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Problems Concerning Spatial Branching Particle Systems with Interaction

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by

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Problems concerning spatial branching particle systems with interaction

Thesis directed by Prof. Janos Englander

In this dissertation, we study the behavior of the particle system of a branching process. In particular, we focus on the branching Brownian motion with some drift. For the center of the system, we describe its asymptotic behavior which depends on the drift. Then we discuss the limiting behavior for the relative system. Combining the center and the relative system, we give the Law of Large Numbers for the particle system.

Dedication

To my parents.

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Contents

Chapter 1

Introduction

1.1 Backgroud

Here we need to give some background and introduction.

The motivation for this dissertation is coming from the paper [5] . To approach our model we need some preliminary knowledge.

Here we give a short overview of the fundamental knowledge, including: Brownian motion, SDE, O-U process, martingale, branching motion, ...

First we give an overview on Brownian motion(sometimes called Wiener process). In this thesis we use B_t or W_t for the standard Brownian motion. Brownian motion is the most fundamental and common stochastic process. We will use it frequently in our model. Brownian Motion is defined in the following way:

Definition 1.1.1. *Brownian motion is a stochastic process* $\{X(t); 0 \leq t < \infty\}$ *with the following properties:*
(a) Every increment $X(t+s) - X(s)$ *is normallly distributed with mean 0 and variance* t.
(b) For every par of disjoint time intervals $[t_1, t_2], [t_3, t_4]$*,say* $t_1 < t_2 \leq t_3 < t_4$*, the increments* $X(t_4) - X(t_3)$ *and* $X(t_2) - X(t_1)$ *are independent random variables with distributions given in (a), and similarly for* n *disjoint time intervals where* n *is an arbitrary positive integer.*
(c) $X(0) = 0$ *and* $X(t)$ *is continuous at time* $t = 0$.

Another very frequently used concept is that of a Stochastic Differential Equation (SDE in short). In fact, SDEs will be interpreted as integral equations involving "stochastic integrals". In general an SDE will be in the following form:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

where B_t is a Brownian motion. It means the following integral equation:

$$X_t = \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s$$

For understanding the integral, it means "stochastic integral". There are several books where one can find more material on SDEs. Here I just skip the details about how the stochastic integral works.

Brownian motion is well studied. There are a lot of well known theorems about Brownian motion. Next we will introduce another fundamental stochastic process which is a little more complicated which is called Ornstein-Uhlenbeck process (O-U process , in short).
To start, we need some knowledge about Stochastic Differential Equation. Consider the following SDE:

$$dX_t = \frac{1}{2}\sigma^2 dB_t - \mu X_t dt$$

where $\sigma, \mu > 0$. Then the solution for this SDE is called an O-U process (sometimes called "mean-reverting process"). As we see, the drift term $-\mu X_t dt$ will pull the motion back to origin. So in our model, we also call it an "inward O-U" process. In our model we also consider the $\mu < 0$ case which means the drift tends to push the motion to infinity. And we call it an "outward O-U" process. For inward O-U process, it is positive recurrent. For outward O-U process, it behaves differently as it is a transient process. As the linearly growing drift suggests, it has a large radial speed.

Next we give a review on martingales. Martingales are very useful tools in stochastic analysis.

Definition 1.1.2. *Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ where $0 \le t < \infty$. Then a stochastic process X is called a submartingale if*
(1) X is adapted (by which we mean $\sigma(X_t) \subset \mathcal{F}_t$) for $0 \le t < \infty$;
(2) $E|X_t| < \infty$ for any $0 \le t < \infty$;
(3) $X_s \le E(X_t|\mathcal{F}_s)$ a.s. for any $0 \le s < t$

The process X is called a supermartingale if $-X$ is a subartingale. Finally, we can give the definition of a martingale.

Definition 1.1.3. *A stochastic process X is a martingale if X is a submartingale and a supermartingale at the same time.*

We also need to give some overview on branching motions. Suppose we start with a single ancestor and individual has $X = 0, 1, 2...$ offspring with corresponding probabilities $p_0, p_1, p_2, ...$ For the branching motion, each particle will be a stochastic process and each particle will split into several

particles.
In our model, we are working on unit time, dyadic branching which means one particle will split into exactly 2 particles at each unit time.

1.2 Essential knowledge

Here we need to introduce some knowledge we will need for the following chapters. We put some theorems here. Some of them are simplified for easy understanding and the way we use in our work. The original version of the theorem may be stronger or more general.

First we need the famous Borel-Cantelli Lemma to get the main results.

Theorem 1.2.1. *Borel-Cantelli Lemma If* $\{A_n\}$ *is a sequence of events on* Ω*, and* $\sum_{n=1}^{\infty} P(A_n) < \infty$*, then* $P(\{A_n \text{ i.o.}\}) = 0$

The following theorem is a direct consequence from Borel-Cantelli lemma to obtain almost sure convergence.

Theorem 1.2.2. $X_n \to 0$ *a.s. if for any* $\epsilon > 0$*,* $\sum P(A_n) < \infty$ *where* $A_n = \{\omega || X_n(\omega)| > \epsilon\}$*.*

Proof. Firstly, $X_n \to 0$ a.s. if and only if for any $\epsilon > 0$, $P(\{|X_n| > \epsilon \text{ i.o.}\}) = 0$. (See details on [4]). From the Borel-Cantelli lemma, we know that if $\sum P(A_n) < \infty$ where $A_n = \{\omega || X_n(\omega)| > \epsilon\}$, then $P(\{|X_n| > \epsilon \text{ i.o.}\}) = 0$. Thus $X_n \to 0$ a.s. □

Now we give a more useful theorem, in terms of variance to give the result of almost sure convergence.

Theorem 1.2.3. $X_n \to 0$ *a.s. if* $\sum_{n=1}^{\infty} \mathtt{Var}(X_n) < \infty$ *and* $\mathtt{E}(X_n) = 0$ *.*

Proof. For any fixed $\epsilon > 0$, let $A_n = \{\omega || X_n(\omega)| > \epsilon\}$. Then from Chebyshev's inequality, we will have

$$P(A_n) \leq \frac{\mathtt{Var}(X_n)}{\epsilon}$$

Thus,

$$\sum P(A_n) \leq \frac{1}{\epsilon} \sum_{n=1}^{\infty} \mathtt{Var}(X_n) < \infty$$

Then from the theorem above, $X_n \to 0$ a.s. □

The following theorem is a famous one about stochastic integration and it is crucial to compute the variance of some random variables in our model.

Theorem 1.2.4. *(Itô-isometry) Denote by B_t standard Brownian motion and let $f(t)$ be a function which is integrable with respect to B_t. Let $I_t = \int_0^t f(s)\,dB_s$. Then*

$$Var(I_t) = \int_0^t f^2(s)ds$$

Next, we have a fundamental inequality.

Theorem 1.2.5. *(Chebysev's Inequality) If X is a random variable and $\epsilon > 0$, then*

$$P(|X| > \epsilon) \leq \frac{E(X^2)}{\epsilon^2}$$

Proof. From definition,

$$E(X^2) = \int_{|X|>\epsilon} X^2 dP + \int_{|X|>\epsilon} X^2 dP$$

Then,

$$\epsilon^2 P(|X| > \epsilon) = \int_{|X|>\epsilon} \epsilon^2 dP \leq \int_{|X|>\epsilon} X^2 dP + \int_{|X|>\epsilon} X^2 dP$$

Thus,

$$P(|X| > \epsilon) \leq \frac{E(X^2)}{\epsilon^2}$$

□

The following theorem is a consequence of Chebysev's Inequality and the property of submartingales.

Theorem 1.2.6. *(Doob's Inequality) Let $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a non-negative submartingale. We have*

$$P(sup\{X_s; 0 \leq s \leq t\} > \epsilon) \leq \frac{E(X^2)}{\epsilon^2}$$

The following theorem provides a relation between continuous martingale and Brownian motion

Theorem 1.2.7. *(Time-Change for Martingale [Dambis,Dubins&Schwarz]) Let M_t to be a continuous martingale and satisfy $\lim_{t\to\infty} < M >_t = \infty$, a.s. Define, for each $0 \leq s < \infty$, the stopping time*

$$T(s) = inf\{t; < M >_t > s\}$$

Then the time-changed process $B_s = M_{T(s)}, = 0 \leq s < \infty$ is a standard one -dimensional Brownian motion. We will also have a.s.

$$M_t = B_{<M>_t}; 0 \leq t < \infty$$

Proof. For the proof, one can refer [10]. □

1.3 Notation

Here we want to introduce the notations we will use in this thesis. Most notations here are usual ones.

Throughout the thesis the symbol $\overset{w}{\Rightarrow}$ (or just $\Rightarrow$) will denote *weak* convergence of finite measures; the symbol $\overset{v}{\Rightarrow}$ will denote *vague* convergence.

By a *bounded rational rectangle* we will mean a set $B \subset \mathbb{R}^d$ of the form $B = I_1 \times I_2 \times \cdots \times I_d$, where I_i is a bounded interval with rational endpoints for each $1 \leq i \leq d$. The family of all bounded rational rectangles will be denoted by $\mathcal{R}$.

The symbols $X \oplus Y$ will denote the independent sum of the random variables X and Y.

As usual, $\mathcal{N}(\mu, \sigma^2)$ will denote the normal distribution with mean μ and variance σ^2.

For $z \in \mathbb{R}$, $\lfloor z \rfloor$ will denote the largest integer which is less than or equal to z.

For a random variable X and a Borel set B, $1_B(X)$ is the indicator function, which means $1_B(X) = 1$ if $X \in B$ and $1_B(X) = 0$ otherwise. Moreover, we can also look at X as a random measure. In that sense, we denote $X(B)$ as the measure on B and define it as $X(B) = 1_B(X)$. Furthermore, if Z is a particle system which contains particles $\{Z_i\}$, $i = 1, 2, \cdots, n$, then we can similarly define the empirical measure, as $Z(B) = \sum_{i=1}^{n} Z_i(B)$.

W_t or B_t denotes the standard Brownian Motion.

$r(A)$ denotes the rank of matrix A.

Chapter 2

Model

2.1 Model

We consider a branching diffusion in $\mathbb{R}^d$, where the motion component is an Ornstein-Uhlenbeck (O-U) process, and dyadic branching occurs in each time unit. (Dyadic branching means precisely two offspring.) In addition, we introduce interaction between particles, that is, either *attraction* or *repulsion*.

Let Z denote the process and Z_t^i the i^{th} particle[1] in time $[m, m+1)$, where $m = 0, 1, 2, \ldots$. As branching is unit time, in the time interval $[m, m+1)$ there are 2^m particles in total. Without interaction from other particles, Z_t^i is an Ornstein-Uhlenbeck process with drift parameter $b \in \mathbb{R}$, corresponding to the operator

$$\frac{1}{2}\Delta - bx \cdot \nabla$$

on $\mathbb{R}^d$. (Note the negative sign of the drift. It is somewhat unusual, but it fits our setup better, because of the sign of the interaction parameter γ, introduced below.) If $b > 0$, then we have an 'inward' O-U process; if $b < 0$, then we have an 'outward' O-U process. If $b = 0$, then it is a Brownian motion.

As far as the aforementioned interaction is concerned, let us fix the *interaction parameter* $\gamma \neq 0$. We assume that the i^{th} particle Z_t^i, on the time interval $[m, m+1)$, 'feels' a drift caused by attraction/repulsion of all other particles as

$$\frac{1}{2^m} \sum_{j=1}^{2^m} \gamma \cdot (Z_t^j - \cdot) dt,$$

and so Z_t^i satisfies the following stochastic differential equation:

$$dZ_t^i = dW_t^{i,m} - bZ_t^i dt + \frac{1}{2^m} \sum_{j=1}^{2^m} \gamma \cdot (Z_t^j - Z_t^i) dt.$$

If $\gamma > 0$, then this means that particles attract each other, whereas if $\gamma < 0$, then this means that they repulse each other.

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[1] We can use an arbitrary labeling, as long as it is independent of the spatial motion.

In the stochastic differential equation above, the $\{W_t^{i,m}\}_{1\le i\le 2^m}$ are independent standard Brownian motions on $[m, m+1)$. In other words, the infinitesimal generator of the ith particle in the time interval $[m, m+1)$ is

$$\frac{1}{2}\Delta + \left(\frac{1}{2^m}\sum_{j=1}^{2^m}\gamma\cdot(Z_t^j - x) - bx\right)\cdot\nabla$$

where $-bx$ is the drift part and $\gamma\cdot(Z_t^j - x)$ is the interaction between particles, $\frac{1}{2}\Delta$ means associated standard Brownian motion part.

2.2 Motivation

Our motivation is coming from the paper [5], where a similar model was studied. There the motion was Brownian motion ($b = 0$), and it has been shown that the center of the system is a Brownian motion, being slowed down such that it tends to a 'terminal position' N almost surely, and N is a d-dimensional, normally distributed random variable, with mean zero. If P^x denotes the probability conditioned on $N = x$, $x \in \mathbb{R}^d$, then the following theorem was presented for $\gamma > 0$ (attraction):

$$2^{-n}Z_n(dy) \Rightarrow \left(\frac{\gamma}{\pi}\right)^{d/2}\exp\left(-\gamma|y-x|\right)\, dy,\ \ P^x - a.s.,$$

as $n \to \infty$ for almost all $x \in \mathbb{R}^d$, where $Z(dy)$ denotes the discrete measure-valued process corresponding to the interacting branching particle system. For $\gamma < 0$, a conjecture was stated.

A similar model for superdiffusions has been introduced and studied by Gill recently [8] and results analogous to those in [5], were obtained. The toolsets used in the two papers are very different though. Gill's paper utilizes the so-called *historical calculus* of E. Perkins.

For classical results on limit theorems for branching particle systems (without interaction), see the fundamental monograph [1], and the more recent article [7].

2.3 Existence and uniqueness

In this section, we show the existence and uniqueness for this system (process). Actually, it is easy to see[2] that we only need to show that, given the initial value in time interval $[m, m+1)$, the system exists and is unique.

Here we will need the existence and uniqueness of a general SDE for high dimensional space. One can refer [10] to get more materials about that. In our model the coefficients are 'nice' enough. So we can apply the well known result to get the existence and uniqueness of our model.

Now, in the time interval $[m, m+1)$, we can look at the 2^m interacting particles (diffusions) as a single $2^m d-$dimensional Brownian motion with a drift $\mathbf{d} : \mathbb{R}^{2^m} \to \mathbb{R}^{2^m}$defined as

$$\mathbf{d}(x_1, x_2, \cdots, x_{d-1}, x_d, \cdots, x_{2^m d}) := \gamma(\beta_1, \beta_2, \cdots, \beta_{d-1}, \beta_d, \cdots, \beta_{2^m d})^T,$$

[2] Otherwise use 'concatenating' for the processes.

where $\beta_k = 2^{-m}\gamma(x_{\overline{k}} + x_{d+\overline{k}} + \cdots + x_{(2^m-1)d+\overline{k}}) - (\gamma + b)x_k$. Here $\overline{k} - k$ is a multiple of d and $0 < \overline{k} \leq d$. As the drift $\mathbf{d}$ is Lipschitz, the existence and uniqueness of our system follows from the uniqueness/existence theorem for stochastic differential equations in high dimensions.

Chapter 3

The center of Mass

3.1 Definition

Definition 3.1.1 (Center of mass)**.** For $t \in [m, m+1)$, there are 2^m particles, denoted by $\{Z_t^i\}, i = 1, 2, \cdots, 2^m$, moving in space. Hence, letting $m := \lfloor t \rfloor$, we define the center of mass (C.O.M.) as

$$\overline{Z_t} := \frac{1}{2^m} \sum_{1}^{2^m} Z_t^i.$$

In this section we are going to show that as $t \to \infty$:

- if $b > 0$, then the center of mass converges to the origin, no matter if attraction or repulsion holds;
- if $b < 0$, then it will tend to infinity with 'speed' e^{-bt}.

The significance of this result is that the attraction/repulsion for Z_i^t is given by

$$\frac{1}{2^m} \sum_{j=1}^{2^m} \gamma(Z_t^j - Z_t^i) dt = \gamma(\overline{Z_t} - Z_t^i) dt,$$

where $\overline{Z_t}$ is as above. Hence, one can replace the interaction between particles by the interaction with the center of mass. Therefore, as a first step, we will study the large time behavior of the center of mass $\overline{Z_t}$.

Before stating our first result, we note that in this section, we will be interested in $a.s.$ and L^2 convergence of the center of mass. Since, it is easy to see that these limits can be verified coordinate-wise, we assume $d = 1$ for this section. (The reader should keep in mind that the results work for any $d \geq 1$.)

3.2 Attraction

Our first result concerns the behavior of the center of mass in the attractive case.

Theorem 3.2.1 (COM; Attraction)**.** *If* $b > 0$, *then* $\lim_{t \to \infty} \overline{Z_t} = 0$ $a.s.$

To prove this theorem, we need some preliminary lemmas first. Below we give two lemmas regarding a general one-dimensional stochastic differential equation

$$\begin{cases} dX_t &= \beta(t)dW_t - bX_t dt, \\ X_0 &= 0, \ a.s., \end{cases} \tag{2.1}$$

where we assume that $b > 0$ and that $\beta(\cdot) > 0$ is locally Lipschitz. Here $\beta(t)$ can be considered a time change of the Brownian part. We assume that $\beta(t)$ converges to 0 as $t \to \infty$, that is that the Brownian motion is slowing down completely. We then want to determine the limiting distribution.

Lemma 3.2.1. *If* $\lim_{t\to\infty} \beta(t) = 0$*, then* $\lim_{t\to\infty} X_t = 0$ *in* L^2*.*

Proof. We can solve the equation (2.1) in the following way. We first assume, that the solution $X_t = X(t)$ is of the form $X(t) = X_1(t)X_2(t)$, with $X_1(0) = 0$, and we can apply the product rule when we differentiate $X(t)$. We then have

$$dX(t) = dX_1(t)X_2(t) + X_1(t)dX_2(t).$$

Next, we can set

$$dX_1(t)X_2(t) = \beta(t)dW_t$$

and

$$X_1(t)dX_2(t) = -bX(t)dt = -bX_1(t)X_2(t)dt,$$

which means that $dX_2(t) = -bX_2(t)dt$.

We obtain $X_2(t) = Ce^{-bt}, \ C \neq 0$, and thus $dX_1(t) = C^{-1}\beta(t)e^{bt}dW_t$, which means $X_1(t) = C^{-1}\int_0^t \beta(s)e^{bs}dW_s$.

Then

$$X(t) = X_1(t)X_2(t) = e^{-bt}\int_0^t \beta(s)e^{bs}dW_s$$

satisfies (2.1). Since the coefficients $\beta(t)$ and bx in (2.1) are both locally Lipschitz functions, the solution exists and it is unique, that is, the solution above is the only solution to the equation.

Clearly, X_t is a mean zero normal random variable, as $\beta(t)$ is a deterministic function. Thus, $\lim_{t\to\infty} X_t = 0$ in L^2 is tantamount to the convergence of the variance of X_t to 0, where the Itô-isometry yields the variance: $\texttt{Var}(X_t) = e^{-2bt}\int_0^t \beta^2(s)e^{2bs}ds$. We may assume that $\lim_{t\to\infty}\int_0^t \beta^2(s)e^{2bs}ds = \infty$, otherwise the statement is trivial. Then, from L'Hospital's rule,

$$\lim_{t\to\infty} \texttt{Var}(X_t) = \lim_{t\to\infty} \frac{\int_0^t \beta^2(s)e^{2bs}ds}{e^{2bt}} = \lim_{t\to\infty} \frac{\beta^2(t)e^{2bt}}{2be^{2bt}} = \lim_{t\to\infty} \frac{\beta^2(t)}{2b}.$$

Therefore, $\lim_{t\to\infty} \beta(t) = 0$ implies $\lim_{t\to\infty} X_t = 0$ in L^2, completing the proof. □

In order to obtain almost sure convergence, we need a stronger condition on β.

Lemma 3.2.2. *Assume in addition, that $\beta(t)$ is decreasing, and that $\sum_{m=1}^{\infty} m\beta^2(m) < \infty$. Then* $\lim_{t\to\infty} X_t = 0$ *a.s.*

Remark 3.2.1. Regarding the last two lemmas, our intuition is as follows. As the Brownian motion is slowed down, for large t, the drift part will dominate. (That is, the system's behavior becomes qualitatively similar to the corresponding dynamical system's behavior.) Since the drift always 'pulls the process back' to the origin, it will return to the origin. ◇

Proof. We have seen that the solution to the stochastic differential equation (2.1) is $X_t = e^{-bt} \int_0^t \beta(s)e^{bs}dW_s$. To prove $\lim_{t\to\infty} X_t = 0$ *a.s.* we need to show that for any $\epsilon > 0$, we have

$$P(\sup\{|X_t| : m \le t < m+1\} > \epsilon, \text{ i.o.}) = 0.$$

Let

$$A_m := \{\sup\{|X_t| : m \le t < m+1\} > \epsilon\}.$$

Then, by the Borel-Canteli lemma, it is sufficient to show that

$$\sum_{m=1}^{\infty} P(A_m) < \infty. \tag{2.2}$$

Denote $Y_t := e^{bt}X_t = \int_0^t \beta(s)e^{bs}dW_s$, and note that Y_t is a martingale as it is an Itô integral. Thus $|Y_t|$ is a submartingale.

We have $P(A_m) < P(\sup\{|Y_t| : m \le t < m+1\} > \epsilon e^{bm})$. By Doob's inequality,

$$P(\sup\{|Y_t| : m \le t < m+1\} > \epsilon e^{bm}) < \frac{E(|Y_{m+1}|^2)}{\epsilon^2 e^{2bm}}.$$

As $E(Y_t) = 0$, we have

$$E(|Y_t|^2) = \mathtt{Var}(Y_t) = \int_0^t \beta^2(s)e^{2bs}ds,$$

and thus

$$\frac{E(|Y_{m+1}|^2)}{\epsilon^2 e^{2bm}} = \frac{\int_0^{m+1} \beta^2(s)e^{2bs}ds}{\epsilon^2 e^{2bm}}.$$

It remains to prove that

$$\sum_{m=1}^{\infty} e^{-2bm} \int_0^{m+1} \beta^2(s)e^{2bs}ds < \infty.$$

We need estimate $e^{-2bm} \int_0^{m+1} \beta^2(s)e^{2bs}ds$. To do that, we break up the expression into two parts:

$$\begin{aligned} e^{-2bm} \int_0^{m+1} \beta^2(s)e^{2bs}ds &= e^{-2bm} \int_0^{(m+1)/2} \beta^2(s)e^{2bs}ds + e^{-2bm} \int_{(m+1)/2}^{m+1} \beta^2(s)e^{2bs}ds \\ &=: I_1^m + I_2^m. \end{aligned}$$

We show now that both $\sum\limits_{m=1}^{\infty} I_1^m$ and $\sum\limits_{m=1}^{\infty} I_2^m$ are finite.

<u>I_1^m summable:</u>

$$I_1^m = e^{-b(m-1)} \int\limits_0^{(m+1)/2} \beta^2(s) e^{2bs-bm-b} ds \leq e^{-(m-1)} \int\limits_0^{(m+1)/2} \beta^2(s) ds.$$

As β decreases to 0, there is a constant C such that $\beta(s) < C$ for all s. Then $\int\limits_0^{(m+1)/2} \beta^2(s)ds < C^2(m+1)/2$. For large m, $m-1 > (m+1)/2$, and so

$$I_1^m \leq C^2 e^{-b(m-1)}(m-1),$$

yielding that $\sum\limits_{m=1}^{\infty} I_1^m \leq C^2 \sum\limits_{m=0}^{\infty} me^{-bm} = C^2(e^b + e^{-b} - 2)^{-1} < \infty$.

<u>I_2^m summable:</u>

$$\begin{aligned} I_2^m &= e^{-2bm} \int\limits_{(m+1)/2}^{m+1} \beta^2(s) e^{2bs} ds \leq e^{2b} \int\limits_{(m+1)/2}^{m+1} \beta^2(s) ds \\ &\leq e^{2b}(m+1)/2 \cdot \beta^2((m+1)/2). \end{aligned}$$

Note that $(m+1)/2 \leq 2\lfloor (m+1)/2 \rfloor$ for $m \geq 1$, and, since β is a decreasing function, one has

$$\begin{aligned} \sum_{m=1}^{\infty} I_2^m &\leq \sum_{m=1}^{\infty} e^{2b}(m+1)/2 \cdot \beta^2((m+1)/2) \\ &\leq 2e^{2b} \sum_{m=1}^{\infty} \lfloor (m+1)/2 \rfloor \beta^2(\lfloor (m+1)/2 \rfloor) \leq 4e^{2b} \sum_{n=1}^{\infty} n\beta^2(n). \end{aligned}$$

Since, by assumption, $\sum\limits_{m=1}^{\infty} m\beta^2(m) < \infty$, we have $\sum\limits_{m=1}^{\infty} I_2^m < \infty$.

Since I_1^m and I_2^m are summable, the summability condition (2.2) indeed holds. □

We now have sufficient preparation to prove Theorem 3.2.1.

Proof of Theorem 3.2.1

Consider the time interval $[m, m+1)$, and recall the definition of the center of mass: $\overline{Z_t} = 2^{-m} \sum\limits_{i=1}^{2^m} Z_t^{i,m}$. For each i, the particle $Z_t^{i,m}$'s motion satisfies the stochastic differential equation

$$dZ_t^{i,m} = dW_t^{i,m} + \left(\gamma 2^{-m} \sum_{j=1}^{2^m} (Z_t^{j,m} - Z_t^{i,m}) - bZ_t^{i,m} \right) dt,$$

where γ is the interaction coefficient, and b is the drift part of the Brownian motion. In our case, we consider $b > 0$.

Taking averages on both sides, the center of mass $\overline{Z_t}$ will thus satisfy

$$d\overline{Z_t} = 2^{-m} \sum_{i=1}^{2^m} dW_t^{i,m} - b\overline{Z_t}\, dt.$$

As the Brownian components of different particles are independent, Brownian scaling yields that

$$2^{-m} \sum_{i=1}^{2^m} dW_t^{i,m} = 2^{-m/2} d\widetilde{W}_t,$$

where $\widetilde{W}_t$ is standard Brownian motion in the time interval $[m, m+1)$. We thus have

$$d\overline{Z_t} = 2^{-m/2} d\widetilde{W}_t - b\overline{Z_t} dt.$$

Hence, in the time interval $[m, m+1)$, we have an Ornstein-Uhlenbeck process, while on $[0, \infty)$, the process $\overline{Z_t}$ satisfies the general stochastic differential equation $d\overline{Z_t} = \beta(t) dW_t - b\overline{Z_t} dt$ with $\beta(t) = 2^{-m/2}$, for $t \in [m, m+1)$.

In order to prove the theorem, it is sufficient to check that the conditions of Lemma 3.2.2 are satisfied.

Clearly, β is decreasing and locally Lipschitz, as $\beta(t) = 2^{-m/2}$ for $t \in [m, m+1)$, and furthermore,

$$\sum_{m=1}^{\infty} m\beta^2(m) = \sum_{m=1}^{\infty} m 2^{-m} < \infty$$

Thus, $\lim_{t\to\infty} \overline{Z_t} = 0$ *a.s.*, completing the proof of Theorem 3.2.1 □

3.3 Repulsion

For the repulsive case ($b < 0$), we have the following theorem.

Theorem 3.3.1 (Exponential escape of the COM for repulsion)**.** *For $b < 0$, $\lim_{t\to\infty} e^{bt}\overline{Z_t} = \mathcal{N}$ a.s., where $\mathcal{N}$ is a normal variable with mean zero and*

$$\texttt{Var}(\mathcal{N}) = \frac{1 - e^{2b}}{|b|(2 - e^{2b})} \cdot I_d.$$

(Here I_d is the identity matrix.)

Proof. By the independence of the coordinate processes, it is enough to consider $d = 1$. In order to show the existence of the limit and to identify it, we are going to utilize the Dambis-Dubins-Schwarz Theorem. We will use the shorthand $\overline{X_t} := e^{bt}\overline{Z_t}$.

More precisely, we are going to show that there exists a Brownian motion B on the same probability space where Z is defined, such that $\overline{X_t} = B_{s(t)}$, P-a.s. Here $t \mapsto s(t)$ is a deterministic time-change of t, mapping $[0, \infty)$ to a finite interval, satisfying that $\lim_{t\to\infty} s(t) = T$, where

$$T = T(b) := \frac{1 - e^{2b}}{|b|(2 - e^{2b})}. \tag{3.1}$$

Consequently, we will have that

$$\lim_{t\to\infty} \overline{X_t} = \lim_{t\to\infty} B_{<\overline{X}>_t} = B_{\lim\limits_{t\to\infty} <\overline{X}>_t} = B_T.$$

To achieve all these, recall first that $\overline{X_t} = \int\limits_0^t \beta(s)e^{bs}dW_s$, and thus, it is a continuous martingale. Therefore by the Dambis-Dubins-Schwarz Theorem (see e.g. Theorem V.1.6 in [13]), $\overline{X_t}$ is a time-changed Brownian motion:

$$\overline{X_t} = B_{<\overline{X}>_t},\ a.s.$$

where $< \overline{X} >$ denotes the increasing process for $\overline{X}$. Since the increasing process is deterministic in this case, we have that

$$s(t) :=< \overline{X} >_t = \mathtt{Var}(\overline{X}_t) = \int_0^t \beta^2(s)e^{2bs}ds,$$

where $\beta(s) := 2^{-m/2}$ for $s \in [m, m+1)$. Thus, $\overline{X_t} = B_{s(t)}$, almost surely, and furthermore,

$$\lim_{t\to\infty} s(t) = \sum_{m=0}^{\infty} \int_m^{m+1} 2^{-m}e^{2bs}ds = \sum_{m=0}^{\infty} 2^{-m} \cdot \frac{e^{2b(m+1)} - e^{2bm}}{2b}.$$

To evaluate the infinite sum, use Abel's (summation by part) formula:

$$\begin{aligned}
\lim_{t\to\infty} s(t) &= \sum_{m=0}^{\infty} 2^{-m} \cdot \frac{e^{2b(m+1)} - e^{2bm}}{2b} \\
&= \frac{1}{2b}(\sum_{m=1}^{\infty} e^{2bm}2^{-m}) - \frac{1}{2b} \\
&= \frac{1}{2b}(\sum_{m=1}^{\infty} (\frac{e^{2b}}{2})^m) - \frac{1}{2b} \\
&= \frac{1}{2b}\left(\frac{1}{1-\frac{e^{2b}}{2}} - 2\right) \\
&= \frac{e^{2b}-1}{b(2-e^{2b})} = T.
\end{aligned}$$

□

Remark 3.3.1. Without the Dambis-Dubins-Schwarz Theorem, one can still prove the existence of the almost sure limit along certain 'discrete time skeletons,' using a much more elementary argument. Indeed, $\overline{Z_t} = e^{-bt}\int\limits_0^t \beta(s)e^{bs}dW_s$, with $\beta(t) = 2^{-m/2}$, and $\overline{X_t} = \int\limits_0^t \beta(s)e^{bs}dW_s$ is a martingale.

Assume that $\sum_{n=0}^{\infty} e^{2bt_n} < \infty$, for example $t_{n+1} - t_n = \delta > 0$ for all $n \geq 0$. Since $\overline{X}_z - \overline{X}_t = \int_t^z \beta(s) e^{bs} dW_s$, and $\beta < 1$, for any $z > t > 0$, the Itô-isometry implies that

$$\mathtt{Var}(\overline{X}_z - \overline{X}_t) = \int_t^z \beta^2(s) e^{2bs} ds \leq \int_t^z e^{2bs} ds = \frac{1}{2b}(e^{2bz} - e^{2bt}) \leq -\frac{1}{2b} e^{2bt}.$$

Fix $\epsilon > 0$. By Doob's inequality,

$$P\left(\sup_{0 \leq k \leq L} |\overline{X}_{t_{N+k}} - \overline{X}_{t_N}| > \epsilon\right) < \frac{E(\overline{X}_{t_{N+L}} - \overline{X}_{t_N})^2}{\epsilon^2}.$$

Since $E(\overline{X}_t) = 0$ and $b < 0$, we can continue with

$$= \epsilon^{-2} \mathtt{Var}(\overline{X}_{t_{N+L}} - \overline{X}_{t_N}) \leq \frac{1}{2|b|\epsilon^2} e^{2bt_N}.$$

Letting $L \to \infty$,

$$P\left(\sup_{k \geq 0} |\overline{X}_{t_{N+k}} - \overline{X}_{t_N}| > \epsilon\right) \leq \frac{1}{2|b|\epsilon^2} e^{2bt_N}.$$

Then, summing up in N, we have

$$\sum_{N=0}^{\infty} P\left(\sup_{k \geq 0} |\overline{X}_{t_{N+k}} - \overline{X}_{t_N}| > \epsilon\right) \leq \sum_{N=0}^{\infty} \frac{1}{2|b|\epsilon^2} e^{2bt_N} < \infty,$$

and by the Borel-Canteli Lemma, on a set of P-probability one,

$$\sup_{k \geq 0} |\overline{X}_{t_{N+k}} - \overline{X}_{t_N}| \leq \epsilon,$$

for all large enough N's. Since in the above argument $\epsilon_k := 1/k, k = 1, 2, ...$ can be chosen, it follows that $\{\overline{X}_{t_n}\}$ is Cauchy, and thus has a limit, with probability one. $\diamond$

Remark 3.3.2 (Exponential speed of C.O.M.)**.** As $\lim_{t \to \infty} e^{bt} \overline{Z_t}$ exists a.s. and $e^{|b|t} \to \infty$, the point $\overline{Z_t}$ will tend to infinity, almost surely, with 'speed' $e^{|b|t}$ in the sense that $\overline{Z_t} \approx e^{|b|t} \cdot \mathcal{N}$. Furthermore, even in higher dimensions, it is clear that the angular component of $\overline{Z_t}$ will be uniformly distributed, by symmetry considerations.

Finally, it is easy to see that $\lim_{b \to 0} T(b) = 2$, in accordance with the already studied driftless case. $\diamond$

Chapter 4

Distribution of Relative System

4.1 The system as viewed from the center ('relative system')

Having described the motion of the center of mass $\overline{Z_t}$, in order to study the whole system, we need to investigate the 'relative system', that is the system as viewed from $\overline{Z_t}$.

Definition 4.1.1 (Relative system)**.** Denote $Y_t^i := Z_t^i - \overline{Z_t}$. The particle system $\{Y_t^i\}_{i=1}^{2^{\lfloor t \rfloor}}$ will be called the **relative system**, or **the system, as viewed from the center of mass**.

We focus on the behavior of the relative system in this section. We will use the shorthand $\sigma_m^2 := 1 - 2^{-m}$.

First, we want to determine the stochastic differential equation for Y_t^i. It can be obtained by direct computation, as follows. One has

$$dY_t^i = \sigma_m^2 dW_t^{i,m} \bigoplus_{j \neq i} -2^{-m} dW_t^{j,m} - (\gamma + b) Y_t^i dt.$$

As $\{W_t^{i,m}\}$ are independent Brownian motions, we can write

$$dY_t^i = \sigma_m d\widetilde{W}_t^i - (\gamma + b) Y_t^i dt,$$

where $\widetilde{W}_t^i$ is a driving Brownian motion for Y_t^i, such that

$$\sigma_m d\widetilde{W}_t^i = \sigma_m^2 dW_t^{i,m} \bigoplus_{j \neq i} -2^{-m} dW_t^{j,m}.$$

When $t \to \infty$, (i.e., $m \to \infty$), the process Y^i will asymptotically satisfy the equation

$$dY_t^i = d\widetilde{W}_t^i - (\gamma + b) Y_t^i dt,$$

yielding that, asymptotically,

(1) If $\gamma + b > 0$, Y_t^i becomes an inward O-U process;

(2) if $\gamma + b < 0$, Y_t^i becomes an outward O-U process;

(3) if $\gamma + b = 0$, Y_t^i becomes Brownian motion.

As a next step, we need to study the correlation between the particles of $\{Y_t^i\}$. As $\sum \widetilde{W}_t^i = 0$, they are obviously not independent.

First we determine the 'degree of freedom' of $\{\widetilde{W}_t^i\}$. Similarly to [5], one can show that the degree of freedom of $\{\widetilde{W}_t^i\}$ is $2^m - 1$. To explain what this means, fix $m \geq 1$ and for $t \in [m, m+1)$ let $Y_t := (Y_t^1, ..., Y_t^{2^m})^T$, where $()^T$ denotes transposed. (This is a vector of length 2^m where each component itself is a d dimensional vector; one can actually view it as a $2^m \times d$ matrix too.) We then have

$$\mathrm{d}Y_t = \sigma_m \mathrm{d}\widetilde{W}_t^{(m)} - \gamma Y_t \mathrm{d}t,$$

where

$$\widetilde{W}^{(m)} = \left(\widetilde{W}^{m,1}, ..., \widetilde{W}^{m,2^m}\right)^T$$

and

$$\widetilde{W}_\tau^{m,i} = \sigma_m^{-1}\left(W_\tau^{m,i} - 2^{-m}\bigoplus_{j=1}^{2^m} W_\tau^{m,j}\right), \ i = 1, 2, ..., 2^m$$

are mean zero, correlated Brownian motions.

Just like in subsection 2.3, here we can also consider Y as a single $2^m d$-dimensional diffusion. Each of its components is an Ornstein-Uhlenbeck process with asymptotically unit diffusion coefficient.

By independence, it is enough to consider the $d = 1$ case.

Let us first describe the distribution of $\widetilde{W}_t^{(m)}$ for $t \geq 0$ fixed. Recall that $\{W_s^{m,i}, \ s \geq 0;\ i = 1, 2, ..., 2^m\}$ are independent Brownian motions. By definition, $\widetilde{W}_t^{(m)}$ is a 2^m-dimensional multivariate normal:

$$\widetilde{W}_t^{(m)} = \sigma_m^{-1} \cdot \begin{pmatrix} 1-2^{-m} & -2^{-m} & ... -2^{-m} \\ -2^{-m} & 1-2^{-m} & ... -2^{-m} \\ . & & \\ . & & \\ . & & \\ -2^{-m} & -2^{-m} & ... \ 1-2^{-m} \end{pmatrix} W_t^{(m)} =: \sigma_m^{-1}\mathbf{A}^{(m)} W_t^{(m)}, \qquad (1.1)$$

where $W_t^{(m)} = (W_t^{m,1}, ..., W_t^{m,2^m})^T$, yielding

$$\mathrm{d}Y_t = \mathbf{A}^{(m)}\mathrm{d}W_t^{(m)} - \gamma Y_t \mathrm{d}t.$$

Since we are viewing the system from the center of mass, $\widetilde{W}_t^{(m)}$ is a **singular** multivariate normal and thus Y is a degenerate diffusion. The 'true' dimension of $\widetilde{W}_t^{(m)}$ is $\mathrm{r}(\mathbf{A}^{(m)})$. Then the same argument as in [5], yields that $\mathrm{rank}(\mathbf{A}^{(m)}) = 2^m - 1$, and the above comment about the degrees of freedom should be understood in this sense.

Moreover, the driving Brownian motions $\{\widetilde{W}_t^i\}$ will be exactly the same as in [5], and thus they have asymptotically vanishing correlation (see Remark 12 in [5]).

The relative system thus coincides with the driftless one in [5], if γ is replaced by $\gamma + b$.

4.2 A useful transformation: the process Z^{Δ}

We first make an important observation, making the last sentence of the previous section more general: we notice that γ and b are 'interchangeable' in the following sense.

Lemma 4.2.1 (Interchangeable coefficients)**.** Suppose that we have two branching particle systems and Y and $\mathcal{Y}$ represent the relative systems for them. Denote by b_1, γ_1 and b_2, γ_2 the corresponding coefficients of Y and $\mathcal{Y}$. Assume that $b_1 + \gamma = b_2 + \gamma_2$. Then the laws of Y and $\mathcal{Y}$ are the same.

Proof. Fix $m \geq 0$ and $i = 1, 2, \cdots, 2^m$. Then Y^i and $\mathcal{Y}^i$ satisfy the same stochastic differential equation

$$dY_t^i = (1 - 2^{-m})^{1/2} d\widetilde{W}_t^i - \gamma Y_t^i dt$$

in the interval $t \in [m, m+1)$, where $\gamma = b_1 + \gamma_1 = b_2 + \gamma_2$. Thus, the fact that single particles have the same law in the two systems, follows by induction, along with the existence and uniqueness of the solution for stochastic differential equations. (We know that for $m = 0$ they start with the same initial value.)

The fact that even the *joint* distributions of the two particle systems agree, follows the same way as we proved existence and uniqueness for the model in subsection 2.3, except that now the independent driving Brownian motions must be replaced by $\sigma_m^{-1} \mathbf{A}^{(m)} W_t^{(m)}$ in $[m, m+1)$ (recall (1.1)). Since the piecewise Lipschitz-ness of the coefficients is preserved, the existence and uniqueness theorem is still in force. $\square$

Definition 4.2.1 (Δ-transformation)**.** Consider Z with γ and b given and let $\mathcal{Z} = Z^{\Delta}$ be another system with parameters

$$\begin{aligned} \gamma^{\Delta} &:= \gamma - \Delta; \\ b^{\Delta} &:= b + \Delta. \end{aligned}$$

Since $b + \gamma = b^{\Delta} + \gamma^{\Delta}$, we know by Lemma 4.2.1 that the corresponding relative systems Y and $\mathcal{Y}$ have the same law.

Consider Z and $\mathcal{Z}$ on the same probability space. Then

$$Z_t(B) = Y_t(B - \overline{Z}_t) \stackrel{d}{=} \mathcal{Y}_t(B - \overline{Z}_t) = \mathcal{Z}_t(B - (\overline{Z}_t - \overline{\mathcal{Z}}_t)), \tag{2.1}$$

for $B \subset \mathbb{R}^d$ Borel and $t \geq 0$. In fact,

$$Z \text{ has the same law as the process } t \mapsto \mathcal{Z}_t(\cdot - D_t), \tag{2.2}$$

where $D_t := \overline{Z}_t - \overline{\mathcal{Z}}_t$. The behavior of D_t for large times depends on the signs of b and $b + \gamma$. E.g. if they are both positive, then D_t tends to the origin a.s.

In particular, with appropriate transformations we can 'knock out' either the interaction or the O-U drift:

(a) Representation with non-interactive system: pick $\Delta := \gamma$. That is, let the non-interactive process $\mathcal{Z}$ correspond to $\gamma^{\Delta} = 0$ and $b^{\Delta} := b + \gamma$. Then (2.2) gives a remarkable link between the interactive and the non-interactive models. This connection is reminiscent to the one found in [8] (see Remark 3.2 there).

(b) Eliminating the O-U drift: pick $\Delta := -b$. That is, the motion component of Z^{Δ} is just a Brownian motion, similarly to [5]. Then (2.2) gives a link between our model and the driftless one studied in [5].

4.3 Outline of the strategy of the rest of the proofs

In light of the previous section, we could choose to base the analysis of the relative system on the corresponding result in [5] when $b + \gamma > 0$ (by eliminating the drift – see part (b) in the previous section), or on the results on the global system in Example 11 of [7] when $b + \gamma < 0$ (by eliminating the interaction – see part (a) in the previous section).

In the second case, we should handle the problem that the setting is different in [7] in that the branching is not unit time but rather exponential.

On top of that, the method of the proof in both [5] and [7] requires the introduction of two non-trivial auxiliary functions, related to the model.

Instead of choosing one of the paths alluded to above, we decided to give a completely elementary proof in the next section for the Strong Law for the relative system in our case, when $b + \gamma > 0$. It does not use the complicated machinery of [5] or [7], and it is done in unit time. The proof only uses some calculations involving the most recent common ancestors of particles and some covariance estimates.

In particular, it gives a new, elementary proof for the Strong Law for the global system, for the case of a non-interactive branching (inward) Ornstein-Uhlenbeck process. See Example 5.2.1.

4.4 Proof of SLLN for the relative system when $b + \gamma > 0$

4.4.1 General comments

Recall that $\sigma_m^2 := 1 - 2^{-m}$, and that for the relative system Y,

$$dY_t^i = \sigma_m d\widetilde{W}_t^i - (b + \gamma) Y_t^i dt$$

for $i = 1, 2, 3, \cdots, 2^m$ in the interval $t \in [m, m+1)$. Recall also that $\sigma_m d\widetilde{W}_t^i = \sigma_m^2 dW_t^{i,m} \bigoplus_{j \neq i} -2^{-m} dW_t^{j,m}$.

Assuming $b + \gamma > 0$, our goal is to find $\lim_{m\to\infty} \frac{1}{2^m} \sum_{i=1}^{2^m} Y_t^i(B)$ for a generic Borel set $B \subset \mathbb{R}^d$. (Here, we consider Y_t as a random measure, and that is why we may write $Y_t^i(B) = 1_B(Y_t^i), i = 1, 2, 3, \cdots, 2^m$.) For $d = 1$, this will be achieved in Theorem 4.4.3 and subsequently, it will be upgraded to higher dimensions; before these, we will prove several preparatory results.

Remark 4.4.1. When taking the limit, we will just consider integer times. This is somewhat weaker than considering continuous times, however, since the model is about unit time branching, we did not have sufficient motivation to go into the technical details as to how one upgrades the limit along integer times to a limit along continuous times. (There are existing techniques though, going back to the work of Assmussen and Hering [1].) $\diamond$

As mentioned above, for simplicity we will first treat $d = 1$, and then show that the main result we got also works for high dimensions.

Next, let us sketch our *strategy* of the investigation:

(1) Find the correlation between the particle positions $Y_m^i, 1 \le i \le 2^m; m \ge 1$.

(2) Use the correlations in (1) to control the correlation between $Y_m^i(B)$, where $B \subset \mathbb{R}^d$.

(3) Establish the Strong Law of Large Numbers for Y, that is, find a measurable function $0 \le f$ such that for $B \subset \mathbb{R}^d$ Borel,

$$\lim_{m\to\infty} \frac{1}{2^m} \sum_{i=1}^{2^m} Y_m^i(B) = \int_B f(x)dx, a.s.$$

Assumption 4.4.1 (No drift)**.** In this section, when studying the relative system, we will assume that $b = 0$ and $\gamma > 0$. We can do this without the loss of generality, since given γ and b, one may apply a Δ-transformation with $\Delta := -b$, that is, eliminate the O-U drift. Then $b^{\Delta} = 0$ and $\gamma^{\Delta} = b + \gamma$, and Y and $\mathcal{Y}$ have the same law by Lemma 4.2.1.

4.4.2 Crucial estimates

Now let us focus on the distribution of the random variables $Y_t^i, 1 \le i \le 2^{\lfloor t \rfloor}, t \ge 0$. Since $dY_t^i = \sigma_m d\widetilde{W}_t^i - \gamma Y_t^i dt$, and since for the general differential equation

$$dY_t = a dW_t - rY_t\, dt,$$

$(a, r > 0)$ with initial value Y_0, the solution is $Y_t = e^{-rt}\left[\int_0^t e^{rs} d(aW_s) + Y_0\right]$, it follows, by conditioning on Y_m^i, that

$$Y_{m+1}^i = e^{-\gamma}\left[\int_m^{m+1} e^{\gamma(s-m)} d(\sigma_m d\widetilde{W}_s^i) + Y_m^i\right], a.s.$$

We know that $\{\widetilde{W}_s^i\}_{0\le s<1}$ is a Brownian motion on time interval $[m, m+1]$ and that it is independent of Y_m^i.

By symmetry, the distributions of Y_t^i and Y_t^j are the same, so we will just write Y_t^i.

We thus have $Y_{m+1}^i = e^{-\gamma}(Y_m^i + X_m^i)$, where

$$X_m^i = \int_m^{m+1} e^{\gamma(s-m)} d(\sigma_m d\widetilde{W}_s^i)$$

is a normal variable with distribution $\mathcal{N}\left(0, \sigma_m(2\gamma)^{-1}\left(e^{2\gamma} - 1\right)\right)$, and $Y_0^i = 0$.

Theorem 4.4.1. *For any $m \geq 0$,*

$$\texttt{Var}(Y^i_{m+1}) = \sigma^2(Y^i_{m+1}) = \frac{e^{2\gamma}-1}{2\gamma}\, e^{-2m\gamma}\left(\frac{1-e^{2m\gamma}}{1-e^{2\gamma}} - \frac{1-(\frac{1}{2}e^{2\gamma})^m}{2-e^{2\gamma}}\right).$$

Proof. As we know, $Y_{m+1} = e^{-\gamma}(Y_m + X_m)$, and $Y_0 = 0$. So we can compute the variance of Y_{m+1} step by step, due to the independence of Y_m and X_m. First, we have

$$\begin{aligned}\sigma^2(Y_{m+1}) &= e^{-2\gamma}\left(\sigma^2(Y_m) + \sigma^2(X_m)\right)\\ &= e^{-2\gamma}\left(\sigma^2(Y_m) + \frac{1}{2\gamma}\left(1-\frac{1}{2^m}\right)(1-e^{-2\gamma})\right),\end{aligned}$$

and $\sigma^2(Y_0) = 0$.
Thus, for convenience, we denote $a_m := \sigma^2(Y_m)$ and

$$b_m := \sigma^2(X_m) = \frac{1}{2\gamma}\left(1-\frac{1}{2^m}\right)\left(1-e^{-2\gamma}\right).$$

Since $a_{m+1} = e^{-2\gamma}(a_m + b_m)$, we use recursion:

$$\begin{aligned}a_{m+1} &= e^{-2\gamma}(e^{-2\gamma}(a_{m-1}+b_{m-1}) + b_m)\\ &= (e^{-2\gamma})^2 a_{m-1} + (e^{-2\gamma})^2 b_{m-1} + e^{-2\gamma} b_m\\ &= (e^{-2\gamma})^3 a_{m-2} + (e^{-2\gamma})^3 b_{m-2} + (e^{-2\gamma})^2 b_{m-1} + e^{-2\gamma} b_m\\ &= (e^{-2\gamma})^{m+1} a_0 + (e^{-2\gamma})^{m+1} b_0\\ &\qquad + \cdots + (e^{-2\gamma})^3 b_{m-2} + (e^{-2\gamma})^2 b_{m-1} + e^{-2\gamma} b_m.\end{aligned}$$

We have $a_0 = b_0 = 0$, and thus,

$$a_{m+1} = \sum_{n=1}^{m} (e^{-2\gamma})^{m+1-n} \frac{1}{2\gamma}\left(1-\frac{1}{2^n}\right)\left(1-e^{-2\gamma}\right).$$

Computation yields that

$$a_{m+1} = \frac{e^{-2(m+1)\gamma}(1-e^{-2\gamma})}{2\gamma}\left(\sum_{n=1}^{m} e^{2n\gamma} + \sum_{n=1}^{m}\left(\frac{e^{2\gamma}}{2}\right)^n\right),$$

and summing the series,

$$a_{m+1} = \frac{e^{2\gamma}-1}{2\gamma}\, e^{-2m\gamma}\left[\frac{1-e^{2m\gamma}}{1-e^{2\gamma}} - \frac{1-(\frac{1}{2}e^{2\gamma})^m}{2-e^{2\gamma}}\right].$$

Using the definition of a_{m+1}, the proof is complete. □

We now need to analyze the covariance between Y^i_m and Y^j_m for $1 \leq i, j \leq 2^m$ and $m \geq 1$. Here the notion of the *most recent common ancestor* (MRCA) becomes important.

Definition 4.4.1 (MRCA)**.** Consider $m \geq 0$ and the particles $Y_m^i, Y_m^j, 1 \leq i \neq j \leq 2^m$. A common ancestor is a particle considered at some integer time interval $[a-1, a)$ with $0 < a < m$, which is an ancestor to both Y_m^i and Y_m^j. The MRCA of the particles is a common ancestor such that a is maximal. This a is then called the *splitting time* of the particles.

Note that in this definition, we consider particles living in different time intervals $[m, m+1)$ as different particles.

As different pairs of particles (Y_m^i, Y_m^j) with different MRCA's may have different covariances, we need to take into account the MRCA of (Y_m^i, Y_m^j). Hence, with splitting time $a > 0$, we will write $(Y_{m,a}^i, Y_{m,a}^j)$ for (Y_m^i, Y_m^j).

Theorem 4.4.2 (A covariance bound)**.** *For any $m \geq 1$ and $1 \leq i \neq j \leq 2^m$,*

$$\mathtt{Cov}(Y_{m,a}^j, Y_{m,a}^i) < Cm(e^{-2\gamma(m-a)} + 2^{-m}),$$

where C is a constant which only depends on γ.

Proof. As we know $Y_{m+1}^i = e^{-\gamma}(Y_m^i + X_m^i)$ for any 'relative particle,' it follows that

$$Y_m^i = e^{-\gamma(m-a)} Y_a^i + \sum_{n=a}^{m-1} e^{-\gamma(m-n)} X_n^i.$$

Here X_n^i is the associated Brownian motion on the unit time interval $[n, n+1]$. From the formula we can easily compute $\mathtt{Cov}(Y_m^i, Y_m^j)$, once we know the corresponding $\mathtt{Cov}(X_n^i, X_n^j)$. For the ith particle (i can be replaced by j, of course),

$$X_n^i = \int_n^{n+1} e^{\gamma(s-n)} d(\sigma_m d\widetilde{W}_s^i),$$

where $\sigma_n d\widetilde{W}_s^i = -\frac{1}{2^n} \sum_{k \neq i} W_s^k + \sigma_n^2 W_s^i$ and $\{W_s^k, k = 1, 2, 3, \cdots, 2^n\}$ are standard independent Brownian motions.

Hence $\mathtt{Cov}(X_n^i, X_n^j)$ can be computed as $\{W_s^k, k = 1, 2, \cdots, 2^n\}$ are all independent. Furthermore, $\mathtt{Var}\left(\int_n^{n+1} e^{\gamma(s-n)} dW_s^k\right) = \frac{e^{2\gamma}-1}{2\gamma}$. Then the X_n^i, X_n^j are just a combination of $\int_n^{n+1} e^{\gamma(s-n)} dW_s^k$. Denote $A_k := \int_n^{n+1} e^{\gamma(s-n)} dW_s^k$. Then

$$X_n^i = -\tfrac{1}{2^n} \textstyle\sum_{k \neq i} A_k + \sigma_n A_i;$$
$$X_n^j = -\tfrac{1}{2^n} \textstyle\sum_{k \neq j} A_k + \sigma_n A_j.$$

One has

$$\begin{aligned}
&\texttt{Cov}(X_n^i, X_n^j)\\
&= \texttt{Cov}(-\frac{1}{2^n}\sum_{k\neq i} A_k + (1-\frac{1}{2^n})A_i, -\frac{1}{2^n}\sum_{k\neq j} A_k + (1-\frac{1}{2^n})A_j)\\
&= (\frac{1}{2^n})^2 \sum_{k\neq i,j} \texttt{Var}(A_k) - \frac{1}{2^n}(1-\frac{1}{2^n})\texttt{Var}(A_i) - \frac{1}{2^n}(1-\frac{1}{2^n})\texttt{Var}(A_j)\\
&= ((\frac{1}{2^n})^2(2^n-2) - \frac{2}{2^n}(1-\frac{1}{2^n}))\frac{e^{2\gamma}-1}{2\gamma}\\
&= -\frac{1}{2^n}\frac{e^{2\gamma}-1}{2\gamma}.
\end{aligned}$$

Since the $Y_m^i = e^{-\gamma(m-a)}Y_a^i + \sum_{n=a}^{m-1} e^{-\gamma(m-n)}X_n^i$, $\{X_k^i\}$ are independent for different k and since $Y_a^i = Y_a^j$, it follows that[1]

$$\texttt{Cov}(Y_m^i, Y_m^j) = e^{-2\gamma(m-a)}\texttt{Var}(Y_a^i) + \sum_{k=a}^{m-1} e^{-2\gamma(m-k)}\texttt{Cov}(X_k^i, X_k^j).$$

From the last theorem, we know that

$$\texttt{Var}(Y_a^i) = \frac{e^{2\gamma}-1}{2\gamma}\, e^{-2\gamma(a-1)}\left[\frac{1-e^{2\gamma(a-1)}}{1-e^{2\gamma}} - \frac{1-(\frac{1}{2}e^{2\gamma})^{a-1}}{2-e^{2\gamma}}\right].$$

Thus,

$$\begin{aligned}
\texttt{Cov}(Y_m^i, Y_m^j) = \frac{e^{2\gamma}-1}{2\gamma}\Bigg(&e^{-2\gamma(m-1)}\left[\frac{1-e^{2\gamma(a-1)}}{1-e^{2\gamma}} - \frac{1-(\frac{1}{2}e^{2\gamma})^{a-1}}{2-e^{2\gamma}}\right]\\
&+\sum_{k=a}^{m-1} e^{-2\gamma(m-k)}\left(-\frac{1}{2^k}\right)\Bigg).
\end{aligned}$$

It then follows that

$$\texttt{Cov}(Y_m^i, Y_m^j) < C_0(I_1 + I_2),$$

where

$$I_1 = e^{-2\gamma(m-1)}\left[\frac{1-e^{2\gamma(a-1)}}{1-e^{2\gamma}} - \frac{1-(\frac{1}{2}e^{2\gamma})^{a-1}}{2-e^{2\gamma}}\right],$$

$$I_2 = \sum_{k=a}^{m-1} e^{-2\gamma(m-k)}(-\frac{1}{2^k}),$$

[1] In this formula, i could be greater than 2^k, in that case X_k^i stand for the normal distribution associated with the ancestor of i at the time k. The normal distribution is defined in the same way as X_n^i.

and $C_0 = \frac{e^{2\gamma}-1}{2\gamma}$. As $\gamma > 0$, $|I_1| < C_1 e^{-2\gamma(m-a)}$ which C_1 is a constant which only depends on γ. Moreover, $I_2 = e^{-2m\gamma} \sum_{k=a}^{m-1} (-(\frac{e^{2\gamma}}{2})^k)$ By easy computation, we can see $|I_2| < C_2(m\, e^{-2m\gamma} + \frac{1}{2^m})$ which C_2 is also a constant which only depends on γ.

Hence, $\texttt{Cov}(Y_m^i, Y_m^j)$ is bounded from above by

$$C_0\left(C_1 e^{-2\gamma(m-a)} + C_2\left(m\, e^{-2m\gamma} + \frac{1}{2^m}\right)\right) < C\, m(e^{-2\gamma(m-a)} + 2^{-m}),$$

and C is a constant that only depends on γ. □

Recall that our goal is to prove the existence of $\lim_{m\to\infty} 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i)$ and identify it. To achieve this, we will use the Borel-Cantelli Lemma in conjunction with the Chebysev inequality, and so we need to estimate $\texttt{Var}\left(2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i)\right)$.

Clearly,

$$\texttt{Var}\left(\sum_{i=1}^{2^m} 1_B(Y_m^i)\right) = \sum_{i=1}^{2^m} \texttt{Var}(1_B(Y_m^i)) + \sum_{\substack{i=1 \\ i\neq j}}^{2^m} \sum_{j=1}^{2^m} \texttt{Cov}(1_B(Y_m^i), 1_B(Y_m^j)),$$

and

$$\texttt{Var}(1_B(Y_m^i)) = P(Y_m^i \in B) - P^2(Y_m^i \in B) \le 1/4.$$

Now we just need to compute $\texttt{Cov}(1_B(Y_m^i), 1_B(Y_m^j))$ for $i \neq j$, and to do that, we need to know the splitting time for the particles Y_m^i and Y_m^j, which we will still denote by a. We now need the following lemma.

Lemma 4.4.1 (Covariance for indicators)**.** *Let (X, Y) be a joint normal vector such that its marginals X and Y are standard normal, and denote $\rho := \texttt{Cov}(X, Y)$. Then there exists an absolute constant $C > 0$ such that*

$$\texttt{Cov}(1_B(X), 1_B(Y)) \le C|\rho|$$

holds for all Borel sets $B \subset \mathbb{R}$ and all $|\rho| < 1/2$.

Proof. Plugging in the joint and marginal densities

$$\frac{1}{2\,\pi\sqrt{1-\rho^2}} e^{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}}\;;\;\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

$$\begin{aligned}
\psi(\rho, B) &:= \texttt{Cov}(1_B(X), 1_B(Y)) \\
&= P(X \in B, Y \in B) - P(X \in B)\cdot P(Y \in B) \\
&= \frac{1}{2\pi}\iint_{B\times B} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}} - e^{\frac{-x^2-y^2}{2}} dxdy \\
&=: \frac{1}{2\pi}\iint_{B\times B} f(x, y, \rho) dxdy.
\end{aligned}$$

Clearly, $\psi(0,B)=0$. Since $f \in C^\infty(\mathbb{R}\times\mathbb{R}\times(-1/2,1/2))$,

$$\psi'(\rho,B)=\frac{1}{2\pi}\iint_{B\times B}\frac{\partial f(x,y,\rho)}{\partial\rho}dxdy,$$

where

$$\frac{\partial f(x,y,\rho)}{\partial\rho}=e^{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}}\cdot\left(\frac{\rho}{(1-\rho^2)^{\frac{3}{2}}}-\frac{1}{\sqrt{1-\rho^2}}\frac{k(x,y,\rho)}{4(1-\rho^2)^2}\right),$$

with $k(x,y,\rho):=2xy2(1-\rho^2)+4\rho(x^2+y^2+2\rho xy)$. For $|\rho|\le\frac{1}{2}$, we have $\frac{x^2+y^2}{2}\le x^2+y^2+2\rho xy$, i.e. $x^2+y^2+4\rho xy\ge 0$, and thus,

$$\exp\left\{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}\right\}\le\exp\left\{-\frac{x^2+y^2}{4}\right\}.$$

Also, $k(x,y,\rho)\le 2(x^2+y^2)+2(2(x^2+y^2))\le 6(x^2+y^2)$. Hence, with some $C>0$ constant,

$$\left|\frac{\partial f(x,y,\rho)}{\partial\rho}\right|\le Ce^{-\frac{x^2+y^2}{4}}(1+x^2+y^2),\ \forall|\rho|\le 1/2.$$

Consequently,

$$\begin{aligned}|\psi'(\rho,B)| &\le \tfrac{1}{2\pi}\iint_{B\times B}Ce^{-\frac{x^2+y^2}{4}}(1+x^2+y^2)dxdy\\ &\le \tfrac{1}{2\pi}\iint_{\mathbb{R}\times\mathbb{R}}Ce^{-\frac{x^2+y^2}{4}}(1+x^2+y^2)dxdy\\ &\le C\int_0^\infty e^{-r^2/4}r(r^2+1)dr=:C'<\infty,\end{aligned}$$

yielding

$$|\psi(\rho,B)|=\left|\int_0^\rho\psi'(s,B)ds\right|\le\int_0^{|\rho|}|\psi'(s,B)|ds\le C'|\rho|.$$

□

Remark 4.4.2. More generally, let both X and Y be $\mathcal{N}(0,\sigma^2)$-distributed. If (X,Y) is joint normal with $\texttt{Cov}(X,Y)=\rho$, then we can scale X and Y to use Lemma 4.4.1. Indeed, $\frac{X}{\sigma}$ and $\frac{Y}{\sigma}$ are then standard normal variables, and $\texttt{Cov}(\frac{X}{\sigma},\frac{Y}{\sigma})=\frac{\rho}{\sigma^2}$. From Lemma 4.4.1, if $\frac{|\rho|}{\sigma^2}<1/2$ $(2|\rho|<\sigma^2)$, then

$$\texttt{Cov}(1_B(X),1_B(Y))=\texttt{Cov}\left(1_{\frac{B}{\sigma}}\left(\frac{X}{\sigma}\right),1_{\frac{B}{\sigma}}\left(\frac{Y}{\sigma}\right)\right)\le\frac{C|\rho|}{\sigma^2}. \tag{4.1}$$

◇

Returning to the question of covariances with splitting time a, note that $\{Y_m^i\}$ are linear combinations of a number of underlying Brownian motions; hence (Y_m^i,Y_m^j) will be joint normal. From Theorem 4.4.2, we know that $\texttt{Cov}(Y_m^i,Y_m^j)\le Cm(e^{-2\gamma(m-a)}+2^{-m})$. From (4.1), we then have the following corollary.

Corollary 4.4.1. *With some constant $C > 0$,*

$$\mathtt{Cov}(1_B(Y_m^i), 1_B(Y_m^j) \leq \frac{Cm(e^{-2\gamma(m-a)} + 2^{-m})}{\sigma^2(Y_m^i)},$$

provided $2m(e^{-2\gamma(m-a)} + 2^{-m}) < \sigma^2(Y_m^i)$. *(Of course, $\sigma^2(Y_m^i) = \sigma^2(Y_m^j)$.)*

Remark 4.4.3. From Theorem 4.4.1, we can easily compute $\lim\limits_{m\to\infty} \sigma^2(Y_m^i) = \frac{1}{2\gamma}$. So if we write $a(m)$ in place of a, and $\lim_{m\to\infty}(m - a(m)) = \infty$, then the condition in Corollary 4.4.1 will always be true for large enough m's. $\diamond$

4.4.3 Strong Law for the relative system; $d = 1$

Now we are ready to prove the following theorem.

Theorem 4.4.3 (SLLN)**.** *Assume that $d = 1, \gamma + b > 0$ and let $B \subset \mathbb{R}^d$ be a Borel set. Then, almost surely,*

$$\lim_{m\to\infty} \left(2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) - P(Y_m^i \in B) \right) = 0.$$

Here, of course $\lim_{m\to\infty} P(Y_m^i \in B)$ exists. That is, similarly to [5], $\lim\limits_{m\to\infty} 2^{-m} Y_m(dy) = f(y)dy$ a.s. in the weak topology, where f is the density for $\mathcal{N}\left(0, \frac{1}{2\gamma}\right)$.

Proof. We need to show that for any $\epsilon > 0$, a.s., only finitely many of the events

$$A_m = \left\{ \left| 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) - E(1_B(Y_m^i)) \right| > \epsilon \right\}$$

will occur. By the Borel-Cantelli Lemma, we just need to show that for some large N, $\sum\limits_{N}^{\infty} P(A_m) < \infty$.

From Chebyshev's inequality,

$$P(A_m) < \frac{\mathtt{Var}\left(2^{-m} \sum\limits_{i=1}^{2^m} 1_B(Y_m^i)\right)}{\epsilon^2},$$

since $E\left(2^{-m} \sum\limits_{i=1}^{2^m} 1_B(Y_m^i)\right) = E(1_B(Y_m^i))$. Thus, we just need to show that

$$\sum_{m=N}^{\infty} \mathtt{Var}\left(2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i)\right) < \infty. \tag{4.2}$$

To this end, note that

$$\texttt{Var}\left(2^{-m}\sum_{i=1}^{2^m}1_B(Y_m^i)\right) = \tfrac{1}{2^{2m}}\Big(\sum_{i=1}^{2^m}\texttt{Var}(1_B(Y_m^i)) + \textstyle\sum_{i=1}^{2^m}\sum_{j=1 i\neq j}^{2^m}\texttt{Cov}(1_B(Y_m^i),1_B(Y_m^j))\Big) \tag{4.3}$$

Since $\texttt{Var}(1_B(Y_m^i)) \leq 1/2 - 1/4 = 1/4$, we have

$$\texttt{Var}\left(2^{-m}\sum_{i=1}^{2^m}1_B(Y_m^i)\right) \leq \frac{1}{2^{2m}}\left(\frac{2^m}{4} + 2^m\sum_{j\neq i}\texttt{Cov}(1_B(Y_m^i),1_B(Y_m^j))\right),$$

for a fixed i, since i and j are symmetric.

Hence, we need to study $\sum\limits_{j\neq i}\texttt{Cov}(1_B(Y_m^i),1_B(Y_m^j))$ for a fixed i. We know that $\texttt{Cov}(1_B(Y_m^i),1_B(Y_m^j))$ depends on the time a when the MRCA of these particles splits. We thus need to differentiate between 'close relatives' and other pairs.

Notice that there are 2^k particles which have the MRCA at time $m-k$ with Y_m^i. From Theorem 4.4.2, we know that

$$\texttt{Cov}(Y_m^j, Y_m^i) < Cm(e^{-2\gamma(m-a)} + 2^{-m}),$$

if the MRCA of i and j is a. If $a = a(m) = m/2$, (or just $m-a$ tends to ∞), then the righthand side converges to zero as $m\to\infty$. We also know $\lim\limits_{m\to\infty}\texttt{Var}(Y_m^i) = \frac{1}{2\gamma}$, and so we may apply Lemma 4.4.1 and the remark following it for $a = a(m) \leq \frac{m}{2}$ and large m. That is, we could choose a large N such that for all $m > N$, the condition

$$\frac{\texttt{Cov}(Y_m^i, Y_m^j)}{\texttt{Var}(Y_m^i)} \leq \frac{1}{2}, \tag{4.4}$$

is satisfied.

Now, the important point is that *the majority of particle-pairs have $a \leq m/2$, that is, they are not 'close' relatives.* Indeed, the number of pairs with $a > m/2$ (close relatives) is $2^m(1+2+4+\cdots+2^{\lfloor\frac{m-1}{2}\rfloor})$. Simple computation yields that

$$2^m(1+2+4+\cdots+2^{\lfloor\frac{m-1}{2}\rfloor}) \leq 2^m(2^{\lfloor\frac{m-1}{2}\rfloor+1}) \leq 2^{m+\frac{m}{2}+1}.$$

As we know, for any pair, we have $\texttt{Cov}(Y_m^i, Y_m^j) \leq \texttt{Var}(Y_m^i) \leq \frac{1}{4}$. Thus, for all of those pairs with $a > m/2$, the total covariance will be controlled by $2^{m+\frac{m}{2}+1}\cdot\frac{1}{4} = 2^{m+\frac{m}{2}-1}$. Moreover, as discussed above, for the pairs with $a \leq m/2$, we may apply Lemma 4.4.1 and the remark following it, yielding

$$\begin{aligned}\sum_{j\neq i}\texttt{Cov}(1_B(Y_m^i),1_B(Y_m^j)) &\leq \sum_{k=\lfloor\frac{m+1}{2}\rfloor}^{m-1}2^kCm(e^{-2\gamma k}+2^{-m})\\ &\leq Cm\sum_{k=1}^{m-1}\left(\left(\frac{2}{e^{2\gamma}}\right)^k + 2^{k-m}\right),\end{aligned}$$

where C only depends on N, γ, B.
We thus have

$$\begin{aligned}
&\mathtt{Var}\left(2^{-m}\sum_{i=1}^{2^m} 1_B(Y_m^i)\right)\\
&\leq \frac{1}{4\times 2^m} + \frac{2^{\frac{m}{2}-1}}{2^m} + \frac{Cm}{2^m}\left(\frac{1-(\frac{2}{e^{2\gamma}})^m}{1-\frac{2}{e^{2\gamma}}} + (1-2^{-m})\right)\\
&\leq \frac{C_1 m}{2^m} + \frac{C_2 m}{e^{2\gamma m}} + \frac{C_3 m^2}{2^m} + \frac{1}{2^{\frac{m}{2}+1}}\\
&\leq \frac{C_0 m}{e^{2\gamma m}} + \frac{C_0 m^2}{2^m} + \frac{1}{2^{\frac{m}{2}+1}}.
\end{aligned}$$

Here C_0 is a constant which only depends on N, γ, B. Since, clearly, $\frac{C_0 m}{e^{2\gamma m}}$, $\frac{C_0 m^2}{2^m}$ and $\frac{1}{2^{\frac{m}{2}+1}}$ are all summable,

$$\mathtt{Var}\left(2^{-m}\sum_{i=1}^{2^m} 1_B(Y_m^i)\right) < \infty.$$

Thus (4.2) holds and the proof is complete. □

4.4.4 Strong Law for the relative system; $d > 1$

We now show that the limit in Theorem 4.4.3 holds for any $d \geq 1$. In fact, the proof carries through, as long as the covariance in high dimensions is controlled by its coordinates. The following lemma shows that the covariance between two indicator variables is controlled by the covariance between the coordinate indicator variables.

Lemma 4.4.2. *Consider an open rectangle B in $\mathbb{R}^d$, that is, $B = B_1 \times B_2 \times B_3 \cdots \times B_d$, where B_i is an open interval in $\mathbb{R}$ for $i = 1, 2, ..., d$. Let $X = (X_1, X_2, \cdots, X_d)$ and $Y = (Y_1, Y_2, \cdots, Y_d)$ be two random vectors in $\mathbb{R}^d$ such that the pairs $(X_1, Y_1), (X_2, Y_2), ..., (X_d, Y_d)$ are independent. Then*

$$|\mathtt{Cov}(1_B(X), 1_B(Y))| \leq \sum_{i=1}^{d} |\mathtt{Cov}(1_{B_i}(X_i), 1_{B_i}(Y_i))|.$$

Proof. One has

$$\begin{aligned}
&\mathtt{Cov}(1_B(X), 1_B(Y))\\
&= P(X \in B, Y \in B) - P(X \in B)P(Y \in B)\\
&= P(X_1 \in B_1, \cdots, X_d \in B_d, Y_1 \in B_1, \cdots, Y_d \in B_d)\\
&\quad - P(X_1 \in B_1, \cdots, X_d \in B_d)P(Y_1 \in B_1, \cdots, Y_d \in B_d).
\end{aligned}$$

Using the assumption,

$$\begin{aligned}
&P(X_1 \in B_1, \cdots, X_d \in B_d) \\
&\qquad = P(X_1 \in B_1)P(X_2 \in B_2)\cdots P(X_d \in B_d); \\
&P(Y_1 \in B_1, \cdots, Y_d \in B_d) \\
&\qquad = P(Y_1 \in B_1)P(Y_2 \in B_2)\cdots P(Y_d \in B_d),
\end{aligned}$$

and

$$\begin{aligned}
&P(X_1 \in B_1, \cdots, X_d \in B_d, Y_1 \in B_1, \cdots, Y_d \in B_d) \\
&= P(X_1 \in B_1, Y_1 \in B_1)P(X_2 \in B_2, Y_2 \in B_2)\cdots P(X_d \in B_d, Y_d \in B_d).
\end{aligned}$$

Using the shorthands $a_i := P(X_i \in B_i, Y_i \in B_i), b_i := P(X_i \in B_i)P(Y_i \in B_i)$, one has $\mathtt{Cov}(1_{B_i}(X_i), 1_{B_i}(Y_i)) = a_i - b_i$, and from the computation above,

$$\mathtt{Cov}(1_B(X), 1_B(Y)) = a_1 a_2 \cdots a_d - b_1 b_2 \cdots b_d.$$

Then the statement becomes

$$|a_1 a_2 \cdots a_d - b_1 b_2 \cdots b_d| \le |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_d - b_d|.$$

Use that $0 \le a_i \le 1$ and $0 \le b_i \le 1$ and induction on d as follows. The statement is true for $d = 1$, and if it is true for some $d \ge 1$, then

$$\begin{aligned}
&|a_1 a_2 \cdots a_d a_{d+1} - b_1 b_2 \cdots b_d b_{d+1}| \\
=&|(a_1 a_2 \cdots a_d - \cdots - b_1 b_2 \cdots b_d)a_{d+1} + (a_{d+1} - b_{d+1})b_1 b_2 ... b_d| \\
\le&|(a_1 a_2 \cdots a_d - \cdots - b_1 b_2 \cdots b_d)a_{d+1}| + |(a_{d+1} - b_{d+1})b_1 b_2 ... b_d| \\
\le&|(a_1 a_2 \cdots a_d - \cdots - b_1 b_2 \cdots b_d)| + |a_{d+1} - b_{d+1}| \\
\le&|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_d - b_d| + |a_{d+1} - b_{d+1}|,
\end{aligned}$$

and so it is also true for $d + 1$. □

Our conclusion is that we can upgrade the proof of Theorem 4.4.3 to $d > 1$.

Chapter 5

The distribution of the particle system

Now we have collected enough information to describe the large time behavior the system as a whole.

5.1 Preparation

Below we describe the system's behavior as it depends on the parameters γ, b. The statements about the large time behavior of the branching particle system will follow from the behavior of the center of mass (Theorem 3.2.1 and Theorem 3.3.1), that of the relative system, and finally, from the following proposition.

Proposition 5.1.1 (Independence)**.** *The tail σ-algebra $\mathcal{T}$ of $\overline{Z}$ is independent of the relative system Y.*

Proof. The proof of this proposition is exactly the same as the corresponding proof of Lemma 14 in [5]. □

Recall that $Z(dy)$ denotes the discrete measure-valued process corresponding to the interacting branching system. The following notion will be important.

Definition 5.1.1 (Local extinction)**.** We say that Z suffers **local extinction**, if

$$Z_n(\mathrm{d}y) \stackrel{v}{\Rightarrow} 0, \text{ as } n \to \infty, \ a.s. \tag{1.1}$$

Since Z is a discrete particle system, (1.1) is tantamount to the property that there exists an almost surely finite random time T such that

$$P(Z_n(B) = 0, \ \forall n \geq T, \ \forall B \subset \mathbb{R}^d \text{ ball}) = 1.$$

Finally, recall that the SLLN for the relative system was proven for $\gamma^\Delta = b + \gamma$ and $b^\Delta = 0$.

5.2 Inward drift in the motion component

We now turn to the first results about the behavior of Z, starting with $b > 0$. In fact, we distinguish between three further sub-cases.

Case 1: $b > 0,\ b + \gamma > 0$.
As we have demonstrated, the center of mass converges to zero as $t \to \infty$, and the relative system will be an inward O-U process with parameter $\gamma + b$. Putting Theorem 4.4.3 together with the a.s. converge of the C.O.M. (Theorem 3.2.1), and finally, with Proposition 5.1.1, we arrive at the following conclusion.

Theorem 5.2.1. *Assume that $b > 0$, and $b + \gamma > 0$. Then*

$$2^{-n} Z_n(\mathrm{d}y) \Rightarrow \left(\frac{\gamma + b}{\pi}\right)^{d/2} \exp(-(\gamma + b)|y|^2)\,\mathrm{d}y,$$

almost surely.

Remark 5.2.1. Since $\mathcal{R}$ is a countable family, the weak limit in the previous theorem is actually equivalent to the statement that for all $B \in \mathcal{R}$,

$$\lim_{n\to\infty} 2^{-n} Z_n(B) = \int_B \left(\frac{\gamma + b}{\pi}\right)^{d/2} \exp(-(\gamma + b)|y|^2)\,\mathrm{d}y,\ a.s.$$

(See the appendix for more elaboration.) ◇

Example 5.2.1 (Non-interactive branching O-U process)**.** Consider the case $\gamma = 0, b > 0$, that is, the case of a non-interacting branching (inward) O-U process with parameter b. The proof goes through in this case as well, and we obtain that for all $B \in \mathcal{R}$,

$$\lim_{n\to\infty} 2^{-n} Z_n(B) = \int_B \left(\frac{b}{\pi}\right)^{d/2} \exp(-b|y|^2)\,\mathrm{d}y,\ a.s.,$$

complementing the exponential-clock results in [7].

Case 2: $b > 0,\ b + \gamma = 0$.

Theorem 5.2.2. *In this case,*

$$\lim_{n\to\infty} C n^{d/2} 2^{-n} Z_n(B) = \mathtt{Leb}(B),\ a.s., \tag{2.1}$$

for any bounded Borel set B, where `Leb` *denotes Lebesgue measure. Here* $C := (2\pi)^{d/2}$.

Proof. We are going to utilize the lemma on interchangeability (Lemma 4.2.1). Namely, we match the relative system with that of another system without interaction.

This other system is the one with $b = \gamma = 0$ (branching Brownian motion without interaction). As far as the behavior of this second system is concerned, it is well known (see [14, 2]), that (2.1) holds.

Even though in [14, 2], the decomposition into C.O.M. and a relative system was not considered, we do it now. That's useful, because by Lemma 4.2.1, the relative system is the same for the two processes, even though the behavior of the C.O.M. is not: for the original system it converges to the origin almost surely (Theorem 3.2.1), and for the non-interacting BBM it has an almost sure (Gaussian) limit (see [5]).

Now use the fact that Lebesgue measure is translation invariant. In both systems, one has

$$Z_t(B) = Y_t(B + \overline{Z}_t),$$

hence

$$Cn^{d/2}2^{-n}Z_n(B) = Cn^{d/2}2^{-n}Y_n(B + \overline{Z}_n).$$

By conditioning on the almost sure limit of $\overline{Z}_n$, and using Proposition 5.1.1, the translation invariance of the Lebesgue measure yields (2.1) for the *relative* system in the $b = \gamma = 0$ case.

But then, by Lemma 4.2.1, the same holds for the relative system in the original model. Since the C.O.M. converges to the origin almost surely for the original model, using Proposition 5.1.1, we conclude that the scaling limit (2.1) also holds for the original system. □

Case 3: $b > 0,\ b + \gamma < 0$.
As the relative system behaves asymptotically like an outward O-U process, we have a conjecture similar to the one in [5]. In our case, however, the center of mass tends to 0 as $t \to \infty$. Thus, the conjecture will take the following form:

Conjecture 5.2.1. *The following dichotomy holds for the long term behavior of the process:*

(1) If $\frac{\log 2}{d} \le |b + \gamma|$*, then* Z *suffers local extinction.*

(2) If $\frac{\log 2}{d} > |b + \gamma|$*, then*

$$2^{-n}e^{d|b+\gamma|n}Z_n(\mathrm{d}y) \overset{v}{\Rightarrow} \mathrm{d}y.$$

Remark 5.2.2. The intuitive explanation of the dichotomy in the conjecture is as follows. Even though the motion component has a strong inward component (forcing the center of mass to tend to the origin, according to Theorem 3.2.1), this is offset by the even stronger repulsion term.

This combined effect is then competing with the mass creation (the 'rate' of mass creation in this case can be taken $\log 2$): if mass creation is stronger, then the Law of Large Numbers is still in force; otherwise the mass creation is no longer able to compensate the fact that particles are 'being pushed away.' ◇

5.3 Outward drift in the motion component

This case is more difficult than the case of the inward drift, and the result below may be somewhat surprising, without keeping in mind the decomposition of the process (C.O.M. plus relative system) we are exploiting.

Case 4: $b < 0$ (Outward drift)

In this case, according to Theorem 3.3.1, the center of mass converges to infinity a.s. as $t \to \infty$, and so the question is, intuitively, whether this effect will be compensated by the large number of particles.

The next result is somewhat counterintuitive in that it says that for outward drift and attraction, the system always exhibits local extinction, no matter what the relationship is between $|b|$ and γ.

Theorem 5.3.1. *For $b < 0$ and $\gamma > 0$ (outward O-U with attraction), there is local extinction:*

$$Z_n(\mathrm{d}y) \stackrel{v}{\Rightarrow} 0 \ a.s.$$

Proof. The proof is based on comparing the speed of the C.O.M. with that of the relative system. Just like before, we distinguish between three sub-cases.

(1) When $b = \gamma = 0$, bounding the union by the sum, and using Borel-Cantelli, it is easily seen that

$$P(Z_n(B(0, cn)^c) > 0 \ i.o.) = 0,$$

for $c > \sqrt{2 \log 2}$. (The asymptotic speed is actually exactly $\sqrt{2 \log 2}$, but we don't need this here. Cf. [11].) Therefore, if $b + \gamma = 0$, the spread of the relative system cannot be more than linear either. To see why this is true, recall that by Δ-transform invariance, the relative system has the same law as a non-interactive branching Brownian motion ($b = \gamma = 0$). That latter one cannot have more than linear spatial spread, because this is true for the global system and the C.O.M. has an a.s. finite limit.

Turning back to the original system, if $b + \gamma = 0$, its relative system spreads at most linearly by the previous paragraph, however, the C.O.M. escapes to infinity exponentially fast, according to our Theorem 3.3.1. Therefore the system stops charging any given ball after some finite random time.

(2) If $b + \gamma > 0$, the speed of the relative system is even smaller, in fact $\mathcal{O}(t)$ as $t \to \infty$ (see [7]), and the conclusion is the same as in (1) above.

(3) Finally, for the case when $b + \gamma < 0$ and $\gamma > 0$, we still have the same conclusion, since the logarithmic rate of escape of the relative system is $-(b + \gamma)t$ (see [7]), which is less than $-bt$, the logarithmic escape rate of the C.O.M.

Hence, in all three cases, local extinction occurs. □

We conclude with posing an open problem:

Open problem (outward O-U with repulsion). Describe the large-time behavior of the system for $b < 0$ and $\gamma < 0$.

Chapter 6

The behavior of the C.O.M. for a drift $b(\cdot)$ bounded between positive constants

So far we have been working under the assumption that the drift is linear: $b(x) = bx$. Assume now instead, that the drift satisfies that

$$0 < a < b(x) < b.$$

For simplicity, we still start with $d = 1$. As we have

$$dZ_t^i = dW_t^i + b(Z_t^i)dt + \frac{1}{2^m}\sum_{j=1}^{2^m}\gamma(Z_t^j - Z_t^i)dt,$$

the motion of the center of mass $\overline{Z_t}$ satisfies the equation

$$d\overline{Z_t} = \frac{1}{2^m}\sum_{i=1}^{2^m} dW_t^i + \frac{1}{2^m}\sum_{i=1}^{2^m} b(Z_t^i)dt.$$

As $a < b(x) < b$, we know that $a < \frac{1}{2^m}\sum_{i=1}^{2^m} b(Z_t^i) < b$. Thus we have

$$\frac{1}{2^m}\sum_{i=1}^{2^m} dW_t^i + a dt < d\overline{Z_t} < \frac{1}{2^m}\sum_{i=1}^{2^m} dW_t^i + b\, dt.$$

Integration on both sides yields

$$\frac{1}{2^m}\sum_{i=1}^{2^m} W_t^i + at < \overline{Z_t} < \frac{1}{2^m}\sum_{i=1}^{2^m} W_t^i + bt.$$

Since (as we have discussed above) $\frac{1}{2^m}\sum_{i=1}^{2^m} W_t^i$ is a slowed down Brownian motion and $b > a > 0$, the center $\overline{Z_t}$ will tend to $+\infty$ with an essentially constant speed a.s. More precisely, we have the following result.

Theorem 6.0.2. *Assume that the drift satisfies $0 < a < b(x) < b$. Then*

$$0 < a < \liminf_{t\to\infty} t^{-1}\overline{Z_t} \le \limsup_{t\to\infty} t^{-1}\overline{Z_t} < b.$$

Remark 6.0.1. Similarly, if we have $a < b(x) < b$ with $a < b < 0$, then the center $\overline{Z_t}$ tends to $-\infty$ with essentially constant speed a.s. For $d > 1$, we could have a similar result by replacing $|\overline{Z_t}|$ with $\overline{Z_t}$, as we could consider the statement coordinate-wise.

Remark 6.0.2. *We can see for the drift $b(x)$ we may have more other choices. It is more interesting for $b(x)$ is bounded by two linear functions if one wants to study it.* $\diamond$

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Appendix A

Appendix

The following criterion was used in this thesis. Let $\texttt{Leb}$ denote d-dimensional Lebesgue measure. Recall that by a *bounded rational rectangle* we mean a set $B \subset \mathbb{R}^d$ of the form $B = I_1 \times I_2 \times \cdots \times I_d$, where I_i is a bounded interval with rational endpoints for each $1 \le i \le d$. The family of all bounded rational rectangles is $\mathcal{R}$.

Lemma A.0.1. *Let $\mu_1, \mu_2, ...$ be finite measures on $\mathbb{R}^d$ and $\mu << \texttt{Leb}$. Then $\mu_n \Rightarrow \mu$ if and only if $\lim_{n\to\infty} \mu_n(B) = \mu(B)$ for all $B \in \mathcal{R}$.*

Proof. The necessity follows from the well-known Portmanteau Theorem for weak convergence, and the absolute continuity of μ.

The sufficiency follows immediately from the theorem on p.14 of [3], as it is easy to check that $\mathcal{R}$ is closed under finite intersections, and each open set in $\mathbb{R}$ can be obtained as a countable (or finite) union of sets in $\mathcal{R}$. □