in SO^+(2, 1), b \in \mathbb{M}^{2,1} \right\}$$

Returning to the method of moving frames, we now have a metric and can begin to think about the basis we would like to assign to the tangent spaces of our surface.

**Definition 2.3.3** (Orthonormal frame). An (oriented) orthonormal frame in $\mathbb{M}^{2,1}$ is a list of vectors $f = (x; e_1, \ldots, e_n)$ with $x \in V$ and $\{e_1, \ldots, e_n\}$ an oriented, orthonormal (with respect to the Minkowski metric) basis for the tangent space $T_x V$.

Because we are working in $\mathbb{M}^{2,1}$ we will need 3 vectors to form a basis, $e_0, e_1, \text{ and } e_2$.

**Definition 2.3.4** (Principal bundle). A principal bundle consists of a total space $P$, a base space $B$ and a projection map $\pi : P \to B$ such that for each point $b \in B$, the inverse image $\pi^{-1}(b) \subset P$ is isomorphic to a Lie group.

Recalling the previous discussion of the method of moving frames and principal bundles, we now see that each fiber of the projection map $\pi(x; e_0, e_1, e_2) = x$ is isomorphic to $SO^+(2, 1)$, which is isomorphic to all the orthonormal frames at $x$. Hence, $M(2, 1)$ forms a principle bundle over $\mathbb{M}^{2,1}$ which is called the frame bundle, denoted by $F(\mathbb{M}^{2,1})$. 

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3. NULL SURFACES IN MINKOWSKI SPACE

We will be considering null surfaces in the 3-dimensional Minkowski space $\mathbb{M}^{2,1}$.

**Definition 3.0.5** (Null surface). A null surface, $\Sigma$, is a 2-D manifold whose first fundamental form is degenerate of rank 1 at every point of $\Sigma$.

**Definition 3.0.6** (First fundamental form). The first fundamental form is $I(v) = \langle dx(v), dx(v) \rangle$ for $v \in T\Sigma$.

The first fundamental form contains information about the metric and restricts the metric on $\mathbb{M}^{2,1}$ to tangent vectors in $T\Sigma$.

3.1 Choosing a Frame

Our first task is to pick a basis, $\{e_0, e_1, e_2\}$, for each tangent space which we can adapt to our surface, $\Sigma$. When picking the basis we will choose $e_0$ and $e_1$ to span the tangent space $T_x\Sigma$, and $e_2$ to be linearly independent of $e_0$ and $e_1$, for all points $x \in \Sigma$. The condition on $e_2$ means $e_2 \notin T_x\Sigma$. Because $\Sigma$ is null, the tangent space, $T_p\Sigma$, at each point, $p$, contains a unique null direction, $\nu$. We will choose our first basis vector, $e_0$, to be in this direction, $e_0 = \nu$. Note that because $e_0$ is null its length will always be zero, and we need not worry about normalizing it to unit length. We want the next basis vector, $e_1$, to be orthogonal to $e_0$, $\langle e_0, e_1 \rangle = 0$. However, because every vector in $T_x\Sigma$ which is linearly independent of $e_0$ is also orthogonal to $e_0$, we simply need to choose a vector which is linearly independent of $e_0$ and in $T_x\Sigma$. This means $e_1$ will be spacelike (i.e. $\langle e_1, e_1 \rangle > 0$) because of the degenerate metric on $T_x\Sigma$. We will normalize $||e_1|| = 1$. The main criteria for picking $e_2$ is that it be linearly independent of $e_0$, orthogonal to $e_1$, and not be in the surface $\Sigma$. In choosing $e_2$, let us turn our attention to the plane which is normal to $e_1$, call it $\Psi$. $\Psi$, by construction, will only contain vectors perpendicular to $e_1$, so we only need to worry about $e_2$’s relationship with $e_0$. Also, because $\Psi$ is orthogonal to a spacelike vector ($e_1$) it will be a timelike plane and contain exactly two linearly independent null directions.
Let us, therefore, restrict our choice of \( e_2 \) to be in \( \Psi \). For \( e_2 \notin T_x \Sigma \) we must choose it linearly independent of \( e_0 \). \( e_0 \) already contains one of the null directions, so we will choose \( e_2 \) to be the other null direction \( \Psi \) contains, \( \langle e_2, e_2 \rangle = 0 \). Because the inner product of two linearly independent null vectors is non-zero, \( \langle e_0, e_2 \rangle \neq 0 \), we can normalize \( e_2 \) such that \( \langle e_0, e_2 \rangle = 1 \).

We now have

\[
\langle e_0, e_0 \rangle = \langle e_0, e_1 \rangle = \langle e_1, e_2 \rangle = \langle e_2, e_2 \rangle = 0
\]

\[
\langle e_0, e_2 \rangle = \langle e_1, e_1 \rangle = 1
\]

These relations give the restricted metric on each tangent plane to our surface as

\[
\hat{g} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

A frame satisfying these conditions will be called \( \theta \)-adapted.

### 3.2 Finding Transformations

We will now examine how our basis vectors transform. We will begin with a general invertible transformation matrix

\[
A = \begin{bmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{bmatrix}
\]

When we apply \( A \) to our basis we transform the original basis to a new basis

\[
\{ \tilde{e}_0, \tilde{e}_1, \tilde{e}_2 \} = \{ e_0, e_1, e_2 \}A
\]

We still want the transformed basis to be a \( \theta \)-adapted basis, so the relations between the vectors \( \{ \tilde{e}_0, \tilde{e}_1, \tilde{e}_2 \} \) must be the same as the relations between \( \{ e_0, e_1, e_2 \} \).

The main thing to notice here is that our new basis can be expressed as a linear combination of the original basis vectors.

\[
\tilde{e}_0 = a_0 e_0 + b_0 e_1 + c_0 e_2
\]

\[
\tilde{e}_1 = a_1 e_0 + b_1 e_1 + c_1 e_2
\]

\[
\tilde{e}_2 = a_2 e_0 + b_2 e_1 + c_2 e_2
\]
We see immediately that $c_0 = c_1 = 0$ because $e_2 \notin T_x \Sigma$. If either $c_0 \neq 0$ or $c_1 \neq 0$ then $\tilde{e}_0$ and $\tilde{e}_1$ would not span $T_x \Sigma$. So $\tilde{e}_0$ and $\tilde{e}_1$ are linear combinations of $e_0$ and $e_1$ only. This also forces $c_2 \neq 0$, otherwise $A$ would not be invertible. Checking the basis relations gives us

\[
\langle \tilde{e}_0, \tilde{e}_0 \rangle = \langle e_0, e_0 \rangle = 0
\]
\[
\langle \tilde{e}_0, \tilde{e}_0 \rangle = a_0^2 \langle e_0, e_0 \rangle + b_0^2 \langle e_1, e_1 \rangle + 2a_0 b_0 \langle e_0, e_1 \rangle
\]
\[
\Rightarrow 0 = b_0^2
\]

Hence $b_0 = 0$, which means $\tilde{e}_0 = a_0 e_0$ for some $a_0 \in \mathbb{R}$.

\[
\langle \tilde{e}_1, \tilde{e}_1 \rangle = \langle e_1, e_1 \rangle = 1
\]
\[
\langle \tilde{e}_1, \tilde{e}_1 \rangle = a_1^2 \langle e_0, e_0 \rangle + b_1^2 \langle e_1, e_1 \rangle + 2a_1 b_1 \langle e_0, e_1 \rangle
\]
\[
\Rightarrow 1 = b_1^2
\]

So $b_1 = \pm 1$. We will choose $b_1 = +1$ for convenience. This gives $\tilde{e}_1 = a_1 e_0 + b_1 e_1 + c_1 e_2 = a_1 e_0 + e_1$ for some $a_1 \in \mathbb{R}$.

\[
\langle \tilde{e}_2, \tilde{e}_2 \rangle = \langle e_2, e_2 \rangle = 0
\]
\[
\langle \tilde{e}_2, \tilde{e}_2 \rangle = a_2^2 \langle e_0, e_0 \rangle + b_2^2 \langle e_1, e_1 \rangle + c_2^2 \langle e_2, e_2 \rangle + 2a_2 b_2 \langle e_0, e_1 \rangle + 2a_2 c_2 \langle e_0, e_2 \rangle + 2b_2 c_2 \langle e_1, e_2 \rangle
\]
\[
\Rightarrow 0 = b_2^2 + 2a_2 c_2
\]

So $b_2^2 = -2a_2 c_2$.

Similarly

\[
\langle \tilde{e}_0, \tilde{e}_1 \rangle = \langle e_0, e_1 \rangle = 0
\]
\[
\Rightarrow 0 = 0
\]

\[
\langle \tilde{e}_0, \tilde{e}_2 \rangle = \langle e_0, e_2 \rangle = 1
\]
\[
\langle \tilde{e}_0, \tilde{e}_2 \rangle = a_0 c_2
\]
\[
\Rightarrow 1 = a_0 c_2
\]
So \( c_2 = \frac{1}{a_0} \)

Finally

\[
\langle \tilde{e}_1, \tilde{e}_2 \rangle = \langle e_1, e_2 \rangle = 0 \\
\langle \tilde{e}_1, \tilde{e}_2 \rangle = a_1 c_2 + b_1 b_2 \\
\Rightarrow 0 = a_1 c_2 + b_1 b_2 \\
0 = a_1 c_2 + b_2
\]

We now have the equations

\[
b_0 = 0, b_1 = 1 \\
c_0 = 0, c_1 = 0, c_2 = \frac{1}{a_0}
\]

\[
b_2^2 = -2a_2 c_2 \\0 = a_1 c_2 + b_2
\]  \hspace{1cm} (3.2.1) \hspace{1cm} (3.2.2)

Plugging \( c_2 = \frac{1}{a_0} \) into (3.2.2) we get \( b_2 = -\frac{a_1}{a_0} \). Solving for \( a_2 \) in (3.2.1) gives \( a_2 = -\frac{b_2^2}{2c_2} = -\frac{a_1^2}{2a_0} \).

Let \( a_0 = \mu \) and \( a_1 = \lambda \). With these results we now have the transformation

\[
A = \begin{bmatrix}
\mu & \lambda & -\frac{\lambda^2}{2\mu} \\
0 & 1 & -\frac{\lambda}{\mu} \\
0 & 0 & \frac{1}{\mu}
\end{bmatrix}
\]

These transformations form a group called the *structure group*, \( G_0 \). This transformation is associated with the 0-adapted frame and will be referred to as \( A_0 \).

Now that we know what our transformations look like, let’s start exploring the invariants. We will do this by restricting the forms to our surface by examining the pullbacks of the dual forms and connections forms to our surface \( \Sigma \). (Note that there is some abuse of notation here, although we are now examining the pullbacks of the dual and connection forms, we will be using the same notation for them.)

### 3.3 Finding Invariants

Let us first examine the dual forms.
\[ dx = e_i \omega^i = e_0 \omega^0 + e_1 \omega^1 + e_2 \omega^2 \]

Because \( dx(v) \in \mathcal{T}_x \Sigma \) for \( v \in \mathcal{T}_x \Sigma \), and the fact that \( e_2 \notin \mathcal{T}_x \Sigma \) while \( \omega^0 \) and \( \omega^1 \) form a basis of \( \mathcal{T}_x \Sigma \), \( dx(v) \) must be a linear combination of \( e_0 \) and \( e_1 \).

Hence, \( \omega^2(v) = 0 \) \( \forall v \in \mathcal{T}_x \Sigma \). Keep this in mind for later.

Turning to the connection forms, \( de_i = e_j \omega^i_j \).

\begin{align*}
    de_0 &= e_0 \omega^0_0 + e_1 \omega^1_0 + e_2 \omega^2_0 \quad (3.3.1) \\
    de_1 &= e_0 \omega^0_1 + e_1 \omega^1_1 + e_2 \omega^2_1 \quad (3.3.2) \\
    de_2 &= e_0 \omega^0_2 + e_1 \omega^1_2 + e_2 \omega^2_2 \quad (3.3.3)
\end{align*}

Differentiating \( \langle e_0, e_0 \rangle = 0 \) and utilizing (3.3.1) gives

\[
0 = d\langle e_0, e_0 \rangle = \langle de_0, e_0 \rangle + \langle e_0, de_0 \rangle = 2\langle de_0, e_0 \rangle = 2\langle e_0 \omega^0_0 + e_1 \omega^1_0 + e_2 \omega^2_0, e_0 \rangle = 2(\omega^0_0 \langle e_0, e_0 \rangle + \omega^1_0 \langle e_0, e_1 \rangle + \omega^2_0 \langle e_0, e_2 \rangle) = 2(0 + 0 + \omega^2_0) = 2\omega^2_0
\]

So \( \omega^0_0 = 0 \).

Repeating this process for the other basis relations gives the following results

\begin{align*}
    d\langle e_1, e_1 \rangle &= 0 \rightarrow \omega^1_1 = 0 \\
    d\langle e_2, e_2 \rangle &= 0 \rightarrow \omega^0_2 = 0 \\
    d\langle e_0, e_1 \rangle &= 0 \rightarrow \omega^0_1 = -\omega^1_0 \\
    d\langle e_1, e_2 \rangle &= 0 \rightarrow \omega^0_1 = -\omega^1_0
\end{align*}

This gives us

\[
\Omega = \begin{bmatrix}
    \omega^0_0 & \omega^0_1 & \omega^0_2 \\
    \omega^1_0 & \omega^1_1 & \omega^1_2 \\
    \omega^2_0 & \omega^2_1 & \omega^2_2
\end{bmatrix} = \begin{bmatrix}
    \omega^0_0 & \omega^0_1 & 0 \\
    \omega^1_0 & 0 & -\omega^1_0 \\
    0 & -\omega^0_1 & -\omega^0_0
\end{bmatrix}
\]
Now that we have found some basic relations between the dual and connection forms, let’s see what happens to them when we transform our frame using $A$. We will begin by examining the dual forms.

\[ dx = e_0 \omega^0 + e_1 \omega^1 \quad \text{and} \quad dx = \tilde{e}_0 \tilde{\omega}^0 + \tilde{e}_1 \tilde{\omega}^1 \]

\[ \Rightarrow e_0 \omega^0 + e_1 \omega^1 = \tilde{e}_0 \tilde{\omega}^0 + \tilde{e}_1 \tilde{\omega}^1 \]

\[ = (\mu e_0) \tilde{\omega}^0 + (\lambda e_0 + e_1) \tilde{\omega}^1 \]

\[ = e_0 (\mu \tilde{\omega}^0 + \lambda \tilde{\omega}^1) + e_1 \tilde{\omega}^1 \]

This implies that

\[ \omega^0 = \mu \tilde{\omega}^0 + \lambda \tilde{\omega}^1 \quad \text{and} \quad \omega^1 = \tilde{\omega}^1 \]

so

\[ \tilde{\omega}^0 = \frac{1}{\mu} \omega^0 + \frac{1}{\lambda} \omega^1 \quad \text{and} \quad \tilde{\omega}^1 = \omega^1 \]

We can also view this in matrix form. Let $e$ denote the $3 \times 3$ matrix of frame vectors $[ e_0 \quad e_1 \quad e_2 ]$, where the $e_i$'s are column vectors.

\[ e = eA \]

\[ dx = e \omega = e \tilde{\omega} \]

\[ e \omega = e \tilde{\omega} \]

\[ = eA \tilde{\omega} \]

\[ \omega = A \tilde{\omega} \]

\[ \Rightarrow \tilde{\omega} = A^{-1} \omega \]

Let’s see what happens to the connection forms. (Remember that $A$ is a constant matrix.)

\[ d \omega = -\Omega \wedge \omega \]

\[ d \tilde{\omega} = -\tilde{\Omega} \wedge \tilde{\omega} \]

\[ d (A^{-1} \omega) = -\tilde{\Omega} \wedge (A^{-1} \omega) \]

\[ A^{-1} d \omega = -\tilde{\Omega} A^{-1} \wedge \omega \]

\[ d \omega = -A \tilde{\Omega} A^{-1} \wedge \omega \]

\[ -\Omega \wedge \omega = -A \tilde{\Omega} A^{-1} \wedge \omega \]

\[ -\Omega = -A \tilde{\Omega} A^{-1} \]

\[ \tilde{\Omega} = A^{-1} \Omega A \]
We now have
\[
\begin{bmatrix}
\tilde{\omega}_0 \\
\tilde{\omega}_1 \\
\tilde{\omega}_2
\end{bmatrix} =
\begin{bmatrix}
\frac{\omega^0 - \lambda \omega^1}{\mu} \\
\omega^1 \\
0
\end{bmatrix}
\]
and
\[
\tilde{\Omega} =
\begin{bmatrix}
\tilde{\omega}_0' & \tilde{\omega}_0'' & 0 \\
\tilde{\omega}_0' & 0 & -\tilde{\omega}_0'' \\
0 & -\tilde{\omega}_0' & -\tilde{\omega}_0''
\end{bmatrix} =
\begin{bmatrix}
\omega^0_0 - \lambda \omega^1_0 & \frac{2\lambda \omega^0_0 + 2\omega^0_0 - \lambda^2 \omega^1_0}{2\mu} & 0 \\
\mu \omega^1_0 & 0 & -2\lambda \omega^0_0 + 2\omega^0_0 - \lambda^2 \omega^1_0 \\
0 & -\mu \omega^1_0 & -\omega^0_0 + \lambda \omega^1_0
\end{bmatrix}
\]

From this point we look to further adapt the frame to the surface, \(\Sigma\). We will do this by examining how the forms change as we choose different frames. This will reveal certain parts of the transformation matrix which we can restrict.

Now, with Cartan’s lemma in mind, let’s differentiate \(\omega^2 = 0\).

\[
d(0) = d\omega^2 = 0 = -\omega^2_0 \wedge \omega^0 - \omega^2_1 \wedge \omega^1 - \omega^2_2 \wedge \omega^2 = -\omega^2_1 \wedge \omega^1 = \omega^1_0 \wedge \omega^1
\]

Invoking Cartan’s lemma gives us
\[
\omega^1_0 = c_1 \omega^1
\]
for some function \(c_1\) on \(\Sigma\). Let’s see what happens to \(c_1\) if we transform our frame.

\[
\begin{align*}
\tilde{\omega}_0' &= \tilde{c}_1 \tilde{\omega}_0' \\
\mu \omega^1_0 &= \tilde{c}_1 \omega^1_0 \\
\omega^1_0 &= \frac{\tilde{c}_1}{\mu} \omega^1 \\
c_1 \omega^1 &= \frac{\tilde{c}_1}{\mu} \omega^1 \\
c_1 &= \frac{\tilde{c}_1}{\mu} \\
\tilde{c}_1 &= \mu c_1
\end{align*}
\]

Where \(c_1\) is an arbitrary function on \(\Sigma\). \(c_1\) is a \textit{relative invariant}, meaning for any point \(\mathbf{x} \in \Sigma\), \(c_1\) is either zero for every frame at \(\mathbf{x}\), or nonzero for
every frame at $x$. However, the method of moving frames does not work well if $c_1$ is zero at some points and non-zero at others. For this reason we will use the constant type assumption which states that if a relative invariant is zero at any point of the surface it is identically zero (i.e. if $c_1(p_0) = 0$ for some $p_0 \in \Sigma$ then $c_1(p) = 0$, $\forall p \in \Sigma$ or if $c_1(p_0) \neq 0$ for some $p_0 \in \Sigma$ then $c_1(p) \neq 0$, $\forall p \in \Sigma$). If $c_1 \neq 0$ then we can choose $c_1$ to normalize our transformations. If $c_1 = 0$ then we can not normalize the transformations and will have to take another approach. For now let’s assume $c_1 \neq 0$; we will revisit the case of $c_1 = 0$ in the Examples section.

Let us choose a frame such that $c_1 = 1$ and restrict our attention to frames which satisfy this condition. This has two results. The first is $\omega_0^1 = c_1 \omega^1 = \omega^1$. The second is $\tilde{c}_1 = \mu c_1 \rightarrow 1 = \mu \cdot 1 \rightarrow \mu = 1$. So the transformation group between frames of this type form a subgroup of $G_0$, which we will call $G_1$, is defined by $\mu = 1$. The transformation matrix of this type is given by

$$A_1 = \begin{bmatrix} 1 & \lambda & -\frac{\lambda^2}{2} \\ 0 & 1 & -\lambda \\ 0 & 0 & 1 \end{bmatrix}$$

We will call this new frame 1-adapted and denote its associated transformation matrix by $A_1$.

Let us now differentiate the new equation $\omega_0^1 = \omega^1 \rightarrow \omega_0^1 - \omega^1 = 0$.

$$d\omega_0^1 = -\omega_0^1 \wedge \omega_0^0$$

$$d\omega^1 = -\omega_0^1 \wedge \omega^0$$

$$d\omega_0^1 - d\omega^1 = 0$$

$$( -\omega_0^1 \wedge \omega_0^0 ) - ( -\omega_0^1 \wedge \omega^0 ) = 0$$

$$-\omega^1 \wedge \omega_0^0 + \omega^1 \wedge \omega^0 = 0$$

$$\omega^1 \wedge ( -\omega_0^0 + \omega^0 ) = 0$$

Invoking Cartan’s lemma, this gives

$$c_2 \omega^1 = \omega_0^0 - \omega^0$$

for some function $c_2$ on $\Sigma$.

Again, let us see how $c_2$ changes under transformation (remember we are now using the 1-adapted transformation matrix).

$$\omega_0^0 = c_2 \omega^1 + \omega^0$$

$$\tilde{\omega}_0^0 = \tilde{c}_2 \omega^1 + \tilde{\omega}^0$$
\[
\omega_0^0 - \lambda \omega_0^1 = (\tilde{c}_2 \omega^1) + (\omega^0 - \lambda \omega^1)
\]
\[
\omega_0^0 = \tilde{c}_2 \omega^1 + \omega^0
\]
\[
c_2 \omega^1 + \omega^0 = \tilde{c}_2 \omega^1 + \omega^0
\]
\[
c_2 = \tilde{c}_2
\]

Evidently, \(c_2\) does not change under transformation. So \(c_2\) is a well-defined function of \(\Sigma\) which is an invariant of the surface. Here again we use the constant type assumption with \(c_2 \neq 0\) for all points \(x \in \Sigma\). We will examine the case of \(c_2 = 0\) in the Examples section. Moving forward, we will differentiate \(\omega_0^0 = c_2 \omega^1 + \omega^0\).

\[
d\omega_0^0 = d(c_2 \omega^1 + \omega^0)
\]
\[
d\omega_0^0 = dc_2 \wedge \omega^1 + c_2 \wedge d\omega^1 + d\omega^0
\]
\[
-\omega_1^0 \wedge \omega_0^1 = dc_2 \wedge \omega^1 + c_2 \wedge (-\omega_0^1 \wedge \omega^0) + (-\omega_0^0 \wedge \omega^0 - \omega_0^0 \wedge \omega^1)
\]
\[
0 = dc_2 \wedge \omega^1 - c_2 \omega_1^1 \wedge \omega^0 - \omega_0^0 \wedge \omega^0
\]
\[
0 = dc_2 \wedge \omega^1 - c_2 \omega_1^0 \wedge \omega^0 - (c_2 \omega^1 + \omega^0) \wedge \omega^0
\]
\[
0 = dc_2 \wedge \omega^1 - c_2 \omega_1^1 \wedge \omega^0 - c_2 \omega^1 \wedge \omega^0
\]
\[
0 = dc_2 \wedge \omega^1 + 2c_2 \omega^0 \wedge \omega^1
\]
\[
0 = (dc_2 + 2c_2 \omega^0) \wedge \omega^1
\]

Again invoking Cartan’s lemma gives

\[
c_3 \omega^1 = dc_2 + 2c_2 \omega^0
\]

for some function \(c_3\) on \(\Sigma\). Calculating how \(c_3\) changes under a transformation of frames

\[
\tilde{c}_2 = c_2
\]
\[
\Rightarrow d\tilde{c}_2 = dc_2
\]
\[
\Rightarrow \tilde{c}_3 \omega^1 - 2\tilde{c}_2 \omega^0 = c_3 \omega^1 - 2c_2 \omega^0
\]
\[
\tilde{c}_3 \omega^1 - 2c_2 (\omega^0 - \lambda \omega^1) = c_3 \omega^1 - 2c_2 \omega^0
\]
\[
\tilde{c}_3 \omega^1 - 2c_2 \omega^0 + 2c_2 \lambda \omega^1 = c_3 \omega^1 - 2c_2 \omega^0
\]
\[
\tilde{c}_3 \omega^1 + 2c_2 \lambda \omega^1 = c_3 \omega^1
\]
\[
\tilde{c}_3 \omega^1 = c_3 \omega^1 - 2c_2 \lambda \omega^1
\]
\[
\tilde{c}_3 \omega^1 = (c_3 - 2\lambda c_2) \omega^1
\]
\[
\Rightarrow \tilde{c}_3 = c_3 - 2\lambda c_2
\]
So $c_3$ is translated by a transformation of the frame. Let us choose a frame such that $c_3$ is 0.

$$\tilde{c}_3 = 0 \rightarrow \lambda = \frac{c_3}{2c_2} = 0$$

This gives us a new frame which will be called 2-adapted. Any two 2-adapted frames differ by a transformation with $\lambda = 0$. Again the 2-adapted frames form a subgroup of $G_1$ which we will call $G_2$, defined by $\lambda = 0$. The associated transformation matrix is

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

So $G_2$ is the identity subgroup. With this adaptation we have fully adapted the frame. This means at each point of our surface we have a unique choice of frame which is adapted to the surface. Let us turn to equations we have not yet examined.

Our next goal is to express the connection forms in terms of the dual forms. We will start with $\omega_1^0$. From our newly adapted frame we have

$$c_3 = 0 \rightarrow 0 = dc_2 + 2c_2 \omega^0$$

$$-dc_2 = 2c_2 \omega^0$$

$$-d(d(c_2)) = d(2c_2 \omega^0)$$

$$-d^2c_2 = 2(dc_2 \wedge \omega^0 + c_2 \wedge d \omega^0)$$

$$0 = (-2c_2 \omega^0 \wedge \omega^0) \wedge \omega^0 + c_2 \wedge (-\omega_0^0 \wedge \omega^0 - \omega_1^0 \wedge \omega^1)$$

$$0 = c_2 \omega_0^0 \wedge \omega^0 + c_2 \omega_1^0 \wedge \omega^1$$

$$0 = c_2 (c_2 \omega^1 + \omega^0) \wedge \omega^0 + c_2 \omega_0^0 \wedge \omega^1$$

$$0 = c_2 \omega^0 \wedge \omega^0 + c_2 \omega_0^0 \wedge \omega^1$$

$$0 = -c_2 \omega^0 \wedge \omega^1 + \omega_1^0 \wedge \omega^1$$

$$0 = (c_2 \omega^0 - \omega_1^0) \wedge \omega^1$$

From this, Cartan’s lemma gives

$$c_4 \omega^1 = c_2 \omega^0 - \omega_1^0$$

$$\omega_1^0 = c_2 \omega^0 + c_4 \omega^1$$

(3.3.4)
for some function $c_4$ on $\Sigma$.

Let us now compute $d\omega^0$; we will need it when we examine equation (3.3.4).

\[
d\omega^0 = -\omega^0_0 \land \omega^0 - \omega^0_1 \land \omega^1 = -(\omega^0 + c_2\omega^1) \land \omega^0 - (c_2\omega^0 + c_4\omega^1) \land \omega^1 = c_2\omega^1 \land \omega^0 - c_2\omega^0 \land \omega^1 = 0
\]

Let’s now differentiate equation (3.3.4).

\[
d\omega_0^1 = dc_2 \land \omega^0 + c_2 \land d\omega^0 + dc_4 \land \omega^1 + c_4 \land d\omega^1
-\omega_0^0 \land \omega_1^0 = -2c_2\omega^0 \land \omega^0 + 0 + dc_4 \land \omega^1 + c_4 \land (\omega^1 \land \omega^0)
-(c_2\omega^1 + \omega^0) \land (c_2\omega^0 + c_4\omega^1) = dc_4 \land \omega^1 + c_4\omega^1 \land \omega^0
-c_2^2\omega^1 \land \omega^0 - c_4\omega^0 \land \omega^1 = dc_4 \land \omega^1 + c_4\omega^1 \land \omega^0
(\omega_0^1 - c_4^2)\omega^0 \land \omega^1 = dc_4 \land \omega^1
\Rightarrow ((-2c_4 + c_2^2)\omega^0 - dc_4) \land \omega^1 = 0
\]

By Cartan’s lemma then, $dc_4 = (-2c_4 + c_2^2)\omega^0 + c_5\omega^1$ for some function $c_5$ on $\Sigma$.

Let’s pause here and recap our results before we move on. We began by choosing an orthonormal frame, \{e_0, e_1, e_2\}, such that $e_0$ and $e_1$ span the surface $\Sigma$, and $e_2$ linearly independent of $e_0$ and $e_1$. This led us to the frame equations

\[
\langle e_0, e_0 \rangle = \langle e_0, e_1 \rangle = \langle e_1, e_2 \rangle = \langle e_2, e_2 \rangle = 0
\]

\[
\langle e_0, e_2 \rangle = \langle e_1, e_1 \rangle = 1
\]

From these equations we arrived at our original transformation

\[
A_0 = \begin{bmatrix}
\mu & \lambda & -\frac{\lambda^2}{2\mu} \\
0 & 1 & \frac{\lambda}{mu}
0 & 0 & \frac{1}{\mu}
\end{bmatrix}
\]

After differentiating the frame equation we found

\[
\begin{bmatrix}
\tilde{\omega}^0 \\
\tilde{\omega}^1 \\
\tilde{\omega}^2
\end{bmatrix} = \begin{bmatrix}
\frac{\omega^0 - \lambda\omega^1}{\mu} \\
\frac{\omega^1}{\mu} \\
0
\end{bmatrix}
\]

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\[ \Omega = \begin{bmatrix} \omega_0^0 & \omega_0^1 & 0 \\ \omega_1^0 & 0 & -\omega_0^0 \\ 0 & -\omega_1^0 & -\omega_0^0 \end{bmatrix} \]

with
\[ \tilde{\omega}_0^0 = \omega_0^0 + c_2 \omega_1^1 \quad \tilde{\omega}_0^1 = \omega_1^1 \quad \tilde{\omega}_1^0 = c_2 \omega_0^0 + c_4 \omega_1^1 \]

From this we discovered \( \omega_0^1 = c_1 \omega_1^1 \). We moved on by assuming \( c_1 \neq 0 \) and normalizing \( c_1 = 1 \), which gave \( \mu = 1 \) and led to our first adaptation

\[ A_1 = \begin{bmatrix} 1 & \lambda & -\frac{\lambda^2}{2} \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix} \]

We then found \( c_2 \omega_1^1 = \omega_0^0 - \omega_0^1 \). However, we found that \( c_2 \) is invariant. We moved on by finding \( c_3 = \omega_1^1 = dc_2 + 2c_2 \omega_0^0 \) with \( \tilde{c}_3 = c_3 - 2\lambda c_2 \). Then we normalized our frame so \( c_3 = 0 \) which resulted in \( \lambda = 0 \) and led to our 2-adapted frame

\[ A_2 = I_3 \]

From here we examined the equation \( c_3 = 0 = dc_2 + 2c_2 \omega_0^0 \) which resulted in \( \omega_1^0 = c_2 \omega_0^0 + c_4 \omega_1^1 \). We then took a detour and found \( d\omega_0^0 = 0 \). After, we differentiated the equation for \( \omega_1^0 \) we found \( dc_4 = (-2c_4 + c_2^2) \omega_0^0 + c_5 \omega_1^1 \).

We now have everything we need to know to construct these null surface.

### 3.4 Constructing Surfaces

Because \( dx = e_0 \omega_0^0 + e_1 \omega_1^1 \), let’s begin by examining \( \omega_0^0 \) and \( \omega_1^1 \). We have already shown that \( d\omega_0^0 = 0 \), so \( \omega_0^0 \) is a closed form. Where a differential form, \( \alpha \) is closed if its exterior derivative is 0, \( d\alpha = 0 \). Poincaré gives us a useful lemma for dealing with closed forms.

**Lemma 3.4.1** (Poincaré’s lemma). Let \( X \) be a contractible open subset of \( \mathbb{R}^n \). Then any smooth closed \( p \)-form \( \alpha \) defined on \( X \) is exact, for \( p > 0 \).

So Poincaré’s Lemma tells us that \( \omega_0^0 \) is also exact. And for an exact form, \( \alpha \), locally there exists a differentiable function \( g \) such that \( \alpha = dg \). Hence there exists a function \( u \) on \( \Sigma \) such that \( \omega_0^0 = du \).

Turning to \( d\omega_1^1 \), we have already found that
\[
\begin{align*}
    d\omega_1^1 &= -\omega_1^1 \wedge \omega_0^0 \\
    &= -\omega_1^0 \wedge \omega_0^1 \\
    &= \omega_0^0 \wedge \omega_1^1
\end{align*}
\]
Theorem 3.4.2 (Frobenius’ theorem). Let $I$ be a differential ideal generated by the 1-form $\phi$ on the manifold $\Sigma$. Suppose $I$ is also generated algebraically by $\phi$. Then for any $p \in \Sigma$ there exists a 1-dimensional integral manifold of $I$. Furthermore, in a sufficiently small neighborhood of $p$ there exists a coordinate $y$ such that $I$ is generated by $dy$.

What the Frobenius theorem is telling us is that if a 1-form, $\phi$, generates an ideal on a surface, $\Sigma$, then there exists a coordinate function, $y$, on $\Sigma$ such that $d\phi$ is, locally, a multiple of the coordinate function at each point, $d\phi = \sigma y$ for some function $\sigma$ on $\Sigma$. Because $d\omega^1 \equiv 0 \pmod{\omega^1}$ it satisfies the conditions of the Frobenius theorem. Hence, there exists a function $\hat{v}$ on $\Sigma$ such that $\omega^1 = \sigma(u, \hat{v})d\hat{v}$ for some function $\sigma$ on $\Sigma$. So

$$d\omega^1 = \omega^0 \wedge \omega^1$$
$$= du \wedge \sigma(u, \hat{v})d\hat{v}$$
$$= \sigma(u, \hat{v})du \wedge d\hat{v}$$
$$\Rightarrow \omega^1 = e^u g(\hat{v})d\hat{v}$$

for some function $g$ on $\Sigma$. Calculating $d(g(\hat{v})d\hat{v})$.

$$d(g(\hat{v})d\hat{v}) = dg(\hat{v}) \wedge d\hat{v} + g(\hat{v})d(d\hat{v})$$
$$= g'(\hat{v})d\hat{v} \wedge d\hat{v} + g(\hat{v})d^2\hat{v}$$
$$= 0$$

So we can apply Poincaré’s lemma to $g(\hat{v})d\hat{v}$ giving

$$g(\hat{v})d\hat{v} = dv$$

for some function $v$ on $\Sigma$. Hence, $\omega^1 = e^u dv$.

Because $\omega^0$ and $\omega^1$ are linearly independent, $u$ and $v$ are functionally independent, and can therefore be used as local coordinates on our surface, $\Sigma$.

Examining the functions $c_2$ and $c_4$.

$$d(c_2) = -2c_2\omega^0$$
$$= -2c_2du$$
$$\Rightarrow c_2 = ae^{-2u}$$

for constant $a \in \mathbb{R}$. (The reason $a$ is not a function of $v$, as would normally be the case, is that the original equation for $dc_2$ admits no $v$-component, so $c_2$ is independent of $v$.)
\[ d(c_4) = (-2c_4 + c_2^2)\omega^0 + c_5\omega^1 \]
\[ = (-2c_4 + a^2e^{-4u})du + c_5e^u dv \]

Examining the \(u\)-component gives
\[ (c_4)_u = (-2c_4 + a^2e^{-4u}) \]
\[ \Rightarrow c_4 = -\frac{1}{2}a^2e^{-4u} + f(v)e^{-2u} \]

where \(f(v)\) is an arbitrary function of \(v\).

We now have everything we need to solve for our frame basis \(e_0, e_1, \) and \(e_2\). We first examine the differential equations for these basis vectors, putting them in terms of our local coordinates \(u\) and \(v\)

\[ de_0 = e_0\omega^0_0 + e_1\omega^1_0 \]
\[ \omega^0_0 = \omega^0 + c_2\omega^1, \quad \omega^1_0 = \omega^1 \]
\[ = e_0(\omega^0 + c_2\omega^1) + e_1\omega^1 \]
\[ = e_0\omega^0 + (c_2e_0 + e_1)\omega^1 \]
\[ = c_2e_0du + (c_2e_0 + e_1)e^u dv \]

\[ de_1 = e_0\omega^0_1 - e_2\omega^1_0 \]
\[ \omega^0_1 = c_2\omega^0 + c_4\omega^1, \quad \omega^1_0 = \omega^1 \]
\[ = e_0(c_2\omega^0 + c_4\omega^1) - e_2\omega^1 \]
\[ = c_2e_0\omega^0 + (c_4e_0 - e_2)\omega^1 \]
\[ = c_2e_0du + (c_4e_0 - e_2)e^u dv \]

\[ de_2 = -(e_1\omega^0_1 + e_2\omega^0_0) \]
\[ \omega^0_1 = c_2\omega^0 + c_4\omega^1, \quad \omega^0_0 = \omega^0 + c_2\omega^1 \]
\[ = -[e_1(c_2\omega^0 + c_4\omega^1) + e_2(\omega^0 + c_2\omega^1)] \]
\[ = -c_2e_1\omega^0 - c_4e_1\omega^1 - e_2\omega^0 - c_2e_2\omega^1 \]
\[ = (-c_2e_1 - e_2)\omega^0 + (-c_4e_1 - c_2e_2)\omega^1 \]
\[ = -c_2(e_1 + e_2)du - (c_4e_1 + c_2e_2)e^u dv \]

We can now begin to integrate these differential equations to construct our frame in terms of the local coordinates \(u\) and \(v\). We begin by examining
the $u$-components of these equations.

\[(e_0)_u = e_0\]
\[\Rightarrow e_0 = e^uF_0(v)\]

where $F_0(v)$ is an arbitrary, vector-valued function of $v$.

\[(e_1)_u = c_2e_0\]
\[= ae^{-2u} \cdot e^uF_0(v)\]
\[= ae^{-u}F_0(v)\]
\[\Rightarrow e_1 = -aF_0(v)e^{-u} + F_1(v)\]

where $F_1(v)$ is an arbitrary, vector-valued function of $v$.

Turning to the $u$-components

\[(e_2)_u = -c_2e_1 - e_2\]
\[= -(ae^{-2u}(ae^{-u}F_0(v) + F_1(v)) - e_2\]
\[= a^2e^{-3u}F_0(v) - aF_1(v)e^{-2u} - e_2\]
\[\Rightarrow e_2 = -\frac{1}{2}a^2e^{-3u}F_0(v) + ae^{-2u}F_1(v) + e^{-u}F_2(v)\]

where $F_2(v)$ is an arbitrary, vector-valued function of $v$.

Turning to the $v$-components

\[(e_0)_v = (c_2e_0 + e_1)e^u\]
\[e^uF'_0(v) = (ae^{-2u}e^uF_0(v) - aF_0(v)e^{-u} + F_1(v))e^u\]
\[F'_0(v) = ae^{-u}F_0(v) - ae^{-u}F_0(v) + F_1(v)\]
\[\Rightarrow F'_0(v) = F_1(v)\]

\[(e_1)_v = (c_2e_0 - e_2)e^u\]
\[(-ae^{-u}F_0(v) + F_1(v)) = ((-\frac{1}{2}a^2e^{-4u} + e^{-2u}f(v))(e^uF_0(v)) -\]
\[\frac{1}{2}a^2e^{-3u}F_0(v) + ae^{-2u}F_1(v) + e^{-u}F_2(v))]e^u\]
\[-ad^{-u}F'_0(v) + F'_1(v) = (-\frac{1}{2}a^2e^{-3u}F_0(v) + e^{-uf(v)F_0(v)} -\]
\[e^{-u}F_2(v) + \frac{1}{2}a^2e^{-3u}F_0(v) - ae^{-2u}F_1(v))e^u\]
\[-ad^{-u}F'_0(v) + F'_1(v) = f(v)F_0(v) - F_2(v) - ae^{-u}F_1(v)\]
\[F'_1(v) = f(v)F_0(v) - F_2(v)\]

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\((e_2)_v = -(c_4e_1 + c_2e_2)e^u\)

\[
\begin{align*}
(-\frac{1}{2}a^2e^{-3u}F_0(v) + ae^{-2u}F_1(v) + e^{-u}F_2(v))_v &= ((-\frac{1}{2}a^2e^{-4u} + e^{-2u}f(v)) \\
&\quad (-ae^{-u}F_0(v) + F_1(v)) + \frac{1}{2}a^2e^{-3u}F_0(v))e^u \\
&\quad (ae^{-2u})e^{-u}F_2(v) + ae^{-2u}F_1(v) - \\
&\quad \frac{1}{2}a^2e^{-3u}F_0(v))e^u \\
&\frac{1}{2}a^2e^{-3u}F_1(v) + ae^{-2u}(f(v)F_0(v) - F_2(v)) = (-\frac{1}{2}a^3e^{-5u}F_0(v) + \frac{1}{2}a^2e^{-4u}F_1(v) + \\
&\quad ae^{-3u}f(v)F_0(v) - e^{-2u}f(v)F_1(v) - \\
&\quad ae^{-3u}F_2(v) - a^2e^{-4u}F_1(v) + \\
&\quad \frac{1}{2}a^3e^{-5u}F_0(v))e^u \\
&\frac{1}{2}a^2e^{-3u}F_1(v) + ae^{-2u}(f(v)F_0(v) - \frac{1}{2}a^2e^{-3u}F_2(v) = (-\frac{1}{2}a^3e^{-4u}F_0(v) + \frac{1}{2}a^2e^{-3u}F_1(v) + \\
&\quad ae^{-2u}f(v)F_0(v) - e^{-u}f(v)F_1(v) - \\
&\quad ae^{-2u}F_2(v) - a^2e^{-3u}F_1(v) + \\
&\quad \frac{1}{2}a^3e^{-4u}F_0(v) \\
&\frac{1}{2}a^2e^{-3u}F_1(v) + ae^{-2u}(f(v)F_0(v) - ae^{-2u}F_2(v) = (-\frac{1}{2}a^2e^{-3u}F_1(v) - ae^{-2u}f(v)F_0(v) - \\
&\quad e^{-u}f(v)F_1(v) - \\
&\quad ae^{-2u}F_2(v) \\
&\quad \Rightarrow F'_2(v) = -e^{-u}f(v)F_1(v) \\
&\quad \Rightarrow F'_2(v) = -f(v)F_1(v)
\end{align*}
\]

These equations give us the system of equations

\[
\begin{align*}
\mathbf{e}_0 &= e^uF_0(v) \quad (3.4.1) \\
\mathbf{e}_1 &= -ae^{-u}F_0(v) + F_1(v) \quad (3.4.2)
\end{align*}
\]

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\[ e_2 = -\frac{1}{2}a^2 e^{-3u} F_0(v) + ae^{-2u} F_1(v) + e^{-u} F_2(v) \]  

(3.4.3)

where

\[ F'_0(v) = F_1(v) \]  

(3.4.4)

\[ F'_1(v) = f(v) F_0(v) - F_2(v) \]  

(3.4.5)

\[ F'_2(v) = -f(v) F_1(v) \]  

(3.4.6)

We can solve these equations, forming a third order ODE for \( F_0(v) \) giving

\[ 0 = F_0(v)^{''' - 2f(v) F'_0(v) + f'(v) F_0(v)} \]  

(3.4.7)

We now have our frame expressed in terms of arbitrary functions, \( f(v), F_0(v), F_1(v), \) and \( F_2(v) \) which means we can choose \( f(v) \) to be any arbitrary function as long as \( F_0(v), F_1(v), \) and \( F_2(v) \) satisfy equations (3.4.4), (3.4.5), and (3.4.6), and see what type of surface they produce using the differential equation for our surface

\[
\begin{align*}
\,d\mathbf{x} & = e_0 \omega^0 + e_1 \omega^1 \\
& = e_0 du + e_1 e^u dv \\
& = e^u F_0(v) du + (ae^{-u} F_0(v) + F_1(v)) e^u dv \\
& = e^u F_0(v) du + (-ae^{-u} F_0(v) + F'_0(v)) e^u dv \\
& = e^u F_0(v) du + (e^u F'_0(v) - aF_0(v)) dv
\end{align*}
\]

**Theorem 3.4.3** (Characterization of null surfaces in \( \mathbb{M}^{2,1} \)). Let \( \Sigma \) be a regular, differentiable null surface in \( \mathbb{M}^{2,1} \). Then \( \Sigma \) is a plane, cone, or can be locally characterized by coordinates \( u \) and \( v \) and arbitrary function \( f(v) \). With

\[
\begin{align*}
\,d\mathbf{x} & = e^u F_0(v) du + (e^u F'_0(v) - aF_0(v)) dv \\
\end{align*}
\]

satisfying the differential equation

\[ 0 = F_0(v)^{''' - 2f(v) F'_0(v) - f'(v) F_0(v)} \]
4. EXAMPLES

Before we start trying to reconstruct surface using equations (3.4.1) - (3.4.6), recall that we arrived equations (3.4.1), (3.4.2), and (3.4.3) by assuming $c_1 \neq 0$ and $c_2 \neq 0$. So let’s go back and see what happens if $c_1 = 0$ or $c_2 = 0$.

4.1 $c_1 = 0$

As a quick reminder, we got $c_1$ by differentiating $\omega^2 = 0$ and applied Cartan’s lemma, giving $\omega^0_1 = c_1^1$. When transformed $\tilde{c}_1 = \mu c_1$. Letting $c_1 = 0$ we get $\omega^0_1 = 0$, and can no longer adapt our frame to be 1-adapted. Let’s now turn to the equations of the frame.

$$d e_0 = e_0^{0} \omega^0 + e_1 \omega^1_0$$

$$= e_0 \omega^0_0$$

$$\Rightarrow d e_0 \equiv 0 \pmod{e_0}$$

So $d e_0$ is a multiple of $e_0$ meaning the line spanned by $e_0$ is constant. Looking how $e_1$ changes gives

$$d e_1 = e_0 \omega^0_1$$

$$\Rightarrow e_1 \equiv 0 \pmod{e_0, e_1}$$

So the plane spanned by $e_0$ and $e_1$ is constant. Because $e_0$ and $e_1$ span $T_x \Sigma$ for every $x \in \Sigma$, the tangent plane to $\Sigma$ is the same at each point. Hence, $\Sigma$ must be contained in the plane spanned by $e_0$ and $e_1$, therefore $\Sigma$ must be a plane.

4.2 $c_1 \neq 0, c_2 = 0$

Recall we arrived with $c_2$ after adapting our frame transformations for the first time, then differentiating $\omega^1_0 = \omega^1$. After applying Cartan’s lemma we arrived at $c_2 \omega^1 = \omega^0_0 - \omega^0$. Letting $c_2 = 0$ we have

$$\omega^0_0 = \omega^0$$
From this we have

\[ de_0 = e_0^0 \omega^0_0 + e_1^1 \omega^1_0 \]
\[ = e_0^0 \omega^0_0 + e_1^1 \omega^1_0 \]

We also know that \( dx = e_0^0 \omega^0_0 + e_1^1 \omega^1_1 \), hence

\[ dx = de_0 \]
\[ \Rightarrow 0 = dx - de_0 
\]
\[ = d(x - e_0) \]

So \( x - e_0 = p \rightarrow x - p = e_0 \) for some constant \( p \in M^{2,1} \). Hence, \( x \) is contained in the null cone defined by \( \langle x - p, x - p \rangle = 0 \).

Now let’s see some examples if both \( c_1 \neq 0 \) and \( c_2 \neq 0 \).
4.3 $f(v) = 0$

$$x(u, v) = \begin{bmatrix}
-1 - av - \frac{a^2v^2}{2} + \frac{av^3}{6} - \frac{a^3v^3}{12} + \frac{1}{4}e^u (4 + 4av - 2v^2 + a^2v^3) \\
-1 - av - \frac{a^2v^2}{2} - \frac{a^3v^3}{12} + \frac{1}{4}e^u (2 + av)^2 \\
-2av^2 + \frac{av^3}{6} - \frac{a^2v^3}{6} + \frac{1}{2}e^u v(2 + (-1 + a)v)
\end{bmatrix}$$
4.4 \( f(v) = 1 \)
4.5 \( f(v) = -1 \)
5. WHERE TO GO FROM HERE

- From equation (3.4.7), in the simple case of \( f(v) = a \in \mathbb{R} \) is constant we have
  \[
  0 = F'''_0(v) - 2aF'_0(v)
  \]
  Letting \( G_0(v) = F'_0(v) \) we have
  \[
  0 = G''_0(v) - 2aG_0(v)
  \]
  This is a simple Sturm-Liouville operator with the weight function \( w(x) = 1 \). Because Sturm-Liouville operators are so well studied it would be interesting to see what things this theory can tell us about the null surfaces in this case.

- The next obvious generalization is to increase the dimensionality of the space to \( M^{3,1} \), then to \( M^{n,k} \).

- Because Minkowski space’s main application is in physics relating to Maxwell’s equations and Einstein’s special theory of relativity, it would be interesting to examine if there are any implications regarding the application of these surfaces to physics. Perhaps there are physical interpretations of the invariants we found. Also, because we examined null surfaces, if a photon is on one of these surfaces then it is restricted to that surface as long as it is traveling through free space. It would be interesting to examine the physical interpretations of this result.

- The next generalizations to make would be to see if these results can be generalized to DeSitter space and applied to the general theory of relativity.
A. MATHEMATICA CODE USED FOR CREATING EXAMPLES

```mathematica
getRelations[func_, verbose_: False, veryVerbose_: False] := Module[
{dSol, f0, f1, f2, func0, func1, func2, f0Vec, f1Vec, f2Vec, e0Rel, e1Rel, e2Rel},
If[veryVerbose == True, verbose = True];
Print["Using f(v) = ", func[v], \\
If[verbose == True, 
Print["Finding the v-Functions"];
]
dSol = DSolve[{f0'[v] == f1[v], 
  f1'[v] == func[v]*f0[v] - f2[v], 
  f2'[v] == -func[v]*f1[v]}, 
  {f0, f1, f2}, v];
func0 = f0 /. dSol[[1]]; 
func1 = f1 /. dSol[[1]]; 
func2 = f2 /. dSol[[1]]; 
If[veryVerbose == True, 
Print["f0: ", func0[v], \\
Print["f1: ", func1[v], \\
Print["f2: ", func2[v], 
];
If[verbose == True, 
Print["Finding the basis vectors"];
e0Rel := Function[{u, v}, Exp[u]*func0[v]]; 
e1Rel := Function[{u, v}, -a*Exp[-u]*func0[v] + func1[v]]; 
e2Rel := Function[{u, v}, -a^2/2*Exp[-3u]*func0[v] + a*Exp[-2u]*func1[v] + Exp[-u]*func2[v]]; 
If[veryVerbose == True, 
Print["e0: ", e0Rel[u, v], 
Print["e1: ", e1Rel[u, v], 
Print["e2: ", e2Rel[u, v], 
]
```
Print["e1: " e1Rel [u,v]"\n"]
Print["e2: " e2Rel [u,v]"\n"]
;

Return[{e0Rel,e1Rel,e2Rel}];
)]

makeExample[func_,verbose_:False,veryVerbose_:False]:= Module[{eRel,constSol0,
constSol1,constSol2,e0,e1,e2,surfSol0,surfSol1,surfSol2,surf0,surf1,surf2,
constSurf0,constSurf1,constSurf2,surf,x},(

eRel=getRelations[func];
If[veryVerbose==True,verbose=True];

If[verbose==True,
Print["\nSolving for constants using 
0th Component:",Print[constSol0];
Print["1st Component:",Print[ constSol1];
Print["2nd Component:",Print[constSol2];
];

If[verbose==True,
Print["\nApplying initial conditions to the basis vectors"];

e0:=Function[{u,v},{eRel[[1]][u,v]/.constSol0[[1]],
eRel[[1]][u,v]/.constSol1[[1]],
eRel[[1]][u,v]/.constSol2[[1]]}];
e1:=Function[{u,v},{eRel[[2]][u,v]/.constSol0[[1]],
eRel[[2]][u,v]/.constSol1[[1]],
eRel[[2]][u,v]/.constSol2[[1]]}];
e2:=Function[{u,v},{eRel[[3]][u,v]/.constSol0[[1]],
eRel[[3]][u,v]/.constSol1[[1]],
eRel[[3]][u,v]/.constSol2[[1]]}];

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If[veryVerbose==True,
Print["e0:"];Print[e0[u,v]];
Print["e1:"];Print[e1[u,v]];
Print["e2:"];Print[e2[u,v]];
];
If[verbose==True,
Print["\nCreating the surface"];
]

surfSol0=DSolve[{D[x[u,v],u]==e0[u,v][[1]],
D[x[u,v],v]==Exp[u]e1[u,v][[1]]},x,{u,v}];
surf0=x/.surfSol0[[1]];

surfSol1=DSolve[{D[x[u,v],u]==e0[u,v][[2]],
D[x[u,v],v]==Exp[u]e1[u,v][[2]]},x,{u,v}];
surf1=x/.surfSol1[[1]];

surfSol2=DSolve[{D[x[u,v],u]==e0[u,v][[3]],
D[x[u,v],v]==Exp[u]e1[u,v][[3]]},x,{u,v}];
surf2=x/.surfSol2[[1]];

If[veryVerbose==True,
Print["0th Component:"];Print[surf0[u,v]];
Print["1st Component:"];Print[surf1[u,v]];
Print["2nd Component:"];Print[surf2[u,v]];
];

If[verbose==True,
Print["\nSolving for the constants, placing X(0,0)={0,0,0}"];
]

constSurf0=Solve[surf0[0,0]==0,C[1]];
constSurf1=Solve[surf1[0,0]==0,C[1]];
constSurf2=Solve[surf2[0,0]==0,C[1]];

If[veryVerbose==True,
Print["0th Component:"];Print[constSurf0];
Print["1st Component:"];Print[constSurf1];
Print["2nd Component:"];Print[constSurf2];
];
If[verbose==True,
Print["\nApplying the initial conditions" ];];

surf=Function[{u,v},{surf0[u,v]/.constSurf0[[1]],
surf1[u,v]/.constSurf1[[1]],
surf2[u,v]/.constSurf2[[1]]}];

Print["X(u, v) = "]; Print[MatrixForm[surf[u,v]]];

Return[surf];
"

plotExample[ex_,c_,range_:2]:=Module[{fx, fy, fz},
ParametricPlot3D[{ex[u,v][[1]]/.a->c,
ex[u,v][[2]]/.a->c,ex[u,v][[3]]/.a->c},
{u,-range,range},{v,-range,range}]]
BIBLIOGRAPHY


