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Viscoelastic Response to Surface and Tidal Loading – Applications to Glacial Isostatic Adjustment of the Earth and Tidal Deformation of the Icy Satellites

Geruo A
University of Colorado at Boulder, geruo.a@colorado.edu

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VISCOELASTIC RESPONSE TO SURFACE AND TIDAL LOADING - APPLICATIONS TO GLACIAL ISOSTATIC ADJUSTMENT OF THE EARTH AND TIDAL DEFORMATION OF THE Icy SATELLITES

by

GERUO A

B.S., Peking University, China, 2006

A thesis submitted to the
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This thesis entitled:
Viscoelastic Response to Surface and Tidal Loading – Applications to Glacial Isostatic
Adjustment of the Earth and Tidal Deformation of the Icy Satellites
written by Geruo A
has been approved for the Department of Physics

_______________________________________
John Wahr

_______________________________________
Shijie Zhong

_______________________________________
Michael Ritzwoller

_______________________________________
Steve Nerem

_______________________________________
Waleed Abdalati

The final copy of this thesis has been examined by the signatory, and I find that both the
content and the form meet acceptable presentation standards of scholarly work in the above
mentioned discipline.
Observations of glacial isostatic adjustment (GIA), the viscoelastic relaxation of the Earth induced by deglaciation following the last glacial maximum, have provided valuable constraints on late Pleistocene ice history and on the internal viscoelastic structure of the solid Earth. The GIA signal is also a significant source of noise for other applications. For example, errors in GIA models due to errors in the assumed ice deglaciation history and mantle viscosity structure, are generally assumed to be the largest source of uncertainty when using GRACE time-variable satellite gravity data to estimate present-day thinning rates of the Antarctic ice sheet.

The same physical law that governs the Earth’s viscoelastic deformation is also applicable to the tidal deformation on Jupiter’s icy moons. One of the long-sought objectives of an orbiter or fly-by mission to one of Jupiter’s icy moons, has been to use observations of tides on the moon to help determine the existence of a liquid ocean and characteristics of the overlying icy shell.

For the first part of this study, we develop a 3-D finite-element model to study the viscoelastic response of a compressible Earth to surface loads. By forcing our model with the ICE-5G global ice loading history, and computing GIA results for a 3-D viscosity profile derived from a realistic seismic tomography model, we study the effects of 3-D viscosity structure on several GIA observables, including relative sea level measurements in Canada, and present-day time-variable gravity and uplift rates in Antarctica. We also apply our semi-
analyze method to the southern Greenland ice sheet (sGrIS). Using a newly developed ice elevation change history along with different 1-D viscosity structures, we study the influence of GIA effects caused by the Post-Little Ice Age (Post-LIA) deglaciation for the last century on GRACE and GPS present-day observables in sGrIS. In general, we find that the effects of a 3-D viscosity profile and the Post-LIA deglaciation play a minor role when using GRACE to study the present-day ice loss, but they could have a significant impact on GPS present-day surface motion estimates.

For the second part of this study, we apply the same finite-element model to solve for the response of Ganymede and Europa to Jupiter’s tidal forcing, using various icy shell models with 3-D structure. We find that the presence of 3-D shell thickness and shear modulus would probably not affect future attempts to detect a liquid ocean and to determine the mean shell thickness. The inference of a possible 3-D shell structure from the tidal measurements would be challenging. Grounded ice, if existed, might be detected from tidal measurements, but its existence might lead to an overestimate of the floating icy shell’s thickness.
Dedication

To my parents

Yingying and Renfu
Acknowledgement

First of all, I wish to thank John Wahr and Shijie Zhong for giving me the opportunity to join the exciting field of geophysics. Without their guidance and patience, it would be impossible for me to finish this thesis. During my Ph.D. study, what John and Shijie have been offering me is more than a wealth of knowledge on various interesting subjects of geophysics, but also the invaluable idea of thinking critically and creatively. Through numerous discussions with them, on topics ranging from GIA to mountaineering, my whole way of thinking about career and life has evolved.

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Chapter 1
Introduction

The main motivations of this study are to better understand the on-going Glacial Isostatic Adjustment (GIA) processes on the Earth, and to gain insights on the relation between the tidal observables and the internal structures of Jupiter’s icy moons. In this section, we present background information regarding the techniques we have developed and the scientific aspects we have investigated in the thesis.

The first subject we investigate is the glacial isostatic adjustment (GIA) of the solid Earth. GIA describes the viscoelastic relaxation of the Earth induced by deglaciation following the last glacial maximum. GIA observations have provided valuable constraints on late Pleistocene ice history and on the internal viscoelastic structure of the solid Earth. Arguably the most useful GIA observations are those provided by relative sea level (RSL) measurements (e.g., Mitrovica and Forte, 2004; and Peltier, 1998), particularly from near the locations of the Laurentide ice sheet in northern Canada. These RSL measurements have been used to calibrate GIA models in a global scale. Among these models, the ICE-5G loading history compound with VM2 viscosity profile (Peltier, 2004) are probably the most widely used global GIA model. So, for most part of this study, we use ICE-5G/VM2 GIA model as our background model. Compared to the results using this standard GIA model, the effects of different viscosity structures and ice loading history are examined.

For the last decade, geodetic observations, including those that monitor secular trends in the Earth’s time-variable gravity field (e.g. from the GRACE satellite gravity mission (Tapley et al., 2004)) and in surface deformation (e.g. GPS crustal motion measurements),
have also proven useful for constraining GIA models. For instance, since RSL constraints are scarce in Antarctica, GPS observations of present-day bedrock uplift rates are proving to be especially useful for assessing GIA Antarctic modeling results (see, e.g., Argus et al., 2011; Bevis et al., 2009; and Thomas et al., 2011). However, the GIA signal is also a significant source of noise for other applications. For example, errors in GIA models due to errors in the assumed ice deglaciation history and mantle viscosity profile, are generally assumed to be the largest source of uncertainty when using GRACE time-variable satellite gravity data to estimate present-day thinning rates of the Antarctic ice sheet (Chen et al., 2006, 2008, 2009; Velicogna and Wahr, 2006; Velicogna, 2009).

To estimate the GIA uncertainty in GRACE-derived Antarctic ice loss rate, the GIA contribution is often computed for different ice histories (e.g. ICE-5G from Peltier, 2004; IJ05 from Ivins and James, 2005; IJ05_R2 from Ivins et al., 2013; and W12 from Whitehouse et al., 2012), and for different lower mantle and upper mantle viscosity profiles (Velicogna and Wahr, 2006), using a semi-analytic model that is applicable to a 1-D spherically symmetric Earth. In this study, the semi-analytic model we have developed is evolved from several pioneering works (Dahlen, 1974; Han and Wahr, 1995; Mitrovica and Peltier, 1992; Peltier, 1974; Wu and Peltier, 1982). By using collocation method, this model circumvents the problem related to infinite amount of modes arisen from a typical compressible media, and it can solve for the viscoelastic response of a compressible Earth. The implementation of this semi-analytic model only requires very little amount of computer resources, so it can be used to investigate a large amount of Earth structures in relatively short CPU time.

Although the vast majority of GIA modeling studies assumes spherically symmetric Earth models, the real Earth certainly possesses laterally inhomogeneous (3-D) structure; particularly in its viscosity, which is likely to be strongly temperature-dependent. It is thus
natural to ask whether GIA results for a 3-D Earth can be adequately represented by the predictions from 1-D models. And, if so, what 1-D viscosity profile will best predict a set of GIA observations, such as RSL measurements in northern Canada? Similarly, if a 1-D viscosity profile is used to make GIA predictions for Antarctica, what errors might be introduced?

To answer these questions, comprehensive GIA models must be built to handle a realistic Earth structure. Over the last decade, various numerical schemes have been developed to address the GIA modeling problem for a 3-D spherical Earth (e.g. Latychev, 2005 and Whitehouse et al., 2006 for finite-volume modeling of a compressible Earth; Martinec, 2000 and Tanaka et al., 2011 for spectral-finite element modeling of a compressible Earth; Martinec, 1999 for spectral, initial value approach for a compressible Earth; Paulson et al., 2005; Wang and Wu, 2006; Wu and van der Wal, 2003; Wu, 2004; Zhong et al., 2003 for finite-element modeling of an incompressible Earth). In this study, we adopt the methodology discussed in Zhong et al., (2003) and Paulson et al., (2005), and develop a 3-D finite-element model in which the effect of compressibility is included, the gravitational potential arising from internal density perturbations is properly treated, and an arbitrary 3-D viscosity structure can be processed. We develop this finite-element model primarily to study effects of 3-D viscosity structure on GIA predictions, on regions where these 3-D effects are presumably important, e.g. Antarctica. For places where the lateral distribution of the mantle properties is believed to be relatively uniform, e.g. Greenland, we only use the semi-analytic model to investigate pertinent GIA effects.

Similar to global GIA models, such as ICE-5G/VM2, Greenland GIA models were also calibrated by fitting relative sea level measurements (Fleming and Lambeck, 2004; Simpson et al., 2009; Tarasov and Peltier, 2002). However, the relative sea level data used in
those GIA modeling studies can not resolve ice elevation changes over decadal timescale (Bennike et al., 2002; Long et al., 2006, 1999, 2003, 2008, 2010; Sparrenbom et al., 2006). So the GIA effect caused by the deglaciation in Greenland for the last century has not been well constrained in the past.

Recently, improved ice elevation change analysis is available through, for instance, surface mass balance (SMB) calculations (e.g. Hanna et al., 2011 and Wake et al., 2009), total SMB calculations including estimates of ice discharge (Rignot et al., 2008; Yang, 2011), and other estimates that adopt a geometric approach by measuring glacier length changes (Leclercq et al., 2011; Oerlemans et al., 2007; Zuo and Oerlemans, 1997). These improved models of Greenland ice sheet evolution allow us to investigate the impact of ice mass changes over the past century on the present-day Earth’s response. In this study, we use the ice elevation change estimated through a novel geometric approach (Kjeldsen et al., 2013) to study the influence of LIA deglaciation on GRACE and GPS present-day observables.

The study of the tidal loading problem of Jupiter’s icy moons, is mainly driven by the recent advance in space exploration programs. JUpiter ICy moons Explorer (JUICE), operated by the European Space Agency (ESA), is planned for launch in 2022 and arrival at Jupiter in 2030. One of the main scientific objectives of the JUICE mission is to detect the subsurface ocean layers on Jupiter’s icy satellites. Similar space mission specifically designed for Europa has been under study by the National Aeronautics and Space Administration (NASA), aiming to determine the thickness of the icy crust and to investigate whether the icy moon could harbor conditions suitable for life. On board of an orbiter or fly-by spacecraft, the radio science component can be used to determine the tidal potential Love number $k_2$ for gravity, and the laser altimeter can be used to estimate $h_2$ for the tidal deformation (Smith et al., 2001 and Neumann et al., 2001). Knowledge of either of those two
Love numbers can provide information on the presence of an ocean beneath the icy outer shell (e.g., Yoder and Sjogren, 1996; Edwards et al., 1996; Moore and Schubert, 2000; Wu et al., 2001; and Wahr et al., 2006).

If a subsurface ocean exists, complications could arise if the icy outer shell has significant lateral variations. As shown in previous studies, thermal conduction in the ice could produce laterally inhomogeneous temperature field (Ojakangas and Stevenson, 1989, Nimmo et al., 2007), which would further induce laterally varying shell thickness and shear modulus. Meanwhile, the study of surface radiation and bombardment processes on Europa suggests that there might exist hemispherical difference in the composition of its icy shell (e.g. Alvarellos et al., 2008; Brown and Hand, 2013; Carlson et al., 2009; Zahnle et al., 1998, 2003, 2008). In those cases, the tidal deformation pattern would not be solely represented as the sum of degree 2 harmonics, and the accurate modeling of the tidal solutions requires a numerical model that can solve for laterally inhomogeneous satellite structure.

As shown in Zhong et al., (2012), our finite-element model, originally developed for the GIA application of the Earth, is capable of addressing the tidal loading problem of a laterally inhomogeneous Moon. In the same study, good match is obtained when comparing the finite-element results with the semi-analytic results for 1-D spherically symmetric Moon models. In Qin et al., 2013, the same numerical model is benchmarked against the analytic results from perturbation theory, for 3-D Moon structures. In this thesis, we apply the finite-element model to study the tidal response of Ganymede and Europa. By solving a set of tidal loading problems with laterally varying icy shell structures, we investigate how the lateral variable structures might complicate the interpretation of the tide measurements, and we discuss how to extract information regarding the interior structure of Ganymede and Europa from their tidal response.
The thesis is organized as follows. In Chapter 2, firstly, we lay out the details regarding the physical formulation and the numerical method for our finite-element model. And then for the GIA application of the Earth, we discuss in detail the implementation of the effects of center of mass motion, polar wander feedback, and self-consistent ocean loading. Secondly, by computing the response of a spherically symmetric Earth to both a Heaviside loading history and the ICE-5G loading history, we produce a benchmark comparison between the finite-element result and a semi-analytic solution. We assess the performance of the numerical model and study the relationship between accuracy and spatial resolution. Using the ICE-5G benchmark results, we quantify the numerical errors associated with the finite-element model in terms of RSL measurements, GRACE and Altimeter mass trend estimates in Antarctica, and Antarctic GPS uplift rates measurements. Finally, we apply the finite-element model to a plausible 3-D viscosity structure, and discuss its possible implications on Antarctic mass loss and surface motion estimates. This part of the thesis has been published in A et al., (2013).

In Chapter 3, we use our semi-analytic model and load the Earth with a recently developed Post-LIA loading history to investigate the influence of LIA deglaciation on present-day geodetic observables in southern Greenland. This part of the thesis is in preparation for publication.

In Chapter 4, we apply the same finite-element model developed in Chapter 2 to the tidal loading problem of Ganymede and Europa. Firstly, we briefly discuss the modification that is required in our numerical model for solving the icy satellite’s tidal loading problem. Secondly, we discuss in detail our test structures for Ganymede and Europa’s icy outer shells. Icy shell structures with laterally varying shell thickness are derived from a thermal conduction model. 3-D shear modulus profiles of the ice are obtained either from a
conduction model, or, for Europa, by assuming a compositional difference between the leading and trailing hemispheres. We also take Ganymede as an example and build test structures for the case of a partially grounded icy shell. Using these shell structures, we compute the tidal response of the icy satellites and examine their implications on tidal observations. The material discussed in this chapter has been submitted for review (A et al., 2013).

We summarize the main results of this study in Chapter 5. In the appendix, we present the detail information on the numerical treatment of the equation of motion and the implementation of the semi-analytic solutions.
Chapter 2

Computations of the viscoelastic response of a 3-D compressible Earth to surface loading: an application to Glacial Isostatic Adjustment in Antarctica and Canada

2.1 Introduction

Observations of glacial isostatic adjustment (GIA), the viscoelastic relaxation of the Earth induced by deglaciation following the last glacial maximum, have provided valuable constraints on late Pleistocene ice history and on the internal viscoelastic structure of the solid Earth. Arguably the most useful GIA observations are those provided by relative sea level (RSL) measurements (e.g., Mitrovica and Forte, 2004; and Peltier, 1998), particularly from near the locations of the Laurentide ice sheet in northern Canada. But geodetic observations have also proven useful for this purpose, including those that monitor secular trends in the Earth’s time-variable gravity field (e.g. from the GRACE satellite gravity mission (Tapley et al., 2004)) and in surface deformation (e.g. GPS crustal motion measurements). For instance, since RSL constraints are scarce in Antarctica, GPS observations of present-day bedrock uplift rates are proving to be especially useful for assessing GIA Antarctic modeling results (see, e.g., Argus et al., 2011; Bevis et al., 2009; and Thomas et al., 2011).

The GIA signal is also a significant source of noise for other applications. For example, errors in GIA models due to errors in the assumed ice deglaciation history and mantle viscosity profile, are generally assumed to be the largest source of uncertainty when using GRACE time-variable satellite gravity data to estimate present-day thinning rates of the Antarctic ice sheet (Chen et al., 2006, 2008, 2009; Velicogna and Wahr, 2006; Velicogna,
To estimate the GIA uncertainty in GRACE-derived Antarctic ice loss rate, the GIA contribution is often computed for different ice histories (e.g. ICE-5G from Peltier, 2004; IJ05 from Ivins and James, 2005; and W12 from Whitehouse et al., 2012), and for different lower mantle and upper mantle viscosity profiles (Velicogna and Wahr, 2006).

The vast majority of all these efforts employ GIA models that assume a 1-D (i.e. radially dependent) viscosity profile. And yet the Earth certainly possesses laterally inhomogeneous (3-D) structure; particularly in its viscosity, which is likely to be strongly temperature-dependent. It is thus natural to ask whether GIA results for a 3-D Earth can be adequately represented by the predictions from 1-D models. And, if so, what 1-D viscosity profile will best predict a set of GIA observations, such as RSL measurements in northern Canada? Similarly, if a 1-D viscosity profile is used to make GIA predictions for Antarctica, what errors might be introduced?

To answer these questions, comprehensive GIA models must be built to handle a realistic Earth structure. Over the last decade, various numerical schemes have been developed to address the GIA modeling problem for a 3-D spherical Earth (e.g. Latychev, 2005 and Whitehouse et al., 2006 for finite-volume modeling of a compressible Earth; Martinec, 2000 and Tanaka et al., 2011 for spectral-finite element modeling of a compressible Earth; Martinec, 1999 for spectral, initial value approach for a compressible Earth; Paulson et al., 2005; Wang and Wu, 2006; Wu and van der Wal, 2003; Wu, 2004; Zhong et al., 2003 for finite-element modeling of an incompressible Earth). In this study, we adopt the methodology discussed in Zhong et al., (2003) and Paulson et al., (2005), and develop a 3-D finite-element model in which the effect of compressibility is included, the gravitational potential arising from internal density perturbations is properly treated, and an arbitrary 3-D viscosity structure can be processed. We develop this finite-element model
primarily to study load-induced deformation of the Earth, such as that produced by the GIA process. But with minor modifications, this model can also be used to study the body tide response of the Earth or any other planetary body (Zhong et al., 2012).

This chapter is structured into three parts. Firstly, we present the physical model for the viscoelastic response of a laterally inhomogeneous Earth. We discuss in detail the numerical methods we employ to solve the equation of motion, and to implement the effects of center of mass motion, polar wander feedback, and self-consistent ocean loading. Secondly, by computing the response of a spherically symmetric Earth to both a Heaviside loading history and the ICE-5G loading history, we produce a benchmark comparison between the finite-element result and a semi-analytic solution. We assess the performance of the numerical model and study the relationship between accuracy and spatial resolution. Using the ICE-5G benchmark results, we quantify the numerical errors associated with the finite-element model for different observation types. Finally, we apply the finite-element model to a plausible 3-D viscosity structure, and discuss the possible errors that might be introduced into Antarctic mass loss and uplift rate estimates, if we approximate the 3-D viscosity profile with its 1-D average. We also consider the difference between 3-D and various 1-D model predictions of RSL observations in northern Canada, to assess whether viscosity models provided by those observations are likely to represent truly global averages, or are more apt to be regional representations.

2.2 Physical models for viscoelastic deformation

2.2.1 Governing equations

Our model of viscoelastic deformation assumes a compressible self-gravitational Earth. The Earth’s mantle is treated as a Maxwell solid overlying an inviscid fluid core. For a
spherically symmetric density distribution, the governing equations of mass and momentum conservation, along with Poisson’s equation for the perturbation in gravity, can be written as (e.g. Tromp and Mitrovica, 1999; Zhong et al., 2003)

\[
\rho^E_i = - (\rho_0 u_i)_r, \tag{1}
\]

\[
\sigma_{ij, j} + \rho_0 \phi_i - \rho^E_i g_i - (\rho_0 g u_i)_j = 0, \tag{2}
\]

\[
\phi_{,ii} = 4\pi G \rho^E_i, \tag{3}
\]

where \( \rho^E_i \) is the Eulerian density perturbation, \( u_i \) is the displacement, \( u_r \) is the radial component of the displacement, \( \sigma_{ij} \) is the stress tensor, \( \rho_0 \) and \( g \) are the unperturbed density and gravitational acceleration, \( \phi \) is the perturbation of the gravitational potential, \( G \) is the gravitational constant, the notation \( A_{,i} \) represents the derivative of the variable \( A \) with respect to \( x_i \), and the repeated index implies summation.

2.2.2 Boundary conditions

The boundary conditions for normal tractions at the surface and the core-mantle boundary (CMB) are given by (see equations (52), (53), (60) of Dahlen, 1974)

\[
\sigma_y n_j = - \Gamma(t, \theta, \phi)n_i, \text{ at the outer surface,} \tag{4}
\]

\[
\sigma_y n_j = - \rho_c \phi n_i + \rho_c g n_i n_j, \text{ at the CMB,} \tag{5}
\]

where \( n_i \) represents the normal vector of the boundary surface, \( \Gamma(t, \theta, \phi) \) is the surface load, and \( \rho_c \) is the density of the core.

2.2.3 Mechanical properties

The Earth’s mantle is treated as a compressible Maxwell solid, and so the constitutive equation can be written as (Wu and Peltier, 1982)

\[
\dot{\sigma}_{ij} + \frac{\mu}{\eta} (\sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}) = 2\mu \dot{\varepsilon}_{ij} + \lambda \varepsilon_{kk} \delta_{ij}, \tag{6}
\]
where $\eta$ is the viscosity, and $\lambda$ and $\mu$ are the Lamé parameters. The strain $\varepsilon_{ij}$ is related to the deformation by
\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}).
\] (7)

Our numerical model can process a layered or spherically symmetric density distribution, along with a fully 3-D structure for the mantle viscosity and Lamé parameters.

### 2.3 Numerical analysis of viscoelastic deformation

To solve the viscoelastic deformation problem for a compressible mantle, we follow a strategy similar to that discussed in Zhong et al., (2003) for incompressible finite-element modeling. Differences from Zhong et al., (2003) arise because $u_{i,j}$ is in general non-zero for a compressible medium, which means the constitutive equation does not now include pressure terms. And the Eulerian density perturbation is now non-zero, which complicates our gravitational potential solution and the momentum equation. For the remainder of this section, we address these issues, build up a matrix equation, and discuss how to solve the resulting equations numerically.

#### 2.3.1 Constitutive equation

We use a formulation that employs incremental displacements and strains, and discretize the constitutive equation (Martinec, 1999; Zhong et al., 2003). Let $u_i^n$ and $u_i^{n+1}$ be the displacements at time $t^n$ and $t^{n+1} = t^n + \Delta t$. An incremental displacement $v_i^n$ and an incremental strain $\Delta \varepsilon_{ij}^n$ can be defined as
\[
v_i^n = u_i^{n+1} - u_i^n,
\] (8)
\[
\Delta \varepsilon_{ij}^n = \frac{1}{2} (v_{i,j}^n + v_{j,i}^n).
\] (9)
Integrating Eq. (6) from $t^n$ to $t^{n+1}$ with the second-order trapezoid rule, we get

$$\sigma_{ij}^{n+1} = \tau_{ij}^{pre} + 2\tilde{\mu}\Delta\varepsilon_{ij}^n + \tilde{\lambda}\Delta\varepsilon_{ikl}^n\delta_{ij},$$  \hspace{1cm} (10)

where

$$\tau_{ij}^{pre} = \frac{1 - \frac{\mu}{\eta} \frac{\Delta t}{2} \sigma_{ij}^n}{1 + \frac{\mu}{\eta} \frac{\Delta t}{2}} \sigma_{ij}^n + \frac{\mu}{3\eta} \frac{\Delta t}{2} \sigma_{ikl}^n \delta_{ij},$$  \hspace{1cm} (11)

$$\tilde{\mu} = \frac{\mu}{1 + \frac{\mu}{\eta} \frac{\Delta t}{2}},$$  \hspace{1cm} (12)

$$\tilde{\lambda} = \frac{\lambda + (\lambda + 2\mu) \frac{\mu}{3} \frac{\Delta t}{2}}{1 + \frac{\mu}{\eta} \frac{\Delta t}{2}}$$  \hspace{1cm} (13)

and where $\tau_{ij}^{pre}$ is a parameter related to the total stress, $\sigma_{ij}^n$, at the previous time step. $\tilde{\mu}$ and $\tilde{\lambda}$ depend on the Lamé parameters, the viscosities, and the time interval $\Delta t$. With Eq’s (8) – (13), we can express the stress tensor in terms of the incremental displacements. One of the principal differences with Zhong et al.,’s (2003) incompressible study, is that here we do not have pressure terms in Eq. (10), and the only unknowns are the incremental displacements.

2.3.2 Gravitational potential

To find the gravitational potential, we assume a layered density structure, and write the density as

$$\rho_0(r') = \sum_{i=1}^{N} \Delta\rho_i H(r_i' - r'),$$  \hspace{1cm} (14)

where $H$ is the Heaviside function ($H(x) = 0$ for $x < 0$, and $H(x) = 1$ for $x \geq 0$), and a density jump of $\Delta\rho_i$ appears at $r'_i = r_i$, for $i = 1, \ldots, N$ from the CMB to the surface of the
Earth. Here $i$ denotes the $i$th layer and $N$ is the number of density layers. Using Eq. (14), we can rewrite Eq. (1) as

$$\rho_i^E = \sum_{i=1}^{N} \Delta \rho_i \mu_i \delta(r_i - r') - \rho_0 u_{i,i}.$$  \hspace{1cm} (15)

In Eq. (15), the first term on the right hand side is the surface density perturbation at the boundary of each density layer, and the second term is a volumetric density perturbation due to non-zero $u_{i,i}$ in the compressible mantle.

Expanded into the spherical harmonic domain, the general solution to Eq. (3) can be written as

$$\phi = \sum_{l,m} Y_l^m \phi_{lm},$$ \hspace{1cm} (16)

where

$$\phi_{lm}(\vec{r}) = \frac{4\pi G}{2l+1} \int_{r_{<}}^{r_{>}} r_{l+1}^l Y_l^m(\theta', \phi') \rho_i^E(r')d^3r',$$ \hspace{1cm} (17)

and where $r_{<}$ and $r_{>}$ are the smaller and larger of $r$ and $r'$, respectively. $Y_l^m$ is given by

$$Y_l^m(\theta, \phi) = p_{lm}(\theta)(\cos(m\phi) + i\sin(m\phi)),$$ \hspace{1cm} (18)

where $p_{lm}(\theta)$ is a normalized associated Legendre polynomial as described in Zhong et al. (2008). Later we may reference to the cosine and sine part of $Y_l^m$, which are computed through the linear combination of $Y_l^m$ and $Y_{l}^{-m}$.

With $\rho_i^E$ given in Eq. (15), and taking into account the load induced surface mass density $\Gamma / g$, we can express Eq. (17) in terms of surface and volumetric contributions $\phi_s$ and $\phi_v$ (see Wu, (2004) for a similar representation of Eq. (19a)),

$$\phi_s = \frac{4\pi G}{2l+1} \sum_{i=1}^{j} \frac{r_{l+2}^i}{r_{l+1}^i} \Delta \rho_i \mu_{i,i}(r_i) + \sum_{i=1}^{N} \frac{r_{l+1}^i}{r_{l+1}^i} \Delta \rho_i \mu_{i,i}(r_i) + \frac{r_{l+1}^i}{r_{N}^i} \frac{\Gamma_{lm}}{g},$$ \hspace{1cm} (19a)
where \( r_i \) denotes the radius of the \( i \)-th boundary and \( r_N \) is the surface radius, \( u_{ir_m} \) and \( \Gamma_{ir_m} \) are the spherical harmonic coefficients of the radial displacements and the surface loads, respectively, and without losing generality, we assume the field point \( r \) satisfies \( r_j < r \leq r_{j+1} \).

It is notable that the surface contribution is caused by the radial displacements at density discontinuities, and the volumetric contribution is due to the compressibility. Using Eq. (19a) and Eq. (19b), we can compute the gravitational potential as long as we know the displacements.

2.3.3 Momentum equation

We follow the usual strategy in finite-element methods (Hughes, 2000; Zhong et al., 2003), and reformulate the momentum equation into its weak form. Eq. (2) can be written as

\[
(\sigma_{ij} + \rho_0 \phi \delta_{ij} - \rho_0 g u_0 \delta_{ij})_{,j} = \rho_0 \phi + \rho \Gamma, \quad (20)
\]

Multiplying Eq. (20) by a weighting function \( w_j \), integrating over the entire mantle, and substituting Eq. (4), (5), (8), (9), (10), (15) and the derivative of Eq. (14) into the integral, leads to the weak form of the momentum equation

\[
\int w_{ij} [\tilde{\mu}(v_{ij} + v_{ij}) + \tilde{\tau}_{ij} \delta_y] dV - \int (w_{ij} \rho_0 g v_{ij} + w_0 \rho_0 g v_{ij}) dV + \sum_i \int w_i \Delta \rho_i g v_i dS_i = -\int w_{ij} \tau_{ij}^{pre} dV + \sum_i \int (w_{ij} \rho_0 g U_i + w_0 \rho_0 g U_{ij}) dV
\]

where \( v_i \) is the incremental displacement that we solve for, and \( U_i \) is the total displacement from the previous time step. In the summations, \( l \) denotes the \( l \)th density boundary.

Eq. (21) can be converted into a matrix equation, using standard finite-element methods (Hughes, 2000). The second and third terms on the left hand side of Eq. (21) lead to
terms in the stiffness matrix that depend on gravity, which is not usually included in non-
geophysical finite element formulations. The right hand side of the equation contributes the
forcing terms for the matrix equation. Most of the forcing terms enter either through the
boundary conditions or through the stress and displacement resulting from the previous time
step.

There are terms on the right hand side of Eq. (21) that depend on the gravitational
potential $\phi$. When we solve Eq. (21), we decompose $\phi$ into $\phi = \phi^0 + \Delta \phi(v_i)$, where $\phi^0$ is the
initial potential (total potential from the previous time step plus the potential induced by the
load itself) and $\Delta \phi(v_i)$ is the incremental potential that depends on the unknown incremental
displacements. We find $\Delta \phi(v_i)$ by using Eq’s (19a) and (19b), replacing $u_i$ in those equations
by $v_i$.

2.3.4 Matrix equation and its solution

2.3.4.1 Matrix equation

To construct a matrix equation from Eq. (21), we use brick elements, with
displacements defined on the eight nodes at the corners, and with the elements arranged into
the same finite-element grid as in Zhong et al., (2003). After including compressibility,
significant changes appear in the stiffness matrix and the forcing terms in the momentum
equation, compared with those in Zhong et al., (2003) for incompressible media. Following
the procedure described in Appendix A, we can convert Eq. (21) into

$$KV = F_0 + F(\Delta \phi(V)).$$

(22)

where $K$ is the total stiffness matrix, $V$ is the incremental displacement vector containing $\vec{v}$
at all the nodes, and $F_0$ is the force vector that depends on the surface load $\Gamma$, the pre-
stresses $\tau_{ij}^{pre}$, the displacements from the previous time step $U_i$ and the initial gravitational
potential $\phi^0$ (i.e., the total gravitational potential at the previous time step plus the gravitational potential induced by the incremental load itself). $F$ is the force vector that depends on the incremental gravitational potential $\Delta\phi(V)$, which in turn depends on the incremental displacements $V$.

2.3.4.2 Solution to the matrix equation

Eq. (22) needs to be solved iteratively, because the incremental gravitational potential $\Delta\phi(V)$ depends on the unknown displacements. For a given time step, an initial guess $V_0$ (chosen as the solution from the previous time step) is assigned to $V$, and the incremental gravitational potential is computed as $\Delta\phi(V_0)$. So the force vector can be obtained as $F_0 + F(\Delta\phi(V_0))$. Eq. (22) is solved with the guess force vector, and the result $V_1$ is assigned to $V$. This process is repeated until we obtain a convergent solution for $V$. For later use, we call this process the self-gravity iteration (see also Wu, 2004 and Zhong et al., 2003).

2.4 Solution methods for degree one deformation, polar wander feedback, and ocean loading

Our compressible model includes the effects of center of mass motion, polar wander feedback, and self-consistent ocean loading, similar to those in Paulson et al., (2005).

2.4.1 Degree-one deformation

The CM frame is the coordinate system with origin at the center of mass of the Earth-plus-load system. Different from the study for incompressible media in Paulson et al., (2005) where the CM is only affected by the load and displacements at the surface and CMB, the CM in the current model formulation for compressible media is also affected by displacements and compressibility within the mantle. We compute deformation in the CM frame, by implementing the following two steps.
(a) Center of mass change induced by the load itself. Each time a new incremental load is applied to the Earth’s surface, the center of mass position changes. We transform our old frame (the CM frame for the previous time step) to the new CM frame, by fixing the finite-element grids and changing the values of the physical quantities defined on those grids. The physical quantities that concern us are the incremental surface loads at the current time step, the total displacements and the stress tensors obtained from the previous time step. Suppose we find the center of mass change induced by the incremental load is \( \Delta \mathbf{r}_{cm} \). We then shift the total displacement field by \(- \Delta \mathbf{r}_{cm}\). Correspondingly, we add an additional surface mass of \(- \Delta \rho \mathbf{r}_{cm} \cos \theta\) at each boundary, where \( \Delta \rho \) is the density jump at that boundary and \( \theta \) is the angular distance between \( \mathbf{r}_{cm} \) and the position vectors at the boundary. Now the loads are identical to those that would be experienced in the new CM frame, and the degree-one term of the gravitational potential induced by the new loads vanishes at the Earth’s surface. The stress tensors are invariant under the degree-one translation, and so we do not modify them.

(b) Center of mass change induced by the deformation. Using the total displacements, the loads, and the stress tensors, all written in the new CM frame, we solve for the incremental displacements for the current time step. In general, the incremental displacements will shift the center of mass position, so we have a new CM frame again. We perform the degree-one translation as described in (a), except that we shift the incremental displacements rather than the total displacements. Adding the incremental displacements to the total displacements, we get the degree-one deformation in the CM frame for the current time step.

2.4.2 Polar wander feedback

Deformation of the Earth and changes in surface loading can perturb the Earth’s
moment of inertia and consequently the orientation of the Earth’s rotation axis. Changes in Earth rotation in turn deform the Earth via the resulting perturbation of the centrifugal potential. To model this process, we write the Earth’s angular velocity vector as

\[ \Omega(m_1, m_2, 1 + m_3), \]

where \( \Omega \) is the unperturbed magnitude of the angular velocity and the \( m_i \) are the dimensionless Cartesian components of the perturbation. We combine the equatorial components of the perturbation into the complex form, \( m_\pm = m_1 \pm i m_2 \). Similarly, for the perturbation to the Earth’s inertia tensor, we use \( I_{\pm} = I_{13} \pm i I_{23} \). For periods much longer than the Chandler Wobble, we have (Mitrovica et al., 2005; Paulson et al., 2005):

\[ m_\pm = \frac{I_\pm}{(C - A)_{hyd}(1 + \delta)}, \tag{23} \]

where \( C \) and \( A \) are the unperturbed principal polar and equatorial moments of inertia, and \( (C - A)_{hyd} \) denotes the purely hydrostatic oblateness, the value of which is determined by the Earth’s response to rotation, in the fluid limit. \( \delta \) is the parameter that describes the non-hydrostatic oblateness, and is set here to be 0.8%, as described by Mitrovica et al. (2005).

The perturbation in the rotation axis will induce a perturbation in the centrifugal potential

\[ \Delta \phi_c = \sqrt{\frac{2\pi}{15}} \Omega^2 r_s^2 (m_1 Y_{21} + m_2 Y_{21}^*), \tag{24} \]

where \( r_s \) is the Earth’s mean surface radius. \( \Delta \phi_c \) will deform the Earth and change the Earth’s rotation axis again. This feedback process is implemented through the following two steps.

(a) When a new incremental load is applied to the Earth’s surface at each time step, we compute the change in the Earth’s moments of inertia, which are proportional to the \( Y_{21} \) components of the load-induced gravitational potential. Using Eq. (23), we find \( m_\pm \). The perturbation in the centrifugal potential induced by the load itself can be obtained using these
$m_z$ in Eq. (24). We denote this potential as $\Delta \phi^0_c$, and we add it to the initial gravitational potential $\phi^0$. We are now ready to start the self-gravity iteration.

(b) $F_0$ in Eq. (22) is built up using $\phi^0 + \Delta \phi^0_c$, $\Gamma$, $U_i$, and $\tau^{pre}_i$. An initial guess $V_0$ is assigned to $V$. To take polar wander feedback into account, we compute the change in the moments of inertia induced by the deformation $V_0$. The $m_z$ are then computed using Eq. (23). With Eq. (24), we get the deformation-induced perturbation of the centrifugal potential $\Delta \phi_c (V_0)$, and the force vector can be computed as $F_0 + F(\Delta \phi(V_0) + \Delta \phi_c (V_0))$. Eq. (22) is solved with the guess force vector and the result $V_1$ is assigned to $V$. This process is repeated until we obtain converged results for both $V$ and $m_z$.

2.4.3 Ocean loading

Changes in ocean loading have two sources: increased volume from melted ice, and the response of the fluid ocean to changes in the topography and in the geoid. Using the same method discussed in Paulson et al., (2005), we include ocean loading via the sea level equation

$$L_0(\theta, \varphi, t) = (N(\theta, \varphi, t) - U(\theta, \varphi, t) + c(t))O(\theta, \varphi, t),$$  \hspace{1cm} (25)

where $L_0$ is the change in height of the ocean load, $N$ and $U$ are changes in the geoid and in the surface topography, $c$ is a spatial constant needed to conserve mass, and $O$ is the ocean function (1 over the ocean, and 0 elsewhere). Integrating Eq. (25) over the ocean surface, we have

$$c(t) = \frac{1}{A_0(t)} \left( - \frac{M_{\text{ice}}(t)}{\rho_w} - \int (N - U)Od\Omega \right)\hspace{1cm} (26),$$

where $A_0(t)$ is the area of the oceans at time $t$, $M_{\text{ice}}$ is the change in ice mass, $\rho_w$ is the water density, and $d\Omega$ is the differential element of solid angle. In principle, $A_0(t)$ depends...
on time because of three effects: (a) as the ice melts off regions that lie below sea level (e.g. Hudson Bay), those regions become ocean and add to the ocean area; (b) as meltwater is added to the oceans, the area will increase as the water flows up over land in regions with sloping bathymetry adjacent to shore; (c) as land uplifts (or subsides) adjacent to the coast, the adjacent ocean coverage decreases (or increases). In this study, we include only effect (a).

Combining Eq. (25) and Eq. (26), we have

\[
L_0(\theta, \varphi, t) = -\frac{M_{in}(t)}{\rho_w A_0(t)} O(\theta, \varphi, t) \\
+ \left[ N(\theta, \varphi, t) - U(\theta, \varphi, t) - \frac{1}{A_0} \int (N - U) \partial \Omega \right] O(\theta, \varphi, t)
\]  

(27)

The first term on the right hand side of Eq. (27) represents the static ocean load, the value of which can be determined from knowledge of the ice loading history and the ocean function. The second term represents the dynamic ocean load. In our finite-element model, these two terms are treated as follows.

(a) Static ocean load. Each time an incremental ice load is applied to the surface of the Earth, the corresponding change in the static ocean load is computed according to the first term on the right hand side of Eq. (27). That change in ocean load is added to the ice load to build the complete \( F_0 \) term.

(b) Dynamic ocean load. The dynamic ocean load is implemented through the self-gravity iteration (see also Paulson et al., 2005 and Wu, 2004). Within an arbitrary iteration, the incremental gravitational potential \( \Delta \phi \) is computed based on \( V \) and the dynamic ocean load from the previous iteration using Eq. (19a) and (19b). \( U \) is the surface value of \( V \), and \( N \) is proportional to the surface value of \( \Delta \phi \). We use the second term on the right hand side of Eq. (27) to compute the updated dynamic ocean load for this iteration. The contribution of this load is added to the surface terms of \( F \). Eq. (22) is solved and a new displacement field is
obtained. This process is repeated until we obtain convergence. For the ICE-5G ice history and VM2 viscosity profile, it takes 6 to 8 self-gravity iterations to reach convergence.

2.5 Benchmark results

Using the finite-element model, we generate numerical solutions and benchmark them against a semi-analytic solution (Appendix B) for a spherically symmetric Earth model. To build the finite-element grids, we divide the Earth’s mantle into 12 caps that have approximately equal size, and each cap is further divided into \( p \) cells in each of the horizontal direction and \( q \) cells in the radial direction. So the total element number is \( 12 \times p \times p \times q \) (see e.g. Zhong et al., 2008). The Earth model we use is available online (Peliter’s website), where the viscosity profile is based on VM2 (Peltier, 2004), and the elastic parameters are derived from PREM (Dziewonski and Anderson, 1981). Two types of loading history are used in the benchmark runs: (1) single harmonic loads with Heaviside loading history, and (2) the realistic ice loading history ICE-5G (Peltier, 2004). The details are shown in the next two subsections.

2.5.1 Single harmonic loads with Heaviside loading history

We apply a single harmonic load with Heaviside time dependence

\[
\sigma(\theta, \varphi, t) = Y^{m_0}_{l_0}(\theta, \varphi) H(t),
\]

where \( Y^{m_0}_{l_0}(\theta, \varphi) \) is the spherical harmonic function for degree \( l_0 \) and order \( m_0 \), and \( H(t) \) is the Heaviside function. For a spherically symmetric Earth, we expect to see the following two features in the Earth’s response: (a) the response should have the same \( Y^{m_0}_{l_0}(\theta, \varphi) \) angular dependence as the load, and (b) the ratio of the response to the surface load should be independent of the order \( m_0 \) chosen for the load (though it should depend on the degree, \( l_0 \)). To verify that the finite-element solution displays these features and matches the semi-
analytic solution, we adopt two quantitative measures of success: the amplitude error $\varepsilon_a$, which measures the deviation of the finite-element solution from the semi-analytic solution for degree $l_0$ and order $m_0$; and the dispersion error $\varepsilon_d$, which measures the combined contributions of all harmonics other than $(l_0, m_0)$. These errors are defined as

$$\varepsilon_a = \frac{1}{T} \int_0^T \frac{S_n(l_0, m_0, t) - S_g(l_0, m_0, t)}{S_g(l_0, m_0, t)} \, dt ,$$

$$\varepsilon_d = \sqrt{\frac{1}{T} \int_0^T \sum_{l \neq l_0, m m_0} \left| S_n(l, m, t) \right|^2 \, dt} .$$

These errors are defined as

$$\varepsilon_a = \frac{1}{T} \int_0^T \frac{S_n(l_0, m_0, t) - S_g(l_0, m_0, t)}{S_g(l_0, m_0, t)} \, dt ,$$

$$\varepsilon_d = \sqrt{\frac{1}{T} \int_0^T \sum_{l \neq l_0, m m_0} \left| S_n(l, m, t) \right|^2 \, dt} .$$

where $S_n$ and $S_g$ are the response from the finite-element model and the semi-analytic solution, respectively. The $Y^m_\ell (\theta, \phi)$ used in Eq. (28) has $\cos(m_0 \phi)$ and $\sin(m_0 \phi)$ parts. In our case, the cosine part of Eq. (28) is taken as the input load, and correspondingly the cosine coefficient of the surface uplift is used as $S_n$ and $S_g$. $T$ is the time duration for which we compute the solutions. Table 2.1 shows the amplitude and dispersion errors for the vertical surface displacement for benchmark calculations with loads at different harmonics and grid size, and Fig 2.1 compares the Love numbers derived from the vertical surface displacement for the numerical and semi-analytic solutions.

For cases A1-A7 with loading harmonics from degree 1 to 4 on a 12x48x48x48 grid, the dispersion errors over 300 Maxwell times are less than 0.2%, which means the surface displacement field from the finite-element model has the same angular dependence as the surface load. The amplitude errors for these cases are smaller than 1.0%. This shows a good match between the finite-element solutions and the semi-analytic solutions, which is evident in Fig 2.1(a). To examine the order dependence in the response, we include $m \neq 0$ cases (A2,
Table 2.1 Amplitude and dispersion errors* for cases with Heaviside loading history

<table>
<thead>
<tr>
<th>Case</th>
<th>(l,m)</th>
<th>Grid</th>
<th>amplitude error (%)</th>
<th>dispersion error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1</td>
<td>(1,0)</td>
<td>12x48x48x48</td>
<td>0.0639</td>
<td>0.0508</td>
</tr>
<tr>
<td>A2</td>
<td>(1,1)</td>
<td>12x48x48x48</td>
<td>0.0726</td>
<td>0.0430</td>
</tr>
<tr>
<td>A3</td>
<td>(2,0)</td>
<td>12x48x48x48</td>
<td>0.2431</td>
<td>0.0691</td>
</tr>
<tr>
<td>A4</td>
<td>(2,2)</td>
<td>12x48x48x48</td>
<td>0.2644</td>
<td>0.0656</td>
</tr>
<tr>
<td>A5</td>
<td>(3,0)</td>
<td>12x48x48x48</td>
<td>0.3104</td>
<td>0.0994</td>
</tr>
<tr>
<td>A6</td>
<td>(3,3)</td>
<td>12x48x48x48</td>
<td>0.3427</td>
<td>0.0847</td>
</tr>
<tr>
<td>A7</td>
<td>(4,0)</td>
<td>12x48x48x48</td>
<td>0.4775</td>
<td>0.1468</td>
</tr>
<tr>
<td>A8</td>
<td>(2,1)</td>
<td>12x48x48x48</td>
<td>2.8895</td>
<td>0.1630</td>
</tr>
<tr>
<td>A9</td>
<td>(2,1)</td>
<td>12x64x64x48</td>
<td>1.1255</td>
<td>0.0936</td>
</tr>
<tr>
<td>A10</td>
<td>(2,1)</td>
<td>12x80x80x48</td>
<td>0.4703</td>
<td>0.0781</td>
</tr>
</tbody>
</table>

* The error results are computed for $T = 300\tau_0$, where $\tau_0$ is 443 years, for both the finite-element model and the semi-analytic method. The incremental time step used in the finite element model is set to be $\Delta t = 0.2\tau_0$. 


Figure 2.1 (a) Time dependent Love number from semi-analytic solutions (dashed lines) and from the finite-element solution (solid lines) in response to single harmonic Heaviside loading. The elapsed time is normalized by $\tau_0 = 443$ yrs (a typical value of the Maxwell relaxation times for the viscosity model). The figure only includes data points with elapsed time $\leq 50$, because the nearly flat viscous tails show consistent agreement between the two solutions. (b) Convergence of the degree 2 order 1 numerical results to the semi-analytic result with increasing spatial resolution in the finite-element model.
A4 and A6) in our calculations. Their errors are consistent with the corresponding $m = 0$
cases (A1, A3 and A5). Similar to Zhong et al., (2003), we find the errors increase with
increasing spherical harmonic degree (i.e. with decreasing spatial scales), because resolving
shorter wavelengths requires a finer mesh.

As shown in cases A8-A10, for the degree 2 order 1 term, the error increases with
time (Fig 2.1(b)), and higher resolution is required to reach an accuracy of better than 1.0%.
This special feature is because of polar wander feedback, which for a spherical Earth affects
only the degree 2, order 1 term, and can be understood with the aid of the analytic expression
given in Eq. (B9) and (B10) in Appendix B. At long periods, $k_f^2$ approaches $k_f$, and the
denominators of Eq. (B9) and (B10) are close to 0. Therefore, to obtain reliable results, $k_f^2$
needs to be resolved to high accuracy. Using the language of the finite-element model, this
means we need to resolve the Earth’s response induced by the perturbed centrifugal potential
to high accuracy. As shown in Table 2.1, by increasing the resolution, we correspondingly
decrease the amplitude error for the degree 2 order 1 response.

2.5.2 ICE-5G loading history

The ICE-5G loading history is a realistic ice model that describes the temporal and
spatial distribution of ice on the Earth’s surface during the last 122000 years. We apply the
ICE-5G load to the same Earth model used in the last section. We use the 12x80x80x48 grids
for our computation (Table 2.2). To compare with the semi-analytic solution, the surface
displacement from our finite-element model is expanded into a set of spherical harmonic
coefficients. And the benchmark results are obtained for $l \leq 32$. For each harmonic, the
amplitude errors are computed using Eq. (29) either for the cosine coefficients or for the sine
coefficients, whichever has the dominant amplitude. Fig 2.2 shows the magnitudes of the
vertical displacement at different spherical harmonic degrees and Table 2.2 shows the
Table 2.2 Amplitude errors* for different spherical harmonics for the ICE-5G loading history.

<table>
<thead>
<tr>
<th>(l,m)</th>
<th>Coefficients</th>
<th>amplitude error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,0)</td>
<td>Cosine</td>
<td>0.6112</td>
</tr>
<tr>
<td>(2,0)</td>
<td>Cosine</td>
<td>0.2908</td>
</tr>
<tr>
<td>(7,0)</td>
<td>Cosine</td>
<td>1.108</td>
</tr>
<tr>
<td>(9,0)</td>
<td>Cosine</td>
<td>1.866</td>
</tr>
<tr>
<td>(12,0)</td>
<td>Cosine</td>
<td>4.7758</td>
</tr>
<tr>
<td>(2,1)</td>
<td>Sine</td>
<td>2.9156</td>
</tr>
<tr>
<td>(4,1)</td>
<td>Sine</td>
<td>0.4551</td>
</tr>
<tr>
<td>(5,1)</td>
<td>Sine</td>
<td>0.3752</td>
</tr>
<tr>
<td>(8,1)</td>
<td>Sine</td>
<td>2.3207</td>
</tr>
<tr>
<td>(10,1)</td>
<td>Sine</td>
<td>2.8064</td>
</tr>
</tbody>
</table>

* The error results are computed using the cosine coefficients for the $m = 0$ terms and the sine coefficients for the $m = 1$ terms. The time duration is $T = 122000$ years. In the finite-element model, the incremental time step $\Delta t$ is 200 years from 122000 years ago to 17000 years ago, and is 25 years from 17000 years ago to the present day.
Figure 2.2 Magnitudes of the surface displacement from the semi-analytic solution (dashed lines) and from the finite-element solution (solid lines) in response to the ICE-5G loading. The resolution in the numerical code is set to be 12x80x80x48. (a) The cosine coefficients for $m = 0$ terms. (b) The sine coefficients for $m = 1$ terms.
amplitude errors.

For the longest wavelength terms, the amplitude errors are less than 1.0% (except for the degree 2 order 1 term). At shorter wavelengths, for instance \( l = 10 \) or 12, larger errors can be observed. To further investigate these errors, we compute the following root mean square values in the spatial domain:

\[
\text{rms}_{\delta u}(\theta, \phi) = \sqrt{\frac{\int_0^T (u_{\text{n}}(\theta, \phi, t) - u_{\text{s}}(\theta, \phi, t))^2 \, dt}{T}},
\]

\[
\text{rms}_u(\theta, \phi) = \sqrt{\frac{\int_0^T u_{\text{s}}(\theta, \phi, t)^2 \, dt}{T}},
\]

where \( u_{\text{n}} \) and \( u_{\text{s}} \) denote the vertical surface displacement from the finite-element model and the semi-analytic solution, respectively. \( \text{rms}_u \) is the RMS value of the surface uplift, and \( \text{rms}_{\delta u} \) is the RMS value of the difference between the finite element and semi-analytic solutions for the surface uplift. As shown in Fig 2.3, the errors are localized to the major loading regions. If we include spherical harmonics up to degree 32, comparing the top left panel with the bottom left panel in Fig 2.3, an error as large as 3.0% can be observed in Canada. On the other hand, if we truncate both solutions to degree 8, comparing the top right panel and the bottom right panel in Fig 2.3, we have an error of less than 1.0% even in Canada. To understand the source of the relatively large error for the high degree terms, we compute the degree amplitude for each \( l \):

\[
a_l = \frac{1}{T} \int_0^T \sqrt{\frac{1}{2l+1} \sum_{m=0}^{l} (C(l,m,t)^2 + S(l,m,t)^2)} \, dt,
\]

where \( C \) and \( S \) are the cosine and sine coefficients for the vertical surface displacement. As shown in Fig 2.4, the degree amplitude decreases rapidly with increasing \( l \). Because the short wavelength components have relatively small amplitudes, their high amplitude errors do not
Figure 2.3  The left two panels show the root mean square values of the surface uplift and of the difference in surface uplift between the finite element and semi-analytic solutions, for degree $l \leq 32$; the right two panels show the rms value of the uplift and of the difference in uplift for degree $l \leq 8$. 
translate to the same percentage error in Fig 2.3. On the other hand, also due to their smaller amplitude, the short wavelength components in the finite element approach are more susceptible to leakage from the components with larger amplitudes. Our numerical experiments suggest that this error becomes nearly 2 times smaller when increasing the resolution from 12x48x48x48 to 12x80x80x48.

![Figure 2.4](image)

**Figure 2.4** degree amplitude for each degree $l$.  

Similar to the Heaviside loading case, the degree 2 order 1 term has a larger error (2.9%) than other large-scale terms (see Table 2.2). For the ICE-5G case, the forcing includes harmonics of all degrees and orders; and in fact the (2, 1) term has a generally smaller amplitude than other large-scale terms (see Figure 2.2). This suggests that the relatively large (2, 1) error could be due to leakage into the (2, 1) term caused by the response to the larger-amplitude harmonics, compounded by the difficulties introduced by polar wander feedback as described above for the Heaviside case.

We investigate this through the following numerical experiments. Since the calculation of the dynamic ocean load involves the integration of the Earth’s response over the ocean, it couples responses at different spherical harmonics. We remove this physical coupling by turning off the sea level iteration and considering only the static ocean load (we consider the static load alone, only for this (2, 1) test; the inclusion of the dynamic ocean load is the default setup unless otherwise specified). We set up three cases with resolution of 12x48x48x48, 12x64x64x48, and 12x80x80x48, respectively (Table 2.2), and run the numerical code to examine its performance. As shown in Fig 2.5 (a), the result converges to the semi-analytic solution with increasing resolution. This suggests that the relatively large (2, 1) error shown in Table 2.2 is probably a resolution issue, and could be improved if we had sufficient memory resources to compute on a finer grid. To determine the effects of leakage caused by the other harmonics in the forcing, we extract the (2, 1) component from the ICE-5G ice load, and compute the response to this single harmonic load. Fig 2.5 (b) shows that the error is reduced significantly compared to the 12x80x80x48 full ICE-5G case, which does suggest that the large (2, 1) error is mainly due to leakage from components with larger amplitudes.
Figure 2.5 Convergence of the degree 2 order 1 numerical results to the semi-analytic solution with (a) increasing resolution, and (b) using only the single harmonic (2,1) load extracted from the ICE-5G loading history. The figures show the elapsed time from 60000 years ago to 15000 years ago.
So far we have discussed benchmark results only for surface uplift. By averaging the difference between the finite-element and semi-analytic results over the entire ice loading history (Eq. (31)), we obtain a modeling error of 3.0% when including \( l \geq 32 \) (less than 1.0% for \( l \leq 8 \)) (See Fig. 3). This also provides a reasonable estimate of the modeling error for any GIA observation that depends mainly on surface uplift. For example, we thus expect a modeling error of about 3% for our finite element predictions (based on a 12x80x80x48 mesh) of historic relative sea level (RSL) measurements, where RSL is defined as the geoid minus the surface topography.

Similar benchmark results can be used to quantify modeling errors related to present day variability. For example, suppose we are interested in using time-variable gravity measurements from the GRACE satellite gravity mission, or the ice elevation measurements from a satellite altimeter mission, to determine ongoing mass changes of the Antarctic ice sheet. This requires that we model and remove GIA-induced secular trends from the total mass change. To model the GIA process on a spherically symmetric Earth, we can use either the semi-analytic method or the finite-element method, and the finite element modeling error can be defined as the difference between the results from these two methods. For the GRACE estimates, we expand our predicted present day GIA geoid rates into spherical harmonics. The harmonic coefficients are processed using the same spatial averaging technique used by Velicogna and Wahr (2006) to determine Antarctic mass loss from GRACE data. The semi-analytic result shows that the GIA effect for ICE-5G contributes 145.40 Gton/yr ice mass gain over Antarctica. Using the finite-element model, we obtain 147.74 Gton/yr. Our estimate of the finite element modeling error for the GRACE GIA correction is obtained by differencing these two numbers: 2.34 Gton/yr (an error of 1.6% relative to the semi-analytic result). For the altimeter estimates, the present day GIA uplift rates are integrated over the
Table 2.3 The impact of GIA on GRACE and altimeter estimates, of the present-day rate of increase in Antarctic mass (results in Gton/yr) for a spherically symmetric Earth and for the ICE-5G deglaciation history and VM2 viscosity profile (Peltier, 2004). Results computed using both the semi-analytic and finite element models are shown. The difference is interpreted as the error in the finite element model.

<table>
<thead>
<tr>
<th></th>
<th>semi-analytic result</th>
<th>finite-element result</th>
<th>modeling error</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRACE</td>
<td>145.40</td>
<td>147.74</td>
<td>2.34</td>
</tr>
<tr>
<td>Altimeter</td>
<td>25.02</td>
<td>24.46</td>
<td>0.56</td>
</tr>
</tbody>
</table>

Antarctic ice sheet, and multiplied by the density of ice to obtain the apparent change in ice mass. The results from the semi-analytic model and the finite-element model are 25.02 Gton/yr and 24.46 Gton/yr, respectively. The implied finite element modeling error is then 0.56 Gton/yr (2.2%) for the Antarctic altimeter estimate. These estimates are summarized in Table 2.3.

Another common geodetic data type used in GIA studies is GPS monitoring of crustal uplift rates. For example, in Antarctica, where RSL observations are scarce, GPS observations provide a means to assess and improve the accuracy of GIA models. In these studies, the elastic response to the contemporaneous ice mass change is modeled and removed from the GPS uplift rates, and the trend of the residuals is interpreted as an estimate of the GIA signal (see, e.g., Thomas et al., 2011). This trend is compared with the appropriate GIA model result, to assess that model. To determine the computational accuracy of our finite element predictions of Antarctic uplift rates, we compute those uplifts using ICE-5G and a spherical Earth, for both the semi-analytic and finite-element methods. The finite element modeling error is taken to be the difference between the results of the two models. The present day uplift rates are computed on a spatial grid and the difference between the two models is shown in Fig 2.6. The modeling error increases with the amplitude of the uplift
rate. The largest error we observe for the uplift rate is 0.98 mm/yr (8.0%). And the largest error observed at an existing GPS station is 0.73 mm/yr (7.0%), the details of which are shown in the next section.

![Diagram showing present day uplift rates in Antarctica from semi-analytic method and difference between finite-element model and semi-analytic model.](image)

**Figure 2.6** The left panel shows the present day uplift rates in Antarctica from the semi-analytic method. The right panel shows the difference in the present day uplift rates between the finite-element model and the semi-analytic model.

2.6 3-D viscosity structure, with implications for Canadian RSL observations and Antarctic geodetic measurements

Most GIA modeling efforts employ 1-D viscosity models. Viscoelastic Green functions (Love number solutions) are computed using the 1-D models and are convolved with an ice model to predict present day observables, such as RSL observations, GRACE mass estimates, altimeter elevation estimates, etc. What errors might this introduce, given that there is almost certain to be lateral variability in the Earth’s viscosity structure? The answer
to this question depends both on the true viscosity structure of the Earth and on the 1-D model used to approximate the Earth.

For the former, we choose a single plausible 3-D viscosity structure, determined from a global seismic tomography model (S20RTS shear wave model of Ritsema, 1999), using assumed values of the activation energy (Paulson et al., 2005). We include an elastic lithosphere that has lateral variations in thickness derived from a compilation of elastic plate thicknesses from Watts (2001) (see Zhong et al., 2003). For our 1-D approximation, we initially employ a 1-D model that best represents the average of the 3-D viscosities under Canada, where the most prominent deglaciation takes place and where most of the relative sea level observations are recorded. The rationale is that it is likely that a 1-D viscosity profile determined by fitting to real GIA observations is most likely a representative of the average viscosity structure beneath Canada. When GIA modelers use the VM2 (Peltier, 2004) 1-D viscosity model, for instance, it is likely that they are using the Canadian average, rather than the global average, of the Earth’s true 3-D structure. To build our 1-D model, at each depth we average together the logarithms of the 3-D viscosities under Canada to get the 1-D value at that depth (Paulson et al., 2005).

To help verify our hypothesis that a 1-D viscosity model like VM2 is likely to be a Canadian average, we use our 3-D and 1-D models to predict relative sea level observations for the last 8,000 years at four prominent RSL observing sites: Churchill, Cape Henrietta Maria, Ottawa Island and Ungava Peninsula. As a comparison, we also include results computed for a globally averaged 1-D viscosity profile. The 1-D viscosity models are plotted in Fig 2.7, and the RSL results are shown in Fig 2.8. As anticipated the Canadian average results are in good agreement with the 3-D results. The global average does not do as good a job. Our result is consistent with previous studies that considered other major loading areas
Figure 2.7 1-D viscosity structures
Figure 2.8 Relative sea level estimates at Churchill (top left), Cape Henrietta Maria (top right), Ottawa Island (bottom left) and Ungava Peninsula (bottom right).
(e.g. see Kaufmann and Wu, 2002 for a 3-D inversion study for Fennoscandia).

We next look at the possible impact of using a 1-D viscosity profile when interpreting geodetic observations of Antarctica. Considering that there are significant lateral variability in the viscosity and the lithosphere thickness beneath Antarctica (Fig 2.9), we suspect that the effect of 3-D structure might be important for Antarctic observations (see Kaufmann et al., 2005 for a 3-D study for Antarctica). We compute present day GIA signals for the 3-D model

*Figure 2.9* The viscosity structure at 372 km depth beneath Antarctica (left) and the lithosphere thickness for Antarctica (right).
and the 1-D models using the finite-element method and the semi-analytic method, respectively. The GRACE and altimeter Antarctic estimates are shown in Table 2.5 (The case IDs are summarized in Table 2.4). These numbers should not be used to correct real GRACE or altimeter measurements, because none of the 1-D averages of our assumed viscosity structure are in particularly close agreement with a real GIA-based viscosity profile, such as VM2. For the GRACE estimates, the “Canada” result (i.e., the calculation using 1-D viscosity averaged for the Canadian region) is 11.1 Gton/yr (5.5%) larger than the 3-D result. For the altimeter estimates, the “Canada” result is 0.6 Gton/yr (1.8%) larger. Because the finite element modeling error given in the last section is 1.6% for the GRACE estimates and 2.2% for the altimeter estimates, we conclude that the use of the 1-D Canadian average viscosity model rather than the 3-D profile introduces an error of 5.5% ± 1.6% for the GRACE estimates. For the altimeter estimates, the 1-D model reproduces the 3-D result quite well, and the error we obtain (1.8%) is not large enough to be significant. As a comparison, Table 2.5 also shows the result from the 1-D model using a globally averaged viscosity. The “Global” result is 7.1% larger than the 3-D result for the GRACE estimates and 8.0% larger for the altimeter estimates, implying that the 1-D Canadian average provides a better match to the 3-D result.

Table 2.4: The viscosity profiles considered for the 3-D comparisons. For each 1-D case, we find the lateral average the logarithms of the 3-D viscosities, at each depth.

<table>
<thead>
<tr>
<th>Case ID</th>
<th>Viscosity Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>3-D</td>
<td>Full 3-D viscosity profile</td>
</tr>
<tr>
<td>Canada</td>
<td>1D viscosity profile derived from the average under Canada.</td>
</tr>
<tr>
<td>Global</td>
<td>1D viscosity profile derived from the global average</td>
</tr>
<tr>
<td>Antarctica</td>
<td>1D viscosity profile derived from the average under Antarctica</td>
</tr>
</tbody>
</table>
Table 2.5 Our ICE-5G Antarctic GRACE and altimeter estimates using different viscosity profiles. Note: these numbers should not be used to correct real GRACE or altimeter estimates (see text).

<table>
<thead>
<tr>
<th></th>
<th>3-D</th>
<th>Canada</th>
<th>Global</th>
<th>Antarctica</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRACE Entire Ant</td>
<td>201.02</td>
<td>212.07</td>
<td>215.29</td>
<td>221.29</td>
</tr>
<tr>
<td>GRACE West Ant</td>
<td>91.34</td>
<td>88.10</td>
<td>90.97</td>
<td>93.40</td>
</tr>
<tr>
<td>GRACE East Ant</td>
<td>108.26</td>
<td>122.90</td>
<td>123.54</td>
<td>133.37</td>
</tr>
<tr>
<td>Altimeter Entire Ant</td>
<td>33.10</td>
<td>33.69</td>
<td>35.73</td>
<td>36.06</td>
</tr>
<tr>
<td>Altimeter West Ant</td>
<td>13.75</td>
<td>14.01</td>
<td>15.23</td>
<td>15.24</td>
</tr>
<tr>
<td>Altimeter East Ant</td>
<td>19.35</td>
<td>19.68</td>
<td>20.50</td>
<td>20.82</td>
</tr>
</tbody>
</table>

The Antarctic GPS uplift rate results are shown in Table 2.6 and Fig 2.10. Comparing the 3-D result and the 1-D “Canada” result, differences of up to 0.5 to 1.6 mm/yr (a relative error ranging from 7% to 60%) in uplift rate can be observed at selected stations, which are significantly larger than the corresponding finite element modeling errors at those stations. Although the “Canada” case produces good agreement with the 3-D results for the altimeter estimates (an integration of the uplift rates), it does not provide a good prediction of the 3-D result at each GPS station. Similarly for the “Global” 1-D case, the 3-D/1-D differences are as large as 1.78 mm/yr (46%). This suggests that the effects of 3-D viscosity structure can be important for localized measurements.

One more useful 1-D viscosity average is the average of the logarithms of the 3-D viscosities under Antarctica, referred to in the tables and figures as “Antarctica”. Given that the “Canada” case does a good job reproducing 3-D results in Canada, it is natural to ask whether the 1-D Antarctic model provides a good match to the 3-D results in Antarctica. The GRACE and altimeter results are presented in the last column of Table 2.5, and the GPS uplift results are shown in Table 2.6 and Fig 2.10. Compared to the 3-D results, the “Antarctica” results are 10.0% larger for the GRACE estimates and 9.0% larger for the
Table 2.6 The ICE-5G Antarctic GPS uplift rate predictions: columns 1-3 specify the station information; columns 4-7 show the uplift rate results from four different cases; columns 8-10 present the difference between the three 1-D case results and the 3-D case result; the 11th column shows the modeling error. Note: these numbers should not be used to correct or interpret real GPS measurements (see text).

<table>
<thead>
<tr>
<th>lon</th>
<th>lat</th>
<th>Station</th>
<th>3-D</th>
<th>Canada-3D</th>
<th>Globe-3D</th>
<th>Antarc-3D</th>
<th>1-D</th>
<th>3-D</th>
<th>Antarc-3D</th>
<th>Modeling error</th>
</tr>
</thead>
<tbody>
<tr>
<td>162.57</td>
<td>-78.93</td>
<td>FTP1</td>
<td>2.75</td>
<td>2.26</td>
<td>2.10</td>
<td>2.19</td>
<td>0.50</td>
<td>0.65</td>
<td>0.57</td>
<td>0.12</td>
</tr>
<tr>
<td>163.19</td>
<td>-77.03</td>
<td>ROB1</td>
<td>2.99</td>
<td>1.32</td>
<td>1.53</td>
<td>1.46</td>
<td>1.67</td>
<td>1.46</td>
<td>1.52</td>
<td>0.12</td>
</tr>
<tr>
<td>164.1</td>
<td>-74.7</td>
<td>TNB1</td>
<td>2.65</td>
<td>1.05</td>
<td>1.88</td>
<td>1.66</td>
<td>1.61</td>
<td>0.77</td>
<td>1.00</td>
<td>0.12</td>
</tr>
<tr>
<td>166.67</td>
<td>-77.85</td>
<td>CRAR</td>
<td>2.77</td>
<td>1.63</td>
<td>1.42</td>
<td>1.48</td>
<td>1.14</td>
<td>1.35</td>
<td>1.28</td>
<td>0.12</td>
</tr>
<tr>
<td>204.98</td>
<td>-78.03</td>
<td>MBL1</td>
<td>5.06</td>
<td>6.16</td>
<td>6.68</td>
<td>6.68</td>
<td>-1.10</td>
<td>-1.62</td>
<td>-1.62</td>
<td>0.20</td>
</tr>
<tr>
<td>215.7</td>
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<td>MBL2</td>
<td>2.47</td>
<td>3.04</td>
<td>2.49</td>
<td>2.74</td>
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<td>-0.03</td>
<td>-0.27</td>
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</tr>
<tr>
<td>218.13</td>
<td>-77.34</td>
<td>MBL3</td>
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<td>5.64</td>
<td>5.75</td>
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<td>-1.78</td>
<td>-1.89</td>
<td>0.15</td>
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<td>279.44</td>
<td>-80.04</td>
<td>W05A</td>
<td>8.93</td>
<td>7.94</td>
<td>8.93</td>
<td>8.82</td>
<td>0.98</td>
<td>-0.01</td>
<td>0.11</td>
<td>0.53</td>
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<tr>
<td>291.45</td>
<td>-85.61</td>
<td>W02B</td>
<td>7.38</td>
<td>6.82</td>
<td>6.75</td>
<td>6.89</td>
<td>0.55</td>
<td>0.62</td>
<td>0.49</td>
<td>0.37</td>
</tr>
<tr>
<td>302.1</td>
<td>-63.32</td>
<td>OHIG</td>
<td>2.64</td>
<td>1.50</td>
<td>1.99</td>
<td>1.80</td>
<td>1.15</td>
<td>0.65</td>
<td>0.84</td>
<td>0.12</td>
</tr>
<tr>
<td>296.97</td>
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<td>BREN</td>
<td>6.09</td>
<td>4.72</td>
<td>6.12</td>
<td>5.78</td>
<td>1.38</td>
<td>-0.03</td>
<td>0.31</td>
<td>0.26</td>
</tr>
<tr>
<td>306.8</td>
<td>-82.86</td>
<td>W04A</td>
<td>10.46</td>
<td>8.95</td>
<td>11.1</td>
<td>10.7</td>
<td>1.51</td>
<td>-0.59</td>
<td>-0.21</td>
<td>0.73</td>
</tr>
</tbody>
</table>
**Figure 2.10** The Antarctic uplift rate results for: the 3-D case (top left), the “global” case result (bottom left), the “Canada” case result (top right), and the “Antarctica” case result (bottom right). The GPS stations are labeled by name.
altimeter estimates. The error in GPS uplift rate estimates can be as large as 1.89 mm/yr (49%). So we do not get a better match using the Antarctic average. We suspect that the reason is related to the complex viscosity structure under Antarctica. Unlike in the mantle beneath Canada, where viscosities are relatively uniform across a large region, the mantle under Antarctica exhibits considerable lateral variations in viscosity. As a result, the 1-D viscosity model averaged under Antarctica might not be expected to produce the best match to the 3-D results, suggesting that it is important to use 3-D models in Antarctica.

However we should note that we cannot formulate this as a general conclusion, because we only consider one possible 3-D viscosity profile in this study. We cannot rule out the possibility that for the Earth’s true viscosity structure, the use of an Antarctic average might better match the 3-D results.

2.7 The effects of 3-D viscosity structure on GPS horizontal motion estimates in Antarctica

In this section, we investigate the effect of 3-D viscosity structure on GPS horizontal motion estimates in Antarctica. For the modeling results discussed here, we choose a reference frame where there is no net rotation of the Earth’s mantle. The horizontal displacement rates using the 3-D case, referred to as 3D, and the 1-D Canadian average of the 3-D case, referred to as 1D, are shown in the top panels in Fig 2.11. For the 3D case (top left panel in Fig 2.11), there is a uniform plate motion across the entire Antarctic plate. For the 1D case (top right panel in Fig 2.11), the horizontal motion shows a similar uniform motion for east Antarctica, however, for the western part, we obtain significant outward motions around the two major rebounding centers. Based on these observations, we separate the effect of 3-D viscosity structure into two components: (a) the effect on the uniform plate motion
observed for the entire Antarctica; and (b) the effect on the deformation within the Antarctic plate.

For (a), we model the uniform plate motion by a rigid rotation along an Euler pole
\[ \vec{v} = \vec{\omega} \times \vec{x}, \]  
\[ \text{(34)} \]
where \( \vec{v} \) denotes the displacement rate at a given position \( \vec{x} \), and \( \vec{\omega} \) is the angular velocity that describes the rigid rotation of the plate. By fitting \( \vec{\omega} \) for Antarctica, we find the 3D case introduces a larger uniform plate motion than the 1D case does. As shown in Table 2.7, the angular velocity from the 3D case is \( \sim 0.01 \) deg/m.y., nearly 2 times larger than that from the 1D case (\( \sim 0.006 \) deg/m.y.).

For (b), we remove the uniform plate motion as given by Eq. (34), and the deformations within the plate are shown in the bottom panels of Fig 2.11, for the 3D case (bottom left), and the 1D case (bottom right), respectively. Comparing the two results, we find that the patterns of the horizontal motion are now similar for both cases, and the deformations within the Antarctic plate only show slight difference in their magnitudes.

A nature question to ask is whether this GIA-induced uniform plate motion would affect the GPS plate motion estimates. In Table 2.8, we show the model results using case 3D, case 1D and also the GPS observed plate motion and the GPS observation error (Sella et al., 2002) for Antarctica. For the 3D case, the magnitude of the GIA-induced plate motion is comparable to the GPS observation error, and only accounts for \( \sim 4\% \) of the observed plate

<table>
<thead>
<tr>
<th></th>
<th>3D</th>
<th>1D</th>
<th>3Da</th>
<th>1Da</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lithosphere</td>
<td>0.0013</td>
<td>8.05E-08</td>
<td>0.0066</td>
<td>1.25E-07</td>
</tr>
<tr>
<td>Antarctica</td>
<td>0.0097</td>
<td>0.0060</td>
<td>0.0038</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

Table 2.7 Angular velocity of the lithosphere and the Antarctic plate in deg/m.y.
Figure 2.11 Antarctic surface displacements for case 3D (top left), and case 1D (top right). After removing the uniform plate motion in Antarctica, the surface displacements for case 3D (bottom left), and case 1D (bottom right).
motion. And the model results from case 1D are even smaller than the 3D results. So, for the 3-D viscosity structure we have investigated, the GIA-induced uniform plate motion would not affect Antarctic plate motion estimates in a significant way. In fact, a 3-D viscosity structure could also induce a uniform plate motion in a global scale. As shown in Table 2.7, using case 3D, instead of case 1D, introduces a net rotation of the Earth’s lithosphere. Similar to what we find for Antarctica, the magnitude of the uniform motion of the lithosphere is also small (0.0013 deg/m.y.) compared to those from the tectonic plate motion (on order of ~ 0.1 deg/m.y., see values given in, e.g. Sella et al., 2002 and Klemann et al., 2008).

As an additional test, we build another two viscosity structures, referred to as 3Da and 1Da, respectively. To build case 3Da, we rescale the upper mantle viscosity of case 3D in the following way: for each vertical column in the finite-element grid (given by a combination of latitude and longitude), we reduce the viscosity from case 3D by a factor of 100 at the bottom of the lithosphere, and keep the viscosity value the same as in case 3D at the top of the lower mantle (at the depth of 670 km); for the rest part of the upper mantle, the logarithm of the viscosity value is linearly interpolated. Case 1Da is then obtained by taking the 1-D Canadian average of case 3Da. Apparently, case 3Da and 1Da has relatively small upper mantle viscosity, especially in the asthenosphere. As shown in Fig 2.12, Table 2.7 and 2.8, there is no significant uniform plate motion in Antarctica for both case 3Da and 1Da. The major 3-D effects we could observe in Fig 2.12 are in the magnitudes of the horizontal and vertical motions. In general, case 3Da introduces a larger horizontal motion and smaller vertical motion than case 1Da does. Similar to case 3D, case 3Da induces a uniform net rotation of the lithosphere, but the magnitude of this rotation is also small (Table 2.7).
Table 2.8 Antarctic plate motion in $10^{-3}$ rad/m.y. in Cartesian coordinates. The last two rows show the GPS observed plate motion in Antarctica, and the observation error.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D</td>
<td>-0.165</td>
<td>0.026</td>
<td>-0.031</td>
</tr>
<tr>
<td>1D</td>
<td>-0.099</td>
<td>-0.034</td>
<td>0</td>
</tr>
<tr>
<td>3Da</td>
<td>0.054</td>
<td>0.036</td>
<td>0.012</td>
</tr>
<tr>
<td>1Da</td>
<td>0.015</td>
<td>-0.002</td>
<td>0</td>
</tr>
<tr>
<td>Observed</td>
<td>-1.431</td>
<td>-1.482</td>
<td>3.358</td>
</tr>
<tr>
<td>Obs.Error</td>
<td>0.047</td>
<td>0.071</td>
<td>0.167</td>
</tr>
</tbody>
</table>

Comparing the results from case 3D and case 3Da, we find that by reducing the upper mantle viscosity, we obtain (a) a larger net rotation of the lithosphere (Table 2.7), but smaller net rotation for the Antarctic plate; and (b) smaller vertical uplifts in Antarctica. The implication from both (a) and (b) could be complicated. For (a), we know that there are two time scales in the GIA problem: the time scale determined by the mantle viscosity structure, and the time scale given by the deglaciation history. For the latter, the time scale associated with ice deglaciation always depends on the regions or spatial wavelength that we look into. If the mantle viscosity is 3-D, then the time scale determined by the viscosity structure would also be sensitive to the spatial regions that we investigate. The present-day displacement rates are determined by both time scales. So, it is likely that we would obtain different uniform plate motion by fitting the horizontal rates for different regions on Earth. For (b), reducing the upper mantle viscosity would lead to a large viscous relaxation rate, which would typically introduce a large surface displacement rate. However, driven by the large viscous rate, the mantle material might have mostly relaxed by present day. If this were the case, then we would only observe a small surface displacement rate. So, reducing the upper mantle viscosity would introduce two competing effects, and whether the present-day surface rates would be increased or decreased depends on which effect is dominant.
Figure 2.12 Antarctic surface displacements for case 3Da (top left), and case 1Da (top right). After removing the uniform plate motion in Antarctica, the surface displacements for case 3Da (bottom left), and case 1Da (bottom right).
2.8 Discussion and conclusions

A 3-D finite-element model is developed to study the viscoelastic response of a compressible Earth to surface loads. The mantle in the model can have a layered density distribution, along with 3-D structure for the mantle viscosity and Lamé parameters. Though here, we consider 3-D inhomogeneities only in the viscosity and assume the Lamé parameters are spherically symmetric. The effects of center of mass motion, polar wander feedback, and self-consistent ocean loading are implemented. The numerical method is discussed in detail. Solutions for spherically symmetric structure are benchmarked against a semi-analytic solution for both a Heaviside loading history and the ICE-5G loading history. In general, accuracies of better than 1% can be obtained in the surface uplift solution for a Heaviside loading history. For ICE-5G, we use surface uplift as a proxy for relative sea level, which is a key observation type when constructing ice deglaciation and mantle viscosity models, and we obtain a modeling error of better than 1% for long wavelength terms ($l \leq 8$), and of about 3% when including degrees up to 32. Our numerical experiments suggest that the error for the short wavelength terms is mainly induced by leakage from terms with larger amplitudes, which can be reduced if we increase the resolution by decreasing the grid size. When the effects of polar wander are included, we find that for ICE-5G, the degree 2 order 1 solution shows a smaller amplitude and a larger error than other global-scale terms. The error is mainly induced by leakage from components with larger amplitudes, and it can be reduced significantly by increasing the resolution. Throughout these discussions, we quantify the numerical errors in the finite-element model by computing the differences between the finite-element results and the semi-analytic solutions. There can, of course, also be errors in the semi-analytic results. However, given that the two solutions agree with each other to high accuracy, it is unlikely that the uncertainties in the finite-element solutions could notably
exceed the error estimates we have provided here.

The GIA signal is a source of noise for GRACE gravity and altimeter elevation estimates of present-day Antarctic ice mass loss. It is important to be able to accurately model and remove GIA effects from those observations and to assess the error in those GIA model results. Using the ICE-5G benchmark results, we investigate the numerical error in our Antarctic GIA estimates. When computing the GIA contribution in Antarctic mass trend using the finite-element model given a 12x80x80x48 resolution setup, we obtain a 1.6% relative error for the GRACE results, and a 2.2% relative error for ice altimeter results.

Antarctic GPS observations have been used to constrain Antarctic GIA models, and it is useful to know how large the GIA modeling errors are when comparing with GPS observations. Our ICE-5G benchmark results show that if we use the finite-element model to compute the GIA-induced present-day uplift rates at existing Antarctic GPS sites, we obtain a maximum numerical error of 7%. This is a significantly larger relative error than is obtained for the altimeter estimates, despite the fact that both are based on uplift rate results. The reason that the GPS errors tend to be larger than the altimeter errors, is that the altimeter error is an integrated value over the entire ice sheet while the uplift rate at a specific GPS station is more sensitive to short wavelength features. This issue can be mitigated in the near future if we can further refine both the spatial and the temporal resolution in our calculation, given increased capacity in parallel computing.

Since there are a large number of GIA studies that employ incompressible Earth models, it is useful to briefly discuss the effect of compressibility on the GIA estimates. We use our semi-analytic method to compute two sets of GIA estimates, one using the original PREM structure and the other using an incompressible version of PREM. We find that making the Earth model incompressible reduces the GIA-induced Antarctic GRACE mass
gain estimates by roughly 2%, and reduces the present day Antarctic uplift rates by about 5%.

We apply the finite-element model to a plausible 3-D viscosity structure. The 3-D viscosity is determined from a global seismic tomography model along with a reasonable model of elastic lithosphere thickness. Using the model results, we study the effects of 3-D viscosity structure on various GIA observables. Relative sea level results from northern Canada show that the 3-D case is better predicted using the 1-D Canadian average than the global average. This suggests that a GIA viscosity model based on relative sea level data is more likely to represent a Canadian average than a true global average.

We investigate the error that might be introduced into GRACE and altimeter estimates of total Antarctic ice mass loss by approximating the 3-D viscosity structure using its 1-D Canadian average. The 1-D model produces a small difference compared to the 3-D results for both the GRACE and the altimeter estimates (5.5% and 1.8%, respectively). For uplift rates computed at the GPS stations, the 3-D results and the results for the 1-D Canadian average show differences ranging from 7% to 60%, which indicates that 3-D effects might be more important for localized measurements. We also compute the GIA estimates using the Antarctic average of the 3-D viscosity profile. The results suggest that the Antarctic average does not provide a better match to the 3-D case. We suspect this may be due to the complex viscosity structure beneath Antarctica.

In addition, we also investigate the effect of 3-D viscosity structure on the GPS horizontal motion estimates in Antarctica. We find that 3-D viscosities could induce a uniform plate motion in Antarctica. After removing this uniform plate motion, the deformation within the Antarctic plate shows similar pattern for the 3-D and the 1-D cases, implying that using GPS observations of the Antarctic horizontal motion to constrain 1-D GIA models would probably require the removal of the uniform plate motion in Antarctica.
We also find that the GIA-induced uniform plate motion in Antarctica is of similar magnitude as the GPS observation error, and is unlikely to affect the GPS plate motion estimates. Different from the GIA solutions using 1-D viscosity structure, where there is no degree-1 toroidal field, GIA results using 3-D viscosity structures show net rotation of the entire lithosphere. Based on the viscosity structures we have tested, we find that reducing the upper mantle viscosity would increase the net rotation of the lithosphere. But the magnitude of this GIA-induced net rotation is significantly smaller than those from the tectonic plate motion.

In this study, we explore the effects of only one 3-D viscosity profile (along with another one that is derived through rescaling the original 3-D profile). So it is difficult to make specific, quantitative estimates of the likely effects of realistic 3-D structure. However, it is still useful to compare our results for the effects of 3-D structure based on this one plausible 3-D viscosity model, with the results from a recent Antarctic mass balance study (Shepherd et al., 2012) that are based on GIA models with different ice deglaciation histories. Shepherd et al., (2012) use two, newly developed GIA models, W12a (Whitehouse et al., 2012) and IJ05_R2 (Ivins et al., 2013), to compute GIA corrections for GRACE Antarctic mass loss estimates. They find that those two corrections differ by roughly 20%. This relative difference is much larger than the relative effects of 3-D structure we obtain here (5.5% for GRACE estimates). Shepherd et al., (2012) also compare the W12a and IJ05_R2 corrections with corrections based on ICE-5G, and find absolute differences that are 5-10 times larger than the differences between the W12a and IJ05_R2 corrections. So, in general, we conclude that the effects of 3-D viscosity structure on GRACE estimates of present-day Antarctic mass loss are probably smaller than the differences between GIA models based on different Antarctic deglaciation histories. On the other hand, the effects of 3-D viscosity structure on Antarctic GPS observations of present-day uplift rate can be significant, and can complicate
efforts to use GPS observations to constrain 1-D GIA models.
Chapter 3

GIA effects on GRACE and GPS observations caused by ice loss in southern Greenland since the Little Ice Age

3.1 Introduction

The on-going deformation of the solid Earth in southern Greenland is determined by the elastic response to present-day ice mass changes and the continuing viscous relaxation of the Earth’s mantle in response to past ice mass changes. The viscous component can be further decomposed into the response of the Earth to the ice evolution during late Pleistocene, from approximately 120,000 to 100 years ago, and the response to the ice mass change occurred during the Little Ice Age (LIA) deglaciation for the last 100 years. The long term GIA effects caused by the late Pleistocene deglaciation have been modeled using the ICE-5G loading history and the VM2 viscosity structure (Peltier, 2004). In this study, our goal is to examine the possible GIA effect caused by the ice mass evolution over the last century on present-day mass trend and surface motion. The accurate modeling of this short term GIA effect requires high resolution ice elevation models. Recently, a novel geometric approach has been developed to estimate mass loss for the southern Greenland ice sheet (sGrIS) (Kjeldsen et al., 2013). This approach is founded in Glen’s Flow law, and it allows us to derive distributed mass changes, both along the ice margin and in the interior of the GrIS where information regarding ice surface changes are limited and associated with great uncertainty (Lecavalier et al., 2013; Vinther et al., 2009). This approach is also independent of historical climate- and ice discharge data, and not spatially confined to outlet glaciers or limited by an estimated upper boundary. Using this approach, an estimate of the ice elevation
changes of the entire sGrIS has been obtained for the last century. Based on this estimate, we compute the LIA viscous results using our GIA model, and then compare the results with present-day GRACE and GPS observables.

In this chapter, section 3.2 describes the Post-LIA loading history, the Earth models and the computational models we use. In section 3.3, we present the LIA results in terms of GRACE mass trend estimates and GPS surface motion estimates. These results are then compared with the observed present-day signals and the other model results, including predicted elastic results, and ICE-5G/VM2 GIA model results. We conclude in section 3.4.

3.2 Model Description

The surface deformation in sGrIS is decomposed into a viscous component, caused by the Earth’s viscous relaxation in response to the ice loss in the past 100 years, and an elastic component, induced by the present-day ice loss. We discuss the computation models for each of the components in this section.

3.2.1 GIA model for the viscous component

To find the viscous component, we use a GIA model that computes the visoelastic response of a spherically symmetric Earth in the spectral domain (A et al., (2013); Han and Wahr, (1995)). The input of the GIA model contains two components: an ice loading history that describes the ice height evolution after the LIA; and a rheological model that specifies the elastic property and the viscosity structure of the Earth. Each of these two components is described in the next two subsections.

3.2.1.1 The Post-LIA loading history

Mass balance estimates of the sGrIS since retreat commence from the maximum extent of the LIA to 2010 has been derived by Kjeldsen et al., (2013) using a newly
developed geometric approach. High quality aerial stereo photogrammetric imagery recorded between 1978 and 1987 is used to map morphological features such as trim lines (boundary between freshly eroded and non-eroded bedrock) and end moraines marking the ice extent of the LIA. Vertical point-based differences associated with former ice extent are obtained, and these point measurements are combined with contemporary ice surface differences derived using NASA’s Airborne Topographic Mapper (ATM) from 2002-2010, NASA’s Ice, Cloud, and land Elevation Satellite (ICESat) from 2003-2009, and NASA’s Land, Vegetation, and Ice Sensor (LVIS) from 2010, to estimate mass loss throughout the 20th and early 21st Century. Based on the mass loss estimates in sGrIS, linear trends are derived on a 1 km by 1 km grid for three time intervals, 1900 - 1981, 1981 - 2002, and 2002 - 2010. For instance, Fig 3.1 shows the linear trend for the three time intervals in terms of ice height change in meters/yr. Rapid ice loss can be observed near Jakobshavn Isbrae, Helheim Glacier, Kangerdlugssuaq Glacier and the Southeast Glaciers (see Kjeldsen et al., 2013 for details).

We assume the ice height is 0 at 1900, and we build a piecewise linear Post-LIA ice loading history, from 1900 to 1981, 1981 to 2002 and 2002 to 2010 using the mass loss rate associated with each corresponding time period.

3.2.1.2 The Earth model

We assume a spherically symmetric, self-gravitating, compressible Earth. The Earth model we use is available online (Peliter’s website). The elastic parameters are derived from PREM (Dziewonski and Anderson, 1981). For the mantle viscosity structure, we use three profiles in this study: (1) the VM2 viscosity structure (Peltier, 2004), referred to as VM2 hereafter; (2) VM2 viscosity structure, but with the upper mantle viscosity (90km to 670km depth) reduced by a factor of 10, referred to as VM2a (see Fig 2); and (3) VM2 viscosity structure, but with the asthenosphere viscosity (90km to 420km depth) reduced by a factor of
10, referred to as VM2b (Fig 3.2). For VM2a, the viscosity for the entire upper mantle is on order of $10^{19}$, which is probably unrealistically small for a region far from plate boundaries, but it could provide an upper bound estimate of the GIA effect. For VM2b, the asthenosphere viscosity is large, but plausible, and we expect the viscous results from VM2b might lie between those from VM2 and VM2a.

Using the Post-LIA loading history along with the Earth model, we compute the viscous response in southern Greenland up to degree and order 200, for which we find the viscous results have converged. In the GIA model, the effects of center of mass motion, polar wander feedback, and self-consistent ocean loading are implemented (see e.g. Paulson et al., (2005) and A et al., (2013)). Because we only include the ice loading in southern Greenland, those effects are small. However, we suspect that those effects might be larger if the Post-LIA deglaciation occurred over the entire globe were accounted in the GIA model.
Figure 3.2 Mantle viscosity structure for VM2 (solid line), VM2a (dashed line) and VM2b (dashdot line). The upper mantle viscosity of VM2a (depth from 90 km to 670 km) is smaller than that of VM2 by a factor of 10. The asthenosphere viscosity of VM2b (depth from 90km to 420km) is smaller than that of VM2 by a factor of 10.

3.2.2 Computations of the elastic component

The linear trend of ice loss for the period of 2002 to 2010 are used as the present-day mass change rates, and are converted into geoid rates using the load Love numbers of potential (Wahr et al., 1998)). To compute the elastic deformation induced by the present-day ice loss in sGrIS, we use the loading displacement Green’s functions for Earth model PREM (with a continental crust), as tabulated by Jentzsch, (1997). We convolve the present-day mass change rates with the Green’s functions to obtain the elastic crustal displacements in both vertical and horizontal directions.
Table 3.1 GRACE estimates of the mass change rate in Gton/yr. Column 1 shows the elastic component due to the present-day ice loss; columns 2-4 are the viscous components caused by the Post-LIA loading history, using VM2, VM2a, and VM2b, respectively.

<table>
<thead>
<tr>
<th>Elastic component</th>
<th>Post-LIA (VM2)</th>
<th>Post-LIA (VM2a)</th>
<th>Post-LIA (VM2b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-96.7</td>
<td>1.6</td>
<td>7.8</td>
<td>4.5</td>
</tr>
</tbody>
</table>

3.3 Results

We present our results in southern Greenland in terms of two geodetic observables:

1) GRACE ice loss estimates; and (2) GPS surface motion estimates.

3.3.1 GRACE estimates of the present-day ice loss

Using the method described in the last section, we compute the present-day geoid rates for the elastic response and the viscous response, respectively. For the GRACE estimates, we expand our predicted present-day geoid rates into spherical harmonics. The harmonic coefficients are processed using the same spatial averaging technique used by Velicogna and Wahr (2006). As shown in Table 3.1, for the elastic component, the GRACE estimate is -96.7 Gton/yr for sGrIS, where the negative sign indicates mass loss. This value would be the present-day mass loss rate inferred from GRACE data, if the linear trend of ice loss for the period of 2002 to 2010 is an accurate representative of the present-day ice loss rate. In addition, GRACE observations would also see mass change caused by the viscous relaxation in response to the deglaciation since LIA. Using VM2, this viscous component is 1.6 Gton/yr. So, not correcting the viscous component from the GRACE results would lead to an underestimate of the present-day ice loss rate by 1.7%. When we reduce the VM2 upper mantle viscosity by a factor of 10 (i.e. using VM2a), the viscous component increases to 7.8 Gton/yr, implying that not removing the viscous response would cause GRACE to
underestimate the present-day ice loss rate by 8.1%. If we only reduce the viscosity within
the asthenosphere (VM2b), the viscous component is 4.5 Gton/yr, corresponding to an
underestimate of the present-day ice loss rate by 4.7%.

To compare our predicted LIA results with the observed present-day signal, we show
the GRACE mass trend for the LIA results using different viscosity structures in Fig 3.3(a) -
(c), and for the apparent GRACE mass trend fit to April, 2002 through June, 2013 in Fig
3.3(d). The maximum LIA mass trend obtained in southern Greenland is ~1cm/yr when using
VM2a (see Fig 3.3(b)), while the observed GRACE trend near the same region is ~ -14cm/yr
(Fig 3.3(d)). So the LIA results only account for ~ 7% of the observed GRACE trend.
Considering that viscosities on order of $10^{19}$ for the entire upper mantle are likely to be too
small in reality, our results using VM2a probably provide an upper-bound estimate of the
Post-LIA viscous response. And even this upper-bound estimate is small compared to the
observed GRACE trend. In Fig 3.3(e), we show the mass trend predicted by the ICE-
5G/VM2 GIA model (Peltier, 2004). In our calculation, we stop the ice mass change in ICE-
5G deglaciation history for the last 100 years, and the result shown here describes the GIA
contribution caused by the global ice mass change from 120,000 years ago to 100 years ago.
In southern Greenland, the magnitude of the ICE-5G/VM2 result is similar to what we find
for the LIA results using VM2a and VM2b. And these results are all significantly smaller
than the observed GRACE trend, so they might not affect the interpretation of the GRACE
data in a remarkable way.

3.3.2 GPS present-day surface motion estimates

The surface displacement rates are shown in Fig 3.4. The pattern of the elastic
response (Fig 3.4(a)) closely resembles the ice height change shown in Fig 3.1. Large uplift
rates are localized around the centers of the glaciers. Significant outward horizontal flux can
Figure 3.3 GRACE estimates of the mass change rate in water equivalent (cm/yr). (a) The GRACE estimates from the LIA results, using VM2; (b) the LIA results using VM2a; (c) the LIA results using VM2b; (d) The apparent GRACE mass trend fit to April, 2002 through June, 2013; and (e) The GRACE estimates using ICE-5G/VM2.

be observed near these centers. The viscous response using VM2 is shown in Fig 3.4(b). The spatial pattern of the displacement rates are similar to the elastic results, but have less short wavelength features due to the presence of an elastic lithosphere. The surface displacement rates in the viscous results are about an order of magnitude smaller than those from the elastic results. When we reduce the upper mantle viscosity by a factor of 10 (VM2a), the viscous rates increase significantly. Except for areas in close proximity to the centers of the major glaciers, both the uplift and horizontal rates, as shown in Fig 3.4(c), are comparable in magnitude to the elastic results. Surface deformation results using VM2b are similar to what
Figure 3.4 The surface displacements in southern Greenland from (a) the elastic deformation induced by the present-day ice loss; (b) the viscous deformation in response to the Post-LIA deglaciation, using VM2; (c), same as (b) but using VM2a; and (d), same as (b) but using VM2b (note the different scaling bars used).
we find using VM2a, only with slightly smaller magnitudes.

In Table 3.2, we compare the predicted LIA uplift rates at three GPS stations in southern Greenland with GPS observations, model results using ICE-5G/VM2, and the predicted elastic uplift rates. The LIA results using VM2 are about an order magnitude smaller than the observed GPS rates and the predicted elastic rates. At KULU, the magnitude of the LIA uplift rates is similar to the model result using ICE-5G/VM2. At the KELY and QAQ1, the LIA uplift rates are significantly smaller than the ICE-5G/VM2 results. When using VM2a and VM2b, the magnitudes of the LIA uplift rates become comparable to those of the observed uplift rates and of the predicted elastic rates. And they are of similar magnitude as the ICE-5G/VM2 results.

Table 3.2 Observed and predicted uplift rates in mm/yr. Column 1 specifies the site names; columns 2-4 are the predicted uplift rates caused by the Post-LIA loading history, using VM2, VM2a and VM2b, respectively; column 5 shows the observed uplift rates using the IGS05 reference frame; column 6 are the uplift rates in response to the ICE-5G loading history, using VM2; and column 7 shows and the predicted elastic uplift rates.

<table>
<thead>
<tr>
<th>Site</th>
<th>Post-LIA (VM2)</th>
<th>Post-LIA (VM2a)</th>
<th>Post-LIA (VM2b)</th>
<th>Observed uplift rates</th>
<th>ICE-5G (VM2)</th>
<th>Elastic uplift rates</th>
</tr>
</thead>
<tbody>
<tr>
<td>KELY</td>
<td>0.3</td>
<td>1.6</td>
<td>0.8</td>
<td>2.1 ± 1.1</td>
<td>-1.8</td>
<td>2.2</td>
</tr>
<tr>
<td>KULU</td>
<td>0.5</td>
<td>3.5</td>
<td>2.3</td>
<td>8.3 ± 1.1</td>
<td>-0.7</td>
<td>3.7</td>
</tr>
<tr>
<td>QAQ1</td>
<td>0.3</td>
<td>1.8</td>
<td>1.3</td>
<td>3.4 ± 1.1</td>
<td>2.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>

The observed uplift rate at a GPS site contains two components: the elastic response to the present-day surface mass change, and the viscous component caused by the historical ice mass change. As mentioned in the last section, we stop the ice change in ICE-5G loading history for the past 100 years, so presumably the viscous component observed by GPS would
be a combination of the ICE-5G/VM2 result and the LIA result. And the GPS observed uplift rate is supposed to match the summation of the LIA result, the ICE-5G/VM2 result and the predicted elastic result. As can be inferred from Table 3.2, we do not have a consistent match at the three GPS stations, for the viscosity structures investigated here. Apparently, there are uncertainties associated with each model component in the uplift rate results. For the LIA results, we do not have a well constrained viscosity structure. When using VM2, the LIA uplift rates at all three GPS sites are smaller than the GPS observation uncertainty (1.1 mm/yr). For the results using VM2b, where the asthenosphere viscosity is large, but plausible, the magnitudes of the uplift rates increase significantly, and can account for ~30-40% of the observed uplift rates. Similar results are obtained from VM2a, only with a moderate increase in the uplift rates. So using different viscosity structures, the effect of LIA signal could range from negligible (VM2) to fairly important (VM2a, VM2b). For the ICE-5G/VM2 GIA model, it is determined mostly by fitting the Canadian relative sea level observations, and it might not provide best estimates for Greenland GIA observables. For the predicted elastic rates, the results are very sensitive to the present-day deformation localized around the GPS site. In our calculation, the present-day deformation is derived from the estimated ice elevation change rates for 2002-2010 (Fig 3.1(c)), and these estimates are determined through laser altimeter measurements. Those measurements might not be able to sample the real surface mass change at a given GPS site, due to limited spatial and temporal resolution.

3.4 Conclusions and discussion

In this study, we use a newly developed ice loss estimates of the sGrIS to investigate the GIA effect caused by the ice mass change in the past century to present-day GRACE
mass trend estimates and GPS surface motion estimates. A piecewise linear Post-LIA
deglaciation history is built using these ice loss estimates, and the GIA-induced surface mass
trend and surface displacements are computed.

In general, the major error sources of GIA modeling are associated with the
uncertainty in the mantle viscosity structure and the uncertainty in the ice loading history. In
this study, we have investigated the LIA results using various 1-D viscosity structures, and
we find that the LIA signals are generally small when we use VM2 viscosity structure. If we
decrease the viscosity in the upper mantle, the LIA signals could increase dramatically. These
results are consistent with those from previous studies for the same region (e.g. Simpson et
al., 2009; Simpson et al., 2011). For the real Earth, there could be 3-D variability in the mantle
viscosity structure. Though for Greenland, due to its cratonic nature, we suspect its lateral
distribution of viscosity is relatively uniform, and including thinner crust or lower viscosity in
the nearby oceanic region might not change the LIA-induced surface mass trend and the
vertical uplift results in southern Greenland significantly.

One thing we did not examine in this study is the uncertainty associated with the Post-
LIA ice loading history. The loading history we use is built from the surface elevation change
results for the last century, and its uncertainty can by determined by considering ice mass
change estimates for the same period based on different techniques. For instance, ice
elevation change can be derived from surface mass balance (SMB) calculations (see, e.g.
Hanna et al., 2011 and Wake et al., 2009), total SMB calculations including estimates of ice
discharge (Rignot et al., 2008; Yang, 2011), and other estimates that adopt a geometric
approach by measuring glacier length changes (Leclercq et al., 2011; Oerlemans et al.,
2007; Zuo and Oerlemans, 1997). An uncertainty estimate might be obtained in future, by
comparing LIA model results using ice loading history constructed based on these different
ice loss estimates.

By comparing our predicted LIA results with GRACE and GPS present-day observations, ICE-5G/VM2 GIA model results and the predicted elastic results, we conclude that (a) assuming VM2 viscosity structure, the GIA effect caused by the Post-LIA deglaciation would not affect the GRACE present-day mass trend estimates, or the GPS present-day surface motion estimates in a significant way. (b) If we reduce the upper mantle (or the asthenosphere) viscosity in VM2 by a factor of 10, the GRACE mass trend caused by the Post-LIA deglaciation is still not important compared to the observed GRACE mass trend, but the LIA signals are of similar magnitude as those from ICE-5G/VM2 model results. For GPS estimates, both the vertical uplifts and the horizontal displacements from the LIA results are large, and the magnitudes of the LIA uplift rates are comparable to the observed GPS uplift rates.
Chapter 4

The effects of laterally varying icy shell structure on the tidal response of Ganymede and Europa

4.1 Introduction

One of the long-sought objectives of an orbiter or fly-by mission to one of Jupiter’s icy moons, has been to use observations of tides on the moon to help determine the existence of a liquid ocean and characteristics of the overlying icy shell. The radio science component of such a mission could be used to estimate the degree-2 tidal potential Love number $k_2$. And if there is an on-board laser altimeter, it could be used to map out tidal displacements (see, e.g. Smith et al., 2001 and Neumann et al., 2001) and to determine the degree-two radial displacement Love number $h_2$. Knowledge of either of those Love numbers could provide information on the presence of an ocean beneath the icy outer shell (see, e.g., Yoder and Sjogren, 1996; Edwards et al., 1996; Moore and Schubert, 2000; Wu et al., 2001; and Wahr et al., 2006), and the two Love numbers could be combined to place constraints on the thickness of the icy shell (Wahr et al., 2006).

But, if a subsurface ocean exists, complications could conceivably arise if the icy outer shell has significant lateral variations in either its thickness or its shear modulus, or if the ocean is not global in extent so that the shell is grounded in places but floating in others. In those cases, the tidal deformation pattern would not be solely represented as the sum of degree 2 harmonics, and so neither the gravity nor the uplift could be completely described by the Love numbers $k_2$ and $h_2$.

In this study, simple but plausible assumptions are used to build various icy shell
models with laterally varying (3-D) structures. The 3-D variations in the models are chosen to be large, so that an upper-bound estimate of 3-D effects can be investigated. By solving a set of tidal loading problems using these laterally variable icy shell structures, we examine how those structures might complicate the interpretation of tidal measurements, and we discuss how to extract information regarding the interior structure of Ganymede and Europa from measurements of their tidal response.

Section 4.2 briefly discusses the numerical model we use to solve the tidal loading problem. In section 4.3, we introduce our test structures for Ganymede and Europa’s icy outer shells. Icy shell structures with laterally varying shell thickness are derived from a thermal conduction model. 3-D shear modulus profiles of the ice are obtained either from a conduction model, or, for Europa, by assuming a compositional difference between the leading and trailing hemispheres. We also take Ganymede as an example and build test structures for the case of a partially grounded icy shell. Using these shell structures, we compute the tidal response of the icy satellites. The results and their implications are discussed in section 4.4 and summarized in section 4.5.

4.2 Numerical model

To solve the tidal loading problem, we use finite-element model CitcomSVE that was designed to compute the viscoelastic response of a 3-D compressible Earth to surface loading (A et al., 2013). That same numerical model has been used to address the tidal response of the Earth’s moon with 3-D internal structure (Zhong et al., 2012). CitcomSVE was developed originally for modeling glacial isostatic adjustment process for an incompressible Earth with 3-D viscoelastic structures (Zhong et al., 2003; Paulson et al., 2005). The physical model and the numerical method for tidal deformation applications, and the benchmark results for
spherically symmetric models have been discussed in detail by A et al., (2013) and Zhong et al., (2012). For the finite-element grid, we divide each satellite’s mantle into 12 caps that have approximately equal size, and each cap is further divided into 48 cells in each of the horizontal direction and 80 cells in the radial direction. So the total element number is 12×48×48×80 (see e.g. Zhong et al., 2008).

The tidal forcing of the icy satellites comes from the gravitational attraction of Jupiter. To first order in the satellite’s eccentricity, and assuming the satellite’s inclination is 0, Jupiter’s gravitational potential can be written as (see e.g. Kaula, 1964; Wahr et al., 2009)

\[
V_T(r, \theta, \phi, t) = A \left( \frac{r}{R} \right)^2 \left[ 1 - 3 \cos^2 \theta \cos(nt) + \sin^2 \theta \left[ 3 \cos(nt) \cos(2\phi) + 4 \sin(nt) \sin(2\phi) \right] \right]
\]

where \( A = \frac{3GM\epsilon R^2}{4a^3} \). Here \( r, \theta, \phi \) are the radius, co-latitude, and eastward longitude; \( M \) is Jupiter’s mass; \( R, a, \) and \( n \) are the satellite’s radius, semi-major axis, and mean motion; and \( \epsilon \) is the orbital eccentricity. \( V_T \) is the sum of spherical harmonics of degree 2, and varies periodically with time at a period of 7.15 days for Ganymede and 3.55 days for Europa. In our numerical model we assume the icy satellite deforms elastically in response to the tidal potential. In Eq. (1), \( V_T \) consists of both (2, 0) forcing (the \( 1 - 3 \cos^2 \theta \) term) and (2, 2) forcing (the \( \sin^2 \theta \) term) (we use the notation \((l, m)\) to represent degree \( l \) and order \( m \) throughout this paper). For simplicity, we include forcing from only the (2, 0) term in our numerical calculations, with the assumption that our conclusions about the general importance of 3-D structure would be the same for (2, 2) forcing.

As input to the numerical model, we build a simplified layered satellite structure for both Ganymede and Europa (see Table 4.1). Both satellites contain an inviscid fluid core, a rocky mantle, an overlying liquid ocean, and an outer icy shell. The structure of the icy satellites’ core is not well constrained. For Ganymede, the observed intrinsic magnetic field
Table 4.1 The spherically symmetric satellite structure used as the default models for Ganymede and Europa. Both satellites contain a fluid core, a rocky mantle, and an ice/ocean layer. The density of the ocean is the same as for the ice.

<table>
<thead>
<tr>
<th></th>
<th>Ganymede</th>
<th>Europa</th>
</tr>
</thead>
<tbody>
<tr>
<td>radius of the satellite (km)</td>
<td>2638</td>
<td>1565</td>
</tr>
<tr>
<td>radius to the top of the rocky mantle (km)</td>
<td>1913</td>
<td>1265</td>
</tr>
<tr>
<td>radius to the top of the core (km)</td>
<td>630</td>
<td>700</td>
</tr>
<tr>
<td>density of the ice (kg/m³)</td>
<td>1040</td>
<td>1000</td>
</tr>
<tr>
<td>density of the rocky mantle (kg/m³)</td>
<td>3300</td>
<td>5150</td>
</tr>
<tr>
<td>density of the core (kg/m³)</td>
<td>5147</td>
<td>5150</td>
</tr>
<tr>
<td>bulk modulus in the ice (GPa)</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>bulk modulus in the rocky mantle (GPa)</td>
<td>110</td>
<td>120</td>
</tr>
<tr>
<td>shear modulus in the ice (GPa)</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>shear modulus in the rocky mantle (GPa)</td>
<td>10</td>
<td>120</td>
</tr>
</tbody>
</table>

suggests the existence of a liquid core (Schubert et al., 1996), while for Europa, the structure and composition of the core are still largely uncertain (see e.g. Schubert et al., 2009).

However, because the existence of a subsurface liquid ocean dramatically reduces the coupling between the surface and the layers below the ocean, the structure of the core has relatively little effect on the pattern or amplitude of the tidal response. In its present form, the 3-D numerical code can not include an inviscid fluid layer (e.g. a liquid ocean) unless it extends to the center of the satellite. To circumvent this problem, we assume that the rocky mantle, liquid ocean, and icy shell are all Maxwell viscoelastic solids, and we assign a low viscosity ($10^{11}$ Pa·s) to the ocean layer and a high viscosity ($10^{20}$ Pa·s) to the ice and rocky mantle. We denote the ocean layer’s characteristic relaxation time as $\tau_o$. $\tau_o$ is determined by the low viscosity in the ocean layer, and is orders of magnitude smaller than the relaxation time for the solid layers. We force the system with a tidal potential that is held constant in
time, and let the system relax viscously. After hundreds of $\tau_0$'s, the ocean layer reaches its fluid limit while the ice and rocky mantle still behave elastically.

In general, the presence of a liquid ocean tends to decouple the mantle deformation below the ocean from that above the ocean, and as a result the surface tidal response is largely insensitive to the depth of the ocean. Although it is likely that, at least for Ganymede, there is an icy layer between the ocean and the rocky mantle (Anderson et al., 1996), in our models (see Table 4.1) we assume the ocean extends downward to lie directly on top of the rocky mantle, which leads to a relatively thick ocean for both satellites. This model configuration does not change the tidal response solutions significantly, but it does help increase the convergence speed of the numerical calculation. As shown in Table 4.1, we use simplified density structures for both satellites. We assume that the density of the ocean is the same as for the ice. And for Europa, we use the same density for the rocky mantle as for the fluid core. These assumptions simplify the density structure and lead to relatively large surface gravity accelerations, but they do not impact our general conclusions regarding the impact of 3-D shell structures. In the following section, we build laterally varying structures for the outer icy shell, which are incorporated into the layered satellite models and are used to obtain the tidal response.

### 4.3 Outer icy shell models

We consider three possible sources of 3-D structure for the outer icy shell: lateral variations in temperature that lead to variations in shell thickness and shear modulus; hemispherically dependent compositional differences that cause 3-D variations in shear modulus; and a shell that is partially grounded and partially floating. Our models for these types of structures are described in the following three subsections.
4.3.1 Icy shell structure from a conduction model

The thickness and elastic properties of an icy shell are directly related to its thermal state. The ice thickness could be significantly non-uniform if the temperature distribution of the ice is determined by thermal conduction. In contrast, if the ice is convecting (Pappalardo et al., 1998), or is undergoing significant lateral flow, there should be little lateral variability in shell thickness (see e.g. Nimmo and Manga, 2009). The lateral variability of elastic properties (e.g. the shear modulus) of the ice also depends on how much lateral variability is present in the temperature field. In this study, one of our primary goals is to determine the maximum, plausible effects of laterally varying shell structure on tidal observables. To achieve this end, we assume that thermal conduction is the dominant heat transfer mechanism, and we choose parameters that lead to large lateral variations in the temperature field of the icy shell. Using this temperature field, we build hypothetical shell models where the lateral variations in shell thickness and/or elastic properties are plausible, but large.

Theory and method

In a conduction model, there are three factors that can influence the lateral variations in temperature: the surface temperature, the tidal dissipation within the icy shell, and the heat flux at the bottom of the shell. The surface temperature of the satellite is determined by the distribution of solar insulation. Its value depends on latitude, and is relatively uniform near the equator (Ojakangas and Stevenson, 1989, Nimmo et al., 2007). The spatial pattern of the tidal dissipation is directly related to the degree-2 tidal potential, and causes the temperature field to vary with both latitude and longitude. Although the basal heat flux can vary with position as well, that variation is not well constrained. So, we assume the basal heat flux is uniform. In general, large basal heating tends to reduce the mean thickness of the icy shell and to eliminate lateral variations induced by internal heating (i.e. the tidal dissipation). In
In this study we choose a value of the basal heat flux that causes the shell thickness to be reasonably small and the lateral variations from the tidal dissipation to be large, which gives more relative lateral variability to the shell structure, especially at low latitudes.

The surface temperature variations are computed based on the theory given in Appendix A of Ojakangas and Stevenson, (1989). Tidal dissipation rates are computed using the method given in Appendix B of Ojakangas and Stevenson (1989), and in Nimmo et al. (2007). Following the finite-difference method described in Nimmo et al. (2007), and using the parameters and methods described below, we obtain a laterally varying temperature field and the 3-D variations in icy shell thickness and shear modulus caused by that temperature field.

**Parameters and the resulting models**

For Europa, we follow Ojakangas and Stevenson (1989) and set the obliquity to 1°, the surface albedo to 0.5, and the orbital eccentricity to 0.0094. For Ganymede, we set the obliquity to 0.3°, the surface albedo to 0.43, and the eccentricity to 0.0013. For the basal heat flux, a wide range of values have been used in different studies (see, e.g. Ojakangas and Stevenson, 1989; O’Brien et al., 2002; and Nimmo et al., 2007). We choose values that lead to significant lateral variations in the temperature field and shell thickness structure, while making sure the mean shell thickness is not large. For Ganymede, the tidal dissipation is relatively weak due to a small orbital eccentricity (0.0013), and its effects on the spatial pattern of the temperature can be easily overwhelmed by a large basal heat flux. So, to maximize the lateral variability from the tidal dissipation we prefer small basal heating. However, if we choose a basal heat flux that is too small (e.g. 0.5 mW/m²), we obtain an icy shell that is more than 300 km thick, which would presumably trigger convection in the ice (Barr and Showman, 2009). To avoid convection and to obtain an upper-bound estimate for
the effects of lateral variability, we use 1 mW/m$^2$ for the basal heat flux and increase the tidal dissipation rate by a factor of 25. By doing so, we have a test structure where the lateral variations in the icy shell are significantly amplified and the mean shell thickness (109 km) is reasonably small.

Lateral variations in ice thickness are determined by assuming the ocean-ice interface occurs along the 250° K isotherm. We show the shell thickness in Fig 4.1(a), and we refer to this model as “GA”. The only 3-D variations in model GA are in its icy shell thickness. Although a temperature-dependent shear modulus profile can also be derived based on the same conduction model, we do not include it in model GA. Instead, we will discuss the shear modulus models later in this section. For Europa, the tidal dissipation rate is relatively large (because of Europa’s large 0.0094 eccentricity), so a larger basal heat flux (4.0 mW/m$^2$) is chosen. The shell thickness is shown in Fig 4.1(b). We refer to this 3-D thickness model as “EA”.

As shown in Fig 4.1, at low and middle latitudes the lateral variability in each satellite’s shell thickness is dominated by the tidal dissipation pattern. The tidal dissipation rate is proportional to the square of the tidal strain rate. Since the tidal strain rate is a linear combination of the (2, 0) and (2, 2) harmonics, the non-spherical components of the shell thickness profile mainly contain terms at (2, 0), (2, 2), (4, 2), (4, 4). In a small region near the poles, the ice thickness increases with latitude. This polar thickness variability is determined by the surface temperature pattern, and is small compared to the lateral variability induced by tidal dissipation. Results of the tidal response for models GA and EA will be discussed in section 4.4.1.

Using the temperature field derived from the conduction models, we also build temperature-dependent shear modulus profiles for the icy shells of each moon. We assume a
Figure 4.1 The icy shell thickness models: (a) model GA for Ganymede, and (b) model EA for Europa. The color bar shows the ice thickness in km.
linear relation between temperature and shear modulus (Gammon et al., 1983, Gagnon et al., 1988, Gagnon et al., 1990), given by

\[ \mu = 34.1 + 0.07 \times (273 - T), \]

where \( T \) is the temperature in Kelvin and \( \mu \) is the shear modulus in GPa. Applying Eq. (2) to the temperature field determined by the conduction model, we build shear modulus profiles for Ganymede and Europa. To investigate the effects of the laterally varying shear modulus alone, we assume constant shell thicknesses of 141 km for Ganymede and 27 km for Europa, with each value corresponding to the maximum shell thickness in models GA and EA, respectively. We refer to the resulting models as “GB” for Ganymede and “EB” for Europa, both with laterally varying shear modulus but constant shell thickness. The results from model GB and EB are presented in section 4.4.2.1.

4.3.2 Degree-1 variability in the shear modulus of Europa’s outer shell

The surface of Europa features a hemispherical dichotomy, where the bright leading side contains relatively pure ice while the reddish trailing side is mixed with ice and hydrated materials. The non-ice material in Europa’s outer shell could have been emplaced from the subsurface ocean, and/or created by exogenous radiation processing (see, e.g., Carlson et al., 2009). In the case of exogenic sources, impact gardening and micrometeorite deposition occur mainly on the leading hemisphere (Zahnle et al., 1998, Zahnle et al., 2003), and the thermal plasma from Io deposits material mostly on the trailing hemisphere (Alvarellos et al., 2008, Zahnle et al., 2008). A recent spectroscopic study of Europa’s surface composition suggests that MgSO\(_4\) is present on Europa’s trailing side, but is absent from its leading side (Brown and Hand, 2013).

A hemispherical difference in composition would presumably cause lateral variations
in the elastic properties of the icy shell. To investigate whether information about this lateral variability could be gained through tidal measurements, and whether the presence of this variability could complicate attempts to learn about the icy shell thickness using tidal observables, we construct a 3-D model where a degree-1 perturbation is added to the shear modulus of Europa’s icy shell. To maximize the possible effects of the lateral variability, we assume the perturbation is present throughout the entire depth of the shell. For simplicity, we assume the other parameters, including the density, Lame parameters, and shell thickness, are spherically symmetric.

The degree-1 structure can be written as:

\[ d\mu / \mu = -\delta \sin \theta \sin \varphi, \]

where \( \mu \) is the shear modulus in the icy shell; \( \delta \) is a positive constant that quantifies the amplitude of the perturbation; and \( \theta \) and \( \varphi \) are the co-latitude and longitude (\( \varphi = 0 \) is the sub-Jovian meridian). Eq. (3) represents a \((1, 1)\) variation in \( \mu \), where the trailing hemisphere has a larger \( \mu \) than the leading hemisphere. In this study, we build degree-1 shear modulus profiles using \( \delta \) = 0.5\%, 1\%, 2\%, and 4\%, and we refer to all these models as “EC”. The minimum value of \( \delta \) (0.5\%) is chosen so that our numerical method is still able to resolve the differences caused by the small variation in shear modulus. In fact, if the hemispherical difference in composition is mainly caused by exogenic processes, the low bombardment rate \((\text{Carlson et al., 2009})\), coupled the fact that adding MgSO\(_4\) to the ice is not likely to dramatically change the shell’s rheological properties \((\text{Brown and Hand, 2013})\), means that even a value as small as \( \delta = 0.5\% \) might be an overestimate of the lateral variability in the shear modulus. As we will demonstrate below, results corresponding to smaller, and probably more realistic values of \( \delta \) can be obtained by linearly scaling the results from model “EC”. Details regarding this are discussed in section 4.4.2.2. For the maximum value, we choose \( \delta = \)
4% mainly to verify the linear relation between the (1, 1) perturbation and the degree-3 tidal solutions (see section 4.4.2.2).

4.3.3 Grounded ice on Ganymede

Suppose that the liquid ocean on Europa or Ganymede is non-global in extent, so that the icy outer shell is grounded in places and floating in others. What impact would that have on the tidal response? To address that question, we take Ganymede as an example. We continue to assume that the rocky mantle is a spherical shell, with its surface at a depth of 2638 - 1913 = 725 km (see Table 4.1) beneath Ganymede’s outer surface. As a result, a location where the outer shell is grounded means that the ice there extends down all the way to that depth. We offer no physical justification for such a model. Our intent is only to determine whether a partially grounded shell could be detected in tidal observations, and how it might complicate attempts to determine the thickness of the floating shell.

We test two hypothetical icy shell structures with grounded ice. For the first model, referred to as GC1, we assume that the icy shell is grounded at longitudes between 0° and 30°. For the second model, referred to as GC2, the icy shell is grounded at longitudes between 0° and 60°. For both models the thickness of the floating shell is kept uniform (109 km). This grounded ice geometry has been chosen to be symmetric about the equator. This makes the tidal results relatively simple and instructive, and we suspect the general conclusions we infer from the results would be applicable for other geometries as well. Those results are discussed in section 4.4.3.

4.4 Results

In this section, geoid and uplift results for 3-D models are presented in the spatial
domain in Figs 4.2-4.7, and in terms of spherical harmonic coefficients in Tables 4.2-4.4. Fig 4.2 shows uplift results over the entire surface, whereas the other figures (Figs 4.3-4.7) show results only along the equator. For all models we use a forcing tidal potential

\[ V_T(\theta, \phi) = V_T^0 Y_{2}^0(\theta, \phi), \]

where \( Y_{2}^0(\theta, \phi) \) is the (2,0) spherical harmonic and the amplitude, \( V_T^0 \), is a constant.

The results in the figures and tables are normalized as follows. First, we note that for a spherically symmetric satellite the tidal perturbation in the geoid, \( N(\theta, \phi) \), and the tidal uplift of the surface, \( U(\theta, \phi) \), are given by

\[ N(\theta, \phi) = k_2 V_T^0 Y_{2}^0(\theta, \phi)/g \quad \text{and} \quad U(\theta, \phi) = h_2 V_T^0 Y_{2}^0(\theta, \phi)/g \tag{4} \]

where \( k_2 \) and \( h_2 \) are the degree-2 Love numbers, and \( g \) is the surface gravitational acceleration. Throughout this section, we use the term “geoid” to refer to the gravitational contributions caused by tidal deformation in the satellite’s interior; and “total geoid” to refer to the “geoid” plus contributions from the direct tidal potential from Jupiter. The total geoid, for example, would be \( N_{tot}(\theta, \phi) = (1 + k_2) V_T^0 Y_{2}^0(\theta, \phi)/g \) for a spherical satellite.

Motivated by Eq. (4), we normalize the 3-D geoid and uplift solutions along the equator (Figs 4.3-4.7), by dividing the geoid and uplift at every location by \( V_T^0 Y_{2}^0(\theta, \phi)/g \). The harmonic \( Y_{2}^0(\theta, \phi) \) is independent of longitude, \( \phi \), and so is constant along the equator. With this normalization, the results for \( N(\theta, \phi) \) and \( U(\theta, \phi) \) for a spherical model would equal \( k_2 \) and \( h_2 \), respectively, at all \( (\theta, \phi) \). For a 3-D model, the normalized results can thus be compared directly with numerical values of \( k_2 \) and \( h_2 \) (0.47 and 1.38 for Ganymede, and 0.23 and 1.1 for Europa, using the spherical structural models described in Table 4.1), to gain a sense of the relative importance of 3-D effects.

For Fig 4.2, which shows the uplift over the entire surface, dividing by \( Y_{2}^0(\theta, \phi) \) would cause problems because \( Y_{2}^0 \) vanishes at certain latitudes. Instead, for Fig 4.2 we divide
the uplift by $V_t^0/g$. We also subtract the dominant $h_2 Y_2^0(\theta, \phi)$ term to obtain Fig 4.2, so that the 3-D effects can be seen more clearly.

For the harmonic coefficients shown in Tables 4.2-4.4, we divide each coefficient by $V_t^0/g$: i.e. the amplitude of the $Y_2^0(\theta, \phi)$ coefficient of the tidal potential (divided by $g$). The spherical harmonic functions are normalized as described in Zhong et al., (2008). For a spherically symmetric model, the $(2, 0)$ components of $N$ and $U$ would thus equal the Love numbers $k_2$ and $h_2$, respectively; while the other $(l, m)$ coefficients would vanish, in principle – although, in practice, numerical noise tends to cause small but non-zero leakage into other harmonics. To construct Tables 4.2-4.4 we subtract the coefficient values obtained for the spherical models described in Table 4.1. This not only removes the spherical Love numbers, $k_2$ and $h_2$, from the $(2,0)$ coefficients, but it also removes the effects of numerical leakage into the other coefficients. The coefficients shown in the Tables, including the $(2, 0)$ terms, thus represent the effects of 3-D structure.

**4.4.1 Results for laterally varying icy shell thickness**

Using the laterally varying shell thickness models GA and EA, and keeping the other satellite properties equal to the spherically symmetric values shown in Table 4.1, we force the satellites with a $(2, 0)$ tidal potential and compute the tidal response. To offset numerical errors related to leakage from the large $(2, 0)$ response into other harmonics, and to highlight the effects of laterally varying shell thickness, we also compute the tidal response using fully spherical satellite models with shell thicknesses of 109 km for Ganymede and 19.6 km for Europa (the mean shell thicknesses of models GA and EA, respectively), and subtract the results for these spherical models from the GA and EA results. The geoid and uplift results in the spherical harmonic domain are shown in Tables 4.2 and 4.3 for Ganymede and Europa,
Table 4.2 The ten largest spherical harmonic coefficients of the geoid and surface uplift for Ganymede (forced by a (2, 0) tidal potential). The spherical harmonic functions are defined as in Zhong et al., 2008. The coefficients are the differences between the results of model GA (a model of non-uniform ice thickness) and the results using a spherical model with an icy shell thickness equal to the mean thickness of model GA (109 km). The values of the geoid and uplift are normalized so that the (2, 0) components correspond to perturbations in the degree-2 Love numbers (see text). Before removing the spherical model result, the (2, 0) components are 0.47 and 1.38 for the geoid and uplift, respectively.

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Table 4.3 The ten largest spherical harmonic coefficients of the geoid and the surface uplift for Europa (forced by a (2, 0) tidal potential). The coefficients are the differences between the results using model EA (a model of non-uniform ice thickness) and the results using a spherical model with the icy shell thickness equal to the mean thickness of model EA (19.6 km). The results are normalized as described in the text. Before subtracting the spherical model result, the (2, 0) components are 0.23 and 1.1 for the geoid and uplift, respectively.

<table>
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respectively. Before subtracting the spherical model result, the dominant response is at (2, 0) for both satellites, and is about 3 orders of magnitude (or more) larger than the response at other degrees. After removing the spherical model result, the largest coefficients occur at even-numbered degrees and orders. These results are due to first-order, spherical harmonic coupling between the satellite structure and the (2, 0) tidal potential. As mentioned in section 4.3.1, the non-spherical components of the icy shell thickness are largely determined by the tidal dissipation rates, which mainly contain terms at (2, 0), (2, 2), (4, 2) and (4, 4). So when this structure is coupled with a (2, 0) tidal potential, the selection rules for products of spherical harmonics (Edmonds, 1957) require that the first-order response can occur only at even-numbered degrees and orders.

In reality, the exact tidal potential from Jupiter also contains a large (2, 2) term, in addition to the (2, 0) forcing considered here (see Eq. (1)). If we were to couple (2, 2) forcing with the non-spherical components of the shell thickness profile, we would expect to obtain a tidal response at even-numbered degrees and orders in that case, too.

Using the results shown in Tables 4.2 and 4.3, we investigate whether the mean shell thickness could be accurately estimated using observations of degree-2 Love numbers, in the presence of lateral variations in shell thickness. As described in Wahr et al., (2006), for a spherically symmetric satellite the combination of Love numbers,

$$\Delta_2 = 1 + k_2 - h_2,$$

(5)

is proportional to the shell thickness (in the case of a thin shell), where the proportionality constant depends on the density distribution of the satellite and the shear modulus of the ice. If we know this constant, then an estimate of $\Delta_2$ based on observations of $k_2$ and $h_2$ from radio tracking and altimetric observations, respectively, could be used to determine the shell thickness. When the shell thickness is laterally varying, one question to ask is whether an
observational determination of $\Delta_2$ would still provide a good estimate of the mean shell thickness.

To address this question, suppose the icy shell has laterally varying thickness, but that $\Delta_2$ is computed using observations of the $(2, 0)$ spherical harmonic coefficients of the geoid and uplift, just as in the spherical case. For Ganymede (model GA), the value obtained for $\Delta_2^{GA}$ would be 0.0911; whereas the value for a spherical satellite with the same mean shell thickness is $\Delta_2^{mean} = 0.0869$. The relative difference between these two numbers is 4.8%, implying there would be a 4.8% error in the estimated thickness if $\Delta_2$ was used to determine that thickness under the assumption of a spherically symmetric shell. For Europa, $\Delta_2^{EA} = 0.1270$, and $\Delta_2^{mean} = 0.1260$, where “EA” represents the result from model EA, and “mean” represents the result from the corresponding spherical model. The relative difference between these two results is 0.8%, implying there would be only a 0.8% error if $\Delta_2$ was used to determine the mean shell thickness. The error estimate for Europa is smaller than for Ganymede because model EA has smaller lateral variability than model GA (the magnitudes of the coefficients in Table 4.3 are smaller than those in Table 4.2). For both Ganymede and Europa, the relative error is much smaller than the observational errors expected for future missions (see, e.g., Wahr et al., 2006), implying that the presence of conduction driven 3-D variability in shell thickness is not likely to limit attempts to use $\Delta_2$ to constrain the mean icy shell thickness.

In Fig 4.2, we plot the surface uplift of Ganymede in the spatial domain, after removing the spherical model results. The uplift is normalized by the amplitude of the $(2, 0)$ tidal potential, but not by its $(\theta, \phi)$ dependence. Fig 4.2 thus shows the pattern of the non-spherical response that could be observed by an altimeter. Clearly there are complicated oscillation patterns, especially along the equator, that have short wavelengths and are not
Figure 4.2 The surface uplift of Ganymede in response to a $(2, 0)$ tidal potential. Shown is the difference between the results using model GA (a model of non-uniform ice thickness) and the results using the mean icy shell thickness of model GA (109 km). The uplift has been normalized by the amplitude of the $(2, 0)$ tidal potential (see text).
present in the ice thickness itself (Fig 4.1(a)).

To investigate in detail the relation between the laterally varying shell thickness and the spatial pattern of the tidal response to (2, 0) forcing, we plot the ice thickness, the geoid, the surface uplift, and the difference between the total geoid and the uplift, all along the equator, in Figs 4.3 and 4.4 for Ganymede and Europa, respectively. Panels b-d in each figure are normalized as described above in the first paragraphs of section 4.4. To show the relative amplitude of the spatial variability we do not remove the results from the spherically symmetric model. Because the spherically-symmetric model results consist solely of a (2, 0) component, those results are constant along the equator, and so their contributions are to add a longitude-independent constant to each figure. The shell thickness structures of the two satellites share a similar shape (compare Fig 4.3(a) with Fig 4.4(a)), and the geoid results (Fig 4.3(b) and Fig 4.4(b)) clearly show order-2 and order-4 patterns (i.e. $\cos^2 \phi$ and $\cos^4 \phi$ longitude-dependence, respectively) that are consistent with the ice thickness structure. In contrast, the uplift shows more short-wavelength features and much larger spatial variability. As a result, the difference between the total geoid and the uplift (that difference would be equal to $\Delta_2 = 1 + k_2 - h_2$ for a spherical satellite) mostly reflects the negative of the uplift. We define this difference as $\Delta(\theta, \phi)$. In contrast to the spherically symmetric case, where $\Delta$ is linearly proportional to the thickness of the icy shell, results from our laterally varying shell thickness model do not show a linear relation between $\Delta(\theta, \phi)$ and the ice thickness that is valid for every $(\theta, \phi)$ (compare Figs 4.3(d) and 4.4(d) with 4.3(a) and 4.4(a)). For example, both $\Delta$ and the surface uplift show significant short-wavelength oscillations that are not present in the ice thickness.

Summarizing the results for laterally varying shell thickness, results in the spherical harmonic domain show that the non-degree 2 harmonics in the tidal solution are determined
Figure 4.3 The (a) ice thickness, (b) geoid, (c) surface uplift, and (d) difference between total geoid and uplift, for Ganymede model GA (a model of non-uniform ice thickness). Results are from (2, 0) tidal forcing, and are plotted along Ganymede’s equator. The normalization is described in the text.
Figure 4.4 Same as Figure 4.3, but for Europa model EA (a model of non-uniform ice thickness).
by first-order coupling between the tidal potential and lateral variations of the icy shell thickness. Gravity observations of the geoid or altimetric observations of the uplift could be used to detect the presence of 3-D shell structure, if those observations were sufficiently accurate. Inverting for details of that structure, though, would likely be difficult. For example, if laterally varying shell thickness exists, we find that the spatial pattern of the tidal deformation exhibits oscillations across the entire surface. A smaller value of uplift does not necessarily correspond to a locally thicker shell, which would complicate efforts to use detailed altimeter measurements of the spatially dependent uplift to infer the pattern of 3-D icy shell thickness. On a positive note, if radio tracking and altimetric observations could be used to resolve the degree-2 gravity and uplift coefficients, the mean shell thickness could be estimated from $\Delta_2$ using an approach developed for a spherical satellite, even in the presence of variable thickness.

4.4.2 Results for a laterally varying shear modulus in the icy shell

4.4.2.1 Temperature-dependent shear modulus derived from the conduction model

We compute the response to a unit amplitude (2, 0) tidal potential using models GB for Ganymede and EB for Europa. Similar to the results for models GA and EA, only even degrees and orders appear in the solutions, and the (2, 0), (2, 2), (4, 2) and (4, 4) terms dominate. This is not surprising since the laterally varying shear modulus profiles are derived from the same temperature fields used to generate the shell thickness models.

The difference between the total geoid and uplift (equal to $\Delta_2 = 1 + k_2 - h_2$ for a spherically symmetric satellite) along the equator is plotted in Figs 4.5(a) and 4.5(b) for models GB and EB, respectively. Similar to the GA and EA results, order-2 and order-4 patterns are evident, but their amplitudes are much smaller than those for models GA and EA.
Figure 4.5 The difference between the total geoid and the uplift along the equator, of (a) Ganymede using model GB, and (b) Europa using model EB. Both models assume 3-D variations in the icy shell’s shear modulus, caused by conduction in the shell. Results are for (2, 0) tidal forcing. The normalization is described in the text.
(compare with Figs 4.3(d) and 4.4(d)). This suggests that a laterally varying shear modulus is likely to have a smaller impact on the tidal response than laterally varying icy shell thickness. We suspect that this general conclusion is true if the shear modulus profile and the shell thickness structure are determined using any conduction model. However, caution needs to be exercised if there is viscous flow in the icy shell, since that tends to reduce the lateral variability in shell thickness, which would cause the effects of laterally varying shear modulus to become relatively more important.

4.4.2.2 Compositional-dependent degree-1 variability in the shear modulus of Europa’s icy shell

Using the degree-1 shear modulus structure from model EC, we compute Europa’s response to (2, 0) tidal forcing for different 3-D amplitudes: i.e. for $\delta = 0.5\%, 1\%, 2\%$, and $4\%$. Again, we remove the spherically symmetric solution from the results. Table 4.4 shows the 10 largest spherical harmonic coefficients when $\delta = 4\%$, for both the geoid and the uplift. The largest non-degree 2 response is at (3, 1). This can be understood in terms of first-order coupling between the (2, 0) forcing and (1, 1) structure. To first order, the response is at (3, 1) and (1, 1), and their magnitudes should be proportional to the amplitude of the (1, 1) perturbation in shear modulus (i.e. to $\delta$). Since our geoid results are computed in the center of mass frame, the degree-1 potential vanishes. For the uplift, both the (1, 1) and (3, 1) terms are among the largest, with the (1, 1) component being about 4 times smaller than the (3, 1) component. In Fig 4.6(a), the magnitude of the (3, 1) term in the geoid is plotted versus the amplitude of the perturbation. A linear relation is evident, which is additional evidence that the 3-D compositional effects are first-order, at least for this range of 3-D amplitudes. There is a similar linear dependence for the uplift (not shown here).
Table 4.4 The ten largest spherical harmonic coefficients of the geoid and uplift for Europa (forced by a (2, 0) tidal potential). The coefficients are the differences between the results for one of the EC models, where compositional-driven (1, 1) structure of amplitude $\delta = 4\%$ is included in the shear modulus of the icy shell, and the results for a uniform icy shell. Before removing the spherical background, the (2, 0) components are 0.23 and 1.1 for the geoid and uplift, respectively.

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Note that the (2, 0) terms in Table 4.4 are considerably smaller than the corresponding (2, 0) terms computed for the laterally varying shell thickness models (see Table 4.2). This means that the effects of compositionally driven shear modulus variability on measurements of $\Delta_2 = 1 + k_2 - h_2$, would be smaller than those described for model EA above. And even the EA effects are too small to have a significant impact on estimates of mean shell thickness obtained from observations of $\Delta_2$.

Using these results, we investigate whether (1, 1) structure in the shear modulus could be detected using observations of the degree-3 tidal response. One complication is that the exact (3, 1) tidal response should contain two components: (a) the (3, 1) response caused by first-order coupling between the (2, 0) tidal potential and the (1, 1) perturbation in shear modulus (i.e. the (3, 1) response computed in this section); and (b) the direct response to the
Figure 4.6 The magnitude of the (3, 1) harmonic coefficient of (a) the geoid and (b) the difference between geoid and uplift, versus the magnitude of the degree-1 perturbation ($\delta$) in shear modulus, for model EC (hemispherical differences in the icy shell’s shear modulus, caused by variations in composition). Results are for (2, 0) tidal forcing. The circles correspond to the results for $\delta = 0.5\%$, 1\%, 2\%, and 4\%. The best fitting line (also included in the figure) passes through the circles, indicating that there is a near-linear dependence of the (3, 1) tidal response on the magnitude of the perturbation. The normalization is described in the text.
(3, 1) tidal potential. To expand on (b): the exact tidal potential from Jupiter includes spherical harmonics of all degrees, including a (3, 1) term. To lowest order, Europa’s response to this or any other tidal forcing term (including the (2, 0) term considered in this paper) can be estimated by assuming Europa is spherically symmetric.

For (a), the (3, 1) term in the geoid, shown in Fig. 6(a), has a magnitude of \( \sim 4.3 \times 10^{-5} \) when \( \delta = 4\% \), and of \( 5.3 \times 10^{-6} \) if \( \delta = 0.5\% \). For the contribution from (b), the amplitude of a degree-\( l \) component of the tidal potential is smaller than the degree-2 amplitude by a factor of \( \left[ \text{Europa’s radius/Europa-Jupiter distance} \right]^{-2} \) (e.g., Agnew, 2008, equation (4)), implying that the degree-3 forcing terms are smaller than the degree-2 terms by a factor of \( \sim 430 \). Using our spherically symmetric model, we find that the degree-3 potential Love number is 0.17. This suggests that the contribution from (b) is \( \sim 3.8 \times 10^{-4} \). So even if the degree-1 perturbation in the shear modulus is as large as 4\%, the effect of lateral variability on the geoid is nearly one order of magnitude smaller than the spherical response to the degree-3 tidal forcing. The conclusions are similar for the (3, 1) term in the uplift, implying that it might be challenging to use (3, 1) geoid or uplift alone to detect a possible (1, 1) structure in the shear modulus. However, if we choose the difference between the (3, 1) geoid and the (3, 1) uplift as our observable, we find that for (a), the difference between geoid and uplift, shown in Fig 4.6(b), is \( \sim 2.5 \times 10^{-4} \) if \( \delta = 4\% \), and is about \( 3.2 \times 10^{-5} \) if \( \delta = 0.5\% \). For (b), we find that the difference between the degree-3 potential Love number and the uplift Love number (i.e. the difference between the geoid and the uplift) to be \( \sim 0.025 \), implying that the contribution from (b) is \( \sim 5.8 \times 10^{-5} \). This value is larger than the contribution from (a) by a factor of \( \sim 1.8 \), if \( \delta = 0.5\% \); and is \( \sim 23\% \) of the contribution from (a), if \( \delta = 4\% \). So, when we choose the difference between the (3, 1) geoid and the (3, 1) uplift as our observable, the effect of lateral variability is amplified, and is now comparable to the response to the degree-3 tidal forcing for the range
of 3-D amplitudes we have investigated.

For a future Europa mission, if an estimate of the mean shell thickness, e.g. through a determination of $\Delta_3$, is available, the response to the degree-3 tidal forcing could be computed using a spherically symmetric Europa model with that estimated mean shell thickness. By removing this contribution from the total degree-3 geoid and uplift measurements, and by estimating the difference between the degree-3 geoid and uplift, the effect of the degree-1 perturbation in the icy shell structure could then conceivably be extracted, leading to an observational constraint on that structure. As argued in section 4.3.2, the amplitude of the degree-1 structure in the shear modulus is probably small, and even $\delta = 0.5\%$ might be an overestimate of the amplitude of the lateral variability. This implies that the detection of the corresponding 3-D effects would likely be challenging. Its success would require accurate modeling of the direct response to degree-3 tidal forcing and a precise measurement of the degree-3 tidal terms.

### 4.4.3 Results using grounded icy shell models

Using the grounded icy shell models GC1 and GC2, we compute Ganymede’s response to (2, 0) tidal forcing. Equatorial values of the geoid, uplift, and difference between the total geoid and uplift, are shown in Fig 4.7 for both models. Where the shell is grounded the geoid and uplift are consistently smaller than where it is floating. For the uplift there is a sharp delineation between grounded and floating regions, with floating values being more than three times larger than grounded values. The geoid shows more large-scale variability: there is no sharp delineation between grounded and floating regions, and the relative variation is smaller (less than a factor of 2). This is because surface gravity is determined by the deformation of the entire satellite. The mass related to the large uplift in the floating region,
Figure 4.7 (a) Geoid, (b) surface uplift, and (c) difference between the total geoid and uplift, for Ganymede models with a non-global liquid ocean. The results are shown as solid and dashed curves for models GC1 (the icy shell is grounded for longitudes between 0° and 30°), and GC2 (the icy shell is grounded for longitudes between 0° and 60°), respectively. The vertical lines show the grounding line positions. The results are for (2, 0) tidal forcing, and are plotted along Ganymede’s equator. The normalization is described in the text.

for instance, can have an impact on the gravity signal above the grounded ice.

The large-scale geoid variability also shows up to some extent in the uplift, because of the tendency of the liquid ocean to lie along a surface of constant potential (the geoid). So, when the uplift is removed from the total geoid to obtain $\Delta(\theta, \phi)$, the results (Fig 4.7(c)) show an even more pronounced floating/grounded difference than the uplift.

The spatial variability obtained here for models GC1 and GC2 is significantly larger
than that for the other 3-D models, in which the subsurface ocean is global. The variability is
dramatic enough to suggest that observations of spatial variations in the tidal observables,
especially in the surface uplift and in the difference between the total geoid and uplift, could
be used to identify regions where the icy shell is grounded.

Suppose that the icy shell is, indeed, partially grounded, and altimeter tidal
observations were able to roughly outline the floating region. Could combined altimeter/radio
tracking observations of $\Delta(\theta, \varphi)$ over just the floating region, be able to determine the floating
shell’s thickness, as has been proposed for a spherical satellite (see section 4.4.1)? Averaging
our GC1 results for $\Delta(\theta, \varphi)$ over the floating region, we find $\Delta^{\text{mean}} = 0.1031$, and $\Delta^{\text{min}} =
0.0903$, where “mean” and “min” denote the mean and minimal values, respectively, for only
the floating region. When we compute $\Delta^{\text{mean}}$, we only consider longitudes larger than 70° and
less than 350° to avoid the transition region between the high and low values. As a
comparison, we also calculate the tidal response of a fully spherical satellite model with a
shell thickness of 109 km (which equals the floating shell’s thickness), and find $\Delta = 0.0869$.
Thus, if $\Delta^{\text{mean}}$ were used to estimate the floating shell’s thickness, the thickness would be
overestimated by ~ 19%. If $\Delta^{\text{min}}$ were used instead, a 4% overestimation would be obtained.
For model GC2, $\Delta^{\text{mean}} = 0.1117$, and $\Delta^{\text{min}} = 0.0941$, which would lead to overestimates of the
thickness by 29% and 8%, respectively. In summary, the existence of grounded ice tends to
increase the difference between the total geoid and uplift over the floating region. If we use
this difference to estimate the thickness of the floating icy shell, we are likely to obtain an
estimate that is too large. An estimate that is closer to the true value may be obtained by
restricting the observations to regions farther away from the grounded ice. But, even then, the
inferred ice thickness would probably be somewhat overestimated.
4.5 Conclusions and discussion

In this study we use a finite-element model to solve for the tidal response of Ganymede and Europa using various icy shell models with laterally variable structures. In all cases, the shell is assumed to lie above a liquid water ocean. Icy shell structures with significant laterally varying shell thickness are derived from a thermal conduction model. 3-D structures of the shear modulus within the shell are built either from a conduction model or, for Europa, by postulating a hemispherical difference in composition. For each model we choose amplitudes of the lateral variability that are plausible, but large. We also build icy shell structures for Ganymede where we assume a non-global ocean, so that the shell is partially grounded in places and floating in others. Results based on these test models provide upper-bound estimates for the effects of 3-D shell structure on the tidal response, and help to determine the likely limitations of future efforts to infer shell structures from tidal measurements.

Relevant to those measurements, the most pertinent general questions are: (1) would the presence of any of these structures complicate efforts to determine the existence or absence of an underlying liquid ocean? (2) Would the effects of those structures on the degree-2 Love numbers \( k_2 \) and \( h_2 \) be small enough that the mean shell thickness could still be accurately estimated from measurements of \( \Delta_2(=1+k_2-h_2) \), as has been argued in the past for a spherically symmetric icy satellite? (3) Could tidal measurements be used to learn anything about those structures? Here, we summarize the answers to those questions for each type of 3-D structure considered in this paper.

Results from the 3-D shell thickness models show that the non-degree 2 harmonics in the geoid are determined by first-order coupling between the tidal potential and lateral variations in the thickness. The effects of variable thickness on the degree-2 Love numbers
are considerably smaller than the effects of a liquid ocean. For example, *Moore and Schubert* (2000) used a spherically symmetric model to conclude that for Europa $k_2$ and $h_2$ would equal $\sim 0.25$ and $\sim 1.20$ if there is a liquid ocean (the value of $h_2$ is slightly larger than our spherically symmetric result due to difference in model parameters), and $\sim 0.015$ and $\sim 0.026$ if there is not. Those differences are far larger than the effects of lateral thickness variations on the $(2, 0)$ geoid and uplift components shown in Table 4.3 for model EA. Because the parameter values used to construct model EA were chosen to maximize those thickness variations, we conclude that the presence of lateral variability in thickness would not obscure the effects of a liquid ocean in radio tracking observations of tidal gravity perturbations, or in altimeter observations of tidal uplift. The conclusion is the same for Ganymede. In fact, the $(2, 0)$ parameters for models GA and EA are small enough that the mean shell thickness could be meaningfully estimated from observations of $\Delta_2 = 1 + k_2 - h_2$, as described above. It is possible that, given sufficient measurement accuracy, radio tracking observations of non-degree 2 tidal harmonics, or altimeter observations of non-degree 2 tidal surface displacements, could be used to infer that lateral variability in shell thickness does exist. Using those observations to place constraints on that variability, though, would probably be difficult, because the spatial pattern of the tidal deformation includes oscillations across the entire surface that do not necessarily mirror the thickness variability. For example, locations that exhibit a smaller value of uplift than average, do not necessarily correspond to locations with a thicker-than-average shell.

Using shear modulus profiles for the icy shell derived from the same conduction models used to estimate the lateral thickness variations, we find similar first-order harmonic coupling effects as for the models with 3-D shell thickness. We also find that unless there are additional mechanisms (such as viscous flow in the icy shell) that act to reduce the lateral
variations in thickness, the effects of a laterally varying shear modulus on the tidal response are likely to be smaller than those of a laterally varying shell thickness. We conclude that, as for the case of laterally varying thickness, the presence of a conductive-driven 3-D variations in shear modulus would not degrade attempts to deduce the existence or absence of a liquid ocean using tidal observations, would not significantly impact attempts to infer mean shell thickness from observations of \( \Delta_2 \), and could probably not be mapped out with confidence using non-degree 2 gravity or altimeter observations.

To represent possible hemispherical differences in the composition of Europa’s icy shell, we include \((1, 1)\) structure in the shear modulus of the ice. Forcing this 3-D structure with a \((2, 0)\) tidal potential, we find a \((3, 1)\) pattern in the surface response, with an amplitude that is proportional to the amplitude of the \((1, 1)\) perturbation in the shear modulus. In general, the tidal potential of Europa contains large \((2, 0)\) and \((2, 2)\) terms, along with smaller terms at other degrees and orders. If a \((1, 1)\) structure in the shear modulus is coupled with the exact tidal potential, the dominant response caused by that structure occurs at harmonics \((3, 1)\) (for \((2, 0)\) and \((2, 2)\) forcing) and \((3, 3)\) (for \((2, 2)\) forcing). This structure would not impact attempts to infer the existence of a liquid ocean, nor would it have a significant effect on mean shell thickness inferred from observations of \( \Delta_2 \). The question of whether the degree-3 response caused by any such \((1, 1)\) structure could be extracted from observed degree-3 tidal solutions depends on the magnitude of the structure, the accuracy of the measurements, and whether there are other types of lateral heterogeneities that could introduce a degree-3 tidal response. As described in Section 4.4.2.2, any future interpretation of observed degree-3 tidal terms would require accurate modeling of the satellite’s spherically symmetric response to degree-3 tidal forcing. One encouraging conclusion that can be inferred from our results, is that the conductive-driven 3-D structures induce
significant tidal response only at even degrees and orders, so they would not affect attempts to detect (1, 1) structure through degree-3 tidal measurements. However, if there exist other sources of order-1 lateral variability in Europa’s icy shell, probably also caused by compositional heterogeneity (for instance, a high degree order-1 structure in the shear modulus, or a (1, 1) structure in satellite rheological properties other than the shear modulus), they would also induce (3, 1) and (3, 3) responses in the tidal solutions. So for a future Europa mission, the accurate interpretation of the degree-3 tidal measurements requires careful investigation of all possible lateral heterogeneities caused by compositional difference in the icy shell.

The generation of a degree-3 tidal response for a planetary body with degree-1 elastic structure, is similar to what Zhong et al. (2012) found for a possible degree-1 mantle structure on the Earth’s Moon using the same finite-element modeling method as in this study. As in Zhong et al. (2012), we find that degree-1 structure also causes changes in the degree-2 tidal response (Table 4.4). The linear dependence of the degree-3 tidal response on the amplitude of the perturbed structure (Fig 4.6) is the outcome of first order coupling between the degree-2 tidal force and the assumed degree-1 structure, as predicted by perturbation theory (Qin et al., 2013). However, the mode-coupling rules imply that changes in the degree-2 response cannot possibly be due to first order coupling (e.g., Zhong et al., 2012). Instead, Qin et al., (2013) show that any change in the degree-2 response must be a consequence of second order coupling. We anticipate that Qin et al.’s (2013) perturbation approach could be applied in the future to study problems similar to those explored here.

Using models with a non-global ocean, we find that the tidally induced surface uplift is much smaller (more than 3 times smaller for our assumed geometries) in the grounded region than in the floating region. In general, the uplift in floating regions tends to have a
slightly smaller amplitude than the uplift of a spherically symmetric satellite with a liquid ocean; and the uplift in grounded regions tends to have a slightly larger amplitude than the uplift of a spherically symmetric satellite without a liquid ocean. Because of this dramatic difference in tidal uplift across the surface, it would presumably be possible to infer the existence of a non-uniform ocean from uplift observations, and even to roughly map out its areal extent. The same might be true for tidal gravity observations, though the fact that tidal gravity observations would almost certainly be made in the spherical harmonic domain would make this more challenging. A grounded icy shell tends to increase the difference between total geoid and uplift, even in the floating region; so that an estimate of the mean thickness of the floating icy shell based on this difference (i.e. on $\Delta(\theta, \varphi)$) would likely be too large.
Chapter 5

Conclusion

The main scope of this study is centered on addressing the viscoelastic deformation of the Earth in response to surface ice loading, and the elastic deformation of Ganymede and Europa in response to Jupiter’s tidal loading. For the surface loading problem of the Earth, GIA solutions have been developed based on either spherically symmetric or laterally varying Earth models, and their impact on different geodetic observables has been investigated. For the tidal loading problem of Jupiter’s icy moons, surface tidal responses are computed, and the relation between the icy shell structure and the tidal measurements has been examined. In this chapter, we summarize this study as follows.

Firstly, we develop a 3-D finite-element model to study the viscoelastic response of a compressible Earth to surface loads. The effects of center of mass motion, polar wander feedback, and self-consistent ocean loading are implemented. To assess the model’s accuracy, we benchmark the numerical results against a semi-analytic solution for spherically symmetric structure. We force our model with the ICE-5G global ice loading history to study the effects of laterally varying viscosity structure on several GIA observables, including RSL measurements in Canada, and present-day time-variable gravity and uplift rates in Antarctica. Canadian RSL observations have been used to determine the Earth’s globally averaged viscosity profile. Antarctic GPS uplift rates have been used to constrain Antarctic GIA models. And GIA time-variable gravity and uplift signals are error sources for GRACE and altimeter estimates of present-day Antarctic ice mass loss, and must be modeled and removed from those estimates. Computing GIA results for a 3-D viscosity profile derived from a
realistic seismic tomography model, and comparing with results computed for 1-D averages of that 3-D profile, we conclude that: (a) a GIA viscosity model based on Canadian relative sea level data is more likely to represent a Canadian average than a true global average; (b) the effects of 3-D viscosity structure on GRACE estimates of present-day Antarctic mass loss are probably smaller than the difference between GIA models based on different Antarctic deglaciation histories; and (c) the effects of 3-D viscosity structure on Antarctic GPS observations of present-day surface displacement rate can be significant, and can complicate efforts to use GPS observations to constrain 1-D GIA models.

Secondly, we use our GIA model to study the impact of Little Ice Age deglaciation on GRACE and GPS present-day observables in southern Greenland. The on-going deformation of the solid Earth in southern Greenland is determined by the elastic response to present-day ice mass changes and the continuing viscous relaxation of the Earth’s mantle in response to past ice mass changes. To estimate the present-day ice mass variability, the viscous response of the Earth has to be computed using GIA model, and removed from GRACE mass change observations, and from GPS surface deformation observations. The accuracy of the GIA model mainly depends on the uncertainties in the ice loading history and in the Earth’s mantle viscosity structure. In this study, mass balance estimates of the southern Greenland ice sheet since the Little Ice Age have been obtained using high quality aerial stereo photogrammetric imagery, combined with contemporary ice surface differences derived using NASA’s laser altimeter measurements. Linear trends for the mass loss of sGrIS are derived for three time intervals, 1900 - 1981, 1981 - 2002, and 2002 - 2010, and are used to build a post-LIA ice loading history. We compute the GIA effects on GRACE and GPS observables using the post-LIA loading history along with different mantle viscosity structures, and we find that (a) the GIA effects on GRACE present-day mass loss estimates and on GPS present-
day surface emotion estimates are probably small; and (b) the GIA effect on GPS present-day surface displacement measurements can be significant, if the upper mantle viscosity of the Earth is small.

For the last part of this study, we use our finite-element model to solve for the response of Ganymede and Europa to tidal forcing from Jupiter, using various icy shell models with laterally variable structure. In all cases, the shell is assumed to be underlain by a liquid water ocean. Icy shells with laterally varying thickness are derived from a thermal conduction model. 3-D shear modulus profiles for the shell are built either from a conduction model or, for Europa, by assuming a hemispherical difference in composition. Icy shell structures with a non-global ocean, so that the shell is grounded in some places and floating in others, are built for Ganymede. Using these shell structures to calculate the tidal response of Ganymede and Europa, we conclude: (a) the presence of lateral variations in thickness or in shear modulus would not degrade future attempts to use tidal observations to decide on the existence or absence of a liquid ocean. (b) The effects of lateral variations in thickness and shear modulus on the degree-2 Love numbers \( (k_2 \text{ and } h_2) \) would be small enough that the mean shell thickness could still be accurately estimated from measurements of \( \Delta_2 = 1 + k_2 - h_2 \), as has been argued in the past for spherically symmetric icy satellites. (c) Given accurate enough observations, the presence of lateral variations in thickness or in shear modulus could be determined by searching for non-degree-2 components in the tidal response. (d) In the absence of significant viscous convective flow in the shell, the effects of a laterally varying shear modulus on the tidal response would be smaller than those of a laterally varying shell thickness. (e) If the shell is partially grounded, tidal perturbations in the geoid and, especially, the surface uplift, would be significantly smaller above the grounded region than above the floating region, so that tidal observations of either gravity or uplift would be able to roughly
differentiate regions where the ice is grounded from those where it is floating.


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Appendix A

The finite-element formalism of the momentum equation

In this appendix, we discuss details regarding the finite-element treatment of the momentum equation for the viscoelastic deformation problem, i.e. Eq. (21) in Chapter 2, later referred as Eq. (21) for simplicity.

A.1 The stiffness matrix

The first term on the left hand side of Eq. (21) represents the ordinary stiffness matrix that does not depend on gravity. It can be written as

\[
\int w_{i,j} [\ddot{\mu}(v_{i,j} + v_{j,i}) + \ddot{\lambda}v_{k,k} \delta_{i,j}] dV = \int \ddot{\varepsilon}(\ddot{w})^T D \ddot{\varepsilon}(\ddot{v}) dV, \tag{A1}
\]

where \( \ddot{\varepsilon} \) and \( D \) are given by

\[
\ddot{\varepsilon}(\ddot{w}) = \begin{bmatrix}
\varepsilon_{\theta\theta} \\
\varepsilon_{\varphi\varphi} \\
\varepsilon_{rr} \\
\varepsilon_{\theta r} \\
\varepsilon_{r \varphi} \\
\varepsilon_{\theta \varphi}
\end{bmatrix} = \begin{bmatrix}
\frac{w_{\theta,\theta}}{r} + \frac{w_r}{r} \\
\frac{w_{\varphi,\varphi}}{r} + w_{\theta,\theta} \cot \theta + \frac{w_r}{r} \\
\frac{1}{r} w_{r,r} \\
-\frac{\cot \theta w_{\theta,\varphi}}{r} + \frac{w_{\varphi,\varphi}}{r} + \frac{w_{r,\varphi}}{r} \\
\frac{w_{\theta,r}}{r} + \frac{w_{\varphi,\varphi}}{r} \\
\frac{w_{\varphi,\theta}}{r} + \frac{w_{r,\varphi}}{r} \sin \theta
\end{bmatrix}, \tag{A2}
\]

\[
D = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda \\
\lambda & \lambda + 2\mu & \lambda \\
\lambda & \lambda & \lambda + 2\mu \\
\mu & 0 & \mu \\
0 & \mu & \mu
\end{bmatrix}, \tag{A3}
\]
and \( \varepsilon \) can be represented by the nodal displacements

\[
\varepsilon_i(\tilde{w}) = \sum_{a,j} B_i^a w_j^a ,
\]

where \( a \) goes through all the nodes and \( i = 1,\ldots,6, j = 1,2,3 \). \( w_j^a \) denotes the \( j \) th component of \( \tilde{w} \) at the \( a \) th node, and \( B^a \) is a 6 by 3 matrix defined at the \( a \) th node. The form of \( B^a \) depends on the coordinate system, which in this case is spherical coordinates, and \( B^a \) is found through the calculation of the nodal positions and the shape functions (see Zhong et al., 2000). Eq. (A1) can be rewritten as

\[
\int \varepsilon(\tilde{w})^T D \tilde{\varepsilon}(\tilde{v}) dV = \sum_{a,b} (\tilde{w}^a)^T K_{ab}^1 \tilde{v}^b
\]

where \( K_{ab}^1 = \int (B^a)^T DB^b dV \). (A5)

The volume integration over the Earth’s entire mantle can be decomposed into sums of elemental contributions, which can be computed directly.

The second term on the left hand side of Eq. (21) represents another symmetric matrix. The radial displacement at a given point can be expressed as the linear combination of the nodal displacements

\[
v_r(\tilde{x}) = v_3(\tilde{x}) = \sum_{b,j} N_b(\tilde{x}) C^b_j(\tilde{x}) v_j^b ,
\]

where we denote the radial component as the third component, and \( N_b(\tilde{x}) \) is the shape function at the \( b \) th node. \( C^b \) is a 3 by 3 matrix, whose form also depends on the coordinate system (Zhong et al., 2000). Using Eq. (A2) and (A4), we have

\[
\nabla \cdot \tilde{w} = \sum_{a,j} (B_i^a + B^a_{2i} + B^a_{3i}) w_j^a .
\]

(A7)

Using Eq. (A6) and (A7), the second term on the left hand side of Eq. (21) can be written as
\[
\int (w_{ij} \rho_0 g v_i + w_r \rho_0 g v_{r,i}) dV = \sum_{a,b} (\vec{v}^r)^\top K_{ab}^2 \vec{v}^b
\]

where \((K_{ab}^2)_{ij} = \left[\left(B_{bi}^a + B_{2i}^a + B_{3i}^a\right)N_b C_{3j}^b + \left(B_{1j}^b + B_{2j}^b + B_{3j}^b\right)N_a C_{3i}^a\right] \rho_0 g dV . \quad (A8)

It is evident that
\[(K_{ab}^2)_{ij} = (K_{ma}^2)_{ji} . \quad (A9)\]

The third term on the left hand side of Eq. (21) represents the restoring forces induced by displacements at every boundary that has a density discontinuity. These can be computed through surface integration.

A.2 The forcing terms

The right hand side of Eq. (21) is composed of the forcing terms. Following the same procedure that we use for the stiffness matrix, the forcing terms can be expressed as the sum of two force vectors, and Eq. (21) becomes
\[KV = F_0 + F(\Delta \phi(V)) . \quad (A10)\]

where \(K\) is the total stiffness matrix, \(V\) is the incremental displacement vector containing \(\vec{v}\) at all the nodes, and \(F_0\) is the force vector that depends on the surface load \(\Gamma\), the pre-stresses \(\tau_{ij}^{pre}\), the displacements from the previous time step \(U_i\), and the initial gravitational potential \(\phi^0\) (i.e., the total gravitational potential at the previous time step plus the gravitational potential induced by the incremental load itself). \(F\) is the force vector that depends on the incremental gravitational potential \(\Delta \phi(V)\), which in turn depends on the incremental displacements \(V\) .
Appendix B

Semi-analytic method

For a spherically symmetric, compressible Earth, we use a spectral method (Han and Wahr, 1995) to find the Love numbers in the Laplace transformation domain for each spherical harmonic degree \( l \) and order \( m \). Following Wu and Peltier, 1982, we expand the Love numbers as a Laurent series of first-order poles:

\[
\tilde{h}_l(s) = h_l^E + \sum_j r_j^h \frac{1}{s + s_j}, \quad (B1)
\]

\[
\tilde{k}_l(s) = k_l^E + \sum_j r_j^k \frac{1}{s + s_j}, \quad (B2)
\]

where \( \tilde{h}_l(s) \) is the Love number for the surface vertical displacement, \( \tilde{k}_l(s) \) is the Love number for the surface potential, \( h_l^E \) and \( k_l^E \) are the elastic Love numbers, \( s \) is the Laplace transform variable, \( -s_j \) denotes the poles, and \( r_j^h \) and \( r_j^k \) are the residues for \( \tilde{h}_l \) and \( \tilde{k}_l \), respectively. Instead of determining the modes \( -s_j \), and the residues \( r_j^h \) and \( r_j^k \) directly, we apply the collocation technique (Peltier, 1974). Evenly spaced points \( \{s_j \mid j = 1, 2, ..., N\} \) are chosen so that they cover the expected range of the negatives of the poles, and the Earth’s response \( \tilde{h}_l(s_j) \) and \( \tilde{k}_l(s_j) \) are computed at those points. Using least-square fitting, we solve for \( r_j^h \) and \( r_j^k \) such that

\[
\tilde{h}_l(s_k) = h_l^E + \sum_{j=1}^N \frac{r_j^h}{s_k + s_j}, \quad (B3)
\]
The robustness of this application of the collocation technique has been verified in Zhong et al., (2003) for an incompressible Earth model. For a compressible Earth, because we are approximating an infinite series (since we have an infinite number of normal modes) by a finite summation, care needs to be taken when choosing $s_j$. We define

$$\delta^h_l = \frac{\tilde{h}_l(s \to 0) - h^E_l}{\sum_{j=1}^{N} \frac{r_j^h}{s_j}},$$  \hspace{1cm} (B5)$$

$$\delta^k_l = \frac{\tilde{k}_l(s \to 0) - k^E_l}{\sum_{j=1}^{N} \frac{r_j^k}{s_j}}.$$  \hspace{1cm} (B6)

To obtain reliable results, we adjust the spacing and the range of the $s_j$'s so that the numerical results for $\delta^h_l$ and $\delta^k_l$ are close to 1 (Mitrovica and Peltier, 1992). We compute the numerators in Eq.'s (B5) and (B6) directly through our code, by finding $\tilde{h}_l$ and $\tilde{k}_l$ at the smallest $s_j$; we compute the denominators by summing the residues. Once we have chosen the $s_j$'s so that the results for $\delta^h_l$ and $\delta^k_l$ differ from 1 by less than 3%, we use the model results computed with those values to obtain Love number solutions for impulse forcing in the time domain:

$$h_l(t) = h^E_l \delta(t) + \sum_{j=1}^{N} r_j^h \exp(-s_j t),$$  \hspace{1cm} (B7)$$

$$k_l(t) = k^E_l \delta(t) + \sum_{j=1}^{N} r_j^k \exp(-s_j t),$$  \hspace{1cm} (B8)

where $h_l(t)$ and $k_l(t)$ are the time dependent Love number solution, and $\delta(t)$ is the Dirac delta function. Since the time dependence of the loading we apply is either a Heaviside
function or a piecewise-linear function (ICE-5G), it can be analytically convolved with the impulse response in the time domain.

For the semi-analytic model, the treatments for degree one deformation, polar wander feedback, and ocean loading have been discussed in Paulson et al., 2005. To find the correction to the degree 2 order 1 Love numbers induced by polar wander feedback, we compute

\[ h_2^{±1}(s) = h_2^L(s) + h_2^T(s) \left( \frac{1 + k_2^L(s)}{k_f - k_2^L(s)} \right), \]  

(B9)  

\[ k_2^{±1}(s) = k_2^L(s) + k_2^T(s) \left( \frac{1 + k_2^L(s)}{k_f - k_2^L(s)} \right), \]  

(B10)  

where the superscripts \( L \) and \( T \) denote the load and tide love numbers, respectively. Here, the fluid love number is defined as

\[ k_f = (1 + \delta) \cdot k_{\text{hyd}}. \]  

(B11)  

where \( \delta \) is set to be 0.8% (Mitrovica et al., 2005), and \( k_{\text{hyd}} \) is the degree-2 tide Love number, \( k_2^T \), computed in the fluid limit. That is, \( k_{\text{hyd}} \) represents the tidal response of an Earth that has the same density profile as the model Earth, but where the material is all assumed to be fluid. (Mitrovica et al., 2005 refer to \( k_{\text{hyd}} \) as \( k_2^T (s \rightarrow 0; LT = 0) \)). At long temporal scales (i.e. as \( s \rightarrow 0 \)), \( k_2^T \) approaches \( k_f \), which makes the denominators in Eq.’s (B9) and (B10) close to 0. In fact, if the lithosphere were modeled as viscoelastic (presumably with a very large viscosity), then \( k_2^T (s \rightarrow 0) \rightarrow k_{\text{hyd}} \) exactly, and so the dominators would converge exactly to 0 if \( \delta \) had been chosen to be 0. So the estimation of these two equations is very sensitive to the value of \( k_f \). Numerical errors that occur at very small values of \( s \) prohibit us from computing \( k_{\text{hyd}} \) by taking the \( s \rightarrow 0 \) limit of \( k_2^T(s) \).
Instead, to obtain an accurate solution we calculate $k_{hyd}$ by computing $k^T_2$ for a fluid Earth using the boundary conditions for the gravitational potential and its gradient (Dahlen, 1974).

The same fluid love number is also used to obtain $(C - A)_{hyd}$ in the finite-element model.