Preview Scheduled Model Predictive Control For Horizontal Axis Wind Turbines

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Preview Scheduled Model Predictive Control
For Horizontal Axis Wind Turbines

by

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B.S.E.E., University of Minnesota, 1986
B.S. Mathematics, University of Minnesota, 1986

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has been approved for the Department of Electrical, Computer, and Energy Engineering

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Lucy Y. Pao

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Dr. Alan Wright

Date ________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
Laks, Jason H. (Electrical Engineering)

Preview Scheduled Model Predictive Control
For Horizontal Axis Wind Turbines

Thesis directed by Prof. Lucy Y. Pao

This research investigates the use of model predictive control (MPC) in application to wind turbine operation from start-up to cut-out. The studies conducted are focused on the design of an MPC controller for a 650 KW, three-bladed horizontal axis turbine that is in operation at the National Renewable Energy Laboratory’s National Wind Technology Center outside of Golden, Colorado. This turbine is at the small end of utility scale turbines, but it provides advanced instrumentation and control capabilities, and there is a good probability that the approach developed in simulation for this thesis, will be field tested on the actual turbine.

MPC is an active area for turbine control research, because wind turbine operation is complicated by multiple factors that are intrinsic to harvesting power from the wind resource:

- Since the goal of the turbine is to produce as much energy as possible from the available power in the air flow passing through the turbine’s rotor plane, either the turbine’s blade pitch (used to regulate aerodynamic torque), or the generator load torque (used to regulate rotor speed at the optimal tip-speed-ratio) are routinely set at the limits of their operating range.

- There is a significant variation in the gain from perturbations in blade pitch to perturbations in bending moments and torque. This variation is dependent on the relative speed between the blade and wind, and the nominal blade pitch. As a result, gain scheduling techniques are found to be necessary in order to obtain adequate speed regulation, and optimal load mitigation.

- The three individual pitch (IP) commands and the generator load command, along with structural loads that can be in conflict with speed regulation objectives, make the turbine
control problem inherently multi-input-multi-output (MIMO) in nature.

- Advanced measurement technologies like LIDAR (light detection and ranging) make the use of preview control plausible in the near future.

Standard formulations of MPC accommodate each of these issues. Also, a common MPC technique provides integral-like control to achieve offset-free operation [9]. At the same time in wind turbine applications, multiple studies [38, 5, 73] have developed “feed-forward” controls based on applying a gain to an estimate of the wind speed changes obtained from an observer incorporating a disturbance model. These approaches are based on a technique that can be referred to as disturbance accommodating control (DAC) [32]. In this thesis, it is shown that offset-free tracking MPC [52] is equivalent to a DAC approach when the disturbance gain is computed to satisfy a regulator equation. Although the MPC literature has recognized that this approach provides “structurally stable” [20] disturbance rejection and tracking, this step is not typically divorced from the MPC computations repeated each sample hit. The DAC formulation is conceptually simpler, and essentially uncouples regulation considerations from MPC related issues. This thesis provides a self-contained proof that the DAC formulation (an observer-controller and appropriate disturbance gain) provides structurally stable regulation.
Dedication

This thesis is dedicated to my parents Henry Laks and Katherine Eliot who unfortunately could not be here to witness the accomplishment.
Acknowledgements

I would like to thank my advisor Professor Lucy Pao for having the wisdom and patience to accept an unusually old graduate student, and my advisor Dr. Alan Wright for his guidance and expertise in wind energy. I want to also give special mention to Dr. Jason Marden for his unending patience and faith.
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Chapter 1

Introduction

1.1 Overview of Standard Turbine Control

At present, wind energy is still one of the most promising renewable energy resources under active development. According to the American Wind Energy Association (AWEA), the total installed capacity in the U.S. was 51.6 GW as of the second quarter of 2012, with 6.8 MW newly installed during 2011. And, according to an AWEA fact sheet [4], this represents “35% of all new generating capacity in the U.S. since 2007– more than new coal and nuclear combined.” However, there are significant issues related to transmission from wind rich locations to energy markets, and also related to the intermittency of the wind resource that need to be addressed in order to achieve further penetration of wind energy into the electrical grid. Whether or not it is possible for wind technology to mitigate these issues is an active area of research [2].

The role of control in wind turbine operation is becoming more important as the size of utility scale wind turbines continues to increase, and the flexibility of tower and blade structures increases in order to reduce material costs. Advanced control techniques can significantly reduce fatigue loads [10] and increase component life, thereby decreasing the cost of energy (COE). Among the different turbine configurations that have been explored (vertical axis versus horizontal, upwind rotor versus downwind rotor, two bladed versus three bladed) the three bladed upwind horizontal axis wind turbine (HAWT) has come to dominate utility scale installations. This configuration has come to be viewed as the most economical way to harvest wind energy while addressing the noise issues that come with faster rotor speeds (as often occur in a two bladed machine with comparable rating) and
the whooping noise problem that occurs with a downwind configuration. Modern turbines employ power electronics to facilitate grid connection and can generate power over a range of rotor speeds, whereas early utility scale turbines often operated with a direct grid connection that required the rotor to turn at a fixed speed to produce power at the grid frequency. The newer variable speed configuration allows the rotor speed to track with wind speed to optimize power capture. Finally, turbines can be fixed pitch in which case the blade is designed to stall in high wind speeds to regulate speed, or variable pitch in which case speed is typically regulated by pitching the blades to feather (to decrease aerodynamic torque and thrust). Variable pitch has become predominate, most likely because of the ability to optimize efficiency for a wider range of wind speeds and decrease blade loads at high wind speeds.

Fig. 1.1 provides a schematic of a typical HAWT. The incoming wind encounters the turbine’s blades (rotor assembly) first and generates aerodynamic torque which is transferred through the hub to the low speed shaft. Most utility scale turbines then multiply the shaft speed up to the
generator with a gearbox. Traditionally, the only measurement available to the control system was generator or rotor speed, and the actuator inputs consist of blade pitch and generator load torque. The objective in below rated wind conditions is to optimize energy capture, and in above rated wind conditions (when the turbine is producing maximum power) the objectives are to mitigate structural loading and regulate speed to prevent over-speed and power faults that would trigger a shut-down of the turbine.

As explained in Chapter 7, there is an optimal ratio between rotor speed (typically quantified in terms of the blade tip speed) and wind speed at which the aerodynamic power developed across the rotor is maximized. Near the optimal tip-speed-ratio, the power captured is relatively independent of the blade pitch once as long as it is near the optimal (often referred to as fine-pitch). Hence, in below rated conditions, blade pitch is held constant at optimal, and the desired tip-speed-ratio is achieved by balancing generator load torque with aerodynamic torque. Also, as further explained in Chapter 7, it turns out that in steady wind conditions, this balance can be achieved by setting load
torque proportional to the square of the rotor speed. The constant of proportionality is determined by the turbine’s rotor geometry using blade-element-moment theory as employed in turbine design analysis codes like WT_Perf [16] developed at the National Renewable Energy Laboratory.

The result is an operating profile like the one depicted in Fig. 1.2. In region 2 (below rated wind speed conditions), the generator torque is set according to the square law. In region 3 (in above rated wind conditions), generator torque is held constant and rotor speed is regulated using blade pitch; ostensibly, region 3 operation occurs only at the intersection of rated speed and torque in Fig. 1.2. In reality, speed is not regulated perfectly, and there may be a speed threshold less than rated before returning to torque control. Also, in some systems, the load torque may be adjusted to keep power (speed×torque) constant as rotor speed varies about rated. In addition, it is unusual that the square law operating points bring the turbine to rated speed at the same time as rated torque, so there is often transition from region 2 to region 3 referred to as region 2.5.

The fact that blade pitch and then load torque are routinely set near their operating limits, is a motivation for using a control technique like model predictive control. This is explained in more detail in the next section. In addition, for over a decade [38], the idea of measuring wind speed changes before they reach the turbine has been explored for use in aiding turbine control. More recently [27], there has been significant interest in advanced measurement technologies like LIDAR (light detection and ranging), which make the previewed wind speed concept seemingly within reach. The work in this thesis endeavors to implement a model predictive controller that utilizes preview measurements and operates the turbine from start-up to cut-out.

1.2 Contributions and Scope

1.2.1 Overview Of Thesis and Contributions

The work in this thesis is mostly applied in terms of getting a model predictive controller (MPC) to operate the wind turbine from start-up to cut-out and to use preview measurements of wind speed changes approaching the turbine. MPC is typically implemented in discrete time and is
also known as receding horizon control, although the latter term is more common when the models and optimization are done in continuous time. Roughly speaking, MPC is the computation of an optimal control sequence based on the dynamics of the plant and an estimate of its initial state. The sequence is optimized for a finite length of time into the future (a finite horizon), and then only the first control in the sequence is applied to the plant and the optimization is repeated again at the next sample hit with a new estimate of the plant state. If the plant model used to predict the system response is non-linear, then the technique is known as non-linear MPC (NMPC). However, because the optimization usually includes explicit constraints on the control actions and/or plant variables, the method normally produces non-linear control actuation whether or not the model employed is non-linear, and is therefore correctly viewed as a non-linear control technique.

The motivation for using an advanced control technique is improved load mitigation and optimal power point tracking. Modelling issues aside, from the perspective of MPC, the wind turbine application should be relatively straightforward; the problem is easily posed in terms of a finite horizon objective and MPC will handle the constraints on pitch and generator load torque. The issue becomes one of load mitigation and tracking performance in an under-actuated system. In below rated conditions, the blade pitch needs to be near a minimum where control authority (i.e. the gain) is greatly reduced (and possibly highly non-linear); and in above rated conditions, the generator torque is set near the maximum safe operating level and rotor/generator speed needs to be tightly regulated. Also in below rated operation where the objective is to maintain the optimal tip-speed-ratio and only the generator load torque has any significant command authority, the system is under-actuated in that running the generator as a motor in order to maintain tip-speed-ratio is not a viable option.

MPC is a good fit in terms of the under-actuated aspects of turbine control, but in above rated conditions it is particularly important to regulate speed without an offset. Normally, regulating to a set-point without an offset is simply a matter of using integral control with anti-windup, but as shown in Section 4.4, there is no straightforward way to combine such an approach with MPC. Instead, it is necessary to employ a method commonly referred to as offset-free MPC. This
provides regulation of system outputs in the presence of constant disturbances and set-points, and is closely related to a disturbance accommodating control (DAC) method that was possibly first proposed in [32]. The idea is to include a model of a system that can generate a persistent disturbance (sometimes called an exosystem) and estimate the state of the generator as part of a state-observer. Then a gain can be computed to “feed-forward” the disturbance state to the control inputs. In its original formulation DAC only achieved output regulation (disturbance rejection) in special cases, because the disturbance gain was computed to mitigate the disturbance effect on the system state. If instead, the gain is computed to mitigate the disturbance effect at the system output as in Chapter 4, then DAC will provide output regulation (e.g., integral control).

The original idea for the thesis was to extend the offset-free MPC techniques to disturbances other than constant offsets by using a slight modification of the DAC approach. However, a generalization of the MPC offset-free methods was recently demonstrated in [52]. Nevertheless, there is still an advantage in reworking their results from the perspective of DAC. In the standard offset-free approach the MPC literature implies that state and control target “shifts” should be computed each sample hit. In Chapter 4 it is shown that from the DAC perspective, most of the computation only needs to be done off-line once and this computation is completely separate from any constraint issues. Further, for the case that regulation is not feasible at all measurements, it has been viewed as complicated [9] to get selective regulation and the methods [51] for this case with constant offsets are not altogether transparent. However, in Chapter 4 a straight forward modification to the DAC approach is provided that handles this case for a large class of arbitrary disturbances, and does not increase the complexity of the model used by the MPC algorithm over that used in [51] (although, the complexity of the observer is increased). With regard to the wind turbine application, the wind preview measurements are completely uncontrollable and so it is absolutely necessary to handle the case that regulation is not possible at all measurements.

The technical contribution of this thesis is in providing a straight forward method to use preview measurements with MPC to obtain preview actuation, and also so that the system achieves offset-free tracking. This requires a marriage of the techniques in [51] for the case where regulation
is not possible at all measurements, and the techniques in [52] for regulation in the presence of disturbances more general than constant offsets. The DAC approach accomplishes this in a way that is fairly intuitive and separates the regulation issues almost completely from the MPC constrained optimization. In particular, the use of an external reference (wind measurements) not generated by a disturbance model is considered; if the reference to be tracked is generated by a disturbance model, then the DAC method is easily modified to be completely equivalent to the approach described in [52].

A second contribution is in the application of $\mathcal{H}_2$ optimal preview control. Since standard formulations of MPC require an estimate of the system state, methods for observer and observer-controller design are used extensively. In particular, this thesis makes use of the approach in Hazell [28] for the optimization of preview gains. The $\mathcal{H}_2$ methods developed by Hazell assume that the preview is a noise free measurement of a known reference, and this is essentially different than a noisy measurement of wind speed. In the final analysis, Hazell shows that the related Riccati equations are based on the system dynamics without the augmentations necessary to model storage of the preview measurements. Further, in deriving his results he is able to side-step issues related to forming an optimal estimate of the preview “state” from the noisy measurements. In Chapter 5, this thesis extends the approach slightly by showing that if the preview measurement noise is white, then the structure of the controller is still that obtained by Hazell where there is partition between preview and state gains.

However, once the observer dynamics are augmented to achieve offset-free tracking, this simple partition is no longer obtained. In this case, using the partitioned architecture is suboptimal, but this is still the approach used in this thesis. A sub-optimal configuration is demonstrated in which the observer is provided the preview measurements and the feed-forward controls (as one would expect to obtain a good state estimate) in a way that still achieves offset-free tracking. Without using the sub-optimal partition, the implication is that to compute the optimal solution, it is necessary that the order of the system used in forming the algebraic Riccati equation include states modelling storage of the preview measurements. This seems to contradict the recent work
of Kristalny and Mirkin [40] where they retain the lower order Riccati equations of the original system augmented while using unstable weights to achieve regulation and tracking. However, their approach does not synthesize an observer-controller architecture, and the method of achieving regulation is essentially the same as output augmentation. As shown in Chapter 4, such an approach does not achieve regulation when used in combination with MPC.

This thesis is not a work in dynamics modelling, nor does it attempt to develop an analytical model of the turbine’s non-linear dynamics. There are also numerous trade-offs to be considered in optimizing for load mitigation, and this thesis does not attempt an exhaustive study trading one load against another. Instead, the focus is on mitigation of blade loads. The controls designed and the metrics used to evaluate performance are heavily weighted towards this goal. However, tower and drive train loads are also evaluated, and some discussion in terms of next steps that address tower loads are provided in the closing chapters of the thesis.

The non-linearity of the turbine’s response is handled indirectly through the design of scheduled controls. The starting point for these designs are numerical linearizations provided by the turbine modelling code FAST [35] developed at the National Renewable Energy Laboratory (NREL). However, there are some minor adjustments that are required based on the assumed configuration of the preview wind measurements, and this is explained in Chapter 2. This chapter also introduces multi-blade coordinates (MBC), the use of which has real advantages in terms of the choice of disturbance models used for offset-free regulation. Chapter 3 provides an overview of MPC and the method implemented for constrained optimization. Chapter 4 develops the DAC approach for offset-free MPC that depends on the internal model principle; this chapter provides an abridged version of the principle that is developed fully in Appendix C. Chapter 5 provides an overview of $\mathcal{H}_2$ optimal preview control, that is used to design the nominal scheduled feed-forward and feedback gains. Chapter 6 details the choice of disturbance models for the turbine application and provides an example of the controller design process for a region 3 operating point. Then Chapter 7 explains the configuration of the preview scheduled MPC scheme and the method of scheduling operating points. Chapter 8 provides simulation results that highlight the important features of the
MPC architecture and benchmark its performance against standard controllers. Finally, Chapter 9 discusses ideas for further research and issues left unresolved in this thesis.

1.2.2 Existing Literature

As in most control fields, the body of existing literature is quite vast and covers numerous aspects of wind turbine control. However, to a large extent, the literature can be divided into two main bodies roughly separated by generator and power electronics control at the back end of the turbine (e.g. [53]), and structural load and aerodynamic operating point issues at the front end, wherein the generator is viewed as a nearly ideal actuator [55]. Studies that encompass both areas are significantly less prevalent, or are only now beginning to appear [2]. Although advanced control techniques based on state-space methods [73] that encompass LQG [68], $\mathcal{H}_\infty$ [24, 26], LPV [8, 7], as well as adaptive [33, 22] and non-linear methods like sliding mode [50, 6] and feedback linearization [14, 42], have all been prevalent in the literature for over a decade, it is unclear that any of these methods have been adopted by industry, even when implementing controllers that pitch the blades individually [12]. One possible reason is that a majority of these studies are done at a single operating point and usually do not address the actuator saturations that occur routinely in wind turbine operation.

Studies on the use of MPC in application to wind turbine control have become more prevalent in the last five years. One of the earliest studies [13] is also one of the most sophisticated since it applies NMPC, uses a neural net to adaptively adjust the model parameters, and simulates the turbine response in conditions that encompass rated and above rated operation. Most of the MPC wind turbine studies can be grouped according to the range of operating conditions under which the turbine is simulated. The studies can be further classified by what loads the control system addresses (e.g., drive train [43], tower [39], blades [46], etc.), by whether or not the control uses generator torque, blade pitch, or both for actuation, and whether or not the blades are pitched individually or collectively. The MPC controller developed in this thesis uses both individual blade pitch and generator load for control actuation.
MPC has also been applied to other aspects of the wind energy conversion (WEC) problem in addition to individual turbine control. In [74] the use of MPC is studied in application to the dispatch of resources in a power generation system that includes a wind farm; [37] investigates the use of MPC for integrating battery storage systems with a WEC system (ostensibly at the wind farm level); the use of plug-in hybrid-electric vehicles as a storage method for balancing wind farm power production is studied in [23]; [57] applies MPC at the supervisory level to manage a hybrid wind-solar stand-alone system; and finally, [17] uses a fast set-membership technique for NMPC to develop and actually implement controls for a prototype high altitude kite WEC system.

In less exotic applications to the horizontal axis upwind turbine (HAWT), there are also numerous existing MPC studies in the literature. Henriksen et al. [29], do not use preview information and only a single linearized model serves as a basis for the MPC algorithm which is demonstrated in all load conditions; the present rotor speed and turbine power determine (schedule) the MPC cost minimized as well as the control actuations available to the MPC algorithm. In their study, the goal was to demonstrate an MPC architecture that operates the turbine in all load conditions while adhering to actuation constraints, and so the performance was not bench-marked relative to other controllers in terms of structural load mitigation.

Closer to the method developed in this thesis are the studies by Soliman et. al. [66] and Kumar and Stol [43] that use more than one linearization. Both studies use collective pitch, constraints on actuation, and evaluate load performance in terms of drive train and power fluctuations. The former designs the controller for operation in all regions and assumes that a measurement of the present wind speed (no preview) is available for scheduling, while the latter demonstrates operation only in above rated conditions and is self scheduled based on present collective blade pitch. It is interesting that the former uses only one MPC optimization each sample hit, but runs multiple observers in parallel, and the latter runs several MPC optimizations in parallel (forming a weighted average of the controls produced), but uses a single extended Kalman filter to estimate the turbine state. Korber and King [39], design an MPC controller for a single operating point in above rated conditions, but explicitly use preview of wind speeds. Their study demonstrated the
trade-off between speed regulation and tower load mitigation; also, relative to a well tuned PID controller, they show a reduction of 17% in tower damage equivalent load during an extreme wind gust.

More recently, Schlipf et. al. [63] demonstrate a NMPC for collective pitch and generator torque control in all operating conditions. Tower loads were addressed in the cost function by penalizing the tower top velocity. This controller operates throughout all operating regions and applies constraints to control effort, rotor speed and tip-speed-ratio. In addition, it explicitly estimates a preview of rotor effective wind speed based on a specific pattern of LIDAR measurements, and uses this information in the NMPC algorithm. The study reports significant reductions in tower loads and speed fluctuations in both extreme gust wind variation and in turbulent wind variation generated using TurbSim [34].

The work in this thesis is distinguished in part, because it addresses blade loads; investigations focusing on mitigation of blade loads are still very under-represented in the wind turbine literature. Among the contributions made by the work in this thesis to the existing body of literature are the following:

- The use of preview measurements in a multi-blade coordinate based linear-controller optimized for blade load mitigation is demonstrated in the 2011 Mechatronics article [48].

- The blade load mitigation capability of an MPC system is compared against the performance of the controllers from the Mechatronics study and presented at the 2011 AIAA Aerospace Sciences Conference Wind Symposium [46].

- Achievable trajectories for the wind turbine are studied in [44] through the use of constrained iterative learning control.

- The tracking ability of a preview scheduled MPC system is studied in [45] and the results presented at the 2012 AIAA Aerospace Sciences Conference Wind Symposium.

- The aspects of the internal model principle that directly apply to the DAC approach are
derived in a paper submitted to Automatica in June of this year and, as of the writing of this thesis, is currently still under review.
Chapter 2

Turbine Modeling

The method for modelling the turbine is inexorably tied with the method of providing wind measurements into the turbine model, and to the controller. This thesis, is not a work in turbine or wind modelling. The studies presented rely on FAST’s methods for keeping track of wind speeds and their effects on the turbine dynamics. This chapter explains how the linearizations provided by FAST are modified to be compatible with the configuration assumed for preview measurements. The working assumption is that three point measurements can be placed at desired locations in front of the turbine, and that these locations can be coordinated with rotor speed and position. Ideally, a preview controller would make use of a form of blade-effective wind-speed, that best matches the transfer of energy or force distribution to the turbine’s blades. Many others in the field make use of various levels of sophistication in this approach (e.g. [64, 63]), in addition to methods for estimating the necessary wind information from LIDAR measurements. The studies in this thesis simply utilize three point measurements.

In general, the turbine-controller system response is evaluated through monte-carlo type simulation using a collection of turbulent wind fields that produce significant variations in wind speeds across the rotor plane. This means that the effective shear seen down the length of a one blade can be significantly different than that sensed by another. However, the turbine tends to respond to the effective shear sensed across the rotor as a whole for the present rotor azimuth (the angle at which each blade is positioned). In this regard, the turbulent wind fields produce effective shear that can vary significantly with rotor position. When shear is uniform across the rotor plane,
large 1P variation is evident in the wind speed seen by an individual blade; when the shear is not uniform, higher frequency components can be evident for transient periods of time.

This suggests that it is beneficial to be able to measure the effective shear as a function of rotor position as the blades rotate. As explained in the next section, this is equivalent to knowing the blade effective wind speed, local to each blade as they rotate. In lieu of a blade effective wind speed, these studies use the three point measurements in front of the turbine at 75% span, to estimate the wind speed that each blade will see, given that the velocity distribution travels to the turbine at a constant mean wind speed that is known exactly. This provides the fairly unrealistic result that the time of transit for a measured velocity to reach the turbine blade is known exactly. Hence, in order to determine the measurement azimuths, all that remains is to estimate how much the blades will rotate from their present azimuth during the time of transit, and then take the three point measurement at locations determined by that future azimuth.

This is somewhat unrealistic given presently available LIDAR systems, but it does offset performance degradation that might be incurred due to not using a blade effective wind speed. More importantly, it is sophisticated enough to explore the effect of non-uniform shear. That this sort of variation is present in the wind fields used for simulation, is demonstrated by comparing the wind speed encountered by each blade to the wind predicted using three stationary point measurements, and interpolating to each blade based on the assumption that the shear is uniform. The method for doing this is explained in the next section. In this section we present the statistics for this comparison over an ensemble of 31 random wind fields, but all representative of the atmospheric conditions present in the row labeled “AR4” of Table 2.1.

The turbine is simulated in each of these fields using stationary point measurements, and then again using rotating measurements with the measurement azimuth based on rotor speed and position. In each case these measurements are compared against the wind speeds that the blades actually encounter. Typical results are displayed in Fig. 2.1. In the top plot of Fig. 2.1a, the actual wind speed seen by a single blade at 75% span, and the corresponding rotating measurement appear as a single waveform, because the accuracy is excellent. The center plot shows that using a sta-
Table 2.1: Meteorological input parameters for TurbSim: a three dimensional wind vector is generated over a 31x31 point grid in the vertical Y-Z plane, centered so that it encompasses the rotor disk. Over time the grid is sampled at 20Hz for a total duration of 630 seconds. The wind profile within the grid is varied by the vertical stability parameter $Ri_t$ and the mean friction velocity (shearing stress) $u_s D$; a power law variation of the vertical wind speed profile is specified by the listed shear exponent $\alpha_0$.

<table>
<thead>
<tr>
<th>Ensemble</th>
<th>Mean Wind [m/s]</th>
<th>$Ri_t$</th>
<th>$\alpha_0$</th>
<th>$U_0$ [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR4</td>
<td>18</td>
<td>-0.18</td>
<td>0.110</td>
<td>0.682</td>
</tr>
<tr>
<td>AR5</td>
<td>18</td>
<td>0.021</td>
<td>0.134</td>
<td>0.860</td>
</tr>
<tr>
<td>AR6</td>
<td>18</td>
<td>0.043</td>
<td>0.125</td>
<td>0.688</td>
</tr>
</tbody>
</table>

Stationary measurement and then interpolating to the blade location (blue), differs significantly from the actual wind speeds (green). The amplitude spectrum (lower plot) shows that the interpolated measurements (blue) are missing energy at harmonics of the 1P (once-per-rev) frequency that the blade encounters as it rotates through different azimuths. Over the ensemble of wind fields, the rms error in rotating measurements is less than 0.1 m/s, while that for stationary interpolation is well over 1 m/sec-- the histogram of errors is provide in Fig. 2.1b. This implies that in turbulent conditions, the ability of a preview control system to mitigate loads will be dependent on the measurement system’s ability to sense the higher harmonic variation in wind speeds. This is born out by the simulation results presented in Chapter 8.

In order to utilize preview measurements for load mitigation, it is necessary that the model predict load perturbations from perturbations in wind speeds. Since these studies rely on the FAST turbine code to obtain linearized turbine models, it is necessary to coordinate the preview measurements with the coefficients that FAST linearization produces. This does not occur automatically. FAST linearizes the turbine model relative to perturbations in uniform wind speed, and to perturbations in vertical and horizontal shear. Hence, either the point measurements need to be scaled into equivalent perturbations in uniform, vertical and horizontal shear, or the linearization needs to be transformed to accept perturbations in wind speeds local to each blade. This essentially means transforming between rotating (individual) and fixed reference frames, and so uses the multi-blade coordinate (MBC) transformation introduced and used in the next several sections.
Figure 2.1: Turbulent wind fields create shear profiles that vary significantly with rotor position: (a) typical waveforms for wind speeds at 75% span; (b) histogram of measurement errors comparing stationary+interpolation accuracy with that of rotating measurements.

2.1 Wind Measurements and MBC

In this section, the basic MBC transformation is developed by working through its application to wind measurements. If the wind speeds throughout the rotor plane vary linearly with vertical and horizontal position, with a possible non-zero average or “uniform” offset, then the speeds seen by a blade as it rotates around the plane will vary sinusoidally. In the most conceptually simple interpretation, the MBC transformation can be viewed as a means to compute the amplitude and phase (or equivalent sine and cosine amplitudes) of the sinusoidal variation and the uniform offset, given the measurement at each blade. If the variation across the rotor is truly planar, then the computation produces constant component amplitudes, otherwise these values can vary with azimuth, and/or also with time if the underlying speeds across the rotor plane are changing with time.

FAST quantifies shear in terms of the change in wind speed seen at the blade tips as a (unitless) fraction of the spatial average wind speed $w_0$ across the rotor disk. If the horizontal wind speed is perturbed away from the nominal $w_0$ due to a (spatially) uniform amount $w_u$, as well as
by horizontal $\Delta_h$ and vertical $\Delta_v$ shear components, then the total perturbation $\Delta$ at a location with horizontal and vertical coordinates $(y, z)$ within the rotor disk is

$$\Delta(y, z) = \frac{\Delta_v}{2} w_0 \frac{z}{R} + \frac{\Delta_h}{2} w_0 \frac{y}{R} + w_u,$$

(2.1)

where $R$ is the radius of the rotor disk. The factor of a half appears, because for example, FAST defines a vertical shear of $\Delta_v$, as occurring positive $\Delta_v/2$ at the top of the rotor, and negative $\Delta_v/2$ at the bottom of the rotor; $\Delta_v$ is the total change from top-to-bottom as a fraction of average wind speed over the rotor disk.

Now assume three blades, and a measurement from each blade. Let $\theta$ represent the clockwise angle (the convention used by FAST) of blade one from vertical and $r$ represent the radius at which the wind speed is measured. Then the measurement position at radius $r$ along blade 1 is $(y, z) = (-r \sin(\theta), r \cos(\theta))$ so that as a function of azimuth $\theta$, the perturbation in wind speed is

$$\Delta(\theta) = \frac{\Delta_v}{2} w_0 \frac{r \cos(\theta)}{R} - \frac{\Delta_h}{2} w_0 \frac{r \sin(\theta)}{R} + w_u$$

(2.2)

$$= w_c \cos(\theta) + w_s \sin(\theta) + w_u.$$

The MBC (cos and sin) component amplitudes seen at the measurement radius $r$,

$$w_{mbc} = \begin{bmatrix} w_u \\ w_c \\ w_s \end{bmatrix},$$

$$M_{m2s}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w_0 \frac{r}{2R} & 0 \\ 0 & 0 & -w_0 \frac{r}{2R} \end{bmatrix},$$

$$w_{sh} = \begin{bmatrix} w_u \\ \Delta_v \\ \Delta_h \end{bmatrix},$$

(2.3)

are determined by the shear $w_{sh}^T = [w_u, \Delta_v, \Delta_h]$ encountered by the blades; the matrix $M_{m2s}$ scales from MBC perturbations to equivalent shear perturbations.

Wind speeds seen at three blades always define a planar, or MBC variation that will fit three point measurements exactly. However, because the variation throughout the rotor plane is never
The matrix can be determined as a simple sum of cosines, sines and an offset

\[
\begin{bmatrix}
  w_1(\theta) \\
  w_2(\theta) \\
  w_3(\theta)
\end{bmatrix} = \begin{bmatrix}
  \Delta(\theta) \\
  \Delta(\theta + \frac{2\pi}{3}) \\
  \Delta(\theta + \frac{4\pi}{3})
\end{bmatrix},
\]

(2.4)

can be determined as a simple sum of cosines, sines and an offset

\[
w(\theta) \triangleq \begin{bmatrix}
w_1(\theta) \\
w_2(\theta) \\
w_3(\theta)
\end{bmatrix} = \begin{bmatrix} 1 & \cos(\theta) & \sin(\theta) \\ 1 & \cos(\theta + \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \\ 1 & \cos(\theta + \frac{4\pi}{3}) & \sin(\theta + \frac{4\pi}{3}) \end{bmatrix} \begin{bmatrix} w_u(\theta) \\ w_c(\theta) \\ w_s(\theta) \end{bmatrix}.
\]

(2.5)

The matrix \(T(\theta)\) is always invertible (it is actually an orthogonal matrix). Hence, given the blade-local wind speeds, the MBC component amplitudes can be computed using the MBC transformation \(T(\theta)^{-1}\)

\[
\begin{bmatrix}
w_{mbc}(\theta)
\end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} \cos(\theta) & \frac{2}{3} \cos(\theta + \frac{2\pi}{3}) & \frac{2}{3} \cos(\theta + \frac{4\pi}{3}) \\ \frac{2}{3} \sin(\theta) & \frac{2}{3} \sin(\theta + \frac{2\pi}{3}) & \frac{2}{3} \sin(\theta + \frac{4\pi}{3}) \end{bmatrix} \begin{bmatrix} w_1(\theta) \\ w_2(\theta) \\ w_3(\theta) \end{bmatrix}.
\]

(2.6)

If the wind speed variation across the rotor is truly planar, then the MBC component amplitudes are constant (with rotor position), and the blade-local wind speeds \(w(\theta_1)\) at one rotor position \(\theta_1\) can be used to compute the wind speeds at another rotor position \(\theta_2\) (at the same time instant) as

\[
w(\theta_2) = T(\theta_2)T(\theta_1)^{-1}w(\theta_1).
\]

(2.7)

The fact that in turbulent conditions the wind speed variation is never truly planar, is the reason
using fixed point measurements to interpolate speeds at other locations results in the larger errors seen in Fig. 2.1.

### 2.2 Linear Models

The starting point for obtaining a linear turbine model is simulation and linearization using FAST. There are options to produce coefficients for various input perturbations, but for the purposes of this discussion, the focus is on generator torque perturbations $\tau_g$, individual blade pitch perturbations

$$p(t) = \begin{bmatrix} p_1(t) & p_2(t) & p_3(t) \end{bmatrix}^T,$$

and uniform wind speed and linear vertical and horizontal wind shear perturbations

$$w_{sh}(t) = \begin{bmatrix} w_u(t) & \Delta_v(t) & \Delta_h(t) \end{bmatrix}^T.$$

In addition, FAST will derive coefficients to determine perturbations in out of plane bending moments,

$$m_r(t) = \begin{bmatrix} m_{r1}(t) & m_{r2}(t) & m_{r3}(t) \end{bmatrix}^T,$$

the torque $\tau_r$ applied to the hub side of the rotor shaft, and generator speed $\Omega_g$.

FAST provides the second order dynamics (displacement and velocity) for each mode shape modelling the flexure of the turbine structure, and these various degrees of freedom (DOF) can be switched on or off. For the purposes of controller design, a reduced set of DOF’s is enabled, that includes a first order damped mode for generator speed, a second order drive-train compliance, and a second order compliance describing the first out-of-plane bending mode of each blade.

It is arranged that the components of the state $x = [x_t^T \ x_r^T \ v_r^T]^T$ consist of the fixed frame quantities $x_t^T$ (e.g., drive train compliance) and the degrees of freedom (DOF) that rotate with the blades, organized into displacements $x_r$ and velocities $v_r$. This ordering is convenient in applying the dynamic MBC transformation to the rotating degrees of freedom as in Section 2.2.2. However, as described in Section 2.2.1, FAST linearizes the turbine with respect to the wind perturbations $w_{sh}$ that occur in fixed (non-rotating) reference frame. To obtain models for preview control with
point measurements of wind speed that rotate with the blades, the FAST linearizations are modified to obtain a model that represents the response to blade-local wind perturbations. In Section 2.2.2, the blade-local models are transformed so that all dynamics and wind perturbations occur in the fixed frame, in which case preview control can be done using perturbations in the MBC component amplitudes.

### 2.2.1 Standard Blade Local Linear Models

The FAST turbine model is simulated in steady, uniform wind conditions at a specified blade pitch and rotor speed until the time response reaches steady state, and then FAST computes a linearized model at a specified number of rotor positions. At each rotor position, the state-space model obtained from FAST is of the form

\[
\dot{x}(t) = A_F(\theta)x(t) + B_{Fsh}(\theta)w_{sh}(t) + \begin{bmatrix} B_{Fp}(\theta) & B_{Fg}(\theta) \end{bmatrix} \begin{bmatrix} p(t) \\ \tau_g(t) \end{bmatrix}, \tag{2.11a}
\]

\[
\begin{bmatrix} m_r(t) \\ \Omega_g(t) \\ \tau_r(t) \end{bmatrix} = \begin{bmatrix} C_{Fm}(\theta) \\ C_{Fg}(\theta) \\ C_{Fr}(\theta) \end{bmatrix} x(t) + \begin{bmatrix} D_{Fms}(\theta) \\ D_{Fgs}(\theta) \\ D_{Fts}(\theta) \end{bmatrix} w_{sh}(t) + \begin{bmatrix} D_{Fmp}(\theta) \\ D_{Fgp}(\theta) \\ D_{Ftp}(\theta) \end{bmatrix} \begin{bmatrix} p(t) \\ \tau_g(t) \end{bmatrix}, \tag{2.11b}
\]

where all coefficients are potentially dependent on the rotor position \( \theta \). In order to directly apply perturbations in the point wind measurements described in the previous section to the turbine model, we make the substitution

\[
w_{sh}(t) = M_{m2s}T(\theta)^{-1}w(t). \tag{2.12}
\]

This results in a linear model

\[
\dot{x}(t) = A_F(\theta)x(t) + B_{Fsh}(\theta)M_{m2s}T(\theta)^{-1}w(t) + \begin{bmatrix} B_{Fp}(\theta) & B_{Fg}(\theta) \end{bmatrix} \begin{bmatrix} p(t) \\ \tau_g(t) \end{bmatrix}, \tag{2.13a}
\]

\[
\begin{bmatrix} m_r(t) \\ \Omega_g(t) \\ \tau_r(t) \end{bmatrix} = \begin{bmatrix} C_{Fm}(\theta) \\ C_{Fg}(\theta) \\ C_{Fr}(\theta) \end{bmatrix} x(t) + \begin{bmatrix} D_{Fms}(\theta)M_{m2s}T(\theta)^{-1} \\ D_{Fgs}(\theta)M_{m2s}T(\theta)^{-1} \\ D_{Fts}(\theta)M_{m2s}T(\theta)^{-1} \end{bmatrix} w(t) + \begin{bmatrix} D_{Fmp}(\theta) \\ D_{Fgp}(\theta) \\ D_{Ftp}(\theta) \end{bmatrix} \begin{bmatrix} p(t) \\ \tau_g(t) \end{bmatrix}. \tag{2.13b}
\]
Presently, FAST does not model torsional dynamics along the pitch axis of each blade, and the blade pitch inputs set the individual blade positions instantaneously. In order to model the delay between commanded pitch and actual pitch, as well as to provide pitch rate outputs, the model above is augmented to include simple first order pitch actuator models

\[
\dot{x}_{pi}(t) = -a_p x_{pi}(t) + a_p p_{ci}(t),
\]

\[
\begin{bmatrix}
p_i \\
p_{ri}
\end{bmatrix} =
\begin{bmatrix}
1 \\
-a_p
\end{bmatrix} x_{pi}(t) + 
\begin{bmatrix}
0 \\
ap
\end{bmatrix} p_{ci}(t)
\]

where \(p_{ci}\) is the \(i\)th commanded pitch. Defining \(p_c = [p_{c1} \ p_{c2} \ p_{c3}]^T\), \(p_r = [p_{r1} \ p_{r2} \ p_{r3}]^T\), and \(x_p^T = [p_1 \ p_2 \ p_3]\), the parallel pitch actuator dynamics

\[
\dot{x}_p(t) = A_{Fpc} x_p(t) + B_{Fpc} p_c(t),
\]

\[
\begin{bmatrix}
p(t) \\
p_r(t)
\end{bmatrix} =
\begin{bmatrix}
C_{Fpc} \\
C_{Fpr}
\end{bmatrix} x_p(t) +
\begin{bmatrix}
0 \\
D_{pr}
\end{bmatrix} p_c(t),
\]

can be augmented to the turbine dynamics (eq. (2.14)) to obtain

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{x}_p(t)
\end{bmatrix} =
\begin{bmatrix}
A_F(\theta) & B_{Fp}(\theta) C_{Fpc} \\
0 & A_{Fpc}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
x_p(t)
\end{bmatrix} +
\begin{bmatrix}
B_{Fsh}(\theta) M_{m2s} T(\theta)^{-1} w(t) + [0 \ B_{Fg}(\theta) B_{Fpc} 0]
\end{bmatrix}
\begin{bmatrix}
p_c(t) \\
\tau_g(t)
\end{bmatrix},
\]

\[
\begin{bmatrix}
m_r(t) \\
\Omega_g(t) \\
\tau_r(t) \\
p_r(t)
\end{bmatrix} =
\begin{bmatrix}
C_{Fm}(\theta) & D_{Fmp}(\theta) C_{Fpc} \\
C_{Fg}(\theta) & D_{Fgp}(\theta) C_{Fpc} \\
C_{Ft}(\theta) & D_{Ftp}(\theta) C_{Fpc} \\
0 & C_{Fpr}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
x_p(t)
\end{bmatrix} +
\begin{bmatrix}
D_{Fmsh}(\theta) M_{m2s} T(\theta)^{-1} \\
D_{Fgsh}(\theta) M_{m2s} T(\theta)^{-1} \\
D_{Ftsh}(\theta) M_{m2s} T(\theta)^{-1} \\
0
\end{bmatrix}
\begin{bmatrix}
w(t) \\
p_c(t) \\
\tau_g(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
D_{pr}
\end{bmatrix}
\begin{bmatrix}
p_r(t) \\
\tau_r(t)
\end{bmatrix}.
\]

Now, if the state is redefined as \(x^T = [x_t^T, x_r^T, v_r^T, x_p^T]\), then with obvious definitions for sub-blocks,
the system can be represented more simply as

$$\dot{x}(t) = A(\theta) x(t) + B_w(\theta) w(t) + \begin{bmatrix} B_{pc} & B_g(\theta) \end{bmatrix} \begin{bmatrix} p_c(t) \\ \tau_g(t) \end{bmatrix},$$

(2.17a)

$$\begin{bmatrix} m_r(t) \\ \Omega_g(t) \\ \tau_r(t) \\ p_r(t) \end{bmatrix} = \begin{bmatrix} C_m(\theta) \\ C_g(\theta) \\ C_t(\theta) \\ C_{pr} \end{bmatrix} x(t) + \begin{bmatrix} D_{mw}(\theta) \\ D_{gw}(\theta) \\ D_{tw}(\theta) \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_c(t) \\ \tau_g(t) \end{bmatrix}.$$  

(2.17b)

Finally, for the purposes of designing for the blade-local turbine model, the models at each of $$N_g$$ equally spaced rotor azimuths $$\theta_i$$ are averaged together to obtain

$$\dot{x}(t) = A x + B_w w(t) + \begin{bmatrix} B_{pc} & B_g \end{bmatrix} \begin{bmatrix} p_c(t) \\ \tau_g(t) \end{bmatrix},$$

(2.18a)

$$\begin{bmatrix} m_r(t) \\ \Omega_g(t) \\ \tau_r(t) \\ p_r(t) \end{bmatrix} = \begin{bmatrix} C_m \\ C_g \\ C_t \\ C_{pr} \end{bmatrix} x + \begin{bmatrix} D_{mw} \\ D_{gw} \\ D_{tw} \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ \tau_g(t) \end{bmatrix}.$$  

(2.18b)

where, for example,

$$A = \frac{1}{N_\theta} \sum_{i=0}^{N_g} A(\theta_i).$$

(2.19)

### 2.2.2 Standard Multi-Blade Coordinate Models

The MBC transform is used on each degree of freedom (DOF) that rotates with the turbine blades as a function of azimuth $$\theta$$. Loosely speaking, the transform computes the cos/vertical and sin/horizontal amplitude of the 1P variation in the rotating variables. The basic transform is defined in terms of displacements; each DOF that rotates with or is inherently part of a blade has a displacement/state $$x_i$$. The goal is to express the differential equation (2.17) for the linear system in states/coordinates that do not rotate with the blades by expressing the rotating states as a function of their MBC/non-rotating counter parts. The rotating state $$x_r^T = [x_1, x_2, x_3]$$ is given by
the inverse MBC transform of corresponding non-rotating components \( x^T_{nr} = [x_u, x_c, x_s] \) according to

\[
x_r = T(\theta)x_{nr}.
\] (2.20)

The velocity \( v_i = \dot{x}_i \) associated with each DOF in a mechanical system requires computation of

\[
\dot{x}_r = \dot{T}(\theta) \Omega x_{nr} + T(\theta) \dot{x}_{nr}.
\] (2.21)

where \( \Omega = \dot{\theta} \) is the rotor speed. So, the second-order transformation to the rotating DOFs is given by

\[
\begin{bmatrix}
  x_r \\
  v_r \\
  x_p
\end{bmatrix} =
\begin{bmatrix}
  T(\theta) & 0 & 0 \\
  \dot{T}(\theta)\Omega & T(\theta) & 0 \\
  0 & 0 & T(\theta)
\end{bmatrix}
\begin{bmatrix}
  x_{nr} \\
  x_{nr} \\
  b_{nr}
\end{bmatrix},
\] (2.22)

where \( v_{nr} = \dot{x}_{nr} \) and

\[
\dot{T}(\theta) =
\begin{bmatrix}
  0 & -\sin(\theta) & \cos(\theta) \\
  0 & -\sin(\theta + \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\
  0 & -\sin(\theta + \frac{4\pi}{3}) & \cos(\theta + \frac{4\pi}{3})
\end{bmatrix}.
\] (2.23)

The six dimensional transformation (2.22) must be replicated for each rotating degree of freedom. For the sake of simplicity, we assume there is only one rotating DOF in the turbine model state, but we add the three, parallel, first-order models of the pitch actuators that rotate with the blades. Including these as part of the rotating system requires that we modify the transformation to

\[
\begin{bmatrix}
  x_r \\
  v_r \\
  x_p
\end{bmatrix} =
\begin{bmatrix}
  T(\theta) & 0 & 0 \\
  \dot{T}(\theta)\Omega & T(\theta) & 0 \\
  0 & 0 & T(\theta)
\end{bmatrix}
\begin{bmatrix}
  x_{nr} \\
  b_{nr} \\
  x_{pnr}
\end{bmatrix},
\] (2.24)

where \( x_p \) contains the pitch angles produced by the three actuators and \( x_{pnr} \) contains the collective, cosine and sine components of the pitch in the non-rotating frame. This completes the transformation between non-rotating and rotating states. Now, recalling that all the other (non-rotating) turbine states are lumped into the vector \( x_t \), the complete (non-dynamic) MBC state
transformation is given by
\[
\begin{align*}
\begin{bmatrix} x_t \\ x_r \\ v_r \\ x_p \end{bmatrix} &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & T(\theta) & 0 & 0 \\ 0 & \dot{T}(\theta)\Omega & T(\theta) & 0 \\ 0 & 0 & 0 & T(\theta) \end{bmatrix} \begin{bmatrix} x_t \\ x_{nr} \\ v_{nr} \\ x_{pnr} \end{bmatrix} \\
x &= M(\theta)\dot{x}.
\end{align*}
\] (2.25)

We now make use of (2.25) to derive the state-space representation of the linear differential equation in non-rotating/MBC coordinates. This is accomplished by first taking the time derivative of both sides of (2.25),

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \dot{T}(\theta)\Omega & 0 & 0 \\ 0 & \dot{T}(\theta)\Omega^2 & 2\dot{T}(\theta)\Omega & 0 \\ 0 & 0 & 0 & \dot{T}(\theta)\Omega \end{bmatrix} \begin{bmatrix} x_t \\ x_{nr} \\ v_{nr} \\ x_{pnr} \end{bmatrix} + \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & T(\theta) & 0 & 0 \\ 0 & 0 & T(\theta) & 0 \\ 0 & 0 & 0 & T(\theta) \end{bmatrix} \begin{bmatrix} \dot{x}_t \\ \dot{x}_{nr} \\ \dot{v}_{nr} \\ \dot{x}_{pnr} \end{bmatrix} \\
&= \bar{M}(\theta)\dot{x} + M_d(\theta)\frac{d}{dt}\bar{\dot{x}}
\end{align*}
\] (2.26)

where
\[
\bar{T}(\theta) = \begin{bmatrix} 0 & -\cos(\theta) & -\sin(\theta) \\ 0 & -\cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \\ 0 & -\cos(\theta + \frac{4\pi}{3}) & -\sin(\theta + \frac{4\pi}{3}) \end{bmatrix},
\] (2.27)

and simplifications (to get the diagonal matrix \(M_d(\theta)\)) are obtained by assuming \(\dot{\Omega} = 0\) and utilizing the fact that \(\dot{x}_{nr} = v_{nr}\). Equating this result with the right hand side of (2.17a), substituting in
(2.25) for \( x \) and solving for the time derivative of \( \dot{x} \) gives

\[
\frac{d}{dt} \dot{x} = M_d(\theta)^{-1}(A(\theta)M(\theta) - \bar{M}(\theta))\dot{x} + M_d^{-1}(\theta)[B_w(\theta) \ B_p \ B_g(\theta)] \begin{bmatrix} w \\ p_c \\ \tau_g \end{bmatrix},
\]

\[
= M_d(\theta)^{-1}(A(\theta)M(\theta) - \bar{M}(\theta))\dot{x} + M_d^{-1}(\theta)[B_w(\theta) \ B_p \ B_g(\theta)] \begin{bmatrix} T(\theta) & 0 & 0 \\ 0 & T(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_mbc \\ p_{mbc} \\ \tau_g \end{bmatrix},
\]

\[
\dot{x} = A_{MBC}(\theta)\dot{x} + [B_{wMBC}(\theta) \ B_{pMBC}(\theta) \ B_g(\theta)] \begin{bmatrix} w_mbc \\ p_{mbc} \\ \tau_g \end{bmatrix}. \tag{2.28}
\]

In a similar fashion for the outputs, we obtain

\[
\begin{bmatrix} m_{mbc} \\ \Omega_g \\ \tau_r \\ p_{trimbc} \end{bmatrix} = \begin{bmatrix} T(\theta)^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & T(\theta)^{-1} \end{bmatrix} \begin{bmatrix} C_m(\theta) \\ C_g(\theta) \\ C_l(\theta) \\ C_{pr} \end{bmatrix} \begin{bmatrix} w_mbc \\ p_{mbc} \\ \tau_g \end{bmatrix}
\]

\[
\begin{bmatrix} T(\theta)^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & T(\theta)^{-1} \end{bmatrix} \begin{bmatrix} D_{mw}(\theta) & 0 & 0 \\ 0 & D_{gw}(\theta) & 0 \\ 0 & 0 & D_{tw}(\theta) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T(\theta) & 0 & 0 \\ 0 & T(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_mbc \\ p_{mbc} \\ \tau_g \end{bmatrix},
\]

\[
\begin{bmatrix} C_{mMBC}(\theta) \\ C_{gMBC}(\theta) \\ C_{rMBC}(\theta) \end{bmatrix} \dot{x} + \begin{bmatrix} D_{nwMBC}(\theta) & 0 & 0 \\ 0 & D_{gwMBC}(\theta) & 0 \\ 0 & 0 & D_{twMBC}(\theta) \end{bmatrix} \begin{bmatrix} w_mbc \\ p_{mbc} \\ \tau_g \end{bmatrix}. \tag{2.29b}
\]

An azimuth-averaged MBC state-space system is obtained from the complete set of linearizations.
in the same manner as for the non-MBC case to obtain

\[
\frac{d}{dt} \dot{x}(t) = A_{MBC} \dot{x}(t) + B_{wMBC} w_{mbc}(t) + [B_{pMBC} B_g] \begin{bmatrix} p_{mbc}(t) \\ \tau_g(t) \end{bmatrix},
\]

(2.30a)

\[
\begin{bmatrix} m_{mbc}(t) \\ \Omega_g(t) \\ \tau_r(t) \\ p_{rmbc}(t) \end{bmatrix} = \begin{bmatrix} C_{mMBC} & D_{mwMBC} \\ C_{gMBC} & D_{gwMBC} \\ C_{rMBC} & D_{twMBC} \\ D_{prwMBC} & D_{prMBC} \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ w_{mbc}(t) + \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{mbc}(t) \\ \tau_g(t) \end{bmatrix}.
\]

(2.30b)
Model predictive control (MPC) refers to a technique also known as receding horizon control, although the latter term seems to be more common where continuous time controllers are employed. MPC can fairly accurately be described as the optimization of a sequence of controls over a finite time horizon, and then only the beginning of the sequence is applied to the system while the optimization is repeated with a new set of measurements. Methods vary depending on
the cost function employed (this thesis uses primarily a quadratic cost), the type of model used for prediction, and various trade-offs between prediction and control horizons. For a historical perspective see [58], and for background of a more tutorial nature see [59].

One of the advantages of MPC is that most formulations explicitly address constraints on control actuation and, because the prediction horizon in usually finite, standard formulations also handle time varying and LPV system descriptions. The objective in this thesis is to develop the system in Fig. 3.1. In this application it is assumed that wind speeds approaching the turbine are measured using an advanced technology like light detection and ranging (LIDAR). The wind speed then determines operating points and a schedule of linearized models that are valid over the prediction horizon used to compute optimal, constrained control actions. In this regard, the MPC method proposed is akin to a scheduled linear controller where the time variation is determined by an exogenous input; therefore, the scheduling variable does not constitute feedback as is often the case in linear parameter varying (LPV) systems [3]. Hence, the problem is similar to the design of a time varying controller applied along a known trajectory. The selection of a plausible trajectory for the turbine is discussed in Chapter 7. In this chapter, the basic cost functions and the method of their optimization is provided. The approach developed in this chapter is then combined with the offset-free methods developed in Chapter 4, and a simple example is provided in Section 4.4 before applying the methods to the wind turbine in Chapter 7.

As described in Section 3.1, the primary optimization uses (time varying) linear models and a quadratic objective function with inequality constraints, but the method of solution requires that the initial guess at a solution to satisfy the inequalities (it must be feasible). However, since it is desirable to constrain the turbine’s rotor speed below an acceptable maximum (and above a minimum), it is necessary to consider the case that the turbine may end up in a state at which an over speed fault occurs. In this case, inequality constraints are violated (infeasible) from the outset. This event is handled by employing, if necessary, a pre-optimization that brings control actions and rotor speed as close as possible to acceptable limits. So, if the initial guess is not feasible, the \( \ell_\infty \) pre-optimization described in Section 3.2 is performed to try and find a feasible starting point. If
this step fails to bring all variables within prescribed limits, then it is assumed that the result of the pre-optimization is the best possible solution, and its solution is used to provide the control input to the plant.

### 3.1 Linear Quadratic MPC with Inequality Constraints

In the linear-quadratic MPC method employed in this thesis, the plant model is a known time-varying linear system

\[
x(k + 1) = A(k)x(k) + B_u(k)u(k) + B_w(k)w(k), \quad \text{(3.1a)}
\]

\[
z(k) = C_z(k)x(k) + D_{zu}(k)u(k) + D_{zw}(k)w(k), \quad \text{(3.1b)}
\]

with \(x(1)\) given, where \(u(k)\) is the control to be determined, and where \(w(k)\) is a vector of exogenous inputs that are known, but not controllable. The models (e.g., \(\{A(k)\}\)) are scheduled according to the preview of mean wind speeds available over the MPC prediction horizon.

The controls (and system response) are optimized to minimize a quadratic cost

\[
J_0(u, x) = \frac{1}{2} x(N + 1)^T \Pi_{N+1} x(N + 1) + x(N + 1)^T r_x(N + 1) + \frac{1}{2} \sum_{k=1}^{N} z(k)^T Q_z(k) z(k), \quad \text{(3.2a)}
\]

\[
= x(N + 1)^T \Pi_{N+1} x(N + 1) + r_x(N + 1)^T x(N + 1)
\]

\[
+ \sum_{k=1}^{N} \frac{1}{2} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} C_z(k)^T Q_z(k) C_z(k) & C_z(k)^T Q_z(k) D_{zu}(k) \\
D_{zu}(k)^T Q_z(k) C_z(k) & D_{zu}(k)^T Q_z(k) D_{zu}(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} Q_x(k) & S_{xz}(k) \\
S_{xz}(k)^T & R_u(k) \end{bmatrix} \begin{bmatrix} w(k) \\ u(k) \end{bmatrix} + w(k)^T D_{zw}(k)^T Q_z(k) D_{zw}(k) w(k), \quad \text{(3.2b)}
\]
The terms in red are constant, independent of the optimization variables \( \{u(k), k \in [1, N]\} \) and \( \{x(k), k \in [2, N+1]\} \), and are dropped so that the cost to be optimized becomes

\[
J(u, x) = \frac{1}{2} u(1)^T R_u(1)(1) + u(1)^T \left( S_{xu}(1)^T x(1) + S_{uw}(1) w(1) \right) + \frac{1}{2} x(N+1)^T \Pi_{N+1} x(N+1) + x(N+1)^T r_x(N+1) + \sum_{k=2}^{N} \frac{1}{2} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q_x(k) & S_{xu}(k) \\ S_{xu}(k)^T & R_u(k) \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + \begin{bmatrix} x(k)^T \\ u(k)^T \end{bmatrix} \begin{bmatrix} S_{xw}(k) \\ S_{uw}(k) \end{bmatrix} w(k). \tag{3.3a}
\]

It is assumed that the cost is to be minimized is subject to constraints on control effort

\[
y_{ctl}(k) = C_{ctl} x(k) + D_{ctlu}(k) u(k) + D_{ctlw}(k) w(k), \tag{3.4}
\]

where \( D_{ctlu} \) is injective (e.g., is tall and has full column rank). These “control specific” constraints provide for limits on control effort, rate, etc.; in normal operation, if these are feasible at one sample hit, then they are guaranteed to be feasible at the next sample hit. The form of eq. (3.4) provides for limits on total control effort where \( C_{ctl} \) may be determined by nominal state-feedback and \( w(k) \) can include exogenous controls like those from preview compensation.

In addition, there may also be constraints on system outputs or states

\[
y_e(k) = C_c x(k) + D_{cu}(k) u(k) + D_{cw}(k) w(k), \tag{3.5}
\]

The constraints on these quantities may or may not be feasible and typically include \( y_{ctl} \) as a subset
of $y_c$. The constraints are expressed as

\begin{align}
C_{ctl}(1)x(1) + D_{ctlu}(1)u(1) + D_{ctlw}(k)w(1) & \leq Y_{ctlmx}(1), & (3.6a) \\
-(C_{ctl}(k)x(1) + D_{ctlu}(1)u(1) + D_{ctlw}(1)w(1)) & \leq -Y_{ctlmn}(1), & (3.6b) \\
C_c(k)x(k) + D_{cu}(k)u(k) + D_{cw}(k)w(k) & \leq Y_{max}(k), & (3.6c) \\
-(C_c(k)x(k) + D_{cu}(k)u(k) + D_{cw}(k)w(k)) & \leq -Y_{min}(k), & (3.6d) \\
C_f x(N+1) & \leq f, & (3.6e)
\end{align}

where the last inequality typically requires the final state to be in some acceptable polygon. The limits are themselves functions of the time index $k$, because, as explained in Chapter 7, they are computed relative to the scheduled set-points.

The optimization variables can be combined into a single optimization vector

$$z \triangleq [u(1)^T \ x(2)^T \ldots \ u(N)^T \ x(N+1)^T]^T,$$

so that the cost can be expressed as

$$f_0(z) = \frac{1}{2} z^T \left[ Q_u \ 0 \ 0 \ldots \ 0 \\
0 \ 0 \ Q_x \ S_{xu} \ldots \ 0 \\
0 \ S_{xu}^T \ Q_u \ \ldots \ 0 \\
\vdots \ \vdots \ \vdots \ \ddots \ \vdots \\
0 \ 0 \ 0 \ \ldots \ \Pi_{N+1}
\right] z + \left[ g_u(1)^T \ g_x(2)^T \ldots \ g_u(N)^T \ g_x(N+1)^T \right] g^T z,$$

$$= \frac{1}{2} z^T Hz + g^T z,$$

where

$$
\begin{bmatrix}
g_u(1) \\
g_x(2) \\
\vdots \\
g_u(N) \\
g_x(N+1)
\end{bmatrix} =
\begin{bmatrix}
S_{xu}(1)x(1) + S_{wu}(1)w(1) \\
S_{xw}(2)w(2) \\
\vdots \\
S_{wu}(N)w(N) \\
r_x(N+1)
\end{bmatrix}.$$
In similar fashion, the state dynamics (3.1a) and inequality constraints (3.6) can be expressed in terms of the optimization vector as

\[
\begin{bmatrix}
-B_u(1) & I & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & -A(2) & -B_u(2) & I & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -A(N) & -B_u(N) & I \\
\end{bmatrix}
\begin{bmatrix}
u(1) \\
x(2) \\
\vdots \\
x(N + 1) \\
\end{bmatrix}
= \begin{bmatrix} A(1) x(1) + B_u(1) w(1) \\
B_u(2) w(2) \\
\vdots \\
B_u(N) w(N) \\
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
D_{ctlw}(1) & 0 & 0 & \ldots & 0 \\
-D_{ctlw}(1) & 0 & 0 & \ldots & 0 \\
0 & C_c(2) & D_{cu}(2) & \ldots & 0 \\
0 & -C_c(2) & -D_{cu}(2) & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & C_f \\
\end{bmatrix}
\begin{bmatrix}
u(1) \\
x(2) \\
\vdots \\
x(N + 1) \\
\end{bmatrix}
\leq \begin{bmatrix} Y_{ctlmax}(1) - D_{ctlw}(1) w(1) - C_{ctl}(1) x(1) \\
-Y_{ctlmin}(1) + D_{ctlw} w(1) + C_{ctl}(1) x(1) \\
Y_{max}(2) - D_{cw}(2) w(2) \\
-Y_{min}(2) + D_{cw}(2) w(2) \\
\vdots \\
f \\
\end{bmatrix}
\]

So, the optimization to be performed is

\[
\min_z \ f(z) = \frac{1}{2} z^T H z + g^T z \\
\text{subj: } C_{eq} z = b \\
P z \leq h.
\]

Since the cost function derives from \( z(k)^T Q z(k) \), \( H \) is at least semi-positive definite. This also insures that the objective is convex, as are the constraints, and so a minimizer is given by any
solution \( \{ z, \nu, \lambda \} \) of the KKT conditions [15]

\[
Hz + g + C_{eq}^T \nu + P^T \lambda = 0,
\]

\( C_{eq} z = b \) \hspace{1cm} (3.13a)

\[
P z - h \leq 0, \hspace{1cm} (3.13b)
\]

\[
\Lambda (P z - h) = 0, \hspace{0.5cm} \Lambda \equiv \text{diag}(\lambda),
\]

\[
\lambda \geq 0.
\]

A solution to this problem can be obtained using the simple primal-dual approach provided in [15]. The KKT conditions are modified to

\[
Hz + g + C_{eq}^T \nu + P^T \lambda = 0
\]

\[
C_{eq} z - b = 0
\]

\[
\Lambda (P z - h) + t_{pd} = 0
\]

where \( t_{pd} > 0 \). A solution of these modified KKT conditions is found using a gradient search, and approaches a solution to the original conditions as \( t_{pd} \to 0 \), provided that

\[
d \equiv P z - h < 0,
\]

\[
\lambda > 0.
\]

During the gradient search, care is taken to insure that these inequalities hold, but the initial guess for \( z \) must insure \( d < 0 \) (it must be feasible).

Given a value for \( t_{pd} \), the problem is essentially reduced to that of finding a root to the modified KKT conditions (3.14). Define

\[
r(z, \nu, \lambda) = \begin{bmatrix} r_z \\ r_\nu \\ r_\lambda \end{bmatrix} = \begin{bmatrix} Hz + g + C_{eq}^T \nu + P^T \lambda \\ C_{eq} z - b \\ \Lambda (P z - h) + t_{pd} \end{bmatrix},
\]

\[(3.16)\]
then a linear approximation at the present value of \( \{ z, \nu, \lambda \} \) is

\[
\begin{bmatrix}
r_z \\
r_\nu \\
r_\lambda \\
\end{bmatrix} \approx \begin{bmatrix} H & C_{eq} & P^T \\ C_{eq} & 0 & 0 \\ \Lambda P & 0 & D \\
\end{bmatrix} \begin{bmatrix}
\Delta z \\
\Delta \nu \\
\Delta \lambda \\
\end{bmatrix}, \quad D = \text{diag}(d),
\]

(3.17)

and a “descent” direction \( \{ \Delta z, \Delta \nu, \Delta \lambda \} \) is found by solving

\[
\begin{bmatrix} H & C_{eq}^T & P^T \\ C_{eq} & 0 & 0 \\ \Lambda P & 0 & D \\
\end{bmatrix} \begin{bmatrix}
\Delta z \\
\Delta \nu \\
\Delta \lambda \\
\end{bmatrix} = \begin{bmatrix}
-r_z \\
-r_\nu \\
-r_\lambda \\
\end{bmatrix}.
\]

(3.18)

So, the primal-dual algorithm suggested in [15] proceeds as follows.

- Given \( z \) such that \( d < 0 \), choose \( \lambda > 0 \).

- Repeat:

  1. Set \( t_{pd} = -d^T \lambda / \mu \) (typically \( \mu \approx 10 \)).
  2. Solve eq. (3.18) to obtain a search direction \( \{ \Delta z, \Delta \nu, \Delta \lambda \} \).
  3. With \( \gamma < 1 \), backtrack line search using parameter \( t_{bl} \leq 1 \) until

\[
P(z + t_{bl} \Delta z) - h < 0,
\]

(3.19a)

\[
\lambda + t_{bl} \Delta \lambda > 0,
\]

(3.19b)

\[
\|r(z + t_{bl} \Delta z, \nu + t_{bl} \Delta \nu, \lambda + t_{bl} \Delta \lambda)\| \leq (1 - t_{bl} \gamma) \|r(z, \nu, \lambda)\|.
\]

(3.19c)

  4. Then set

\[
z := z + t_{bl} \Delta z,
\]

(3.20a)

\[
\nu := \nu + t_{bl} \Delta \nu,
\]

(3.20b)

\[
\lambda := \lambda + t_{bl} \Delta \lambda.
\]

(3.20c)

- Stop when

\[
\|r_z(z, \nu, \lambda)\| + \|r_\nu(z, \nu, \lambda)\| < \epsilon_{feas},
\]

(3.21a)

\[
-d^T \lambda < \epsilon.
\]

(3.21b)
As long as $d < 0$, then $D^{-1}$ exists and simple row and column operations can be used to show that a descent direction can be found by solving

$$
\begin{bmatrix}
H - P^T D^{-1} \Lambda P & C_{eq}^T \\
C_{eq} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta z \\
\Delta \nu
\end{bmatrix}
= \begin{bmatrix}
P^T D^{-1} r_\lambda - r_z \\
-r_\nu
\end{bmatrix}.
$$

(3.22)

This allows $\{\Delta z, \Delta \nu\}$ to be obtained as solutions of

$$
\begin{bmatrix}
H - P^T D^{-1} \Lambda P & C_{eq}^T \\
C_{eq} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta z \\
\Delta \nu
\end{bmatrix}
= \begin{bmatrix}
P^T D^{-1} r_\lambda - r_z \\
-r_\nu
\end{bmatrix},
$$

(3.23)

and then used to compute

$$
\Delta \lambda = -D^{-1} r_\lambda - D^{-1} \Lambda P \Delta z.
$$

(3.24)

As shown in Appendix B, there is a significant advantage in that eq. (3.23) can be solved efficiently using a Riccati recursion. Since this method requires an initial $z$ that is feasible ($d < 0$), the next section presents a method for finding such a $z$ when one exists.

### 3.2 Feasibility and $\ell_\infty$ MPC

To determine the combined feasibility of eq. (3.11) and eq. (3.10), the primal-dual approach is applied to the problem

$$
\min_{s,z} f(s,z) = s,
$$

(3.25a)

subj:

$$
C_{eq} z - b = 0,
$$

(3.25b)

$$
P z - h - cs \leq 0.
$$

(3.25c)

where typically $c = 1$ (the all-ones vector). The initial value for $s$ can always be chosen so that $P z - h - cs \leq 0$. This slack variable essentially transforms hard/required constraints into (soft) objectives that are not necessarily satisfied in a solution of this new problem. If at any time $s < 0$ during the search for a solution, then there is a feasible $z$ for the original LQ inequality constraints.
This feasibility problem is solved in two passes. The first pass breaks the problem up into shorter horizons and solves for control actions that satisfy only the constraints that are guaranteed feasible by the assumed form for $D_{ctlu}$. This provides a starting control sequence that is feasible with respect to $\{Y_{ctlmx}, Y_{ctlmn}\}$, but not necessarily with respect to $\{Y_{max}, Y_{min}\}$. In the second pass, the vector $c$ is then chosen so that the control constraints are hard and remain feasible while a search determines the feasibility of the state and/or output constraints that are not guaranteed.

The motivation for using two passes is that it may not be possible to satisfy system output constraints (e.g., where $D_{cu}$ is not surjective). In such an event the minimal value of $s$ may be quite large, possibly leaving considerable violations of the control constraints even when they are perfectly feasible. So, given $x(1)$, the first pass solves the simpler problems

$$\min_{s, z(k)} \ f(s, z(k)) = s, \quad (3.26)$$

subject to

$$C_{eq}(k) \begin{bmatrix} z(k) \\ -B_u(k) I \end{bmatrix} + \begin{bmatrix} u(k) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} b(k) \\ A(k) x(k) + B_w(k) w(k) \end{bmatrix}, \quad (3.27a)$$

$$\begin{bmatrix} D_{ctlu}(k) & 0 \\ -D_{ctlu}(k) & 0 \end{bmatrix} \begin{bmatrix} u(k) \\ x(k+1) \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} s \leq \begin{bmatrix} Y_{ctlmx}(k) - D_{ctlw}(k) w(k) - C_{ctl}(k) x(k) \\ -Y_{ctlmn}(k) + D_{ctlw}(k) w(k) + C_{ctl}(k) x(k) \end{bmatrix}, \quad (3.27b)$$

and then the next stage is initialized using $x(k+1)$ from this stage. This gives a control sequence which satisfies the control specific constraints.

With a feasible control sequence in hand, a second pass solves the larger problem

$$\min_{s, z} \ f(s, z) = s, \quad (3.28)$$
subject to

$$\begin{bmatrix}
D_{ctlu}(1) & 0 & 0 & \cdots & 0 \\
-D_{ctlu}(1) & 0 & 0 & \cdots & 0 \\
0 & C_c(2) & D_{cu}(2) & \cdots & 0 \\
0 & -C_c(2) & -D_{cu}(2) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & C_f
\end{bmatrix} \begin{bmatrix}
u(1) \\
x(2) \\
\vdots \\
x(N+1)
\end{bmatrix} = \begin{bmatrix}
0 \\
-\bar{c} \\
-\bar{c} \\
\vdots \\
1
\end{bmatrix},$$

$$C_{eq}z = b, \quad (3.29a)$$

$$\begin{bmatrix}
Y_{ctlmx}(1) - D_{ctlw}(1)w(1) - C_{ctl}(1)x(1) \\
-Y_{ctlmx}(1) + D_{ctlw}(1)w(1) + C_{ctl}(1)x(1) \\
Y_{max}(2) - D_{cw}(2)w(2) \\
-Y_{min}(2) + D_{cw}(2)w(2) \\
f
\end{bmatrix},$$

$$\begin{bmatrix}
0 \\
-\bar{c} \\
-\bar{c} \\
\vdots \\
1
\end{bmatrix} \begin{bmatrix}
Pz \\
-h \\
-s
\end{bmatrix} < 0. \quad (3.29b)$$

where $\bar{c}$ applies the variable $s$ only to those constraints that are not guaranteed feasible; in effect, the control specific constraints are “hard” during this second pass optimization. Since the first pass insures that the control specific constraints are feasible from the outset, they remain so as the controls are adjusted to minimize $s$. If a positive value for $s$ is found to be optimal, this only implies that the output/state constraints are infeasible.

The initial value for $s$ can always be chosen large enough to insure $d \geq Pz - h - cs < 0$. Hence, with an easily computable initial guess that is feasible, the task is to find a solution to the modified KKT conditions

$$\begin{bmatrix}
r_s \\
r_z \\
r_\nu \\
r_\lambda
\end{bmatrix} = \begin{bmatrix}
1 - c^T \lambda \\
C_{eq}^T \nu + P^T \lambda \\
C_{eq}z - b \\
\Lambda d + t_{pd}
\end{bmatrix} = 0. \quad (3.30)$$

When the simple primal-dual algorithm is applied, the termination condition is modified to also exit if at any point it is found that $s < 0$. Upon exit, if $s > 0$, then a feasible starting condition for the LQ optimization of Section 3.1 cannot be found.
For the $\ell_\infty$ objective, a descent direction is found by solving
\[
\begin{bmatrix}
0 & 0 & 0 & -c^T \\
0 & 0 & C_{eq}^T & P^T \\
0 & C_{eq} & 0 & 0 \\
-Ac & Ap & 0 & D
\end{bmatrix}
\begin{bmatrix}
\Delta_s \\
\Delta_z \\
\Delta_\nu \\
\Delta_\lambda
\end{bmatrix}
= \begin{bmatrix}
r_s \\
r_z \\
r_\nu \\
r_\lambda
\end{bmatrix},
\] (3.31)

which, using elementary row and column operations, can be written in the equivalent form
\[
\begin{bmatrix}
-c^T D^{-1} Ac & c^T D^{-1} Ap & 0 \\
P^T D^{-1} Ac & -P^T D^{-1} Ap & C_{eq}^T \\
0 & C_{eq} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta_s \\
\Delta_z \\
\Delta_\nu
\end{bmatrix}
= \begin{bmatrix}
r_s + c^T D^{-1} r_\lambda \\
r_z - P^T D^{-1} r_\lambda \\
r_\nu
\end{bmatrix},
\] (3.32a)

\[\Delta_\lambda + D^{-1} Ap \Delta_z - D^{-1} Ac \Delta_s = -D^{-1} r_\lambda.\] (3.32b)

Unfortunately, there is no block manipulation that will simplify eq. (3.32a). However, it is possible to solve for a search direction using two iterations of the Riccati recursion as follows. Solving the lower two rows
\[
\begin{bmatrix}
-P^T D^{-1} Ap & C_{eq}^T \\
C_{eq} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta_z \\
\Delta_\nu
\end{bmatrix}
= \begin{bmatrix}
r_z - P^T D^{-1} r_\lambda - P^T D^{-1} Ac \Delta_s \\
r_\nu
\end{bmatrix}
\] (3.33)

results in expressions that are affine in $\Delta_s$:
\[
\begin{bmatrix}
\Delta_z \\
\Delta_\nu
\end{bmatrix}
= \begin{bmatrix}
z_0 \\
v_0
\end{bmatrix}
+ \begin{bmatrix}
z_1 \\
v_1
\end{bmatrix} \Delta_s.
\] (3.34)

The vector coefficients are obtained by using the Riccati recursion to solve
\[
\begin{bmatrix}
z_0 \\
v_0
\end{bmatrix}
= -\begin{bmatrix}
-P^T D^{-1} Ap & C_{eq}^T \\
C_{eq} & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
r_z - P^T D^{-1} r_\lambda \\
r_\nu
\end{bmatrix},
\] (3.35)

and then again to solve
\[
\begin{bmatrix}
z_1 \\
v_1
\end{bmatrix}
= -\begin{bmatrix}
-P^T D^{-1} Ap & C_{eq}^T \\
C_{eq} & 0
\end{bmatrix}^{-1}
\begin{bmatrix}
r_z - P^T D^{-1} r_\lambda - P^T D^{-1} Ac \\
r_\nu
\end{bmatrix}
- \begin{bmatrix}
z_0 \\
v_0
\end{bmatrix},
\] (3.36)
The former equation is obtained from eq. (3.33) using $\Delta_s = 0$, and the latter is obtained using $\Delta_s = 1$. That the Riccati recursion can be applied follows from the assumed form for $D_{ctlu}$ as long as it is also a sub-block of $D_{cu}$ (see Appendix B; this insures that $R_u > 0$).

The top row of eq. (3.32a) shows that

$$-c^T D^{-1} \Lambda c \Delta_s = -r_s - c^T D^{-1} r_\lambda - c^T D^{-1} \Lambda P \Delta z.$$  \hfill (3.37)

Now it is possible to substitute in $\Delta z = z_0 + z_1 \Delta_s$ and solve for $\Delta_s$

$$\Delta_s = \frac{1}{c^T D^{-1} \Lambda c} (r_s + c^T D^{-1} r_\lambda + c^T D^{-1} \Lambda P(z_0 + z_1 \Delta_s))$$  \hfill (3.38a)

$$\Rightarrow \left(1 - \frac{c^T D^{-1} \Lambda P z_1}{c^T D^{-1} \Lambda c}\right) \Delta_s = \frac{1}{c^T D^{-1} \Lambda c} (r_s + c^T D^{-1} r_\lambda + c^T D^{-1} \Lambda P z_0)$$  \hfill (3.38b)

$$\Rightarrow \Delta_s = \frac{(r_s + c^T D^{-1} r_\lambda + c^T D^{-1} \Lambda P z_0)}{c^T D^{-1} \Lambda c - c^T D^{-1} \Lambda P z_1}$$  \hfill (3.38c)
Chapter 4

Disturbance Accommodating Control for Output Regulation and MPC

The version of output regulation considered in this work characterizes the conditions under which system outputs asymptotically go to zero in response to certain classes of persistent inputs, and where the system is internally stable. Other variations of the problem can be considered where the system is stable only in an input-output sense [71], but this case is not pursued in this work. The standard technique for achieving regulation in MPC systems, known as “offset-free MPC” [9], is closely related to disturbance accommodating control (DAC) [32] in that both use an observer to estimate the state of an exogenous system that generates a persistent excitation. DAC was originally formulated to address any persistent signal that can be generated by a marginally stable linear system (e.g., a continuous time system with poles on the imaginary axis, or a discrete-time system with poles on the unit circle), by application of a feedback gain to the state of the disturbance model in the observer. However, in its original formulation, DAC did not achieve output regulation except in special cases. Offset-free MPC was originally formulated to mitigate persistent offsets at system outputs, but applied a computation of a “target” or “reference shift” each sample hit, instead of using a static gain. This technique was recently [52] generalized to handle the same general class of disturbances as DAC. In this chapter, we show that the reference shifting computation can be replaced with a static gain. In the course of characterizing the necessary conditions of the resulting observer-controller, an internal model principle is derived. This chapter provides an abridged development of the principle, and Appendix C gives a full formal proof.
4.1 DAC and the Regulator Equation

4.1.1 The Regulator Equation

In order to determine the features of the controller that are necessary to achieve regulation without exact knowledge of how the disturbance couples into the system, we consider a state space system with control input \( u(k) \),

\[
\begin{bmatrix}
  x_t(k+1) \\
  x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
  A & B_d \\
  0 & A_d
\end{bmatrix}
\begin{bmatrix}
  x_t(k) \\
  x_d(k)
\end{bmatrix} +
\begin{bmatrix}
  B_u \\
  0
\end{bmatrix} u_t(k) +
\begin{bmatrix}
  B_w \\
  0
\end{bmatrix} w_t(k),
\]

\( (4.1a) \)

where \( A_d \) represents an unstable exogenous system with \( n_\lambda \) distinct eigenvalues \( \{ \lambda_1, \ldots, \lambda_{n_\lambda} \} \triangleq \sigma(A_d) \), and the “\( t \)” denotes “total” response, as apposed to the asymptotic state response which is derived shortly. The input \( w_t(k) \) represents other exogenous inputs to the system that may be measurable, but for the majority of this chapter, it is assumed that \( w_t(k) = 0 \). If we let \( \mathbb{C}_g \) represent the region of the complex plane in which stable poles reside (e.g., for discrete-time systems, the interior of the unit circle), then \( \sigma(A_d) \notin \mathbb{C}_g \). It is also assumed that all vectors take values in finite dimensional spaces

\[
\begin{align*}
  u_t \in \mathcal{U}, & \quad \dim(\mathcal{U}) = N_u, & & w_t \in \mathcal{W}, & \quad \dim(\mathcal{W}) = N_w, \\
  y_t \in \mathcal{Y}, & \quad \dim(\mathcal{Y}) = N_y, & & x_t \in \mathcal{X}, & \quad \dim(\mathcal{X}) = N_x, \\
  x_d \in \mathcal{X}_d, & \quad \dim(\mathcal{X}_d) = N_d, & & x_tc \in \mathcal{X}_c, & \quad \dim(\mathcal{X}_c) = N_c.
\end{align*}
\]

\( (4.2a) \)
The control $u(k)$ is assumed to be generated using feedback according to

$$x_{tc}(k) = A_c x_{tc}(k) + \begin{bmatrix} B_{c1} & B_{c2} \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix},$$

(4.3a)

$$u_t(k) = C_c x_{tc}(k) + \begin{bmatrix} D_{c1} & D_{c2} \end{bmatrix} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}.$$  

(4.3b)

Direct feed-through $D_c \neq 0$ is included, because as explained in Chapter 5, this is required in a discrete-time $\mathcal{H}_2$ optimal controller. Without other exogenous inputs, the closed loop is autonomous with dynamics determined by

$$\begin{bmatrix} x_t(k+1) \\ x_{tc}(k+1) \\ x_d(k+1) \end{bmatrix} = \begin{bmatrix} A + B_u D_c C_y & B_u C_c & B_d + B_u D_c C_d \\ B_c C_y & A_c & B_c D_c \\ 0 & 0 & A_d \end{bmatrix} \begin{bmatrix} x_t(k) \\ x_{tc}(k) \\ x_d(k) \end{bmatrix} \triangleq \begin{bmatrix} A_L & B_Ld \end{bmatrix} \begin{bmatrix} x_{tL}(k) \\ x_d(k) \end{bmatrix},$$

(4.4a)

$$y_t(k) = \begin{bmatrix} C_y & 0 & C_d \end{bmatrix} \begin{bmatrix} x_t(k) \\ x_{tc}(k) \\ x_d(k) \end{bmatrix} \triangleq \begin{bmatrix} C_{Ly} & C_d \end{bmatrix} \begin{bmatrix} x_{tL}(k) \\ x_d(k) \end{bmatrix}.$$  

(4.4b)

The system in eq. (4.4) is considered stable when the loop subsystem $A_L$ is stable.

**Definition** The system in eq. (4.4) is stable if and only if $\sigma(A_L) \subset \mathbb{C}_g$.

We stipulate this as a condition for output regulation.

**Definition** Output regulation is achieved if and only if the system in eq. (4.4) is stable, and the output satisfies

$$\lim_{k \to \infty} y_t(k) = 0,$$

(4.5)

for any initial condition.

So we require that $\sigma(A_L) \subset \mathbb{C}_g$ and $\sigma(A_d) \notin \mathbb{C}_g$ are disjoint, and this also means that the asymptotic response depends only on the state “$x_d(k)$” of the disturbance model.
Lemma 4.1.1 If the system in eq. (4.4) is stable \( (\sigma(A_L) \subset \mathbb{C}_g) \), then

\[
\lim_{k \to \infty} x_{tL}(k) = \lim_{k \to \infty} x_L(k),
\]

(4.6)

where \( x_L(k) \) is the asymptotic state response of the loop, and is dependent on the disturbance state \( x_d(k) \) according to

\[
x_L(k) = \Pi V^{-1} x_d(k) = \begin{bmatrix} \Pi_x V^{-1} & \Pi_c V^{-1} \end{bmatrix} \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}
\]

(4.7)

where \( \Pi = [\Pi_x^T \Pi_c^T]^T \) are determined by a Jordan decomposition of the autonomous system. This then also means that the asymptotic output is given by

\[
y(k) \triangleq \begin{bmatrix} C_L y & C_d \end{bmatrix} \begin{bmatrix} \Pi V^{-1} & I \\ I & I \end{bmatrix} x_d(k) = \begin{bmatrix} C_y & 0 \\ C_d \end{bmatrix} \begin{bmatrix} \Pi_x V^{-1} \\ \Pi_c V^{-1} \end{bmatrix} x_d(k).
\]

(4.8)

Proof If the loop is stable, then \( \sigma(A_L) \cap \sigma(A_d) = \emptyset \), and the system has a Jordan decomposition such that

\[
\begin{bmatrix} A_L & B_{Ld} \\ 0 & A_d \end{bmatrix}\begin{bmatrix} U & \Pi \\ 0 & V \end{bmatrix} = \begin{bmatrix} U & \Pi \\ 0 & V \end{bmatrix}\begin{bmatrix} J & 0 \\ 0 & J_d \end{bmatrix},
\]

(4.9a)

where \( \sigma(A_L) = \sigma(UJU^{-1}) \subset \mathbb{C}_g \), and hence,

\[
\lim_{k \to \infty} J^k = 0.
\]

(4.10)
This shows that the asymptotic response is then

\[
\lim_{k \to \infty} \begin{bmatrix} x_{tL}(k) \\ x_d(k) \end{bmatrix} = \lim_{k \to \infty} \begin{bmatrix} A_L & B_{Ld} \\ 0 & A_d \end{bmatrix}^k \begin{bmatrix} x_{tL}(0) \\ x_d(0) \end{bmatrix},
\]

(4.11a)

\[
= \lim_{k \to \infty} \begin{bmatrix} U & \Pi \\ 0 & V \end{bmatrix} \begin{bmatrix} J^k & 0 \\ 0 & J_d^k \end{bmatrix}^{-1} \begin{bmatrix} U & \Pi \\ 0 & V \end{bmatrix} \begin{bmatrix} x_{tL}(0) \\ x_d(0) \end{bmatrix},
\]

(4.11b)

\[
= \lim_{k \to \infty} \begin{bmatrix} 0 & \Pi J_d^k \\ 0 & V J_d^k \end{bmatrix} \begin{bmatrix} U^{-1} & -U^{-1} \Pi V^{-1} \\ 0 & V^{-1} \end{bmatrix} \begin{bmatrix} x_{tL}(0) \\ x_d(0) \end{bmatrix} = \lim_{k \to \infty} \begin{bmatrix} \Pi V^{-1} V J_d^k V^{-1} \\ V J_d^k V^{-1} \end{bmatrix} x_d(0),
\]

(4.11c)

\[
= \lim_{k \to \infty} \begin{bmatrix} \Pi V^{-1} \\ I \end{bmatrix} \begin{bmatrix} \Pi x V^{-1} \\ I \end{bmatrix} x_d(0) = \lim_{k \to \infty} \begin{bmatrix} \Pi x V^{-1} \\ I \end{bmatrix} x_d(k),
\]

(4.11d)

where we have made the partition \( \Pi = [\Pi_x^T \Pi_c^T]^T \) compatibly with the dimensions of \( [x_t^T \ x_c^T]^T \).

This shows that the asymptotic state response always stays within the invariant subspace determined by the columns of \( [V^{-1} \Pi^T \ I]^T \) from the Jordan decomposition in (4.9). Regulation is achieved if the asymptotic response in (4.8) is zero. Putting these two results together we have the following regulator lemma.

**Lemma 4.1.2** Regulation occurs if and only if the system in eq. (4.4) is stable, and there is a solution \( X = \Pi V^{-1} \) to the regulator equation

\[
\begin{bmatrix} A_L & B_{Ld} \\ C_{Ly} & C_d \end{bmatrix} \begin{bmatrix} X \\ I \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix} A_d.
\]

(4.12)

**Proof** Necessity follows for a stable loop, by setting \( y(k) = 0 \) in eq. (4.8) for arbitrary disturbance states \( x_d(k) \), adjoining it with the Sylvester equation obtained (via the Jordan decomposition (4.9))
from the top row of

\[
\begin{bmatrix}
A_L & B_{Ld} \\
0 & A_d
\end{bmatrix}
\begin{bmatrix}
U & \Pi \\
0 & V
\end{bmatrix}
\begin{bmatrix}
0 \\
V^{-1}
\end{bmatrix} =
\begin{bmatrix}
A_L & B_{Ld} \\
0 & A_d
\end{bmatrix}
\begin{bmatrix}
\Pi V^{-1} \\
I
\end{bmatrix}
\]

(4.13a)

which leads to a recursion using the disturbance model eigenvectors \(\{d_{i,j}\}\) and computes a basis \(\{x_{i,j}, x_{c,i,j}\}\) for the invariant subspace \(\mathcal{R}([V^{-1}\Pi^T I]^T)\) in which the asymptotic state response evolves. The development of this recursion then leads to a fairly intuitive statement of the internal model principle (IMP). To this end, it is assumed that \(A_d\) is similar to a Jordan form wherein each eigenvalue \(\lambda_i \in \sigma(A_d), i \in [1, n_\lambda]\) is contained in only one

\[y_t(k) =
\begin{bmatrix}
C_L y & C_d
\end{bmatrix}
\begin{bmatrix}
I & X \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\bar{x}_{tL}(k) \\
x_d(k)
\end{bmatrix} =
\begin{bmatrix}
C_L y & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x}_{tL}(k) \\
x_d(k)
\end{bmatrix}.
\]

(4.15b)
Jordan block. So

\[ A_d = V_d J_d V_d^{-1}, \]  

(4.16a)

\[
J_d = \begin{bmatrix}
J_{d1} & 0 & \ldots & 0 \\
0 & J_{d2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & J_{dn_\lambda}
\end{bmatrix},
\]

(4.16b)

\[ V_d = \begin{bmatrix}
V_1 & 0 & \ldots & 0 \\
0 & V_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & V_{n_\lambda}
\end{bmatrix}, \]

(4.16c)

where

\[ V_d = \begin{bmatrix}
\bar{x}_{d_{i,0}} & \ldots & \bar{x}_{d_{i,n_i-1}}
\end{bmatrix}, \]

(4.17)

and

\[ \bar{x}_{d_{i,j}} = \lambda_i \bar{x}_{d_{i,j}} + \bar{x}_{d_{i,j-1}}, \]

(4.18)

This means that each eigenvalue \( \lambda_i \) is associated with only one chain of \( n_i \) generalized eigenvectors \( \{x_{d_{i,j}}, j_i \in [0, n_i - 1]\} \), where

\[ A_d x_{d_{i,0}} = \lambda_i x_{d_{i,0}}, \]

(4.19a)

\[ A_d x_{d_{i,j}} = \lambda_i x_{d_{i,j}} + x_{d_{i,j-1}}, \]

(4.19b)

and the order of the disturbance model is \( N_d = \sum n_i \). In terms of the class of disturbances represented (offsets, ramps, sinusoids, etc.), this assumption of single Jordan blocks presents no loss of generality [30, 20]. Finally, all generalized eigenvectors are referred to as simply “eigenvectors”, and where the standard eigenvectors are singled out (i.e., \( x_{i,0} \in N(A_d - I \lambda_i) \)), they are referred to as the zero-order eigenvectors. Now, it is possible to give the recursion determining the asymptotic response; regulation occurs if the vectors obtained are in the null space of \([C_L y C_d]\).

Lemma 4.1.3 Regulation occurs if and only if the system in eq. (4.4) is stable, and the set of
vectors \( \{x_{i,j}, x_{c_i,j}\}, j_i \in [0, n_i - 1], i \in [1, n_\lambda] \) obtained from the recursion

\[
(A_L - I \lambda_i) \begin{bmatrix} x_{i,0} \\ x_{c_i,0} \end{bmatrix} = \begin{bmatrix} A + B_u D_c C_y - I \lambda_i & B_u C_c \\ B_c C_y & A_c - I \lambda_i \end{bmatrix} \begin{bmatrix} x_{i,0} \\ x_{c_i,0} \end{bmatrix} = -\begin{bmatrix} B_d x_{d,i,0} \\ B_c C_d x_{d,i,0} \end{bmatrix},
\]

(4.20a)

\[
\begin{bmatrix} A + B_u D_c C_y - I \lambda_i & B_u C_c \\ B_c C_y & A_c - I \lambda_i \end{bmatrix} \begin{bmatrix} x_{i,j_i} \\ x_{c_i,j_i} \end{bmatrix} = -\begin{bmatrix} B_d x_{d,i,j_i} \\ B_c C_d x_{d,i,j_i} \end{bmatrix},
\]

(4.20b)

are such that

\[
y_{i,j_i} = \begin{bmatrix} C_y & 0 & C_d \end{bmatrix} \begin{bmatrix} x_{i,j_i} \\ x_{c_i,j_i} \\ x_{d,i,j_i} \end{bmatrix} = 0.
\]

(4.21)

**Proof** By lemma C.1.2, regulation occurs if and only if we have a solution \( X \) to eq. (4.12). Multiply this equation from the right by any eigenvector \( x_{d,i,j_i} \) of the disturbance model, and arrange the two rows as separate equations

\[
A_L x_{L_i,j_i} - \lambda_i x_{L_i,j_i} = x_{L_i,j_i,-1} - B_{Ld} x_{d,i,j_i},
\]

(4.22a)

\[
\begin{bmatrix} C_{Ly} & C_d \end{bmatrix} \begin{bmatrix} x_{L_i,j_i} \\ x_{d,i,j_i} \end{bmatrix} = 0,
\]

(4.22b)

where we have used the fact that

\[
X A_d x_{d,i,j_i} = PV^{-1}(\lambda_i x_{d,i,j_i} + x_{d,i,j_i,-1}) = \lambda_i x_{L_i,j_i} + x_{L_i,j_i,-1},
\]

(4.23)

so that each \( x_{L_i,j_i} \) is a column of \( P \). Now, expanding \( \{A_L, C_{Ly}, B_{Ld}\} \) as defined in eq. (4.4), establishes necessity. Conversely, given a set \( \{x_{i,j_i}, x_{c_i,j_i}\}, j_i \in [0, n_i - 1], i \in [1, n_\lambda] \) satisfying eq. (4.20) and eq. (4.21), it is immediate that the set also satisfies eq. (4.22) with \( x_{L_i,j_i} = [x_{i,j_i}^T, x_{c_i,j_i}^T]^T \). If we now let

\[
\Pi = \begin{bmatrix} x_{L_1,0} & \cdots & x_{L_1,n_1-1} & \cdots & x_{L_{\lambda},0} & \cdots & x_{L_{\lambda},n_\lambda-1} \end{bmatrix},
\]

(4.24)
then eq. (4.22) can be written as

\[ A_L \Pi = \Pi J_d - B_{Ld}V, \tag{4.25a} \]

\[
\begin{bmatrix}
C_{Ly} & C_d
\end{bmatrix}
\begin{bmatrix}
\Pi \\
V
\end{bmatrix} = 0. \tag{4.25b}
\]

Multiplying this result through from the right by \( V^{-1} \), noting that \( \Pi J_d V^{-1} = \Pi V^{-1} A_d \), and letting \( X = \Pi V^{-1} \), it should now be clear that this is equivalent to the regulator equation.\( \square \)

We note that the matrix on the left in eq. (4.20) is non-singular if the loop is stable, in which case the recursion determines the asymptotic subspace uniquely.

### 4.1.2.1 The IMP for Non-Selective Regulation

In this section, a property of the controller is obtained that is necessary to have \( y = [y_1^T \ y_2^T]^T = 0 \) for any possible disturbance coupling \( \{B_d, C_d\} \). The next section considers the case that only \( y_1 = 0 \) is required.

The recursion (4.20) and eq. (4.21) can be written as

\[
\begin{bmatrix}
A - I \lambda_i & B_u C_c & B_u D_c \\
0 & A_c - I \lambda_i & B_c \\
C_y & 0 & -I
\end{bmatrix}
\begin{bmatrix}
x_{i,j} \\
x_{ci,j} \\
y_{i,j}
\end{bmatrix}
= \begin{bmatrix}
x_{i,j,-1} \\
x_{ci,j,-1} \\
0
\end{bmatrix}
- \begin{bmatrix}
B_d \\
0
\end{bmatrix} x_{di,j}. \tag{4.26}
\]

A solution \( \{x_{i,j}, x_{ci,j}, y_{i,j}\} \) to eq. (4.26) automatically gives a solution \( \{x_{i,j}, x_{ci,j}\} \) to eq. (4.20). Further, because of the “\(-I\)” in the lower right hand corner, the matrix on the left in eq. (4.26) cannot have a vector of the form \( \{0, 0, y \neq 0\} \) in its null space. It then follows that this matrix has a non-trivial null space only if the matrix \( A_L - I \lambda_i \) on the left in eq. (4.20) has a non-trivial null space. Therefore, if the loop is stable \( (\sigma(A_L) \in \mathbb{C}_g) \), then the matrix on the left in eq. (4.26) is non-singular.

The implication is that for a stabilizing controller, eq. (4.26) uniquely determines the modal output \( y_{i,j} \) for each unique disturbance coupling \( \{B_d x_{di,j}, 0, C_d x_{di,j}\} \). Conversely, take any solu-
tion \{x_{ci,ji}, y_{i,ji} \neq 0\} to the second row

\[(A_c - I\lambda_i)x_{ci,ji} + B_c y_{i,ji} = x_{ci,ji} - 1,\] (4.27)

and any arbitrary \(x_{i,ji}\), then there is a disturbance coupling \{\(B_d x_{di,ji}, 0, C_d x_{di,ji}\}\) that makes \(\{x_{i,ji}, x_{ci,ji}, y_{i,ji} \neq 0\}\) a modal output for the system. That is, for this coupling, if at any time \(x_d(k) = x_{di,ji}\), then the asymptotic output lies along \(y_{i,ji} \neq 0\), and regulation is not achieved.

Therefore, to have regulation without exact knowledge of the way the disturbance couples into the system, the controller must be designed so that \(y_{i,ji} = 0\) in every solution to the second row. For the zero-order case, we need

\[(A_c - I\lambda_i)x_{ci,0} + B_c y_{i,0} = 0 \Rightarrow y_{i,0} = 0.\] (4.28)

In order for this to be the case, then it is certainly necessary that \(\mathcal{R}(A_c - I\lambda_i) \neq \mathcal{R}_c\); otherwise, \(\forall y_{i,0}\) there would always be a \(x_{ci,0}\) such that

\[(A_c - I\lambda_i)x_{ci,0} = -B_c y_{i,0}.\] (4.29)

This shows that for each \(\lambda_i \in \sigma(A_d)\), it is necessary that \(\lambda_i \in \sigma(A_c)\) as well. Appendix C shows that in fact, to have rejection at all \(N_y\) outputs used by the controller, the multiplicity of each \(\lambda_i \in \sigma(A_c) \cap \sigma(A_d)\), must be \(n_i \times N_y\). Further, it is also necessary that

\[N(B_c) = 0,\] (4.30a)

\[\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_c) = 0,\] (4.30b)

since otherwise, there are specific \(\{x_{ci,0}, y_{i,0} \neq 0\}\) satisfying eq. (4.29); and again, in this case there is a disturbance coupling that will generate a non-zero output when the disturbance state is \(x_d(k) = x_{di,0}\).

Finally, Appendix C shows that each chain of disturbance model eigenvectors \(\{x_{di,ji}, j_i \in [0, n_i - 1]\}\) must be duplicated in the controller with multiplicity \(N_y\). This means that there exists a restriction of the controller to an invariant subspace, such that the restriction contains
\( \{ \lambda_i \in \sigma(A_d), i \in [1, n_\lambda] \} \) as eigenvalues, and the dimension of this subspace is \( N_y \) times the dimension of the disturbance model. Further, it is also shown that the plant cannot have a transmission zero at \( \lambda_i \in \sigma(A_d) \), with an associated input direction that is coincident with any control direction that might be generated when the controller state lies within the invariant subspace(s) that duplicate the disturbance model. Simply put, when the plant has a transmission zero at a disturbance mode \( \lambda_i \), it is not possible to regulate the mode if the controller generates controls in the directions associated with the zero. In fact, when this is the case, the controller mode \( \lambda_i \) is unobservable through the controller-plant cascade, and the loop is then unstable.

### 4.1.2.2 The IMP for Selective Regulation

For the case that we only require a subset \( y_1 \) of the measurements (used by the controller) to be regulated, eq. (4.26) becomes

\[
\begin{bmatrix}
A - I\lambda_i & B_u C_c & B_u D_{c1} & B_u D_{c2} \\
0 & A_c - I\lambda_i & B_{c1} & B_{c2} \\
C_{y1} & 0 & -I & 0 \\
C_{y2} & 0 & 0 & -I
\end{bmatrix}
\begin{bmatrix}
x_{i,j_i} \\
x_{ci,j_i} \\
y_{1i,j_i} \\
y_{2i,j_i}
\end{bmatrix}
= \begin{bmatrix}
x_{i,j_i-1} \\
x_{ci,j_i-1} \\
y_{1i,j_i-1} \\
y_{2i,j_i-1}
\end{bmatrix}
- \begin{bmatrix}
B_d \\
0 \\
C_{d1} \\
C_{d2}
\end{bmatrix}x_{di,j_i}.
\]  

(4.31)

This selective rejection goal is addressed in detail in Appendix C as well. Following arguments similar to those used in the non-selective case, we find that it is necessary to have \( y_1 = 0 \) in all solutions to the second row. For the zero-order case, this requires

\[
(A_c - I\lambda_i)x_{ci,0} + B_{c1}y_{1i,0} + B_{c2}y_{2i,0} = 0 \Rightarrow y_{1i,0} = 0.
\]  

(4.32)

Now, however, the implications regarding the multiplicity of the modes \( \lambda_i \) in the controller are somewhat more complicated. In Appendix C, following arguments similar to those in the previous section, it is shown that the controller must satisfy

\[
N(B_{c1}) = 0,
\]

(4.33a)

\[
\mathcal{R}([A_c - I\lambda_i B_{c2}]) \cap \mathcal{R}(B_{c1}) = 0,
\]

(4.33b)
and it also follows that the required multiplicity in the controller is

\[ n_s \equiv n_1 + \dim (\mathcal{R}(B_{c2})) - \dim (\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_{c2})) \cdot \tag{4.34} \]

This configuration encompasses preview of exogenous inputs and feed-forward controls as in Chapter 5. For example, with wind preview measurements available to an observer-controller, \( y_2 \) may be comprised of collective, vertical, and horizontal wind components \([w_{cy}, w_{vy}, w_{hy}]^T \equiv w_y\) and eq. (4.35) becomes

\[
\begin{bmatrix}
    A - I\lambda_i & B_u C_c & B_u D_{c1} & B_u D_{cw} \\
    0 & A_c - I\lambda_i & B_{c1} & B_{cw} \\
    C_{y1} & 0 & -I & 0 \\
    0 & 0 & 0 & -I
\end{bmatrix}
\begin{bmatrix}
    x_{i,j_i} \\
    x_{ci,j_i} \\
    y_{1i,j_i} \\
    w_{yi,j_i}
\end{bmatrix}
= \begin{bmatrix}
    x_{i,j_i-1} \\
    x_{ci,j_i-1} \\
    0 \\
    0
\end{bmatrix}
- \begin{bmatrix}
    B_d \\
    0 \\
    C_{d1} \\
    C_{dw}
\end{bmatrix}
\begin{bmatrix}
    x_{di,j_i}
\end{bmatrix}.
\tag{4.35}
\]

The subscript \( y \) is used to emphasize that this is a measurement, and may not be the same as the wind actually impacting the turbine. It is necessary to consider arbitrary disturbance content \( (= C_{dw} x_{di,j_i}) \) in the measurement used by the controller; the disturbance content in the wind actually hitting the turbine is encompassed by arbitrary (and unknown) couplings \( = B_d x_{di,j_i} \). Neither the controller nor the plant dynamics has a path to effect the value of the measurement \( w_y \), and so this case is accurately described as one where it is truly only possible to obtain rejection at \( y_1 \).

Finally, note that it is fairly straightforward to verify that the minimal required multiplicity \( n_s = n_1 \), can always be achieved using output augmentation. That is, augment the plant with the disturbance dynamics at measurements where rejection is feasible, and then design stabilizing feedback for the augmented plant. This is a standard approach for obtaining regulating control [30]. It is certainly possible to design an observer for the augmented system and thereby obtain a state estimate as required for MPC. And, it is even possible to supply the state of the augmented dynamics directly, thereby omitting their estimation and obtaining an efficient implementation with a reduced order observer.

However, in Section 4.4, it is demonstrated that simply combining output augmentation and anti-windup with MPC does not provide regulation. Also, the DAC approach (a disturbance model
in the observer, and no direct augmentation) can provide wind-up free regulation as a byproduct without any additional modifications. This is certainly the case for SISO applications where the plant is open loop stable, because when the control input saturation is correctly reflected in the observer, the system simplifies to the cascade of the stable plant and a stable observer. Further, wind-up free operation is achieved no matter how complex the disturbance model is (e.g., ramps and/or arbitrary sums of sinusoids with amplitudes that may increase as polynomial functions of time). For MIMO applications, it is necessary to carefully consider stability for all possible combinations of saturated and unsaturated control inputs, but this is a complicating factor no matter how a regulating controller is obtained.

4.2 Designing (DAC) Observer-Controllers for Output Regulation

Design of observer-controllers that achieve robust output regulation is relatively straightforward, and the design of the state-feedback and disturbance feedback can be done separately. This is particularly useful in the design of $H_2$ optimal control as in Chapter 5, where it is shown that inclusion of the disturbance model does not affect the part of the cost that is optimized with the design of the feedback gains. However, including the disturbance model, does affect the part of the cost optimized with the design of observer gains.

This section provides a brief overview of the DAC technique (Section 4.2.1), the iterative method for designing a disturbance gain that provides robust rejection (Section 4.2.1.1), and a proof that the resulting controller satisfies the IMP (Section 4.2.1.2). Then it is shown how the approach can be modified to handle selective rejection (Section 4.2.2), and finally a simple example is provided (Section 4.4) that demonstrates wind-up free regulation.
4.2.1 DAC Background and Non-Selective Regulation

Since we are constructing an observer-controller, we must have that the nominal model

\[
\begin{bmatrix}
           x(k+1) \\
x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
\hat{A} & \hat{B}_d \\
0 & \hat{A}_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k)
\end{bmatrix} +
\begin{bmatrix}
\hat{B}_u \\
0
\end{bmatrix} u(k),
\]

(4.36a)

\[
y(k) =
\begin{bmatrix}
\hat{C}_y & \hat{C}_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k)
\end{bmatrix},
\]

(4.36b)

is detectable, and that \(\{\hat{A}, \hat{B}_u\}\) is stabilizable. The observer-controller gains \(\{B_c, C_c\}\) are partitioned into state and disturbance sections, where

\[
B_c =
\begin{bmatrix}
L_x \\
L_d
\end{bmatrix},
C_c =
\begin{bmatrix}
K_x & K_d
\end{bmatrix}.
\]

(4.37)

The design of the state-feedback \(K_x\) can be done separately, and it is presumed that this is already done, and that it is stabilizing

\[
\sigma(\hat{A} + \hat{B}_u K_x) \in \mathcal{C}_g
\]

(4.38)

for the nominal plant model. As required by a discrete-time \(\mathcal{H}_2\) optimal controller, the observer-controller is implemented with a direct feed-through gain \(D_c = L_\Delta\), so that it is implemented as
\[
\begin{align*}
\mathbf{u}(k) &= \begin{bmatrix} K_x & K_d \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{x}_d(k) \end{bmatrix} - L\Delta \begin{bmatrix} \bar{C}_y & \bar{C}_d \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{x}_d(k) \end{bmatrix} - y(k) \\
\begin{bmatrix} \bar{x}(k+1) \\ \bar{x}_d(k+1) \end{bmatrix} &= \begin{bmatrix} \bar{A} & \bar{B}_d \\ 0 & \bar{A}_d \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{x}_d(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_u \\ 0 \end{bmatrix} \begin{bmatrix} K_x & K_d \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{x}_d(k) \end{bmatrix} - L\Delta \begin{bmatrix} \bar{C}_y & \bar{C}_d \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{x}_d(k) \end{bmatrix} - y(k) \\
&+ \begin{bmatrix} L_x \\ L_d \end{bmatrix} \left( \begin{bmatrix} \bar{C}_y & \bar{C}_d \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{x}_d(k) \end{bmatrix} - y(k) \right) \\
&= \begin{bmatrix} \bar{A} + \bar{B}_u K_x + L_x \bar{C}_y - \bar{B}_u L\Delta \bar{C}_y & \bar{B}_d + \bar{B}_u K_d + L_x \bar{C}_d - \bar{B}_u L\Delta \bar{C}_d \\ L_d \bar{C}_y & \bar{A}_d + L_d \bar{C}_d \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{x}_d(k) \end{bmatrix} + \begin{bmatrix} (\bar{B}_u L\Delta - L_x) \bar{C}_y \\ -L_d \bar{C}_y \end{bmatrix} x(k) \\
&+ \begin{bmatrix} (\bar{B}_u L\Delta - L_x) \bar{C}_d \\ -L_d \bar{C}_d \end{bmatrix} \begin{bmatrix} x(k) \\ \bar{x}_d(k) \end{bmatrix}
\end{align*}
\]
With the feedback from the observer state, and the observer-controller direct feed-through gain, the nominal system then evolves according to

$$
\begin{align*}
\begin{bmatrix}
x(k+1) \\
x_d(k+1)
\end{bmatrix} &= \begin{bmatrix}
\bar{A} & \bar{B}_d \\
0 & \bar{A}_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k)
\end{bmatrix} + \begin{bmatrix}
\bar{B}_u \\
0
\end{bmatrix}
\begin{bmatrix}
[K_x & K_d] \\
[\bar{C}_y & \bar{C}_d]
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} - L\Delta
\begin{bmatrix}
\bar{C}_y & \bar{C}_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} - y(k)
\end{align*}
$$

(4.40a)

$$
\begin{align*}
\begin{bmatrix}
x(k+1) \\
x_d(k+1) \\
\bar{x}(k+1) \\
\bar{x}_d(k+1)
\end{bmatrix} &= \begin{bmatrix}
\bar{A} + \bar{B}_u L\Delta \bar{C}_y & \bar{B}_d + \bar{B}_u L\Delta \bar{C}_d & \bar{B}_u K_x - \bar{B}_u L\Delta \bar{C}_y & \bar{B}_u K_d - \bar{B}_u L\Delta \bar{C}_d \\
0 & \bar{A}_d & 0 & 0 \\
(\bar{B}_u L\Delta - L_x)\bar{C}_y & (\bar{B}_u L\Delta - L_x)\bar{C}_d & \bar{A} + \bar{B}_u K_x + L_x \bar{C}_y - \bar{B}_u L\Delta \bar{C}_y & \bar{B}_d + \bar{B}_u K_d + L_x \bar{C}_d - \bar{B}_u L\Delta \bar{C}_d \\
-L_d \bar{C}_y & -L_d \bar{C}_d & L_d \bar{C}_y & \bar{A}_d + L_d \bar{C}_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k) \\
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix}
\end{align*}
$$

(4.40b)

With no other exogenous inputs, the closed loop system becomes autonomous

$$
\begin{align*}
\begin{bmatrix}
x(k+1) \\
x_d(k+1) \\
\bar{x}(k+1) \\
\bar{x}_d(k+1)
\end{bmatrix} &= \begin{bmatrix}
\bar{A} + \bar{B}_u L\Delta \bar{C}_y & \bar{B}_d + \bar{B}_u L\Delta \bar{C}_d & \bar{B}_u K_x - \bar{B}_u L\Delta \bar{C}_y & \bar{B}_u K_d - \bar{B}_u L\Delta \bar{C}_d \\
0 & \bar{A}_d & 0 & 0 \\
(\bar{B}_u L\Delta - L_x)\bar{C}_y & (\bar{B}_u L\Delta - L_x)\bar{C}_d & \bar{A} + \bar{B}_u K_x + L_x \bar{C}_y - \bar{B}_u L\Delta \bar{C}_y & \bar{B}_d + \bar{B}_u K_d + L_x \bar{C}_d - \bar{B}_u L\Delta \bar{C}_d \\
-L_d \bar{C}_y & -L_d \bar{C}_d & L_d \bar{C}_y & \bar{A}_d + L_d \bar{C}_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k) \\
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix}
\end{align*}
$$

(4.41a)

and transforming to the error system gives

$$
\begin{align*}
\begin{bmatrix}
x(k+1) \\
x_d(k+1) \\
\bar{x}(k+1) - x(k+1) \\
\bar{x}_d(k+1) - x_d(k+1)
\end{bmatrix} &= \begin{bmatrix}
\bar{A} + \bar{B}_u K_x & \bar{B}_d + \bar{B}_u K_d & \bar{B}_u K_x - \bar{B}_u L\Delta \bar{C}_y & \bar{B}_u K_d - \bar{B}_u L\Delta \bar{C}_d \\
0 & \bar{A}_d & 0 & 0 \\
0 & 0 & \bar{A} + L_x \bar{C}_y & \bar{B}_d + L_x \bar{C}_d \\
0 & 0 & L_d \bar{C}_y & \bar{A}_d + L_d \bar{C}_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k) \\
\bar{x}(k) - x(k) \\
\bar{x}_d(k) - x_d(k)
\end{bmatrix}
\end{align*}
$$

(4.42a)
So, it is apparent that system stability is only dependent on the gains \( \{K_x, L_x, L_d\} \), and that when the estimation error goes to zero, the direct feed-through gain has no effect. In this latter case, the closed loop dynamics are determined by

\[
\begin{bmatrix}
    x(k+1) \\
    x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
    \bar{A} + \bar{B}_d K_x & \bar{B}_d + \bar{B}_d K_d \\
    0 & \bar{A}_d
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    x_d(k)
\end{bmatrix},
\] (4.43a)

\[
y(k) = \begin{bmatrix}
    \bar{C}_y & \bar{C}_d
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    x_d(k)
\end{bmatrix}.
\] (4.43b)

According to lemma C.1.2, regulation is achieved if and only if there is a solution \( \bar{X} \) to the (state-feedback) regulator equation

\[
\begin{bmatrix}
    \bar{A} + \bar{B}_d K_x & \bar{B}_d + \bar{B}_d K_d \\
    \bar{C}_y & \bar{C}_d
\end{bmatrix}
\begin{bmatrix}
    \bar{X} \\
    I
\end{bmatrix} =
\begin{bmatrix}
    \bar{X} \\
    0
\end{bmatrix} \bar{A}_d.
\] (4.44)

Now, the nominal disturbance coupling \( \bar{C}_d \) is constructed at the same time as the disturbance gain \( K_d \) so that

- control actuation for disturbance rejection is not in conflict with transmission zeros,
- the nominal model eq. (4.36) is detectable,
- and there is a solution \( \bar{X} \) to the state-feedback regulator equation (4.44).

Since it is necessary that the observer-controller have \( N_y \) disturbance models, in the most straightforward construction, the disturbance model \( A_d \) is extended to a block diagonal form

\[
\bar{A}_d =
\begin{bmatrix}
    A_{d1} & 0 & \ldots & 0 \\
    0 & A_{d2} & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & A_{dN_y}
\end{bmatrix},
\]

\[
\bar{B}_d = \begin{bmatrix} B_{d1} & \ldots & B_{dN_y} \end{bmatrix},
\]

\[
\bar{C}_d = \begin{bmatrix} C_{d1} & \ldots & C_{dN_y} \end{bmatrix},
\] (4.45)

where \( A_{d\ell} = A_d, \ \ell \in [1, N_y] \). Since \( A_d = V J_d V^{-1} \) where

\[
V = \begin{bmatrix}
    x_{d1,0} & \ldots & x_{d1,n_1-1} & \ldots & x_{dn_\lambda,0} & \ldots & x_{dn_\lambda,n_\lambda-1}
\end{bmatrix},
\] (4.46)
we have that

$$\hat{A}_d = \hat{V} \hat{J}_d \hat{V}^{-1},$$  \hspace{1cm} (4.47a)

$$\hat{V} =
\begin{bmatrix}
V_1 & 0 & \ldots & 0 \\
0 & V_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & V_{N_y}
\end{bmatrix}, \quad \hat{V}_\ell = V, \quad \ell \in [1, N_y],$$  \hspace{1cm} (4.47b)

so that

$$x_{di,ji,\ell} = \begin{bmatrix} 0^T_{N_d(\ell-1)} & x_{di,ji,\ell}^T & 0^T_{N_d(N_y-\ell)} \end{bmatrix}^T,$$  \hspace{1cm} (4.48)

where $0^T_{N_d(m)}$ is a vector of $N_d \times m$ zeros. As we’ll show, the controller eigenvectors $x_{ci,ji,\ell}$ required by the IMP will then be related to the extended disturbance model eigenvectors $x_{di,ji,\ell}$ according to

$$x_{ci,ji,\ell} = \begin{bmatrix} 0 \\ x_{di,ji,\ell} \end{bmatrix}.$$  \hspace{1cm} (4.49)

It is now possible to unambiguously define

$$\mathcal{Y}_i \triangleq \bigcup_{j, i \in [1, n]} \text{span} \{x_{ci,ji,\ell}, \quad \ell \in [1, N_y]\}.$$  \hspace{1cm} (4.50)

It can be shown that this is the controller invariant subspace that generates control directions $\mathcal{U}_i \triangleq C_c \mathcal{Y}_i$ that regulate the mode $\lambda_i \in \sigma(A_d)$.

### 4.2.1.1 Design of The Disturbance Gain and The Nominal Disturbance Coupling

The computations in this section are nearly identical to the reference shifting computations used in MPC [52] (in particular, if $K_x = 0$ is chosen). However, we include computations for $\bar{C}_d$, and include a final gain computation for $K_d$. Similar to the derivation of eq. (4.20), we begin by multiplying the state-feedback regulator eq. (4.44) from the right by $x_{di,ji,\ell}$, to obtain an equivalent
bases are independent of the order of the eigenvector so that input (or set of controls) can be actuated to reject disturbances at each output. Typically, the bases for \( U \) are chosen to be

\[
\begin{bmatrix}
\tilde{A} + \tilde{B}_u K_x \\
\tilde{C}_y
\end{bmatrix}
\begin{bmatrix}
\tilde{B}_d + \tilde{B}_u K_d \\
\tilde{C}_d
\end{bmatrix}
\begin{bmatrix}
\hat{X}_{x_{i,j,i}} \\
x_{i,j,i}
\end{bmatrix}
= \begin{bmatrix}
\hat{X}(\lambda; x_{i,j,i} + x_{i,j,i-1}) \\
0
\end{bmatrix}
\]

Defining

\[
\tilde{A} = \tilde{A} + \tilde{B}_u K_x,
\]

and

\[
u_{i,j,i} = K_d x_{i,j,i}, \quad x_{i,j,i} = \hat{X} x_{i,j,i},
\]

this result can be arranged as

\[
\begin{bmatrix}
\tilde{A} - I \lambda_i & \tilde{B}_u \\
\tilde{C}_y & 0
\end{bmatrix}
\begin{bmatrix}

0 & \tilde{B}_d x_{i,j,i} \\
\tilde{C}_d x_{i,j,i}
\end{bmatrix}
\begin{bmatrix}

x_{i,j,i} \\
x_{i,j,i-1}
\end{bmatrix}
= \begin{bmatrix}
0
\end{bmatrix}.
\]

Now, it is not difficult to show

\[
\begin{bmatrix}
\tilde{A} - I \lambda_i & \tilde{B}_u \\
\tilde{C}_y & 0
\end{bmatrix}
\begin{bmatrix}
X \\
\mathcal{V}
\end{bmatrix}
= \begin{bmatrix}
X \\
\mathcal{V}
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
\tilde{A} - I \lambda_i & \tilde{B}_u \\
\tilde{C}_y & 0
\end{bmatrix}
\begin{bmatrix}
X \\
\mathcal{V}
\end{bmatrix}
= \begin{bmatrix}
X \\
\mathcal{V}
\end{bmatrix}
\]

so that if the chosen control directions are not in conflict with transmission zeros as required by the IMP, then there is always a solution \([x^T_{i,j,i} u^T_{i,j,i}]^T\) to eq. (4.54). The construction of the gains \( \{K_d, \tilde{C}_d\} \) is now accomplished by finding solutions to eq. (4.54) where \( u_{i,j,i} \) are constrained to be bases for \( \mathcal{U}_{i,j} \) that satisfy eq. (4.55).

We begin with selection of bases for the control-direction spaces \( \mathcal{U}_{i,j} \). In practice, this is usually not difficult, since it is known (through physical modeling, analysis, etc.) which control input (or set of controls) can be actuated to reject disturbances at each output. Typically, the bases are independent of the order of the eigenvector so that

\[
\mathcal{U}_{i,j} = \mathcal{U}_i = \text{span} \{ u_i, \ell \in [1, N_y] \},
\]
where the set \( \{ u_{i,\ell} \} \) spans the input combinations that can achieve rejection for the mode \( \lambda_i \in \sigma(A_d) \). For complex eigenvalues, the set \( \{ u_{i,\ell} \} \) must contain the appropriate complex conjugates. Often, we can choose \( u_{i,\ell} = e_{m_{\ell}} \) where \( e_{m_{\ell}} \) is the standard unit vector corresponding to the \( m_{\ell} \)th input that is responsible for rejection of the mode \( \lambda_i \) at the \( \ell \)th output.

Next, we find solutions \( \{ u_{i,j_i,\ell}, x_{i,j_i,\ell} \} \) to the vector relations in eq. (4.54). Such solutions exist if and only if they are also solutions for

\[
\begin{bmatrix}
\bar{A} - I\lambda_i & \bar{B}_u \\
0 & \bar{C}_y(I\lambda_i - \bar{A})^{-1}\bar{B}_u
\end{bmatrix}
\begin{bmatrix}
x_{i,j_i,\ell} \\
u_{i,j_i,\ell}
\end{bmatrix}
= \begin{bmatrix}
x_{i,j_i-1,\ell} \\
u_{i,j_i-1,\ell}
\end{bmatrix}
- \begin{bmatrix}
\bar{B}_d x_{i,j_i-1,\ell} \\
(\bar{C}_y(I\lambda_i - \bar{A})^{-1}\bar{B}_d + \bar{C}_d) x_{i,j_i-1,\ell}
\end{bmatrix}
\]

(4.57)

where \( (I\lambda_i - \bar{A})^{-1} \) exists, because \( K_x \) is stabilizing. The upper row determines \( x_{i,j_i,\ell} \) via

\[
x_{i,0,\ell} = - (\bar{A} - I\lambda_i)^{-1} (\bar{B}_u u_{i,0,\ell} + \bar{B}_d x_{i,0,\ell}) \quad (4.58a)
\]

\[
x_{i,j_i,\ell} = - (\bar{A} - I\lambda_i)^{-1} (\bar{B}_u u_{i,j_i,\ell} + \bar{B}_d x_{i,j_i,\ell} - x_{i,j_i-1,\ell}) \quad (4.58b)
\]

while the lower row defines \( \bar{C}_d \) completely via

\[
\bar{C}_d x_{i,j_i,\ell} = - (I\lambda_i - \bar{A})^{-1} (\bar{B}_u u_{i,j_i,\ell} + \bar{B}_d x_{i,j_i,\ell}) \quad (4.59a)
\]

\[
= \bar{C}_y x_{i,j_i,\ell} \quad (4.59b)
\]

\[
\bar{C}_d x_{i,j_i-1,\ell} = - \bar{C}_y (I\lambda_i - \bar{A})^{-1} (\bar{B}_u u_{i,j_i,\ell} + \bar{B}_d x_{i,j_i,\ell} - x_{i,j_i-1,\ell}) \quad (4.59c)
\]

\[
= \bar{C}_y x_{i,j_i,\ell} \quad (4.59d)
\]

If we collect these vector solutions into matrices

\[
X_{i,\ell} = \begin{bmatrix}
x_{i,0,\ell} & \ldots & x_{i,n_{i-1},\ell}
\end{bmatrix}, \quad X_{\ell} = \begin{bmatrix}
x_{1,\ell} & \ldots & X_{n,\ell}
\end{bmatrix}, \quad (4.60a)
\]

\[
U_{i,\ell} = \begin{bmatrix}
u_{i,0,\ell} & \ldots & u_{i,n_{i-1},\ell}
\end{bmatrix}, \quad U_{\ell} = \begin{bmatrix}
u_{1,\ell} & \ldots & U_{n,\ell}
\end{bmatrix}, \quad (4.60b)
\]

\[
Y_{i,\ell} = \begin{bmatrix}
y_{i,0,\ell} & \ldots & y_{i,n_{i-1},\ell}
\end{bmatrix}, \quad Y_{\ell} = \begin{bmatrix}
y_{1,\ell} & \ldots & Y_{n,\ell}
\end{bmatrix}, \quad (4.60c)
\]
we can then write
\[
\hat{X}_t = X_t V^{-1}, \quad \hat{X} = [\hat{X}_1 \ldots \hat{X}_{N_y}], \tag{4.61a}
\]
\[
K_t = U_t V^{-1}, \quad K_d = [K_1 \ldots K_{N_y}], \tag{4.61b}
\]
\[
C_t = Y_t V^{-1}, \quad \hat{C}_d = [C_1 \ldots C_{N_y}] \tag{4.61c}
\]

We note that with this construction, all disturbance modes in the controller are observable as long as \( \hat{B}_d \) is chosen so that the right hand side of eq. (4.59a) is not 0 (e.g., choosing \( \hat{B}_d = 0 \) always works). And,

\[
\mathcal{U}_i = \bigcup_{j_i \in [1, n_i]} \mathcal{U}_{i,j_i} = C_c \mathcal{V}'_i = [K_x \ K_d] \mathcal{V}'_i, \tag{4.62}
\]

where \( \mathcal{U}_{i,j_i} \) are chosen to satisfy eq. (4.55). We still need to choose the observer gain so that \( B_c = [L_x^T \ L_d^T]^T \) satisfies eq. (4.30), but we show in the next section, that any stabilizing observer gain will work.

### 4.2.1.2 Proof of Robust Regulation

In this section, we show that the observer-controller constructed in the previous section satisfies the IMP for the nominal model \( \{A, \hat{B}_u, \hat{C}_y, \hat{D} = 0\} \). This then guarantees regulation for any possible disturbance coupling \( \{B_d, C_d\} \) into the nominal model. In fact, as long as the actual plant combined with the observer-controller is stable, and arbitrary modeling errors are not so great that the actual plant violates eq. (4.55), then we get robust regulation for the actual plant as well.

As already noted, by construction we have computed \( K_d \) such that

\[
C_c \mathcal{V}'_i = [K_x \ K_d] \mathcal{V}'_i = \bigcup_{j_i \in [1, n_i]} \mathcal{U}_{i,j_i}, \tag{4.63}
\]

where \( \mathcal{U}_{i,j_i} \) are chosen/specified so that

\[
\begin{bmatrix}
\hat{A} - I \lambda_i & \hat{B}_u \\
\hat{C}_y & 0
\end{bmatrix}
\begin{bmatrix}
\mathcal{X}'_i \\
\mathcal{U}_{i,j_i}
\end{bmatrix}
= 
\begin{bmatrix}
\mathcal{X}'_i \\
\mathcal{U}_{i,j_i}
\end{bmatrix} \tag{4.64}
\]

This satisfies the conditions of eq. (4.55) for the nominal plant. Further, as long as the constructed \( \mathcal{U}_i = C_c \mathcal{V}'_i \) satisfy eq. (4.55) for the actual plant and the loop remains stable, then we will obtain
robust regulation for the actual plant as well. What remains, is to show that the correct multiplicity of the disturbance model exists, and that \( B_c = [L_x^T L_d^T]^T \) satisfies eq. (4.30) as well.

Since the controller is constructed so that there is a solution \( \hat{X} \) to the state-feedback regulator equation (4.44), there is a similarity transformation

\[
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} = \begin{bmatrix} I & -\hat{X} \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix},
\]

(4.65)

that, when applied to the observer-controller dynamics,

\[
\begin{bmatrix}
\bar{x}(k+1) \\
\bar{x}_d(k+1)
\end{bmatrix} = \begin{bmatrix}
\hat{A} + \hat{B}_u K_x & \hat{B}_d + \hat{B}_u K_d \\
0 & \hat{A}_d
\end{bmatrix} \begin{bmatrix} \bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} - \begin{bmatrix}
\hat{B}_u \\
0
\end{bmatrix} \begin{bmatrix} L \Delta \left[ \hat{C}_y \quad \hat{C}_d \right] \begin{bmatrix} \bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} + L_x - \hat{B}_u L \Delta \left[ \hat{C}_y \quad \hat{C}_d \right] \begin{bmatrix} \bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} - L_x - \hat{B}_u L \Delta \end{bmatrix} y(k),
\]

(4.66a)

shows that we can implement the controller as

\[
\begin{bmatrix}
\bar{x}(k+1) \\
\bar{x}_d(k+1)
\end{bmatrix} = \begin{bmatrix}
\hat{A} + \hat{B}_u K_x & 0 \\
0 & \hat{A}_d
\end{bmatrix} \begin{bmatrix} \bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} + \begin{bmatrix} L_x - \hat{B}_u L \Delta \\
L_d \hat{C}_y \quad \hat{A}_d
\end{bmatrix} \begin{bmatrix} \bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} - \begin{bmatrix} L_x - \hat{B}_u L \Delta \\
L_d \hat{C}_y \quad \hat{A}_d
\end{bmatrix} y(k),
\]

(4.67a)

where \( \bar{L}_x = L_x - \hat{X} L_d \). This form shows that in fact \( x_{c_{i,j},\ell} = [0^T x_{d_{i,j},\ell}^T]^T \) are eigenvectors for the observer-controller with the multiplicity \( n_i \times N_y \) required by the IMP. We still need to verify that the multiplicity of any mode \( \lambda_i \in \sigma(A_d) \cap \sigma(A_c) \) is not greater than \( n_i \times N_y \), but this follows using an intermediate result that also allows us to verify that the observer gain \( B_c \) satisfies eq. (4.30).
Lemma 4.2.1 ([9, 52]) If the observer is stable and \( \dim\left( N\left( \tilde{A}_d - I\lambda_i \right) \right) = N_y \), then

\[
\dim\left( \mathcal{R}\left( L_d \right) \right) = N_y, \tag{4.68a}
\]

\[
\mathcal{R}\left( \tilde{A}_d - I\lambda_i \right) \cap \mathcal{R}\left( L_d \right) = 0. \tag{4.68b}
\]

**Proof** If the observer is stable, it means that the disturbance model modes \( \lambda_i \in \sigma(\tilde{A}_d) \) are stabilizable through \( [L^T_x \ L^T_d]^T \) using state-feedback \([\tilde{C}_y \ \tilde{C}_d]\). Hence, by the PBH rank test, we must have full row rank in

\[
\begin{bmatrix}
\tilde{A} - I\lambda_i & \tilde{B}_d & L_x \\
0 & \tilde{A}_d - I\lambda_i & L_d
\end{bmatrix}, \tag{4.69}
\]

and this can only be the case if

\[
\mathcal{R}\left( \begin{bmatrix} \tilde{A}_d - I\lambda_i & L_d \end{bmatrix} \right) = \mathcal{X}_d. \tag{4.70}
\]

However, by construction, we have that \( \dim\left( N\left( \tilde{A}_d - I\lambda_i \right) \right) = N_y \), so that the rank deficiency in \( \tilde{A}_d - I\lambda_i \) is precisely the number of measurements. Therefore, it must be that in \( [\tilde{A}_d - I\lambda_i \ L_d] \), the rank deficiency is recovered through the columns of \( L_d \), and this implies that \( \dim\left( \mathcal{R}\left( L_d \right) \right) = N_y \) and \( \mathcal{R}\left( \tilde{A}_d - I\lambda_i \right) \cap \mathcal{R}\left( L_d \right) = 0. \)

Using this result, we can now show that the multiplicity of any mode \( \lambda_i \in \sigma(\tilde{A}_d) \cap \sigma(\tilde{A}_c) \) is exactly \( n_i \times N_y \), and that eq. (4.30) is satisfied as well. With the observer-controller realization in eq. (4.67), assume there exists \( \{x, x_d\} \) with \( x \neq 0 \), such that

\[
\begin{bmatrix}
\tilde{A} + \tilde{B}_d K_x + (\tilde{L}_x - \tilde{B}_d L_\Delta) \tilde{C}_y - I\lambda_i & 0 \\
L_d \tilde{C}_y & \tilde{A}_d - I\lambda_i
\end{bmatrix} \begin{bmatrix} x \\ x_d \end{bmatrix} = 0, \tag{4.71}
\]

which would imply that the controller has more than \( N_y \) zero-order eigenvectors associated with the mode \( \lambda_i \). Note that, for the lower row to hold, \( \mathcal{R}\left( \tilde{A}_d - I\lambda_i \right) \cap \mathcal{R}\left( L_d \right) = 0 \) implies that \( (\tilde{A}_d - I\lambda_i)x_d = 0 \), and then \( \dim\left( \mathcal{R}\left( L_d \right) \right) = N_y \) requires that \( \tilde{C}_y x = 0 \). Applying this to the upper row, it means that \( (\tilde{A} - I\lambda_i)x = 0 \), but then since the gain \( K_x \) is stabilizing, it must be that \( x = 0 \) (\( \Rightarrow \)). Therefore, the multiplicity of any mode \( \lambda_i \in \sigma(\tilde{A}_d) \cap \sigma(\tilde{A}_c) \) is exactly \( n_i \times N_y \).
We now assume that there exists \( \{x, x_d, y\} \) with \( y \neq 0 \), such that

\[
\begin{bmatrix}
 L_x - \tilde{B}_u L_{\Delta} \\
 L_d
\end{bmatrix} y = \begin{bmatrix}
 \tilde{A} + \tilde{B}_u K_x + (L_x - \tilde{B}_u L_{\Delta}) \tilde{C}_y - I \lambda_i & 0 \\
 L_d \tilde{C}_y & \tilde{A}_d - I \lambda_i
\end{bmatrix} \begin{bmatrix}
 x \\
 x_d
\end{bmatrix},
\]

which would imply that \( \mathcal{R}(A_c - I \lambda_i) \cap \mathcal{R}(B_c) \neq 0 \). Again, since \( \mathcal{R}(\tilde{A}_d - I \lambda_i) \cap \mathcal{R}(L_d) = 0 \) and \( N(L_d) = 0 \), the lower row shows that \( (\tilde{A}_d - I \lambda_i) x_d = 0 \) and \( \tilde{C}_y x = y \). This then implies we must have

\[
\begin{bmatrix}
 L_x - \tilde{B}_u L_{\Delta} \\
 L_d
\end{bmatrix} y = \begin{bmatrix}
 \tilde{A} + \tilde{B}_u K_x + (L_x - \tilde{B}_u L_{\Delta}) \tilde{C}_y - I \lambda_i \\
 L_d \tilde{C}_y
\end{bmatrix} x
\]

\[
\Rightarrow 0 = (\tilde{A} - I \lambda_i) x.
\]

But again, since \( \tilde{A} \) is stable, it must be the case that \( x = 0 \), and hence \( y = \tilde{C}_y x = 0 \) (\( \Rightarrow \Leftarrow \)). Therefore, we have

\[
N(B_c) = N \left( \begin{bmatrix}
 L_x - \tilde{B}_u L_{\Delta} \\
 L_d
\end{bmatrix} \right) = 0, \quad \therefore N(L_d) = 0,
\]

\[
\mathcal{R}(A_c - I \lambda_i) \cap \mathcal{R}(B_c) = \mathcal{R} \left( \begin{bmatrix}
 \tilde{A} + \tilde{B}_u K_x + (L_x - \tilde{B}_u L_{\Delta}) \tilde{C}_y - I \lambda_i & 0 \\
 L_d \tilde{C}_y & \tilde{A}_d - I \lambda_i
\end{bmatrix} \right) \cap \mathcal{R} \left( \begin{bmatrix}
 L_x - \tilde{B}_u L_{\Delta} \\
 L_d
\end{bmatrix} \right) = 0,
\]

and our construction of the observer-controller satisfies the IMP.

### 4.2.2 Design of DAC Type Observer-Controllers for Selective Rejection

In this section, we show how the observer-controller approach in Section 4.2.1.1 can be modified to achieve selective rejection. In the most straightforward construction, the observer is again augmented with \( N_y = n_1 + n_2 \) copies of the disturbance model. However, even though the model is augmented with \( N_y \) copies of \( A_d \), the end result is that the resulting observer-controller dynamics only retains a multiplicity of \( n_1 \) in those dynamics. We show below, that the extra copies allow
us to arrange for \( \dim(\mathcal{R}(A_c - \lambda_i) \cap \mathcal{R}(B_{c2})) = n_2 \) as required by the IMP. Also, as shown at the end of Section 4.2.1.2, using \( N_y \) copies of the disturbance model in the observer dynamics results in a full rank \( B_{c1} \), and hence, it will also provide \( \mathcal{R}([A_c - \lambda_i B_{c2}]) \cap \mathcal{R}(B_{c1}) = 0 \). To facilitate the construction, we further partition the nominal model as

\[
\begin{bmatrix}
  x_t(k + 1) \\
  x_{d1}(k + 1) \\
  x_{d2}(k + 1)
\end{bmatrix} =
\begin{bmatrix}
  \hat{A} & \hat{B}_{d1} & \hat{B}_{d2} \\
  0 & \hat{A}_{d1} & 0 \\
  0 & 0 & \hat{A}_{d2}
\end{bmatrix}
\begin{bmatrix}
  x_t(k) \\
  x_{d1}(k) \\
  x_{d2}(k)
\end{bmatrix} +
\begin{bmatrix}
  \hat{B}_u \\
  0 \\
  0
\end{bmatrix} u(k) \tag{4.75a}
\]

\[
\begin{bmatrix}
  y_1(k) \\
  y_2(k)
\end{bmatrix} =
\begin{bmatrix}
  \hat{C}_{y1} & \hat{C}_{d11} & \hat{C}_{d12} \\
  \hat{C}_{y2} & \hat{C}_{d21} & \hat{C}_{d22}
\end{bmatrix}
\begin{bmatrix}
  x_t(k) \\
  x_{d1}(k) \\
  x_{d2}(k)
\end{bmatrix} \tag{4.75b}
\]

The modified construction is summarized as follows:

- Design a stabilizing state-feedback \( K_x \) for the nominal model \( \{\hat{A}, \hat{B}_u\} \).

- The disturbance model \( \hat{A}_d \) is partitioned into a block diagonal form with matrices \( \hat{A}_{d1} \) and \( \hat{A}_{d2} \) that contain \( n_1 \) and \( n_2 \) copies, respectively, of the disturbance model \( A_d \).

- Choose \( \hat{B}_d = [\hat{B}_{d1} \hat{B}_{d2}] \) with \( \hat{B}_{d2} = 0 \).

- Choose \( \hat{C}_{d22} \) so that the disturbance model \( A_{d2} \) is observable at \( y_2 \), and set \( \hat{C}_{d12} = 0 \).

- Design the disturbance gain \( K_{d1} \) and disturbance coupling \( \hat{C}_{d11} \) for the nominal subsystem \( \{\hat{A}, \hat{B}_u, \hat{B}_{d1}, \hat{C}_{y1}, \hat{C}_{d11}\} \), and set \( \hat{C}_d = [K_{d1} \ 0] \).

- With the solution of the regulator equation \( \hat{X}_1 \) used to design \( K_{d1} \), set

\[
\hat{C}_{d21} = -(\hat{C}_{y2}\hat{X}_1 + \hat{D}_{y2}K_{d1}) \tag{4.76}
\]

- Design a stabilizing observer gain for the nominal system augmented with the disturbance model \( \hat{A}_d \) and disturbance model couplings chosen above.
We now show that these choices satisfy the IMP for selective rejection. With the additional partition introduced earlier, the observer-controller dynamics take the form

\[
\begin{bmatrix}
    \bar{x}(k+1) \\
    \bar{x}_{d1}(k+1) \\
    \bar{x}_{d2}(k+1)
\end{bmatrix} = 
\begin{bmatrix}
    \hat{A} + \hat{B}_u K_x & \hat{B}_{d1} + \hat{B}_u K_{d1} & 0 \\
    0 & \hat{A}_{d1} & 0 \\
    0 & 0 & \hat{A}_{d2}
\end{bmatrix}
\begin{bmatrix}
    \bar{x}(k) \\
    \bar{x}_{d1}(k) \\
    \bar{x}_{d2}(k)
\end{bmatrix} + 
\begin{bmatrix}
    L_{x1} - \hat{B}_u L_{\Delta 1} & L_{x2} - \hat{B}_u L_{\Delta 2} \\
    L_{d11} & L_{d12} \\
    L_{d21} & L_{d22}
\end{bmatrix}
\begin{bmatrix}
    y_1(k) \\
    y_2(k)
\end{bmatrix}.
\] (4.77a)

In this scheme, \( \hat{A}_{d2} \) does not represent disturbances that are to be rejected at the output \( y_1 \), so we chose \( \hat{B}_{d2} = 0 \) from the start. Here we have also assumed that \( K_d = [K_{d1} \ 0] \), since it is only required to achieve rejection at the outputs in \( y_1 \), and the multiplicity of the disturbance model in \( \hat{A}_{d1} \) should be adequate to do so. Therefore, as in Section 4.2.1.1, \( K_{d1} \) and \( \hat{C}_{d11} \) are designed so that there is a solution to the regulator equation

\[
\begin{bmatrix}
    \hat{A} & \hat{B}_{d1} + \hat{B}_u K_{d1} \\
    \hat{C}_{y1} & \hat{C}_{d11}
\end{bmatrix}
\begin{bmatrix}
    \bar{X}_1 \\
    I
\end{bmatrix} = 
\begin{bmatrix}
    \bar{X}_1 \\
    0
\end{bmatrix} \hat{A}_{d1}.
\] (4.78)

We can now apply the regulator transformation

\[
\begin{bmatrix}
    \bar{x}(k) \\
    \bar{x}_{d1}(k) \\
    \bar{x}_{d2}(k)
\end{bmatrix} = 
\begin{bmatrix}
    I & -\bar{X}_1 & 0 \\
    0 & I & 0 \\
    0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
    \bar{x}(k) \\
    \bar{x}_{d1}(k) \\
    \bar{x}_{d2}(k)
\end{bmatrix},
\] (4.79)
to the observer-controller dynamics to obtain

$$A_c = \begin{bmatrix} \bar{A} & 0 & 0 \\ 0 & \bar{A}_{d1} & 0 \\ 0 & 0 & \bar{A}_{d2} \end{bmatrix} + \begin{bmatrix} L_{x1} - \bar{B}_u L_{\Delta 1} & L_{x2} - \bar{B}_u L_{\Delta 2} \\ L_{d11} & L_{d12} \\ L_{d21} & L_{d22} \end{bmatrix} \begin{bmatrix} \bar{C}_{y1} & 0 & 0 \\ \bar{C}_{y2} & 0 & \bar{C}_{d22} \end{bmatrix}$$

(4.80a)

$$= \begin{bmatrix} \bar{A} + (\bar{L}_{x1} - \bar{B}_u L_{\Delta 1}) \bar{C}_{y1} + (\bar{L}_{x2} - \bar{B}_u L_{\Delta 2}) \bar{C}_{y2} & 0 & (\bar{L}_{x2} - \bar{B}_u L_{\Delta 2}) \bar{C}_{d22} \\ L_{d11} \bar{C}_{y1} + L_{d12} \bar{C}_{y2} & \bar{A}_{d1} & L_{d12} \bar{C}_{d22} \\ L_{d21} \bar{C}_{y1} + L_{d22} \bar{C}_{y2} & 0 & \bar{A}_{d2} + L_{d22} \bar{C}_{d22} \end{bmatrix}$$

(4.80b)

$$B_c = \begin{bmatrix} \bar{L}_{x1} - \bar{B}_u L_{\Delta 1} & \bar{L}_{x2} - \bar{B}_u L_{\Delta 2} \\ L_{d11} & L_{d12} \\ L_{d21} & L_{d22} \end{bmatrix},$$

(4.80c)

where we used the fact that the choice for \( \bar{C}_{d21} \) results in \( 0 = \bar{C}_{y2} \bar{X}_1 + \bar{C}_{d21} + \bar{D}_y K_{d1} \). It should be clear now, that the construction contains at least the minimum multiplicity \( n_1 \) of disturbance models required.

Also, by lemma C.1.5, any stabilizing observer gain gives

$$\dim \left( \mathcal{R} \left( \begin{bmatrix} L_{d11} & L_{d12} \\ L_{d21} & L_{d22} \end{bmatrix} \right) \right) = n_1 + n_2 = N_y,$$

(4.81a)

$$\mathcal{R} \left( \begin{bmatrix} A_{d1} - I \lambda_i & 0 \\ 0 & A_{d2} - I \lambda_i \end{bmatrix} \right) \cap \mathcal{R} \left( \begin{bmatrix} L_{d11} & L_{d12} \\ L_{d21} & L_{d22} \end{bmatrix} \right) = 0,$$

(4.81b)

and it will then follow (as in Section 4.2.1.2) that \( \mathcal{R} ([A_c - \lambda_i B_{c2}]) \cap \mathcal{R} (B_{c1}) = 0 \). Let \( \{x_{d1i,0,\ell_1}, \ell_1 \in [1, n_1]\} \) and \( \{x_{d2i,0,\ell_2}, \ell_2 \in [1, n_2]\} \) denote the eigenvectors of \( \bar{A}_{d1} \) and \( \bar{A}_{d2} \), respectively. We can now show that \( \dim (\mathcal{R} (A_c - I \lambda_i) \cap \mathcal{R} (B_{c2})) = n_2 \), and only the modes in \( \bar{A}_{d1} \) are retained in the
observer-controller. Because we set $\hat{C}_{d2} = 0$, the vectors $[0^T \; 0^T \; x_{d2i,0,\ell_2}^T]^T$ are such that

$$
\begin{bmatrix}
\hat{A} - I\lambda_i & 0 & 0 \\
0 & \hat{A}_{d1} - I\lambda_i & 0 \\
0 & 0 & \hat{A}_{d2} - I\lambda_i
\end{bmatrix}
\begin{bmatrix}
L_{x1} - \bar{B}_u L_{\Delta_1} \\
L_{d11} \\
L_{d21}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
\bar{C}_{y1} & 0 & 0 \\
\bar{C}_{y2} & 0 & \bar{C}_{d22}
\end{bmatrix}
\begin{bmatrix}
x_{d2i,0,\ell_2}
\end{bmatrix}
$$

Since $\bar{C}_{d22}$ is chosen so that all modes $\lambda_i \in \sigma(\hat{A}_{d2})$ are observable at $y_2$,

$$\bar{C}_{d22} x_{d2i,0,\ell_2} \neq 0,$$

(4.83)

generates $n_2$ linearly-independent vectors in eq. (4.82a), so that

$$n_s = n_1 + \dim(\mathcal{R}(B_{c2})) - \dim(\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_{c2})),
$$

(4.84a)

$$= n_1.
$$

(4.84b)

It remains to show that the multiplicity of each mode $\lambda_i \in \sigma(A_d) \cap \sigma(A_c)$ is exactly $n_i \times n_1$.

By way of contradiction, assume that there exists $\{x, x_{d1}, x_{d2}\}$, with $\{x, x_{d2}\} \neq \{0, 0\}$, such that

$$
(A_c - \lambda_i)\begin{bmatrix}
x^T \\
x_{d1}^T \\
x_{d2}^T
\end{bmatrix}^T = 0,
$$

(4.85)

which then implies the controller has more than $n_1$ zero-order eigenvectors associated with the mode $\lambda_i$. However, eq. (4.80a) shows that this can be written as

$$
\begin{bmatrix}
\bar{A} - I\lambda_i & 0 & 0 \\
0 & \bar{A}_{d1} - I\lambda_i & 0 \\
0 & 0 & \bar{A}_{d2} - I\lambda_i
\end{bmatrix}
\begin{bmatrix}
x \\
x_{d1} \\
x_{d2}
\end{bmatrix}
= 
\begin{bmatrix}
L_{x1} - \bar{B}_u L_{\Delta_1} \\
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix}
\begin{bmatrix}
\bar{C}_{y1} & 0 & 0 \\
\bar{C}_{y2} & 0 & \bar{C}_{d22}
\end{bmatrix}
\begin{bmatrix}
x \\
x_{d1} \\
x_{d2}
\end{bmatrix}.
$$

(4.86)

Since eq. (4.81) holds, the lower two rows in eq. (4.86) show that both sides must in fact be zero. From the left side, this then implies that $x = 0$ (because $\bar{A}$ is stable), and that $x_{d1}$ and $x_{d2}$ are eigenvectors of $\bar{A}_{d1}$ and $\bar{A}_{d2}$, respectively. Applying this to the right side, we then have
that $\tilde{C}_{d22}x_{d2} = 0$, so that $x_{d2}$ is an unobservable eigenvector of $\tilde{A}_{d2}$. But, this is impossible if the disturbance model is chosen to be observable at $y_2 (\Rightarrow \Leftarrow)$. Therefore, the multiplicity of each mode $\lambda_i \in \sigma(\tilde{A}_d) \cap \sigma(\tilde{A}_c)$ is exactly $n_i \times n_1$, and the observer-controller satisfies the IMP.

4.3 DAC for Achieving Disturbance-Free MPC

In this closing section of Chapter 4, it is shown that the DAC approach can be combined with an MPC controller to achieve disturbance rejection and that the approach is equivalent to the offset-free methods in the MPC literature. In the next section, a brief argument is presented showing that offset-free MPC computations are equivalent to computing the disturbance gain $K_d$ of the previous section. Then the following section details the combination of DAC and MPC that will be referred to as MPDAC.

4.3.1 Equivalence of Offset-free MPC and DAC

The standard method for achieving offset free regulation in model predictive systems is to use an augmented observer to estimate the disturbance causing system offsets, and then “shift” the control and state targets for the MPC optimization [9]. For example, if the objective is to achieve a constant output $y_s$ in spite of a constant disturbance $d_s$ that couples into the system through $\{\tilde{B}_d, C_d\}$, then in steady state the system and constant control $u_s$ must satisfy

$$\begin{bmatrix} x_s \\ y_s \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} + \begin{bmatrix} \tilde{B}_d \\ C_d \end{bmatrix} d_s.$$  (4.87)

In the MPC literature, this equation is usually written

$$\begin{bmatrix} \tilde{A} - I & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} -\tilde{B}_d d_s \\ y_s - C_d d_s \end{bmatrix},$$  (4.88)

and it may as well be assumed that $y_s - C_d d_s = -\tilde{C}_d d_s$ so that the disturbance model generates the desired constant reference ($y_s = (C_d - \tilde{C}_d)d_s$). In quadratic MPC, the cost function optimized is
then

\[
J(x_0, u) = \sum_{k=1}^{N} \|Q(y(k) - y_s)\| + \|R(r(k) - u_s)\|, \tag{4.89a}
\]

\[
= \sum_{k=1}^{N} \|QC_y(x(k) - x_s)\| + \|R(r(k) - u_s)\|. \tag{4.89b}
\]

In [52] this approach is generalized to any disturbance/reference that can be generated via \(d(k) = \tilde{A}_d^k d_0\), where the spectrum \(\sigma(\tilde{A}_d)\) is typically taken from the unit circle so that the disturbances are persistent and bounded. In this case, eq. (4.88) becomes a modal relationship

\[
\begin{bmatrix}
\tilde{A} - I\lambda_i & \tilde{B}_i \\
\tilde{C}_y & 0
\end{bmatrix}
\begin{bmatrix}
x_i \\
u_i
\end{bmatrix}
= \begin{bmatrix}
-B_d \\
-C_d
\end{bmatrix} d_i, \tag{4.90}
\]

where \(\lambda_i d_i = \tilde{A}_d d_i\), and for the sake of simplicity it is assumed that \(\tilde{A}_d\) has a complete set of zero-order eigenvectors.

In contrast, the regulator equation (4.44) (with state-feedback gain \(K_x = 0\)) can be multiplied from the right by any eigenvector \(d_i\)

\[
\begin{bmatrix}
\tilde{A} & \tilde{B}_u K_d + \tilde{B}_d \\
\tilde{C}_y & \tilde{C}_d
\end{bmatrix}
\begin{bmatrix}
\tilde{X} \\
I
\end{bmatrix} d_i = \begin{bmatrix}
\tilde{X} \\
0
\end{bmatrix} \tilde{A}_d d_i = \begin{bmatrix}
\tilde{X} \\
0
\end{bmatrix} \lambda_i d_i, \tag{4.91}
\]

and then rearranged as

\[
\begin{bmatrix}
\tilde{A} & I\lambda \\
\tilde{C}_y & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{X} d_i \\
K_d d_i
\end{bmatrix} = \begin{bmatrix}
-B_d d_i \\
-C_d d_i
\end{bmatrix}. \tag{4.92}
\]

So, using

\[
\begin{bmatrix}
x_i \\
u_i
\end{bmatrix}
= \begin{bmatrix}
X d_i \\
K_d d_i
\end{bmatrix} \tag{4.93}
\]

it is apparent that solving the regulator equation for \(\{K_d, X\}\) provides a set of solutions \(\{x_i, u_i\}\) as required in the standard offset-free MPC approach. Conversely, the set of modal solutions obtained in the offset-free approach can be used to compute \(\{K_d, X\}\) (this is essentially the computation in eq. (4.61)). So, in effect, both approaches compute a basis for the space \(\mathcal{R}([\tilde{X}^T I]^T)\) in which the asymptotic response resides when the control \(u_s(k) = K_d x_d(k)\) is such that the asymptotic output is zero. In the next section, the DAC observer controller is combined with an MPC controller so as to obtain disturbance rejection.
4.3.2 Model Predictive Disturbance Accommodating Control (MPDAC)

The cost function for the MPC algorithm is set up so that there is a penalty on any control perturbation contributed by the MPC algorithm, while constraining the total control (MPC+state feedback+disturbance feedback) to be within acceptable limits. The model for the MPC computations is that of the plant with state and disturbance feedback in place. In order to minimize tracking error, it is then the job of the MPC controller to perturb the plant state into the invariant subspace that produces zero-output while keeping the total control constrained.

For the case that the MPC algorithm is not using exogenous measurements, the model it uses will indicate that it can bring its control effort to zero when the observer state (which includes a disturbance model) is in the invariant subspace that produces zero output. In this case, the MPC control effort will not contain disturbance content and the multiplicity in the disturbance model only needs to be that required without MPC.

However, if the MPC algorithm does use exogenous measurements like preview, then when the exogenous signals contain disturbance content, the model used by the MPC algorithm will in general not indicate zero output even if the observer state is in the desired subspace. In this event, its control effort will have persistent disturbance content and it must be treated as a plant measurement at which it is not possible to achieve disturbance rejection. In short, the multiplicity of the disturbance model in this case is increased as discussed in Section 4.2.2. So, the MPDAC observer proposed here is not as efficient as the constrained observer design method described in [51]. However, as shown in this section, these extra dynamics do not need to be included in the model used by the MPC algorithm for prediction.

Hence, when the MPC algorithm uses exogenous measurements, the model for DAC observer
design is
\[
\begin{bmatrix}
  x(k + 1) \\
  x_{d1}(k + 1) \\
  x_{d2}(k + 1)
\end{bmatrix} =
\begin{bmatrix}
  \hat{A} & B_{d1} & 0 \\
  0 & \hat{A}_{d1} & 0 \\
  0 & 0 & \hat{A}_{d2}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_{d1}(k) \\
  x_{d2}(k)
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  u(k) \\
  u_{mpc}(k)
\end{bmatrix},
\]
(4.94a)
\[
\begin{bmatrix}
  y_1(k) \\
  u_{mpc}(k)
\end{bmatrix} =
\begin{bmatrix}
  \hat{C}_{y1} & \hat{C}_{d11} & 0 \\
  0 & \hat{C}_{d21} & \hat{C}_{d22}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_{d1}(k) \\
  x_{d2}(k)
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  I
\end{bmatrix}
\begin{bmatrix}
  u_{mpc}(k)
\end{bmatrix},
\]
(4.94b)
where \( y_1 \) is the output where regulation is feasible, and the multiplicity of the disturbance model in \( \hat{A}_{d2} \) is \( N_u = \dim(u_{mpc}) \). Note that the model for the measurement \( [y_1^T \ u_{mpc}^T]^T \) includes disturbance content from \( A_{d2} \) in the MPC control, but that this content does not enter the system dynamics with the actual MPC control \( u_{mpc} \).

State feedback \( K_x \), if desired, is designed based on the subsystem
\[
\begin{align*}
  x(k + 1) &= \hat{A}x(k) + u(k), & (4.95a) \\
  y_1(k) &= \hat{C}_{y1}x(k), & (4.95b)
\end{align*}
\]
while the disturbance gains \( \{K_{d1}, \hat{C}_{d11}, \hat{C}_{d21}, \hat{C}_{d22}\} \) are computed using the selective-rejection procedure discussed in Section 4.2.2.

When the observer-controller estimation error goes to zero (see equations (4.42) and (4.43)), the system response is determined by
\[
\begin{bmatrix}
  x(k + 1) \\
  x_{d1}(k + 1) \\
  x_{d2}(k + 1)
\end{bmatrix} =
\begin{bmatrix}
  \hat{A} + BuK_x & \hat{B}_{d1} + BuK_{d1} & 0 \\
  0 & \hat{A}_{d1} & 0 \\
  0 & 0 & \hat{A}_{d2}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_{d1}(k) \\
  x_{d2}(k)
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  u_{mpc}(k)
\end{bmatrix},
\]
(4.96a)
\[
\begin{bmatrix}
  y_1(k) \\
  u_{total}(k)
\end{bmatrix} =
\begin{bmatrix}
  \hat{C}_{y1} & \hat{C}_{d11} & 0 \\
  K_x & K_{d1} & 0
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_{d1}(k) \\
  x_{d2}(k)
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  I
\end{bmatrix}
\begin{bmatrix}
  u_{mpc}(k)
\end{bmatrix},
\]
(4.96b)
and, ostensibly, this is the model that the MPC algorithm bases its computations on. However, the state \( x_{d2} \) of the extra disturbance model plays no role in the system state \( \{x, x_{d1}\} \) and output
\[ \begin{bmatrix} x(k+1) \\ x_{d1}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A} + BuK_x & \bar{B}_{d1} + B_dK_{d1} \\ 0 & \bar{A}_{d1} \end{bmatrix} \begin{bmatrix} x(k) \\ x_{d1}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_u \\ 0 \end{bmatrix} u_{mpc}(k), \] 

(4.97a)

\[ \begin{bmatrix} y_1(k) \\ u_{total}(k) \end{bmatrix} = \begin{bmatrix} \bar{C}_{y1} & \bar{C}_{d11} \\ K_x & K_{d1} \end{bmatrix} \begin{bmatrix} x(k) \\ x_{d1}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_{mpc}(k). \] 

(4.97b)

The output \( u_{total} \) is provided so that constraints can be applied to the total control effort. If there are additional system outputs (or state combinations) at which constraints are necessary, the output can be augmented to provide these as well.

### 4.4 Simple and Complete Examples

In the next two sections the proposed MPDAC method is applied to output regulation and then to (exogenous) reference tracking. The regulation example shows that offset free operation can’t be obtained by simply adding MPC to a system that already provides regulation via output augmentation. The reference tracking example demonstrates that providing preview to the MPC algorithm requires additional modifications to the basic offset-free configuration.

#### 4.4.1 Output Regulation with Anti-windup and MPC

Figure 4.1: (a) The feedback \( K_{aw} \) stabilizes the integrator during saturation; (b) simply providing the saturation function with MPC does not provide regulation; (c) providing the MPC an estimate of a disturbance acting on the system achieves regulation as long as the portion of \( K_x \) applied to the disturbance satisfies the appropriate regulator equation.

This section considers the addition of MPC to an existing system that uses integral control...
with anti-windup as depicted in Fig. 4.1a. In this case the plant

\[ p(z) = \frac{1 - 0.9}{z - 0.9} \]  

(4.98)

is realized as

\[ x(k + 1) = Ax(k) + Bu(k) + Bw w(k), \]  

(4.99a)

\[ y(k) = Cy x(k) + Dyu u(k) + Dyw w(k), \]  

(4.99b)

where

\[ A = 0.9, \quad Bu = 0.25, \quad Bw = 0.25, \]  

(4.100a)

\[ Cy = 0.4, \quad Dyu = 0, \quad Dyw = 0. \]  

(4.100b)

The output augmentation is a simple accumulator \((A_d = 1)\) so that the combined dynamics are

\[
\begin{bmatrix}
  x(k + 1) \\
  y_i(k + 1)
\end{bmatrix} =
\begin{bmatrix}
  A & 0 \\
  Cy & A_d
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  y_i(k)
\end{bmatrix} +
\begin{bmatrix}
  Bu & 0 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  u(k) \\
  u_{aw}(k)
\end{bmatrix}.
\]  

(4.101)

State feedback

\[ K = \begin{bmatrix} K_x & K_i \end{bmatrix} \]  

(4.102)

is designed to place the close loop poles at \(\{0.5, 0.51\}\).

If the control is not saturated, then in Fig. 4.1a

\[ u(k) = u_{aw}(k) = K_i y_i(k) + K_x x(k) \]  

(4.103)

and the effect of the anti-wind up gain is not present. When the control \(u\) saturates, the only remaining loop is around the anti-windup gain and the accumulator:

\[ y_i(k + 1) = y_i(k) + K_{aw} K_i y_i(k) + (K_{aw} K_x x(k) + y(k)). \]  

(4.104)

The anti-windup gain is set so that \(K_{aw} K_i = -1\), thereby placing the pole of the anti-windup loop at the origin. So, given that the control remains saturated \((u = u_{sat} \neq u_{aw})\), and assuming that the disturbance is constant, the output of the integrator goes to a constant

\[ y_{isat} = K_{aw} K_x x_{dc} + (u_{sat} + d_{in}), \]  

(4.105)
since the DC gain of the plant is unity \((y \rightarrow u_{\text{sat}} + d_{\text{in}})\), and where \(x_{dc}\) is the steady state plant response to \(u_{\text{sat}} + d_{\text{in}}\). This is the function of the anti-windup gain— it keeps the integrator from accumulating/“winding up” a constant tracking error into a ramp going to infinity.

In simulation the saturation limits are set at ±1 and the input is subjected to a disturbance that goes from 0.5 to 1.5 and then back as shown in the top plot of Fig. 4.2. To mitigate the input disturbance, the total control integrates to the opposite magnitude; when the disturbance goes beyond the saturation limit it can’t be cancelled completely and the plant output (center plot) becomes non-zero. Integral control with anti-windup, operating without MPC (labeled “AW” in Fig. 4.1), regulates the output to zero until the disturbance goes to 1.5, and then it prevents the integrator (lower/plot) from going off to infinity. This is in contrast to removing anti-windup (setting \(K_{aw} = 0\, \text{“No AW” in Fig. 4.1}).

In an MPC system, the function of the saturation block is provided by the control perturbation \(u_{\text{mpc}}\) as shown in Fig. 4.1b. The response using anti-windup and MPC is shown as “AW+MPC” in Fig. 4.2; in this case, the center plot shows that the output is not regulated to zero. The MPC algorithm uses the closed loop model

\[
\begin{align*}
\begin{bmatrix}
x(k+1) \\
y_i(k+1)
\end{bmatrix} &= \begin{bmatrix} A & 0 \\ C_y & A_d \end{bmatrix} \begin{bmatrix} B_w \\ K_x & K_i \end{bmatrix} \begin{bmatrix} x(k) \\ y_i(k) \end{bmatrix} + \begin{bmatrix} B_w \\ -K_{aw} \end{bmatrix} u_{\text{mpc}}(k), \\

u_{total}(k) &= \begin{bmatrix} K_x & K_i \end{bmatrix} \begin{bmatrix} x(k) \\ y_i(k) \end{bmatrix} + u_{\text{mpc}}(k),
\end{align*}
\]

where the state is provided to the MPC algorithm directly from the system (without using an observer), and the saturation limits ±1 are applied to \(u_{\text{total}}(k)\) during optimization. The issue is that there is no information to indicate that the integrator state \(y_i(k)\) should be non-zero. The MPC algorithm balances a penalty on its control effort with the output produced by its model with a non-zero state. Without information that there is a disturbance acting on the system, it appears to the MPC algorithm that it should allocate some effort to bringing the integrator state to 0.

In closing this section on explicit integral control, it should be noted that the configuration in Fig. 4.1b does provide regulation when \(K_{aw} = 0\). However, in this case the MPC controller...
does not have the ability to mitigate wind-up that will occur when the control saturates. This can be remedied to some extent by giving the MPC controller an additional input summing into the integrator, but doing so will then prevent regulation without an offset. Pursuing this final idea further will show that there is a trade-off in that fast mitigation of wind-up can only be obtained to the detriment of regulation.
### 4.4.2 Output Regulation with MPDAC

Using the same plant and input disturbance configuration, but without integral control, output regulation is obtained using a DAC observer-controller based on the model

\[
\begin{bmatrix}
    x(k+1) \\
    x_d(k+1)
\end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x(k) \\
    x_d(k)\end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix} u(k),
\]

\[y(k) = \begin{bmatrix} C_y & C_d \end{bmatrix} \begin{bmatrix} x(k) \\
    x_d(k)\end{bmatrix},\]

as shown in Fig. 4.1c. In this case the disturbance is incorrectly modelled as coupling directly into the output. This will demonstrate that it does not matter how the disturbance actually enters the system, which in this case happens to be through the plant input.

The nominal state-feedback gain \( K_x \) is chosen to place the closed loop pole at 0.5 and this results in a closed (state-feedback) loop gain from input to \( y \) of 0.2. Then \( K_d \) is set to -5 so that the closed loop DC gain from \( x_d \) to the \textit{model} output is \( 0.2 \times -5 = -1 \) and \( C_d \) is computed so that there is a solution to the state-feedback regulator equation (4.44). Since the control amplitude needs to track with the input disturbance magnitude, the result is that the observer-controller state \( x_d \) (for the output disturbance model) will track with \( y - u_{total} \). This is evident in the black/square line of the lower plot in Fig. 4.2.

The observer gain \( L = [L_x\ L_d]^T \) is chosen to place the observer poles at \{0.3, 0.31\}. The observer-controller dynamics are then

\[
\begin{bmatrix}
    x(k+1) \\
    x_d(k+1)
\end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x(k) \\
    x_d(k)\end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix} \begin{bmatrix} K_x & K_d \end{bmatrix} \begin{bmatrix} x(k) \\
    x_d(k)\end{bmatrix} - \begin{bmatrix} L_x \\ L_d \end{bmatrix} y(k) + \begin{bmatrix} B_u \\ 0 \end{bmatrix} u_{mpc}(k),
\]

\[u_{fbk}(k) = \begin{bmatrix} K_x & K_d \end{bmatrix} \begin{bmatrix} x(k) \\
    x_d(k)\end{bmatrix},\]
and the closed loop MPC model is
\[
\begin{bmatrix}
  x(k+1) \\
  x_d(k+1)
\end{bmatrix} = \left( \begin{bmatrix}
  A & 0 \\
  0 & A_d
\end{bmatrix} + \begin{bmatrix}
  B_u \\
  0
\end{bmatrix} \begin{bmatrix}
  K_x & K_d
\end{bmatrix} \right) \begin{bmatrix}
  x(k) \\
  x_d(k)
\end{bmatrix} + \begin{bmatrix}
  B_w \\
  K_{uw}
\end{bmatrix} u_{mpc}(k),
\] (4.109a)

\[
z(k) = \begin{bmatrix}
  C_y & C_d \\
  0 & 0
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  x_d(k)
\end{bmatrix} + \begin{bmatrix}
  0 \\
  1
\end{bmatrix} u_{mpc}(k),
\] (4.109b)

\[
u_{total}(k) = \begin{bmatrix}
  K_x & K_d
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  x_d(k)
\end{bmatrix} + u_{mpc}(k),
\] (4.109c)

where the MPC algorithm uses the state estimate obtained in the observer. A horizon of 20 samples is used for the MPC algorithm and the cost is set by defining

\[
Q_x = C_z^T Q_z C_z = \begin{bmatrix}
  C_y \\
  C_d
\end{bmatrix} \begin{bmatrix}
  C_y & C_d
\end{bmatrix},
\] (4.110a)

\[
R_u = D_{zu}^T Q_z D_{zu} = 1/10,
\] (4.110b)

\[
S_{zu} = C_z^T Q_z D_{zu} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix},
\] (4.110c)

\[
S_{uw} = D_{zu}^T Q_z D_{zw} = 0,
\] (4.110d)

\[
\Pi_{N+1} = \Pi_f = Q_x + A^T \Pi_f A - A^T \Pi_f B_u \left( R_u + B_u \Pi_f B_u \right)^{-1} B_u^T \Pi_f A,
\] (4.110e)

Now, during simulation the plant is perturbed to

\[
p(z) = \frac{1 - .9}{z - .8},
\] (4.111)

so that it has different dynamics and its DC gain is reduced by a factor of 2. The results ("DAC+MPC") are provided in Fig. 4.2 for comparison with the anti-windup results of the previous section. In the absence of MPC, the internal model principle guarantees that the output asymptotically goes to zero. And, when this occurs, the observer state is in the subspace that provides zero output through \([C_y C_d]\) – this is the only way the observer output estimate matches the actual
(a) MPDAC Reference Tracking

(b) MPDAC Reference Tracking w/ Feed-forward and Preview

Figure 4.3: (a) Reference tracking without preview requires the same complexity as disturbance rejection, but does not regulate if the MPC algorithm is provided the reference \( w \); (b) If the MPC algorithm is provided a preview then the multiplicity of the disturbance model must be increased— in this case it is also possible to add feed-forward compensation with the same increase in complexity.

Plant output in steady state. Since the MPC algorithm is based on the combined plant-disturbance model, it can minimize its control cost (eventually to zero) by helping to nudge the model state into the invariant subspace that provides zero output. This occurs as long as the MPC algorithm is not given exogenous information in addition to the observer state. The next section shows that additional modifications are required if the MPC algorithm is given preview information.

### 4.4.3 Reference Tracking with MPDAC

In this section, the system configuration is changed as shown in Fig. 4.3 so that the exogenous input \( w(\text{k}) \) represents a reference that the plant output is supposed to track. The system is simulated without using a preview of the reference, then using the preview only in the MPC algorithm, and then also adding preview feed-forward control. The plant model for reference tracking uses

\[
A = 0.9, \quad B_u = 0.25, \quad B_w = 0, \quad (4.112a)
\]

\[
C_y = 0.4, \quad D_{yu} = 0, \quad D_{yw} = -1, \quad (4.112b)
\]

so that the output \( y \) is now the tracking error relative to the reference \( w \).
4.4.3.1 Offset-free Tracking without Preview

Since the plant is stable, the state-feedback is chosen to be $K_x = 0$ and regulation is provided by the DAC observer-controller, but transient performance is largely determined by the plant’s time constant $0.9^k$. In this case $C_d$ is computed to be compatible with $K_d = 1$, and instead of using simple pole placement to design the observer, the methods discussed in Chapter 5 are employed. So, the observer gains are designed for the system

$$
\begin{bmatrix}
    x(k+1) \\
    x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
    A & 0 \\
    0 & A_d
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    x_d(k)
\end{bmatrix} +
\begin{bmatrix}
    B_u \\
    B_{xn}
\end{bmatrix} u(k) +
\begin{bmatrix}
    0 & 0 & 0 \\
    0 & B_{dn} & 0
\end{bmatrix}
\begin{bmatrix}
    n_x(k) \\
    n_dx(k) \\
    n_y(k) \\
    w(k)
\end{bmatrix},
$$

(4.113a)

$$
\Delta(k) = u(k) - K_x \begin{bmatrix}
    x(k) \\
    x_d(k)
\end{bmatrix} - \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    n_x(k) \\
    n_dx(k) \\
    n_y(k) \\
    w(k)
\end{bmatrix},
$$

(4.113b)

$$
y(k) = C_y \begin{bmatrix}
    x(k) \\
    x_d(k)
\end{bmatrix} + \begin{bmatrix}
    0 & 0 & D_{yn} & -1
\end{bmatrix}
\begin{bmatrix}
    n_x(k) \\
    n_dx(k) \\
    n_y(k) \\
    w(k)
\end{bmatrix},
$$

(4.113c)

to minimize the $H_2$ norm to deviations $\Delta(k)$ away from the desired state-feedback. The $n_{(j)}$ terms represent state and measurement noise sources. The observer gains are obtained by designing the
In this case, the corresponding LQR weighting matrices are

\[
\begin{bmatrix}
Q_x = C_z^T Q_z C_z = & \begin{bmatrix}
B_{xn} Q_{zz} B_{xn}^T & 0 \\
0 & B_{dn} Q_{zd} B_{dn}^T
\end{bmatrix}, \\
R_y = D_{zy}^T Q_z D_{zy} = & Q_{zw} + D_{yn} Q_{zy} D_{yn}^T, \\
S_{xy} = C_z^T Q_z D_{zy} = & \begin{bmatrix}
0 \\
0
\end{bmatrix},
\end{bmatrix}
\]

where \( Q_z \) is diagonal (or at least block diagonal) formed from \( \{ Q_{xx}, Q_{zd}, Q_{zy}, Q_{zu} \} \). Normally, the matrices \( \{ B_{xn}, B_{dn} \} \) that correspond to the amount of noise expected to be acting on the system states, are never actually formed. Instead, \( Q_x \) is chosen to be diagonal such that its terms can be tuned on a per-state basis. Making a component of \( Q_x \) large relative to \( R_y \) tends to make the time constants associated with estimation of that state faster. For the example in this section

\[
Q_x = \begin{bmatrix}
1 & 0 \\
0 & 100
\end{bmatrix},
\]

\[
R_y = 1.
\]

Given this choice of LQR weighting, the estimation of offsets acting on the plant is fast. Then, in the absence of MPC, the tracking performance is dominated by the time constant of the plant, because we have chosen \( K_x = 0 \). Although this design process is somewhat more involved, the implementation of the observer is still (with \( K_x = 0 \)) exactly the same as in eq. (4.108).
Figure 4.4: Referencing tracking using DAC and MPC with and without preview. With MPC and no preview, the tracking performance of the plant can be improved (red) over that obtained from the DAC controller (green). Keeping only the multiplicity required for regulation and giving the MPC controller a preview of the reference, regulation is lost (dashed blue). This can be remedied by increasing the multiplicity of the disturbance model in the observer (black).

The fast estimation of offsets and slow system response can be observed in Fig. 4.4 which shows the result of applying a reference ("REF", top plot) that goes from 0 to 1 at sample hit 21, and then from 1 to -1 at sample hit 59. Without the use of MPC, the feedback control ("DAC", lower plot) goes from 0 to 1 in essentially one step at sample hit 22, and then from 1 to -1 at sample hit 60. The output ("DAC", upper plot) responds as $1 - 0.9^k$ with the slow time constant of the plant.
When MPC is enabled, the algorithm uses the closed loop model (with $K_x = 0$)

\[
\begin{bmatrix}
 x(k+1) \\
 x_d(k+1)
\end{bmatrix} = \begin{bmatrix}
 [A & 0] + [B_u & [K_x & K_d]] [x(k)] + [B_u] u_{mpc}(k),
\end{bmatrix}
\]

\[
z(k) = \begin{bmatrix}
 C_x \\
 0
\end{bmatrix} \begin{bmatrix}
 x(k) \\
 x_d(k)
\end{bmatrix} + \begin{bmatrix}
 0 \\
 1
\end{bmatrix} u_{mpc}(k) + \begin{bmatrix}
 -1 \\
 0
\end{bmatrix} w(k),
\]

\[
u_{total}(k) = \begin{bmatrix}
 K_x & K_d
\end{bmatrix} \begin{bmatrix}
 x(k) \\
 x_d(k)
\end{bmatrix} + u_{mpc}(k) + 0 \times w(k),
\]

and cost defined by

\[
Q_x = C_x^T Q_z C_z = \begin{bmatrix}
 C_x^T Q_z C_z
\end{bmatrix} 5 \begin{bmatrix}
 C_x \\
 C_d
\end{bmatrix},
\]

\[
R_u = D_{zu}^T Q_z D_{zu} = 1/5,
\]

\[
S_{zu} = C_x^T Q_z D_{zu} = \begin{bmatrix}
 0 \\
 0
\end{bmatrix},
S_{zw} = C_x^T Q_z D_{zw} = \begin{bmatrix}
 C_x \\
 C_d
\end{bmatrix} 5 \times -1,
\]

\[
S_{uw} = D_{zu}^T Q_z D_{zw} = 0,
\]

\[
\Pi_{N+1} = \Pi_f = Q_x + A^T \Pi_f A - A^T \Pi_f B_u (R_u + B_u \Pi_f B_u)^{-1} B_u^T \Pi_f A.
\]

During simulation, the plant model is not perturbed, and the MPC algorithm is not supplied a preview of the reference so that $w(k) = 0$ in the MPC model. This results in the “DAC+MPC” waveforms in Fig. 4.4. The transient response is greatly improved and the system is still regulating the output to the reference with zero-offset.

However, since the reference $w(k)$ is available, it is desirable to have preview actuation. This should occur if the observer and MPC algorithm are left as is and the actual reference $w(k)$ is provided to the MPC algorithm instead of 0. The results (“DAC+MPC w/Prev”) are shown in Fig. 4.4. In this case, there is actuation (lower plot) before changes in the reference occur, but the system no longer regulates without an offset.
4.4.3.2 Offset-free Tracking With Preview

In order to achieve offset free tracking with preview, the control perturbations not generated by feedback from the observer are treated as a noisy system measurement. In this case, the observer is designed based on

\[
\begin{bmatrix}
  x(k+1) \\
  x_d(k+1) \\
  x_{du}(k+1)
\end{bmatrix}
= \begin{bmatrix}
  A & 0 & 0 \\
  0 & A_d & 0 \\
  0 & 0 & A_d
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_d(k) \\
  x_{du}(k)
\end{bmatrix}
+ \begin{bmatrix}
  B_u \\
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  u(k) \\
  w(k)
\end{bmatrix}
+ \begin{bmatrix}
  B_{xn} & 0 & 0 & 0 & 0 & B_u \\
  0 & B_{dn} & 0 & 0 & 0 & 0 \\
  0 & 0 & B_{un} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  n_x(k) \\
  n_{dx}(k) \\
  n_{ux}(k) \\
  n_y(k) \\
  n_u(k) \\
  u_{ext}(k)
\end{bmatrix},
\]

\[
(4.119a)
\]

\[
\begin{bmatrix}
  y_c(k) \\
  \bar{u}_{ext}(k)
\end{bmatrix}
= \begin{bmatrix}
  C_y & C_{d11} & 0 \\
  0 & C_{d21} & C_{d22}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_d(k) \\
  x_{du}(k)
\end{bmatrix}
+ \begin{bmatrix}
  0 & 0 & D_{yn} \\
  0 & 0 & 0 & 0 & D_{un} & 1
\end{bmatrix}
\begin{bmatrix}
  n_x(k) \\
  n_{dx}(k) \\
  n_{ux}(k) \\
  n_y(k) \\
  n_u(k) \\
  u_{ext}(k)
\end{bmatrix},
\]

\[
(4.119b)
\]

where \( \bar{u}_{ext}(k) \) is the measured external-control perturbation and \( u_{ext}(k) \) is the actual control perturbation being added to the observer-controller feedback and being input to the plant. The additional copy of the dynamics \( A_d \) represents disturbance content in the measurement \( \bar{u}_{ext}(k) \).
In contrast to eq. (4.108), the observer-controller dynamics are implemented as

\[
\begin{bmatrix}
  x(k+1) \\
  x_d(k+1) \\
  x_u(k+1)
\end{bmatrix} =
\begin{bmatrix}
  A & 0 & 0 \\
  0 & A_d & 0 \\
  0 & 0 & A_d
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_d(k) \\
  x_u(k)
\end{bmatrix} +
\begin{bmatrix}
  B_u \\
  0 \\
  0
\end{bmatrix} \begin{bmatrix}
  K_x & K_d & 0 \\
  0 & L_{dd} & L_{du} \\
  0 & L_{ud} & L_{uu}
\end{bmatrix}
\begin{bmatrix}
  C_y & C_{d11} & 0 \\
  0 & C_{d21} & C_{d22}
\end{bmatrix}
\begin{bmatrix}
  y(k) \\
  x_d(k) \\
  x_u(k)
\end{bmatrix} \\
- \begin{bmatrix}
  L_x & L_{xu} \\
  L_{dd} & L_{du} \\
  L_{ud} & L_{uu}
\end{bmatrix}
\begin{bmatrix}
  y_e(k) \\
  u_{ext}(k)
\end{bmatrix},
\]

(4.120a)

\[
u_{sfbk}(k) = \begin{bmatrix}
  K_x & K_d & 0 \\
  0 & L_{dd} & L_{du} \\
  0 & L_{ud} & L_{uu}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_d(k) \\
  x_u(k)
\end{bmatrix}.
\]

(4.120b)

The distinction, besides the extra disturbance dynamics, is that control perturbations \( u_{ext}(k) \) generated exogenously enter through the gains \( \{L_{xu}, L_{du}, L_{uu}\} \) instead of through \( B_u \). The effect of the latter gain is still taken into account in the observer model (4.119) for design of gains, but it is not present in the observer-controller implementation as would normally be the case without treating \( u_{ext}(k) \) as a noisy measurement.

The observer gains are obtained by designing the LQR optimal full information “control” \([y_e(k) \ \bar{u}_{ext}(k)]\) for the adjoint system

\[
\begin{bmatrix}
  x(k+1) \\
  x_d(k+1) \\
  x_u(k+1)
\end{bmatrix} =
\begin{bmatrix}
  A & 0 & 0 \\
  0 & A_d & 0 \\
  0 & 0 & A_d
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_d(k) \\
  x_u(k)
\end{bmatrix} +
\begin{bmatrix}
  C_y & 0 \\
  C_{d11}^T & C_{d21}^T \\
  0 & C_{d22}^T
\end{bmatrix}
\begin{bmatrix}
  y_e(k) \\
  \bar{u}_{ext}(k)
\end{bmatrix} -
\begin{bmatrix}
  K_x^T \\
  K_d^T
\end{bmatrix}\Delta(k),
\]

(4.121a)
In this case, the values used for design of observer gains are

\[ Q_x = C_z^T Q_z C_z = \begin{bmatrix} B_{xu} Q_{xx} B_{xu}^T + B_u Q_{zu} B_u^T & 0 & 0 \\ 0 & B_{du} Q_{zd} B_{du}^T & 0 \\ 0 & 0 & B_{un} Q_{zu} B_{un}^T \end{bmatrix}, \]  

(4.122a)

\[ R_y = D_{zy}^T Q_z D_{zy} = \begin{bmatrix} Q_{zw} + D_{yn} Q_{zy} D_{yn}^T & 0 \\ 0 & Q_{zu} + D_{un} Q_{zun} D_{un}^T \end{bmatrix}, \]  

(4.122b)

\[ S_{xy} = C_z^T Q_z D_{xy} = \begin{bmatrix} 0 & B_u Q_{zu} \\ 0 & 0 \end{bmatrix}. \]  

(4.122c)

In this case, the values used for design of observer gains are

\[ Q_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} B_u \\ 0 \\ 0 \end{bmatrix}^T, \]  

(4.123a)

\[ R_y = \begin{bmatrix} 1 & 0 \\ 0 & 1/10 \end{bmatrix}, \]  

(4.123b)

\[ S_{xy} = \begin{bmatrix} 0 & 10 B_u \\ 0 & 0 \end{bmatrix}. \]  

(4.123c)
In addition to the state-feedback control \([K_x(= 0) \ K_d \ 0]\), a feed-forward control is designed as discussed in Chapter 5 and situated as shown in Fig. 4.3b. The delays are arranged so that the MPC is provided a preview of both the reference and the feed-forward control in a manner that insures these previews are coincident with each other.

The model used by the MPC algorithm is that of the closed loop when the observer estimation error goes to zero. As explained in Section 4.3.2, this does not include the extra dynamics that model the disturbance content in the measurement of exogenous control inputs, but in this case does include the feed-forward command

\[
\begin{bmatrix}
    x(k+1) \\
    x_d(k+1)
\end{bmatrix} = \begin{bmatrix} A & 0 \\
0 & A_d \end{bmatrix} + \begin{bmatrix} B_u \\
0 \end{bmatrix} \begin{bmatrix} K_x & K_d \end{bmatrix} \begin{bmatrix} x(k) \\
x_d(k) \end{bmatrix} + \begin{bmatrix} B_u \\
0 \end{bmatrix} u_{mpc}(k) + \begin{bmatrix} 0 & B_u \\
0 & 0 \end{bmatrix} \begin{bmatrix} w(k) \\
u_{ff}(k) \end{bmatrix}, \tag{4.124a}
\]

\[
z(k) = \begin{bmatrix} C_z \\
D_{zu} \\
D_{zw} \end{bmatrix} \begin{bmatrix} x(k) \\
x_d(k) \\
0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} w(k) \\
u_{ff}(k) \end{bmatrix}, \tag{4.124b}
\]

\[
u_{total}(k) = \begin{bmatrix} K_x & K_d \\
0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\
x_d(k) \end{bmatrix} + u_{mpc}(k) + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} w(k) \\
u_{ff}(k) \end{bmatrix}. \tag{4.124c}
\]

The simulation results using this configuration are provided in black in Fig. 4.4. The system now provides significant preview actuation and the desired regulation is reacquired.
This chapter derives formulae for designing $\mathcal{H}_2$ optimal, output-feedback controllers that use preview. It is shown that the problem partitions according to the standard separation principle for $\mathcal{H}_2$ control. That is, the state of the system can be augmented to model preview information, and then optimal state-feedback and output injection gains can be designed for the augmented
system. As shown in Hazell [28], the optimal full-information gains are independent of the amount of preview, and the order of the discrete-time algebraic Riccati equation (DARE) can be reduced to that of the plant without preview augmentations.

In this chapter, it is shown that in the absence of augmentations for output regulation, dual results are obtained for the design of the observer output injection gains. If the plant and preview measurements (in Fig. 5.1, \( y_y(k) \) and \( w_y(k) \), respectively) are viewed as corrupted by white noise, and an optimal observer is designed for the preview augmented system, then the dimension of the observer DARE again reduces to that of the un-augmented plant. In this case the optimal output injection to the plant state-estimates is independent of the amount of preview. Further, the optimal estimate of the preview is simply to attenuate the preview measurement (in inverse proportion to its measurement noise) and store it in a series of delays. The resulting optimal output feedback controller with noisy preview measurements has the block partitioned structure shown in Fig. 5.1.

Unfortunately, when the observer model is augmented to provide regulation, the gains \([L_y \ L_{pr}]\)
no longer partition as shown in Fig. 5.1. Instead, output injection is required into every state of
the storage for the preview from both $y_g(k)$ and $w_y(k)$, and injection into the plant states must
be handled similarly. Further, the dimension of the associated DARE cannot be reduced— the
states augmented in order to model the preview must be included. For this reason, a sub-optimal
approach is taken for the final design. The optimal estimate of the feed-forward control $u_{ff}(k)$ is
assumed to be done separately using only the preview measurement, and without use of the plant
output measurement $y_g(k)$. The feedback controller is optimized assuming that it has access to
the noisy plant measurement $y_g(k)$, a noisy measurement $w_y(k)$ of the present disturbance, and a
noisy measurement $u_y(k)$ of the desired feed-forward command $u_{ext}(k)$. This partitioning is illus-
trated in Fig. 5.2. In this case, “AUGMENTED PLANT” refers to the extra dynamics required
for regulation as discussed in Chapter 4.

This chapter begins with a review of $\mathcal{H}_2$ performance (Section 5.1), feedback/feedforward
gain computation (Section 5.1.3), and then the separation property as it applies to the $\mathcal{H}_2$ cost
for output feedback (Section 5.2). In these first sections, it is shown that for discrete-time, the
optimal control includes feed-forward (preview or not) from the disturbance inputs to the system.
This is of interest primarily, because it shows how to handle the preview gains (Section 5.1.4), and
also when applied to the dual observer design problem (Section 5.3), it shows that there is a direct
feed-through term that is not present in the continuous time case.

5.1 $\mathcal{H}_2$ System Performance and State-Feedback

5.1.1 $\mathcal{H}_2$ State-Feedback Standing Assumptions

For the purposes of computing full information (i.e. state-feedback) gains, the system $\Sigma_z$ of
interest is

$$x(k+1) = Ax(k) + B_w w(k) + B_u u(k),$$

(5.1a)

$$z(k) = C_z x(k) + D_z w(k) + D_z u(k).$$

(5.1b)
It is assumed that any weighting $Q_z$ like that used in Appendix A has been appropriately absorbed into the matrices associated in the forming of $z(k)$. Including such weighting explicitly, would change the results of this chapter only where products between the matrices $\{C_z, D_{zw}, D_{zu}\}$ occur in the design of full information gains; and further, $Q_z$ does not change the formulae for computing observer output injection gains.

Also, the feed-through term $D_{zw}$ is included, because in contrast to continuous-time, the auto-correlation

$$E\{w(k)w(k+j)^T\} = I_w\delta(j)$$

of zero-mean white noise is well defined outside the context of an integral. This thesis considers only the case that the control effort is fully penalized in the performance $z(k)$, all modes on the unit circle are detectable at $z(k)$, and the system is stabilizable. So, the following standing assumptions are made.

**Assumption 5.1.1**

\[ R([A - I\lambda B_u]) = \mathcal{D}, \quad \forall \lambda \notin \mathbb{C}_g \] \hfill (5.3a)

\[ D_{zu}^T D_{zu} \preceq R_u > 0, \] \hfill (5.3b)

\[ D_{zu}^T [C_z D_{zw}] = 0, \] \hfill (5.3c)

\[ N \left( \begin{bmatrix} A - I e^{j\omega} \\ C_z \end{bmatrix} \right) = 0. \] \hfill (5.3d)

That the system is stabilizable (assumption (5.3a)) is obviously necessary, assumption (5.3b) penalizes all directions of control effort and guarantees that the optimization is not singular, and assumption (5.3c) greatly simplifies the resulting formulae. Assumption (5.3d) is required to guarantee that there is a unique, positive-definite (stabilizing) solution to the DARE [75]. Arbitrary systems satisfying assumption (5.3b) can be transformed to satisfy (5.3c), by application of the affine control shift described in Appendix A. Use of this control transformation requires additional assumptions on the transmission zeros from the disturbance input to the performance output, in lieu of assumption (5.3d), and this is explained in Appendix A as well.
5.1.2 \( \mathcal{H}_2 \) Performance

Temporarily assume that the system \( \Sigma_z \) is stable and that \( w(k) \) is a zero-mean white-noise input. Then it is possible to characterize performance by computing the variance of the output

\[
\| \Sigma_z \|^2_{\mathcal{H}_2} \triangleq \mathbb{E} \left\{ \left( z(k) - \mathbb{E}\{z(k)\} \right)^T (z(k) - \mathbb{E}\{z(k)\}) \right\},
\]

\[
= \mathbb{E} \left\{ z(k)^T z(k) \right\}, \quad : \mathbb{E}\{w(k)\} = 0 \Rightarrow \mathbb{E}\{z(k)\} = 0,
\]

\[
= \text{trace} \left( \mathbb{E}\left\{ z(k) z(k)^T \right\} \right).
\]

(5.4a) (5.4b) (5.4c)

Now, assuming \( u(k) = 0 \) and substituting

\[
z(k) = \sum_{i=0}^{\infty} D_{zw} w(k) + C_z A^i B_w w(k - i - 1),
\]

(5.5)

into the definition for the \( \mathcal{H}_2 \) norm and simplifying we obtain

\[
\| \Sigma_z \|^2_{\mathcal{H}_2} = \mathbb{E} \left\{ \left( D_{zw} w(k) + \sum_{i=0}^{\infty} C_z A^i B_w w(k - i - 1) \right)^T \left( D_{zw} w(k) + \sum_{j=0}^{\infty} C_z A^j B_w w(k - j - 1) \right) \right\}
\]

\[
= \mathbb{E} \left\{ w(k)^T D_{zw}^T D_{zw} w(k) \right\} + \mathbb{E} \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (w(k - i - 1))^T B_w^T A^{iT} C_z C_z A^j B_w w(k - j - 1) \right\}
\]

\[
+ 2 \mathbb{E} \left\{ \sum_{j=0}^{\infty} w(k)^T D_{zw} C_z A^j B_w w(k - j - 1) \right\}
\]

(5.6a)

\[
= \text{trace} \left( D_{zw} \left( \mathbb{E}\{w(k) w(k)^T\} \right) D_{zw}^T \right)
\]

\[
+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \text{trace} \left( C_z A^i B_w \left( \mathbb{E}\{w(k - j - 1) w(k - i - 1)^T\} \right) B_w^T A^{iT} C_z \right),
\]

(5.6b)

\[
= \text{trace} \left( D_{zw} D_{zw}^T \right) + \sum_{i=0}^{\infty} \text{trace} \left( C_z A^i B_w B_w^T A^{iT} C_z^T \right)
\]

(5.6c)

\[
= \text{trace} \left( D_{zw} D_{zw}^T \right) + \text{trace} \left( B_w^T \left( \sum_{i=0}^{\infty} A^{iT} C_z^T C_z A^i \right) B_w \right)
\]

(5.6d)

In the derivations for eq. (5.6), the double sum becomes a single sum and the single sum becomes zero, because for white noise, \( \mathbb{E}(w(k - j) w(k - i)) = 0, \forall j \neq i \). The matrix \( X \) is the observability Grammian

\[
X = \sum_{j=0}^{\infty} (A^T)^j C_z^T C_z A^j > 0,
\]

(5.7)
and satisfies the Lyapunov equation

\[ X = C_z^T C_z + A^T X A. \]  

(5.8)

If we denote the impulse response matrix of \( \Sigma_z \) with \( u(k) = 0 \), by \( \Sigma_z(k) \), then the \( \mathcal{H}_2 \) norm can be expressed as

\[ \| \Sigma_z \|_{\mathcal{H}_2}^2 = \text{trace} \left( \sum_{k=0}^{\infty} \Sigma_z(k)^T \Sigma_z(k) \right) = \sum_{k=0}^{\infty} \text{trace} \left( (\Sigma_z(k))^T \Sigma_z(k) \right). \]

(5.9)

This shows, that feedback controls resulting in the same closed-loop impulse response, have the same \( \mathcal{H}_2 \) performance.

### 5.1.3 \( \mathcal{H}_2 \) Full Information Feedback/Feedforward

Using the expression (5.6d), it is possible to deduce the \( \mathcal{H}_2 \) optimal full-information control gains. Again, since \( \mathbb{E}\{w(k)w(k)^T\} = I_w \), it is possible to include a feed-forward term in the control

\[ u(k) = \begin{bmatrix} K_x & K_u \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \leftrightarrow \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \]

(5.10)

So, in order to facilitate an organized presentation, and as a prelude to computing preview gains, the derivation is done with the augmented system \( \Sigma_{z0} \)

\[
\begin{bmatrix}
  x(k+1) \\
  w(k+1)
\end{bmatrix} = \begin{bmatrix} A & B_w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{w}(k),
\]

(5.11a)

\[
z(k) = \begin{bmatrix} C_z & D_{zw} \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} + D_{zu} u(k) + 0 \bar{w}(k), \quad \bar{D}_{zw} \bar{w} = 0,
\]

(5.11b)

where we define \( \bar{w}(k) \equiv w(k+1) \). Denote \( \Sigma_{z0} \) with the control (5.10) as \( \Sigma_{Ku0} \), then this system is given by

\[
x(k+1) = (\bar{A} + \bar{B}_u \bar{K}) \bar{x}(k) + \bar{B}_w \bar{w}(k), \quad \bar{D}_{zw} \bar{w} = 0.
\]

(5.12a)

\[
z(k) = (\bar{C}_z + \bar{D}_{zu} \bar{K}) \bar{x}(k) + \bar{D}_{zw} \bar{w}(k).
\]

(5.12b)
Assuming that the gain is stabilizing, then according to eq. (5.6d) and eq. (5.7), the $\mathcal{H}_2$ norm is

$$
\|\Sigma_{Kw0}\|_{\mathcal{H}_2}^2 = \text{trace}(B_{zw}B_{zw}^T) + \sum_{i=0}^{\infty} \text{trace}\left(B_{w}^T(A + BK)^T(\overline{C}_z + D_{zu}\overline{K})(A + BK)^TB_w\right)
$$

$$
= \text{trace}\left(B_{w}^T\overline{X}B_w\right)
$$

(5.13a)

where $\overline{X}$ is the positive semi-definite solution to

$$
\overline{X} = (\overline{C}_z + D_{zu}\overline{K})^T(\overline{C}_z + D_{zu}\overline{K}) + (A + BK)^T\overline{X}(A + BK)
$$

(5.14a)

$$
= \overline{C}_z^T\overline{C}_z + \overline{K}_z^TR_u\overline{K}_z + (A + BK)^T\overline{X}(A + BK), \quad \overline{K}_z^T\overline{C}_z = D_{zu}^T[C_z D_{zu}]= 0.
$$

(5.14b)

Now, from the standard LQR optimization presented in Appendix A, it recognized that if

$$
\overline{K} = \overline{K}_s = -(R_u + B^T\overline{X}_sB)^{-1}B^T\overline{X}_sA
$$

(5.15)

where $\overline{X}_s$ is the stabilizing solution to the DARE

$$
\overline{X}_s = \overline{A}^T\overline{X}_s\overline{A} + \overline{C}_z^T\overline{C}_z - \overline{A}^T\overline{X}_s\overline{B}(R_u + B^T\overline{X}_sB)^{-1}B^T\overline{X}_s\overline{A},
$$

(5.16a)

$$
= \overline{C}_z^T\overline{C}_z + \overline{K}_s^TR_u\overline{K}_s + (\overline{A} + \overline{B}\overline{K}_s)^T\overline{X}_s(\overline{A} + \overline{B}\overline{K}_s),
$$

(5.16b)

then $\overline{X} = \overline{X}_s$ is also a solution to eq. (5.14b). Also, from Appendix A, setting $u(k) = \overline{K}_s\overline{x}(k)$, gives the minimizing control for the cost

$$
\overline{x}_0^T\overline{X}_s\overline{x}_0 = \sum_{k=0}^{\infty} z(k)^Tz(k)
$$

(5.17)

with the dynamics in eq. (5.11) when $\overline{w}(k) = 0$ and $\overline{\pi}(0) = \overline{x}_0$. Now, more significantly, the $\mathcal{H}_2$ norm (5.13b) can be written as

$$
\text{trace}(B_{w}^T\overline{X}_sB_w) = \text{trace}(X_sB_{w}B_{w}^T), \quad \because \text{trace}(MN) = \text{trace}(NM)
$$

(5.18a)

$$
= \text{trace}(X_s \sum_{i=1}^{n_u} \overline{b}_i\overline{b}_i^T) = \text{trace}(\sum_{i=1}^{n_u} X_s\overline{b}_i\overline{b}_i^T)
$$

(5.18b)

$$
= \text{trace}(\sum_{i=1}^{n_u} \overline{b}_i^T X_s\overline{b}_i) = \sum_{i=1}^{n_u} \overline{b}_i^T X_s\overline{b}_i
$$

(5.18c)
where $\bar{b}_i$ is the $i^{th}$ column of $\bar{B}_w$. Therefore, since $\bar{K}_*$ gives the minimal cost $\bar{x}_0^T\bar{X}_s\bar{x}_0$ for arbitrary $ar{x}_0$ (e.g., including $\bar{x}_0 = \bar{b}_i$), it now follows that $\bar{K}_*$ is also the optimal $\mathcal{H}_2$ state-feedback gain.

A closer inspection of the DARE (5.16a) determining $\bar{X}_s$ shows that

$$\bar{X}_s = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} A^T & 0 \\ B_w^T & 0 \end{bmatrix} \bar{X}_s \begin{bmatrix} A & B_w \\ 0 & 0 \end{bmatrix} + \bar{C}_z^T \bar{C}_z,$$

$$- \begin{bmatrix} A^T & 0 \\ B_w^T & 0 \end{bmatrix} \bar{X}_s \begin{bmatrix} B_u \\ 0 \end{bmatrix} (R_u + B_u^T X_{11}^* B_u)^{-1} \begin{bmatrix} B_u^T & 0 \end{bmatrix} \bar{X}_s \begin{bmatrix} A & B_w \\ 0 & 0 \end{bmatrix},$$

(5.19a)

$$= \begin{bmatrix} A^T X_{11} A - A^T X_{11} B_u (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} A + C_z^T C_z & X_{12} \\ X_{21} & X_{22} \end{bmatrix},$$

(5.19b)

where

$$X_{21} = X_{12} = B_w^T X_{11} A - B_u^T X_{11} B_u (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} A + D_{zw}^T C_z,$$

(5.20a)

$$X_{22} = B_w^T X_{11} B_u D_{zw} - B_w^T X_{11} B_u (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} B_u.$$

(5.20b)

Note that, $\bar{X}_s > 0 \Rightarrow X_{11} > 0$, and Assumption 5.1.1 guarantees that the stabilizing solution to

$$X_{11} = A^T X_{11} A - A^T X_{11} B_u (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} A + C_z^T C_z,$$

(5.21)

is unique. Hence, the DARE associated with the optimal feedback/feedforward gain $\bar{K}_*$, depends only on the original un-augmented dynamics. Similar inspection of the optimal gain shows that

$$\bar{K}_* = - \begin{bmatrix} R_u + B_u^T & 0 \\ X_{21} & X_{22} \end{bmatrix}^{-1} \begin{bmatrix} B_u \\ 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}^{-1} \begin{bmatrix} A & B_w \\ 0 & 0 \end{bmatrix}$$

(5.22a)

$$= - (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} A - (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} B_u,$$

(5.22b)

$$\delta [K_u \ K_{u0}].$$

(5.22c)

Finally, note that since $\bar{D}_w = [0 \ I]^T$, the $\mathcal{H}_2$ norm (5.13b) becomes

$$\|\Sigma_{Kw0}\|_{\mathcal{H}_2}^2 = \text{trace} \left( D_{zw}^T D_{zw} + B_w^T X_{11} B_w - B_u^T X_{11} B_u (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} B_u \right).$$

(5.23)
Also, by construction, using only the feedback gain $K_x$ gives the minimum $\mathcal{H}_2$ norm without the feed-forward control $u_{ff}(k) = K_{w0}w(k)$, and this results in the cost
\[
\|\Sigma_K\|_{\mathcal{H}_2}^2 = \text{trace} \left( D_{zw}^T D_{zw} + B_{w}^T X_{11} B_{w} \right).
\]
Hence,
\[
\|\Sigma_{K_{w0}}\|_{\mathcal{H}_2}^2 = \|\Sigma_K\|_{\mathcal{H}_2}^2 - \text{trace} \left( B_{w}^T X_{11} B_{u} \left( R_{u} + B_{u}^T X_{11} B_{u} \right)^{-1} B_{u}^T X_{11} B_{w} \right),
\]
so that using feed-forward is always at least as good as feedback only. Finally, note also that $\Sigma_{K_{w0}}$ can be viewed as applying zero-step ahead preview and feedback directly to the system $\Sigma_z$ without the preliminary step of creating the augmented system $\Sigma_{z0}$. In the next section, it is shown that each additional level of disturbance preview provides a similar reduction in $\mathcal{H}_2$ norm.

### 5.1.4 Recursive Preview Gain Computation

Additional levels of preview gain can be obtained with further augmentation of the system to make future disturbances $w(k + i)$ available as part of the system state. For example, a gain for one-step ahead preview can be obtained by optimizing for the system $\Sigma_{z1}$

\[
\begin{bmatrix}
  x(k + 1) \\
  w(k + 1) \\
  w(k + 2)
\end{bmatrix} =
\begin{bmatrix}
  A & B_w & 0 \\
  0 & 0 & I \\
  0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  w(k) \\
  w(k + 1)
\end{bmatrix} +
\begin{bmatrix}
  B_u \\
  0 \\
  0
\end{bmatrix} u(k) +
\begin{bmatrix}
  0 \\
  0 \\
  I
\end{bmatrix} \bar{w}(k),
\]

\[
z(k) =
\begin{bmatrix}
  C_z & D_{zw} & 0
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  w(k) \\
  w(k + 1)
\end{bmatrix} + D_{zu} u(k) + \bar{D}_{zw} \bar{w}(k), \quad \bar{D}_{zw} = 0,
\]

\[
5.26
\]
where in this case \( \overline{w}(k) = w(k+2) \). Expanding the DARE for this system in similar fashion to that used in the previous section, shows that

\[
\begin{bmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{bmatrix} = \begin{bmatrix}
A^T X_{11} A & A^T X_{11} B_w & A^T X_{12} \\
B_w^T X_{11} A & B_w^T X_{11} B_w & B_w^T X_{12} \\
X_{21} A & X_{21} B_w & X_{22}
\end{bmatrix} + \begin{bmatrix}
C_z^T \\
D_{zw}^T \\
0
\end{bmatrix} \begin{bmatrix}
C_z \\
D_{zw} \\
0
\end{bmatrix} \quad \text{(5.27a)}
\]

From this expression, it is deduced that \( X_{11} \) again satisfies eq. (5.21),

\[
\begin{aligned}
X_{21} &= X_{12} = B_w^T X_{11} A - B_u^T X_{11} B_u \left( R_u + B_u^T X_{11} B_u \right)^{-1} B_u^T X_{11} A + D_{zw} C_z, \\
X_{31} &= X_{21} A - X_{21} B_u \left( R_u + B_u^T X_{11} B_u \right)^{-1} B_u^T X_{11} A,
\end{aligned} \quad \text{(5.28a,b)}
\]

and by induction for \( i > 1 \)

\[
\begin{aligned}
X_{i+1,1} &= X_{i,1} A - X_{i,1} B_u \left( R_u + B_u^T X_{11} B_u \right)^{-1} B_u^T X_{11} A, \\
&= X_{i,1} \left( A - B_u \left( R_u + B_u^T X_{11} B_u \right)^{-1} B_u^T X_{11} A \right), \\
&= X_{i,1} \left( A + B_u K_x \right). \quad \text{(5.29a-c)}
\end{aligned}
\]

Similarly,

\[
\begin{aligned}
X_{22} &= B_w^T X_{11} B_w - B_u^T X_{11} B_u \left( R_u + B_u^T X_{11} B_u \right)^{-1} B_u^T X_{11} B_w + D_{zw} D_{zw} \\
X_{33} &= X_{22} - X_{21} B_u \left( R_u + B_u^T X_{11} B_u \right)^{-1} B_u^T X_{21} \\
X_{i+1,i+1} &= X_{i,i} - X_{i,i} B_u \left( R_u + B_u^T X_{11} B_u \right)^{-1} B_u^T X_{i,i} \quad \text{(5.30a-c)}
\end{aligned}
\]
Similar expansion of the expression for the gain shows that terms not on the diagonal or first row (column) of $\overline{X}$ are irrelevant, since

$$
\overline{K} = - (R_u + B_u^T X_{11} B_u)^{-1} \begin{bmatrix} B_u^T & 0 & 0 \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \begin{bmatrix} A & B_w & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}^{-1}
$$

(5.31a)

$$
= - (R_u + B_u^T X_{11} B_u)^{-1} \begin{bmatrix} B_u^T X_{11} A & B_u^T X_{11} B_w & B_u^T X_{21}^T \end{bmatrix}
$$

(5.31b)

$$
\equiv [K_x \ K_{w0} \ K_{wi}],
$$

(5.31c)

so that $K_x$ and $K_{w0}$ are given as in eq. (5.22b), and for $i > 0$

$$
K_{wi} = - (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{i+1,1}^T.
$$

(5.32)

Finally, since $\overline{B}_w = [0 \ldots 0 I]^T$ always picks off the last element on the diagonal of $\overline{X}$, it is easily shown that the $\mathcal{H}_2$ norm is

$$
||\Sigma_{K_{wi}}||_{\mathcal{H}_2}^2 = ||\Sigma_{K_{wi} - I}||_{\mathcal{H}_2}^2 - \text{trace} \left( X_{i-1,1} B_u (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{i-1,1}^T \right),
$$

(5.33)

so that the cost continually decreases with increasing preview. The system $\Sigma_{K_{wi}}$ can be viewed as applying state-feedback to the system $\Sigma_{zi}$, or it can be viewed as applying $i$-step ahead preview and state-feedback to the system $\Sigma_z$.

Also, note that the recursion (5.29c) produces powers of the stable matrix $A + B_u K_x$, so that $X_{i,1}$ gets continually smaller. This implies that the cost may not go to zero as preview increases. An expression for asymptotic cost as the preview goes to $\infty$ is given by Hazell [28].

### 5.1.5 $\mathcal{H}_2$ State-feedback With Disturbance Augmentations

In closing, we briefly show that the system can be augmented with persistent disturbance dynamics,

$$
\begin{bmatrix} x(k+1) \\ x_d(k+1) \end{bmatrix} = \begin{bmatrix} A & B_d \\ 0 & A_d \end{bmatrix} \begin{bmatrix} x(k) \\ x_d(k) \end{bmatrix} + \begin{bmatrix} B_w \\ 0 \end{bmatrix} w(k) + \begin{bmatrix} B_u \\ 0 \end{bmatrix} u(k),
$$

(5.34a)

$$
z(k) = C_z x(k) + D_{zw} w(k),
$$

(5.34b)
and a disturbance gain $K_d$ without changing the $\mathcal{H}_2$ performance for state-feedback. Specifically, using the control
\[
u(k) = \tilde{u}_*(k) = [K_x \quad K_d]
\begin{bmatrix}
x(k) \\
x_d(k)
\end{bmatrix} + K_{w0}w(k),
\tag{5.35}
\]
results in closed loop dynamics given by
\[
\begin{bmatrix}
x(k+1) \\
x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
A + B_uK_x & B_d + B_uK_d \\
0 & A_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k)
\end{bmatrix}
+ \begin{bmatrix}
B_w + B_uK_x \\
0
\end{bmatrix} w(k),
\tag{5.36a}
\]
\[
z(k) = C_z x(k) + (D_{zw} + K_{w0})w(k).
\tag{5.36b}
\]
This system has the transfer function
\[
\Sigma_{zw}(z) = D_{zw} + K_{w0} + C_z
\begin{bmatrix}
I_z - (A + B_uK_x) & -(B_d + B_uK_d) \\
0 & I_z - A_d
\end{bmatrix}^{-1}
\begin{bmatrix}
B_w + B_uK_x \\
0
\end{bmatrix}
\tag{5.37a}
\]
\[
= D_{zw} + K_{w0} + C_z(I_z - (A + B_uK_x))^{-1}(B_w + B_uK_x),
\tag{5.37b}
\]
which is independent of $K_d$ and is identical to the one obtained using the control
\[
u_*(k) = K_x x(k) + K_{w0}w(k).
\tag{5.38}
\]
Therefore, both controls have the same $\mathcal{H}_2$ performance with respect to $w(k)$. This implies that designing an output feedback controller to approximate $\tilde{u}_*(k)$ is as valid as designing to approximate $u_*(k)$; the two controls have the same cost associated with their use in full-information control.

However, as shown in the next section, output feedback control cost is partitioned between full-information and approximation costs, and it will be shown numerically, that adding the dynamics necessary for regulation to the observer increases the estimation cost. This is expected, since output feedback that provides asymptotic disturbance rejection from a disturbance input to an output of interest, must by definition, result in a closed loop with a different impulse response than output feedback that does not provide regulation (though, a different impulse response does not guarantee that the trace-sum-square of the impulse will be different). And, according to Zhou, Doyle, and Glover [75], the optimal output feedback controller is unique.
5.2 The Separation Principle for $H_2$ Preview Control

This section considers the case where it is not possible to directly measure states, and so the system includes a measured output \( \tilde{y}(k) \) that is used by a feedback controller. In this section, the focus is on the system \( \Sigma y \) obtained from \( \Sigma z \) with the additional output \( \tilde{y}(k) \) and a measurement noise \( \tilde{n}_y(k) \)

\[
x(k + 1) = Ax(k) + \begin{bmatrix} 0 & B_w \end{bmatrix} \begin{bmatrix} \tilde{n}_y(k) \\ w(k) \end{bmatrix} + B_u u(k),
\]

\[
z(k) = C_z x(k) + \begin{bmatrix} 0 & D_{zw} \end{bmatrix} \begin{bmatrix} \tilde{n}_y(k) \\ w(k) \end{bmatrix} + D_{zu} u(k),
\]

\[
\tilde{y}(k) = \bar{C}_y x(k) + \begin{bmatrix} \bar{D}_{y}n \\ \bar{D}_{yu} \end{bmatrix} \begin{bmatrix} \tilde{n}_y(k) \\ w(k) \end{bmatrix} + \bar{D}_{yu} u(k), \quad \bar{D}_{yu} = 0.
\]

The tilde’s on the output measurement variables indicate that in later sub-sections the measurement will be partitioned to include noisy plant and preview measurements, and possibly a noisy measurement of the ideal feed-forward control \( u_{ff}(k) \) (Section 5.4). The terms associated with measurement noise \( \tilde{n}_y(k) \) could have been included in the sections on state-feedback, but since they do not directly effect the cost objective \( z(k) \), the results would have been identical and it would have clouded the development. And, we still maintain Assumption 5.1.1, but will require additional assumptions to find an optimal observer.

5.2.1 $H_2$ Cost Separation

Denote by \( \Sigma_{zcl} \), the closed-loop map from \( \tilde{w}(k) \) to \( z(k) \), obtained from the system \( \Sigma y \) with control \( u(k) \) generated by feedback from the output \( \tilde{y}(k) \) through a controller \( \Sigma_C \). The remainder
Figure 5.3: The output-feedback cost can be partitioned into a state-feedback cost and an observer cost, by adding and subtracting the $\mathcal{H}_2$ optimal feedback/feedforward controls.

of this section demonstrates that the $\mathcal{H}_2$ norm of the closed-loop system $\Sigma_{zcl}$ can be computed as

$$||\Sigma_{zcl}||_{\mathcal{H}_2}^2 = ||\Sigma_{Kw0}||_{\mathcal{H}_2}^2 + ||(R_u + B_u^T X_{11} B_u) \frac{1}{2} \Sigma_{\Delta cl}||_{\mathcal{H}_2}^2$$

(5.40)

where $\Sigma_{Kw0}$ is the full information (state-feedback) system (5.12), and $\Sigma_{\Delta cl}$ is a closed-loop map to be defined with respect to Fig. 5.3. This then shows that the $\mathcal{H}_2$ performance of the full information controller is the best possible outcome of any linear feedback controller.

Because the feedback $K_x x(k)$ and feed-forward $K_{w0} w(k)$ commands in Fig. 5.3 are both added and subtracted to the output feedback $u(k)$, the response is identical to that obtained from $\Sigma_{zcl}$ using output-only feedback through the controller $\Sigma_C$. Note that if the disturbance gain $K_{w0}$ is augmented as

$$K_{w0} \triangleq \begin{bmatrix} 0 & K_{w0} \end{bmatrix},$$

(5.41)

to accommodate $\bar{w}(k) = [\bar{y}(k)^T \, w(k)^T]^T$, then the full information closed-loop map $\Sigma_{Kw0}$ obtained
from $\Sigma_z$ using feedforward/feedback,

$$x(k+1) = (A + B_u K_x) x(k) + (\overline{B}_w + B_u K_w 0) \tilde{w}(k), \quad (5.42a)$$

$$z(k) = (C_z + D_{zu} K_x) x(k) + (\overline{D}_{zw} + D_{zw} K_w 0) \tilde{w}(k), \quad (5.42b)$$

is the same as the map from the internal $\tilde{w}(k)$ to $z(k)$ in the boxed subsystem of Fig. 5.3. Similarly, the map $\Sigma_U$ from the internal variable $\Delta(k)$ to $z(k)$ in the boxed subsystem of Fig. 5.3 is obtained from

$$x(k+1) = (A + B_u K_x) x(k) + B_u \Delta(k), \quad (5.43a)$$

$$z(k) = (C_z + D_{zu} K_x) x(k) + D_{zu} \Delta(k). \quad (5.43b)$$

So, once the closed-loop map $\Sigma_{\Delta cl} : w(k) \rightarrow \Delta(k)$ is known, the closed-loop transfer function can be computed as

$$\Sigma_{zcl}(z) = \Sigma_{Kw 0}(z) + \Sigma_U(z) \Sigma_{\Delta cl}(z). \quad (5.44)$$

Now to obtain eq. (5.40), the $\mathcal{H}_2$ norm is computed in the frequency domain with the use of adjoints. The adjoint of a transfer function $\Sigma(z)$ is $\Sigma(z)^\sim = \Sigma(1/z)^T$ e.g.,

$$\Sigma_U(z)^\sim \triangleq D_{zu}^T + (C_z + D_{zu} K_x)^T (I/z - (A + B_u K_x)^T)^{-1} B_u^T. \quad (5.45)$$

In the time domain, the strict definition of the adjoint system $\Sigma_U^\sim$ is the response is recursed backwards so that

$$x(k-1) = (A + B_u K_x)^T x(k) + (C_z + D_{zu} K_x)^T u(k). \quad (5.46)$$

Also, given two different systems of compatible dimensions that are both stable, it is possible to show that (compare with eq. (5.9))

$$\langle \Sigma_a, \Sigma_b \rangle \triangleq \text{trace} \left( \sum_{k=0}^{\infty} \Sigma_a(k)^T \Sigma_b(k) \right) \quad (5.47)$$

defines and inner product that gives the square of the $\mathcal{H}_2$ norm when $\Sigma_a = \Sigma_b$. Since this is essentially the trace of a matrix of inner products of various sequences, it can be computed in the
frequency domain as

\[
\langle \Sigma_a, \Sigma_b \rangle = \frac{1}{2\pi} \int \text{trace} \left( \Sigma_a \left( \frac{1}{z} \right)^T \Sigma_b(z) \right) \, dz,
\]

(5.48a)

\[
\triangleq \langle \Sigma_a(z), \Sigma_b(z) \rangle,
\]

(5.48b)

Further, the definition of the adjoint is consistent with this inner product in that

\[
\langle \Sigma_a(z) \Sigma_b(z), \Sigma_c(z) \rangle = \langle \Sigma_b(z), \Sigma_a(z)^* \Sigma_c(z) \rangle.
\]

(5.49)

Now, eq. (5.40) is obtained by applying the frequency domain inner product to \( \Sigma_{zcl} \) as defined in eq. (5.44), and making use of the facts [75] that,

\[
\Sigma_U(z)^* \Sigma_U(z) = R_u + B_u^T X_{11} B_u = \Delta_u,
\]

(5.50)

and that

\[
\langle \Sigma(z), \Sigma_U(z)^* \Sigma_{Kw0}(z) \rangle = 0,
\]

(5.51)

for any stable causal system \( \Sigma \). Hence,

\[
\| \Sigma_{zcl} \|_{\mathcal{H}_2}^2 = \langle \Sigma_{Kw0} + \Sigma_U \Sigma_{\Delta cl}, \Sigma_{Kw0} + \Sigma_U \Sigma_{\Delta cl} \rangle,
\]

(5.52a)

\[
= \langle \Sigma_{Kw0}, \Sigma_{Kw0} \rangle + \langle \Sigma_U \Sigma_{\Delta cl}, \Sigma_U \Sigma_{\Delta cl} \rangle + \langle \Sigma_U \Sigma_{\Delta cl}, \Sigma_{Kw0} \rangle + \langle \Sigma_{Kw0}, \Sigma_U \Sigma_{\Delta cl} \rangle,
\]

(5.52b)

\[
= \| \Sigma_{Kw0} \|_{\mathcal{H}_2}^2 + \langle \Sigma_{\Delta cl}, \Sigma_U^* \Sigma_{\Delta cl} \rangle + \langle \Sigma_{\Delta cl}, \Sigma_U^* \Sigma_{Kw0} \rangle + \langle \Sigma_{\Delta cl}, \Sigma_{Kw0} \Sigma_{\Delta cl} \rangle.
\]

(5.52c)

Now by eq. (5.51), it follows that

\[
0 = \langle \Sigma_{\Delta cl}, \Sigma_U^* \Sigma_{Kw0} \rangle = \langle \Sigma_U^* \Sigma_{Kw0}, \Sigma_{\Delta cl} \rangle.
\]

(5.53)

Therefore

\[
\| \Sigma_{zcl} \|_{\mathcal{H}_2}^2 = \| \Sigma_{Kw0} \|_{\mathcal{H}_2}^2 + \langle \Sigma_{\Delta cl}, \Sigma_U^* \Sigma_{\Delta cl} \rangle,
\]

(5.54a)

\[
= \| \Sigma_{Kw0} \|_{\mathcal{H}_2}^2 + \langle \Sigma_{\Delta cl}, \Delta_u \Sigma_{\Delta cl} \rangle,
\]

(5.54b)

\[
= \| \Sigma_{Kw0} \|_{\mathcal{H}_2}^2 + \left\{ \Delta_u^{\frac{1}{2}} \Sigma_{\Delta cl}, \Delta_u^{\frac{1}{2}} \Sigma_{\Delta cl} \right\}
\]

(5.54c)

proving eq. (5.40).
5.2.2 $\mathcal{H}_2$ Observer Design

To close this section, it is shown that the cost $\|\Delta_2^s \Sigma_{\Delta cl}\|_{\mathcal{H}_2}$ can be minimized by designing an observer. This will indicate how the feed-forward gain is handled when the observer is not augmented to provide DAC type regulation. Then Section 5.3 extends the result to the case that noisy preview measurements are available. And finally, Section 5.4 shows that when the observer is augmented to provide regulation, the reduced order and structured result of Section 5.3 is lost.

5.2.2.1 The Dual Observer Design Problem

To begin, note that since the system $\Sigma_{zcl}$ is the same as simply using the output feedback controller $\Sigma_C$, the output $\Delta(k)$ can be computed as an output of the system $\Sigma_{\Delta 0}$

\begin{align}
\bar{B}_w \quad \bar{w}(k) \\
\bar{x}(k+1) &= A\bar{x}(k) + \begin{bmatrix} 0 & B_w \end{bmatrix} \begin{bmatrix} \bar{y}_y(k) \\ w(k) \end{bmatrix} + B_u u(k), \quad (5.55a)
\end{align}

\begin{align}
\Delta(k) &= - K_{xw} x(k) - \begin{bmatrix} 0 & K_{w0} \end{bmatrix} \begin{bmatrix} \bar{y}_y(k) \\ w(k) \end{bmatrix} + u(k), \quad (5.55b)
\end{align}

\begin{align}
\bar{y}(k) &= \bar{C}_y x(k) + \begin{bmatrix} \bar{D}_{yn} & \bar{D}_{yw} \end{bmatrix} \begin{bmatrix} \bar{y}_y(k) \\ w(k) \end{bmatrix} + 0 \times u(k), \quad (5.55c)
\end{align}

where $u(k)$ is generated using output feedback as in Fig. 5.4a. Note that the output scaling $\Delta_2^s$ has been dropped, since in designing for the adjoint system, the scaling plays the same role as $\bar{B}_w$ in eq. (5.13b). That is, the control minimizing $\|\Sigma_{\Delta cl}\|_{\mathcal{H}_2}$ also minimizes the scaled norm, and the scaling is only necessary to correctly compute the cost.

Now, starting from the frequency domain formula (5.48) it is not difficult to show that

\begin{align}
\langle \Sigma_{\Delta 0}(z), \Sigma_{\Delta 0}(z) \rangle = \langle \Sigma_{\Delta 0}(z)^-, \Sigma_{\Delta 0}(z)^- \rangle.
\end{align}

(5.56)
Figure 5.4: Minimizing the norm $||\Sigma_{\Delta cl}||_{H_2}$ can be accomplished by designing for the original system (a) or its adjoint (b).

So, the norm can be minimized by designing an adjoint-controller for $\Sigma_{\Delta 0}$

\[
x(k + 1) = A^T x(k) - K^T_\Delta \Delta(k) + C^T_y \bar{y}(k)
\]  
(5.57a)

\[
\bar{w}(k) = B^T_w x(k) - K^T_w \Delta(k) + D^T_{wy} \bar{y}(k),
\]  
(5.57b)

\[
u(k) = B^T_u x(k) + \Delta(k) + 0 \bar{y}(k),
\]  
(5.57c)

(taking the liberty of running the adjoint forward in time is valid in this context, since there is no mixing between adjoint and non-adjoint systems) as shown in Fig. 5.4b. Then the controller for the original system is obtained by taking the adjoint of the adjoint-controller. In the adjoint system, the role of disturbance is now taken by $\Delta(k)$, while the performance output is in this case $\bar{w}(k)$. Further, the available measurement is $u(k)$ and the control input is $\bar{y}(k)$.

The advantage in working with the adjoint system, is that in the context of transfer functions (where all initial states are zero), the disturbance $\bar{\Delta}(k)$ can be constructed perfectly from the available measurement $u(k)$ by using an observer for the adjoint system. That is, the transfer function $\Sigma_{\Delta cl}^{-} : \Delta(k) \to \bar{w}(k)$ obtained using an observer and “statefeedback/feedforward” gains $[L^T Z^T \bar{L}_0^T]$, gives the same transfer function as that obtained when the gains are applied to the adjoint state and disturbance directly.

That an adjoint-observer gives the same transfer function as adjoint-full-information control,
is shown by writing out the expression for the combined adjoint-observer-system dynamics and transforming to the error system. Also, where the observer-controller requires a measurement of the disturbance $\Delta(k)$, it will instead use

$$\Delta_u(k) = u(k) - B_u^T \bar{x}(k) \approx \Delta(k),$$  \hspace{1cm} (5.58)

so that the observer update is

$$\bar{x}(k+1) = A^T \bar{x}(k) - K_x^T \Delta_u(k) + \bar{C}_y^T \bar{y}(k),$$  \hspace{1cm} (5.59a)

$$= A^T \bar{x}(k) - K_x^T \left( u(k) - B_u^T \bar{x}(k) \right) + \bar{C}_y^T \bar{y}(k),$$  \hspace{1cm} (5.59b)

$$= \left( A^T + K_x^T B_u^T \right) \bar{x}(k) - K_x^T u(k) + \bar{C}_y^T \bar{y}(k),$$  \hspace{1cm} (5.59c)

and the “control” $\bar{y}(k)$ is

$$\bar{y}(k) = L_x^T \bar{x}(k) + \bar{L}_0^T \Delta_u(k) = L_x^T \bar{x}(k) + \bar{L}_0^T (u(k) - B_u^T \bar{x}(k)), \hspace{1cm} (5.60a)$$

$$= \left( L_x^T - \bar{L}_0^T B_u^T \right) \bar{x}(k) + \bar{L}_0^T u(k).$$  \hspace{1cm} (5.60b)

Now, substituting in the expression for the “measurement” $u(k)$ from the adjoint eq. (5.57) gives

$$\bar{y}(k) = \left( L_x^T - \bar{L}_0^T B_u^T \right) \bar{x}(k) + \bar{L}_0^T B_u \Delta(k),$$  \hspace{1cm} (5.61a)

$$\bar{x}(k+1) = \left( A^T + K_x^T B_u^T \right) \bar{x}(k) - K_x^T B_u \Delta(k) + \bar{C}_y^T \bar{y}(k),$$  \hspace{1cm} (5.61b)

$$= \left( A^T + K_x^T B_u^T + \bar{C}_y^T \left( L_x^T - \bar{L}_0^T B_u^T \right) \right) \bar{x}(k)$$

$$+ \left( \bar{C}_y \bar{L}_0^T B_u - K_x^T B_u \right) x(k) + \left( \bar{C}_y \bar{L}_0^T - K_x^T \right) \Delta(k).$$  \hspace{1cm} (5.61c)

Finally, substituting in the expression for the “control” $\bar{y}(k)$ into the expressions for the “performance” output $\bar{w}(k)$ and state $x(k)$ gives

$$\bar{w}(k) = B_w^T \bar{x}_{pd}(k) + D_{yw} \bar{y}(k) - K_w^T \Delta(k),$$  \hspace{1cm} (5.62a)

$$= B_w^T \bar{x}_{pd}(k) + D_{yw} \left( L_x^T \bar{x}(k) + \bar{L}_0^T B_u \Delta(k) \right) - K_w^T \Delta(k)$$

$$= \left( B_w^T + D_{yw} \bar{L}_0^T B_u^T \right) \bar{x}_{pd}(k) + D_{yw} \left( L_x^T - \bar{L}_0^T B_u^T \right) \bar{x}(k) + \left( D_{yw} \bar{L}_0^T - K_w^T \right) \Delta(k).$$  \hspace{1cm} (5.62b)
So, the combined adjoint-observer-system dynamics are given by

\[ x(k + 1) = A^T x(k) - K_x^T \Delta(k) + \overline{C}_y^T \left( \left( L_x^T - L_0^T B_u^T \right) \bar{x}(k) + L_0^T B_u^T x(k) + L_0^T v(k) \right), \]

\[ = \left( A^T + \overline{C}_y^T L_0^T B_u^T \right) x(k) + \overline{C}_y^T \left( L_x^T - L_0^T B_u^T \right) \bar{x}(k) + \left( \overline{C}_y^T L_0^T - K_x^T \right) \Delta(k). \]  

(5.63a)

So, the combined adjoint-observer-system dynamics are given by

\[
\begin{bmatrix}
  x(k+1) \\
  \bar{x}(k+1)
\end{bmatrix} = \begin{bmatrix}
  A^T + \overline{C}_y^T L_0^T B_u^T & \overline{C}_y^T \left( L_x^T - L_0^T B_u^T \right) \\
  \overline{C}_y^T L_0^T B_u^T - K_x^T B_u^T & A^T + K_x^T B_u^T + \overline{C}_y^T \left( L_x^T - L_0^T B_u^T \right)
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  \bar{x}(k)
\end{bmatrix} + \begin{bmatrix}
  \overline{C}_y^T L_0^T - K_x^T \\
  \overline{C}_y^T L_0^T - K_x^T
\end{bmatrix} \Delta(k)
\]

(5.64a)

and transforming to the error system gives

\[
\begin{bmatrix}
  x(k + 1)
  \bar{x}(k + 1) - x(k + 1)
\end{bmatrix} = \begin{bmatrix}
  A^T + \overline{C}_y^T L_x^T & \overline{C}_y^T L_x^T \\
  0 & A^T + \overline{C}_y^T L_x^T
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  \bar{x}(k) - x(k)
\end{bmatrix} + \begin{bmatrix}
  \overline{C}_y^T L_0^T - K_x^T \\
  0
\end{bmatrix} \Delta(k)
\]

(5.65a)

\[
\bar{w}(k) = \begin{bmatrix}
  \overline{B}_u^T + \overline{D}_{yu}^T L_x^T - \overline{D}_{yu}^T L_0^T B_u^T \\
  \overline{D}_{yu}^T L_x^T - \overline{D}_{yu}^T L_0^T B_u^T
\end{bmatrix} \begin{bmatrix}
  x(k) \\
  \bar{x}(k) - x(k)
\end{bmatrix} + \begin{bmatrix}
  \overline{D}_{yu}^T L_0^T - K_w^T \Delta(k)
\end{bmatrix}
\]

(5.65b)

Since the “disturbance” \( \Delta(k) \) has zero-gain into the error dynamics, it is now clear, that in the context of transfer functions, where \( 0 = x(0) = \bar{x}(0) \), that the observer dynamics play no role in the transfer function from \( \Delta(k) \) to \( \bar{w}(k) \). Therefore, the optimal controller for the adjoint system is obtained by designing an observer

\[
\begin{align*}
\bar{x}(k + 1) &= \left( A^T + K_x^T B_u^T + \overline{C}_y^T L_x^T - \overline{C}_y^T L_0^T B_u^T \right) \bar{x}(k) + \left( \overline{C}_y^T L_0^T - K_x^T \right) u(k) \\
\bar{y}(k) &= \left( L_x^T - L_0 B_u^T \right) \bar{x}(k) + L_0^T u(k).
\end{align*}
\]

(5.66a)

(5.66b)
The optimal observer-controller for the original system is then the adjoint of eq. (5.66)

\[
\begin{align*}
 u(k) &= (K_x - \overline{L}_0 \overline{C}_y) \overline{x}(k) + \overline{L}_0 \overline{y}(k) \\
 &= K_x \overline{x}(k) - \overline{L}_0 (\overline{C}_y \overline{x}(k) - \overline{y}(k)) \\
 \overline{x}(k+1) &= (A + B_u K_x + L_x \overline{C}_y - B_u \overline{L}_0 \overline{C}_y) \overline{x}(k) + (B_u \overline{L}_0 - L_x) \overline{y}(k) \\
 &= A \overline{x} + B_u \left( K_x \overline{x}(k) - \overline{L}_0 (\overline{C}_y \overline{x}(k) - \overline{y}(k)) \right) + L_x (\overline{C}_y \overline{x}(k) - \overline{y}(k))
\end{align*}
\]

(5.67a) (5.67b) (5.67c) (5.67d)

where the state transformation \( \overline{x}(k) \leftarrow -\overline{x}(k) \) has also been applied. This observer-controller has the structure shown in Fig. 5.5.

5.2.2.2 \( \mathcal{H}_2 \) Optimal Observer Gains

At this point, it would be nice to use the dual of the state-feedback results, e.g. the DARE eq. (5.21), to design the optimal gains \([L_x^T \; \overline{L}_0^T]\) for the adjoint. Unfortunately, because in general

\[
\begin{bmatrix}
 \overline{D}_{yw} \\
 B_w^T \\
 K_w^T
\end{bmatrix} \begin{bmatrix}
 \overline{D}_{yn} & \overline{D}_{yw} \\
 0 & 0 \\
 B_w^T & K_w^0
\end{bmatrix} \neq 0,
\]

(5.68)
the “performance” output
\[
\bar{w}(k) = B_w^T x(k) - K_w^T \Delta(k) + D_{yw}^T \bar{y}(k),
\]
(5.69)
has cross terms between the “control” \(\bar{y}(k)\) and the “state” \([x(k)^T \Delta(k)^T]^T\). So, it is necessary to apply the affine control transformation described in Appendix A, and use the modified assumptions for the adjoint system.

**Assumption 5.2.1**

\[
\Re \left( [A^T - I\Lambda C_y^T] \right) = \Re^*,
\]
(5.70a)
\[
D_{yw} D_{yw}^T \leq R_y > 0,
\]
(5.70b)
\[
N \left( \begin{bmatrix} A^T - I e^{i\omega} C_2^T \\ B_w^T \\ D_{yw}^T \end{bmatrix} \right) = 0.
\]
(5.70c)

In this case, the DARE
\[
Y_{11} = \bar{A} Y_{11} \bar{A}^T + \bar{B}_w \bar{B}_w^T + \bar{A} Y_{11} \bar{C}_2^T \left( D_{yw} D_{yw}^T + \bar{C}_2 Y_{11} \bar{C}_2^T \right)^{-1} \bar{C}_2 Y_{11} \bar{A}^T,
\]
(5.71a)
\[
= \bar{B}_w \bar{B}_w^T + \bar{L}_x D_{yw} D_{yw}^T + (\bar{A} + \bar{L}_x \bar{C}_2) Y_{11} (\bar{A} + \bar{L}_x \bar{C}_2)^T,
\]
(5.71b)
is the counter-part to eq. (5.21), but is associated with the shifted system
\[
\bar{A}^T x(k + 1) = \left( A^T - C_y^T (D_{yw} D_{yw}^T)^{-1} D_{yw} B_w^T \right) x(k) - \left( K_x^T - C_y^T (D_{yw} D_{yw}^T)^{-1} D_{yw} K_w^T \right) \Delta(k) + C_y^T \bar{y}(k),
\]
(5.72a)
\[
\bar{B}_w^T w(k) = \left( I - D_{yw} (D_{yw} D_{yw}^T)^{-1} D_{yw} \right) \bar{B}_w^T x(k) - \left( I - D_{yw} (D_{yw} D_{yw}^T)^{-1} D_{yw} \right) \bar{K}_w^T \Delta(k) + D_{yw}^T \bar{y}(k).
\]
(5.72b)

So (see Appendix A), the optimal control for the unshifted (adjoint) system is then obtained by making the appropriate substitutions in
\[
[K_x \ K_w] = -(R_u + B_u^T X_{11} B_u)^{-1} \left[ (B_u^T X_{11} A + D_{2u}^T C_z) \ (B_u^T X_{11} B_w + D_{2w}^T D_{2w}) \right],
\]
(5.73)
By duality, the necessary substitutions are

\[ \begin{align*}
K_x &\leftarrow L_x^T, \\
K_{w0} &\leftarrow L_0^T, \\
R_u &\leftarrow \bar{D}_{yu}\bar{D}_{yw}^T, \\
B_u &\leftarrow \bar{C}_y^T, \\
X_{11} &\leftarrow Y_{11}, \\
A &\leftarrow A^T, \\
D_{zu} &\leftarrow \bar{D}_{yw}^T, \\
C_z &\leftarrow \bar{B}_{w}, \\
B_w &\leftarrow -K_x^T, \\
D_{zw} &\leftarrow -K_w^T,
\end{align*} \tag{5.74a,b,c,d,e} \]

so that (note the minus sign on \(K_x^T\) and \(K_w^T\),

\[ \begin{bmatrix} L_x^T & L_0^T \end{bmatrix} = (\bar{D}_{yu}\bar{D}_{yw}^T + \bar{C}_y Y_{11} \bar{C}_y^T)^{-1} \left[ -(\bar{C}_y Y_{11} A^T + \bar{D}_{yu} \bar{B}_{w}^T) (\bar{C}_y Y_{11} K_x^T + \bar{D}_{yu} K_w^T) \right]. \tag{5.75} \]

The subscript on \(Y_{11}\) is used to emphasize that the dimension of the solution is that of the model without preview augmentations. In the next section, where \(\bar{y}(k)\) is defined to include preview measurements and the system state is augmented to include preview, it turns out that the DARE that must be solved, has the same dimension as \(Y_{11}\) and is independent of the amount of preview.

Similarly, making the substitutions

\[ \begin{align*}
R_u &\leftarrow \bar{D}_{yu}\bar{D}_{yw}^T, \\
B_u &\leftarrow \bar{C}_y^T, \\
D_{zu} &\leftarrow \bar{D}_{yw}^T, \\
C_z &\leftarrow \bar{B}_{w}, \\
B_w &\leftarrow -\bar{K}_x^T \Delta_a^\frac{3}{2}, \\
D_{zw} &\leftarrow \bar{K}_w^T \Delta_a^\frac{3}{2}, \\
X_{11} &\leftarrow Y_{11},
\end{align*} \tag{5.76a,b,c,d} \]

in eq. (5.23), shows that the cost associated with the observer is

\[ \begin{bmatrix} \Delta_u^\frac{1}{2} \Sigma \Delta_c \end{bmatrix}_{2\ell_2}^2 = \text{trace} \left( \Delta_u^\frac{1}{2} \left( \bar{K}_x Y_{11} \bar{R}_x^T + \bar{R}_w \bar{R}_w^T - \bar{R}_x Y_{11} \bar{C}_y^T \left( \bar{D}_{yu} \bar{D}_{yw}^T + \bar{C}_y Y_{11} \bar{C}_y^T \right)^{-1} \bar{C}_y Y_{11} \bar{K}_x^T \right) \Delta_a^\frac{1}{2} \right). \tag{5.77} \]
5.3 \( \mathcal{H}_2 \) Optimal Output Feedback With Preview and No Regulation

In this section, we explicitly consider the case that a noisy measurement \( w_y(k) \) of two-step ahead preview \( w(k + 2) \) of \( w(k) \) is available, as in the system \( \Sigma_{\Delta 2} \)

\[
\begin{bmatrix}
    x(k + 1) \\
    w(k + 1) \\
    w(k + 2)
\end{bmatrix} = \begin{bmatrix}
    A & B_w & 0 \\
    0 & 0 & I_w \\
    0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    w(k) \\
    w(k + 1)
\end{bmatrix} + \begin{bmatrix}
    B_u \\
    0 \\
    0
\end{bmatrix} u(k) + \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & I_w \\
    0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
    n_y(k) \\
    n_w(k) \\
    \overline{w}(k)
\end{bmatrix},
\]

\( \Delta(k) = u(k) - \begin{bmatrix}
    K_x & K_{w0} & K_{w1}
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    w(k) \\
    w(k + 1)
\end{bmatrix} - \begin{bmatrix}
    0 & 0 & K_{w2}
\end{bmatrix} \begin{bmatrix}
    n_y(k) \\
    n_w(k) \\
    \overline{w}(k)
\end{bmatrix}, \quad (5.78b) \)

\[
\begin{bmatrix}
    \overline{y}(k) \\
    y_y(k) \\
    w_y(k)
\end{bmatrix} = \begin{bmatrix}
    C_y & D_{yw} & 0 \\
    0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    w(k) \\
    w(k + 1)
\end{bmatrix} + \begin{bmatrix}
    D_{yn} & 0 \\
    0 & D_{wn} & I_w
\end{bmatrix} \begin{bmatrix}
    n_y(k) \\
    n_w(k) \\
    \overline{w}(k)
\end{bmatrix}.
\]

Control gains are designed for the associated adjoint system \( \Sigma_{\Delta 2}^\star \)

\[
\begin{bmatrix}
    x(k + 1) \\
    w(k + 1) \\
    w(k + 2)
\end{bmatrix} = \begin{bmatrix}
    A^T & 0 & 0 \\
    B_{w}^T & 0 & 0 \\
    0 & I_w & 0
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    w(k) \\
    w(k + 1)
\end{bmatrix} - \begin{bmatrix}
    K_{x}^T \\
    K_{w0}^T \\
    K_{w1}^T
\end{bmatrix} \Delta(k) + \begin{bmatrix}
    C_y^T & D_{yw}^T & 0 \\
    D_{yn}^T & 0 \\
    0 & D_{wn}^T
\end{bmatrix} \begin{bmatrix}
    y_y(k) \\
    w_y(k)
\end{bmatrix},
\]

\( \Delta(k) = \begin{bmatrix}
    0 & 0 \\
    0 & 0 \\
    0 & I_w
\end{bmatrix} \begin{bmatrix}
    x(k) \\
    w(k) \\
    w(k + 1)
\end{bmatrix} + \begin{bmatrix}
    D_{yn}^T & 0 \\
    0 & D_{wn}^T
\end{bmatrix} \begin{bmatrix}
    y_y(k) \\
    w_y(k)
\end{bmatrix}.
\]

Section 5.3.1 computes the output-injection and feed-through gains for the augmented systems, and shows that the DARE for the augmented system is diagonal, and that only the \( Y_{11} \) sub-block requires solution. This results in a DARE with dimension equal to that of the un-augmented plant.

Then Section 5.3.1 examines the form of the resulting observer controller.
5.3.1 \( \mathcal{H}_2 \) Preview Observer DARE and Gains

As in Section 5.2.2, it is again the case that the shifted system needs to be computed before using the DARE (5.71) for the dual system. This requires

\[
\hat{A} = \hat{A}^T - \tilde{C}_y^T (\hat{D}_{yw}\hat{D}_{yw}^T)^{-1}\hat{D}_{yw}\hat{B}_w^T
\]

\[
= \begin{bmatrix}
A^T & 0 & 0 \\
B_w^T & 0 & 0 \\
0 & I_w & 0
\end{bmatrix}
- \begin{bmatrix}
C_y^T & 0 \\
D_{yw}^T & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
D_{yn}D_{yn}^T & 0 \\
0 & I + D_{wn}D_{wn}^T
\end{bmatrix}^{-1}
\begin{bmatrix}
D_{yn} & 0 & 0 \\
0 & D_{wn} & I_w \\
0 & 0 & I_w
\end{bmatrix}
\]

\[
\hat{B}_w = \begin{bmatrix}
A^T & 0 & 0 \\
B_w^T & 0 & 0 \\
0 & I_w & 0
\end{bmatrix}
- \begin{bmatrix}
C_y^T & 0 \\
D_{yw}^T & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
(D_{yn}D_{yn}^T)^{-1}D_{yn} & 0 & 0 \\
0 & (I + D_{wn}D_{wn}^T)^{-1}D_{wn} & (I + D_{wn}D_{wn}^T)^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & I_w
\end{bmatrix}
\]

\[
\hat{K}_x = \hat{K}_x^T - \tilde{C}_y^T (\hat{D}_{yw}\hat{D}_{yw}^T)^{-1}\hat{D}_{yw}\hat{K}_w^T
\]

\[
= \begin{bmatrix}
K_x^T & 0 & 0 \\
K_{w0}^T & 0 & 0 \\
K_{w1}^T & 0 & 0
\end{bmatrix}
- \begin{bmatrix}
C_y^T & 0 \\
D_{yw}^T & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
(D_{yn}D_{yn}^T)^{-1}D_{yn} & 0 & 0 \\
0 & (I + D_{wn}D_{wn}^T)^{-1}D_{wn} & (I + D_{wn}D_{wn}^T)^{-1}
\end{bmatrix}
\begin{bmatrix}
K_x^T & 0 & 0 \\
K_{w0}^T & 0 & 0 \\
K_{w1}^T & 0 & 0
\end{bmatrix}
\]

as well as
\[ \overline{B}_w = \left( I - D_{gw}^T (D_{gw} D_{gw}^T)^{-1} D_{gw} \right) \overline{B}_w^T \]  

(5.82a)

\[
\begin{bmatrix}
I & 0 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
- 
\begin{bmatrix}
D_{yn}^T & 0 \\
0 & D_{wn}^T \\
0 & I_w
\end{bmatrix}
\begin{bmatrix}
(D_{yn} D_{yn}^T)^{-1} D_{yn} & 0 & 0 \\
0 & (I + D_{wn} D_{wn}^T)^{-1} D_{wn} & (I + D_{wn} D_{wn}^T)^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & I_w
\end{bmatrix}
\]

(5.82b)

\[
\begin{bmatrix}
I - D_{yn}^T (D_{yn} D_{yn}^T)^{-1} D_{yn} & 0 & 0 \\
0 & I - D_{wn}^T (I + D_{wn} D_{wn}^T)^{-1} D_{wn} & -D_{wn}^T (I + D_{wn} D_{wn}^T)^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & I_w
\end{bmatrix}
\]

(5.82c)

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & I - (I + D_{wn} D_{wn}^T)^{-1} \\
0 & -D_{wn}^T (I + D_{wn} D_{wn}^T)^{-1} & I - (I + D_{wn} D_{wn}^T)^{-1}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \overline{B}_n^T_w \\
0 & 0 & \overline{B}_w^T
\end{bmatrix},
\]

(5.82d)

\[ \overline{R}_w^T = \left( I - D_{gw}^T (D_{gw} D_{gw}^T)^{-1} D_{gw} \right) \overline{K}_w^T, \]  

(5.82e)

\[
\begin{bmatrix}
0 \\
-D_{wn}^T (I + D_{wn} D_{wn}^T)^{-1} K_{w2}^T \\
K_{w2}^T - (I + D_{wn} D_{wn}^T)^{-1} K_{w2}^T
\end{bmatrix}
= \begin{bmatrix}
0 \\
\overline{K}_n^T_w \\
\overline{K}_w^T
\end{bmatrix},
\]

(5.82f)

Note in particular, that the form (5.82c) for \( \overline{B}_w \) depends only on the preview noise magnitude (determined via \( D_{wn} \)), and not on the amount of preview.

These computations result in the shifted (standard form) system

\[
\begin{bmatrix}
x(k+1) \\
w(k+1) \\
w(k+2)
\end{bmatrix}
= 
\begin{bmatrix}
A^T & 0 & 0 \\
B_w^T & 0 & 0 \\
0 & I_w & 0
\end{bmatrix}
\begin{bmatrix}
x(k) \\
w(k) \\
w(k+1)
\end{bmatrix}
- 
\begin{bmatrix}
K_{xw}^T \\
K_{w0}^T \\
K_{w1}^T
\end{bmatrix}
\Delta(k) + 
\begin{bmatrix}
C_{y}^T & 0 \\
D_{gw}^T & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\overline{y}_y(k) \\
\overline{w}_y(k)
\end{bmatrix},
\]

(5.83a)
and

\[
\begin{bmatrix}
  n_y(k) \\
  n_w(k) \\
  \bar{w}(k)
\end{bmatrix}
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & \hat{B}_{nw}^T \\
  0 & 0 & \hat{B}_{w2}^T
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  w(k) \\
  w(k+1)
\end{bmatrix}
\begin{bmatrix}
  0 \\
  -\hat{K}_{nw}^T \Delta(k) + \begin{bmatrix}
  0 & 0 \\
  0 & D_{wn}^T \\
  0 & I_w
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  y(g(k)) \\
  \hat{w}_y(k)
\end{bmatrix}.
\]

(5.84a)

Now, because the last row of \( \hat{A} \), and last column of \( \hat{C}_y \), have remained zero in the shifted system, substitution into the DARE (5.71) results in significant simplifications

\[
Y = \hat{A} Y \hat{A}^T + \hat{B}_w \hat{B}_w^T - \hat{A} Y \hat{C}_y^T \left( \hat{D}_{yw} \hat{D}_{yw}^T + \hat{C}_y \hat{Y} \hat{C}_y^T \right)^{-1} \hat{C}_y \hat{Y} \hat{A}^T
\]

(5.85a)

\[
= \begin{bmatrix}
  A & B_w & 0 \\
  0 & 0 & I \\
  0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  \hat{A}^T & 0 & 0 \\
  0 & \hat{B}_{w2}^T & 0 \\
  0 & 0 & \hat{B}_{nw} \hat{B}_{nw}^T + \hat{B}_{w2} \hat{B}_{w2}^T
\end{bmatrix}
\begin{bmatrix}
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  A^T & 0 & 0 \\
  0 & B_w^T & 0 \\
  0 & I & 0
\end{bmatrix}
\]

(5.85b)

\[
= \begin{bmatrix}
  (AY_{11} + B_{w}Y_{21})A^T + (AY_{12} + B_{w}Y_{22})B_w^T & AY_{13} + B_{w}Y_{23} & 0 \\
  Y_{31}A^T + Y_{32}B_w^T & Y_{33} & 0 \\
  0 & 0 & \hat{B}_{nw} \hat{B}_{nw}^T + \hat{B}_{w2} \hat{B}_{w2}^T
\end{bmatrix}
\]

(5.85c)

\[
= \begin{bmatrix}
  (AY_{11} + B_{w}Y_{21})C_y^T & (AY_{12} + B_{w}Y_{22})D_{yw}^T & 0 \\
  Y_{31}C_y^T + Y_{32}D_{yw}^T & 0 & \left( \hat{D}_{yw} \hat{D}_{yw}^T + \hat{C}_y \hat{Y} \hat{C}_y^T \right)^{-1} \\
  0 & 0 & 0
\end{bmatrix}
\times
\begin{bmatrix}
  C_y(AY_{11} + B_{w}Y_{21})^T + D_{yw}(AY_{12} + B_{w}Y_{22})^T & C_yY_{31}^T + D_{yw}Y_{32} & 0 \\
  0 & 0 & 0
\end{bmatrix}.
\]

(5.85d)

Noting that the off diagonal terms in the third row and column are zero (e.g., \( Y_{13}, Y_{23}, \) etc.), we
get a further simplification

\[
\bar{Y} = \begin{bmatrix}
(AY_{11} + B_w Y_{21}) A^T + (AY_{12} + B_w Y_{22}) B_w^T & 0 & 0 \\
0 & Y_{33} & 0 \\
0 & 0 & B_{nw} \bar{B}_{nw}^T + \bar{B}_{w2} \bar{B}_{w2}^T
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(AY_{11} + B_w Y_{21}) C_y^T + (AY_{12} + B_w Y_{22}) D_{yw}^T & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
- \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
(\bar{D}_{yw} \bar{D}_{yw}^T + \bar{C}_y Y \bar{C}_y^T)^{-1}
\]

\[
\times \begin{bmatrix}
C_y (AY_{11} + B_w Y_{21})^T + D_{yw} (AY_{12} + B_w Y_{22})^T & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

showing that the off diagonal terms in the second row/column are also zero. Hence,

\[
\bar{Y} = \begin{bmatrix}
AY_{11} A^T + B_w Y_{22} B_w^T & 0 & 0 \\
0 & \bar{B}_{nw} \bar{B}_{nw}^T + \bar{B}_{w2} \bar{B}_{w2}^T & 0 \\
0 & 0 & \bar{B}_{nw} \bar{B}_{nw}^T + \bar{B}_{w2} \bar{B}_{w2}^T
\end{bmatrix}
\]

\[
- \begin{bmatrix}
AY_{11} C_y^T + B_w Y_{22} D_{yw}^T & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
(\bar{D}_{yw} \bar{D}_{yw}^T + \bar{C}_y Y \bar{C}_y^T)^{-1}
\]

\[
\times \begin{bmatrix}
C_y Y_{11} A^T + D_{yw} Y_{22} B_w^T & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

where the required inverse turns out to be block diagonal

\[
(\bar{D}_{yw} \bar{D}_{yw}^T + \bar{C}_y Y \bar{C}_y^T)^{-1} = \begin{bmatrix}
D_{yn} & 0 & 0 \\
0 & 0 & D_{wn}^T \\
0 & I_w & 0
\end{bmatrix}
\begin{bmatrix}
D_{yn}^T & 0 \\
0 & 0 \\
0 & I_w
\end{bmatrix}
+ \begin{bmatrix}
C_y & D_{yw} & 0 \\
0 & 0 & Y_{22} \\
0 & 0 & Y_{33}
\end{bmatrix}
\begin{bmatrix}
Y_{11} & 0 & 0 \\
0 & Y_{22} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
C_y^T & 0 \\
0 & D_{yw}^T & 0 \\
0 & 0 & 0
\end{bmatrix}
^{-1}
\]

\[
= \begin{bmatrix}
(C_y Y_{11} C_y^T + D_{yw} Y_{22} D_{yw}^T + D_{yn} D_{yn}^T)^{-1} & 0 \\
0 & (I_w + D_{wn} D_{wn}^T)^{-1}
\end{bmatrix}.
\]
So, only the “$Y_{11}$” block needs to satisfy the DARE

$$Y_{11} = AY_{11}A^T + B_w Y_{22} B_w^T$$

$$-(AY_{11}C_y + B_w Y_{22} D_yw)(C_y Y_{11} C_y^T + D_yw Y_{22} D_yw^T + D_yw D_yw^T)^{-1}(C_y Y_{11} A^T + D_yw Y_{22} B_w^T)$$

(5.89)

where

$$Y_{22} = Y_{33} = \bar{B}_{nw} \bar{B}_{nw}^T + \bar{B}_{w2} \bar{B}_{w2}^T,$$ \hspace{1cm} (5.90a)

$$= (I + D_{wn} D_{wn}^T)^{-1} D_{wn} D_{wn}^T (I + D_{wn} D_{wn}^T)^{-1}$$

$$+ (I - (I + D_{wn} D_{wn}^T)^{-1})(I - (I + D_{wn} D_{wn}^T)^{-1}),$$ \hspace{1cm} (5.90b)

$$= (I + D_{wn} D_{wn}^T)^{-1} D_{wn} D_{wn}^T (I + D_{wn} D_{wn}^T)^{-1},$$

$$+ ((I + D_{wn} D_{wn}^T)^{-1} D_{wn} D_{wn}^T) D_{wn} D_{wn}^T (I + D_{wn} D_{wn}^T)^{-1}),$$ \hspace{1cm} (5.90c)

$$= (I + D_{wn} D_{wn}^T)^{-1} \left( D_{wn} D_{wn}^T + D_{wn} D_{wn}^T D_{wn} D_{wn}^T \right) (I + D_{wn} D_{wn}^T)^{-1},$$ \hspace{1cm} (5.90d)

$$= D_{wn} (I + D_{wn} D_{wn}^{-1}) \left( I + D_{wn} D_{wn}^{-1} \right) (I + D_{wn} D_{wn}^{-1})^{-1} D_{wn}^T,$$ \hspace{1cm} (5.90e)

$$= D_{wn} (I + D_{wn} D_{wn}^{-1})^{-1} D_{wn}^T.$$ \hspace{1cm} (5.90f)

All of the terms in equations (5.89) and (5.90) are independent of the amount of preview. Further, the simplifying property that last row of $\bar{A}$ and $\bar{C}_y$ remain zero, holds for arbitrary levels of preview (with the exception of no preview, in which case the result is the DARE of the previous section).

A similar inspection of the resulting observer gains reveals similar results. In this case, the
The output-injection gain to the augmented plant model is

\[
\bar{L}^T_x = - \left( \bar{D}_{yw} \bar{D}_{yw}^T + \bar{C}_y \bar{Y} \bar{C}_y^T \right)^{-1} \left( \bar{C}_y \bar{Y} \bar{A}^T + \bar{D}_{yn} \bar{B}_w^T \right)
\]

\[
= - \left( \bar{D}_{yw} \bar{D}_{yw}^T + \bar{C}_y \bar{Y} \bar{C}_y^T \right)^{-1} \begin{bmatrix} C_y & D_{yw} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A^T & 0 & 0 \\ B_w^T & 0 & 0 \end{bmatrix} \begin{bmatrix} D_{yn} & 0 & 0 \\ 0 & D_{wn} & I \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_w \end{bmatrix}
\]  

(5.91a)

\[
= - \begin{bmatrix} (C_y Y_{11} C_y^T + D_{yw} Y_{22} D_{yw}^T + D_{yn} D_{yn}^T)^{-1} (C_y Y_{11} A^T + D_{yw} Y_{22} B_w^T) & 0 & 0 \\ 0 & 0 & (I_w + D_{wn} D_{wn}^T)^{-1} \end{bmatrix}
\]  

(5.91b)

\[
= \begin{bmatrix} L_{xy}^T & 0 & 0 \\ 0 & 0 & L_{ww}^T \end{bmatrix}
\]  

(5.91c)

and the feed-through gain is

\[
\bar{L}_0^T = \left( \bar{D}_{yw} \bar{D}_{yw}^T + \bar{C}_y \bar{Y} \bar{C}_y^T \right)^{-1} \left( \bar{C}_y \bar{Y} \bar{K}_x^T + \bar{D}_{yn} \bar{K}_w^T \right)
\]

\[
= \left( \bar{D}_{yw} \bar{D}_{yw}^T + \bar{C}_y \bar{Y} \bar{C}_y^T \right)^{-1} \begin{bmatrix} C_y & D_{yw} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} K_x^T \\ K_{w0}^T \\ K_{w1}^T \end{bmatrix} + \begin{bmatrix} D_{yn} & 0 & 0 \\ 0 & D_{wn} & I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  

(5.92a)

\[
= \begin{bmatrix} (C_y Y_{11} C_y^T + D_{yw} Y_{22} D_{yw}^T + D_{yn} D_{yn}^T)^{-1} (C_y Y_{11} K_x^T + D_{yw} Y_{22} K_{w0}^T) & L_{xy}^T \\ (I_w + D_{wn} D_{wn}^T)^{-1} K_{w2}^T & L_{ww}^T \end{bmatrix}
\]  

(5.92b)

At this point, it is worth noting that there is no output-injection from any measurement into the storage for \( w(k) \). This is a result of the form for \( \bar{C}_y \) and \( \bar{B}_w \), and the fact that \( \bar{Y} \) and \( \left( \bar{D}_{yw} \bar{D}_{yw}^T + \bar{C}_y \bar{Y} \bar{C}_y^T \right) \) are diagonal. Additional preview would not change the diagonal forms, and
for example, three-step ahead preview would result in
\[
L_x^T = - \left( \overline{D}_{yw} \overline{D}_y^T + \overline{C}_y Y \overline{C}_y^T \right)^{-1} \left( \overline{C}_y Y \overline{A}^T + \overline{D}_{yn} \overline{B}_w^T \right)
\]
\[
= - \left( \overline{D}_{yw} \overline{D}_y^T + \overline{C}_y Y \overline{C}_y^T \right)^{-1} \left[ \begin{array}{ccc} C_y & D_{yw} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] Y \left[ \begin{array}{ccc} A^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array} \right] + \left[ \begin{array}{ccc} D_{yn} & 0 & 0 \\ 0 & D_{wn} & 0 \\ 0 & 0 & I_w \end{array} \right]
\]
\[
(5.93a)
\]
\[
= - \left( \overline{D}_{yw} \overline{D}_y^T + \overline{C}_y Y \overline{C}_y^T \right)^{-1} \left[ \begin{array}{ccc} C_y Y_{11} A^T + D_{yw} Y_{22} B_w^T & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]
\]
\[
(5.93b)
\]
\[
= \left[ \begin{array}{ccc} L_x^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_{ww}^T \end{array} \right].
\]
\[
(5.93d)
\]

Hence, there is never injection into internal preview-storage states. Only the first preview-storage state gets input from the noisy preview measurement through $L_{ww}$. Also, as $D_{wn} \to 0$ the preview gain satisfies $L_{ww} \to I$. Similarly, for i-step ahead preview, the feed-through gain for the preview measurement satisfies $L_{\Delta w} \to K_{wi}$ as $D_{wn} \to 0$.

The cost can be computed in terms of the diagonal $Y$ and shifted system matrices, with their substitution into eq. (5.77) in an appropriate way
\[
\left\| \Delta_u^2 \Sigma_\Delta \right\|_{\text{trace}} = \text{trace} \left( \Delta_u^2 \left( \bar{K}_x Y \bar{K}_x^T + \bar{K}_w \bar{K}_w^T - \bar{K}_x Y \bar{C}_y \left( R_y + \bar{C}_y \bar{C}_y^T \right)^{-1} \bar{C}_y Y \bar{K}_x \right) \Delta_u^2 \right), \quad R_y \equiv \overline{D}_{yw} \overline{D}_{yw}^T.
\]
\[
(5.94)
\]

However, the effect of increasing preview is not immediately clear, because the cost contains the term
\[
\bar{K}_w \bar{K}_w^T = \left[ \begin{array}{ccc} 0 \\ \bar{K}_{nw} \bar{K}_{nw}^T \\ \bar{K}_{w2} \bar{K}_{w2}^T \end{array} \right] = \bar{K}_{nw} \bar{K}_{nw} + \bar{K}_{w2} \bar{K}_{w2}
\]
\[
(5.95)
\]
that decreases with preview, but is bounded below by 0; and, the difference

\[
\bar{R}_x \bar{V} \bar{V}_x^T - \bar{R}_x \bar{Y} \bar{Y}_y^T \left( R_y + \bar{C}_y \bar{Y} \bar{C}_y^T \right)^{-1} \bar{C}_y \bar{Y} \bar{K}_x^T
\]

\[
= \left[ \begin{array}{ccc}
K_x Y_{11} & K_{w0}^T Y_{22} & K_{w1} Y_{22} \\
K_{w0}^T & K_{w1}^T & K_{w1} \end{array} \right] \left[ \begin{array}{c}
K_x^T \\
K_{w0}^T \\
K_{w1}^T 
\end{array} \right]^{-1} - \bar{R}_x \left[ \begin{array}{ccc}
Y_{11} C_y^T & 0 & \left( Y_{11} C_y^T \right)^T \\
Y_{22} D_y^T & 0 & 0 \\
0 & 0 & 0 
\end{array} \right]^{-1} \left[ \begin{array}{ccc}
C_y Y_{11} & D_y w Y_{22} & 0 \\
0 & 0 & 0 \\
C_y Y_{11} + D_y w Y_{22} \bar{K}_{w0}^T 
\end{array} \right] \bar{R}_x^T,
\]

(5.96a)

\[
= K_x Y_{11} K_x^T + K_{w0}^T Y_{22} K_{w1}^T + K_{w1} Y_{22} K_{w1}^T - \left( C_y Y_{11} K_x^T + D_y w Y_{22} K_{w0}^T \right)^T \\
\times \left[ \begin{array}{ccc}
D_y n D_y^T + C_y Y_{11} C_y^T & 0 & \left( C_y Y_{11} C_y^T \right)^T \\
0 & I_w & 0 \\
0 & 0 & 0 
\end{array} \right]^{-1} \left( C_y Y_{11} K_x^T + D_y w Y_{22} K_{w0}^T \right),
\]

(5.96b)

\[
= K_x Y_{11} K_x^T + K_{w0}^T Y_{22} K_{w1}^T + K_{w1} Y_{22} K_{w1}^T - \left( C_y Y_{11} K_x^T + D_y w Y_{22} K_{w0}^T \right)^T \left( C_y Y_{11} C_y^T + D_y w Y_{22} D_y^T \right)^{-1} \left( C_y Y_{11} K_x^T + D_y w Y_{22} K_{w0}^T \right),
\]

(5.96c)

that increases with preview, but is bounded above, because \( K_{wi} \to 0 \) as \( (A + B_u K_x)^i \to 0 \) (see equations eq. (5.29c) and eq. (5.32)). Further, when disturbance models are included to achieve regulation, the effect of increasing preview becomes even less clear. Nevertheless, the cost for output-feedback with noisy i-step ahead preview, and no regulation is given by

\[
\left\| \Delta_0^\frac{1}{2} \sum_\Delta \right\|_{2}^2 = \text{trace} \left( \Delta_0^\frac{1}{2} \left( \bar{K}_{nu}^T \bar{K}_{nu} + \bar{K}_{wu}^T \bar{K}_{wu} \\
- \left( C_y Y_{11} K_x + D_y w Y_{22} K_{w0}^T \right)^T \left( D_y n D_y^T + C_y Y_{11} C_y^T + D_y w Y_{22} D_y^T \right)^{-1} \left( C_y Y_{11} K_x + D_y w Y_{22} K_{w0}^T \right) \\
+ K_x Y_{11} K_x^T + \sum_{j=0}^{i-1} K_{wj}^T Y_{22} K_{wj}^T \right) \Delta_0^\frac{1}{2} \right).
\]

(5.97)
5.3.2 $\mathcal{H}_2$ Preview Observer Implementation

The observer-controller, including preview storage, is implemented as

$$
\begin{bmatrix}
\hat{x}(k+1) \\
\hat{w}(k+1) \\
\hat{w}(k+2)
\end{bmatrix} =
\begin{bmatrix}
A & B_w & 0 \\
0 & 0 & I_w \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x}(k) \\
\hat{w}(k) \\
\hat{w}(k+1)
\end{bmatrix} +
\begin{bmatrix}
L_{xy} & 0 \\
0 & 0 \\
0 & L_{ww}
\end{bmatrix}
\begin{bmatrix}
C_y & D_{yw} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x}(k) \\
\hat{w}(k) \\
\hat{w}(k+1)
\end{bmatrix} -
\begin{bmatrix}
y_y(k) \\
w_y(k) \\
w_y(k)
\end{bmatrix}
$$

+ $\begin{bmatrix}
B_u \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
K_x & K_{w0} & K_{w1}
\end{bmatrix}
\begin{bmatrix}
\hat{x}(k) \\
\hat{w}(k) \\
\hat{w}(k+1)
\end{bmatrix} -
\begin{bmatrix}
L_{\Delta y} & L_{\Delta w}
\end{bmatrix}
\begin{bmatrix}
C_y & D_{yw} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{x}(k) \\
\hat{w}(k) \\
\hat{w}(k+1)
\end{bmatrix} -
\begin{bmatrix}
y_y(k) \\
w_y(k) \\
w_y(k)
\end{bmatrix}.$$

This can be partitioned into a feedforward-preview system

$$
\begin{bmatrix}
\hat{w}(k+1) \\
\hat{w}(k+2)
\end{bmatrix} =
\begin{bmatrix}
0 & I_w \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{w}(k) \\
\hat{w}(k+1)
\end{bmatrix} -
L_{wuw} w_y(k)
$$

(5.99a)

$$
\begin{bmatrix}
\hat{w}(k)
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
K_{w0} & K_{w1}
\end{bmatrix}
\begin{bmatrix}
\hat{w}(k) \\
\hat{w}(k+1)
\end{bmatrix} +
\begin{bmatrix}
0
\end{bmatrix} w_y(k)
$$

(5.99c)

$$
\begin{bmatrix}
u_{ff}(k)
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
K_{w0} & K_{w1}
\end{bmatrix}
\begin{bmatrix}
\hat{w}(k) \\
\hat{w}(k+1)
\end{bmatrix} +
\begin{bmatrix}
0
\end{bmatrix} L_{pr} (I_w + D_{wn} D_{wn}^T)^{-1} w_y(k),
$$

(5.99d)

and an observer for the un-augmented system

$$
\begin{align*}
u(k) &= Ax(k) + B_w \hat{w}(k) + B_u (u_{ff}(k) + K_x \hat{x}(k) - L_{\Delta y} (C_y \hat{x}(k) + D_{yw} \hat{w}(k) - y_y(k))) \\
&+ L_{xy} (C_y \hat{x}(k) + D_{yw} \hat{w}(k) - y_y(k)),
\end{align*}
$$

(5.100)

where $u(k)$ is also the control passed to the actual plant. This is still $\mathcal{H}_2$ optimal (for the noise as modeled), and has the partitioning presented in Fig. 5.1, where the preview measurement is scaled.
in inverse proportion to the white noise variance. However, this configuration does not provide regulation, since it does not estimate the disturbance content in the preview measurement. The next section shows that when the disturbance content in the preview measurement is estimated, the product $L_x C_y$ is full, so that the optimal system cannot, in general, be partitioned as above.

5.4 $\mathcal{H}_2$ Sub-Optimal Output Feedback With Preview and Regulation

This final section on $\mathcal{H}_2$ preview control begins by showing that the structure of the observer-controller obtained in the preview section, is lost when the disturbance content in the preview measurement is estimated, as is necessary to obtain regulation using the approach presented in Chapter 4.

5.4.1 $\mathcal{H}_2$ Optimal Output Feedback With Preview and Regulation

The source of the problem that destroys the structure of the previous section, is not the initial multiplicities of the disturbance model that provide regulation. That is, replacing the basic plant model used in the previous section with

$$
\begin{bmatrix}
    x(k+1) \\ x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
    A & B_d \\ 0 & A_d
\end{bmatrix}
\begin{bmatrix}
    x(k) \\ x_d(k)
\end{bmatrix} +
\begin{bmatrix}
    B_w \\ 0
\end{bmatrix} w(k) +
\begin{bmatrix}
    B_u \\ 0
\end{bmatrix} u(k),
$$

(5.102)

will result in exactly the same results, and so, it may be assumed in this section that the plant model already includes the disturbance model $A_d$. The issue arises when the multiplicity of the disturbance model is increased in order to estimate the disturbance content in $w(k)$. In this latter case, for example, when using two-step ahead preview, the augmented model is

$$
\begin{bmatrix}
    x(k+1) \\ w(k) \\ w(k+1) \\ x_{wd}(k+1)
\end{bmatrix} =
\begin{bmatrix}
    A & B_w & 0 & 0 \\ 0 & 0 & I_w & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{wd}
\end{bmatrix}
\begin{bmatrix}
    x(k) \\ w(k) \\ w(k+1) \\ x_{wd}(k)
\end{bmatrix} +
\begin{bmatrix}
    B_u \\ 0 \\ 0 \\ 0
\end{bmatrix} u(k) +
\begin{bmatrix}
    0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    n_y(k) \\ n_w(k) \\ \bar{w}(k)
\end{bmatrix},
$$

(5.103a)
and

\[
\begin{bmatrix}
  y(k) \\
  w_y(k) \\
  x_{wd}(k)
\end{bmatrix} =
\begin{bmatrix}
  C_y & D_{yw} & 0 & 0 \\
  0 & 0 & 0 & C_{wd}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  w(k) \\
  w(k+1) \\
  x_{wd}(k)
\end{bmatrix} +
\begin{bmatrix}
  D_{yn} & 0 & 0 \\
  0 & D_{wn} & 0
\end{bmatrix}
\begin{bmatrix}
  n_y(k) \\
  n_w(k) \\
  \overline{w}(k)
\end{bmatrix},
\tag{5.104a}
\]

where the objective is to minimize

\[
\Delta(k) = u(k) +
\begin{bmatrix}
  -K_x & -K_{w0} & -K_{w1} & 0 \\
  0 & 0 & 0 & -K_{w2}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  w(k) \\
  w(k+1) \\
  x_{wd}(k)
\end{bmatrix} +
\begin{bmatrix}
  0 & 0 & 0 & -K_{w2}
\end{bmatrix}
\begin{bmatrix}
  n_y(k) \\
  n_w(k) \\
  \overline{w}(k)
\end{bmatrix},
\tag{5.105}
\]

The additional disturbance dynamics $A_{dw}$ model the event that the preview measurement contains similar modes. In effect, this can be viewed as estimating disturbance content that arrives in the measurement, but is not actually present in $w(k)$ as it affects the plant dynamics.

In order to design observer gains, we again find that it is necessary to compute the shifted system, and this requires

\[
\bar{A} = \bar{A}^T - C_y^T (D_{yw} D_{yw}^T)^{-1} D_{yw} \bar{B}_w^T
\tag{5.106a}
\]

\[
\begin{bmatrix}
  A^T & 0 & 0 & 0 \\
  B_{w}^T & 0 & 0 & 0 \\
  0 & I_w & 0 & 0 \\
  0 & 0 & A_{wd}^T & 0
\end{bmatrix} =
\begin{bmatrix}
  C_y^T & 0 \\
  D_{yw}^T & 0 \\
  0 & 0 \\
  0 & C_{wd}^T
\end{bmatrix}
\begin{bmatrix}
  D_{yn} D_{yn}^T & 0 & 0 \\
  0 & I + D_{wn} D_{wn}^T & 0 \\
  0 & 0 & I_w
\end{bmatrix}^{-1}
\begin{bmatrix}
  D_{yn} & 0 & 0 \\
  0 & D_{wn} & I_w \\
  0 & 0 & I_w
\end{bmatrix},
\tag{5.106b}
\]
so that

\[
\hat{A} = \begin{bmatrix}
A^T & 0 & 0 & 0 \\
0 & I_w & 0 & 0 \\
0 & 0 & 0 & A_{wd}^T
\end{bmatrix}
\]

(5.107a)

\[
- \begin{bmatrix}
C^T_y & 0 \\
(D_{yw}D_{yw}^T)^{-1}D_{yw} & 0 & 0 \\
0 & (I + D_{wn}D_{wn}^T)^{-1}D_{wn} & (I + D_{wn}D_{wn}^T)^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & I_w & 0
\end{bmatrix}
\]

(5.107b)

\[
= \begin{bmatrix}
A^T & 0 & 0 & 0 \\
0 & I_w & 0 & 0 \\
0 & 0 & -C_{wd}^T(I + D_{wn}D_{wn}^T)^{-1}D_{wn} & A_{wd}^T
\end{bmatrix}
\]

(5.107c)

It is at this point, that the structure obtained previously is lost, because of the presence of a non-zero block in every row and column of \( \hat{A} \). In this case, the first term in the DARE

\[
\bar{Y} = \begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & Y_{14} \\
Y_{21} & Y_{22} & Y_{23} & Y_{24} \\
Y_{31} & Y_{32} & Y_{33} & Y_{34} \\
Y_{41} & Y_{42} & Y_{43} & Y_{44}
\end{bmatrix}

= \begin{bmatrix}
A & B_w & 0 & 0 \\
0 & I_w & 0 & 0 \\
0 & 0 & -D_{wn}^T(I + D_{wn}D_{wn}^T)^{-1}C_{wd} & Y_{31} & Y_{32} & Y_{33} & Y_{34} \\
0 & 0 & 0 & A_{wd} & Y_{41} & Y_{42} & Y_{43} & Y_{44}
\end{bmatrix}

\times

\begin{bmatrix}
A^T & 0 & 0 & 0 \\
B_w^T & 0 & 0 & 0 \\
0 & I_w & 0 & 0 \\
0 & 0 & -C_{wd}^T(I + D_{wn}D_{wn}^T)^{-1}D_{wn} & A_{wd}^T
\end{bmatrix}

+ \ldots
\]

(5.108)

forces a full-order solution to be computed.

However, the more significant problem is that a full Riccati solution \( \bar{Y} \), results in a full
observer gain

\[
L_x^T = - \left( D_{yw}D_{yw}^T + C_y Y C_y^T \right)^{-1} \left( \begin{bmatrix}
C_y & D_{yw} & 0 & 0 \\
0 & 0 & 0 & C_{wd}
\end{bmatrix} Y \begin{bmatrix}
A^T & 0 & 0 & 0 \\
B_w^T & 0 & 0 & 0 \\
0 & I_w & 0 & 0 \\
0 & 0 & 0 & A_{wd}^T
\end{bmatrix} + \begin{bmatrix}
D_{yn} & 0 & 0 \\
0 & D_{wn} & I \\
0 & 0 & I_w & 0
\end{bmatrix} \right)
\]

(5.109a)

\[
= - \left( D_{yw}D_{yw}^T + C_y Y C_y^T \right)^{-1} \begin{bmatrix}
C_y Y_{11} + D_{yw} Y_{21} & C_y Y_{12} + D_{yw} Y_{22} & C_y Y_{13} + D_{yw} Y_{23} & C_y Y_{14} + D_{yw} Y_{24} \\
C_{wd} Y_{41} & C_{wd} Y_{42} & C_{wd} Y_{43} & C_{wd} Y_{44}
\end{bmatrix} Y
\]

(5.109b)

\[
\times \begin{bmatrix}
A^T & 0 & 0 & 0 \\
B_w^T & 0 & 0 & 0 \\
0 & I_w & 0 & 0 \\
0 & 0 & 0 & A_{wd}^T
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & I_w & 0
\end{bmatrix},
\]

and this reduces to

\[
L_x^T = - \left( D_{yw}D_{yw}^T + C_y Y C_y^T \right)^{-1} \begin{bmatrix}
(C_y Y_{11} + D_{yw} Y_{21}) A^T + (C_y Y_{12} + D_{yw} Y_{22}) B_w^T & C_y Y_{13} + D_{yw} Y_{23} & 0 & (C_y Y_{14} + D_{yw} Y_{24}) A_{wd}^T \\
C_{wd} Y_{41} A^T + C_{wd} Y_{42} B_w^T & C_{wd} Y_{43} & I_w & C_{wd} Y_{44} A_{wd}
\end{bmatrix}
\]

(5.110a)

\[
= \begin{bmatrix}
L_{xy}^T & L_{w0y}^T & L_{w1y}^T & L_{wdy}^T \\
L_{xx}^T & L_{w0w}^T & L_{w1w}^T & L_{wdw}^T
\end{bmatrix}
\]

(5.110b)

This shows that there is output injection to all states in the augmented model, from both the \(y_y(k)\) and \(w_y(k)\) prediction errors. Further, if the product \(L_x C_y\) is assimilated into the dynamics of the observer (i.e., \(\bar{A} + L_x C_y + \ldots\)), then there is no structure whatsoever, and no partitioning is possible.

For this reason, in order to obtain regulation, the sub-optimal approach shown in Fig. 5.2 is taken. In this case, the preview controller is optimized without knowledge of the measurement \(y_y(k)\). Then separately, a regulating observer is optimized, viewing both the present/feedforward estimate \(\hat{y}(k)\), and the feed-forward control \(u_{f,f}(k)\), as noisy measurements.
5.4.2 \( H_2 \) Sub-Optimal Output Feedback Design With Regulation

Once the decision is made to use a sub-optimal approach, the design of the observer becomes relatively straightforward, and completely independent of the amount of preview available. The design model is

\[
\begin{bmatrix}
  x(k+1) \\
  x_d(k+1) \\
  x_{dw}(k+1) \\
  x_{du}(k+1)
\end{bmatrix} =
\begin{bmatrix}
  A & B_d & 0 & 0 \\
  0 & A_d & 0 & 0 \\
  0 & 0 & A_{dw} & 0 \\
  0 & 0 & 0 & A_{du}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_d(k) \\
  x_{dw}(k) \\
  x_{du}(k)
\end{bmatrix} +
\begin{bmatrix}
  B_u \\
  0 \\
  0 \\
  0
\end{bmatrix} u(k) +
\begin{bmatrix}
  B_{xn} & 0 & 0 & B_w & B_u \\
  B_{dn} & 0 & 0 & 0 & 0 \\
  B_{dwn} & 0 & 0 & 0 & 0 \\
  B_{dwn} & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  n_x(k) \\
  n_y(k) \\
  n_w(k) \\
  n_u(k) \\
  w(k) \\
  u_{ext}(k)
\end{bmatrix}
\]

\[(5.111a)\]

\[
\begin{bmatrix}
  y_y(k) \\
  w_y(k) \\
  u_y(k)
\end{bmatrix} =
\begin{bmatrix}
  C_y & C_d & 0 & 0 \\
  0 & 0 & C_{dw} & 0 \\
  0 & 0 & 0 & C_{du}
\end{bmatrix}
\begin{bmatrix}
  x(k) \\
  x_d(k) \\
  x_{dw}(k) \\
  x_{du}(k)
\end{bmatrix} +
\begin{bmatrix}
  D_{yu} \\
  0 \\
  0 \\
  0
\end{bmatrix} u(k) +
\begin{bmatrix}
  n_x(k) \\
  n_y(k) \\
  n_w(k) \\
  n_u(k) \\
  w(k) \\
  u_{ext}(k)
\end{bmatrix}
\]

\[(5.111b)\]
\[
\Delta(k) = u(k) - \begin{bmatrix}
K_x & K_d & 0 & 0 \\
K_x & K_d & 0 & 0 \\
x_d(k) & x_d(k) & x_dw(k) & x_du(k)
\end{bmatrix} - \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
w(k) & w(k) & u_{ext}(k)
\end{bmatrix}. \tag{5.111c}
\]

This model includes extra multiplicities \(A_{dw} \) and \(A_{du} \) in the disturbance model as discussed in Chapter 4. The desired control for optimizing observer gains is state-feedback through \([K_x \ K_d] \), that includes the disturbance gain for regulation; the model assumes (correctly) that the disturbance \(w(k)\) and external control \(u_{ext}(k)\) are already applied to the system.

The observer gains are obtained by designing the full-information “control” \([y^T \ w_y^T \ u_y^T]^T \) for the adjoint system

\[
\begin{bmatrix}
x(k+1) \\
x_d(k+1) \\
x_{dw}(k+1) \\
x_{du}(k+1)
\end{bmatrix} =
\begin{bmatrix}
A^T & 0 & 0 & 0 \\
B_d^T & A_d^T & 0 & 0 \\
0 & 0 & A_{dw}^T & 0 \\
0 & 0 & 0 & A_{du}^T
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k) \\
x_{dw}(k) \\
x_{du}(k)
\end{bmatrix} +
\begin{bmatrix}
C_y^T & 0 & 0 \\
C_d^T & 0 & 0 \\
0 & C_{dw}^T & 0 \\
0 & 0 & C_{du}^T
\end{bmatrix}
\begin{bmatrix}
y(k) \\
y_d(k) \\
x_{dw}(k) \\
x_{du}(k)
\end{bmatrix}
- 
\begin{bmatrix}
K_y^T \\
K_d^T \\
w_y(k) \\
0
\end{bmatrix}
\Delta(k), \tag{5.112a}
\]

\[
\begin{bmatrix}
n_x(k) \\
n_y(k) \\
n_w(k) \\
n_u(k) \\
w(k) \\
u_{ext}(k)
\end{bmatrix} =
\begin{bmatrix}
B_x^T & B_{dn}^T & B_{dwn}^T & B_{dun}^T \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
B_w^T & 0 & 0 & 0 \\
B_u^T & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k) \\
x_{dw}(k) \\
x_{du}(k)
\end{bmatrix} +
\begin{bmatrix}
D_{yn}^T & 0 & 0 \\
0 & D_{wn}^T & 0 \\
0 & 0 & D_{un}^T \\
0 & 0 & I_w \\
0 & 0 & I_u
\end{bmatrix}
\begin{bmatrix}
y_y(k) \\
w_y(k) \\
w_{yn}(k) \\
w_{yn}(k)
\end{bmatrix} - \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\Delta(k). \tag{5.112b}
\]
The various noise gains (e.g., $B_{xn}$) are usually not explicitly formed. Instead, the matrices

$$Q_x = B_w Q_{xw} B_w^T = \begin{bmatrix} B_{xn} Q_{xn} B_{xn}^T + B_w Q_w B_w^T + B_u Q_u B_u^T & 0 & 0 & 0 \\ 0 & B_{dn} Q_{xn} B_{dn}^T & 0 & 0 \\ 0 & 0 & B_{dwn} Q_{xn} B_{dwn}^T & 0 \\ 0 & 0 & 0 & B_{dun} Q_{xn} B_{dun}^T \end{bmatrix},$$

(5.113a)

$$R_y = D_{yw} Q_{xw} D_{yw}^T = \begin{bmatrix} D_{yn} Q_y D_{yn}^T + D_{yw} Q_w D_{yw}^T & Q_w D_{yw}^T & 0 \\ D_{yu} Q_w & Q_w + D_{wn} Q_{wn} D_{wn}^T & 0 \\ 0 & 0 & Q_u + D_{un} Q_{un} D_{un}^T \end{bmatrix},$$

(5.113b)

$$S_{xy} = B_w Q_{xw} D_{yw}^T = \begin{bmatrix} B_w Q_w D_{yw}^T & B_w Q_w & B_u Q_u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(5.113c)

are defined directly so that $Q_x$ is diagonal. Making an entry in $Q_x$ large relative to $R_y$ tends to make the time constants associated with estimation of that state faster. Making $\{Q_y, Q_w, Q_u\}$ large makes use of the associated measurement expensive; it implies that the measurement is not accurate. Similar effects hold for the associated measurement noises (e.g., making $Q_{wn}$ large has generally the same effect as making $Q_w$ large).

The resulting output injection gain is of the form

$$[L_x^T \ L_0^T] = (D_{yw} D_{yw}^T + C_y Y_{11} C_y^T)^{-1} \left[ -(C_y Y_{11} A_x^T + D_{yw} B_w^T) \ (C_y Y_{11} K_x^T + D_{yw} K_{w0}^T) \right]$$

(5.114a)
so that

\[
L_x = -\left( D_{yw} D_{yw}^T + C_y Y_{11} C_y^T \right)^{-1} \left[ \begin{array}{c}
D_{yn} \\
0 \\
0 \\
0 \\
D_{yn} \\
0 \\
0 \\
I_w \\
0 \\
I_u
\end{array} \right], \quad (5.115a)
\]

\[
= -\left( D_{yw} D_{yw}^T + C_y Y_{11} C_y^T \right)^{-1} \left[ \begin{array}{c}
B_w D_{yw}^T \\
0 \\
0 \\
0 \\
B_w D_{yw}^T \\
0 \\
0 \\
I_u
\end{array} \right], \quad (5.115b)
\]

and

\[
L_0 = \left( D_{yw} D_{yw}^T + C_y Y_{11} C_y^T \right)^{-1} \left[ \begin{array}{c}
D_{yn} \\
0 \\
0 \\
D_{yn} \\
0 \\
0 \\
I_w \\
0 \\
I_u
\end{array} \right], \quad (5.116a)
\]

\[
= \left( D_{yw} D_{yw}^T + C_y Y_{11} C_y^T \right)^{-1} \left[ \begin{array}{c}
K_x Y_{11} C_y \\
0 \\
0 \\
0 \\
K_x Y_{11} C_y \\
0 \\
0 \\
I_u
\end{array} \right] \quad (5.116b)
\]

The observer-controller is then implemented as in eq. (5.67) using the measurements from the plant and exogenous inputs

\[
u(k) = (K_x - L_0 C_y) x(k) + L_0 w_y(k) + L_y y_y(k) \quad (5.117a)
\]

\[
x(k + 1) = \left( A + L_x C_y \right) x(k) + B_u u(k) - L_x y_y(k) \quad (5.117b)
\]

It is worth noting, for example, that \( u_y \) does not enter the observer directly through \( B_u \) as it would
had we not assume it was corrupted by a measurement disturbance as well as noise. Instead, \( u_y \)
enters through the gain \( L_x \) that is determined in part by \( B_u \) via eq. (5.115b).

5.4.3 \( \mathcal{H}_2 \) Preview Optimal and Sub-Optimal Cost Comparisons

The noise gains \( \{D_{yn}, D_{wn}, D_{un}\} \) are set during observer design, to obtain desirable levels of control effort and frequency and transient response. Their effect on the \( \mathcal{H}_2 \) cost is of secondary importance, and the inclusion regulation further complicates comparisons between sub-optimal and optimal \( \mathcal{H}_2 \) performance. Once an acceptable observer controller is obtained, it is possible to evaluate the \( \mathcal{H}_2 \) performance without measurement noise present, and in terms of the stochastic input \( w(k) \). This facilitates comparison with the full information performance. So, the following are computed as functions of preview:

- \( \mathcal{H}_2 \) performance for full information feedback,
- \( \mathcal{H}_2 \) sub-optimal performance with regulation and no measurement noise.

So, for each level of preview, the performance of the sub-optimal approach is evaluated without measurement noise (e.g., \( L_{pr} = I \)) by evaluating \( \|\Sigma_{zcl}\|_{\mathcal{H}_2}^2 \) directly. This is accomplished, by augmenting the system \( \Sigma_\Delta \) so that it provides \( y(k), w_y(k) \) and \( u_{ff}(k) \), and then closing the loop around the sub-optimal observer. Consider the use of 2-step ahead preview. In this case, \( \Sigma_{\Delta 2} \) is
where this will result in closed loop matrices

\[
\begin{bmatrix}
x(k+1) \\
w(k+1) \\
w(k+2)
\end{bmatrix} = \begin{bmatrix} A & B_w & 0 \\
0 & 0 & I_w \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix} x(k) \\
w(k) \\
w(k+1)
\end{bmatrix} + \begin{bmatrix} B_u \\
0 \\
0
\end{bmatrix} u(k) + \begin{bmatrix} 0 \\
0 \\
I_w
\end{bmatrix} w(k), \tag{5.118a}
\]

\[
z(k) = \begin{bmatrix} C_z & D_{zw} & 0 \\
w(k) \\
w(k+1)
\end{bmatrix} + 0 w(k), \tag{5.118b}
\]

\[
\begin{bmatrix} y_y(k) \\
w_y(k) \\
u_{ff}(k)
\end{bmatrix} = \begin{bmatrix} C_y & D_{gw} & 0 \\
0 & I_w & 0 \\
K_x & K_{w0} & K_{w1}
\end{bmatrix} \begin{bmatrix} x(k) \\
w(k) \\
w(k+1)
\end{bmatrix} + \begin{bmatrix} 0 \\
0 \\
K_w
\end{bmatrix} w(k), \tag{5.118c}
\]

and the loop is closed around the sub-optimal observer

\[
(5.119)
\]

\[
\begin{bmatrix} \bar{x}(k+1) \\
\bar{x}_{du}(k+1) \\
\bar{x}_{du}(k+1)
\end{bmatrix} = \begin{bmatrix} \bar{x}(k) \\
L_{xy} & L_{xw} & L_{xu} \\
L_{dy} & L_{dw} & L_{du}
\end{bmatrix} \begin{bmatrix} C_y & 0 & 0 \\
0 & C_{dw} & 0 \\
0 & 0 & C_{du}
\end{bmatrix} \begin{bmatrix} \bar{x}(k) \\
\bar{x}_{du}(k) \\
\bar{x}_{du}(k)
\end{bmatrix} + \begin{bmatrix} 0 \\
B_u \\
B_u
\end{bmatrix} u(k) + \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix} 0 \\
C_x & 0 & 0 \\
0 & C_{dw} & 0 \\
0 & 0 & C_{du}
\end{bmatrix} \begin{bmatrix} \bar{x}(k) \\
\bar{x}_{du}(k) \\
\bar{x}_{du}(k)
\end{bmatrix} + \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix} 0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix} y_y(k) \\
w_y(k) \\
u_{ff}(k)
\end{bmatrix}. \tag{5.119}
\]

This will result in closed loop matrices \(\{A_{cl}, B_{wcl}, C_{zcl}, D_{zcl}\}\), and the \(\mathcal{H}_2\) performance is then obtained from

\[
\|\Sigma_{zcl}\|^2_{\mathcal{H}_2} = \text{trace} \left( D_{zcl} D_{zcl}^T \right) + \text{trace} \left( B_{wcl}^T X_{cl} B_{wcl} \right), \tag{5.120}
\]

where \(X_{cl}\) satisfies the Lyapunov equation

\[
X_{cl} = C_{zcl}^T C_{zcl} + A_{cl}^T X_{cl} A_{cl}. \tag{5.121}
\]
Although this does not include the measurement noise sources used to optimize the observer-controller, it does allow a more direct comparison with the full information performance, the latter being free of measurement noise as well. And, this is acceptable, since the noise gains \( \{D_n, D_u\} \) are tuned based on considerations other than \( \mathcal{H}_2 \) performance, as described in the next chapter.
Chapter 6

Design of Preview Tracking and Disturbance Rejection for the Wind Turbine

This chapter demonstrates the design of a preview output-feedback controller for the wind turbine. The techniques developed in Chapters 4 and 5 are applied to a wind turbine model based on the three-bladed Controls Advanced Research Turbine (CART3) located at NREL’s National Wind Technology Center located outside of Golden, CO. The CART3 is a 40 m diameter turbine with three blades. The modes summarized in Table 6.1 are included in the model used for the purposes of controller design. FAST is used to obtain linearized models as discussed in Chapter 2. In the final MPC design, linearizations are computed at a series of operating points along a desired trajectory as explained in Chapter 7. However, in this chapter, all results presented, are based on a single operating point corresponding to a uniform wind speed of 18 m/s, a rotor speed of 41.7 rpm, and a blade pitch of 14.3°.

Table 6.1: CART3 modes included for the purposes of controller design.

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<tbody>
<tr>
<td>36.6</td>
<td>40</td>
<td>18.7</td>
<td>3581</td>
<td>41.7</td>
<td>675</td>
</tr>
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A detailed development is provided for the MBC controller, and its characterization based on the linear MBC model; simulation results with the full order non-linear FAST turbine simulator
are provided in Chapter 8.

Figure 6.1: A sample rate of 20 Hz is sufficient to capture the structural modes of the CART3.

As apparent from earlier chapters, all design is done in discrete-time. Therefore, all linearizations obtained from fast are converted to discrete time models using the zero-order hold method [21], and a 20 Hz sample rate. In the case of MBC models, the MBC transformation is applied before conversion to discrete-time. The sample rate chosen is high enough to capture all the resonances in the CART3 model (see Table 6.1). This is confirmed by the frequency responses displayed in Fig. 6.1, where it is observed that the discrete-time response matches the continuous-time response out to 10 Hz (the Nyquist rate) where the response is below peak by at least 20 dB.
6.1 Multi-Blade Coordinate Models for Control With Regulation and Preview

Design of controllers begins with conversion of the MBC model to discrete time,

$$x(k + 1) = A_{MBC}x(k) + [B_{wMBC} \ B_{pMBC} \ B_{g}] \begin{bmatrix} w_{mbc}(k) \\ p_{mbc}(k) \\ \tau_{g}(k) \end{bmatrix},$$

\[ (6.1a) \]

$$\begin{bmatrix} m_{mbc}(k) \\ \Omega_{g}(k) \\ \tau_{r}(k) \\ p_{rmbc}(k) \end{bmatrix} = \begin{bmatrix} C_{mMBC} \\ C_{gMBC} \\ C_{rMBC} \end{bmatrix} x(k) + \begin{bmatrix} D_{mwMBC} & 0 & 0 \\ D_{gwMBC} & 0 & 0 \\ D_{twMBC} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{mbc}(k) \\ p_{mbc}(k) \\ \tau_{g}(k) \end{bmatrix}.$$ \[ (6.1b) \]

Then the discrete time model is augmented and partitioned so that the desired performance metrics \( z(k) \) and feedback measurements \( y(k) \) are available as outputs:

$$z(k) = \begin{bmatrix} \Omega_{eg}(k) & m_{mbc}(k)^T & z_{tr}(k) & p_{rmbc}(k)^T \end{bmatrix}^T,$$

$$\Omega_{eg} = \Omega_{g}(k) - D_{tw} w_u(k),$$

$$z_{tr}(k) = \frac{1}{T_s} \left( \frac{\tau_{r}(k)}{GB Ratio} - \tau_{g}(k) \right),$$

$$y(k) = \begin{bmatrix} \Omega_{eg}(k) \\ m_{v}(k) \\ m_{h}(k) \end{bmatrix}.$$ \[ (6.2d) \]

In this case, the feedback measurements \( y(k) \) consist of perturbations in the generator speed tracking error \( \Omega_{eg} \), and perturbations in the vertical \( m_{v} \) and horizontal \( m_{h} \) component amplitudes of the blade-root bending moments. Cyclic loads appear on these outputs as constant offsets, and so can be mitigated with integral control. In above rated conditions, generator torque control is held constant, and so it is not used as a control input for the results presented in this section \( (\tau_{g}(k) = 0) \).

The generator speed tracking error is computed relative to \( D_{tw} w_u \) as the desired change in
generator speed due to changes in uniform wind speed $w_u$. The constant $D_{\Omega w}$ is determined by the operating region and the slope of the ideal transition profile at each operating point. This is explained in more detail in Chapter 7. For the above rated operating point demonstrated in this section, $D_{\Omega w} = 0$. The tracking error $\Omega_{eg}$ is also a performance measurement, as well as any perturbations in the component amplitudes of the blade-root bending moments, and all component amplitudes of the MBC pitch rates $p_{rmbc}$. In addition, the rate of change of twisting torque $z_{tr}$ applied to the drive train is also penalized.

The delay-state used to compute the change in twisting torque $z_{tr}(k)$ is assimilated and the resulting model is denoted using standard notation

$$x(k+1) = Ax(k) + B_w w(k) + B_u u(k),$$  \hspace{1cm} (6.3a)

$$z(k) = C_z x(k) + D_{zu} w(k) + D_{zu} u(k),$$  \hspace{1cm} (6.3b)

$$y(k) = C_y x(k) + D_{yw} w(k) + D_{yu} u(k).$$  \hspace{1cm} (6.3c)

However, in order to obtain the assumption (5.3c) for the purposes of full-information gain computation, the standard change of variable

$$\bar{u}(k) = u(k) + (D_{zu}^T D_{zu})^{-1} D_{zu}^T (C_z x(k) + D_{zu} w(k))$$  \hspace{1cm} (6.4)

is applied to obtain

$$x(k+1) = \left[ A - B_u (D_{zu}^T D_{zu})^{-1} D_{zu}^T C_z \right] x(k) + \left[ B_w - B_u (D_{zu}^T D_{zu})^{-1} D_{zu}^T D_{zu} \right] w(k) + B_u \bar{u}(k),$$

$$\bar{B}_w$$

so that

$$D_{zu}^T [\bar{C}_z \bar{D}_{zw}] = 0.$$

(6.6)
In this case, the form assumed in Chapter 2 for the pitch actuator model guarantees

\[ Ru = D_{zu}^T D_{zu} > 0. \]  \hfill (6.7)

As in Appendix A, a performance weight

\[ Q_z = \begin{bmatrix} W_g & 0 & 0 & 0 \\ 0 & I_{3 \times 3} W_m & 0 & 0 \\ 0 & 0 & W_{tr} & 0 \\ 0 & 0 & 0 & I_{3 \times 3} W_{pr} \end{bmatrix} \]  \hfill (6.8)

is handled explicitly. The weights \( \{W_g, W_m, W_{tr}, W_{pr}\} \) are hand tuned during the process of feedback gain design. So, the results of Chapter 5 are modified to use \( Q_z \) and the transformed system so that the relevant Riccati equations are

\[
\begin{align*}
X_{11} &= \tilde{A}^T X_{11} A - \tilde{A}^T X_{11} B_u (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} \tilde{A} + \tilde{Q}_z, \\
X_{21} &= X_{12}^T = \tilde{B}_u^T X_{11} A - \tilde{B}_u^T X_{11} B_u (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{11} A + \tilde{B}_u^T Q_z \tilde{C}_z, \\
X_{i+1,1} &= X_{i,1} (\tilde{A} + B_u K_x).
\end{align*}
\]  \hfill (6.9a,b,c)

where

\[ \tilde{Q}_z = C_z^T Q_z C_z - C_z^T (I - Q_z D_{zu} R_u^{-1} D_{zu}^T Q_z) C_z, \]  \hfill (6.10)

and (after simplification as in Appendix A)

\[
\begin{align*}
[K_x \ K_{w0}] &= - (R_u + B_u^T X_{11} B_u)^{-1} \begin{bmatrix} (B_u^T X_{11} A + D_{zu}^T C_z) \ (B_u^T X_{11} B_w + D_{zu}^T D_{zw}) \end{bmatrix}, \\
K_{wi} &= - (R_u + B_u^T X_{11} B_u)^{-1} B_u^T X_{i+1,1}^T.
\end{align*}
\]  \hfill (6.11a,b)

### 6.2 Nominal Feedforward/Feedback (Full Information) and Disturbance Gains

In this section the design of preview and regulation gains are obtained in three steps. First, the \( \mathcal{H}_2 \) optimal feed-forward/feedback gains are obtained. Then the model is augmented with the
Figure 6.2: Full information (state-feedback/preview closed loop) response to 1 m/sec step change in collective wind: the top plot shows collective out-of-plane flap response; the center plot shows the rate of change of twisting torque; and the bottom plot shows the (collective) pitch rate required. The step change hits the turbine at 0.5 sec, while preview actuation (see the pitch rate) starts well before 0.5 sec.

disturbance model required for regulation and disturbance gains are computed. Then the system is augmented with additional disturbance models so that the observer-controller can use the exogenous control ($u_{ff}$ or $u_{ff} + u_{mpc}$) and wind preview and still achieve regulation.

6.2.1 Full Information Gains

The storage of preview measurements are viewed as part of the turbine state so that feed-forward and feedback gains are designed using the $H_2$ approach from Chapter 5; this is depicted in Fig. 6.3. The weights $\{W_g, W_m, W_{tr}, W_{pr}\}$ are tuned in order, by first placing emphasis on blade flap (i.e., increasing $W_m$), then the rate of change of twisting torque, speed regulation, and finally pitch rate. Pitch rate is penalized until the state-feedback/preview response to a 1 m/s step in
collective wind, produces pitch rates less than $20^\circ$/sec; this result can be observed in the responses provided in Fig. 6.2. The resulting collective wind frequency responses are provided in Fig. 6.6.

At the end of this process, optimized values for the state-feedback gain $K_x$ and preview gains $\{K_{w0}, K_{w1}, \ldots\}$ are obtained. As explained in Chapter 5, it is also possible to compute full-information $H_2$ performance as a function of preview. This result is provided in Fig. 6.4. It is apparent that beyond 7-10 samples of preview, there is essentially no performance improvement. The responses presented so far, all use 10 samples (0.5 sec. at the 20Hz sample rate) of preview. Using only full information feedback does not result in regulation. This is apparent in the generator high speed shaft (HSShiftV) response in Fig. 6.6, that does not go to zero at DC.

6.2.2 DAC Regulation Gains

Regulation is achieved by augmenting the observer with a disturbance model $\tilde{A}_d$ and computing the associated disturbance gain $K_d$. In above rated conditions, the goal is to reject offsets in generator speed and cyclic bending moments $m_v$ and $m_h$ that are used as feedback measurements.
And, this should occur to some extent for ramp changes in wind speeds. So, the disturbance model

\[
\begin{bmatrix}
x_s(k+1) \\
x_r(k+1)
\end{bmatrix} =
\begin{bmatrix}
A_d \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
x_s(k) \\
x_r(k)
\end{bmatrix},
\]

\hspace{1cm} (6.12a)

\[
d(k) =
\begin{bmatrix}
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_s(k) \\
x_r(k)
\end{bmatrix},
\]

\hspace{1cm} (6.12b)

is chosen to generate both steps and ramps. The fact that simple double integrators can be used to mitigate cyclic (sinusoidal) loads is a significant advantage of using MBC. The dynamics \(A_d\) are replicated once for each of three feedback measurements to obtain a block diagonal \(\hat{A}_d\). Then disturbance gains \(\{K_d, C_d\}\) are computed so that there is a solution to the regulator equation

\[
\begin{bmatrix}
A + B_u K_x & B_u K_d \\
C_y & C_d
\end{bmatrix}
\begin{bmatrix}
\hat{X} \\
I
\end{bmatrix} =
\begin{bmatrix}
\hat{X} \\
I
\end{bmatrix}
\]

\hspace{1cm} \hat{A}_d

\hspace{1cm} (6.13)
6.2.3 Observer Output-Injection Gains

Figure 6.5: Turbine model for design of observer gains.

In this application, the exogenous controls and preview measurements are treated as if they are noisy measurements that may contain disturbance content. So, the model for observer design is as shown in Fig. 6.5, wherein the basic disturbance model $A_d$ is duplicated three times to obtain regulation at $y$, three times to model disturbance content in each of the wind preview measurements, and then four more times for each of the feed-forward controls $\{p_{mbc}, \tau_g\}$, even though generator torque is not used in above rated conditions. This is so that the order of the resulting controller remains the same at all operating points during preview scheduled control, and this includes those at which generator torque is used.

Observer gains are optimized with all disturbance augmentations in place, so computations are done for the adjoint of the system.
Figure 6.6: Full information and output (observer) feedback frequency response to variations in
collective wind: the top plot shows the generator/high speed shaft response; center plot shows the
collective out-of-plane flap response, and the lower plot shows the response in the rate of change of
twisting torque.

\[
x(k + 1) = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & \mathcal{A}_d & 0 & 0 \\ 0 & 0 & \mathcal{A}_{dw} & 0 \\ 0 & 0 & 0 & \mathcal{A}_{du} \end{bmatrix} \begin{bmatrix} x(k) \\ x_d(k) \\ x_{dw}(k) \\ x_{du}(k) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & B_w \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_y(k) \\ n_w(k) \\ n_u(k) \\ w_{mbc}(k) \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \\ 0 \\ 0 \end{bmatrix} p_{mbc}(k),
\]

\[
\begin{bmatrix} y(k) \\ \bar{w}_{mbc}(k) \\ \bar{u}_{ff}(k) \end{bmatrix} = \begin{bmatrix} C_y & C_d & 0 & 0 \\ 0 & C_{d21} & C_{wd} & 0 \\ 0 & C_{d31} & 0 & C_{ud} \end{bmatrix} \begin{bmatrix} x(k) \\ x_d(k) \\ x_{dw}(k) \\ x_{du}(k) \end{bmatrix} + \begin{bmatrix} D_{gn} & 0 & 0 & D_w \\ 0 & D_{wn} & 0 & I_w \\ 0 & 0 & D_{un} & 0 \end{bmatrix} \begin{bmatrix} n_y(k) \\ n_w(k) \\ n_u(k) \\ w_{mbc}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u(k).$

(6.14a)

(6.14b)
The gains $C_{dw}$ and $C_{du}$ are chosen so that $x_{dw}$ and $x_{du}$ are observable at $w_{mbc}(k)$ and $u_{ff}(k)$, respectively. The gains \{${C_d}_{21}, {C_d}_{31}$\} are computed as discussed in Chapter 4 to satisfy the selective IMP.

The noise gains \{${D_n}_{gn}, {D_n}_{vn}, {D_n}_{hn}$\} affect the speed of the observer dynamics, including the rate at which offsets are rejected; making these small tends to increase speed. Similarly, the noise gains \{${D_n}_{wn}, {D_n}_{un}$\} affect the rate at which change can occur in the states $x_{dw}$ and $x_{du}$. These weights are hand tuned to get the resulting closed loop (original plant model and the preview $u_{ff}$ feeding into the observer-controller) frequency response and pitch rates to match the full information responses as closely as possible. This results in the frequency responses provided in Fig. 6.6.

It is of interest to see the closed loop response over the range of preview used for design. The flap frequency response and speed-regulation step response are plotted in Fig. 6.7 as preview is varied from 0 to 10 samples showing that performance deteriorates as preview decreases. Additional redesign is most likely required if this feedback controller is intended to function both with and without preview, because of the lack of speed regulation in the absence of preview.

Using the approach discussed in Section 5.4.3, the performance of the sub-optimal observer-controller with preview compensation can be compared with the full-information case. This is provided in Fig. 6.4, and shows that the performance of the observer-controller (output-feedback) is always worse than the full-information performance.
Figure 6.7: Output (observer) feedback $\mathcal{H}_2$ performance as preview increases; dark blue indicates feedback only, and preview up to 10 samples (0.5 sec) is indicated as shades transition to red.
Chapter 7

Implementation of Preview Scheduled MPC (PSMPC) for the Wind Turbine

This chapter explains the operation of the architecture shown in Fig. 7.1 for use with NREL’s three-bladed controls advanced research turbine (CART3). A significant part of the design is the development of the scheduling function so that control and speed set-points can be determined as functions of wind speed. In this approach, the choice of operating points determines the profile that the turbine should follow as wind speeds vary over time. The method described in Section 7.1 is
the relatively straightforward approach of computing set-points such that turbine is in equilibrium with the aerodynamic torque developed on the rotor at each wind speed.

The approach developed in this chapter assumes that it is possible to make any necessary control adjustments relative to the scheduled set-points, with the assistance of linear controls generated by feedback from the tracking error. Prior studies using iterative learning control [44] have shown that it is not unreasonable to expect that it is possible to adjust load torque and pitch so that the turbine speed tracks a reasonable speed profile determined by wind speed. With the configuration shown in Fig. 7.1, the MPC algorithm views the turbine (and nominal controls) as a time varying linear system; this is fairly accurate as long as the turbine’s response stays near the scheduled operating points, and control perturbations remain small.

The simulation results presented in Chapter 8 show that this is the case over moderate wind speeds and rates of change of wind speed. However, the natural response of the scheduled linear controllers drives pitch and load torque outside of acceptable limits when the rate of change of wind speeds is high enough. It is the job of the MPC controller to adjust the total control effort to stay within prescribed limits, and with a preview of approaching wind speeds, to make these adjustments in a way that is optimal over the preview horizon.

Section 7.1 explains the method used for computing set-points, and then the configuration of the MPC controller for the selected schedule and the configuration of nominal controls are explained in Section 7.2 and Section 7.3, respectively. The process for design of the nominal controllers is essentially the one demonstrated in Chapter 6. A linearization of the turbine at each set-point is used as the basis for design of feedback/feedforward and observer gains. Since the MPC controller keeps the control actuation very close to prescribed limits, there is some freedom in allowing the nominal feedback to be active even when the scheduled set-point is at a limit (e.g., when the load torque is at maximum in region 3). These trade-offs are discussed in more detail in Section 7.3.
Figure 7.2: (a) The CART3 power coefficient \( C_p \) data is computed using WT_Perf, and indicates the optimal TSR and blade pitch for capturing maximum aerodynamic power. (a) Based on the CART3 geometry and maximum \( C_p \), there are a number of options for transitioning generator torque from the square law to rated.

7.1 Set-Point Scheduling

The turbine’s steady state operating characteristics are determined by it’s power coefficient \( C_p(\lambda, \beta) \) which describes the fraction of available aerodynamic power that the turbine will capture as a function of tip-speed-ratio (TSR) \( \lambda \) and blade pitch \( \beta \). So, given the mean wind speed \( w_u \) and air density \( \rho \) throughout the area \( A \) swept out by the rotor, the aerodynamic power captured by the rotor

\[
P_{aero} = (\rho A w_u) \left( \frac{1}{2} w_u^2 \right) C_p(\lambda, \beta),
\]

(7.1)
is the product of the mass flow rate through the rotor area, the kinetic energy per unit mass, and the turbines power coefficient. Since

\[
\lambda = \frac{R \Omega}{w_u},
\]

(7.2)
where \( R \) is the rotor radius and \( \Omega \) is the rotor speed, the aerodynamic power can be expressed in terms of rotor speed

\[
P_{aero} = \frac{1}{2} \rho A R^3 \frac{C_p(\lambda, \beta)}{\lambda^3} \Omega^3.
\]

(7.3)
Then, because aerodynamic torque $\tau_{\text{aero}}$ is related to power according to

$$P_{\text{aero}} = \tau_{\text{aero}}\Omega,$$  \hspace{1cm} (7.4)

the aerodynamic torque acting on the rotor is

$$\tau_{\text{aero}} = \frac{1}{2} \rho A R^3 \frac{C_p(\lambda, \beta)}{\lambda^3} \Omega^2.$$  \hspace{1cm} (7.5)

So, if the power coefficient is a maximum at $(\lambda_{\text{opt}}, \beta_{\text{opt}})$, then in steady state, the maximum power is obtained if the generator torque is set according to

$$\tau_{\text{gen}} = \frac{1}{2N_{gb}} \rho A R^3 \frac{C_p(\lambda_{\text{opt}}, \beta_{\text{opt}})}{\lambda_{\text{opt}}^3} \Omega^2,$$  \hspace{1cm} (7.6)

where $N_{gb}$ is the ratio of the generator speed to the rotor speed. This is known as the square law, and is the standard method for setting generator load in below rated conditions. Using the TSR definition, any such operating point profile is easily translated into one based on wind speed; e.g., from equations (7.2) and (7.5) it follows that

$$\tau_{\text{aero}} = \frac{1}{2} \rho A C_p(\lambda, \beta) w^2_u,$$  \hspace{1cm} (7.7)

so, in region 2 generator torque can be scheduled according to

$$\tau_{\text{gen}} = \frac{1}{2N_{gb}} \rho A C_p(\lambda_{\text{opt}}, \beta_{\text{opt}}) w^2_u.$$  \hspace{1cm} (7.8)

To schedule set-points for rotor (or generator) speed, load torque, and blade pitch as functions of wind speed in all operating regions, it is necessary to determine the blade pitch and TSR at which aerodynamic torque balances with load torque (or vice-versa) for a given rotor speed. Using NREL’s WT_Perf code [16], steady state operating point data for the CART3 is computed at a rotor speed of 35 rpm, over a range of TSR’s and blade pitches to obtain contours of constant power coefficient and aerodynamic torque as depicted in Fig. 7.2a and Fig. 7.3a, respectively. According to eq. (7.5), the torque magnitude data for other rotor speeds $\Omega$ is obtained from this data if it is scaled by $\Omega/\Omega_{35}$, where $\Omega_{35} = 2\pi 35/60$ rad/sec. Using this contour data, it is possible to determine
the blade pitch and TSR required to balance aerodynamic torque with any desired load torque (or vice-versa).

It is highly unusual that the geometry of the turbine will be such that use of the square law will bring the turbine to rated speed at the same time that the load torque hits rated. So, inevitably, there is a transition, referred to as region 2.5, that is used to switch between constant pitch (the square law) and constant power (or torque) operation. A set-point profile is obtained by choosing a starting point for the transition between region 2 (where pitch is constant) and region 3 (where generator load is constant). Three possible transition profiles are depicted in Fig. 7.2b and have the same starting point, but use different slopes to reach rated. The determination of the schedules as a function of wind speed is explained in detail for profile L (“lower”); the details related to the other options are similar, but vary depending on how the profile departs from the square law.

In any case, the pitch set point is increased above $\beta_{opt}$ as soon as the torque profile deviates from the square law; this insures that there is some head room for speed regulation via pitch when the rotor speed hits rated. The region 3 start point (the end of region 2.5) is set by scaling the torque contours to 41.7 rpm (rated speed), and then searching along the $\lambda_{opt}$ line to find the point at which the aerodynamic torque balances with the rated maximum of $\tau_3 = N_g b^3 3581 \text{ N-m}$ (this is not necessarily a standard method, but proves to be convenient). The resulting pitch start point
for region 3 is found to be $\beta_3 \approx 11^\circ$. Now, the set-points at the start of region 2.5 are known, since pitch is specified and torque and wind speed are related by eq. (7.7); and, the set-points at the end of region 2.5 are known, since wind speed ($w_{u3} \Delta R\Omega_3/\lambda_{opt}$, $\Omega_3 \Delta 2\pi \times 1.7/60$ rad/sec) and torque $\tau_3$ have been specified, and pitch was determined from the torque contour data.

The rest of the set-points for region 2.5 are obtained by taking the load torque and blade pitch as specified by the linear section of Fig. 7.2 so that

$$\tau_2.5(\Omega) = \tau_{2.5} + \Omega \frac{\tau_3 - \tau_{2.5}}{\Omega_3 - \Omega_{2.5}} \quad (7.9a)$$

$$\beta_{2.5}(\Omega) = \beta_{opt} + \Omega \frac{\beta_3 - \beta_{opt}}{\Omega_3 - \Omega_{2.5}} \quad (7.9b)$$

Then the associated wind speed required for scheduling each $\{\tau_{2.5}(\Omega), \beta_{2.5}(\Omega)\}$ pair is obtained from the torque contour data by scaling relative to $\Omega$ and finding the $\lambda$ that gives $\tau_{aero} = \tau_{2.5}(\Omega)$; this then gives $w_u = R\Omega/\lambda$.

In region three, torque is specified as rated $\tau_3$, and $\lambda = R\Omega_3/w_u$ is determined by wind speed, so scheduling pitch is simply a matter of using the scaled torque contour data to find the pitch that balances $\tau_{aero}$ with the rated torque for region 3. The final result for profile L are the torque and pitch schedules depicted in Fig. 7.3, as well as a rotor/generator speed schedule (not shown). Further, using blade bending-moment contour data (also obtained from WT_Perf), the resulting set-point schedules for profiles U,C and L in region 2.5 can be compared as shown in Fig. 7.4. Profile L results in significantly lower blade bending moments ($\approx 30\%$) with only marginal decrease in power ($< 10\%$ on avg.). Since the goal is to mitigate blade loads, the schedule generated from profile L will be used as the basis for the scheduled controller described in the remaining sections of this chapter. FAST simulates the turbine to obtain steady-state operation, and then computes linearizations of the turbine that are converted to MBC equivalents as discussed in Chapter 2. This is repeated at 58 evenly spaced set-points along profile L between wind speeds of 1 m/sec and 30 m/sec. Then nominal controls are designed for each of these 58 operating points, and this provides the basis for the MPC models.
7.2 Configuration of Scheduling and MPC

Since it is assumed that control perturbations added to scheduled set-points keep the turbine at scheduled operating points, the MPC algorithm is configured assuming that it will have a schedule index $s(k)$ into an array of linear models that are valid at each at each sample hit $k$ over the prediction horizon. The schedule $s(k)$ is set by wind speeds obtained from the preview measurements that are available over the horizon; the linearization is that obtained from FAST at the operating point scheduled for each wind speed; and the MPC model is the linearization in closed loop with the nominal feedback control. Section 7.2.1 details configuration of the MPC cost objectives, and then Section 7.2.2 explains the scheduling function.

7.2.1 MPC Model and Cost Function

As discussed in Chapter 4, the MPC optimization is based on the closed loop model of the plant and any known/measured input sequences that are known over the preview horizon as depicted in Fig. 7.5. It is possible to pre-compute the nominal feed-forward commands so that they can be treated in the same manner as the wind preview measurements. As discussed in Chapter 4 the MPC model uses only the disturbance models that have the associated regulation gains $\{K_d, C_d\}$ and takes into account the nominal feedback control. In addition, the MPC model is
augmented with storage for the total control, so that translating between the notation of Chapter 3 and Chapter 6, this means that the closed loop MPC model is

\[
\begin{bmatrix}
    x(k+1) \\
    x_d(k+1) \\
    u_{total}(k)
\end{bmatrix} =
\begin{bmatrix}
    A(s(k)) + B_u(s(k))K_x(s(k)) & B_u(s(k))K_d(s(k)) & 0 \\
    0 & \tilde{A}_d & 0 \\
    K_x(s(k)) & K_d(s(k)) & 0
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    x_d(k) \\
    u_{total}(k-1)
\end{bmatrix}
\]

\[
B_u(k) + \begin{bmatrix}
    0 \\
    u_{mpc}(k) + \\
    I_{4 \times 4}
\end{bmatrix}
\begin{bmatrix}
    w_{mbc}(k) \\
    w_{mbc}(k) \\
    w_{mbc}(k) \\
    w_{mbc}(k)
\end{bmatrix},
\]

(7.10a)
\[
\begin{align*}
\begin{bmatrix}
z_{mpc}(k) \\
z_m(k) \\
p_{mbc}(k) \\
\tau_{gen}(k)
\end{bmatrix} &= \begin{bmatrix}
C_{zp}(s(k)) + D_{zpu}(s(k))K_x(s(k)) & D_{zpu}(s(k))K_d(s(k)) & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k) \\
u_{total}(k-1)
\end{bmatrix} + \\
\begin{bmatrix}
D_{zu}(k) \\
D_{zw}(k)
\end{bmatrix}
\begin{bmatrix}
D_{zpu}(s(k)) \\
I_p \\
I_{gen}
\end{bmatrix}
u_{mpc}(k) + \\
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
w_{mbc}(k) \\
w_{ff}(k)
\end{bmatrix} \\
(7.10b)
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
y_{ctl}(k) \\
y(k) \\
u(k) \\
u_{ctl}(k) \\
\Omega(k)
\end{bmatrix} &= \begin{bmatrix}
M_tK_x(s(k)) & M_tK_d(s(k)) & 0 \\
M_tK_x(s(k)) & M_tK_d(s(k)) & -M_t \\
C_{\Omega}(s(k)) & C_{\Omega d}(s(k)) & 0
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k) \\
u_{total}(k-1)
\end{bmatrix} + \\
\begin{bmatrix}
D_{ctua}(k) \\
D_{ctlw}(k)
\end{bmatrix}
\begin{bmatrix}
M_t \\
\underbrace{M_t} \\
M_t
\end{bmatrix}
u_{mpc}(k) + \\
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
w_{mbc}(k) \\
w_{ff}(k)
\end{bmatrix} \\
(7.10c)
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
y_{c}(k) \\
y_{ctl}(k) \\
y_{ctl}(k) \\
\Omega(k)
\end{bmatrix} &= \begin{bmatrix}
M_tK_x(s(k)) & M_tK_d(s(k)) & 0 \\
M_tK_x(s(k)) & M_tK_d(s(k)) & -M_t \\
C_{\Omega}(s(k)) & C_{\Omega d}(s(k)) & 0
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k) \\
u_{total}(k-1)
\end{bmatrix} + \\
\begin{bmatrix}
D_{cu}(k) \\
D_{cw}(k)
\end{bmatrix}
\begin{bmatrix}
M_t \\
\underbrace{M_t} \\
M_t
\end{bmatrix}
u_{mpc}(k) + \\
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
w_{mbc}(k) \\
w_{ff}(k)
\end{bmatrix}, \\
(7.10d)
\end{align*}
\]

where

\[
u_{mpc}(k) = \begin{bmatrix}
\tau_{gen}(k) \\
p_{mbc}(k)
\end{bmatrix},
(7.11)\]
and

\[ z_{\text{mpc}}(k) = \begin{bmatrix} \Omega_{\text{eg}}(k) & m_{\text{mbc}}(k)^T & z_{\text{tr}}(k) & p_{\text{mbc}}(k)^T \tau_{\text{gen}}(k) \end{bmatrix}^T. \]  

(7.12)

This model keeps the same plant quantities in the performance output \( z(k) \), but replaces control related outputs with penalties on MPC control effort. So, for example, the matrix \( I_{\text{gen}} \) selects out the generator load command from \( u_{\text{mpc}}(k) \). This model also provides for constraints in the change of total control.

The matrix

\[ M_\ell = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & 1 & -\sqrt{2} & 0 \\ 0 & 1 & 0 & \sqrt{2} \\ 0 & 1 & 0 & -\sqrt{2} \end{bmatrix}. \]  

(7.13)

passes \( \tau_{\text{gen}} \) through and then accounts for the way the MBC component amplitudes are combined through the inverse MBC transform to form the individual pitch commands. This means that in lieu of constraining the individual pitch amplitudes

\[ |p| = |p_u + p_v \cos(\theta) + p_h \sin(\theta)| = |p_u + \sqrt{p_v^2 + p_h^2} \cos\left(\theta + \arctan\left(\frac{p_h}{p_v}\right)\right)|, \]  

(7.14)

the linear combinations bound the worst case amplitudes

\[ |p| \leq |p_u| + \sqrt{2}|p_{v,h}|, \]  

(7.15)

that occur when \( p_v = p_h \).

The MPC cost weight is a function of the scheduling variable \( s \)

\[ Q_z(s) = \begin{bmatrix} W_{ge}(s) & 0 & 0 & 0 & 0 \\ 0 & I_{3 \times 3}W_{m}(s) & 0 & 0 & 0 \\ 0 & 0 & W_{rr}(s) & 0 & 0 \\ 0 & 0 & 0 & I_{3 \times 3}W_{p}(s) & 0 \\ 0 & 0 & 0 & 0 & W_{gen}(s) \end{bmatrix}. \]  

(7.16)
Weights are generally made inversely proportional to the amount of head-room available for perturbations. In below rated conditions where the pitch set point results in less pitch authority $W_{gen} \ll W_p$, and in above rated conditions $W_{gen} \gg W_p$. The rate penalty $W_{rr}$ is set to avoid drive train transients and keep pitch rates acceptable. In region 2.5, the weights are linearly interpolated between their below and above rated settings.

### 7.2.2 Scheduler Configuration

The wind speed used for scheduling is a low passed version of the measured uniform component of the previewed wind speed across the rotor disk. A preview of future perturbations is computed relative to the present scheduling wind speed and the set-points it determines; a schematic of the necessary operations is provided in Fig. 7.6. “Delay$_{LP}$” matches the delay through the linear-phase low-pass filter, “Delay$_{FF}$” matches the amount of preview used in the nominal feed-forward compensation, and “Delay$_{MPC}$” provides the additional preview required by the MPC algorithm. The total delay through the scheduling block is coordinated with the total preview time so that the quantities with the greatest delay are coincident with the present inputs to the turbine.

Based on the design example in Chapter 6, 0.5 sec preview is adequate for the feed-forward compensation, and at 41.7 rpm, a 1 sec MPC horizon will provide for more than one revolution of the rotor. Added to this delay is 1 sec worth of filtering delay in the low-pass block, so that

![Figure 7.6: Scheduling of set-points and future perturbations are based on the present value of the low-passed preview wind-speed.](image-url)
the total delay through the scheduling block needs to be at least 2.5 sec. To operate at maximum sustained wind speeds of 25 m/sec (>55 mph), this corresponds to a preview distance of 62.5 m. Keeping the preview distance constant, slower mean wind speeds will require additional latency in the scheduling block.

7.3 Configuration of Observer and Nominal Controls

Since the MPC algorithm adjusts the total control to stay within prescribed limits, in theory it is possible to have the nominal linear controls active even when the set-points are near limits. However, given that the turbine is an under-actuated system (controls are prone to reach acceptable limits in normal operation), additional measures would be necessary during the design of the nominal feedback to insure the closed loop remains stable when plant inputs do not all saturate simultaneously (recall that for an open-loop stable plant, the DAC approach remains stable when all loops are open). Techniques that address saturation of actuators individually (e.g., [19]) are not pursued in this thesis, and therefore the nominal feedback gains generate only pitch or torque commands, but not both simultaneously. Using profile L from Section 7.1, there is a portion of region 2.5 where there is enough pitch head-room to allow the preview/feed-forward and disturbance gains to generate pitch and generator torque at the same time. Since the preview controller does not involve feedback, and according to DAC theory, saturations are not an issue (saturation is equivalent to changing these gains to zero).

The feedback/feed-forward gains can be partitioned according to actuator

\[
\begin{bmatrix}
K_x & K_d
\end{bmatrix} = \begin{bmatrix}
K_{px} & K_{pd} \\
K_{τx} & K_{τd}
\end{bmatrix},
\]

\[
K_w = \begin{bmatrix}
K_{pw} \\
K_{τw}
\end{bmatrix}
\]

(7.17a)

(7.17b)

where \((\cdot)_p\) generates pitch commands and \((\cdot)_τ\) generates load torque commands. As long as the
pitch set point is below $6^\circ$, 

\begin{align}
K_{px} &= 0, \\
K_{pd} &= 0, \\
K_{pw} &= 0,
\end{align}

(7.18a) (7.18b) (7.18c)

and the techniques of Chapter 4 and Chapter 5 are used to compute the gains \( \{ K_{\tau x}, K_{\tau d}, K_{\tau w} \} \).

From the perspective of stability, it is possible to use non-zero \( K_{pw} \), but the MPC constraints are such where set point is at the minimum allowable pitch, the MPC algorithm will squelch all pitch perturbations and so there is no reason to generate preview pitch actuation. In region 2.5 where the pitch set point is \( \geq 6^\circ \), non-zero values are computed for \( \{ K_{pd}, K_{pw} \} \); as mentioned, \( K_{pw} \) does not affect stability, and according to DAC theory, the stability of the feedback controller is independent of \( K_{pd} \) as well. In above rated conditions

\begin{align}
K_{\tau x} &= 0, \\
K_{\tau d} &= 0,
\end{align}

(7.19a) (7.19b)

and non-zero values are computed for the gains \( \{ K_{pd}, K_{px}, K_{pd}, K_{pw}, K_{\tau w} \} \), so that all forms of pitch control are active, and only preview/feed-forward torque control is active.
Chapter 8

Simulation Results

The performance of preview control and preview scheduled MPC (PSMPC) is evaluated in simulation in the following sections. First, configuration of the LIDAR system used in simulations is discussed. Then the CART3 turbine and the configuration of the controllers that use preview is discussed in Section 8.2. The LIDAR system is used in simulation, except in Section 8.3 where the sensitivity of preview control to wind evolution is explored while simulating the turbine in above rated conditions. These preliminary results are done without MPC or LIDAR distortion, and tentatively show that the designed $H_2$ optimal gains appear to be fairly robust to the measurement errors induced by evolution. Then Section 8.4 evaluates the PSMPC system during large, but uniform changes in wind speed that take the turbine between regions 2 and 3. And then finally, Section 8.5 benchmarks MPC load mitigation performance against baseline controller performance in above rated conditions.

8.1 LIDAR Preview Wind Measurements

As discussed in Chapter 2 and Appendix D, it is assumed that the LIDAR is a hub-mounted continuous-wave instrument and there are three focal points that can be placed simultaneously at three different locations that are coordinated with the rotor speed and position. Prior studies [47] based on the CART3, are consistent with the results in Section 8.3; it is found that after more than about 0.5 s the blade load mitigation performance of preview control does not substantially increase. Also, as discussed in this section, the directional bias of the LIDAR measurements in-
creases substantially with shorter preview distances. Hence, there is a tradeoff between keeping the measurement angle as small as possible, and keeping the measurement distance as short as possible to mitigate evolution effects. This tradeoff will be further explored in the final draft of the thesis.

Figure 8.1: A hub mounted LIDAR takes measurements 2 s out in front of the turbine at 75% blade span. Assuming 18 m/s wind speed and a 20 m blade length, this corresponds to a distance of 36 m and a measurement angle of about 23 deg.

The directional bias of the LIDAR motivates the use of longer preview times, since the blade is typically most sensitive to wind speeds at spans greater than 50% [61], and aiming the LIDAR at small angles from horizontal requires larger distances to reach the corresponding blade span. As depicted in Fig. 8.1, a 2 sec preview distance corresponds to 36 m at the average wind speed of 18 m/s used here, and with the 75% span for the CART3 at 15 m from the hub, this also corresponds to a measurement angle of 23 deg. This allows evaluation of the performance of preview control measuring at a distance ahead of the turbine where there is still some accuracy in predicting the rotor position. The LIDAR model employed has been integrated into a version of FAST and its characteristics are discussed in detail in [65].

8.2 Turbine Model and Simulation Cases

As discussed in Chapter 2, the linearized model from FAST is scaled so that individual point measurements of wind at 75% blade-span can be used as disturbance inputs to the model. During
simulation all available degrees of freedom (DOF) provided by FAST are utilized except for teeter and those related to offshore operation. Doing so adds a second blade flap mode, an edge-wise blade flap mode, and two modes each for tower sway in the fore-aft and side-to-side directions, and a single yaw mode (yaw actuation is not used, but this latter DOF approximates a side-to-side compliance in the drive train).

When LIDAR measurements are used, they are focused at 36 m out in front of the turbine so that they provide 2.0 s of preview. However, the preview controllers themselves do not utilize all the available preview. With ideal measurements, the preview distance is set to provide only as much preview as is required by the controllers. As summarized in Table 8.1, the LTI-OFBK controller used for benchmarking in above rate conditions, only uses 0.45 s of the available preview. The nominal state-feedback used in the MPC controller also uses 0.45 s of preview, but the MPC algorithm itself uses 1.0 s of preview. As indicated in Table 8.1, the MPC and LTI-OFBK controllers are simulated with and without the LIDAR model distortions.

Table 8.1: Preview times used by controllers and measurement configurations: the controllers always use the same amount of preview, but the measurement scheme may take measurements further out so that the system has additional preview that is unused by the controller.

<table>
<thead>
<tr>
<th>Controller</th>
<th>Design Preview (sec/samples)</th>
<th>Measurement/Simulation Preview (sec/samples)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LTI</td>
<td>OFBK 0.45/N_{prev}=9</td>
<td>0.45/N_{prev}=9</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LIDAR 2.0/N_{prev}=40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>LIDAR Obs. Est.</td>
</tr>
<tr>
<td>MPC</td>
<td>SFBK 0.45/N_{prev}=9</td>
<td>1.0/N_{prev}=20</td>
</tr>
<tr>
<td></td>
<td>Opt. 1.0/N_{mp}=20</td>
<td>1.0/N_{prev}=20</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.0/N_{prev}=40</td>
</tr>
</tbody>
</table>

8.3 Sensitivity to Wind Evolution (above rated conditions)

The nominal operating point for design (and simulation) is in steady wind of 18 m/sec, a blade pitch of 12.7°, and rotor speed of 41.7 rpm. During simulation, all DOF’s provided by FAST are utilized except for yaw and teeter. To separate issues that arise in the design of output-feedback versus state-feedback controllers, from the effects of wind evolution, this study assumes that the
Figure 8.2: Simple preview control implemented as state-feedback: $x(k)$ is the plant/turbine state; $x_a(k)$ is the state of extra dynamics that provide integral control of speed errors and rejection of 1P loads at each blade individually; and $x_{pr}(k)$ contains the preview measurements of approaching wind speed changes.

entire state of the turbine is available for feedback.

As discussed in Chapter 2, the simulation is set up so that the control system will use individual point measurements $w(k+i) = [w_1(k+i) \ w_2(k+i) \ w_3(k+i)]^T$ of the wind that the blades will encounter after a fixed time delay $i$, so that the point measurements rotate in unison with the blades. State feedback/feedforward gains are designed as discussed in Chapter 5, except that the plant is first augmented with integral control “$P_{aug}$” as shown in Fig. 8.2; output augmentation can be used to obtain offset-free speed regulation since in this case the nominal feedback is not combined with an MPC controller.

In formulating the $\mathcal{H}_2$ cost, emphasis on blade flap and speed are adjusted to get the best attenuation in the flap response to wind perturbations. Then the weight $W_{pr}$ on pitch effort is increased until the (linear) closed loop response, to a step change in collective wind, produces pitch rates on the order of $10^\circ$/sec. Generally, the $\mathcal{H}_2$ performance improves with the number of samples of preview as shown in Fig. 8.3a, but remains bounded below so that there is a diminishing return (zero-preview corresponds to feedback-only). However, the goal is not the precise value of the $\mathcal{H}_2$
norm, but the attenuation of perturbations in blade load due to perturbations in wind speed. This goal evident in the frequency responses provided in Fig. 8.3b.

8.3.1 Simulations with Evolution

The simulation is configured as shown in Fig. 8.4. TurbSim is used to generate wind speeds at the turbine that are consistent with Von Karman spectral model. Then, the technique presented in Appendix D is used to induce evolved wind speeds located at 60 evenly spaced azimuths that are at a measurement radius of approximately 75% rotor radius (≈15 m). The evolution distance $d = d_i$ is chosen to be equivalent to $i$ samples at the control system sample rate of 20 Hz ($= 1/T_s$), at the nominal wind speed of 18 m/sec, so that

$$d_i = 18T_s i. \quad (8.1)$$

The controller is simulated using feedback only, and also with increments of $i$ samples of preview

$$K_{ff} = [K_{w0} \ldots K_{wi} 0 \ldots], \quad (8.2)$$
Figure 8.4: During simulation, FAST reads in wind speeds at the turbine directly from the TurbSim output; the controller estimates blade positions 1/2 second into the future, and interpolates wind speeds at these positions from the annulus of evolved wind speeds.

up to \( i = 10 \) samples. Where \( 0 \leq i \leq 10 \), the controller uses preview gains up through \( K_{wi} \), and where \( i > 10 \), the controller uses only preview gains through \( K_{w10} \). In the latter case, this is equivalent to taking measurements further than 1/2 sec (at 18m/sec average wind speed) ahead of the turbine, and then waiting until those wind speeds are within 1/2 sec of reaching the turbine before storing them in the feed-forward delays.

As depicted in Fig. 8.4, the present rotor position \( \theta_R(k) \) and speed \( \Omega_R(k) \), are used to predict the blade positions after 1/2 sec, and then the evolved wind speeds at the 60 azimuth locations are interpolated to the predicted blade positions. This gives the controller a preview of wind speeds that the blades will encounter over a 1/2 sec horizon. Results typical of this process are provided in Fig.8.5a which shows predicted/evolved wind speeds and the speed that the blade
Figure 8.5: (a) (b) Simulation results: the top plot shows the rms preview-measurement error as the evolution time/distance increases; the bottom plot shows the rms blade load as preview/evolution increases. Blade load mitigation improves with preview at first, but with increasing preview distance/evolution, performance starts to deteriorate.

actually encounters. It is also worth noting that all measurements used, and controls generated, by the controller are relative to the nominal operating point of 18m/sec wind speed, 12.7° blade pitch, and 1800 rpm generator speed (41.7 rpm rotor speed).

The controller is simulated using the same base TurbSim wind field. For each set of evolved wind speeds computed using evolution distances that correspond to preview times in the range 0.05 sec (0.9 m) to 10 sec (180 m), the simulation is repeated, and the rms blade load, and preview measurement error (relative to the wind at the blade) are computed. The results are provided in Fig. 8.5b. As expected from Fig. 8.3a, the benefit of preview bottoms out near 0.2 sec (4 samples) of preview time. After 0.2 sec, blade load mitigation performance deteriorates as the preview errors increase with applied evolution. However, there is a surprisingly large range of preview times (0.2 s to 3 s) where the performance is still significantly better than using feedback only.
Figure 8.6: (a) The PSMPC controller is simulated in uniform winds that vary with rates ranging from 2 m/s/s to 0.08 m/s/s. (b) Somewhere between rates of 1.43 m/s/s and 2 m/s/s the system completely fails to track the desired speed profile and it is not possible to continue the simulation.

8.3.2 Discussion of Evolution Simulation Results

As expected, the performance of the preview controller actually becomes worse than feedback only as the accuracy of the preview measurements declines. However, there may be several mitigating factors in the present work. First, the level of turbulence used in the base wind field (and thereby indirectly in the evolution) is severe, as can be observed in Fig. 8.5a which shows multiple step-like changes well in excess of 1 m/sec, and a total change of 15 m/sec within 4 sec of simulation time. Second, the controller is using point measurements of wind speed at 75% span for preview control instead of using some method of estimating a blade effective wind speed that would tend to average out high frequency content. This makes the present configuration more sensitive to the error introduced by the evolution model, since it produces the most significant error at higher frequencies. It can be expected then, that the sensitivity to evolution effects may decrease significantly in less turbulent conditions, even if the average wind speed remains high. In fact, earlier work indicates that it is the total time of transit, and not the distance that is most significant determining the amount of wind speed change that occurs over a longitudinal separation.
8.4 Simulation of PSMPC During Transitions Between Regions 2 and 3

The results in this section only pertain to the preview scheduled MPC (PSMPC) controller described in the preceding chapter, since the benchmark controllers are not designed to operate outside of region 3. Simulation results are obtained using FAST [35] with the proposed PSMPC method and a model of the CART3. The MPC algorithm computes perturbations to both pitch and torque, independent of operating region and insures that input constraints are satisfied. During simulations, all available degrees of freedom are enabled in FAST’s non-linear model, except for teeter, and FAST’s generalized-dynamic-wake inflow model is used. The system is simulated in uniform wind conditions that vary as shown in Fig. 8.6(a). Each wind profile varies from 7 m/s to 27 m/s and the desired response brings the turbine from region 2, well into region 3 and back.

It can be observed in Fig. 8.6(b), that the system fails to reach rated speed for a rate of change in wind-speed somewhere between 1.43 m/s/s and 2 m/s/s. Because the rotor speed is well below the scheduled rated level during this failure, the turbine is left operating at extremely low TSR’s where FAST apparently has trouble computing. So, the exact nature of system failure (to track on schedule) is not yet known, but there is enough data to discern the trend and possibly suggest some modifications to improve the mode of failure.

8.4.1 The Role of the DAC/Offset-Free Formulation

The necessity of using the DAC/Offset-free component is apparent in Fig. 8.7. These results are obtained without the MPC algorithm active, so that the system is operating as a scheduled linear controller. In the next section focuses on the necessity of the MPC algorithm. On the left in Fig. 8.7(a), the blue curve shows speed reference tracking performance with the DAC gain $K_d$ in place, and the dashed green curve shows the performance with $K_d = 0$. Even at this slow rate of wind-speed variation, the system cannot track the desired speed reference accurately enough to avoid a 3% over-speed fault. On the right, in Fig. 8.7(b) provides the effects of the same change in collective wind-speed, but this time with an added 10% shear at a blade span of 75%. The
Figure 8.7: (a) Scheduled controller (No MPC) response w/ and w/o DAC gain; even at very low rates of wind-speed change, the scheduled controller cannot regulate speed without an offset; with the DAC gain, offset-free regulation is achieved. (b) (No MPC) though not included in subsequent results, the DAC gain achieves similar rejection of cyclic bending moments in the face of wind shear; the insets show that only the 2P component is remains when using the DAC gain.

The bottom plot shows collective blade-root bending-moment with $K_d = 0$ and the top plot shows the same output with the DAC gain active. The insets show that the DAC gain completely removes the 1P component so that the only cyclic variations left are at 2P and above. This is the only simulation considered in which shear is present. All subsequent results are obtained using wind profiles without shear.

8.4.2 The Role of MPC

In the previous section results show that the system can track large changes in wind-speed without the use of the MPC algorithm. Indeed, when the rates of wind-speed change are slow enough that the controls do not saturate, the system performance is nearly identical, with and without the use of MPC. The results presented in Fig. 8.8 demonstrate the necessity of the MPC algorithm at higher rates of wind-speed change. On the left, in Fig. 8.8(a), the center plot shows the resulting total (set-point + controllers) torque command using just the scheduled linear-control in green and the control obtained using PSMPC control in dashed red. Without the MPC component,
Figure 8.8: (a) At higher rates of wind-speed change, the scheduled controller violates constraints on control inputs unless the MPC algorithm is active. (b) The response of the system without MPC exhibits severe overspeed faults and induces ringing in the low speed shaft. The PSMPC controller plans for control saturations and scheduled gain changes and is able to mitigate both speed faults and excessive load transients.

the linear controller motors the turbine as wind-speeds increase and then violates the maximum allowable torque as wind-speeds decrease. In addition, without MPC, the pitch command goes below the minimum allowable.

In the complete PSMPC system response, controls adhere to all input constraints, and as shown on the right, in Fig. 8.8(b), the turbine response is greatly improved. The top plot shows that without MPC, there is a dangerous over-speed fault that is completely absent when using PSMPC. The center plot shows that the level of blade-root bending-moment is also reduced, but
more significant is the difference in low speed shaft torque as shown in the lower plot.

The rapid change in wind-speeds translate into rapid changes in the scheduled nominal gains. The nominal scheduled-linear controllers are optimized to operate at single operating points and make no provision for bumpless transfer from one controller to the next. When the jumps between linear controller gain-sets are large, it induces ringing in the low speed shaft compliance. However, because the PSMPC algorithm knows precisely what gain changes are going to occur, it is able to mitigate this adverse effect completely. This is an advantage of scheduling without the use of turbine measurements; the sample-hit at which gain changes occur are known exactly over the span of time for which preview is available. In effect, the MPC algorithm is serving as a bumpless transfer technique. This would not be possible without preview, and would be difficult to arrange for, if the turbine (feedback) measurements were used to adjust the scheduling to better match actual rotor speed, TSR, and blade pitch.

8.4.3 Discussion of PSMPC Results

The objective of these investigations was to test the robustness of the PSMPC scheme and to see if the quality of the performance could be affected by reasonable adjustments to the schedule of operating points used. The results demonstrate that the PSMPC approach operates well even in the presence of very large and fast wind-speed changes (20 m/s total change in a time span of less than 20 sec). The study has successfully demonstrated the robustness of the proposed approach. However, in order to assess the effect of LIDAR distortion on system viability, it will be necessary to simulate in turbulent wind conditions. Also, to successfully discriminate between different set-point profiles, it will at least be necessary to use a distribution of wind-speeds that accurately reflect those for the turbine site; the triangle variation used here with minimal time at rated speed is not adequate.

The robustness demonstrated by the approach is promising and bodes very well for additional enhancements that are possible without changing the structure of the PSMPC feedback loops or the MPC algorithms. There is a potentially large number of ways that the scheduler function can
be modified. In the PSMPC implementation, the scheduling is done over a preview of 2 s while the MPC horizon is only 1 s. So, the profile outside of the MPC horizon can be adjusted without affecting the bumpless transfer function that the MPC algorithm provides when it has access to a known time variation in linear models. These changes in profile can be based on the rate of change of wind-speed so that the size of scheduled accelerations are reduced. Further, potentially significant, improvements in load mitigation might be achieved by similar adjustments of the set-points to schedule blade pitch consistently with reductions in scheduled rotor acceleration. This would require a wider range of linearizations and two dimensional indexing. However, given the formulation used for a quadratic MPC optimization, this mostly impacts the amount of memory required in the processing system, and only a minimal increase in computational burden.

8.5 Load Performance Relative to Baseline (above rated conditions)

8.5.1 Overview of Baseline Controllers

Figure 8.9: MBC turbine response at the generator shaft speed (left), vertical bending moment (center) and horizontal bending moment (right) to the collective, vertical and horizontal pitch commands.

In above rated conditions, load mitigation is bench marked against several baseline controllers. Representing standard PI approaches to turbine speed regulation are a collective pitch (CP) and a independent pitch (IP) controls that are designed using the approach outlined in [12], but simplified
to operate only in above rated conditions. These controls are based on the qualitative property of the MBC transform in that it tends to decouple the effects of the uniform, collective and vertical pitch commands. This is evident in the open-loop frequency responses provided in Fig. 8.9; the generator speed (left) is essentially unresponsive to the cyclic pitch commands, while the cyclic bending moment amplitudes (center, and right) are unresponsive to the collective pitch command. Also, the cyclic pitch to cyclic bending moment amplitude channels are separated by approximately 10 dB out until approximately 0.3 Hz. This suggests that as long as the loop response is below 0 dB by 0.3 Hz, that the pitch components can be viewed as independent. This leads to relatively conservative design of PI controls for each channel with the resulting frequency responses shown in Fig. 8.10.

In addition, an MBC $\mathcal{H}_\infty$ preview controller is also used to provide benchmark results. This controller is designed using the methods discussed in [48], and the blade load mitigation performance to perturbations in wind speed is evident in the maximum singular value plots shown in Fig. 8.11. This controller is designed by augmenting the plant with integral control on the generator speed, and cyclic bending moment measurements, as well as states to store the preview. Then a $\mathcal{H}_\infty$ optimal controller is designed for the augmented system resulting in an extremely high order controller [48].

8.5.2 Blade Load Mitigation Performance

This section evaluates the load mitigation of the PSMPC system in above rated conditions without the effects of evolution. It is found that a LIDAR measurement angle on the order of $20^\circ$ still provides substantial preview benefits, despite the associated directional bias that results. The MPC performance is compared against an MBC based $\mathcal{H}_\infty$ LTI-OFBK design documented in [46], and also against collective pitch (CP) and individual pitch (IP) controllers that use only feedback and no preview measurements. For the CP feedback-only controller the objective is speed regulation. For all other controllers, the objective is both blade load mitigation and speed regulation. Also, in all controllers except the CP feedback-only design, the additional feedback from measurements of the blade-root bending moments is providing a means to adjust the pitch of each of the three
Figure 8.10: Baseline loop responses. (a) Collective pitch to generator speed; the addition of a notch guarantees attenuation of drive train resonance. (b) The vertical-to-vertical and horizontal-to-horizontal loops are identical; the cross over is very conservative to insure gain is less than 1 where vertical and horizontal channels couple (at frequencies above 1 Hz).

Figure 8.11: Dynamic output feedback, preview controllers. The MBC preview controllers have gains that go to zero at DC reflecting the fact that they use integral control. The baseline controllers have significantly higher peak frequency responses.

blades individually. The CP feedback-only controller uses only a measurement of generator speed error that provides no information as to how the blades might be pitched individually.
8.5.3 Load Mitigation Simulations

8.5.3.1 Blade Load Mitigation

The dynamics of the tower and its loads were not considered in the controller designs, so the results for blade loads are examined first. The cyclic loads encountered over the course of all 31 wind inflows are presented in Fig. 8.12 for each control and preview configuration. The percentage of all load cycles of a given size is plotted along the y-axis and the size of the load cycle is plotted along the x-axis. Curves to the left and lower represent better performance, consistent with smaller loads and less cycles. The CP controller (blue) has the largest percentage of loads over 400 kN-m. Adding individual pitch feedback from blade-root bending moments (IP, green) decreases average blade loads by 20% (c.f. Table 8.2) and adding preview (LTI-OFBK, magenta) provides about another 30% load reduction compared with IP. These three cases are plotted for reference in both the plot displaying results using ideal measurements (Fig. 8.12a) and the plot displaying results using LIDAR (Fig. 8.12b). Further, presented in both cases is the load curve obtained using three stationary point measurements (without LIDAR distortion) that are then interpolated to wind speeds at blade locations for use by the LTI-OFBK controller (as in previous work[45]); in this case, the load performance of the LTI-OFBK preview controller degrades (dashed magenta) to become worse than that of IP without preview.

This degradation occurs, because the error in preview wind speeds increases significantly when interpolating from stationary point measurements so that the maximum error is about 6 m/s. However, this performance degradation does not occur if the LTI-OFBK preview controller uses LIDAR distorted measurements (‘*.-.’ magenta, Fig. 8.12b), even though the maximum measurement error of the LIDAR preview measurements (about 5 m/s) is on the same order of magnitude as the interpolated preview measurements. So, it appears preview control performance has a greater dependence on the RMS error of preview measurements. Preview control may be insensitive to specific features characteristic of LIDAR distortion, or which may be prevalent to the wind fields used here, but a characterization of such features is not a topic of this study. What is apparent at
Figure 8.12: Cyclic loads observed over all 31 inflows used for simulation: in both figures, results obtained using CP, IP, and LTI-OFBK are presented. The LTI-OFBK results shown in both plots are obtained using ideal measurements (solid magenta) and measurements interpolated from wind speeds at stationary locations (dashed magenta). (a) Using ideal-rotating measurements, MPC (red) proves to have significantly smaller peak loads (OUTLIERS) while being comparable to LTI-OFBK preview control (magenta), but adding an over-speed constraint (gold) degrades performance slightly. (b) Using LIDAR measurements degrades performance of all controllers, but MPC still has significantly lower peak loads, and adding an estimate of LIDAR error based on turbine response to the preview measurements degrades MPC performance further; in this case, preview controller performance is still significantly better than that obtained using interpolated wind speeds (dashed magenta) even though interpolated measurement errors are on the same order of magnitude as those produced by LIDAR.

This point, is that the angle at which the LIDAR measurements are made can be quite large (nearly 23 degrees as is used here) and much of the advantages of preview control are retained.

These results are made more precise in Table 8.2. The improvement in load performance of the LTI-OFBK controller relative to IP feedback-only is about the same independent of the use of LIDAR, and whether considering the maximum cyclic load or the average of damage-equivalent-loads (DEL) [54] observed over all 31 inflows. In the former metric, maximum load decreases by 8% and in the latter metric load decreases by at least 22%, independent of whether the measurements are ideal or LIDAR distorted.

Also evident in Table 8.2 is the fact that the main advantage in using MPC at a single oper-
Table 8.2: Load performance metrics: “+” denotes absolute maximum in kN-m; “*” denotes average over each blade and over each of the 31 inflows simulated. Percent improvement (load reduction) is shown for each controller; the IP performance is rated relative to CP; all other controllers are rated relative to IP.

8.5.3.2 Other Metrics

Finally, in order to be complete, results across all 31 inflows for speed, power, and tower loads are provided in the box plots of Figures 8.14 and 8.15. The LTI-OFBK design has more emphasis on speed regulation than do the MPC designs and this is evident in the size of the box (the extent of which depicts +/- 25%) in the plots of both the speed (Fig. 8.14a) and power (Fig. 8.14b) results. Also evident is the effectiveness of the over-speed limit in keeping the outliers below the 3% limit (the gold box in the “IDEAL” section has a 95 percentile dot below the black solid line...
Figure 8.13: Speed regulation for a specific inflow case (AR4_S11). The LTI-OFBK controller (magenta) has tight speed regulation in order to prevent over-speed faults (which can occur anyway due to integral windup); the MPC controller (red) is tuned to provide less speed regulation, but does not suffer from integral wind-up; the over-speed faults of the MPC controller are mitigated with the use of a 3% over-speed constraint (gold).

representing the 3% limit in Figures 8.14a and 8.14b). MPC also shows slightly improved (lower) tower loads relative to the baseline feedback-only designs in that the quartiles have a smaller spread (as long as the observer estimate of LIDAR error is not used). In terms of extreme/outlier tower loads, MPC appears to have done no harm, but there was no consideration of these dynamics in the design of any of the controllers.

8.5.4 Discussion of Load Mitigation Results

MPC has been demonstrated to be an effective method for load mitigation in highly variable wind conditions during which pitch saturation is likely to occur. It explicitly satisfies constraints on pitch magnitude and rate while avoiding integral windup. In the event that wind speeds mo-
Figure 8.14: Box plots for speed regulation and average power using data from all 31 inflows: boxes have lines at the lower quartile, median, and upper quartile values. A solid red line shows the mean (the median is plotted with a black dotted line). The whiskers extend to the 10th and 90th percentiles. Large black dots indicate the extent of the 5th and 95th percentiles.

Figure 8.15: Box plots for tower loads using data from all 31 inflows.

mentarily drop while the speed set point is held constant, MPC with preview can recover without the excessive loads or the over-speed faults that may typically be exhibited by LTI preview control approaches. This is due in part to the fact that with MPC, nominal speed regulation can be relaxed while limiting over-speed faults with application of a hard constraint on maximum speed. However, it was also demonstrated that the application of hard constraints will typically degrade load mitigation performance.
Further, we have found that preview control is robust to the distortions typical of LIDAR measurement systems in turbulent conditions, in the presence of shear, and even at fairly large measurement angles. Measuring wind speeds with a LIDAR directional bias of 23 deg degrades controller performance by only 8% in terms of average DEL. Relative to an IP feedback-only controller, preview using LIDAR still provides an improvement in damage equivalent load of 32%, 25%, and 29% when using MPC without an over-speed constraint, MPC with an over-speed constraint, and a LTI preview controller, respectively.
Chapter 9

Unresolved Issues and Directions for Further Research

This work has endeavored to design an MPC system that utilizes preview measurements of wind speed and operates the turbine from start-up to cut-out. In the course of accomplishing this goal, it was necessary to extend offset-free MPC techniques for arbitrary disturbances (assuming they are generated by linear time-invariant dynamics) and account for the use of exogenous measurements. The resulting system uses preview of wind speeds approaching the turbine for both set-point scheduling and feed-forward control actuation. Simulations based on the dynamic CART3 indicate that the approach is very stable. Blade load mitigation was the focus for this thesis, but since the preview scheduled MPC system is inherently multi-input multi-output, it can address additional objectives without changing its basic formulation. This is an important feature in the even that the additional dynamics of a floating platform are taken into account.

However, at this point, the most significant issue left unaddressed by this work is a proof of stability. As the system is non-linear and subject to unknown disturbances, this is somewhat problematic. Ultimately, what is required is a global Lyapunov function with the underlying assumption that the structure will not fail. Less stringent, is the task of finding a local Lyapunov function or invariant sets given reasonable bounds on the size of disturbances with the assumption that the preview error is not significant. In the case of invariant sets, there is an existing body of literature for their application in MPC [1, 56, 49, 18, 36]. Another possibility is the application of LPV techniques [19]. A reasonable expectation is that both of these should be pursued as next steps. The former would take into consideration the complete system operating under reasonable
bounds on wind speed, and the latter would allow the design of control gains such that the LPV system remains stable for arbitrary combinations of pitch and torque saturation.

The second most significant issue is the multiplicity required for the disturbance models employed and the necessity of some sort optimal trajectory generation. This is an area that may prove fruitful in the short term. It is not necessary for the observer to estimate the disturbance content in the MPC control perturbations as long as the MPC controller does not use exogenous inputs in forming its command. This may seem limiting in that preview actuation is highly desirable, but the remedy is to estimate the wind disturbance state so that it optimally fits the wind speeds over the preview horizon. This would make the disturbance state “predictive” and the disturbance state would be part of the model used by the MPC algorithm. This is in contrast to the present configuration where the extra multiplicities required for the observer to use wind measurements are not part of the MPC model. It is in the configuration yet undeveloped, that the approach can also utilize spectral models of wind energy. By tuning the time constants of the disturbance estimation, this new configuration can address optimal trajectory generation to some extent. In addition, feedback from the turbine itself can “correct” the disturbance state estimate so that when preview measurements become unusable, some form of disturbance tracking control [5, 69] is retained. Also in this scenario, it may be possible to avoid using the feed-forward controls as input to the observer, since in this case the disturbance gain is serving as a form of preview control.

For a system with significant stochastic inputs that are large with respect to control authority, stability is inevitably going to be an ongoing issue. However, the use of invariant sets may also aid in sizing the structure in a way that takes into account controls and extreme loads. And in the short term, there are changes to the configuration of the disturbance models and the scheduling function that are likely to improve performance and robustness, and may have additional unforeseen benefits as well.
Bibliography


[13] Carlo Bottasso, Alessandro Croce, Barbara Savini, Walter Sirchi, and Lorenzo Trainelli. Aero-


Appendix A

Discrete-Time LQR Optimal Control

This appendix derives the formula for the discrete-time infinite-horizon linear-quadratic state-feedback control gain. This is obtained by finding the finite horizon solution and letting the horizon go to infinity. This optimization is done for the linear time-invariant system

\[ x(k + 1) = Ax(k) + B_u u(k), \]
\[ z(k) = C_z x(k) + D_{zu} u(k). \]

with \( x(1) \) given, where \( u(k) \) is the control to be determined, and where the vector \( w(k) \) of exogenous inputs is not present. The controls (and system response) are optimized to minimize a quadratic cost

\[
J_{[1,N]}(u,x) = \frac{1}{2} x(N + 1)^T \Pi_f x(N + 1) + \frac{1}{2} \sum_{k=1}^{N} z(k)^T Q_z z(k),
\]

\[
= x(N + 1)^T \Pi_f x(N + 1) + \sum_{k=1}^{N} \frac{1}{2} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} C_z^T Q_z C_z & C_z^T Q_z D_{zu} \\ D_{zu}^T Q_z C_z & D_{zu}^T Q_z D_{zu} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}
\]

\[
= \begin{bmatrix} Q_x & S_{zu} \\ S_{zu}^T & R_u \end{bmatrix}
\]

where it is assumed that

\[ Q_z > 0, \]
and the matrix $D_{zu}$ is injective. These assumptions imply that

$$R_u > 0.$$  \hfill (A.4)

In addition, it is assumed that $\{A, B_u\}$ is stabilizable and $\{C_z, A\}$ is detectable.

It simplifies the formulae greatly if the cross terms between $u(k)$ and $x(k)$ are removed using a Schur transformation

$$J_{[1,N]}(u, x) = \frac{1}{2} x(N + 1)^T \Pi_f x(N + 1),$$

$$J_{[1,N]}(u, x) = \frac{1}{2} x(N + 1)^T \Pi_f x(N + 1)$$

$$+ \sum_{k=1}^{N} \frac{1}{2} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^{T} \begin{bmatrix} Q_x & S_{xu} \\ S_{xu}^T & R_u \end{bmatrix} \begin{bmatrix} I \\ -R_u^{-1} S_{xu}^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R_u \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix},$$  \hfill (A.5a)

$$= \frac{1}{2} x(N + 1)^T \Pi_f x(N + 1)$$

$$+ \sum_{k=1}^{N} \frac{1}{2} \begin{bmatrix} x(k) \\ u(k) + R_u^{-1} S_{xu} x(k) \end{bmatrix}^{T} \begin{bmatrix} Q_x - S_{xu} R_u^{-1} S_{xu}^T & 0 \\ 0 & R_u \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) + R_u^{-1} S_{xu} x(k) \end{bmatrix}. \hfill (A.5c)

Since this transformation was simply a matter of inserting the equivalent of the identity matrix in order to obtain the diagonal form, the cost has not changed. So if $\tilde{u}_s(k)$ is a minimizer for

$$J_{[1,N]}(\tilde{u}, x) = \frac{1}{2} x(N + 1)^T \Pi_f x(N + 1)$$

$$+ \sum_{k=1}^{N} \frac{1}{2} \begin{bmatrix} x(k) \\ \tilde{u}(k) \end{bmatrix}^{T} \begin{bmatrix} \tilde{Q}_x & 0 \\ 0 & R_u \end{bmatrix} \begin{bmatrix} x(k) \\ \tilde{u}(k) \end{bmatrix},$$  \hfill (A.6)

where

$$\tilde{Q}_x = Q_x - S_{xu} R_u^{-1} S_{xu}^T,$$  \hfill (A.7a)

$$= C_z^T Q_z C_z - C_z^T \left( I - Q_z D_{zu} (D_{zu}^T D_{zu} Q_z D_{zu})^{-1} D_{zu}^T Q_z \right) C_z.$$  \hfill (A.7b)
and
\[ x(k + 1) = Ax(k) + Bu(k), \quad (A.8a) \]
\[ = (A - Bu^1S_{zu}^T)x(k) + Bu(u(k) + R_u^{-1}S_{zu}x(k)), \quad (A.8b) \]
\[ \hat{A} \]
\[ \hat{\alpha}(k) \]
\[ z(k) = C_z x(k) + D_{zu} u(k), \quad (A.8c) \]
\[ = (C_z - D_{zu} R_u^{-1} S_{zu}^T)x(k) + D_{zu} \hat{\alpha}(k), \quad (A.8d) \]
then
\[ u_*(k) = \hat{\alpha}_*(k) - R_u^{-1} S_{zu}^T x(k), \quad (A.9) \]
minimizes the original cost (A.2) with the constraint (A.1). Also, with a little work it is possible to show that
\[ \sigma \left( Q_z D_{zu} (D_{zu}^T Q_z D_{zu})^{-1} D_{zu}^T Q_z \right) \subset \{0, 1\}, \quad (A.10) \]
so that \( \hat{Q}_z \geq 0 \), and the new cost (A.6) has a well defined minimum.

So, the task is now to find the optimal control for the cost (A.6) with the dynamics constraint (A.8). Whatever control is used on the horizon \([1, N - 1]\), the control used at \( k = N \) can only affect the cost of the final stage. So, the optimal choice for the last control is the minimizer for
\[ J_{[N,N]}(\hat{\alpha}, x) = \frac{1}{2} x(N + 1)^T \Pi_j x(N + 1) + x(N)^T \hat{Q}_x x(N) + \frac{1}{2} \hat{\alpha}(N)^T R_u \hat{\alpha}(N), \quad (A.11a) \]
\[ = \frac{1}{2} \left( \hat{A} x(N) + B_u \hat{\alpha}(N) \right)^T \Pi_j \left( \hat{A} x(N) + B_u \hat{\alpha}(N) \right) + \frac{1}{2} u(N)^T R_u u(N) \]
\[ + \frac{1}{2} x(N)^T \hat{Q}_x x(N), \quad (A.11b) \]
\[ = x(N)^T \hat{A}^T \Pi_j B_u u(N) + \frac{1}{2} u(N)^T \left( R_u + B_u \hat{\Pi}_j B_u \right) u(N) \quad (A.11c) \]
\[ + \frac{1}{2} x(N)^T \left( \hat{Q}_x + \hat{A}^T \Pi_j \hat{A} \right) x(N). \quad (A.11d) \]
Taking the derivative of this expression
\[ \frac{\partial}{\partial \hat{\alpha}(N)} J_{[N,N]}(\hat{\alpha}, x) = B_u^T \Pi_j \hat{A} x(N) + R_u u(N), \quad (A.12) \]
setting the result to zero, and solving for \( \tilde{u}_s(N) \) shows that

\[
\tilde{u}_s(N) = - \left( R_u + B_u \tilde{\Pi}_f B_u \right)^{-1} B_u^T \tilde{\Pi} \tilde{A} x(N).
\] (A.13)

This can be substituted back into the cost for the last stage to obtain

\[
J_{[N,N]}(\tilde{u}, x) = - \frac{1}{2} x(N)^T \tilde{A}^T \tilde{\Pi}_f B_u \left( R_u + B_u \tilde{\Pi}_f B_u \right)^{-1} B_u^T \tilde{\Pi}_f \tilde{A} x(N)
\]

A.14a

\[+ \frac{1}{2} x(N)^T \left( \tilde{Q}_x + \tilde{A}^T \tilde{\Pi}_f \tilde{A} \right) x(N), \]

A.14b

\[= \frac{1}{2} x(N)^T \left( \tilde{Q}_x + \tilde{A}^T \tilde{\Pi}_f \tilde{A} - \tilde{A}^T \tilde{\Pi}_f B_u \left( R_u + B_u \tilde{\Pi}_f B_u \right)^{-1} B_u^T \tilde{\Pi}_f \tilde{A} \right) x(N). \] (A.14c)

So, it is now possible to back propagate this result and write the original cost as

\[
J_{[1,N]}(\tilde{u}, x) = \frac{1}{2} x(k)^T \Pi_k x(k) + \sum_{t=1}^{k-1} \frac{1}{2} \begin{bmatrix} x(t)^T & 0 \\ \tilde{u}(t) & R_u \end{bmatrix} \begin{bmatrix} \tilde{Q}_x & 0 \\ 0 & R_u \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{u}(t) \end{bmatrix},
\] (A.15a)

\[= \frac{1}{2} x(1)^T \Pi_1 x(1). \] (A.15b)

where \( \Pi_k \) is generated using the Riccati recursion

\[
\Pi_k = \tilde{Q}_x + \tilde{A}^T \Pi_{k+1} \tilde{A} - \tilde{A}^T \Pi_{k+1} B_u \left( R_u + B_u \tilde{\Pi}_{k+1} B_u \right)^{-1} B_u^T \Pi_{k+1} \tilde{A},
\] (A.16)

with initial condition \( \Pi_{N+1} = \Pi_f \). Further, the expression in eq. (A.13) for the control remains valid at any stage, i.e.

\[
\bar{u}_s(k) = - \left( R_u + B_u \tilde{\Pi}_{k+1} B_u \right)^{-1} B_u^T \Pi_{k+1} \tilde{A} x(k).
\] (A.17)

Now, denote by \( \Pi_{[1,N]} \) the \( \Pi_1 \) that determines the optimal cost over the horizon length \( N \) for the objective \( J_{[1,N]}(\tilde{u}, x) \). If

\[
\Pi = \lim_{N \to \infty} \Pi_{[1,N]},
\] (A.18)

exists, then that limit must satisfy the discrete-time algebraic Riccati equation (DARE)

\[
\Pi = \tilde{Q}_x + \tilde{A}^T \Pi \tilde{A} - \tilde{A}^T \Pi B_u \left( R_u + B_u \tilde{\Pi} B_u \right)^{-1} B_u^T \Pi \tilde{A},
\] (A.19)

and the cost for the infinite horizon objective is finite. We already have that for any finite horizon

\[
J_{[1,N]} = \frac{1}{2} x(1)^T \Pi_1 x(1) \geq 0,
\]

so that the limit is bounded below. Also, denote the optimal control
for a horizon of $K$ by $u_{[1,K]}$, then since
\[
J_{[1,N+1]}(\bar{u}, x) = J_{[1,N]}(\bar{u}, x) + \frac{1}{2} \left( x(N+1)^T (Q_x - \Pi_f) x(N+1) + u(N+1)^T R_u u(N+1) + x(N+2)^T \Pi_f x(N+2) \right),
\] (A.20)
it follows that
\[
J_{[1,N+1]}(\bar{u}_{[1,N+1]}, x) = x(1)^T \Pi_{[1,N+1]} x(1) \geq x(1)^T \Pi_{[1,N]} x(1) = J_{[1,N]}(\bar{u}_{[1,N]}, x),
\] (A.21)
as long as $Q_x \geq \Pi_f$ (e.g., use $\Pi_f = 0$). That is, using the control optimal for a horizon of $N + 1$, accumulates at least as much cost in the first $N$ samples as does the control that is optimal for the horizon of $N$ and therefore
\[
\Pi_{[1,N+k]} \geq \Pi_{[1,N]}, \forall k \geq 0.
\] (A.22)

Now if $\{\bar{A}, B_u\}$ is stabilizable, then $\Pi_{[1,N]}$ is non-decreasing and bounded above (because a stabilizing feedback gives finite cost over any horizon) so that it must have a well-defined finite limit.

At this point, the infinite horizon problem is essentially solved, except for the technical condition that $\{\bar{C}_z, \bar{A}\}$ cannot have any unobservable modes on the unit circle [25]. This holds if and only if
\[
\begin{bmatrix}
\bar{A} - I e^{j\omega} \\
\bar{C}_z
\end{bmatrix} x \neq 0, \forall x \neq 0.
\] (A.23)

But, this is equivalent to
\[
\begin{bmatrix}
A - I e^{j\omega} & B_u \\
C_z & D_{zu}
\end{bmatrix} \begin{bmatrix}
I \\
-R_u^{-1} S_{xu}
\end{bmatrix} x \neq 0, \forall x \neq 0.
\] (A.24)

So it is sufficient to require that
\[
N \begin{bmatrix}
A - I e^{j\omega} & B_u \\
C_z & D_{zu}
\end{bmatrix} = 0.
\] (A.25)

With this assumption in hand, it is possible to show that $\Pi$ computed as in eq. A.18 is positive semi-definite by using the DARE (A.19) re-written as
\[
\Pi = \bar{Q}_x + \bar{R}_x^T \bar{R}_x + (\bar{A} + B_u \bar{R}_x)^T \Pi (\bar{A} + B_u \bar{R}_x),
\] (A.26)
where

\[ \hat{K}_x = - \left( R_u + B_u \hat{\Pi} B_u \right)^{-1} B_u^T \hat{\Pi} \hat{A}. \] (A.27)

Now, if \{\hat{C}_x, \hat{A}\} is detectable, then the converse result that any positive semi-definite solution to the DARE gives a stabilizing control also holds.

Finally, the optimal control for the original problem is

\[
\begin{aligned}
    u_*(k) &= - (R_u + B_u \Pi B_u)^{-1} B_u^T \Pi \hat{A} x(k) - R_u^{-1} S_{xu}^T x(k), \quad (A.28a) \\
    &= - (R_u + B_u \Pi B_u)^{-1} B_u^T \Pi (A - B_u R_u^{-1} S_{xu}^T) x(k) - R_u^{-1} S_{xu}^T x(k), \quad (A.28b) \\
    &= - (R_u + B_u \Pi B_u)^{-1} B_u^T \Pi A x(k) \\
    &\quad + \left( (R_u + B_u \Pi B_u)^{-1} B_u^T \Pi B_u R_u^{-1} S_{xu}^T - R_u^{-1} S_{xu}^T \right) x(k), \quad (A.28c) \\
    &= - (R_u + B_u \Pi B_u)^{-1} B_u^T \Pi A x(k) \\
    &\quad + \left( (R_u + B_u \Pi B_u)^{-1} B_u^T \Pi B_u - I \right) R_u^{-1} S_{xu}^T x(k), \quad (A.28d) \\
    &= - (R_u + B_u \Pi B_u)^{-1} B_u^T \Pi A x(k) \\
    &\quad + (R_u + B_u \Pi B_u)^{-1} (B_u^T \Pi B_u - R_u - B_u \Pi B_u) R_u^{-1} S_{xu}^T x(k), \quad (A.28e) \\
    &= - (R_u + B_u \Pi B_u)^{-1} \left( B_u^T \Pi A x(k) + S_{xu}^T \right) x(k). \quad (A.28f)
\end{aligned}
\]
Appendix B

The Riccati Recursion for Constrained LQR

As discussed in Chapter 3 it is necessary to solve the KKT systems

\[
\begin{bmatrix}
H - P^T D^{-1} \Lambda P & C_{eq}^T \\
C_{eq} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta z \\
\Delta \nu
\end{bmatrix} =
\begin{bmatrix}
- \begin{bmatrix} r_z - P^T D^{-1} r_\lambda \end{bmatrix} \\
- \begin{bmatrix} r_\nu \end{bmatrix}
\end{bmatrix} \triangleq \begin{bmatrix} -\tilde{r}_z \\
-\tilde{r}_\nu
\end{bmatrix},
\] (B.1)

for the quadratic cost function, and

\[
\begin{bmatrix}
-P^T D^{-1} \Lambda P & C_{eq}^T \\
C_{eq} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta z \\
\Delta \nu
\end{bmatrix} =
\begin{bmatrix}
- \begin{bmatrix} r_z - P^T D^{-1} r_\lambda - P^T D^{-1} \Lambda \Delta s \end{bmatrix} \\
- \begin{bmatrix} r_\nu \end{bmatrix}
\end{bmatrix} \triangleq \begin{bmatrix} -\tilde{r}_z \\
-\tilde{r}_\nu
\end{bmatrix},
\] (B.2)

for the $\ell_\infty$ cost function. The (1,1) block for the $\ell_\infty$ case, is

\[
\begin{bmatrix}
-D_{ctlu}(1)^T & -D_{ctlu}(1)^T & 0 & 0 & \ldots & 0 \\
0 & 0 & C_c(2)^T & -C_c(2)^T & \ldots & 0 \\
0 & 0 & -D_{cu}(2)^T & D_{cu}(2)^T & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & C_{f}^T
\end{bmatrix} - P^T D^{-1} \Lambda P =
\begin{bmatrix}
R_{\ell_\infty}(1) & 0 & 0 & \ldots & 0 \\
0 & Q_{\ell_\infty}(2) & S_{\ell_\infty}(2) & \ldots & 0 \\
0 & S_{\ell_\infty}(2)^T & R_{\ell_\infty}(2) & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \Pi_{f \infty}
\end{bmatrix}
\] (B.3a)

(B.3b)
where

\[ R_{\ell_\infty}(1) = -D_{\text{ctlu}}(1)^T \left( \Lambda_p(1) D_p(1)^{-1} + \Lambda_n(1) D_n(1)^{-1} \right) D_{\text{ctlu}}(1), \]  
\[ Q_{\ell_\infty}(k) = -C_c(k)^T \left( \Lambda_p(k) D_p(k)^{-1} + \Lambda_n(k) D_n(k)^{-1} \right) C_c(k), \]  
\[ S_{\ell_\infty}(k) = -C_c(k)^T \left( \Lambda_p(k) D_p(k)^{-1} + \Lambda_n(k) D_n(k)^{-1} \right) D_{\text{cu}}(k), \]  
\[ R_{\ell_\infty}(k) = -D_{\text{cu}}(k)^T \left( \Lambda_p(k) D(k)^{-1} + \Lambda_n(k) D(k)^{-1} \right) D_{\text{cu}}(k), \]  
\[ \Pi_{f_{\infty}} = -C_f^T (\Lambda_f D_f^{-1}) C_f, \]

and \( \Lambda \) and \( D \) have been partitioned as

\[ \Lambda = \begin{bmatrix} \Lambda_p(1) & 0 & \cdots & 0 \\ 0 & \Lambda_n(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda_f \end{bmatrix}, \quad D = \begin{bmatrix} D_p(1) & 0 & \cdots & 0 \\ 0 & D_n(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_f \end{bmatrix}. \]

By definition \( \{\Lambda, D\} \) are diagonal, and it is assumed that \( \Lambda > 0, D < 0 \). Also, it is assumed and that \( D_{\text{ctlu}} \) is injective (e.g., skinny and full column rank) and is a sub-block of \( D_{\text{cu}} \) so that \( R_{\ell_\infty}(k) > 0 \). That \( Q_{\ell_\infty} \geq 0 \) also follows. The quadratic case has a similar form where

\[ R_q(k) = R_u(k) + R_{\ell_\infty}(1) > 0, \]  
\[ Q_q(k) = Q_x(k) + Q_{\ell_\infty}(k) \geq 0, \]  
\[ S_q(k) = S_{rx}(k) + S_{\ell_\infty}(k), \]  
\[ \Pi_{fq} = \Pi_f + \Pi_{f_{\infty}}, \]

It is now possible to show that \( \{\Delta_z, \Delta_\nu\} \) can be found via a Riccati recursion. Consider a horizon of two controls. For either the quadratic or \( \ell_\infty \) cases, it is necessary to solve a system with
the form

\[
\begin{bmatrix}
R_u(1) & 0 & 0 & 0 & -B_u(1)^T & 0 & \Delta_u(1) & -\tilde{\tau}_u(1) \\
0 & Q_x(2) & S_{xu}(2) & 0 & I & -A(2)^T & \Delta_x(2) & -\tilde{\tau}_x(2) \\
0 & S_{xu}(2)^T & R_u(2) & 0 & 0 & -B_u(2)^T & \Delta_u(2) & -\tilde{\tau}_u(2) \\
0 & 0 & 0 & \Pi_f & 0 & I & \Delta_x(3) & -\tilde{\tau}_x(3) \\
-B_u(1) & I & 0 & 0 & 0 & 0 & \Delta_{\nu}(1) & -r_{\nu}(1) \\
0 & -A(2) & -B_u(2) & I & 0 & 0 & \Delta_{\nu}(2) & -r_{\nu}(2)
\end{bmatrix} = \begin{bmatrix}
\Delta_u(1) \\
\Delta_x(2) \\
\Delta_u(2) \\
\Delta_x(3) \\
\Delta_{\nu}(1) \\
\Delta_{\nu}(2)
\end{bmatrix}.
\] (B.7)

The goal is now to uncouple system so that the first stage variables \{\Delta_u(1), \Delta_x(2), \Delta_{\nu}(1)\} can be obtained independent of \{\Delta_u(2), \Delta_x(3), \Delta_{\nu}(2)\}. As in Appendix A it is possible to apply a Schur transformation to obtain

\[
\begin{bmatrix}
R_u(1) & 0 & 0 & 0 & -B_u(1)^T & 0 & \Delta_u(1) & -\tilde{\tau}_u(1) \\
0 & \tilde{Q}_x(2) & 0 & 0 & I & -\tilde{A}(2)^T & \tilde{\Delta}_u(2) & -\tilde{\tau}_u(2) \\
0 & 0 & R_u(2) & 0 & 0 & -B_u(2)^T & \tilde{\Delta}_u(2) & -\tilde{\tau}_u(2) \\
0 & 0 & 0 & \Pi_f & 0 & I & \tilde{\Delta}_x(3) & -\tilde{\tau}_x(3) \\
-B_u(1) & I & 0 & 0 & 0 & 0 & \Delta_{\nu}(1) & -r_{\nu}(1) \\
0 & -\tilde{A}(2) & -B_u(2) & I & 0 & 0 & \Delta_{\nu}(2) & -r_{\nu}(2)
\end{bmatrix} = \begin{bmatrix}
\Delta_u(1) \\
\tilde{\Delta}_u(2) \\
\tilde{\Delta}_u(2) \\
\tilde{\Delta}_x(3) \\
\Delta_{\nu}(1) \\
\Delta_{\nu}(2)
\end{bmatrix}.
\] (B.8)

where

\begin{align}
\tilde{\tau}_x(2) &= \tilde{\tau}_x(2) - S_{xu}(2) R_u(2)^{-1} \tilde{\tau}_u(2), \\
\tilde{Q}_x(2) &= \tilde{Q}_x(2) - S_{xu}(2) R_u(2)^{-1} S_{xu}(2)^T, \\
\tilde{A}(2) &= A(2) - B_u(2) R_u(2)^{-1} S_{xu}(2)^T, \\
\tilde{\Delta}_u(2) &= \Delta_u(2) + R_u(2)^{-1} S_{xu}(2)^T \Delta_x(2).
\end{align} (B.9)

Next, the order of the equations is permuted and the order of the variables is permuted so that the
time indexed quantities appear in blocks

\[
\begin{bmatrix}
0 & -B_u(1) & I & 0 & 0 & 0 & \Delta_\nu(1) & -r_\nu(1) \\
-B_u(1)^T & R_u(1) & 0 & 0 & 0 & 0 & \Delta_u(1) & -\bar{r}_u(1) \\
I & 0 & \bar{Q}_x(2) & -\bar{A}(2)^T & 0 & 0 & \Delta_x(2) & -\bar{r}_x(2) \\
0 & 0 & -\bar{A}(2) & 0 & -B_u(2) & I & \Delta_\nu(2) & -r_\nu(2) \\
0 & 0 & 0 & -B_u(2)^T & R_u(2) & 0 & \bar{\Delta}_u(2) & -\bar{r}_u(2) \\
0 & 0 & 0 & I & 0 & \Pi_f & \Delta_x(3) & -\bar{r}_x(3)
\end{bmatrix} = (B.10)
\]

At this point, the lower four rows are manipulated with elementary row and column operations in such a way that there is no mixing between time indexes, and so they are suppressed for the matrix sub-blocks in the following. Through elementary column operations, the lower two elements of the last column are cancelled to obtain

\[
\begin{bmatrix}
0 & -B_u & I & 0 & 0 & 0 \\
-B_u^T & R_u & 0 & 0 & 0 & 0 \\
I & 0 & \bar{Q}_x & -\bar{A}^T & 0 & \bar{A}^T\Pi_f \\
0 & 0 & -\bar{A} & -B_uR_u^{-1}B_u^T & 0 & I + B_uR_u^{-1}B_u^T\Pi_f \\
0 & 0 & 0 & -B_u^T & R_u & 0 \\
0 & 0 & 0 & I & 0 & 0
\end{bmatrix} \times
\begin{bmatrix}
\Delta_\nu(1) \\
\Delta_u(1) \\
\Delta_x(2) \\
\Delta_\nu(2) + \Pi_f\Delta_x(3) \\
\bar{\Delta}_u(2) + R_u^{-1}B_u^T\Pi_f\Delta_x(3) \\
\Delta_x(3)
\end{bmatrix} = \begin{bmatrix}
-r_\nu(1) \\
-\bar{r}_u(1) \\
-\bar{r}_x(2) \\
-r_\nu(2) - B_uR_u^{-1}\bar{r}_u(2) \\
-\bar{r}_u(2) \\
-\bar{r}_x(3)
\end{bmatrix}. \quad (B.11)
\]

Then, since the matrix inversion lemma shows that

\[
(R_u + B_u^T\Pi_fB_u)^{-1} = R_u^{-1} - R_u^{-1}B_u^T(I + B_uR_u^{-1}B_u^T\Pi_f)^{-1}B_u^T\Pi_fR_u^{-1}, \quad (B.12a)
\]
the fourth row can be scaled and added to the third row cancelling the element in the third row of
the last column

\[
\begin{bmatrix}
0 & -B_u & I & 0 & 0 & 0 \\
-B_u^T & R_u & 0 & 0 & 0 & 0 \\
I & 0 & \tilde{Q}_x & -\tilde{A} & 0 & 0 \\
0 & 0 & -\tilde{A} & -B_u R_u^{-1} B_u^T & 0 & I + B_u R_u^{-1} B_u^T \Pi_f \\
0 & 0 & 0 & -B_u^T & R_u & 0 \\
0 & 0 & 0 & I & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta_\nu(1) \\
\Delta_u(1) \\
\Delta_x(2) \\
\Delta_\nu(2) + \Pi_f \Delta_x(3) \\
\Delta_u(2) + R_u^{-1} B_u^T \Pi_f \Delta_x(3) \\
\Delta_x(3)
\end{bmatrix}
\]

where

\[
\begin{align*}
\tilde{Q}_x &= \tilde{Q}_x + \tilde{A}^T \Pi_f (I + B_u R_u^{-1} B_u^T \Pi_f)^{-1} \tilde{A}, \\
\tilde{A}^T &= \tilde{A}^T (I - \Pi_f (I + B_u R_u^{-1} B_u^T \Pi_f)^{-1} B_u R_u^{-1} B_u^T).
\end{align*}
\]

Then all elements in the fourth column above the last row are removed by adding the scaled versions
of the last row to the rows above it

\[
\begin{pmatrix}
0 & -B_u & I & 0 & 0 & 0 \\
-B_u^T & R_u & 0 & 0 & 0 & 0 \\
I & 0 & \tilde{Q}_x & 0 & 0 & 0 \\
0 & 0 & -\tilde{\mathbf{A}} & 0 & 0 & I + B_u R_u^{-1} B_u^T \Pi_f \\
0 & 0 & 0 & 0 & R_u & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\Delta \nu(1) \\
\Delta u(1) \\
\Delta x(2) \\
\Delta \nu(2) + \Pi_f \Delta x(3) \\
\Delta u(2) + R_u^{-1} B_u^T \Pi_f \Delta x(3) \\
\Delta x(3) \\
\end{pmatrix}
\]

\[
=r_u(1) \\
-\tilde{r}_u(1) \\
-r(2) \\
r_u(2) - B_u R_u^{-1} \tilde{r}_u(2) - B_u R_u^{-1} B_u^T \tilde{\mathbf{r}}_x(3) \\
-\tilde{r}_u(2) - B_u^T \tilde{\mathbf{r}}_x(3) \\
-\tilde{\mathbf{r}}_x(3)
\]

(B.15)

where

\[
\begin{align*}
r(2) &= \tilde{\mathbf{r}}_x(2) - \tilde{\mathbf{A}}^T (I + \Pi_f B_u R_u^{-1} B_u^T)^{-1} \Pi_f (r_u(2) + B_u R_u^{-1} \tilde{r}_u(2)) + \tilde{\mathbf{A}}^T (I + \Pi_f B_u R_u^{-1} B_u^T)^{-1} \tilde{\mathbf{r}}_x(3), \\
&= \tilde{\mathbf{r}}_x(2) - \tilde{\mathbf{A}}^T (I + \Pi_f B_u R_u^{-1} B_u^T)^{-1} (\Pi_f (r_u(2) + B_u R_u^{-1} \tilde{r}_u(2)) - \tilde{\mathbf{r}}_x(3)).
\end{align*}
\]

(B.16a)
Finally, $\hat{A}$ is removed from the fourth row with an elementary column operation so that

\[
\begin{bmatrix}
0 & -B_u & I & 0 & 0 & 0 \\
-B_u^T & R_u & 0 & 0 & 0 & 0 \\
I & 0 & \Pi_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta_\nu(1) \\
\Delta_u(1) \\
\Delta_x(2) \\
\Delta_x(3)
\end{bmatrix}
= \begin{bmatrix}
\Delta_\nu(1) \\
\Delta_u(1) \\
\Delta_x(2) \\
\Delta_x(3) - (I + B_u R_u^{-1} B_u^T \Pi_f)^{-1} \hat{A} \Delta_x(2)
\end{bmatrix},
\tag{B.17a}
\]

where

\[
\Pi_2 = \hat{Q}_x + \hat{A}^T \Pi_f (I + B_u R_u^{-1} B_u^T \Pi_f)^{-1} \hat{A}.
\tag{B.18}
\]

At this point, the first stage can be solved and $\Delta_x(2)$ forward propagated into the second. It should now be clear that the expression for $\Pi_2$ is the recursion sought. Algebraically, it can be
manipulated into its more familiar form as follows:

\[
\Pi_k = \hat{Q}_x(k) + \tilde{A}(k)^T \Pi_{k+1} (I + B_u(k) R_u(k)^{-1} B_u(k)^T \Pi_{k+1})^{-1} \tilde{A}(k),
\]

(B.19a)

\[
= \hat{Q}_x(k) + \tilde{A}(k)^T \Pi_{k+1} (I + B_u(k) (R_u(k) + B_u(k)^T \Pi_{k+1} B_u(k))^{-1} B_u(k)^T \Pi_{k+1}) \tilde{A}(k),
\]

(B.19b)

\[
- K(k) \frac{\hat{Q}_x(k) + \tilde{A}(k)^T \Pi_{k+1} \tilde{A}(k) - 2 \tilde{A}(k)^T \Pi_{k+1} B_u(k) (R_u(k) + B_u(k)^T \Pi_{k+1} B_u(k))^{-1} B_u(k)^T \Pi_{k+1} \tilde{A}(k)}{\tilde{A}(k)^T \Pi_{k+1} B_u(k) (R_u(k) + B_u(k)^T \Pi_{k+1} B_u(k))^{-1} B_u(k)^T \Pi_{k+1} \tilde{A}(k)}
\]

(B.19c)

\[
= \hat{Q}_x(k) + \tilde{A}(k)^T \Pi_{k+1} \tilde{A}(k) + \tilde{A}(k)^T \Pi_{k+1} B_u(k) K(k) + K(k)^T B_u(k)^T \Pi_{k+1} \tilde{A}(k)
\]

(B.19d)

\[
= \hat{Q}_x(k) + (\tilde{A}(k) + B_u(k) K(k))^T \Pi_{k+1} (\tilde{A}(k) + B_u(k) K(k)) + K(k)^T R_u(k) K(k). \quad \text{(B.19e)}
\]

The second line above is obtained with the matrix inversion lemma, the third line is obtained by subtracting and adding the last term, the indicated definition for \( K(k) \) leads to the fourth line, and then finally the last term in the fourth line is expanded and combined with the terms above it to obtain the final form.

This recursion can be back propagated indefinitely along with

\[
r(k) = \tilde{r}_x(k) - \tilde{A}(k)^T (I + \Pi_{k+1} B_u(k) R_u(k)^{-1} B_u(k)^T)^{-1} (\Pi_{k+1} (r_u(k) + B_u(k) R_u(k)^{-1} \tilde{r}_u(k)) - r(k + 1)).
\]

(B.20)

to arrive at the first stage. The first stage variables are then obtained by solving

\[
\begin{bmatrix}
0 & -B_u & I \\
-B_u^T & R_u & 0 \\
I & 0 & \Pi_2
\end{bmatrix}
\begin{bmatrix}
\Delta_u(1) \\
\Delta_u(1) \\
\Delta_x(2)
\end{bmatrix}
= \begin{bmatrix}
-r_u(1) \\
-\tilde{r}_u(1) \\
-r(2)
\end{bmatrix}.
\]

(B.21)

Using elementary row and column operations, this reduces to

\[
\begin{bmatrix}
0 & 0 & I \\
0 & R_u + B_u^T \Pi_2 B_u & 0 \\
I & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta_u(1) + \Pi_2 B_u \Delta_u(1) \\
\Delta_u(1) \\
\Delta_x(2)
\end{bmatrix}
= \begin{bmatrix}
-r_u(1) - B_u (R_u + B_u^T \Pi_2 B_u)^{-1} (\tilde{r}_u(1) + B_u^T (r(2) - \Pi_2 r_u(1))) \\
-\tilde{r}_u(1) - B_u^T (r(2) - \Pi_2 r_u(1)) \\
-r(2) + \Pi_2 r_u(1)
\end{bmatrix},
\]

(B.22)
So, the solution for an \( N \)-point horizon is obtained by back propagating equations \( \text{(B.19e)} \) and

\[
\Delta u(1) = - (R_u + B_u^T \Pi_2 B_u)^{-1} (\overline{\tau}_u(1) + B_u^T (r(2) - \Pi_2 r_\nu(1))),
\]

\[
\Delta_x(2) = - r_\nu(1) - B_u (R_u + B_u^T \Pi_2 B_u)^{-1} (\overline{\tau}_u(1) + B_u^T (r(2) - \Pi_2 r_\nu(1))),
\]

\[
\Delta_\nu(1) = - \Pi_2 B_u \Delta_u(1) - r_\nu(1),
\]

\[
\Delta_\nu(1) = - \Pi_2 \Delta_x(2) - r(2).
\]

The equations for forward propagation (solving for the second stage variables) can be written as

\[
\begin{bmatrix}
\Delta_\nu(2) + \Pi_f \Delta_x(3) \\
\Delta_u(2) + R_u^{-1} B_u^T \Pi_f \Delta_x(3) \\
\Delta_x(3) - (I + B_u R_u^{-1} B_u^T \Pi_f)^{-1} \overline{\Delta} \Delta_x(2)
\end{bmatrix} =
\begin{bmatrix}
-\overline{\tau}_x(3) \\
-R_u^{-1} \overline{\tau}_u(2) - R_u^{-1} B_u^T \overline{\tau}_x(3) \\
-(I + B_u R_u^{-1} B_u^T \Pi_f)^{-1} (r_\nu(2) + B_u R_u^{-1} \overline{\tau}_u(2) + B_u R_u^{-1} B_u^T \overline{\tau}_x(2))
\end{bmatrix},
\]

or equivalently

\[
\begin{bmatrix}
\Delta_\nu(2) \\
\Delta_u(2) \\
\Delta_x(3)
\end{bmatrix} =
\begin{bmatrix}
-\Pi_f \Delta_x(3) - \overline{\tau}_x(3) \\
-R_u^{-1} B_u^T \Pi_f \Delta_x(3) - R_u^{-1} \overline{\tau}_u(2) - R_u^{-1} B_u^T \overline{\tau}_x(3) \\
(I + B_u R_u^{-1} B_u^T \Pi_f)^{-1} \overline{\Delta} \Delta_x(2) - (I + B_u R_u^{-1} B_u^T \Pi_f)^{-1} (r_\nu(2) + B_u R_u^{-1} \overline{\tau}_u(2) + B_u R_u^{-1} B_u^T \overline{\tau}_x(2))
\end{bmatrix}.
\]

At the same time, the last row in eq. \( \text{(B.8)} \) shows that

\[
\Delta_x(3) = \overline{\Delta} \Delta_x(2) + B_u \Delta_u(2) - r_\nu(2).
\]

Substituting this into the third row of eq. \( \text{(B.25)} \) and solving for \( \Delta_u(2) \) gives

\[
\Delta_u(2) = - (R_u + B_u^T \Pi_f B_u)^{-1} B_u^T \Pi_f \overline{\Delta} \Delta_x(2) - (R_u + B_u^T \Pi_f B_u)^{-1} (\overline{\tau}_u(2) + B_u^T (\overline{\tau}_x(3) - \Pi_f r_\nu(2))).
\]

So, the solution for an \( N \)-point horizon is obtained by back propagating equations \( \text{(B.19e)} \) and
\[ (B.20) \text{using the initial conditions} \]

\[ \Pi_{N+1} = \Pi_f, \quad (B.28a) \]
\[ r(N + 1) = \tilde{r}_x(N + 1), \quad (B.28b) \]

and then forward propagating (recalling that \( \Delta u = \bar{\Delta} u - R_u^{-1} S_{xu}^T \Delta x \))

\[ \Delta x(1) = 0, \quad (B.29a) \]
\[ K(k) = - (R_u(k) + B_u(k)^T \Pi_{k+1} B_u(k))^{-1} B_u(k)^T \Pi_{k+1} \bar{A}(k), \quad (B.29b) \]
\[ \Delta u(k) = K(k) \Delta x(k) - R_u(k)^{-1} S_{xu}^T(k) \Delta x(k) \]
\[ - (R_u(k) + B_u(k)^T \Pi_{k+1} B_u(k))^{-1} (\tilde{r}_u(k) + B_u(k)^T (r(k + 1) - \Pi_{k+1} r_t(k))) , \quad (B.29c) \]
\[ \Delta x(k + 1) = \bar{A}(k) \Delta x(k) + B_u(k) \Delta u(k) - r_t(k), \]
\[ = A(k) \Delta x(k) + B_u(k) \Delta u(k) - r_t(k), \quad (B.29d) \]
\[ \Delta r_t(k) = - \Pi_{k+1} \Delta x(k + 1) - r(k + 1). \quad (B.29e) \]
Appendix C

The Internal Model Principle for DAC

This appendix is organized into two main sections. Section C.1 addresses the case where regulation is possible at all measurements used for control. In this section, we provide a complete presentation and proof of the internal model principle for robust regulation, and also prove that observer-controllers can be constructed to satisfy it. Section C.2 specializes the results to the case where regulation is only possible at a subset of the measurements used for control. In this section we state a modified version of the internal model principle, but do not provide a formal proof since it would closely follow that in Section C.1. However, we do provide a detailed construction for an observer-controller in this case, and show that it satisfies the modified internal model principle.

C.1 Non-Selective Regulation

This first section considers the case where the goal is to achieve disturbance rejection at all measurements. The basic theory is most easily laid out for this case. Section C.1.1 serves as a review of standard regulation theory for this goal, which we refer to as non-selective regulation. In this presentation, we take the liberty of dispensing with some considerations that are resolved in the earliest literature on the internal model principle (e.g.,[20]). That is, we assume that all the disturbance modes under consideration are actually observable at the outputs where they are to be rejected; and that the rejection goals have already been translated to equivalent goals at the measurements used for control. That such translation is possible has been shown to be necessary for applications where rejection is desired at outputs that are not directly measurable. This has been
referred to as “readability” in much of the early regulation literature. One final caveat, is that we work through the theory for discrete-time, but except for the precise form for the initial condition response ($A^k$ versus $e^{At}$), and the region $\mathbb{C}_\eta$ of the complex plain in which stable eigenvalues reside, all results hold for continuous-time as well.

After a review of regulation theory, Section C.1.2 presents a self contained statement and proof of the internal model principle (IMP) that is different in form from that usually presented in the literature [20, 60]. With our statement of the IMP, it is relatively straight forward to verify that the DAC-type observer-controller construction described in Section C.1.3 does in fact provide robust output regulation. In Section C.2, these results are specialized to the case where rejection is achievable at only a subset of measured outputs. We refer to this objective as selective rejection, and Section C.2.1 addresses the design of DAC-type observer-controllers for this case.

C.1.1 Overview of Standard Linear Time Invariant Regulator Theory

In this paper, it is assumed that the plant is linear time-invariant, with a response determined by

\[
\begin{bmatrix}
    x_t(k+1) \\
    x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
    A & B_d \\
    0 & A_d
\end{bmatrix}
\begin{bmatrix}
    x_t(k) \\
    x_d(k)
\end{bmatrix} +
\begin{bmatrix}
    B_u \\
    0
\end{bmatrix} u_t(k), \tag{C.1a}
\]

\[
y_t(k) =
\begin{bmatrix}
    C_y & C_d
\end{bmatrix}
\begin{bmatrix}
    x_t(k) \\
    x_d(k)
\end{bmatrix} + D_y u_t(k), \tag{C.1b}
\]

where $A_d$ represents an unstable exogenous system with $n_\lambda$ distinct eigenvalues $\{\lambda_1, \ldots, \lambda_n\} \subseteq \sigma(A_d)$. The control $u(k)$ is assumed to be generated using feedback according to

\[
x_{tc}(k) = A_c x_{tc}(k) + B_c y_t(k), \tag{C.2a}
\]

\[
u_t(k) = C_c x_{tc}(k), \quad D_c = 0, \tag{C.2b}
\]
and all vectors take values in finite dimensional spaces, e.g.

\[ \begin{align*}
    u_t & \in \mathcal{U}, \quad \dim(\mathcal{U}) = n_u, & \text{(C.3a)} \\
    y_t & \in \mathcal{Y}, \quad \dim(\mathcal{Y}) = n_y, & \text{(C.3b)} \\
    x_t & \in \mathcal{X}, \quad \dim(\mathcal{X}) = n_x, & \text{(C.3c)} \\
    x_d & \in \mathcal{X}_d, \quad \dim(\mathcal{X}_d) = n_d, & \text{(C.3d)} \\
    x_{tc} & \in \mathcal{X}_c, \quad \dim(\mathcal{X}_c) = n_c. & \text{(C.3e)}
\end{align*} \]

The following derivations can be modified to account for direct feed-through \( D_c \neq 0 \) in the controller, as long as care is taken to stipulate that the algebraic loop that is introduced is well posed. As an ancillary effect, including direct feed-through in the controller complicates the formulas considerably, and since it is not necessary to have such feed-through, we develop the results without it.

Finally, we denote the range and null space of a map \( M \) by \( \mathcal{R}(M) \) and \( \mathcal{N}(M) \), respectively. If \( M : \mathcal{U} \to \mathcal{X} \), and \( \mathcal{U}_s \subset \mathcal{U} \) is a subspace of \( \mathcal{U} \), then the restriction of \( M \) to \( \mathcal{U}_s \) is indicated by \( M|_{\mathcal{U}_s} \). Perhaps the simplest example would be to start with a multi-input system, and then consider the effect of only using the first input; then \( B|_{\mathcal{U}_s} = B_1 \), where \( B_1 \) is the first column of \( B \).

Another variation would be to consider only using controls that are arbitrary combinations of two input directions, say \( \mathcal{U}_s = \text{span}\{u_1, u_2\} \); in this case \( B|_{\mathcal{U}_s} = BU \) where \( U = [u_1 \ u_2] \). Lastly, a set \( \{x: x = Mu, u \in \mathcal{U}\} \) is denoted simply as \( M\mathcal{U} \).

We assume that the plant \( \{A, B_u, C_y, D_y\} \) and disturbance coupling \( \{B_d, C_d\} \) are not precisely known, but that the controller \( \{A_c, B_c, C_c, 0\} \) and disturbance dynamics \( A_d \) can be stipulated exactly. We also assume that despite the uncertainty in the plant model, the closed loop is stable (as defined below). If we let \( \mathbb{C}_g \) represent the interior of the unit circle in the complex plane, then we also assume that the disturbance model \( A_d \) is unstable so that its spectrum \( \sigma(A_d) \notin \mathbb{C}_g \).

Further, we assume \( A_d \) is similar to a Jordan form wherein each eigenvalue \( \{\lambda_i \in \sigma(A_d), i \in \} \)
\([1, n_\lambda]\) is contained in only one Jordan block. So

\[ A_d = V_d J_d V_d^{-1}, \]  

\begin{equation} \label{C4a} 
J_d = \begin{bmatrix} J_{d1} & 0 & \hdots & 0 \\ 0 & J_{d2} & \hdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \hdots & J_{dn_{\lambda}} \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & 0 & \hdots & 0 \\ 0 & V_2 & \hdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \hdots & V_{n_{\lambda}} \end{bmatrix}, \end{equation}

\begin{equation} \label{C4b} 
V_i = \begin{bmatrix} x_{d1,0} & \hdots & x_{d1,n_i-1} \\ \vdots & \ddots & \vdots \\ 0 & \hdots & x_{dn_{\lambda},0} & \hdots & x_{dn_{\lambda},n_{\lambda}-1} \end{bmatrix}, \quad J_{di} x_{di,j_i} = \lambda_i x_{di,j_i} + x_{di,j_i-1}, \end{equation}

so that

\[ V = \begin{bmatrix} x_{d1,0} & \hdots & x_{d1,n_1-1} & \hdots & x_{dn_{\lambda},0} & \hdots & x_{dn_{\lambda},n_{\lambda}-1} \end{bmatrix}, \]  

\begin{equation} \label{C5} 
\begin{split} 
x_{d1,0} & = \lambda_1 x_{d1,0}, \\
A_d x_{d1,j_i} & = \lambda_1 x_{d1,j_i} + x_{d1,j_i-1}, \\
\end{split} \end{equation}

This means that each eigenvalue \(\lambda_i\) is associated with only one chain of \(n_i\) generalized eigenvectors \(x_{di,j_i}\), where \(j_i \in [0, n_i - 1]\),

\begin{equation} \label{C6} 
x_{d1,0}^T = \begin{bmatrix} 0 & \hdots & 0 & \lambda_1 x_{d1,0} & \hdots & 0 \end{bmatrix}. \end{equation}

and the order of the disturbance model is \(n_d = \sum n_i\). In terms of the class of disturbances represented (offsets, ramps, sinusoids, etc.), this assumption of single Jordan blocks presents no loss of generality [30, 20]. Finally, we will usually refer to all generalized eigenvectors as simply “eigenvectors”, and where we wish to single out the standard eigenvectors (i.e., \(x_{i,0} \in \mathcal{N}(A_d - I\lambda_i)\)), we will refer to them as the zero-order eigenvectors.

Controlling the system in (C.1) with the \(u_t(k)\) generated in (C.2), the closed loop becomes
autonomous

\[
\begin{bmatrix}
x_t(k+1) \\
x_{te}(k+1) \\
x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
A & B_uC_c & B_d \\
B_cC_y & A_c + B_cDC_c & B_cC_d \\
0 & 0 & A_d
\end{bmatrix}
\begin{bmatrix}
x_t(k) \\
x_{te}(k) \\
x_d(k)
\end{bmatrix}
\]  
(C.8a)

\[
\begin{bmatrix}
x_t(k) \\
x_{te}(k) \\
x_d(k)
\end{bmatrix} =
\begin{bmatrix}
A_L & B_{Ld} \\
0 & A_d
\end{bmatrix}
\begin{bmatrix}
x_{L}(k) \\
x_d(k)
\end{bmatrix},
\]  
(C.8b)

\[
y_t(k) =
\begin{bmatrix}
C_y & DC_c & C_d
\end{bmatrix}
\begin{bmatrix}
x_t(k) \\
x_{te}(k) \\
x_d(k)
\end{bmatrix}
\]  
(C.8c)

\[
\begin{bmatrix}
x_{L}(k) \\
x_d(k)
\end{bmatrix} =
\begin{bmatrix}
C_{Ly} & C_d
\end{bmatrix}
\begin{bmatrix}
x_{L}(k) \\
x_d(k)
\end{bmatrix},
\]  
(C.8d)

The system in eq. (C.8) is considered stable when the loop subsystem $A_L$ is stable.

**Definition** The system in eq. (C.8) is stable if and only if $\sigma(A_L) \subset \mathbb{C}_g$.

We stipulate this as a condition for output regulation.

**Definition** Output regulation is achieved if and only if the system in eq. (C.8) is stable, and the output satisfies

\[
\lim_{k \to \infty} y_t(k) = 0,
\]  
(C.9)

for any initial condition.

So we require that $\sigma(A_L) \subset \mathbb{C}_g$ and $\sigma(A_d) \not\subset \mathbb{C}_g$ are disjoint, and this also means that the asymptotic response depends only on the state of the disturbance model.

**Lemma C.1.1** If the system in eq. (C.8) is stable ($\sigma(A_L) \subset \mathbb{C}_g$), then

\[
\lim_{k \to \infty} x_{L}(k) = \lim_{k \to \infty} x_{L}(k),
\]  
(C.10)
where $x_L(k)$ is the asymptotic state response of the loop, and is dependent on the disturbance state $x_d(k)$ according to

$$x_L(k) = \Pi V^{-1} x_d(k), \quad (C.11a)$$

$$= \begin{bmatrix} \Pi_x V^{-1} \\
\Pi_c V^{-1} \end{bmatrix} x_d(k) \begin{bmatrix} x(k) \\
x_c(k) \end{bmatrix} \quad (C.11b)$$

where $\Pi = [\Pi_x^T \Pi_c^T]^T$ are determined by a Jordan decomposition of the autonomous system. This then also means that the asymptotic output is given by

$$y(k) \triangleq [C_L y \ C_d] \begin{bmatrix} \Pi V^{-1} \\
I \end{bmatrix} x_d(k) \quad (C.12a)$$

$$= \begin{bmatrix} C_y & DC_c & C_d \end{bmatrix} \begin{bmatrix} \Pi V^{-1} \\
I \end{bmatrix} x_d(k). \quad (C.12b)$$

**Proof** If the loop is stable, then $\sigma(A_L) \cap \sigma(A_d) = \emptyset$, and the system has a Jordan decomposition such that

$$\begin{bmatrix} A_L & B_{Ld} \\
0 & A_d \end{bmatrix} = U \begin{bmatrix} \Pi & J \\
0 & V \end{bmatrix} \begin{bmatrix} U^{-1} & -U^{-1} \Pi V^{-1} \\
0 & V^{-1} \end{bmatrix}, \quad (C.13)$$

where $\sigma(A_L) = \sigma(UJU^{-1}) \subset \mathbb{C}_g$, and hence,

$$\lim_{k \to \infty} J^k = 0. \quad (C.14)$$
This shows that the asymptotic response is then
\[
\lim_{k \to \infty} \begin{bmatrix} x_{tL}(k) \\ x_d(k) \end{bmatrix} = \lim_{k \to \infty} \begin{bmatrix} A_L & B_{Ld} \\ 0 & A_d \end{bmatrix}^k \begin{bmatrix} x_{tL}(0) \\ x_d(0) \end{bmatrix},
\] (C.15a)
\[
= \lim_{k \to \infty} \begin{bmatrix} U & \Pi \\ 0 & V \end{bmatrix} \begin{bmatrix} 0 & 0 \\ J_d^k & 0 \\ 0 & V \end{bmatrix}^{-1} \begin{bmatrix} x_{tL}(0) \\ x_d(0) \end{bmatrix},
\] (C.15b)
\[
= \lim_{k \to \infty} \begin{bmatrix} \Pi V^{-1} \\ I \end{bmatrix} A_d^k x_d(0)
\] (C.15c)
\[
= \lim_{k \to \infty} \begin{bmatrix} \Pi_x V^{-1} \\ \Pi_c V^{-1} \\ I \end{bmatrix} x_d(k),
\] (C.15d)
where we have made the partition
\[
\Pi = \begin{bmatrix} \Pi_x \\ \Pi_c \end{bmatrix}
\] (C.16)
compatibly with the dimensions of \([x_t^T \ x_{tc}^T]^T\).■

We also have that if the initial state of the loop is \(x_{tL}(0) = \Pi V^{-1} x_d(0)\), then
\[
\begin{bmatrix} x_{tL}(k) \\ x_d(k) \end{bmatrix} = \begin{bmatrix} U & \Pi \\ 0 & V \end{bmatrix} \begin{bmatrix} J_d^k \\ J_d^k \end{bmatrix} \begin{bmatrix} U & \Pi \\ 0 & V \end{bmatrix}^{-1} \begin{bmatrix} \Pi \\ V \end{bmatrix}^{-1} x_d(0),
\] (C.17a)
\[
= \begin{bmatrix} U J_d^k & \Pi J_d^k \\ 0 & V J_d^k \end{bmatrix} I \begin{bmatrix} \Pi \\ V \end{bmatrix}^{-1} x_d(0)
\] (C.17b)
\[
= \begin{bmatrix} \Pi V^{-1} \\ I \end{bmatrix} A_d^k x_d(0) = \begin{bmatrix} x_L(k) \\ x_d(k) \end{bmatrix},
\] (C.17c)
so that the state of system evolves along the asymptotic response. Hence, the asymptotic loop
response $x_L(k)$ is also a solution of the system dynamics in eq. (C.8), and we must have that

$$
\begin{bmatrix}
 x_L(k + 1) \\
 x_d(k + 1)
\end{bmatrix} = \begin{bmatrix} I \\ \Pi V^{-1} \end{bmatrix} A_d x_d(k),
$$

(C.18a)

$$
\begin{bmatrix}
 A_L \\ B_{Ld}
\end{bmatrix} x_L(k) =
\begin{bmatrix}
 0 \\ A_d
\end{bmatrix} x_d(k)
$$

(C.18b)

$$
\begin{bmatrix}
 A_L \\ B_{Ld}
\end{bmatrix} \Pi V^{-1} x_d(k) =
\begin{bmatrix}
 0 \\ A_d
\end{bmatrix} \Pi V^{-1} x_d(k)
$$

(C.18c)

Since this relationship holds for arbitrary disturbance states $x_d(k)$, it follows that we have a solution of the Sylvester equation

$$
\begin{bmatrix}
 A_L \\ B_{Ld}
\end{bmatrix} \begin{bmatrix} \Pi V^{-1} \\ I \end{bmatrix} = \begin{bmatrix} \Pi V^{-1} \end{bmatrix} A_d.
$$

(C.19)

And, $\Pi V^{-1}$ is the unique solution of this Sylvester equation, because $\sigma(A_L) \subset C_y \Rightarrow \sigma(A_L) \cap \sigma(A_d) = \emptyset$.

This result, of course, also follows directly from the Jordan decomposition of the system in eq. (C.8), and that is guaranteed by having a stable loop. However, we wish to stress the role $[(\Pi V^{-1})^T I]^T$ plays in the system response; $[x_{Ld}^T x_d^T]^T$ is the superposition of the transient response and the asymptotic response, and the latter evolves within the subspace $\mathcal{R}([(\Pi V^{-1})^T I]^T)$ that is an invariant subspace of the autonomous system in eq. (C.8). Since,

$$
\mathcal{R} \left( \begin{bmatrix} \Pi V^{-1} \\ I \end{bmatrix} \right) = \mathcal{R} \left( \begin{bmatrix} \Pi_x V^{-1} \\ \Pi_x V^{-1} \end{bmatrix} \right) = \mathcal{R} \left( \begin{bmatrix} \Pi_x \\ \Pi_c \\ V \end{bmatrix} \right),
$$

(C.20)

to have $y(k) = 0$ in eq. (C.12a) for all possible disturbance states $x_d(k)$, this subspace must satisfy

$$
\mathcal{R} \left( \begin{bmatrix} \Pi_x \\ \Pi_c \\ V \end{bmatrix} \right) \subset \mathcal{N} \left( \begin{bmatrix} C_y & DC_c & C_d \end{bmatrix} \right).
$$

(C.21)

Essentially, this proves necessity for the following regulator lemma.
Lemma C.1.2 Regulation occurs if and only if the system in eq. (C.8) is stable, and there is a solution \( X = \Pi V^{-1} \) to the regulator equation

\[
\begin{bmatrix}
A_L & B_{Ld} \\
C_{Ly} & C_d
\end{bmatrix}
\begin{bmatrix}
X \\
I
\end{bmatrix}
= \begin{bmatrix}
X \\
0
\end{bmatrix} A_d. \tag{C.22}
\]

**Proof** Necessity follows for a stable loop, by setting \( y(k) = 0 \) in eq. (C.12a) for arbitrary disturbance states \( x_d(k) \), adjoining the result with eq. (C.19), and then defining \( X = \Pi V^{-1} \). Sufficiency follows, since application of the transformation [30]

\[
\begin{bmatrix}
\bar{x}_{tL}(k) \\
x_d(k)
\end{bmatrix}
= \begin{bmatrix}
I & -X \\
0 & I
\end{bmatrix}
\begin{bmatrix}
x_{tL}(k) \\
x_d(k)
\end{bmatrix}, \tag{C.23}
\]

to the system in eq. (C.8), gives

\[
\begin{bmatrix}
\bar{x}_{tL}(k) \\
x_d(k)
\end{bmatrix}
= \begin{bmatrix}
I & -X \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A_L & B_{dl} \\
0 & A_d
\end{bmatrix}
\begin{bmatrix}
I & X \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\bar{x}_{tL}(k) \\
x_d(k)
\end{bmatrix}, \tag{C.24a}
\]

\[
= \begin{bmatrix}
A_L & 0 \\
0 & A_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}_{tL}(k) \\
x_d(k)
\end{bmatrix}, \tag{C.24b}
\]

\[
y_t(k) = \begin{bmatrix}
C_{Ly} & C_d
\end{bmatrix}
\begin{bmatrix}
I & X \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\bar{x}_{tL}(k) \\
x_d(k)
\end{bmatrix}, \tag{C.24c}
\]

\[
= \begin{bmatrix}
C_{Ly} & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x}_{tL}(k) \\
x_d(k)
\end{bmatrix}. \tag{C.24d}
\]

Clearly, if \( \sigma(A_L) \in C_g \), then the asymptotic output is zero. \( \blacksquare \)

We note that \( \sigma(A_L) \cap \sigma(A_d) = \emptyset \) automatically guarantees a unique solution \( X = \Pi V^{-1} \) in eq. (C.19), and the \( n_d \) columns of

\[
\begin{bmatrix}
XV \\
V
\end{bmatrix}
= \begin{bmatrix}
\Pi_x \\
\Pi_c
\end{bmatrix} \tag{C.25}
\]

each contain an eigenvector \([x_{i,j_1}^T, x_{i,j_2}^T, x_{d,i,j_1}^T]^T\) of the system associated with \( \lambda_i \in \sigma(A_d) \). This brings us to the final lemma of this background section on regulation.
Lemma C.1.3 Regulation occurs if and only if the system in eq. (C.8) is stable, and there is a set of vectors \( \{ \{x_{i,j}, x_{ci,j}\}, \ j_i \in [0, n_i - 1], \ i \in [1, n_\lambda]\} \) that satisfy

\[
\begin{bmatrix}
A - I\lambda_i & B_u C_c \\
B_c C_y & A_c + B_c D C_c - I\lambda_i
\end{bmatrix}
\begin{bmatrix}
x_{i,j} \\
x_{ci,j}
\end{bmatrix}
= 
\begin{bmatrix}
x_{i,j,-1} \\
x_{ci,j,-1}
\end{bmatrix}
- 
\begin{bmatrix}
B_d x_{di,j_i} \\
B_c C_d x_{di,j_i}
\end{bmatrix},
\]

\[x_{i,-1} \neq 0, \ x_{ci,-1} \neq 0,
\]

and are such that

\[
y_{i,j_i} \triangleq \begin{bmatrix}
C_y & D C_c & C_d
\end{bmatrix}
\begin{bmatrix}
x_{i,j_i} \\
x_{ci,j_i} \\
x_{di,j_i}
\end{bmatrix} = 0.
\]  

Proof By lemma C.1.2, regulation occurs if and only if we have a solution \( X \) to eq. (C.22). Multiply this equation from the right by any eigenvector \( x_{di,j_i} \) of the disturbance model, and arrange the two rows as separate equations

\[
A_L x_{Li,j_i} - \lambda_i x_{Li,j_i} = x_{Li,j_i - 1} - B_{Ld} x_{di,j_i},
\]

\[
\begin{bmatrix}
C_{Ly} & C_d
\end{bmatrix}
\begin{bmatrix}
x_{Li,j_i} \\
x_{di,j_i}
\end{bmatrix} = 0,
\]

where we have used the fact that

\[
X A_d x_{di,j_i} = \Pi V^{-1} (\lambda_i x_{di,j_i} + x_{di,j_i - 1}) = \lambda_i x_{Li,j_i} + x_{Li,j_i - 1},
\]

so that each \( x_{Li,j_i} \) is a column of \( \Pi \). Now, expanding \( \{A_L, C_{Ly}, B_{Ld}\} \) as defined in eq. (C.8), establishes necessity. Conversely, given a set \( \{\{x_{i,j_i}, x_{ci,j_i}\}, \ j_i \in [0, n_i - 1], \ i \in [1, n_\lambda]\} \) satisfying eq. (C.26) and eq. (C.27), it is immediate that the set also satisfies eq. (C.28) with \( x_{Li,j_i} = [x_{i,j_i}^T, x_{ci,j_i}^T]^T \). If we now let

\[
\Pi = \begin{bmatrix}
x_{L1,0} & \cdots & x_{L1,n_1-1} & \cdots & x_{Ln_\lambda,0} & \cdots & x_{Ln_\lambda,n_\lambda-1}
\end{bmatrix},
\]
then eq. (C.28) can be written as

\[ A_L \Pi = \Pi J_d - B_{Ld} V, \]  

\[ (C.31a) \]

\[ \begin{bmatrix} C_{Ly} & C_d \end{bmatrix} \begin{bmatrix} \Pi \\ V \end{bmatrix} = 0. \]  

\[ (C.31b) \]

Multiplying this result through from the right by \( V^{-1} \), noting that \( \Pi J_d V^{-1} = \Pi V^{-1} A_d \), and letting \( X = \Pi V^{-1} \), it should now be clear that this is equivalent to the regulator equation. ♦

Note that since \( \sigma(A_L) \in \mathbb{C}_g \), it follows that \( A_L - I\lambda_i \) is non-singular, and the recursion in eq. (C.26) determines the set \( \{ \{ x_{i,j_i}, x_{ci,j_i} \} \}, \ j_i \in [0,n_i - 1], \ i \in [1,n_\lambda] \} \) uniquely, for each unique coupling \( \{B_d x_{di,j_i}, C_d x_{di,j_i}\}, \ j_i \in [0,n_i - 1], \ i \in [1,n_\lambda] \} \).

### C.1.2 The Internal Model Principle

This section addresses necessary and sufficient conditions on the plant \( \{ A, B, C_y, D_y \} \) and controller \( \{ A_c, B_c, C_c, 0 \} \) that guarantee the rejection goal in eq. (C.27) for any possible disturbance coupling \( \{B_d, C_d\} \). Thus far, we have shown that for a stable loop, the asymptotic response is determined by the \( n_d \) system eigenvectors that are solutions of the recursion in eq. (C.26), and that generate the associated modal outputs \( y_{i,j_i} \). This brings us to a statement of the internal model principle.

**Theorem C.1.4** Disturbance rejection occurs for all possible couplings \( \{B_d, C_d\} \), if and only if the loop is stable and:

1. (controller structure for regulation)

   \[ N(B_c) = 0, \]  

   \[ \mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_c) = 0, \]  

   \[ \text{(C.32a) (C.32b)} \]

2. (internal model) For each chain \( \{ x_{di,j_i}, \ j_i \in [0,n_i - 1], \ i \in [1,n_\lambda] \} \) of disturbance model eigenvectors associated with the modes \( \{ \lambda_i, \ i \in [1,n_\lambda] \} \), the controller must have \( n_y \) mutually-independent chains of generalized eigenvectors \( x_{ci,j_i,\ell} \) associated with the same modes. More
precisely, there must be a linearly independent set \[\{x_{ci,ji,\ell}, \ j_i \in [0, n_i - 1], \ i \in [1, n], \ \ell \in [1, n_y]\}\] such that

\[
A_c x_{ci,0,\ell} = \lambda_i x_{ci,0,\ell}, \tag{C.33a}
\]

\[
A_c x_{ci,ji,\ell} = \lambda_i x_{ci,ji,\ell} + x_{ci,ji-1,\ell}, \tag{C.33b}
\]

(3) and (transmission zero) define

\[
\mathcal{Y}_{i,ji} = \text{span}\{x_{ci,ji,\ell}, \ \ell \in [1, n_y]\}, \tag{C.34}
\]

then the plant and controller together must satisfy

\[
\begin{bmatrix}
A - I\lambda_i & B_u \\
C_y & D_y
\end{bmatrix}
\begin{bmatrix}
\mathcal{Y} \\
C_c \mathcal{Y}_{i,ji}
\end{bmatrix}
= \begin{bmatrix}
\mathcal{Y} \\
\mathcal{Y}
\end{bmatrix}. \tag{C.35}
\]

We note that only the requirement that the loop be stable, and the third condition of the IMP theorem, require anything of the plant. The actual plant parameters can vary arbitrarily as long as these two requirements are not violated, and then robust regulation is still achieved. The former requires that the plant is detectable and stabilizable, and the latter requires, as we explain below, the absence of certain transmission zeros. This final point warrants further explanation, that is provided in comments at the close of the proof of necessity.

Before beginning a formal proof, we derive an equivalent form of the basic relationship in eq. (C.26), that emphasizes the interplay between the modal outputs \(y_{i,ji}\) and the disturbance coupling coefficients \(\{B_d, C_d\}\). Note, that eq. (C.27) shows that we can write

\[
\begin{bmatrix}
C_y & DC_c & -I
\end{bmatrix}
\begin{bmatrix}
x_{i,ji} \\
x_{ci,ji} \\
y_{i,ji}
\end{bmatrix}
= -C_d x_{di,ji}, \tag{C.36}
\]
and the lower row in eq. (C.26) can be written as

\[ x_{ci,j,i-1} = (A_c - I\lambda_i)x_{ci,j,i} + B_c(C_yx_{ci,j,i} + DC_cx_{ci,j,i} + C_dx_{di,j,i}), \]

\[ = \begin{bmatrix} 0 & (A_c - I\lambda_i) & B_c \end{bmatrix} \begin{bmatrix} x_{i,j,i} \\ x_{ci,j,i} \\ y_{i,j,i} \end{bmatrix}. \]

(C.37a) (C.37b) (C.37c)

So, the modal output equation (C.27) can be embedded into the recursion (C.26) using \( y_{i,j,i} \) as a new variable, and the combined equations written as

\[
\begin{bmatrix}
A - I\lambda_i & B_c C_c & 0 \\
0 & A_c - I\lambda_i & B_c \\
C_y & DC_c & -I
\end{bmatrix}
\begin{bmatrix}
x_{i,j,i} \\
x_{ci,j,i} \\
y_{i,j,i}
\end{bmatrix}
=
\begin{bmatrix}
x_{i,j,i-1} \\
x_{ci,j,i-1} \\
y_{i,j,i}
\end{bmatrix}
-
\begin{bmatrix}
0 \\
0 \\
C_d x_{di,j,i}
\end{bmatrix},
\]

(C.38a)

\[ x_{i,-1} \neq 0, \quad x_{ci,-1} \neq 0. \]

(C.38b)

For a stable loop, the set of solutions to this recursion (also, being solutions to the recursion in eq. (C.26)), determine the invariant subspace \( \mathcal{R}([\Pi V^{-1} I]^T) \) in which the asymptotic response evolves.

Because of the “\(-I\)” in the lower right corner, if the matrix on the left in eq. (C.38) has a non-trivial null space, then it cannot contain a vector of the form \([0^T 0^T y^T]^T\). And, if \([x^T x_c^T y^T]^T\) is a non-trivial vector in the null space of the matrix on the left in eq. (C.38), then \([x^T x_c^T]^T\) is a non-trivial vector in the null space of the matrix \((A_L)\) on the left in eq. (C.26). Therefore, if the map on the left of eq. (C.38) is singular, then the loop is not stable. This also shows that for a stable loop, each unique disturbance coupling \([B_d x_{di,j,i}, C_d x_{di,j,i}]\) generates a unique solution (modal response) \(\{x_{i,j,i}, x_{ci,j,i}, y_{i,j,i}\}\).

Conversely, take any solution \(\{x_{ci,0}, y_{i,0} \neq 0\}\) of the second row in eq. (C.38a)

\[ (A_c - I\lambda_i)x_{ci,0} + B_c y_{i,0} = 0, \]

(C.39)
and any \( x_{i,0} \), then there is a disturbance coupling \( \{ B_d, C_d \} \) that makes \( \{ x_{i,0}, x_{ci,0}, y_{i,0} \neq 0 \} \) a solution of eq (C.38a). For this disturbance coupling, \( y_{i,0} \neq 0 \) is then the modal output associated with the zero-order eigenvector \( [x_{i,0}^T \ x_{ci,0}^T \ x_{di,0}^T]^T \) for the system as in eq. (C.11a). That is, if the disturbance state is \( x_d(k) = x_{di,0} \), then the asymptotic output will not be zero. Identical arguments apply if any \( y_{i,ji} \neq 0 \) is a solution for the center row in eq (C.38).

**Proof of Theorem C.1.4:** \((\Rightarrow)\) If the loop is stable, the solutions of the recursion eq. (C.38) determine the asymptotic response. And, as just discussed, if rejection is achieved for all possible disturbance couplings, it is necessary that \( y_{i,ji} = 0 \) be the only solution of the second row in eq. (C.38). In particular, we need

\[
(A_c - I\lambda_i) x_{ci,0} + B_c y_{i,0} = 0 \Rightarrow y_{i,0} = 0.
\]

This holds only if eq.'s (C.32a) and (C.32b) hold. This, in turn, requires that

\[
dim(\mathcal{R}(A_c - I\lambda_i)) \leq \dim(\mathcal{X}_c) - n_y, \tag{C.41a}
\]

\[
\Leftrightarrow \dim(N(A_c - I\lambda_i)) \geq n_y, \tag{C.41b}
\]

Hence, the controller must have at least \( n_y \) linearly-independent zero-order eigenvectors \( \{ x_{ci,0,\ell}, \ell \in [1, n_y] \} \) for each unstable mode \( \lambda_i \in \sigma(A_d) \). At the same time, in order that the loop be internally stable, we must also have that these modes are stabilizable

\[
dim(\mathcal{R}(A_c - I\lambda_i) B_c)) = \dim(\mathcal{X}_c), \tag{C.42a}
\]

\[
\Rightarrow \dim(\mathcal{R}(A_c - I\lambda_i)) \geq \dim(\mathcal{X}_c) - n_y, \tag{C.42b}
\]

\[
\Leftrightarrow \dim(N(A_c - I\lambda_i)) \leq n_y, \tag{C.42c}
\]

and this shows that in fact \( \dim(N(A_c - I\lambda_i)) = n_y \) must hold. Further, given that the loop is stable, there are disturbance couplings \( \{ B_d, C_d \} \) that will require arbitrary

\[
x_{ci,0} \in \mathcal{X}_{i,0} \triangleq \text{span} \{ x_{ci,0,\ell}, \ell \in [0, n_y] \} = N(A_c - I\lambda_i) \tag{C.43}
\]

in solution to eq. (C.38).
Now, since we require \( y_{i,j_i} = 0 \) in all solutions of the center row in eq. (C.38a), this row then becomes a relation defining generalized eigenvectors. That is, in order to have

\[
(A_c - I\lambda_i) x_{c_i,j_i} + B_c y_{i,j_i} = x_{c_i,j_i-1} \Rightarrow y_{i,j_i} = 0,
\]

we must have \( x_{c_i,j_i-1} \in \mathcal{R}(A_c - I\lambda_i) \), since we already have eq. (C.32) from the \( j_i = 0 \) case. Immediately, we then have that, in fact, \((A_c - I\lambda_i) x_{c_i,j_i} = x_{c_i,j_i-1}\) so that each \( x_{c_i,j_i} \) is a generalized eigenvector. Since the solution for the zero-order case can be any \( x_{c_i,0} \in V_{i,0} \), it follows that \( x_{c_i,j_i} \in V_{i,j_i} \) (which, along with eq. (C.32), forces \( y_{i,j_i} = 0 \)) it follows that eq. (C.38) then reduces to

\[
x_{c_i,j_i} \in V_{i,j_i},
\]

\[
\begin{bmatrix}
A - I\lambda_i & B_u C_c \\
C_y & D C_c
\end{bmatrix}
\begin{bmatrix}
x_i,j_i \\
x_{c_i,j_i}
\end{bmatrix}
= 
\begin{bmatrix}
x_{i,j_i-1} \\
x_{c_i,j_i-1}
\end{bmatrix} - 
\begin{bmatrix}
B_d x_{d_i,j_i} \\
C_d x_{d_i,j_i}
\end{bmatrix},
\]

\[
x_{i,-1} \nless 0, \quad x_{c_i,-1} \nless 0
\]

If this equation is to hold for all possible disturbance couplings, then it is necessary that eq. (C.35) hold. This requires not only that,

\[
\begin{bmatrix}
A - I\lambda_i & B_u \\
C_y & D_y
\end{bmatrix}
\begin{bmatrix}
\mathcal{X} \\
\mathcal{Y}
\end{bmatrix}
= 
\begin{bmatrix}
\mathcal{X} \\
\mathcal{Y}
\end{bmatrix},
\]

so that necessarily \( n_u \geq n_y \), but also that

\[
\begin{bmatrix}
A - I\lambda_i & B_{i,j_i} \\
C_y & D_{i,j_i}
\end{bmatrix}
\begin{bmatrix}
\mathcal{X} \\
\mathcal{Y}_{i,j_i}
\end{bmatrix}
= 
\begin{bmatrix}
\mathcal{X} \\
\mathcal{Y}
\end{bmatrix},
\]
where \( \mathcal{U}_{i,j} \equiv C_c \mathcal{Y}_{i,j_i} \), and
\[
\begin{bmatrix}
  B_{i,j_i} \\
  D_{i,j_i}
\end{bmatrix} = \begin{bmatrix}
  B_u \\
  D_y
\end{bmatrix}_{\mathcal{U}_{i,j_i}}.
\]
Since,
\[
\dim \left( \mathcal{R} \left( \begin{bmatrix}
  A - I \lambda_i & B_{i,j_i} \\
  C_y & D_{i,j_i}
\end{bmatrix} \right) \right) \leq n_x + \dim (\mathcal{U}_{i,j_i})
\]
(C.49a)
\[
\leq n_x + \dim (\mathcal{Y}_{i,j_i})
\]
(C.49b)
\[
= n_x + n_y,
\]
(C.49c)
and the dimension of the space on the right in eq. (C.47) is \( n_x + n_y \), it must be the case that \( \dim (\mathcal{U}_{i,j_i}) = n_y \). For the \( j_i = 0 \) case, \( \dim (\mathcal{U}_{i,0}) = n_y \) requires that the unstable modes \( \lambda_i \in \sigma(A_d) \cap \sigma(A_c) \) of the controller must be observable. Further, at modes \( \lambda_i \in \sigma(A_d) \cap \sigma(A_c) \), there can be no transmission zeros in the plant restricted to controls generated by the internal model.

The requirements imposed by eq. (C.35), are actually redundant with the requirement that the loop be stable. This is seen from the following argument. Say the controller is designed so that eq. (C.44) holds. Then it is necessary that \( x_{ci,j_i} \in \mathcal{Y}_{i,j_i} \) in any solution to eq. (C.38). However, if the conditions in eq. (C.35) are violated for some \( \mathcal{Y}_{i,j_i} \), then there is a disturbance coupling that produces a vector that is not in the range of the map on the left side of eq. (C.38) \( \forall x_{ci,j_i} \in \mathcal{Y}_{i,j_i} \); that is, the square matrix is not surjective and it therefore has a non-trivial null space. Hence, in this case, the matrix on the left of eq. (C.26) has a non-trivial null space as well, so that \( \lambda_i \in \sigma(A_L) \cap \sigma(A_d) \) is an unstable mode of the closed loop.

**Proof of Theorem C.1.4:** (\( \Leftarrow \)) This follows almost immediately by Lemma C.1.3. The conditions in the theorem guarantee that for every possible disturbance coupling \( \{B_d, C_d\} \), there are \( n_d \) solutions \( \{x_{i,j_i}, x_{ci,j_i}, x_{di,j_i}, y_{i,j_i} = 0\} \) to the recursion in eq. (C.26), and that there is no solution of this recursion such that \( y_{i,j_i} \neq 0 \). These solutions then determine the subspace \( \mathcal{R} \left( [P^T V^T]^T \right) \) in which the asymptotic response evolves, and this subspace is such that \( y_{i,j_i} = 0 \).
C.1.3 Design of DAC Type Observer-Controllers for Non-Selective Rejection

Achieving output regulation using observer-controllers, is based on ideas that were possibly first recognized in [32], in which the DAC approach is presented. The idea is essentially that the state of the disturbance model can be estimated using an observer in the same way that the plant state is, and then the control includes feedback from the estimated disturbance state. The extension to the original DAC approach described here, is that the control computed from the disturbance state insures that a regulator equation is satisfied.

Although [32] did not make use of a regulator equation (and thus, achieved regulation only under special circumstances [30]), it is now generally recognized that regulation can be achieved with this approach when an appropriate regulator equation is satisfied [70]. In this section, we show that this approach can produce a controller that satisfies the conditions in the IMP Theorem C.1.4, and thus achieves regulation that is “robust” to any errors in the assumed form for the disturbance coupling \( \{B_d, C_d\} \).

We note, however, that the most direct approach to satisfying the IMP, is to use output augmentation as in [30, 20]. Designing a reduced-order observer-controller for the augmented system is then an efficient method for obtaining a stabilizing controller. For the case that rejection is to be achieved at all measurements, the DAC-based approach described here, results in a controller of the same order. For the selective case considered in Section C.2, this approach is not as efficient as output augmentation.

Section C.1.3.1 introduces the nominal models used for design and derives the regulator equation satisfied by the resulting observer-controller. Section C.1.3.2 then explains the construction of the disturbance gain and the nominal disturbance coupling \( \bar{C}_d \) that are computed as part of the same recursion. Finally, in Section C.1.3.3 we prove that the constructed observer-controller satisfies the IMP.
C.1.3.1 Models and DAC Background

Since we are constructing an observer, we must have that the nominal model

\[
\begin{bmatrix}
    x(k+1) \\
    x_d(k+1)
\end{bmatrix} =
\begin{bmatrix}
    \hat{A} & \hat{B}_d \\
    0 & \hat{A}_d
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    x_d(k)
\end{bmatrix} +
\begin{bmatrix}
    \hat{B}_u \\
    0
\end{bmatrix} u(k),
\]  

(C.50a)

\[
y(k) =
\begin{bmatrix}
    \bar{C}_y & \bar{C}_d
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    x_d(k)
\end{bmatrix} + \bar{D}_y u(k),
\]  

(C.50b)

is detectable, and that \(\{\hat{A}, \hat{B}_u\}\) is stabilizable. The observer-controller gains \(\{C_c, B_c\}\) are partitioned into state and disturbance sections, where

\[
C_c =
\begin{bmatrix}
    K_x & K_d
\end{bmatrix},
\]  

(C.51a)

\[
B_c =
\begin{bmatrix}
    L_x \\
    L_d
\end{bmatrix},
\]  

(C.51b)

The design of the state-feedback \(K_x\) can be done separately, and it is presumed that this is already done, and that it is stabilizing

\[
\sigma(\hat{A} + \hat{B}_u K_x) \subset \mathcal{C}_g
\]  

(C.52)
for the nominal plant model. Now the observer-controller system \( \{ A_c, B_c, C_c, 0 \} \) is then determined by

\[
\begin{bmatrix}
\bar{x}(k + 1) \\
\bar{x}_d(k + 1)
\end{bmatrix} =
\begin{bmatrix}
\bar{A} & \bar{B}_d \\
0 & \bar{A}_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix} +
\begin{bmatrix}
\bar{B}_u \\
0
\end{bmatrix}
\begin{bmatrix}
K_x & K_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix},
\]

\[
+ \begin{bmatrix}
L_x \\
L_d
\end{bmatrix}
\begin{bmatrix}
y(k)
\end{bmatrix} - \begin{bmatrix}
\bar{C}_y & \bar{C}_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix}
\]

\[-\bar{D}_y \begin{bmatrix}
K_x & K_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix},
\]

\[
= \begin{bmatrix}
\bar{A} + \bar{B}_u K_x & \bar{B}_d + \bar{B}_u K_d \\
0 & \bar{A}_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
L_x \\
L_d
\end{bmatrix}
\begin{bmatrix}
y(k)
\end{bmatrix}
\]

\[-\begin{bmatrix}
\bar{C}_y + \bar{D}_y K_x & \bar{C}_d + \bar{D}_y K_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix}.
\]

\[
u(k) = \begin{bmatrix}
K_x & K_d
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_d(k)
\end{bmatrix}.
\]

If there is no modeling error (an assumption that we eventually dispense with), a standard observer-controller argument shows that the loop is stable, and then when the estimation error goes to zero, the response of the closed loop system is determined by

\[
\begin{bmatrix}
x(k + 1) \\
x_d(k + 1)
\end{bmatrix} = \begin{bmatrix}
\bar{A} + \bar{B}_u K_x & \bar{B}_d + B_u K_d \\
0 & \bar{A}_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k)
\end{bmatrix},
\]

\[
y(k) = \begin{bmatrix}
\bar{C}_y + \bar{D} K_x & \bar{C}_d + \bar{D} K_d
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x_d(k)
\end{bmatrix}.
\]

Now, according to lemma C.1.2, regulation is achieved if and only if there is a solution \( \bar{X} \) to the
(state-feedback) regulator equation
\[ \begin{bmatrix} \check{A} + \check{B}_u K_x & \check{B}_d + \check{B}_u K_d \\ \check{C} + \check{D}_y K_x & \check{C}_d + \check{D}_y K_d \end{bmatrix} \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \tilde{A}_d. \] (C.55)

The nominal disturbance coupling \( \check{C}_d \) is constructed at the same time as the disturbance gain \( K_d \) so that

- the conditions in eq. (C.35) are satisfied,
- the nominal model eq. (C.50) is detectable,
- and there is a solution \( \bar{X} \) to the state-feedback regulator equation (C.55).

Since it is necessary that the observer-controller have \( n_y \) disturbance models, in the most straightforward construction, the disturbance model \( A_d \) is extended to a block diagonal form

\[
\check{A}_d = \begin{bmatrix}
A_{d1} & 0 & \cdots & 0 \\
0 & A_{d2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{dn_y}
\end{bmatrix},
\] (C.56a)

\[
\check{B}_d = \begin{bmatrix}
B_{d1} & \cdots & B_{dn_y}
\end{bmatrix},
\] (C.56b)

\[
\check{C}_d = \begin{bmatrix}
C_{d1} & \cdots & C_{dn_y}
\end{bmatrix},
\] (C.56c)

where \( A_{d\ell} = A_d, \ \ell \in [1, n_y] \). Since \( A_d = VJ_dV^{-1} \) where

\[
V = \begin{bmatrix} x_{d1,0} & \cdots & x_{d1,n_1-1} & \cdots & x_{dn_\lambda,0} & \cdots & x_{dn_\lambda,n_\lambda-1} \end{bmatrix},
\] (C.57)

we have that

\[
\check{A}_d = \tilde{V} \tilde{A}_d \tilde{V}^{-1},
\] (C.58a)

\[
\tilde{V} = \begin{bmatrix}
V_1 & 0 & \cdots & 0 \\
0 & V_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & V_{n_y}
\end{bmatrix}, \quad \tilde{V}_\ell = V, \ \ell \in [1, n_y],
\] (C.58b)
so that

\[
x_{\text{di},j_1,\ell} = \left[0^T_{n_d(m-1)} \ x_{\text{di},j_1}^T \ 0^T_{n_d(m-\ell)}\right]^T,
\]

where \(0_{n_d(m)}\) is a vector of \(n_d \cdot m\) zeros. As we’ll show, the controller eigenvectors \(x_{ci,j_1,\ell}\) required by the IMP will then be related to the extended disturbance model eigenvectors \(x_{\text{di},j_1,\ell}\) according to

\[
x_{ci,j_1,\ell} = \begin{bmatrix} 0 \\ x_{\text{di},j_1,\ell} \end{bmatrix}.
\]

(C.60)

### C.1.3.2 Design of The Disturbance Gain and The Nominal Disturbance Coupling

The computations in this section are nearly identical to the reference shifting computations used in MPC [52] (in particular, if \(K_x = 0\) is chosen). However, we include computations for \(\hat{C}_d\), and include a final gain computation for \(K_d\). Similar to the derivation of eq. (C.26), we begin by multiplying the state-feedback regulator eq. (C.55) from the right by \(x_{\text{di},j_1,\ell}\), to obtain an equivalent set of vector relations

\[
\begin{bmatrix}
\hat{A} + \hat{B}_u K_x & \hat{B}_d + \hat{B}_u K_d \\
\hat{C} + \hat{D}_y K_x & \hat{C}_d + \hat{D}_y K_d
\end{bmatrix}
\begin{bmatrix}
\hat{X} x_{\text{di},j_1,\ell} \\
x_{\text{di},j_1,\ell}
\end{bmatrix}
= \begin{bmatrix}
\hat{X}(\lambda_i x_{\text{di},j_1,\ell} + x_{\text{di},j_1-1,\ell}) \\
0
\end{bmatrix}
\]

(C.61a)

\[
= \begin{bmatrix}
I \lambda_i & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\hat{X} x_{\text{di},j_1,\ell} \\
x_{\text{di},j_1,\ell}
\end{bmatrix}
+ \begin{bmatrix}
\hat{X} x_{\text{di},j_1-1,\ell} \\
0
\end{bmatrix}.
\]

(C.61b)

Defining

\[
\bar{A} \doteq \hat{A} + \hat{B}_u K_x, \quad \bar{C}_y \doteq \hat{C}_y + \hat{D}_y K_x,
\]

and

\[
u_{i,j_1,\ell} \doteq K_d x_{\text{di},j_1,\ell}, \quad x_{i,j_1,\ell} \doteq \hat{X} x_{\text{di},j_1,\ell},
\]

(C.62)
this result can be arranged as
\[
\begin{bmatrix}
\tilde{A} - I\lambda_i & \tilde{B}_u \\
\tilde{C}_y & \tilde{D}_y \\
\end{bmatrix}
\begin{bmatrix}
x_{i,j_i,\ell} \\
u_{i,j_i,\ell} \\
\end{bmatrix}
+ \begin{bmatrix}
\tilde{B}_d x_{d,i,j_i,\ell} \\
\tilde{C}_d x_{d,i,j_i,\ell} \\
\end{bmatrix} = \begin{bmatrix}
x_{i,j_i,1,\ell} \\
0 \\
\end{bmatrix}.
\] (C.64)

Now, it is not difficult to show
\[
\begin{bmatrix}
\tilde{A} - I\lambda_i & \tilde{B}_u \\
\tilde{C}_y & \tilde{D}_y \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{X} \\
\mathcal{Y} \\
\end{bmatrix} = \begin{bmatrix}
\mathcal{X} \\
\mathcal{Y} \\
\end{bmatrix} \iff \begin{bmatrix}
\tilde{A} - I\lambda_i & \tilde{B}_u \\
\tilde{C}_y & \tilde{D}_y \\
\end{bmatrix}
\begin{bmatrix}
\mathcal{X} \\
\mathcal{Y} \\
\end{bmatrix} = \begin{bmatrix}
\mathcal{X} \\
\mathcal{Y} \\
\end{bmatrix}
\] (C.65)

so that if the nominal system satisfies eq. (C.46) as required by the IMP, then there is always a solution \([x_{i,j_i,\ell}^T u_{i,j_i,\ell}^T]^T\) to eq. (C.64). The construction of the gains \(\{K_d, \tilde{C}_d\}\) is now accomplished by finding solutions to eq. C.64 where \(u_{i,j_i,\ell}\) are constrained to be bases for the \(\mathcal{U}_{i,j_i}\) in eq. (C.47).

We begin with selection of bases for the spaces \(\mathcal{U}_{i,j_i}\). In practice, this is usually not difficult, since it is known (through physical modeling, analysis, etc.) which control input (or set of controls) can be actuated to reject disturbances at each output. Typically, the bases are independent of the order of the eigenvector so that

\[
\mathcal{U}_{i,j_i} = \mathcal{U}_i = \text{span}\{u_{i,\ell}, \ell \in [1, n_y]\},
\] (C.66)

where the set \(\{u_{i,\ell}\}\) spans the input combinations that can achieve rejection for the mode \(\lambda_i \in \sigma(A_d)\).

Often, we can choose \(u_{i,\ell} = e_{m_\ell}\) where \(e_{m_\ell}\) is the standard unit vector corresponding to the \(m_\ell^{th}\) input that is responsible for rejection of the mode \(\lambda_i\) at the \(\ell^{th}\) output.

Next, we find solutions \(\{u_{i,j_i,\ell}, x_{i,j_i,\ell}\}\) to the vector relations in eq. (C.64). Such solutions exist if and only if they are also solutions for

\[
\begin{bmatrix}
\tilde{A} - I\lambda_i & \tilde{B}_u \\
0 & \tilde{C}_y(I\lambda_i - \tilde{A})^{-1}\tilde{B}_u + \tilde{D} \\
\end{bmatrix}
\begin{bmatrix}
x_{i,j_i,\ell} \\
u_{i,j_i,\ell} \\
\end{bmatrix}
= \begin{bmatrix}
x_{i,j_i,1,\ell} \\
u_{i,j_i,1,\ell} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tilde{B}_d x_{d,i,j_i,\ell} \\
(\tilde{C}_y(I\lambda_i - \tilde{A})^{-1}\tilde{B}_d + \tilde{C}_d) x_{d,i,j_i,\ell} \\
\end{bmatrix}.
\]
where \((I\lambda_i - \widetilde{A})^{-1}\) exists, because \(K_x\) is stabilizing. The upper row determines \(x_{i,j_1,\ell}\) via

\[
x_{i,0,\ell} = -(\widetilde{A} - I\lambda_i)^{-1}(\widetilde{B}_u u_{i,0,\ell} + \widetilde{B}_d x_{di,0,\ell}),
\]

while the lower row defines \(\widetilde{C}_d\) completely via

\[
\widetilde{C}_d x_{di,0,\ell} = y_{i,0,\ell}, \quad \widetilde{C}_d x_{di,j_1,\ell} = y_{i,j_1,\ell}, \tag{C.68}
\]

where

\[
y_{i,0,\ell} = -\left(\widetilde{C}_y(I\lambda_i - \widetilde{A})^{-1}\widetilde{B}_u + \widetilde{D}\right) u_{i,0,\ell}
- \widetilde{C}_y(I\lambda_i - \widetilde{A})^{-1}\widetilde{B}_d x_{di,0,\ell},
\]

\[
y_{i,j_1,\ell} = -\left(\widetilde{C}_y(I\lambda_i - \widetilde{A})^{-1}\widetilde{B}_u + \widetilde{D}\right) u_{i,j_1,\ell}
+ \widetilde{C}_y(I\lambda_i - \widetilde{A})^{-1}(x_{i,j_1-1,\ell} - \widetilde{B}_d x_{di,j_1,\ell}). \tag{C.69b}
\]

If we collect these vector solutions into matrices

\[
X_i,\ell = \begin{bmatrix} x_{i,0,\ell} & \ldots & x_{i,n_i-1,\ell} \end{bmatrix}, \quad X_\ell = \begin{bmatrix} X_{1,\ell} & \ldots & X_{n_\lambda,\ell} \end{bmatrix}, \tag{C.70a}
\]

\[
U_i,\ell = \begin{bmatrix} u_{i,0,\ell} & \ldots & u_{i,n_i-1,\ell} \end{bmatrix}, \quad U_\ell = \begin{bmatrix} U_{1,\ell} & \ldots & U_{n_\lambda,\ell} \end{bmatrix}, \tag{C.70b}
\]

\[
Y_i,\ell = \begin{bmatrix} y_{i,0,\ell} & \ldots & y_{i,n_i-1,\ell} \end{bmatrix}, \quad Y_\ell = \begin{bmatrix} Y_{1,\ell} & \ldots & Y_{n_\lambda,\ell} \end{bmatrix}, \tag{C.70c}
\]

we can then write

\[
\widetilde{X}_\ell = X_\ell V^{-1}, \quad \widetilde{X} = [\widetilde{X}_1 \ldots \widetilde{X}_{n_y}], \tag{C.71a}
\]

\[
K_\ell = U_\ell V^{-1}, \quad K_d = [K_1 \ldots K_{n_y}], \tag{C.71b}
\]

\[
C_\ell = Y_\ell V^{-1}, \quad \widetilde{C}_d = [C_1 \ldots C_{n_y}]. \tag{C.71c}
\]

We note that with this construction, all disturbance modes in the controller are observable as long as \(\widetilde{B}_d\) is chosen so that the right hand side of eq. (C.69a) is not 0 (e.g., choosing \(B_d = 0\) always works). And, \(C_c [\mathcal{Y}_{i,j_1}] = [K_x \ K_d] [\mathcal{Y}_{i,j_1}] = \mathcal{V}_{i,j_1}\) where \(\mathcal{V}_{i,j_1}\) are chosen to satisfy eq. (C.35) via eq. (C.47). We still need to choose the observer gain so that \(B_c = [L_x^T \ L_d^T]^T\) satisfies eq. (C.32), but we show in the next section, that any stabilizing observer gain will work.
C.1.3.3 Proof of Robust Regulation

In this section, we show that the observer-controller constructed in the previous section satisfies the IMP for the nominal model \(\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}\). This then guarantees regulation for any possible disturbance coupling \(\{B_d, C_d\}\) into the nominal model. In fact, as long as the actual plant combined with the observer-controller is stable, and arbitrary modeling errors are not so great that the actual plant violates eq. (C.35), then we get robust regulation for the actual plant as well.

As already noted, by construction we have computed \(K_d\) such that \(C_{c}Y_{i,j_i} = V_{i,j_i}u_{i,j_i}\), where \(u_{i,j_i}\) are chosen/specified so that

\[
\begin{bmatrix}
\hat{A} - I\lambda_i & \hat{B}_{i,j_i} \\
\hat{C} & \hat{D}_{i,j_i}
\end{bmatrix}
\begin{bmatrix}
\mathcal{X} \\
\mathcal{U}_{i,j_i}
\end{bmatrix}
= \begin{bmatrix}
\mathcal{X} \\
\mathcal{Y}
\end{bmatrix},
\]

(C.72)

and \(\{\hat{B}_{i,j_i}, \hat{D}_{i,j_i}\}\) are defined as in eq. (C.48), but for the nominal model. This satisfies the conditions of eq. (C.35) of the IMP for the nominal plant. Further, as long as the constructed \(u_{i,j_i} = C_{c}Y_{i,j_i}\) satisfy eq. (C.35) for the actual plant and the loop remains stable, then we will obtain robust regulation for the actual plant as well. What remains, is to show that the correct multiplicity of the disturbance model exists (eq. (C.33)), and that \(B_c = [L_x^T \quad L_d^T]^T\) satisfies eq. (C.32) as well.

Since the controller is constructed so that there is a solution \(\hat{X}\) to the state-feedback regulator equation (C.55), there is a similarity transformation

\[
\begin{bmatrix}
\bar{X}(k) \\
\bar{X}_d(k)
\end{bmatrix}
= \begin{bmatrix}
I & -\hat{X} \\
0 & I
\end{bmatrix}
\begin{bmatrix}
X(k) \\
X_d(k)
\end{bmatrix},
\]

(C.73)

that, when applied to the observer-controller dynamics, shows that we can implement the controller
as

\[
\begin{bmatrix}
\tilde{x}(k + 1) \\
\tilde{x}_d(k + 1)
\end{bmatrix} =
\begin{bmatrix}
\bar{A} & 0 \\
0 & \hat{A}_d
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(k) \\
\tilde{x}_d(k)
\end{bmatrix}
\]

\[+
\begin{bmatrix}
\bar{L}_x \\
L_d
\end{bmatrix}
\begin{bmatrix}
y(k) - \begin{bmatrix} \bar{C}_y + \bar{D}_y K_x & 0 \end{bmatrix} \begin{bmatrix}
\tilde{x}(k) \\
\tilde{x}_d(k)
\end{bmatrix}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\bar{A} - \bar{L}_x(\bar{C}_y + \bar{D}_y K_x) & 0 \\
-L_d(\bar{C}_y + \bar{D}_y K_x) & \hat{A}_d
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(k) \\
\tilde{x}_d(k)
\end{bmatrix}
\]

\[A_c
\]

\[+
\begin{bmatrix}
\bar{L}_x \\
L_d
\end{bmatrix} y(k).
\]

(C.74b)

This form shows that in fact \( x_{ci,ji,\ell} = [0^T \ x_{di,ji,\ell}^T]^T \) are eigenvectors for the observer-controller with the multiplicity \( n_i n_y \) required by the IMP. We still need to verify that the multiplicity of any mode \( \lambda_i \in \sigma(\hat{A}_d) \cap \sigma(A_c) \) is not greater than \( n_i n_y \), but this follows using an intermediate result that also allows us to verify that the observer gain \( B_c = [\bar{L}_x L_d^T]^T \) satisfies eq. (C.32).

**Lemma C.1.5 ([9, 52])** If the observer is stable and \( \dim(\mathcal{N}(\hat{A}_d - I \lambda_i)) = n_y \), then

\[
\dim(\mathcal{R}(L_d)) = n_y,
\]

\( \mathcal{R}(\hat{A}_d - I \lambda_i) \cap \mathcal{R}(L_d) = 0. \) (C.75b)

**Proof** If the observer is stable, it means that the disturbance model modes \( \lambda_i \in \sigma(\hat{A}_d) \) are stabilizable through \( [L_x^T \ L_d^T]^T \) using state-feedback \([\bar{C}_y \ \bar{C}_d]\). Hence, by the PBH rank test, we must have full row rank in

\[
\begin{bmatrix}
\hat{A} - I \lambda_i & \bar{B}_d & \bar{L}_x \\
0 & \hat{A}_d - I \lambda_i & L_d
\end{bmatrix},
\]

(C.76)

and this can only be the case if

\[
\mathcal{R} \left( \begin{bmatrix} \hat{A}_d - I \lambda_i & L_d \end{bmatrix} \right) = \mathcal{R}_d.
\]

(C.77)
However, by construction, we have that \( \dim \left( N(\tilde{A}_d - I\lambda_i) \right) = n_y \), so that the rank deficiency in \( \tilde{A}_d - I\lambda_i \) is precisely the number of measurements. Therefore, it must be that in \( [\tilde{A}_d - I\lambda_i \ L_d] \), the rank deficiency is recovered through the columns of \( L_d \), and this implies that \( \dim (R(L_d)) = n_y \) and \( R(\tilde{A}_d - I\lambda_i) \cap R(L_d) = 0 \). 

Using this result, we can now show that the multiplicity of any mode \( \lambda_i \in \sigma(A_d) \cap \sigma(A_c) \) is exactly \( n_i n_y \), and that eq. \( (C.32) \) is satisfied as well. With the observer-controller realization in eq. \( (C.74) \), assume there exists \( \{x, x_d\} \) with \( x \neq 0 \), such that

\[
\begin{bmatrix}
\tilde{A} - I\lambda_i - \tilde{L}_x (\tilde{C}_y + \tilde{D}_y K_x) & 0 \\
-L_d(\tilde{C}_y + \tilde{D}_y K_x) & \tilde{A}_d - I\lambda_i
\end{bmatrix}
\begin{bmatrix}
x \\
x_d
\end{bmatrix} = 0,
\]

which would imply that the controller has more than \( n_y \) zero-order eigenvectors associated with the mode \( \lambda_i \). Note that, for the lower row to hold, \( R(\tilde{A}_d - I\lambda_i) \cap R(L_d) = 0 \) implies that \( (\tilde{A}_d - I\lambda_i)x_d = 0 \), and then \( \dim (R(L_d)) = n_y \) requires that \( (\tilde{C}_y + \tilde{D}_y K_x)x = 0 \). Applying this to the upper row, it means that \( (\tilde{A} - I\lambda_i)x = 0 \), but then since the gain \( K_x \) is stabilizing, it must be that \( x = 0 \) \( (\Rightarrow \Leftarrow) \). Therefore, the multiplicity of any mode \( \lambda_i \in \sigma(A_d) \cap \sigma(A_c) \) is exactly \( n_i n_y \).

We now assume that there exists \( \{x, x_d, y\} \) with \( y \neq 0 \), such that

\[
\begin{bmatrix}
\tilde{L}_x \\
L_d
\end{bmatrix}
\begin{bmatrix}
y \\
x
\end{bmatrix} = \begin{bmatrix}
\tilde{A} - I\lambda_i - \tilde{L}_x (\tilde{C}_y + \tilde{D}_y K_x) & 0 \\
-L_d(\tilde{C}_y + \tilde{D}_y K_x) & \tilde{A}_d - I\lambda_i
\end{bmatrix}
\begin{bmatrix}
x \\
x_d
\end{bmatrix},
\]

which would imply that \( R(A_c - I\lambda_i) \cap R(B_c) \neq 0 \). Again, since \( R(\tilde{A}_d - I\lambda_i) \cap R(L_d) = 0 \) and \( N(L_d) = 0 \), the lower row shows that \( (\tilde{A}_d - I\lambda_i)x_d = 0 \) and \( (\tilde{C}_y + \tilde{D}_y K_x)x = -y \). This then implies we must have

\[
\begin{bmatrix}
\tilde{L}_x \\
L_d
\end{bmatrix}
\begin{bmatrix}
y \\
x
\end{bmatrix} = \begin{bmatrix}
\tilde{A} - I\lambda_i - \tilde{L}_x (\tilde{C}_y + \tilde{D}_y K_x) \\
-L_d(\tilde{C}_y + \tilde{D}_y K_x)
\end{bmatrix}
x,
\]

\[
\Rightarrow 0 = (\tilde{A} - I\lambda_i)x.
\]
But again, since $\tilde{A}$ is stable, it must be the case that $x = 0$, and hence $-y = (\tilde{C}_y + \tilde{D}_y K_x)x = 0 \iff 0$. Therefore, we have

$$N(B_c) = N\left( \begin{bmatrix} \tilde{L}_x \\ L_d \end{bmatrix} \right) = 0, \quad \therefore N(L_d) = 0, \quad (C.81a)$$

$$\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_c)$$

$$= \mathcal{R}\left( \begin{bmatrix} \tilde{A} - I\lambda_i - \tilde{L}_x(\tilde{C}_y + \tilde{D}_y K_x) & 0 \\ -L_d(\tilde{C}_y + \tilde{D}_y K_x) & \tilde{A}_d \end{bmatrix} \right) \cap \mathcal{R}\left( \begin{bmatrix} \tilde{L}_x \\ L_d \end{bmatrix} \right) = 0, \quad (C.81b)$$

and our construction of the observer-controller satisfies the IMP.

### C.2 Selective Rejection

We now consider the case where rejection can only be achieved at a subset $y_1$ of the measurements $y = [y_1^T \ y_2^T]^T$, and where dim$(y_1) = n_1$, dim$(y_2) = n_2$. All the manipulations and considerations (e.g., $\sigma(A_L) \cap \sigma(A_d) = \emptyset$ leading to the recursion in eq. (C.26), and stability implying non-singularity) leading to eq. (C.38) still hold. With the partition $y = [y_1^T \ y_2^T]^T$, eq. (C.38) becomes

$$\begin{bmatrix} A - I\lambda_i & B_u C_c & 0 & 0 \\ 0 & A_c - I\lambda_i & B_{c1} & B_{c2} \\ C_{y1} & D_{y1} C_c & -I & 0 \\ C_{y2} & D_{y2} C_c & 0 & -I \end{bmatrix} \begin{bmatrix} x_{i,0} \\ x_{i,0} \\ y_{i,0} \\ y_{i,0} \end{bmatrix} = \begin{bmatrix} B_d x_{di,0} \\ 0 \\ C_{d1} x_{di,0} \\ C_{d2} x_{di,0} \end{bmatrix}, \quad (C.82a)$$

$$\begin{bmatrix} A - I\lambda_i & B_u C_c & 0 & 0 \\ 0 & A_c - I\lambda_i & B_{c1} & B_{c2} \\ C_{y1} & D_{y1} C_c & -I & 0 \\ C_{y2} & D_{y2} C_c & 0 & -I \end{bmatrix} \begin{bmatrix} x_{i,j} \\ x_{i,j} \\ y_{i,j} \\ y_{i,j} \end{bmatrix} = \begin{bmatrix} x_{i,j-1} \\ x_{i,j-1} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} B_d x_{di,j} \\ 0 \\ C_{d1} x_{di,j} \\ C_{d2} x_{di,j} \end{bmatrix}. \quad (C.82b)$$

Now, we need $y_{1i,0} = 0$ to be the only solution of the second row in eq. (C.82a); i.e.,

$$(A_c - I\lambda_i) x_{ci,0} + [B_{c1} \ B_{c2}] \begin{bmatrix} y_{1i,0} \\ y_{2i,0} \end{bmatrix} = 0 \Rightarrow y_{1i,0} = 0,$$  \quad (C.83)
and this holds if and only if

\[ N(B_{c1}) = 0, \quad (C.84a) \]
\[ \mathcal{R}([A_c - I\lambda_i \ B_{c2}]) \cap \mathcal{R}(B_{c1}) = 0. \quad (C.84b) \]

This means that

\[
\dim(\mathcal{X}_c) - n_1 \geq \dim(\mathcal{R}([A_c - I\lambda_i \ B_{c2}])),
= \dim(\mathcal{R}(A_c - I\lambda_i)) + \dim(\mathcal{R}(B_{c2}))
- \dim(\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_{c2})),
\]
\[
\iff \dim(N(A_c - I\lambda_i)) \geq n_1 + \dim(\mathcal{R}(B_{c2}))
- \dim(\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_{c2})). \quad (C.85)
\]

At the same time for stability, we need these modes to be stabilizable

\[
\dim(\mathcal{X}_c) = \dim(\mathcal{R}([A_c - I\lambda_i \ B_{c1} \ B_{c2}])),
\Rightarrow \dim(\mathcal{X}_c) - n_1 \leq \dim(\mathcal{R}([A_c - I\lambda_i \ B_{c2}])),
= \dim(\mathcal{R}(A_c - I\lambda_i)) + \dim(\mathcal{R}(B_{c2}))
- \dim(\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_{c2})),
\]
\[
\iff \dim(N(A_c - I\lambda_i)) \leq n_1 + \dim(\mathcal{R}(B_{c2}))
- \dim(\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_{c2})), \quad (C.86)
\]

so that the multiplicity of the disturbance model must in fact be

\[ n_s \triangleq n_1 + \dim(\mathcal{R}(B_{c2})) - \dim(\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_{c2})). \quad (C.87) \]

We now have that the controller must have a zero-order eigenspace with dimension \( n_s \), and that all solutions of the second row in eq. (C.82) must have \( y_{1i,j_i} = 0 \). In this case, eq. (C.82b)
becomes

\[
\begin{bmatrix}
A - I\lambda_i & B_u C_c & 0 \\
0 & A_c - I\lambda_i & B_c \\
C_{y1} & D_{y1} C_c & 0 \\
C_{y2} & D_{y2} C_c & -I
\end{bmatrix}
\begin{bmatrix}
x_{i,j_i} \\
x_{ci,j_i} \\
y_{2i,j_i} \\
0
\end{bmatrix}
= \begin{bmatrix}
x_{i,j_i-1} \\
x_{ci,j_i-1} \\
0 \\
0
\end{bmatrix} - \begin{bmatrix}
0 \\
0 \\
C_{d1} x_{di,j_i} \\
C_{d2} x_{di,j_i}
\end{bmatrix},
\] (C.88a)

where, for a stable loop, the matrix on the left must be injective. Hence, there is a particular disturbance coupling that corresponds to \(y_{2i,j_i} = 0\). Achieving regulation for this coupling then leads to considerations essentially identical to the non-selective case, that require \(n_s\) linearly independent chains of eigenvectors for each mode of the disturbance model. So, for this particular case, we find that eq. (C.82) now reduces to

\[
x_{ci,j_i} \in \mathcal{V}_{i,j_i} = \text{span}\{x_{ci,j_i,\ell}, \ell \in [1,n_s]\},
\] (C.89a)

\[
\begin{bmatrix}
A - I\lambda_i & B_u C_c \\
C_{y1} & D_{y1} C_c
\end{bmatrix}
\begin{bmatrix}
x_{i,j_i} \\
x_{ci,j_i}
\end{bmatrix}
= \begin{bmatrix}
x_{i,j_i-1} \\
x_{ci,j_i-1}
\end{bmatrix} - \begin{bmatrix}
B_d x_{di,j_i} \\
C_{d1} x_{di,j_i}
\end{bmatrix}.
\] (C.89b)

And, this holds for all possible disturbance couplings only if

\[
\begin{bmatrix}
A - I\lambda_i & B_u \\
C_{y1} & D_{y1}
\end{bmatrix}
\begin{bmatrix}
\mathcal{X} \\
C_c \mathcal{V}_{i,j_i}
\end{bmatrix}
= \begin{bmatrix}
\mathcal{X} \\
\mathcal{Y}_1
\end{bmatrix},
\] (C.90)

where \(\mathcal{Y}_1\) is the space in which the output subset \(y_1\) can take its values. Hence, we have the following statement of the IMP for selective rejection.

**Theorem C.2.1** Disturbance rejection occurs for a subset \(y_1\) of the measurements \(y\), for all possible couplings \(\{B_d,C_d\}\), if and only if the loop is stable and:

1. (controller structure for regulation)

   \[
   N(B_{c1}) = 0,
   \] (C.91a)

   \[
   \mathcal{R}(\{A_c - I\lambda_i B_c\}) \cap \mathcal{R}(B_{c1}) = 0,
   \] (C.91b)
(2) (internal model) For each chain \( \{x_{di,j_i}, \ j_i \in [0, n_i - 1], \ i \in [1, n_\lambda] \} \) of disturbance model eigenvectors, the controller must have

\[
n_s \geq n_1 + \dim(\mathcal{R}(B_c)) - \dim(\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_c)),
\]

(C.92)

mutually-independent chains of generalized eigenvectors \( x_{ci,j_i,\ell} \). More precisely, there must be a linearly independent set \( \{x_{ci,j_i,\ell}, \ j_i \in [0, n_i - 1], \ i \in [1, n_\lambda], \ \ell \in [1, n_s]\} \) such that

\[
A_c x_{ci,0,\ell} = \lambda_i x_{ci,0,\ell},
\]

(C.93a)

\[
A_c x_{ci,j_i,\ell} = \lambda_i x_{ci,j_i,\ell} + x_{ci,j_i-1,\ell},
\]

(C.93b)

(3) and (transmission zero) define

\[
\mathcal{V}_{i,j_i} = \text{span}\{x_{ci,j_i,\ell}, \ \ell \in [1, n_s]\},
\]

(C.94)

then the controller and plant together must satisfy

\[
\begin{bmatrix}
A - I\lambda_i & B_u \\
C_{y1} & D_{y1}
\end{bmatrix}
\begin{bmatrix}
\mathcal{X} \\
\mathcal{V}_{i,j_i}
\end{bmatrix} =
\begin{bmatrix}
\mathcal{X} \\
\mathcal{Y}_1
\end{bmatrix}.
\]

(C.95)

A formal proof of this theorem, would closely follow that of theorem C.1.4 and so is omitted.

This result shows that the multiplicity of the disturbance model must be precisely \( n_s \), whereas prior results \([60, 20]\) only give the lower bound \( n_1 \). We note that the requirements of the theorem are easily satisfied with the minimum multiplicity \( n_s = n_1 \) by using output augmentation. However, they complicate the construction of a DAC-type observer-controller that will achieve disturbance rejection.

C.2.1 Design of DAC Type Observer-Controllers for Selective Rejection

In this section, we show how the observer-controller approach in Section C.1.3 can be modified to achieve selective rejection. In the most straightforward construction, the observer is again augmented with \( n_y = n_1 + n_2 \) copies of the disturbance model. However, even though the model is augmented with \( n_y \) copies of \( A_d \), the end result is that the resulting observer-controller dynamics
only retains a multiplicity of $n_1$ in those dynamics. We show below, that the extra copies allow us to arrange for \( \dim (\mathcal{R} (A_c - \lambda_i) \cap \mathcal{R} (B_{c2})) = n_2 \) as required by the IMP. Also, as shown at the end of Section C.1.3.3, using \( n_y \) copies of the disturbance model in the observer dynamics results in a full rank \( B_c \), and hence, it will also provide \( \mathcal{R} ([A_c - \lambda_i B_{c2}] \cap \mathcal{R} (B_{c1}) = 0 \). To facilitate the construction, we further partition the nominal model as

\[
\begin{bmatrix}
  x_t(k + 1) \\
  x_d1(k + 1) \\
  x_d2(k + 1)
\end{bmatrix}
= \begin{bmatrix}
  \tilde{A} & \tilde{B}_{d1} & \tilde{B}_{d2} \\
  0 & \tilde{A}_{d1} & 0 \\
  0 & 0 & \tilde{A}_{d2}
\end{bmatrix}
\begin{bmatrix}
  x_t(k) \\
  x_d1(k) \\
  x_d2(k)
\end{bmatrix}
+ \begin{bmatrix}
  \tilde{B}_u \\
  0 \\
  0
\end{bmatrix}
\begin{bmatrix}
  u(k)
\end{bmatrix}
\tag{C.96a}
\]

\[
\begin{bmatrix}
  y_1(k) \\
  y_2(k)
\end{bmatrix}
= \begin{bmatrix}
  \tilde{C}_{y1} & \tilde{C}_{d11} & \tilde{C}_{d12} \\
  \tilde{C}_{y2} & \tilde{C}_{d21} & \tilde{C}_{d22}
\end{bmatrix}
\begin{bmatrix}
  x_t(k) \\
  x_d1(k) \\
  x_d2(k)
\end{bmatrix}
+ \begin{bmatrix}
  \tilde{D}_{y1} \\
  \tilde{D}_{y2}
\end{bmatrix}
\begin{bmatrix}
  u(k)
\end{bmatrix}.
\tag{C.96b}
\]

The modified construction is summarized as follows:

- Design a stabilizing state-feedback \( K_x \) for the nominal model \( \{\tilde{A}, \tilde{B}_u\} \).
- The disturbance model \( \tilde{A}_d \) is partitioned into a block diagonal form with matrices \( \tilde{A}_{d1} \) and \( \tilde{A}_{d2} \) that contain \( n_1 \) and \( n_2 \) copies, respectively, of the disturbance model \( A_d \).
- Choose \( \tilde{B}_d = [\tilde{B}_{d1} \, \tilde{B}_{d2}] \) with \( \tilde{B}_{d2} = 0 \).
- Choose \( \tilde{C}_{d22} \) so that the disturbance model \( A_{d2} \) is observable at \( y_2 \), and set \( \tilde{C}_{d12} = 0 \).
- Design the disturbance gain \( K_{d1} \) and disturbance coupling \( C_{d11} \) for the nominal subsystem \( \{\tilde{A}, \tilde{B}_u, \tilde{B}_{d1}, \tilde{C}_{y1}, \tilde{C}_{d11}, \tilde{D}_{y1}\} \), and set \( K_d = [K_{d1} \, 0] \).
- With the solution of the regulator equation \( \tilde{X}_1 \) used to design \( K_{d1} \), set \( \tilde{C}_{d21} = -(\tilde{C}_{y2}\tilde{X}_1 + \tilde{D}_{y2}K_{d1}) \).
- Design a stabilizing observer gain for the nominal system augmented with the disturbance model \( \tilde{A}_d \) and disturbance model couplings chosen above.
We now show that these choices satisfy the IMP for selective rejection. With the partition $y = [y_1^T \ y_2^T]^T$ introduced earlier, if we define

$$\tilde{C}_{y1} \triangleq \tilde{C}_{y1} + \tilde{D}_{y1} K_x,$$  \hspace{1cm} (C.97a)  
$$\tilde{C}_{y2} \triangleq \tilde{C}_{y2} + \tilde{D}_{y2} K_x,$$  \hspace{1cm} (C.97b)  
$$\tilde{C}_{d11} \triangleq \tilde{C}_{d11} + \tilde{D}_{y1} K_{d1},$$  \hspace{1cm} (C.97c)  

then the observer-controller dynamics take the form

$$\begin{bmatrix}
\tilde{x}(k+1) \\
\tilde{x}_{d1}(k+1) \\
\tilde{x}_{d2}(k+1)
\end{bmatrix} = \begin{bmatrix}
\tilde{A} & \tilde{B}_{d1} + \tilde{B}_u K_{d1} & 0 \\
0 & \tilde{A}_{d1} & 0 \\
0 & 0 & \tilde{A}_{d2}
\end{bmatrix} \begin{bmatrix}
\tilde{x}(k) \\
\tilde{x}_{d1}(k) \\
\tilde{x}_{d2}(k)
\end{bmatrix}$$  
$$- \begin{bmatrix}
L_{x1} & L_{x2} \\
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix} \begin{bmatrix}
\tilde{C}_{y1} & \tilde{C}_{d11} & 0 \\
\tilde{C}_{y2} & \tilde{C}_{d21} + \tilde{D}_{y2} K_{d1} & \tilde{C}_{d22}
\end{bmatrix} \begin{bmatrix}
\tilde{x}(k) \\
\tilde{x}_{d1}(k) \\
\tilde{x}_{d2}(k)
\end{bmatrix}$$  
$$+ \begin{bmatrix}
L_{x1} & L_{x2} \\
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix} \begin{bmatrix}
y_1(k) \\
y_2(k)
\end{bmatrix},$$  \hspace{1cm} (C.98)  

In this scheme, $\tilde{A}_{d2}$ does not represent disturbances that are to be rejected at the output $y_1$, so we chose $\tilde{B}_{d2} = 0$ from the start. Here we have also assumed that $K_{d} = [K_{d1} \ 0]$, since it is only possible to achieve rejection at the outputs in $y_1$, and the multiplicity of the disturbance model in $\tilde{A}_{d1}$ should be adequate to do so. Therefore, as in Section C.1.3, $K_{d1}$ and $\tilde{C}_{d11}$ are designed so that there is a solution to the regulator equation

$$\begin{bmatrix}
\tilde{A} & \tilde{B}_{d1} + \tilde{B}_u K_{d1} \\
\tilde{C}_{y1} & \tilde{C}_{d11}
\end{bmatrix} \begin{bmatrix}
\tilde{X}_1 \\
I
\end{bmatrix} = \begin{bmatrix}
\tilde{X}_1 \\
0
\end{bmatrix} \tilde{A}_{d1},$$  \hspace{1cm} (C.99)  

and so that the modes in $\tilde{A}_{d1}$ are observable at $y_1$, and so that the transmission zero conditions in eq. (C.95) are satisfied.
We can now apply the regulator transformation

\[
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_{d1}(k) \\
\bar{x}_{d2}(k)
\end{bmatrix} =
\begin{bmatrix}
I & -\bar{X}_1 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
\bar{x}_{d1}(k) \\
\bar{x}_{d2}(k)
\end{bmatrix},
\tag{C.100}
\]

to the observer-controller dynamics to obtain

\[
A_c = \begin{bmatrix}
\bar{A} & 0 & 0 \\
0 & \bar{A}_{d1} & 0 \\
0 & 0 & \bar{A}_{d2}
\end{bmatrix} - \begin{bmatrix}
\bar{L}_{x1} & \bar{L}_{x2} \\
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix}
\begin{bmatrix}
\bar{C}_{y1} & 0 & 0 \\
\bar{C}_{y2} & 0 & \bar{C}_{d22}
\end{bmatrix},
\tag{C.101a}
\]

\[
B_c = \begin{bmatrix}
\bar{L}_{x1} & \bar{L}_{x2} \\
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix},
\tag{C.101b}
\]

where we used the fact that the choice for \(\bar{C}_{d21}\) results in 0 = \(\bar{C}_{y2}\bar{X}_1 + \bar{C}_{d21} + \bar{D}_{y2}K_{d1}\). It should be clear now, that the construction contains at least the minimum multiplicity \(n_1\) of disturbance models required.

Also, by lemma C.1.5, any stabilizing observer gain gives

\[
\dim \left( \mathcal{R} \left( \begin{bmatrix}
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix} \right) \right) = n_1 + n_2 = n_y,
\tag{C.102a}
\]

\[
\mathcal{R} \left( \begin{bmatrix}
A_{d1} - I\lambda_i & 0 \\
0 & A_{d2} - I\lambda_i
\end{bmatrix} \right) \cap \mathcal{R} \left( \begin{bmatrix}
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix} \right) = 0,
\tag{C.102b}
\]

and it will then follow (as in Section C.1.3.3) that \(\mathcal{R} ([A_c - \lambda_i B_{c2}] \cap \mathcal{R} (B_{c1}) = 0\). Let \(\{x_{d1i,0,\ell_1}, \ell_1 \in [1, n_1]\}\) and \(\{x_{d2i,0,\ell_2}, \ell_2 \in [1, n_2]\}\) denote the eigenvectors of \(\bar{A}_{d1}\) and \(\bar{A}_{d2}\), respectively. We can now show that \(\dim (\mathcal{R} (A_c - I\lambda_i) \cap \mathcal{R} (B_{c2})) = n_2\), and only the modes in \(\bar{A}_{d1}\) are retained in the
observer-controller. Because we set \( \hat{C}_{d22} = 0 \), the vectors \([0^T \ 0^T \ x_{d2i,0,t_2}^T]\) are such that

\[
(A_c - I\lambda_i) \begin{bmatrix}
0 \\
0 \\
x_{d2i,0,t_2}
\end{bmatrix} = - \begin{bmatrix}
\bar{L}_{x1} & \bar{L}_{x2} \\
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix} \begin{bmatrix}
0 \\
0 \\
\hat{C}_{d22} x_{d2i,0,t_2}
\end{bmatrix}
\]

\(\in \mathcal{R} \left( \begin{bmatrix}
\bar{L}_{x2} \\
L_{d12} \\
L_{d22}
\end{bmatrix} \right) = B_c2.\) \(\text{(C.103b)}\)

Since \( \hat{C}_{d22} \) is chosen so that all modes \( \lambda_i \in \sigma(\bar{A}_{d2}) \) are observable at \( y_2 \),

\[
\hat{C}_{d22} x_{d2i,0,t_2} \neq 0,
\]

generates \( n_2 \) linearly-independent vectors in eq. (C.103a), so that

\[
n_s = n_1 + \dim(\mathcal{R}(B_{c2})) - \dim(\mathcal{R}(A_c - I\lambda_i) \cap \mathcal{R}(B_{c2})) \]

\(= n_1.\) \(\text{(C.105b)}\)

Now note that the observer-controller dynamics in eq. (C.101a) are determined by

\[
A_c = \begin{bmatrix}
\bar{A} - L_{x1} \bar{C}_{y1} - L_{x2} \bar{C}_{y2} & 0 & -\bar{L}_{x2} \bar{C}_{d22} \\
-L_{d11} \bar{C}_{y1} - L_{d12} \bar{C}_{y2} & \bar{A}_{d1} & -L_{d12} \bar{C}_{d22} \\
-L_{d21} \bar{C}_{y1} - L_{d22} \bar{C}_{y2} & 0 & \bar{A}_{d2} - L_{d22} \bar{C}_{d22}
\end{bmatrix}
\]

\(\text{(C.106)}\)

This clearly has at least \( n_s = n_1 \) copies of the disturbance model. It remains to show that the multiplicity of each mode \( \lambda_i \in \sigma(A_d) \cap \sigma(A_c) \) is exactly \( n_i n_1 \).

By way of contradiction, assume that there exists \( \{x,x_{d1},x_{d2}\} \), with \( \{x,x_{d2}\} \neq \{0,0\} \), such that

\[
(A_c - \lambda_i) \begin{bmatrix}
x^T \\
x_{d1}^T \\
x_{d2}^T
\end{bmatrix}^T = 0,
\]

which then implies the controller has more than \( n_1 \) zero-order eigenvectors associated with the
mode $\lambda_i$. However, eq. (C.101a) shows that this can be written as

$$
\begin{bmatrix}
\bar{A} - I\lambda_i & 0 & 0 \\
0 & \bar{A}_{d1} - I\lambda_i & 0 \\
0 & 0 & \bar{A}_{d2} - I\lambda_i
\end{bmatrix}
\begin{bmatrix}
x \\
x_{d1} \\
x_{d2}
\end{bmatrix} =
\begin{bmatrix}
\bar{L}_{x1} & \bar{L}_{x2} \\
L_{d11} & L_{d12} \\
L_{d21} & L_{d22}
\end{bmatrix}
\begin{bmatrix}
\bar{C}_{y1} & 0 & 0 \\
\bar{C}_{y2} & 0 & \bar{C}_{d22}
\end{bmatrix}
\begin{bmatrix}
x \\
x_{d1} \\
x_{d2}
\end{bmatrix}.
$$

(C.108)

Since eq. (C.102) holds, the lower two rows in eq. (C.108) show that both sides must in fact be zero. From the left side, this then implies that $x = 0$ (because $\bar{A}$ is stable), and that $x_{d1}$ and $x_{d2}$ are eigenvectors of $\bar{A}_{d1}$ and $\bar{A}_{d2}$, respectively. From the right side, we then have that $\bar{C}_{d22}x_{d2} = 0$, so that $x_{d2}$ is an unobservable eigenvector of $\bar{A}_{d2}$. But, this is impossible if the disturbance model is chosen to be observable at $y_2$ ($\Rightarrow\Leftarrow$). Therefore, the multiplicity of each mode $\lambda_i \in \sigma(A_d) \cap \sigma(A_c)$ is exactly $n_i n_1$, and the observer-controller satisfies the IMP.
Appendix  D

Spectral Models for Wind Evolution

This appendix describes the spectral method for emulating wind evolution developed by Eric Simley. It appears to be a more spatially complete version of the method first presented in [11]. This technique is used to pre-process TurbSim [34] wind fields to induce changes that are spectrally consistent with the model used by TurbSim to generate the field. The “evolved” wind field is then used as the source for preview measurements. The following is an excerpt provided by Eric Simley from a paper coauthored with myself and submitted to the 2013 American Control Conference [31].

TurbSim provides a number of spectral models to generate wind velocity distributions that are representative of various atmospheric conditions. A relatively simple model, which is used in this thesis, is the von Karman model, defined in [34]. A mean wind speed of $U = 18$ m/s and a turbulence intensity of 17.5% are used for simulation. TurbSim implements this model in a two-dimensional plane that (assuming no yaw error) is parallel with the turbine’s rotor—this is a plane in the transverse and vertical directions, or left-right and up-down, when facing the rotor. At a finite number of grid points within the two-dimensional plane, wind speed time series are generated for the longitudinal $u$ component perpendicular to the rotor plane, the transverse $v$ component, and the vertical $w$ component.

When determining wind speeds at locations upstream of the rotor plane, the FAST aeroelastic simulator [35] uses Taylor’s frozen turbulence hypothesis [62] and simply shifts the rotor-plane wind speeds forward in time by $d/U$ where $d$ is the upstream distance and $U$ is the mean wind speed. In order to create a more realistic preview measurement simulation, we generate a
circle of upstream measurements of the $u$ component that are properly correlated with the original wind speeds to model wind evolution. The resulting wind field is shown in Fig. D.1, where the blue points indicate the $N$ wind speed locations in the original TurbSim wind field and the $M$ red points represent the circle of preview measurements upstream of the rotor. The preview distance is $d$ and measurements are provided at a rotor radius of $r$. In general, preview measurements are determined by azimuthally linearly interpolating between the $M$ measurement locations.

**D.0.2 Spatial Coherence**

The correlation between wind speeds at different locations in a wind field can be described using spatial coherence functions. Coherence describes the correlation between two signals as a function of frequency. Magnitude-squared coherence, the definition used in this thesis, between signals $a$ and $b$ is defined as

$$
\gamma_{ab}^2(f) = \frac{|S_{ab}(f)|^2}{S_{aa}(f) S_{bb}(f)},
$$  \hspace{1cm} (D.1)

where $S_{aa}(f)$ and $S_{bb}(f)$ are the power spectral density (PSD) functions for signals $a$ and $b$ respectively, and $S_{ab}(f)$ is the cross-power spectral density (CPSD) between signals $a$ and $b$. CPSD [67] is defined as

$$
S_{ab}(f) = \int_{-\infty}^{\infty} R_{ab}(\tau) e^{-j2\pi f \tau} d\tau
$$  \hspace{1cm} (D.2)

where $R_{ab}(\tau)$ is the time-domain cross-correlation function between the stochastic signals $a$ and $b$. Cross-correlation is further defined as

$$
R_{ab}(\tau) = E[a(t) b^*(t-\tau)].
$$  \hspace{1cm} (D.3)

The von Karman spectral model contains the spatial coherence function for separations in the transverse and vertical directions, given by the IEC 61400-1 3rd ed. standard, and defined in [34]. To introduce wind evolution, an additional spatial coherence function for wind speeds separated in the longitudinal direction is implemented. The longitudinal coherence function used here is an analytic model for a neutral boundary layer provided by Kristensen [41]. Longitudinal coherence
is given by equation 20 in [41]:

\[
\gamma_{u_i u_j, \text{long}}^2(f) = e^{-2\alpha G(f\ell/U)} \left(1 - e^{-\left(2\alpha^2 (f\ell/U)^2\right)^{-1}}\right)^2
\]  

(D.4)

where

\[
G(f\ell/U) = (33)^{-2/3} \frac{\left(33 f\ell/U\right)^2 (33 f\ell/U + 3/11)^{1/2}}{(33 f\ell/U + 1)^{11/6}}
\]

and

\[
\alpha = \frac{\sigma}{U} \frac{d}{\ell}.
\]

(D.5)

\[\ell\] is the length scale of the turbulence, \(d\) is the longitudinal separation, and \(\sigma\) is related to the total turbulent kinetic energy as

\[
\frac{\sigma^2}{2} = \int_0^\infty E(f) \, df
\]

(D.7)

where \(E(f)\) is the energy spectrum. The length scale \(\ell\) is assumed to be equal to the hub height of the turbine \(HH = 37\) m. \(\sigma\) is set equal to \(3.15\sqrt{3}\), where 3.15 is the standard deviation of the three wind components (corresponding to 17.5% turbulence intensity). The longitudinal coherence in (D.4) is shown in Fig. D.2 for five longitudinal separations between 1 m and 100 m.

The coherence function \(\gamma_{u_i u_j, \text{long}}^2(f)\) in (D.4) describes how wind speeds are correlated along the longitudinal direction. Similarly, the coherence in the transverse and vertical directions \(\gamma_{u_i u_j, \text{tran+vert}}^2(f)\) is defined by the IEC coherence function implemented in TurbSim and defined in [34]. For wind speeds at locations \(i\) and \(j\) separated in both the longitudinal and transverse/vertical directions, the coherence is defined as the product

\[
\gamma_{u_i u_j}^2(f) = \gamma_{u_i u_j, \text{tran+vert}}^2(f) \gamma_{u_i u_j, \text{long}}^2(f)
\]

(D.8)

This simple form of correlation for a general three-dimensional spatial separation is equivalent to assuming that wind speeds are correlated independently in the longitudinal and transverse/vertical directions.
D.0.3 Generating the Wind Field

The wind speed preview measurements are properly correlated with a TurbSim wind field, located at the rotor plane, using an extension of the Veers method [72] implemented in TurbSim. The method implemented in TurbSim involves finding the frequency domain representation of all the wind speed signals, followed by using the inverse Fourier transform to create the time series. For each frequency bin in the Fourier representation of the wind speeds, the frequency components are correlated according to the spatial coherence model. The proper amplitude, given by the PSD of the spectrum model, is then applied. For the von Karman model, in which the PSD $S_{uu}(f)$ of the $u$ component is the same at all locations, the vector of $u$ component frequency components $U_N(f)$ at the $N$ wind speed locations is calculated using the following matrix operation

$$U_N(f) = \sqrt{S_{uu}(f)}L_N(f)z_N(f). \quad (D.9)$$

$L_N(f)$ is a $N \times N$ matrix and $z_N(f)$ is a $N \times 1$ vector of uncorrelated unity-magnitude complex numbers with uniformly distributed random phase. The matrix multiplication $L_N(f)z_N(f)$ properly correlates the random-phase frequency components provided by $z_N(f)$ according to the coherence model. $L_N(f)$ is a lower-triangular matrix obtained by finding the Cholesky decomposition of a correlation matrix $\Gamma_N(f)$:

$$\Gamma_N(f) = L_N(f)L_N^T(f), \quad (D.10)$$

where the components of $\Gamma_N(f)$ are defined as

$$\Gamma_{N_{i,j}}(f) = \sqrt{\gamma_{u_i,u_j,\text{tran+vert}}(f)}, \quad 1 \leq i, j \leq N. \quad (D.11)$$

By multiplying the vector of correlated wind speed components by $\sqrt{S_{uu}(f)}$, the von Karman PSD is introduced. Once $U_N(f)$ is calculated for all frequency bins, the inverse Fourier transform is used to find the wind speed time series at each location. By initializing the frequency-domain calculations with a $z_N(f)$ vector containing random phases, many realizations of the wind field adhering to the proper spectrum and coherence models can be generated. The $v$ and $w$ components are calculated using the same technique.
An extension of this frequency-domain method is used to find $M$ preview measurements that are properly correlated with an existing TurbSim wind field. This extended method involves calculating the random unity-magnitude frequency components $z_N(f)$ in (D.9) for each frequency, given the vector $U_N(f)$:

$$z_N(f) = \sqrt{S_{uu}(f)} L_N^{-1}(f) U_N(f).$$  \hfill (D.12)

Once $z_N(f)$ is found, the vector of frequency components $U_M(f)$ at the $M$ measurement locations is calculated such that all $N+M$ wind speeds are properly correlated according to the coherence relationship given in (D.8). This is achieved using the matrix operation

$$U_M(f) = W(f) L(f) \begin{bmatrix} z_N(f) \\ z_M(f) \end{bmatrix},$$  \hfill (D.13)

where $z_M(f)$ is a $M \times 1$ vector of uncorrelated unity-magnitude complex numbers with uniformly distributed phase. The $(N + M) \times (N + M)$ lower-triangular matrix $L(f)$, used to correlate the frequency components for all $N+M$ wind speeds, is determined by finding the Cholesky decomposition of the correlation matrix

$$\Gamma(f) = L(f) L^T(f)$$  \hfill (D.14)

where the elements of $\Gamma(f)$ are defined as

$$\Gamma_{i,j}(f) = \sqrt{\gamma_{u_i,u_j}^2(f)}, \quad 1 \leq i,j \leq N + M,$$  \hfill (D.15)

using the coherence structure in (D.8). $\Gamma(f)$ is structured such that indices 1 through $N$ represent the original TurbSim wind speed locations and indices $N + 1$ through $N + M$ represent the measurement locations. Thus,

$$\Gamma_{1...N,1...N}(f) = \Gamma_N(f).$$  \hfill (D.16)

Finally, the proper von Karman spectrum magnitude is applied to the correlated wind speed components to yield $U_M(f)$ using the $M \times (M + N)$ matrix

$$W(f) = \begin{bmatrix} 0 \\ \sqrt{S_{uu}(f)} I_M \end{bmatrix},$$  \hfill (D.17)
where $I_M$ is the $M \times M$ identity matrix.

After the inverse Fourier transform is applied, the $M$ wind speed preview measurement signals are properly correlated with each other according to the transverse and vertical coherence function implemented in TurbSim, and correlated with the original wind speed components according to the coherence structure in (D.8) with longitudinal coherence given by (D.4), simulating wind evolution.
Figure D.1: Original wind field with added preview measurements. The blue points represent the grid containing the $N$ wind speed locations, encompassing the rotor plane, in the original wind field. The red points indicate the circle of $M$ preview measurement locations for a preview distance $d$ upstream of the rotor and measurement radius $r$. The original wind field contains $u$, $v$, and $w$ components of the wind, while the measurement locations only contain the $u$ component required by the controller. The wind speed measurements are correlated with the original wind field to simulate wind evolution.

Figure D.2: Coherence curves for five longitudinal separations between 1 m and 100 m using the Kristensen longitudinal coherence formula given by (D.4). The length scale parameter is set equal to the hub height of 37 m and $\sigma$ is $3.15\sqrt{3}$. 