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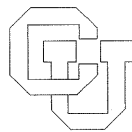
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**A Good Algorithm for Smallest Spanning Trees
With a Degree Constraint**

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DEPARTMENT OF COMPUTER SCIENCE

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A GOOD ALGORITHM FOR SMALLEST SPANNING
TREES WITH A DEGREE CONSTRAINT

by

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Abstract

Given a connected graph with edge costs, we seek a spanning tree having a specified degree at one vertex r , with cost as small as possible. A previous algorithm, using edge exchanges, has run time $O(V^2)$; we improve this to $O(E \log \log V + V \log V)$. Here V and E are the number of vertices and edges. The algorithm uses edge exchanges ordered efficiently on a reduced graph; it also uses efficient algorithms for minimum spanning trees and priority queues.

1. Introduction

Many variations of the minimum spanning tree problem occur in communication networks and computer networks ([H,P]). These problems are often intractable ([G,P]). In contrast, the problem we discuss enjoys an efficient solution.

We are given a connected undirected graph G , a vertex r , and a positive integer b . Each edge e in G has a real-valued cost $c(e)$. If T is a spanning tree, then $d(r)$ denotes the degree of r in T ; the cost of T is $c(T) = \sum_{e \in T} c(e)$. The problem is to find a spanning tree T with $d(r) = b$, having smallest cost among all such trees.

In an application, r might represent a central computing site; the other vertices are terminals which must be linked to r by cable paths. The degree constraint guarantees the computer's load is spread over a given number of ports. The smallest cost property guarantees as little cable as possible is used.

This problem was first discussed by Glover and Klingman [GK]. They sketch an algorithm that can be implemented in $O(V^2)$ time. We improve this to $O(E \log \log V + V \log V)$. Here V and E are the number of vertices and edges. Our algorithm is based on Glover and Klingman's technique of "admissible edge exchanges", coupled with a method for pruning the graph and efficiently ordering the exchanges. It also uses efficient algorithms for minimum spanning trees [CT,Y] and priority queues [AHU]. Section 2 derives some useful properties of spanning trees. Section 3 gives the algorithm and extends it to \geq and \leq degree constraints. Section 4 gives concluding remarks.

2. Optimal Edge Exchanges

The basic step of the algorithm is to make an edge exchange on a tree, i.e., starting with a spanning tree T , we replace an edge $e \in T$ by an edge $f \notin T$, obtaining a new tree $T-e+f$. This section describes how to choose the exchanges optimally. The results are implicit or actually stated in [GK]. Our development is self-contained and more direct.

Let R be the set of all edges incident to r in G . Let \mathcal{T}_k be the set of all spanning trees T with $d(r) = |T \cap R| = k$. We seek a tree in \mathcal{T}_b with smallest cost possible. For example, Figure 1(a) shows a graph, where an edge label both identifies the edge and specifies its cost. Figure 1(b) indicates a smallest tree T_k in each \mathcal{T}_k .

Note in general, for some $u \geq 1$, $\mathcal{T}_k \neq \emptyset$ exactly when $u \leq k \leq |R|$. For if T is a spanning tree with $d(r) = u$, any $k-u$ edges incident to r can be added (and other edges deleted) to get a tree with $d(r) = k$. Note further, it is possible that $u > 1$, if $G-r$ is not connected.

In Figure 1, T_k and T_{k+1} are related by an edge exchange, e.g., $T_2 = T_1 - 4 + 2$. We now prove this is true in general.

Theorem 1: Let T be a smallest tree in \mathcal{T}_k . Then

- (a) If $\mathcal{T}_{k-1} \neq \emptyset$, there are edges $e \in T \cap R$, $f \notin T \cup R$, such that $T-e+f$ is a smallest tree in \mathcal{T}_{k-1} ;
- (b) if $\mathcal{T}_{k+1} \neq \emptyset$, there are edges $e \in T-R$, $f \in R-T$ such that $T-e+f$ is a smallest tree in \mathcal{T}_{k+1} .

Proof: We prove (a) only, since (b) is analogous. Let S be a smallest tree in \mathcal{T}_{k-1} containing as many edges of T as possible. We will find edges e, f so $S = T-e+f$.

Choose f to be an edge in $S-R-T$. Note f exists, since $|S-R| > |T-R|$. To choose e , note there is a path in T joining the ends of f . Let e be an edge in this path that joins the two connected components of $S-f$. Thus both $T-e+f$ and $S-f+e$ are trees.

Suppose $e \notin R$. Then $T-e+f \in T_k$, so $c(e) \leq c(f)$, and $S-f+e \in T_{k-1}$, so $c(f) \leq c(e)$. We conclude $c(e) = c(f)$. Thus $S-f+e$ is a smallest tree in T_{k-1} containing more edges of T than S does. This contradicts the definition of S , and shows $e \in R$.

Thus $T-e+f \in T_{k-1}$, so $c(T-e+f) \geq c(S)$, and $S-f+e \in T_k$, so $c(S-f+e) \geq c(T)$. The two inequalities imply $c(T-e+f) = c(S)$, whence $T-e+f = S$, as desired. \square

It is easy to characterize the exchanges of Theorem 1. We do this for part (a); a similar result holds for part (b), but is not needed here.

Corollary 1: Let T be a smallest tree in T_k . If edges $e \in T \cap R$, $f \notin T \cup R$ are chosen so $T-e+f$ is a tree and $c(f) - c(e)$ is smallest among such edges, then $T-e+f$ is a smallest tree in T_{k-1} . \square

We can also find a reduced set of edges that contains smallest trees in all T_k 's. For this, recall a spanning forest of a graph consists of a spanning tree for each connected component.

Corollary 2: Let U be a minimum cost spanning forest of $G-r$. Then $U \cup R$ contains a smallest tree in T_k , if $T_k \neq \emptyset$.

Proof: Any spanning tree of G contains an edge from r to each tree in U . So u , the number of trees in U , is the least k with $T_k \neq \emptyset$. A smallest tree in T_u consists of U and the smallest edge joining r to each tree. Now proceed by induction on k , using Theorem 1(b). \square

3. The Algorithm

The basic idea is to find a smallest spanning tree containing R ; then execute exchanges as in Corollary 1, until $d(r)$ decreases to b . To speed up the computation, we first find a minimum spanning forest U in $G-r$; then only edges in $U \cup R$ need be considered, as shown by Corollary 2.

To find the exchanges specified by Corollary 1, we use a system of priority queues. Let T be the spanning tree at a given point in the computation. Consider an edge $e \in T \cap R$. An exchange that removes e from T adds an edge f joining the two connected components of $T-e$. A priority queue $F(e)$ stores all such edges f . The priority of f is its cost $c(f)$. Thus the smallest edge f that can replace e is easily found. A priority queue X stores exchanges (e,f) , where for each edge $e \in T \cap R$, f is the smallest edge that can replace e . The priority of (e,f) is $c(f)-c(e)$. Thus the smallest exchange of Corollary 1 is easily found in X .

Now we state the algorithm in Pidgin ALGOL [AHU].

procedure D; begin comment Input to D consists of a graph G , a vertex r , and an integer b , where $T_b \neq \emptyset$. Output is T , a smallest spanning tree with $d(r)=b$;

comment reduce the graph;

1. let U be a minimum spanning forest of graph $G-r$; let R be the set of edges of G incident to r ;
2. remove all edges from G except those in $U \cup R$;
3. let T be a smallest spanning tree containing R ;
comment initialize priority queues;
4. make X an empty priority queue;
5. for each edge $e \in R$ do

```
begin
6. let F(e) be a priority queue containing all edges  $f \in U$  that join
   components of  $T-e$ ;
7. if  $F(e) \neq \emptyset$  then begin let f be the smallest edge in F(e);
   add (e,f) to X end;
   end;
comment reduce r's degree by exchanges;
8. while  $d(r) > b$  do
   begin
9. remove the smallest exchange (e,f) from X; remove f from F(e);
10. let e' be the edge in  $R-e$  such that f joins the components of  $T-e'$ ;
11. remove the exchange (e',f') containing e' from X;
    remove f from F(e');
12. merge F(e) and F(e') into a new priority queue F(e');
13. if  $F(e') \neq \emptyset$  then begin let f' be the smallest edge in F(e');
    add (e',f') to X end;
14.  $T \leftarrow T-e+f$ ;
    end;
end;
```

For Figure 1, in lines 1-3, $U=\{3,4\}, R=T=\{1,2,6\}=T_3$. The first exchange, (6,3), changes T to T_2 ; then exchange (2,4) changes T to T_1 . Figure 1 also shows a "greedy" approach, similar to Kruskal's spanning tree algorithm [K], does not work for our problem: Suppose we form T by choosing edges, smallest first, subject to the constraints that T is acyclic and $d(r) \leq b$. If $b \leq 2$, T_b is found. But for $b=3$, T_2 is found, so the method fails.

Theorem 2: Procedure D finds a smallest spanning tree with $d(r)=b$.

The time is $O(E \log \log V + V \log V)$ and the space is $O(E+V)$.

Proof: Correctness of D follows from Section 2, if we show line 9 selects an exchange with smallest possible cost. To do this, consider any any edge $f \in U-T-R$. The path in T joining the ends of f contains two edges of R, since any cycle in $U \cup R$ contains two edges of R. So for exactly two edges $e \in T \cap R$, (e, f) is a valid exchange. By induction, $f \in F(e)$ for these two edges e and no others. So the F queues record all valid exchanges. It follows easily that X contains a smallest cost exchange, and line 9 works correctly.

Now we analyze the time. The forest U is found (step 1) in $O(E \log \log V)$ time, by using an efficient minimum spanning tree algorithm [CT,Y]. Tree T is found (step 3) by starting with edges $U \cup R$, contracting R into a single vertex, finding a minimum spanning tree, and then restoring R. The time is $O(V \log \log V)$. (This can be reduced to $O(V)$ using the techniques of [PS].).

The priority queue operations are adding an element (steps 6, 7, 13), finding and removing a smallest element (steps 7, 9, 13), removing an arbitrary element (twice in step 11), and merging two queues (step 12). Using 2-3 trees [AHU], a sequence of n such operations can be done in $O(n \log n)$ time. The algorithm does $O(|R|) = O(V)$ such operations, so the time is $O(V \log V)$. Note to do the remove operations in step 11, we maintain pointers from each edge e to its exchange (e, f) in X, and from each edge f to its two occurrences in queues $F(e)$. The pointers for f are also used to do step 10 (once) in $O(\log V)$ time.

The time for finding spanning trees and manipulating priority queues dominates, so the time bound follows. The space bound is obvious. □

The algorithm is easily modified to find a smallest spanning tree with $d(r) \geq b$ (or $d(r) \leq b$). Suppose the test in line 8 is changed to $X \neq \emptyset$. Then a smallest tree in every T_k is found. The desired tree is the smallest in any T_k , $k \geq b$ ($k \leq b$). This tree is easily remembered.

Corollary 3: A smallest spanning tree with $d(r) \geq b$ ($d(r) \leq b$) can be found in time $O(E \log \log V + V \log V)$, and space $O(E+V)$. □

This approach can be slightly improved by observing how the cost of a smallest tree in T_k varies with k . In Figure 1, the minimum occurs in T_2 . This illustrates the situation in general.

Corollary 4: The cost of a smallest spanning tree in T_k is a concave function of k .

Proof: Suppose T is a smallest tree in T_k , and the algorithm finds the exchange (e, f) (in line 9) giving a smallest tree in T_{k-1} . Note e' (in line 10) has $c(e') \leq c(e)$ (otherwise, the exchange (e', f) is chosen instead of (e, f) .) An exchange that is valid for $T - e + f$ but not valid for T has the form (e', g) , where (e, g) is valid for T (see line 12). So for any exchange valid for $T - e + f$, there is a smaller exchange valid for T . Thus the cost of the exchange found in line 9 never decreases as k decreases. □

Corollary 4 can be used to reduce the number of iterations of the loop in lines 8-14, when searching for a smallest tree with $d(r) \geq b$ ($d(r) \leq b$). However the asymptotic time bound does not change.

4. Concluding Remarks

We have presented an algorithm for finding a smallest spanning tree subject to a constraint of the form $d(r)=b$, $d(r)\geq b$, or $d(r)\leq b$.

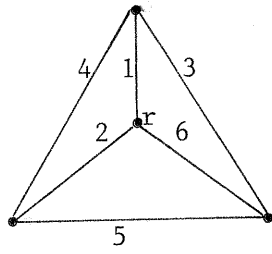
We pose some related questions:

1. Is there a faster algorithm? In particular, can a smallest spanning tree with degree constraint be found as fast as a minimum spanning tree?
2. A minimum spanning tree in a directed graph can be found in $O(E \log V)$ time [T]. Can a smallest spanning tree with degree constraint be found efficiently in a directed graph?

The techniques presented here generalize to handle degree constraints at $O(\log V)$ different vertices efficiently; however $\Omega(n^\epsilon)$ constraints, for any $\epsilon > 0$, lead to a problem that is intractable (i.e. NP-complete). We hope to report on these results and others in a forthcoming paper.

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(a)

k	T_k	$c(T_k)$
1	1,3,4	8
2	1,2,3	6
3	1,2,6	9

(b)

Figure 1: Graph (a) and smallest tree T_k in \mathcal{T}_k (b).

