On Some Min-Max Cardinals on Boolean Algebras

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On some min-max cardinals on Boolean algebras

by

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B.A., University of Colorado, 2008
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A thesis submitted to the
Faculty of the Graduate School of the
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of the requirements for the degree of
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This thesis entitled:
On some min-max cardinals on Boolean algebras
written by Kevin Selker
has been approved for the Department of Mathematics

______________________________
Don Monk

______________________________
Prof. Keith Kearnes

Date ______________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
This thesis is concerned with cardinal functions on Boolean Algebras (BAs) in general, and especially with min-max type functions on atomless BAs. The thesis is in two parts:

(1) We make use of a forcing technique for extending Boolean algebras. The same type of forcing was employed in [BK81], [Kos99], and elsewhere. Using and modifying a lemma of Koszmider, and using CH, we prove some general extension lemmas, and in particular obtain an atomless BA, $A$ such that $f(A) = s_{mm}(A) = \omega < u(A) = \omega_1$. The example answers questions which were raised by [Mon08] and [Mon11].

(2) We investigate cardinal functions of min-max and max type and also spectrum functions on moderate products of Boolean algebras. We prove several theorems determining the value of a function on a moderate product in terms of the values of that function on the factors.
Dedication

To Mom, Dad, Lisa, M.E., Pipin, Eduardo, Alfonsina, and Beawolf.
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Soon after Cantor identified the dichotomy between the cardinality, $c$, of the real numbers and the cardinality of countable sets, mathematicians began noticing many properties of countable sets that fail to hold for sets of size $c$. For a few examples, consider that

1. The union of countably many measure zero sets has measure zero.
2. Countably many nowhere dense sets cannot cover the real line.
3. Given a countable collection of sequences of natural numbers, there is a sequence eventually dominating each of the given ones.
4. Given countably many sets of integers, there is another set having only finitely many elements in common with each given one.

While it is obvious that each of these items fails to hold if the word “countable” is replaced with “cardinality $c$,” a natural question is whether they may hold for some uncountable cardinals. A continuum cardinal is thus defined to be the smallest cardinality at which some property, which is true for countable sets and false for sets of size $c$, fails. Of course, under the continuum hypothesis (CH), every uncountable cardinal is at least $c$ so the question is trivial, but the CH is neither provable nor refutable from the usual axioms (ZFC) of set theory. Thus the motivating question of the theory of continuum cardinals has been to find what possibilities exist in situations where the CH is false.
Many of the characteristics defining these continuum cardinals come from the real numbers, and are naturally defined in terms of the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{fin}$, which is dual its Stone space, the Stone-Čech remainder $\beta\mathbb{N} \setminus \mathbb{N}$. Viewing these characteristics in the more general Boolean algebraic setting raises new questions and gives new possibilities. One can attempt to classify the behavior and relationships among the characteristics for a specific class of Boolean algebras (e.g. atomless, free, superatomic, or interval). Also, by varying the Boolean algebra from $\mathcal{P}(\mathbb{N})/\text{fin}$, one can remove the necessity of assuming that the CH fails, thus many results are possible without forcing or extra set theoretic assumptions.

Boolean algebras also have many natural cardinal characteristics of their own, such as the number of subalgebras, or the maximal size of a free subalgebra. Since these cardinals depend on the Boolean algebra, we call them cardinal functions on Boolean algebras. The category of Boolean algebras is dual to the category of Boolean topological spaces via Stone duality, and so some functions come from topology. For example, given a Boolean algebra $A$,

(1) We say that a collection of open sets $B$ is a local basis at a point $p$ if for every neighborhood $U$ of $p$, there is a $B \in B$ such that $B \subseteq U$. Then $u(A)$ is the minimal size of any local basis for points in the Boolean space $\text{Ult}(A)$.

(2) $s(A)$ is the maximal size of a discrete subspace of $\text{Ult}(A)$.

Another function $\mathfrak{f}$ is closely related to the notion of free sequences in a topological space: A sequence $\langle x_\alpha : \alpha < \xi \rangle$ is free in the topological sense if whenever $\beta < \xi$, the sets $\{x_\alpha : \alpha < \beta\}$ and $\{x_\alpha : \alpha \geq \beta\}$ have disjoint closures.

The majority of this thesis investigates what possibilities exist for cardinal functions of this type. Many of the cardinal functions defined for general algebras are in fact continuum cardinals, by considering the case $A = \mathcal{P}(\mathbb{N})/\text{fin}$, and so an important question is where (consistently) these cardinal invariants lie amongst the other classical continuum cardinals.

The functions $\mathfrak{f}, s, u$ have algebraic definitions for a Boolean algebra $A$:

- We may define $u(A)$ to be the minimal size of a nonprincipal ultrafilter generating set of
A subset \( Y \) of a BA is **ideal-independent** if no element of \( Y \) belongs to the ideal generated by the other elements of \( Y \). Then \( s(A) \) is the maximal size of an ideal-independent family of \( A \).

We define a “small” version of \( s \), \( s_{mm}(A) \) as the minimal size of any ideal-independent family of \( A \) that is maximal with respect to inclusion.

A **free sequence** in a BA is a sequence \( X = \{ x_\alpha : \alpha < \gamma \} \) such that whenever \( F \) and \( G \) are finite subsets of \( \gamma \) such that each member of \( F \) is less than each member of \( G \) then

\[
\left( \prod_{\alpha \in F} x_\alpha \right) \cdot \left( \prod_{\beta \in G} -x_\beta \right) \neq 0
\]

where empty products equal 1 by definition.

We define \( f(A) \) to be the minimal size of a free sequence in \( A \) that is maximal with respect to end-extension.

To avoid triviality we restrict our attention to atomless Boolean algebras, and ask what possible inequalities exist for these functions on an algebra \( A \). In Chapter 2 we construct an example from CH of an atomless Boolean algebra \( A \) with \( s_{mm}(A) = f(A) = \aleph_0 < u(A) = \aleph_1 \), which answers a problem raised by Don Monk. The construction involves a forcing-like approach to extend Boolean algebras.

Another avenue for research is to investigate the behavior of cardinal functions under algebraic operations like homomorphisms, subalgebras and products.

The notion of a **moderate product** of Boolean algebras is due independently to Gurevich [Gur82] and Weese [Wee80]. These products were extensively studied by Heindorf [Hei92], to whom the name is due. The moderate product is a subdirect product that contains an isomorphic copy of the weak product as a subalgebra, and is embeddable into the full direct product. In contrast to the direct product, moderate products retain much of the structure of their factors and behave
more like weak products. Indeed, weak product is special case of moderate product. Moderate products can serve as a source of examples, and Heindorf used the construction to construct large retractive Boolean algebras not embeddable into interval algebras.

In chapter 3 we investigate the behavior of many well known cardinal functions under the moderate product. Given a moderate product $A$ and a cardinal function $k$, we give $k(A)$ in terms of the values of $k$ on the factors.
Chapter 2

Ideal Independence, Free Sequences, and the Ultrafilter Number

Definition 2.0.1.  
(1) A subset $Y$ of a BA is ideal-independent if $\forall y \in Y, y \not\in \langle Y \setminus \{y\} \rangle^{\text{id}}$. By $\langle Y \setminus \{y\} \rangle^{\text{id}}$ we mean the ideal generated by the members of $Y$ other than $y$.

(2) We define $s_{\text{mm}}(A)$ to be the minimal size of an ideal-independent family of $A$ that is maximal with respect to inclusion.

(3) A free sequence in a BA is a sequence $X = \{x_\alpha : \alpha < \gamma\}$ such that whenever $F$ and $G$ are finite subsets of $\gamma$ such that $\forall i \in F \forall j \in G[i < j]$, then
\[
\left(\prod_{\alpha \in F} x_\alpha\right) \cdot \left(\prod_{\beta \in G} -x_\beta\right) \neq 0.
\]
Where empty products equal 1 by definition.

(4) We define $f(A)$ to be the minimal size of a free sequence in $A$ that is maximal with respect to end-extension.

(5) We define $u(A)$ to be the minimal size of a nonprincipal ultrafilter generating set of $A$.

(6) If $A$ is a Boolean algebra and $u$ is a nonprincipal ultrafilter on $A$, let $P(A,u)$ be the partial order consisting of pairs $(p_0, p_1)$ where $p_0, p_1 \in A \setminus u$, and $p_0 \cap p_1 = \emptyset$, ordered by $(p_0, p_1) \leq (q_0, q_1)$ (“$p_0, p_1$ is stronger than $(q_0, q_1)$”) iff $q_i \subseteq p_i$ for $i = 0, 1$.

The main result of this chapter is that under CH there is an atomless BA $A$ such that $\omega = f(A) = s_{\text{mm}}(A) < u(A) = \omega_1$. Theorem 2.10 in [Mon08] asserts the existence of an atomless BA
with $s_{mm}(A) < u(A)$, but the proof is faulty. The existence of an atomless BA $A$ with $f(A) < u(A)$ is a problem raised in [Mon11].

From now on, unless specified otherwise, $A$ will denote a subalgebra of $\mathcal{P}(\kappa)$ for $\kappa$ an infinite cardinal and $u$ will denote a nonprincipal ultrafilter on $A$.

**Lemma 2.0.1.** Suppose that $G$ is a filter that intersects every dense subset of $P(A, u)$. Let $g = \bigcup_{(p_0, p_1) \in G} p_0$. Let $e, f \in A$. Suppose that for some $p = (p_0, p_1) \in G$ we have $e \triangle f \subseteq p_0 \cup p_1$. Then the set $b := (g \cap e) \cup (f \setminus g)$ is a member of $A$.

**Proof.** First we claim $g \cap (e \setminus f) = p_0 \cap (e \setminus f)$. In fact, $\supseteq$ is clear. Now suppose that $q \in G$; we want to show that $q_0 \cap (e \setminus f) \subseteq p_0 \cap (e \setminus f)$. Choose $r \in G$ such that $r \leq q, p$. Note that $r_0 \cap p_1 \subseteq r_0 \cap r_1 = 0$. Hence

$$q_0 \cap (e \setminus f) \subseteq r_0 \cap (p_0 \cup p_1) \cap (e \setminus f) \subseteq p_0 \cap (e \setminus f).$$

Second we claim $(-g) \cap (f \setminus e) = (p_1 \cap (f \setminus e))$.

For $(\supseteq)$ suppose that $q \in G$; we want to show that $q_0 \cap p_1 \cap (f \setminus e) = 0$. Choose $r \in G$ such that $r \leq p, q$. Then $r_0 \cap p_1 \subseteq r_0 \cap r_1 = 0$, so $q_0 \cap p_1 \cap (f \setminus e) \subseteq r_0 \cap p_1 = 0$.

For $(\subseteq)$, we have $f \setminus e \subseteq p_0 \cup p_1$, so $-p_1 \subseteq p_0 \cup -(f \setminus e) \subseteq g \cup -(f \setminus e)$, hence $-q \cap (f \setminus e) \subseteq p_1$.

Now write

$$b = [b \cap (e \cap f)] \cup [b \cap -(e \cap f)] = \{[(g \cap e) \cup (-g \cap f)] \cap (e \cap f)\} \cup \{(g \cap e) \cup (-g \cap f)\} \cap -(e \cap f)\} = (g \cap e \cap f) \cup (-g \cap e \cap f) \cup (g \cap e \cap -f) \cup (-g \cap f \cap -e) \cup (e \cap f) \cup [p_0 \cap (e \setminus f)] \cup [p_1 \cap (f \setminus e)].$$

The last line is a member of $A$. 

We now use a version of Proposition 3.6 from [Kos99]. We include a detailed proof here for completeness.
Lemma 2.0.2. Let \( X \) be maximal ideal-independent in \( A \). Suppose that \( A, X, \) and \( P(A, u) \) are all
subsets of a c.t.m. \( M \) of ZFC. Suppose that \( G \) is \( P(A, u) \)-generic over \( M \), and let \( g = \bigcup \{ p_0 : (p_0, p_1) \in G \} \).
Then \( X \) is still maximal ideal independent in \( \langle A \cup \{ g \} \rangle \) (as viewed in \( M[G] \)).

Proof. Let \( e, f \in A \). For any \( p \in P(A, u) \) we define \( p^* = (e \cap p_0) \cup (f \cap p_1) \), \( a_p = \kappa \setminus (p_0 \cup p_1) \),
\( e_p = a_p \cap e \), and \( f_p = a_p \cap f \). We define a subset \( D_{e,f} \) of \( P(A, u) \) in \( M \) as follows.

\( p \in D_{e,f} \) iff one of the following conditions holds:

1. \( p_0 \cup p_1 \supseteq e \triangle f \)

2. \( \exists n \in \omega \exists x_0, \ldots, x_n \in X \ [x_0 \subseteq p^* \cup x_1 \cup \ldots \cup x_n] \)

3. \( \exists n \in \omega \exists x_0, \ldots, x_n \in X \ [p^* \cup a_p \subseteq x_0 \cup \ldots \cup x_n] \)

We claim

If \( p \in P(A, u) \) and \( x \notin u \), then there is a \( q \leq p \) such that \( x \subseteq q_0 \cup q_1 \) \( (*) \)

In fact, let \( q_0 = p_0 \cup (-p_1 \cap x) \) and let \( q_1 = p_1 \cup (-q_0 \cap x) \).

First we claim that \( D_{e,f} \) is dense in \( P(A, u) \). So let \( p \in P(A, u) \). One of the following holds

(i) \( e_p \cap f_p \in u \)

(ii) \( \kappa \setminus (e_p \cup f_p) \in u \)

(iii) \( e_p \setminus f_p \in u \)

(iv) \( f_p \setminus e_p \in u \).

Note that \( e_p \setminus f_p = a_p \cap (e \setminus f) \), \( f_p \setminus e_p = a_p \cap (f \setminus e) \), and \( e_p \Delta f_p = a_p \cap (e \Delta f) \). If (i) or (ii) is the case, then \( e_p \Delta f_p \notin u \), so also \( e \Delta f \notin u \) (as \( p_0 \cup p_1 \notin u \)). By \( (*) \) there is a \( q \leq p \) such that \( q_0 \cup q_1 \supseteq e \Delta f \), so that (1) of the definition of \( D_{e,f} \) is satisfied.

Next, suppose that (iii) is the case. Then also \( e \setminus f \in u \); by \( (*) \) there is a \( q \leq p \) such that
\( -(e \setminus f) \subseteq q_0 \cup q_1 \), so that \( a_q \subseteq e \setminus f \). Now by maximality of \( X \) in \( A \) we have that for some \( n \in \omega \)
and some \( x_0, \ldots, x_n \in X \),
(v) \( x_0 \subseteq q^* \cup x_1 \cup \ldots \cup x_n, \) or

(vi) \( q^* \subseteq x_0 \cup \ldots \cup x_n. \)

If (v) is the case, then condition (2) in the definition of \( D_{e,f} \) is satisfied. So suppose that (vi) is the case. Again, by maximality of \( \mathcal{X} \) in \( A \), there is an \( m \in \omega \) and some \( y_0, \ldots, y_m \in \mathcal{X} \) such that either:

(vii) \( a_q \subseteq y_0 \cup \ldots \cup y_m, \) or

(viii) \( y_0 \subseteq y_1 \cup \ldots \cup y_m \cup a_q. \)

If (vii) holds then \( q^* \cup a_q \subseteq x_0 \cup \ldots \cup x_n \cup y_0 \cup \ldots \cup y_m, \) so condition (3) of the definition of \( D_{e,f} \) is satisfied. Suppose then that (viii) holds.

- **Case 1.** \( a_q \cap y_0 \in u. \) Then \( a_q \setminus y_0 \notin u. \) Let \( r_0 = q_0 \) and \( r_1 = q_1 \cup (a_q \setminus y_0). \) We claim that \( r^* \cup a_r \subseteq y_0 \cup x_0 \cup \ldots \cup x_n, \) so \( r \) satisfies (3) in the definition of \( D_{e,f}. \) In fact, \( a_r = a_q \cap y_0 \subseteq y_0. \)

Now recall \( r^* = (e \cap r_0) \cup (f \cap r_1). \) Note that \( r_0 \setminus q_0 = \emptyset \) and \( r_1 \setminus q_1 \subseteq a_q. \) In particular, since \( a_q \subseteq e \setminus f, \) \( f \cap r_1 = f \cap q_1. \) Hence \( r^* = q^*, \) and by (vi) \( q^* \subseteq x_0 \cup \ldots \cup x_n. \) So \( r \) satisfies condition (3) of \( D_{e,f}. \)

- **Case 2.** \( a_q \cap y_0 \notin u. \) Then let \( r_0 = q_0 \cup (a_q \cap y_0) \) and let \( r_1 = q_1. \) Now using (viii) we have that \( y_0 \subseteq y_1 \cup \ldots \cup y_m \cup (a_q \cap y_0). \) Also \( a_q \cap y_0 \subseteq a_q \subseteq e, \) so \( a_q \cap y_0 \subseteq r^*. \) Thus we have \( y_0 \subseteq y_1 \cup \ldots \cup y_m \cup r^*. \) So condition (2) in the definition of \( D_{e,f}. \) is satisfied.

The case when \( f_p \setminus e_p \in u \) is treated similarly. Thus we have proved that the sets \( D_{e,f} \) are indeed dense.

Now suppose that \( b \in (A \cup \{g\}); \) we will show that \( \mathcal{X} \cup \{b\} \) is not ideal independent. Write \( b = (e \cap g) \cup (f \cap -g) \) for some \( e, f \in A. \) Now let \( p \in D_{e,f} \) be such that \( p \in G. \) Note that \( p_0 \subseteq g. \) Also \( p_1 \subseteq (-g), \) for, suppose that \( q \in G; \) we want to show that \( p_1 \cap q_0 = 0. \) Choose \( r \in G \) such that \( r \leq p, q. \) Then \( p_1 \cap q_0 \subseteq r_1 \cap r_0 = 0. \) So \( p^* \subseteq b. \) We consider cases according to the definition of \( D_{e,f}. \)
• Case 1. $p_0 \cup p_1 \supseteq e \triangle f$. Then Lemma 2.0.1 gives that $b \in A$, so $\mathcal{X} \cup \{b\}$ is not independent by maximality of $\mathcal{X}$ in $A$.

• Case 2. $\exists n \in \omega \exists x_0, \ldots, x_n \in \mathcal{X} \ [x_0 \subseteq p^* \cup x_1 \cup \ldots \cup x_n]$. Then $x_0 \subseteq b \cup x_1 \cup \ldots \cup x_n$.

• Case 3. $\exists n \in \omega \exists x_0, \ldots, x_n \in \mathcal{X} \ [p^* \cup a_p \subseteq x_0 \cup \ldots \cup x_n]$. Clearly $b \cap (p_0 \cup p_1) = p^*$, so $b \subseteq p^* \cup a_p$. So also $b \subseteq x_0 \cup \ldots \cup x_n$.

\textbf{Proposition 2.0.3 (CH).} Assuming the continuum hypothesis, there is an atomless BA $B$ such that $s_{\text{mm}}(B) = \omega$ and $u(B) = \omega_1$.

\textit{Proof.} First we make some definitions. Suppose that $A$ is a countable atomless subalgebra of $\mathcal{P}(\omega)$ and $u$ is an ultrafilter of $A$. ($A$ can be explicitly constructed by starting with a countable independent subset of $\mathcal{P}(\omega)$ and generating a Boolean algebra.) All ultrafilters are assumed to be nonprincipal. Let $P(A, u)$ be defined as above. For each $a \notin u$, define

$$D_a = \left\{ (p_0, p_1) \in P(A, u) : a \subseteq (p_0 \cup p_1), p_0 \setminus a \neq \emptyset \neq p_1 \setminus a \right\}.$$ 

We claim that each $D_a$ is dense in $P(A, u)$. If $p = (p_0, p_1) \in P(A, u)$, then we have that $b := p_0 \cup p_1 \cup a \notin u$. Because $A$ is atomless, there are disjoint, nonempty $x_0, x_1 \subseteq \omega \setminus b$ such that each $x_i \notin u$. Define $q_0 = p_0 \cup x_0$ and $q_1 = p_1 \cup x_1 \cup (a \setminus p_0)$. We have $q_0 \setminus a \neq 0$ since $x_0 \subseteq \omega \setminus a$, hence $x_0 = x_0 \setminus a \subseteq q_0 \setminus a$. Similarly $q_1 \setminus a \neq 0$. So $(q_0, q_1)$ is an extension of $p$ in $D_a$.

Next, for $i \in \omega$ we define

$$E_i = \{(p_0, p_1) \in P(A, u) : i \in p_0 \cup p_1\}.$$ 

Since $u$ is nonprincipal, $\{i\}$ is not a member of $u$ for any $i \in \omega$. Thus if $p = (p_0, p_1) \notin E_i$ then $(p_0 \cup \{i\}, p_1)$ is an extension of $p$ that is a member of $E_i$.

Fix some maximal ideal-independent set $\mathcal{X}$ in $A$, and for $e, f \in A$ let $D_{e,f}$ be as in the above lemma. Let

$$\mathcal{D} = \{D_a : a \notin u\} \cup \{D_{e,f} : e, f \in A\} \cup \{E_i : i \in \omega\}.$$
Since $\mathcal{D}$ is a countable collection of dense sets, there is a filter $G \subseteq P(A, u)$ such that $G$ has nonempty intersection with every member of $\mathcal{D}$. Given such a $G$ we will say that $x$ is generic for $P(A, u)$ iff $x = \bigcup_{(p_0, p_1) \in G} p_0$. By the proof of the above lemma, $\mathcal{X}$ remains maximal in $\langle A \cup \{x\} \rangle$. We shall show, however, that $x \notin A$, and $u$ does not generate an ultrafilter in $\langle A \cup \{x\} \rangle$. First, suppose for a contradiction that $x \in A$. Then either $x \in u$ or $-x \in u$. If $-x \in u$ then $D_x \in \mathcal{D}$, so choose $p = (p_0, p_1) \in D_x \cap G$. By definition of $x$ we have $p_0 \subseteq x$. But $p \in D_x$, so also $p_0 \setminus x \neq \emptyset$, a contradiction. We reach a contradiction similarly if $x \in u$. In fact, the same argument works since if $p \in D_{-x} \cap G$ then $p_1 \subseteq -x$. For, if $q \in G$, choose $r \in G$ with $r \leq p, q$. Then $q_0 \cap p_1 \subseteq r_0 \cap r_1 = \emptyset$. So $p_1 \cap q_0 = \emptyset$. Hence $p_1 \cap x = \emptyset$.

Next, suppose that $u$ were to generate an ultrafilter in $\langle A \cup \{x\} \rangle$. So there is an $a \in A \setminus u$ such that either $x \leq a$ or $-x \leq a$. If $x \leq a$ then consider $(p_0, p_1) \in G \cap D_a$. We claim that $x = x \cap a = p_0 \cap a \in A$, a contradiction. In fact, clearly $x \cap a \supseteq p_0 \cap a$. For the other inclusion, consider an arbitrary $q \in G$ and let $r \in G$ be such that $r \leq q, p$. Then since $p \in D_a$ we get $q_0 \cap a \subseteq r_0 \cap (p_0 \cup p_1) \cap a \subseteq p_0$, since $r_0 \cap r_1 = 0$ and $p_1 \subseteq r_1$. Thus $x \cap a \subseteq p_0 \cap a$. To carry out a symmetrical argument in case $-x \leq a$ we just need to see that $\neg x = \bigcup_{(p_0, p_1) \in G} p_1$. For $(-)$, suppose that $i \in -x$. Let $p \in G \cap E_i$. So $i \in p_0 \cup p_1$. We must have $i \notin p_0$ or else $i \in x$, so $i \in p_1$. For the opposite inclusion, suppose that $p \in G$ and $i \in p_1$. Letting $q \in G$ be arbitrary it suffices to show that $i \notin q_0$. Find $r \in G$ such that $r \leq p, q$. Then $r_0 \cap r_1 = \emptyset$ implies that $r_0 \cap p_1 = \emptyset$, so $i \notin r_0$. Now, because $r_0 \supseteq q_0$, we see that also $i \notin q_0$.

As a final preliminary, we would like to see that $\langle A \cup \{x\} \rangle$ is atomless (since $A$ is). Suppose for a contradiction that $x \cdot a$ is an atom for some $a \in A$. If $a \notin u$ then $x \cdot a = p_0 \cdot a$ for $(p_0, p_1) \in D_a \cap G$ (as proved and used above). As $p_0 \cdot a \in A$ this contradicts the fact that $A$ is atomless. So $a \in u$. Now, consider $p := (p_0, p_1) \in D_{-a} \cap G$. We have that $p_0 \setminus (-a) = p_0 \cdot a$ is not $\emptyset$. Also $p_0 \cdot a \notin u$. So there is a $q \in D_{a \cdot p_0} \cap G$. Then as above we have $q_0 \cdot (a \cdot p_0) = x \cdot (a \cdot p_0)$. Note that $x \cdot p_0 = p_0$, so the set on the right hand side is equal to $p_0 \cdot a$, hence is nonempty, and is in fact equal to the atom $x \cdot a$. But the set on the left hand side is in $A$, a contradiction. If the $-x \cdot a$ were assumed to be the atom, a symmetric argument yields a contradiction.
Let \( \ell_\alpha : \alpha < \omega_1 \) enumerate the limit ordinals below \( \omega_1 \). Partition \( \omega_1 \) into the sets \( \{ M_i : i \in \omega_1 \} \), with each part of size \( \omega_1 \). For each \( i \in \omega_1 \) let \( \langle k^i_\alpha : \alpha < \omega_1 \rangle \) enumerate \( M_i \setminus (\ell_i + 1) \). Now we construct a sequence \( \langle A_\alpha : \alpha < \omega_1 \rangle \) of countable atomless subalgebras of \( \mathcal{P}(\omega) \) as follows. Let \( A_0 \) be an arbitrary denumerable atomless subalgebra of \( \mathcal{P}(\omega) \). For any limit ordinal \( \alpha = \ell_i \) let \( A_\alpha = \bigcup_{\beta < \alpha} A_\beta \) and let \( \langle u^i_\beta : \beta < \omega_1 \rangle \) enumerate all the nonprincipal ultrafilters on \( A_\alpha \). Now suppose \( \alpha \) is the successor ordinal \( \gamma + 1 \). If \( \gamma = k^i_\beta \), we proceed as follows. Note that \( \ell_i < k^i_\beta \) and so \( u^i_\beta \subseteq A_\gamma \). Let \( \overline{u^i_\beta} \) denote the filter on \( A_\gamma \) generated by \( u^i_\beta \). If \( \overline{u^i_\beta} \) is not an ultrafilter or if \( \gamma \) is not in any of the sets \( M_i \setminus (\ell_i + 1) \) let \( A_\alpha = A_\gamma \). If \( \overline{u^i_\beta} \) is an ultrafilter then we let \( x_\gamma \) be generic for \( P \left( A_\gamma, u^i_\beta \right) \). Define \( A_\alpha = \langle A_\gamma \cup \{ x_\gamma \} \rangle \). Note that \( \overline{u^i_\beta} \) does not generate an ultrafilter on \( A_\alpha \).

Now define \( B = \bigcup_{\alpha < \omega_1} A_\alpha \). \( B \) is atomless as it is a union of atomless algebras. Suppose that some countable \( X \subseteq B \) generates an ultrafilter on \( B \). Then pick a limit ordinal \( \alpha = \ell_i < \omega_1 \) such that \( X \subseteq A_\alpha \). So \( X \) generates an ultrafilter of \( A_\alpha \); say it generates \( u^i_\beta \). Let \( \gamma = k^i_\beta \). Then by construction, \( X \) does not generate an ultrafilter on \( A_{\gamma + 1} \), contradiction. Therefore \( |B| = \omega_1 = u(B) \).

Next we claim that \( s_{\text{mm}}(B) = \omega \) because \( \mathcal{X} \) is still maximal ideal-independent in \( B \). Since any member of \( B \) is a member of \( A_\alpha \) for some \( \alpha < \omega_1 \), it suffices to show that \( \mathcal{X} \) is still maximal independent in each \( A_\alpha \). Suppose to the contrary, and let \( \alpha \) be minimal such that there is some \( x \in A_\alpha \setminus \mathcal{X} \) such that \( \mathcal{X} \cup \{ x \} \) is still ideal-independent. If \( \alpha < \omega_1 \) is a limit ordinal then \( x \in A_\beta \) for some \( \beta < \alpha \). By minimality of \( \alpha \), we see that \( \mathcal{X} \cup \{ x \} \) is not ideal-independent, a contradiction. If \( \alpha = \beta + 1 \), then Lemma 2.0.2 implies that, since \( \mathcal{X} \) is maximal ideal independent in \( A_\beta \), it remains so in \( A_\alpha \), a contradiction. Thus \( s_{\text{mm}}(B) = \omega \) as claimed.

**Lemma 2.0.4.** Suppose that \( A \) is an atomless subalgebra of \( \mathcal{P}(\kappa) \) and suppose that \( u \) is an ultrafilter of \( A \) (nonprincipal as always). Suppose that \( C = \langle c_i : i < \xi \rangle \subseteq A \) is a free sequence of \( A \) that is maximal with respect to end-extension and such that \( c_i \subseteq c_j \) for each \( i > j \in \xi \). There is a family, \( \mathcal{E} \), of dense subsets of \( P(A, u) \) such that \( C \) remains maximal in \( \langle A \cup \{ g \} \rangle \) whenever \( g = \bigcup_{(p_0, p_1) \in G} p_0 \) for a filter \( G \subseteq P(A, u) \) intersecting each member of \( \mathcal{E} \). Moreover \( |\mathcal{E}| \leq |A| \).

**Proof.** First we claim that \( \xi \) is a limit ordinal. In fact suppose that \( \xi = \eta + 1 \). We claim that \( c_\eta \) is
an atom of $A$ (a contradiction). In fact, suppose that $a \in A$ is such that $0 < a < c$. Then $C \smallsetminus \{a\}$ is still a free sequence, which contradicts the maximality of $C$. (Indeed, any strictly decreasing sequence of members of $A$ is free.) We will use the following fact several times:

\begin{align*}
\forall a \in A \left( \exists i \in \xi \left[ a \subseteq (\kappa \setminus c_i) \right] \text{ or } \exists i < j \in \xi \left[ (c_i \setminus c_j) \subseteq a \right] \text{ or } \kappa \setminus c_0 \subseteq a \right)
\end{align*}

(#)\n
To see this, suppose that $a \in A$. Clearly the desired conclusion holds if $a = \emptyset$ or $a = \kappa$; so suppose that $a \neq \emptyset, \kappa$. By maximality of $C$ we have that either

A. $\exists F \in [\xi]^{<\omega}$ such that $(\bigcap_{i \in F} c_i) \cap a = \emptyset$, or

B. $\exists F, G \in [\xi]^{<\omega}$, with $\forall i \in F \forall j \in G[i < j]$ such that $(\bigcap_{i \in F} c_i) \cap (\bigcap_{j \in G(\kappa \setminus c_j)) \cap (\kappa \setminus a) = \emptyset$

If $A$ holds then $F \neq \emptyset$ since $a \neq \emptyset$ and then $c_{\max F} \cap a = \emptyset$ so that $a \subseteq (\kappa \setminus c_{\max F})$, hence the first part of (#) holds.

If $B$ holds then $F \neq \emptyset$ or $G \neq \emptyset$ since $a \neq \kappa$. If $F \neq \emptyset \neq G$ then $(c_{\max F} \setminus c_{\min G}) \subseteq a$, giving the second condition of (#). If $F \neq \emptyset = G$ then $c_{\max F} \subseteq a$, giving the second condition of (#) again. Finally if $F = \emptyset \neq G$ then $(\kappa \setminus c_{\min G}) \subseteq a$, giving the second or third condition of (#).

Let $e, f \in A$. For any $p \in P(A, u)$ we define $p^* = (p_0 \cap e) \cup (p_1 \cap f)$, $a_p = \omega \setminus (p_0 \cup p_1)$, $e_p = a_p \cap e$, and $f_p = a_p \cap f$. We define a subset $E_{e,f}$ of $P(A, u)$ as follows.

$p \in E_{e,f}$ iff one of the following conditions holds:

1. $p_0 \cup p_1 \supseteq e \triangle f$

2. $\exists i < j \in \xi [p^* \supseteq c_i \setminus c_j]$

3. $\exists i \in \xi [p^* \cup a_p \subseteq \kappa \setminus c_i]$

4. $\kappa \setminus c_0 \subseteq p^*$.

We claim that $E_{e,f}$ is dense. Let $p \in P(A, u)$. One of the following holds

(i) $e_p \cap f_p \in u$
(ii) \( \omega \setminus (e_p \cup f_p) \in u \)

(iii) \( e_p \setminus f_p \in u \)

(iv) \( f_p \setminus e_p \in u \).

If (i) or (ii) is the case, then \( e_p \triangle f_p \not\in u \), so also \( e \triangle f \not\in u \) (as \( p_0 \cup p_1 \not\in u \)). Thus as in the proof of Lemma 2.0.2 we can extend \( p \) to a condition \( q \) such that \( q_0 \cup q_1 \supseteq e \triangle f \), so that (1) of the definition of \( E_{e,f} \) is satisfied.

Next, suppose that (iii) is the case. Then also \( e \setminus f \in u \), so again by the proof of Lemma 2.0.2 we can first extend \( p \) to some condition \( q \) so that \( a_q \subseteq e \setminus f \). Now \( q^* \in A \), so, by (\( \ast \)), either

(v) \( \exists i < \xi \ [q^* \subseteq \kappa \setminus c_i] \), or

(vi) \( \exists i < j \in \xi \ [q^* \supseteq c_i \setminus c_j] \), or

(vii) \( \kappa \setminus c_0 \subseteq q^* \).

If (vi) holds then \( q \) is in \( E_{e,f} \) by virtue of condition (2). If (vii), then \( q \) is in \( E_{e,f} \) by virtue of (4). So we assume now that (v) is the case, and fix \( i \in \xi \) as guaranteed by (v). Now also \( a_q \in A \), so either

(viii) \( \exists j < \xi \ [a_q \subseteq \kappa \setminus c_j] \), or

(ix) \( \exists j < k \in \xi \ [a_q \supseteq c_j \setminus c_k] \), or

(x) \( \kappa \setminus c_0 \subseteq a_q \).

First suppose that (viii) holds. Then \( a_q \cup q^* \subseteq (\kappa \setminus c_i) \cup (\kappa \setminus c_j) = \kappa \setminus (c_i \cap c_j) = \kappa \setminus c_{\max\{i,j\}} \), so \( q \in E_{e,f} \) by virtue of condition (3). Next assume that (ix) holds and fix \( j < k \in \xi \) as in that case. We consider two cases.

- Case 1. \( (c_j \setminus c_k) \in u \). Then extend \( q \) to a condition \( r \) such that \( r_0 = q_0 \), and \( r_1 = q_1 \cup (\neg q_0 \cap \neg (c_j \setminus c_k)) \). Then \( -(c_j \setminus c_k) \subseteq r_0 \cup r_1 \), so \( a_r \subseteq c_j \setminus c_k \). Note that \( r_1 \setminus q_1 \subseteq a_q \subseteq e \setminus f \), so \( (r_1 \setminus q_1) \cap f = \emptyset \). Then \( r^* = (r_0 \cap e) \cup (r_1 \cap f) = (q_0 \cap e) \cup (q_1 \cap f) \), and \((r_1 \setminus q_1) \cap f = \emptyset \).
so in fact \( r^* = q^* \). Recall that \( q^* \subseteq (\kappa \setminus c_i) \) so \( r^* \cup a_r \subseteq (\kappa \setminus c_{\text{max}\{i,k\}}) \). Thus condition (3) holds for \( r \).

- **Case 2.** \((c_j \setminus c_k) \not\subseteq u\). Then we extend \( q \) to a condition \( r \) so that \( r_0 = q_0 \cup (c_j \setminus c_k) \) and \( r_1 = q_1 \). Recall that \((c_j \setminus c_k) \subseteq a_q \subseteq e\), so \( r^* \supseteq (r_0 \cap e) \supseteq (c_j \setminus c_k) \cap e = c_j \setminus c_k \). Thus \( r \) satisfies condition (2) in the definition of \( E_{e,f} \).

Finally suppose that (x) is the case. Again we consider two cases.

- **Case 1.** \( a_q \cap c_0 \not\subseteq u \). Then we extend \( q \) to a condition \( r \) where \( r_0 = q_0 \) and \( r_1 = q_1 \cup (a_q \cap c_0) \). Then \( a_r \subseteq (\kappa \setminus c_0) \). Also \( r^* = q^* \) by the same argument as in Case 1 above. So \( a_r \cup r^* \subseteq (\kappa \setminus c_i) \), and \( r \) satisfies condition (3) of the definition of \( E_{e,f} \).

- **Case 2.** \( a_q \cap c_0 \in u \). Then we extend \( q \) to a condition \( r \) by setting \( r_0 = q_0 \cup (a_q \setminus c_0) \) and \( r_1 = q_1 \). Then \( r^* \supseteq r_0 \cap e \supseteq \kappa \setminus c_0 \), so condition (4) in the definition of \( E_{e,f} \) holds.

Thus the sets \( E_{e,f} \) are dense. Let \( \mathcal{E} = \{E_{e,f} : e, f \in A\} \). Clearly \(|\mathcal{E}| \leq |A| \). Suppose that \( G \) is a filter that intersects each member of \( \mathcal{E} \), and let \( g = \bigcup_{(p_0,p_1) \in G} p_0 \). We must show that \( C \) is still maximal in \( \langle A \cup \{g\} \rangle \). Letting \( b \in \langle A \cup \{g\} \rangle \) we can write \( b = (g \cap e) \cup (f \setminus g) \) for some \( e, f \in A \).

Let \( p \in G \cap E_{e,f} \); we will show that \( C \cup \{b\} \) is no longer free, considering cases according to the definition of \( E_{e,f} \).

- **Case 1.** \( p_0 \cup p_1 \supseteq e \triangle f \). By Lemma 2.0.1, in this case \( b \in A \). So \( b \) does not extend \( C \) by maximality in \( A \).

- **Case 2.** \( \exists i < j \in \xi [p^* \supseteq c_i \setminus c_j] \). We have that \( p^* \subseteq b \), so also \( c_i \setminus c_j \subseteq b \). Then \((c_i) \cap (\kappa \setminus c_j) \cap (\kappa \setminus b) = \emptyset \), so \( b \) does not extend \( C \).

- **Case 3.** \( \exists i \in \xi [p^* \cup a_p \subseteq \kappa \setminus c_i] \). Clearly \( b \cap (p_0 \cup p_1) = p^* \), so \( b \subseteq p^* \cup a_p \). So \( b \subseteq \kappa \setminus c_i \).

Thus \( c_i \cap b = \emptyset \), and again \( b \) does not extend \( C \).

- **Case 4.** \( \kappa \setminus c_0 \subseteq p^* \). Since \( p^* \subseteq b \), also \( \kappa \setminus c_0 \subseteq b \) so \((\kappa \setminus c_0) \cap (\kappa \setminus b) = \emptyset \). □
Proposition 2.0.5 (CH). Assuming CH there is an atomless Boolean algebra \( A \) such that \( \text{f}(A) = \omega \) and \( \text{u}(A) = \omega_1 \).

Proof. As above, we construct a sequence of countable, atomless subalgebra of \( \mathcal{P}(\omega) \), \( \langle A_\alpha : \alpha < \omega_1 \rangle \), such that if \( B := \bigcup_{\alpha < \omega_1} A_\alpha \), \( \text{u}(B) = \omega_1 \). The differences are as follows.

1. In \( A_0 \) there is a countable maximal free sequence \( C = \langle c_i : i \in \omega \rangle \) such that \( c_j \subseteq c_i \) for each \( i < j \in \omega \).

2. The sets \( E_{e,f} \) of Lemma 2.0.4 take the place of the \( D_{e,f} \) in the definition of \( D \).

We claim that \( \text{f}(B) = \omega \). Because \( B \) is atomless, it suffices to show that \( C \) is still maximal in \( B \).

Since each member of \( B \) is also a member of some \( A_\alpha \) for \( \alpha < \omega_1 \), it suffices to show that \( C \) is maximal in each \( A_\alpha \). Suppose not and let \( \alpha < \omega_1 \) be minimal such that for some \( x \in A_\alpha \), \( C \setminus \{x\} \) is still a free sequence. Clearly \( \alpha \) is a successor ordinal, say \( \alpha = \beta + 1 \). By minimality of \( \alpha \), \( C \) is maximal in \( A_\beta \). The above lemma guarantees that \( C \) is still maximal in \( A_\alpha \), contradiction. \( \square \)

Corollary 2.0.6 (CH). Assuming CH there is an atomless Boolean algebra \( A \) such that \( \text{s}_{\text{mm}}(A) = \text{f}(A) = \omega < \omega_1 = \text{u}(A) \).
Chapter 3

Cardinal Functions on Moderate Products of Boolean Algebras

3.1 Definition and Basic Facts

Let $I$ be an infinite set and let $B$ be a subalgebra of $\mathcal{P}(I)$ containing all singletons. Let $\langle A_i : i \in I \rangle$ be a system of Boolean algebras.

Given $x \in B$ let $\chi_x \in \prod_{i \in I} A_i$ be defined by setting $\chi_x(i) = 1_{A_i}$ if $i \in x$ and $\chi_x(i) = 0_{A_i}$ otherwise.

For any $a \in \prod_{i \in I} A_i$ we define the following subsets of $I$:

- $1(a) = \{ i \in I : a(i) = 1_{A_i} \}$,
- $0(a) = \{ i \in I : a(i) = 0_{A_i} \}$,
- $p(a) = \{ i : a(i) > 0_{A_i} \}$, and
- $m(a) = \{ i : 0_{A_i} < a(i) < 1_{A_i} \}$.

Also for $j \in I$ and $x \in A_j$ we define $\chi^j_x \in \prod_{i \in I} A_i$ by:

$$
\chi^j_x(i) = \begin{cases} 
  x & \text{if } i = j \\
  0 & \text{if } i \neq j.
\end{cases}
$$

Then we define the moderate product of $\langle A_i : i \in I \rangle$ over $B$, denoted $\prod^B_{i \in I} A_i$, to be the subset of $\prod_{i \in I} A_i$ consisting of $\{ x \in \prod_{i \in I} A_i : 1(x) \in B$ and $|m(x)| < \aleph_0 \}$.

Proposition 3.1.1. As defined above, $\prod^B_{i \in I} A_i$ is in fact a subalgebra of $\prod_{i \in I} A_i$. 

Proof. For brevity, let $A = \prod_{i \in I} A_i$. Clearly $0, 1 \in A$. We will show that $A$ is closed under the operations $+$ and $\cdot$. Suppose that $a, b \in A$. Then $1(a+b) = 1(a) \cup 1(b) \cup d$ for some $d \subseteq m(a) \cup m(b)$. So $d \in [I]^{<\aleph_0} \subseteq B$, hence $1(a+b) \in B$ as well. Now $m(a+b) \subseteq m(a) \cup m(b)$, so still $|m(a+b)| < \aleph_0$. Thus $a + b \in A$. Now if $a \in A$ then $1(-a) = (I \setminus 1(a)) \setminus m(a)$, and $m(a) \in B$, so $1(-a) \in B$. Also $m(-a) = m(a)$, so $|m(-a)| < \aleph_0$. So $-a \in A$, and $A$ does indeed form a subalgebra.

Note that if $B$ is the collection of finite and cofinite subsets of $I$, then $\prod_{i \in I} A_i$ is the same as the weak product.

**Proposition 3.1.2.** Let $I' := \{i \in I : |A_i| > 2\}$. If $I'$ is infinite then $A := \prod_{i \in I} A_i$ is not complete.

Proof. For each $i \in I'$ let $a_i \in A_i \setminus 2$. We claim that $X := \{\chi^i_{a_i} : i \in I'\}$ has no sum in $A$. Suppose to the contrary that $b = \sum^A X$. As $b \in A$ we have that $|m(b)| < \aleph_0$, so take $i \in I' \setminus m(b)$. Then $b(i) = 1$ since $b \geq \chi^i_{a_i}$. Now if we let $b'$ be the element of $A$ defined coordinate wise by setting $b'(j) = b(j)$ for all $j \neq i$, and $b'(i) = a_i$, we have that $b' < b$, but $b$ is still an upper bound for $X$, completing the proof.

**Corollary 3.1.3.** A moderate product is complete iff it is a finite product of complete algebras.

**Proposition 3.1.4.** $\prod_{i \in I} A_i$ is atomless iff $A_i$ is atomless for each $i \in I$. In particular, if $a \in A_i$ is an atom then $\chi^i_a$ is an atom of $A$.

**Proposition 3.1.5.** If $B \leq C$ then $\prod_{i \in I} A_i \leq \prod_{i \in I} A_i$.

**Proposition 3.1.6.** The algebras $B$, and, for any $i \in I$, $A_i$ are retracts of $\prod_{i \in I} A_i$.

Proof. For $B$, the maps $b \mapsto \chi^i_b$ and $a \mapsto 1(a)$ witness the retraction.

Now fix $i \in I$ and let $U \subseteq A_i$ be any ultrafilter of $A_i$ and let $X \subseteq A_i$ be a generating set for $A_i$. Let $\psi : X \to \prod_{i \in I} A_i$ be defined by setting

$$
\psi(a) = \begin{cases} 
\chi^i_a & \text{if } a \notin U \\
\chi^i_a + \chi_{I \setminus \{i\}} & \text{if } a \in U.
\end{cases}
$$
If $\prod F = 0$ for some $F \in [X]^{<\omega}$ then there is an $a \in F \cap U$ and an $b \in F \setminus U$. So $\psi(a) \cdot \psi(b) = 0$; hence $\prod \{\psi(f) : f \in F\} = 0$. So by Sikorski’s extension criterion we can extend $\psi$ to a homomorphism $\widehat{\psi} : A_i \to \prod_{i \in I} A_i$.

Let $\varepsilon : \prod_{i \in I} A_i \to A_i$ be defined by $\varepsilon(a) = a(i)$. Then $\varepsilon \circ \widehat{\psi} = \text{id}_{A_i}$.

Proposition 3.1.7. The algebras $A_i$ and $B$ are isomorphically embedded in $\prod_{i \in I} A_i$. Moreover the images of $A_i$ and $B$ are regular subalgebras of the moderate product.

3.2 Cardinal Functions on Moderate Products

Notation. Henceforth let $A = \prod_{i \in I} A_i$.

3.2.1 Ultrafilter Spectrum

For any Boolean algebra $A$ and any ultrafilter $\mathcal{U}$ on $A$ define

$$\chi(\mathcal{U}) = \min \{|X| : X \subseteq \mathcal{U} \text{ and } \forall u \in \mathcal{U} \exists x \in X[x \leq u]\},$$

and

$$u_{\text{spec}}(A) = \{\chi(\mathcal{U}) : \mathcal{U} \text{ is a nonprincipal ultrafilter on } A\}.$$  

Shelah initiated the study of $u_{\text{spec}}(\mathcal{P}(\omega))$ using the notation $\text{Sp}_\chi$ in [She08].

In what follows, we will say that a set $X$ minimally generates an ultrafilter $\mathcal{U}$ if $X$ generates $\mathcal{U}$, and no set $Y$ with $|Y| < |X|$ generates $\mathcal{U}$.

Proposition 3.2.1. $u_{\text{spec}}(A) = u_{\text{spec}}(B) \cup \bigcup_{i \in I} u_{\text{spec}}(A_i)$.

Proof. Note that if $X$ minimally generates a nonprincipal ultrafilter on $A_i$, then $\{\chi_x^i : x \in X\}$ minimally generates a nonprincipal ultrafilter on $A$.

Now suppose that $X$ minimally generates a nonprincipal ultrafilter on $B$, we claim $Y := \{\chi_x : x \in X\}$ minimally generates a nonprincipal ultrafilter on $A$. For brevity we will denote by $\mathcal{U}_X$ and $\mathcal{U}_Y$ the filters generated respectively by $X$ and $Y$. Clearly $|Y| = |X|$. $Y$ satisfies the FIP because $X$ does.
Now let \( a \in A \). Then take \( x \in X \) with \( x \leq 1(a) \) or \( x \leq -1(a) \). In case \( x \leq 1(a) \) then \( \chi_x \leq a \) so we are done. Suppose that \( x \leq I \setminus 1(a) \). Now \( m(a) \) is a finite subset of \( I \), so there is some \( z \in X \) with \( z \leq -m(a) \cdot x \). Then \( \chi_z \leq -a \), so \( \mathcal{U}_Y \) is an ultrafilter. Suppose that \( \mathcal{U}_Y \) is principal over \( a \), then take \( \chi_x \in Y \) with \( \chi_x \leq a \) (so \( \chi_x = a \)). Then for all \( x' \in X \), \( \chi_{x'} \geq \chi_x \), so that \( x' \geq x \), thus \( \mathcal{U}_X \) is principal over \( x \), a contradiction.

Suppose that \( Z \) is such that \(|Z| < |Y|\) and \( Z \) generates \( \mathcal{U}_Y \). We claim that the collection 
\[ 1"Z := \{1(z) : z \in Z\} \] generates \( \mathcal{U}_X \), which will contradict minimality of \( X \). First we check that 
\[ 1"Z \subseteq \mathcal{U}_X. \] For any \( z \in Z \), \( z \in \mathcal{U}_Y \), so we may take \( x \in X \) with \( \chi_x \leq z \). Then \( x \leq 1(z) \), so that 
\[ 1(z) \in \mathcal{U}_X. \] Next we show \( 1"Z \) generates \( X \), hence generates \( \mathcal{U}_X \). If \( x \in X \), \( \chi_x \in Y \), so there is some \( z \in Z \) with \( z \leq \chi_x \). Then \( 1(z) \leq x \), so \( 1"Z \) is as claimed. Thus there is no such \( Z \), so \( Y \) is minimal, and \( \supseteq \) is proved.

Now suppose that \( X \) minimally generates a nonprincipal ultrafilter \( \mathcal{U}_X \) on \( A \), with \( |X| = \kappa \notin \bigcup_{i \in I} u_{\text{spect}}(A_i) \), we will show that \( \kappa \in u_{\text{spect}}(B) \). Let \( Y = \{1(x) : x \in X\} \), and again let \( \mathcal{U}_Y \) denote the filter generated by \( Y \). We first claim

1. For each \( y \in Y \), \( \chi_y \in \mathcal{U}_X \).

Suppose not, so take \( x \in X \) with \( \chi_{1(x)} \notin \mathcal{U}_X \). Then \( \chi_m(x) \geq -\chi_{1(x)} \cdot x \in \mathcal{U}_X \), and \( m(x) \) is finite, so in fact for some \( i \in m(x) \), \( \chi(i) \in \mathcal{U}_X \). Then \( \left\{ \chi_x^i : x \in X \right\} \) generates \( \mathcal{U}_X \). So \( \pi_i \mathcal{U}_X \) is an ultrafilter on \( A_i \) generated by \( \{x(i) : x \in X\} \), but by our assumption that \( \kappa \notin u_{\text{spect}}(A_i) \), there is some collection \( \{x_\alpha : \alpha < \lambda\} \subseteq A_i \) with \( \lambda < \kappa \) such that \( \pi_i \mathcal{U}_X \) is also generated by \( \{x_\alpha : \alpha < \lambda\} \). But then \( \left\{ \chi_x^\alpha : \alpha < \lambda \right\} \) generates \( \mathcal{U}_X \), a contradiction of the minimality of \( X \). So (1) holds. Next we claim that

2. \( \{\chi_y : y \in Y\} \) generates \( \mathcal{U}_X \).

Let \( u \in \mathcal{U}_X \), and pick \( x \in X \) with \( x \leq u \). Then \( \chi_{1(x)} \leq x \leq u \), so also (2) holds.

By (1) and (2) and the minimality of \( X \), we must have that \(|Y| = |X|\). We now show \( \mathcal{U}_Y \) is a nonprincipal ultrafilter. Suppose that \( Y \) does not satisfy the FIP, so for some \( y_0, \ldots, y_n \in Y \), 
\[ \prod_{k=0}^n y_k = \emptyset. \] But then \( \prod_{k=0}^n \chi_y = 0 \), a contradiction of (2). Now let \( b \in B \), and by (2) take \( y \in Y \) so that \( \chi_y \leq \chi_b \) or \( \chi_y \leq -\chi_b \), then \( y \leq b \) or \( y \leq -b \). Thus \( \mathcal{U}_Y \) is an ultrafilter. If \( \mathcal{U}_Y \) were principal
over \( \{i\} \in I \) then \( \mathcal{U}_X \) is generated by \( \{x(i) : x \in X\} \), a contradiction as in the proof of (1). Thus \( \mathcal{U}_Y \) is nonprincipal.

Finally, suppose that \( Z \subseteq U_Y \) is such that \( |Z| < |Y| \) and \( Z \) generates \( \mathcal{U}_Y \). We claim that \( \{\chi_z : z \in Z\} \) generates \( \mathcal{U}_X \). First we show that \( \{\chi_z : z \in Z\} \subseteq \mathcal{U}_X \). For any \( z \in Z \), there is a \( y \in Y \) with \( y \leq z \), so that \( \chi_y \leq \chi_z \), hence \( \chi_z \in U_X \). Now if \( a \in U_X \) then there is a \( y \in Y \) such that \( \chi_y \leq a \). Choosing \( z \in Z \) with \( z \leq y \), we have that also \( \chi_z \leq a \). Thus \( \{\chi_z : z \in Z\} \) generates \( \mathcal{U}_X \), contradicting the minimality of \( X \).

\[ \square \]

**Corollary 3.2.2.** If \( K \) is an infinite system of infinite cardinals such that either

(i) \( |K| < \sup K \) or \( |K| = \sup K \in K \), or

(ii) \( |K| = \sup K \) is singular,

Then there is an atomless Boolean algebra \( A \) such that \( u_{\text{spect}}(A) = K \).

Note that if \( K \) does not satisfy (i) or (ii) then \( \sup K \) is regular limit. To see that \( |K| \) is a limit cardinal we will actually show that \( \neg(i) \) implies \( |K| = \aleph_{|K|} \). Suppose to the contrary that \( |K| < \aleph_{|K|} \). Then \( |K| = \aleph_\beta \) where \( \beta < |K| \). But then \( K \subseteq \{\aleph_\alpha : \alpha < \beta + 1\} \), so that \( |K| \leq |\{\aleph_\alpha : \alpha < \beta + 1\}| = |\beta| < |K| \), a contradiction.

In what follows we will let \( \text{FinCo}(K) \) denote the set \( \{S \subseteq K : |S| < \aleph_0 \text{ or } |K \setminus S| < \aleph_0\} \).

**Proof of corollary.** First note that for any cardinal \( \mu \), \( u_{\text{spect}}(\text{Fr}(\mu)) = \{\mu\} \). In fact, consider any \( X \in [\text{Fr}(\mu)]^{\aleph_\mu} \); we will show \( X \) cannot generate a nonprincipal ultrafilter. For each \( x \in X \) there is a finite set of free generators \( Y_x \) such that \( x \in \langle Y_x \rangle \). Let \( y \in \text{Fr}(\mu) \) be a free generator not in any \( Y_x \) for \( x \in X \). Then for every \( x \in X \), \( x \cdot y \neq 0 \neq x \cdot -y \), so \( X \) does not generate an ultrafilter.

Now let \( \kappa = |K| \). If (i) holds, then let \( \lambda \in K \) such that \( \lambda \geq \kappa \). Then we may enumerate \( K \) (possibly with repetitions) as \( K = \{\mu_\alpha : \alpha < \lambda\} \). Define \( A = \prod_{\alpha < \lambda}^{\text{FinCo}(\lambda)} \text{Fr}(\mu_\alpha) \). By the above proposition, \( u_{\text{spect}}(A) = \{\lambda\} \cup \bigcup_{\alpha < \lambda} \{\mu_\alpha\} \), as desired.

Now suppose (ii) holds. Let \( \mu \in K \) be such that \( \mu > \text{cf}(\kappa) \). By allowing repeats, we may write \( \kappa = \sup \{\lambda_\alpha : \alpha < \mu\} \), where each \( \lambda_\alpha < \kappa \). Next again, allowing repeats, for each \( \lambda_\alpha \), let \( \pi_\alpha \in K \)
be such that \( \lambda_\alpha < \pi_\alpha \). Each \( \pi_\alpha < \kappa = |K| \), so we may write \( K = \bigcup_{\alpha < \mu} K_\alpha \) where \( |K_\alpha| = \pi_\alpha \) for all \( \alpha < \mu \).

Let \( A_\alpha = \prod_{\beta \in K_\alpha} \text{Fr}(\beta) \), and let \( A = \prod_{\alpha < \mu} A_\alpha \). By the above proposition, \( u_{\text{spect}}(A) = \{ \mu \} \cup \bigcup_{\alpha < \mu} \{|K_\alpha|\} \cup \bigcup_{\alpha < \mu} K_\alpha = K \).

By Theorem 14.2 in [Mon96], \( \kappa \) is weakly compact if and only if for every BA \( A \), \( \kappa = \sup u_{\text{spect}}(A) \Rightarrow \kappa \in u_{\text{spect}}(A) \). The above corollary shows that in case \( \sup K \) is weakly compact, this is the only restriction on possibilities for \( u_{\text{spect}}(A) \). More precisely, if \( K \) is a set of cardinals with \( \sup K \) weakly compact, then \( K = u_{\text{spect}}(A) \) for some BA if and only if \( \sup K \in K \). If \( \kappa \) is regular and not weakly compact, it follows from the above mentioned theorem that there is a BA with \( \sup u_{\text{spect}}A = \kappa \) but \( \kappa \not\in u_{\text{spect}}(A) \). In fact the construction in [Mon96] shows that in this case there is an atomless BA \( A \) with \( u_{\text{spect}}(A) = \kappa \). Thus we are left with the following question.

**Problem 3.2.3.** Suppose that \( \kappa \) is regular limit and not weakly compact. If \( K \) is a cofinal set of cardinals less than \( \kappa \), is there a BA \( A \) such that \( u_{\text{spect}}(A) = K \)?

### 3.2.2 Hereditary Lindelöf degree

For any topological space, \( X \) the **Lindelöf degree** is the smallest cardinal \( L(X) \) such that every open cover of \( X \) has a sib cover with at most \( L(X) \) elements. Then the **hereditary Lindelöf degree** of a Boolean algebra \( A \), denoted \( hL(A) \) is

\[
\sup \{ L(X) : X \text{ is a subspace of Ult}(A) \}
\]

In what follows we use the following from [Mon96].

**Proposition 3.2.4.** \( hL(A) = \sup \{ \kappa : \text{there is an ideal of } A \text{ not generated by fewer than } \kappa \text{ elements} \} \).

For any ideal \( J \) on a BA let \( \chi(J) \) be the minimal size of a generating set for \( J \). Then we define \( hL_{\text{spect}}(A) = \{ \chi(J) : J \text{ is an ideal of } A \} \), so \( hL(A) = \sup hL_{\text{spect}}(A) \).

We have the following for \( A \) a moderate product.
Proposition 3.2.5. $hL_{\text{spect}}(A) = S \cup hL_{\text{spect}}(B) \cup \bigcup_{i \in I} hL_{\text{spect}}(A_i)$, where $S = \{ \kappa : \kappa$ is singular and there are sequences $\langle \lambda_\alpha : \alpha < \text{cf}(\kappa) \rangle$ and $I' = \{ i_\alpha : \alpha < \text{cf}(\kappa) \} \subseteq I$ such that for all $\alpha$, $\lambda_\alpha \in hL_{\text{spect}}(A_{i_\alpha})$, and $\sup_{\alpha < \text{cf}(\kappa)} \lambda_\alpha = \kappa$.

Proof. First we prove $\supseteq$. In fact, if $J \subseteq A_i$ is any ideal not generated by fewer than $\kappa$ elements, then $\{ a \in A : a(i) \in J \}$ is an ideal in $A$ not generated by fewer than $\kappa$ elements. Now suppose that $J \subseteq B$ is an ideal of $B$ not generated by fewer than $\kappa$ elements. Let $K = \{ a \in A : p(a) \in J \}$. We claim that $K$ is not generated by fewer than $\kappa$ elements also. In fact, suppose that $X \in [K]^{<\kappa}$. Then there is some $b \in J$ such that $b$ is not generated by $\{ p(x) : x \in X \}$. So then $\chi_b \in K$ is not generated by $X$. Next suppose that $\kappa \in S$; adopt the notation as in the definition of $S$. Then for each $i_\alpha \in I'$ let $J_\alpha \subseteq A_{i_\alpha}$ be an ideal with $\chi(J_\alpha) = \lambda_\alpha$. Define

$$Y = \bigcup_{\alpha < \text{cf}(\kappa)} \{ \chi_\alpha^{i_\alpha} : a \in J_\alpha \},$$

and let $J = \langle Y \rangle^{id}$. We claim that $J$ is an ideal of $A$ with $\chi(J) = \kappa$. In fact suppose that $J$ is generated by some set $S \subseteq A$ of size $\mu < \kappa$. Then let $\alpha < \text{cf}(\kappa)$ be such that $\lambda_\alpha > \mu$. We must have that $S_\alpha := \{ s(i_\alpha) : s \in S \}$ generates $J_\alpha$, but this is a contradiction since

$$|S_\alpha| \leq \mu < \lambda_\alpha = \chi(J_\alpha).$$

So $\supseteq$ is proved.

Now suppose that $\kappa \in hL_{\text{spect}}(A)$, but $\kappa \notin hL_{\text{spect}}(A_i)$ for any $i \in I$ and $\kappa \notin hL_{\text{spect}}(B)$. We will show $\kappa \in S$. Let $J \subseteq A$ be an ideal such that $\chi(J) = \kappa$; say that $J$ is generated by $X \in [A]^{<\kappa}$. Define $Q = \{ 1(x) : x \in X \}$. So $Q$ generates some ideal $\langle Q \rangle^{id}$ on $B$, and $|Q| \leq \kappa$. Since $\kappa \notin hL_{\text{spect}}(B)$, there is some $R \in [B]^{<\kappa}$ such that $R$ generates the same ideal as $Q$.

Next for each $i \in I$ let $M_i = \{ x(i) : x \in X \}$. Then $M_i$ generates an ideal in $A_i$, so for each such $i$ we have some $R_i \in [A_i]^{<\kappa}$ that generates the same ideal as $M_i$. We assume that these choices were minimal, so that $|R_i| \in hL_{\text{spect}}(A_i)$ for each $i$. Now define

$$Y := \{ \chi_r : r \in R \} \cup \bigcup_{i \in I} \{ \chi_r^{i_\alpha} : r \in R_i \}. $$
We claim that $Y$ generates $J$. First we check that each member of $Y$ is a member of $J$. In fact if $r \in R$ then we have $r \in \langle Q \rangle^{\mathrm{id}}$ and each member of $Q$ is a set $1(x)$ for some $x \in X$. So for some $F \in [X]^{<\omega}$ we have $r \leq \sum_{x \in F} 1(x)$. Then also $\chi_r \leq \sum F$, so $\chi_r \in J$. Likewise for any $r \in R_i$ we have some $F \in X$ such that $r \leq \sum_{x \in F} x(i)$. Then also $\chi_r^i \leq \sum F$.

So it suffices now to show that each member of $X$ is generated by $Y$. Let $x \in X$. We have

$$x = \chi_1(x) + \sum_{i \in m(x)} \chi_x^i,$$

Now $1(x) \in Q$, so there is some $F \in [R]^{<\omega}$ such that $1(x) \leq \sum F$. Also for each $i \in m(x)$ we have $x(i) \in M_i$ so there is some $F_i \in [R_i]^{<\omega}$ such that $x(i) \leq \sum F_i$. So then

$$x \leq \sum_{r \in F} \chi_r + \sum_{i \in m(x)} \left( \sum \{ \chi_s^i : s \in F_i \} \right),$$

which is a finite sum of members of $Y$.

Thus $Y$ does generate $J$, and thus $|Y| \geq \kappa$. Recall that $|R| < \kappa$ and $|R_i| < \kappa$ for all $i$. Also $|I| < \kappa$ because we assumed $\kappa \not\in h\text{L}_{\text{spect}}(B)$; the ideal generated by $\kappa$ singletons has character $\kappa$.

Thus there must be some sequence $\langle \mu_\alpha : \alpha < \cf(\kappa) \rangle$ and $\langle \lambda_\alpha : \alpha < \cf(\kappa) \rangle$ such that $|R_{i\alpha}| = \lambda_\alpha$ and $\sup_{\alpha < \kappa} \lambda_\alpha = \kappa$, as desired. \hfill \Box

**Corollary 3.2.6.** $h\text{L}(A) = \max \{ h\text{L}(B), \sup_{i \in I} h\text{L}_{\text{spect}}(A_i) \}$.

**3.2.3 Cellularity Spectrum**

We follow the notation from [Mon01b], where $PT(A)$ is defined as the set

$$\{|P| : P \text{ is a partition of 1 in } A\}.$$

**Proposition 3.2.7.** $PT(A) = \bigcup_{i \in I} PT(A_i) \cup PT(B) \cup S$, where $S := \{ \kappa : \kappa \text{ is singular and there exists a sequence } \langle \mu_\alpha : \alpha < \cf \kappa \rangle \text{ of cardinals, and a sequence } \langle i_\alpha : \alpha < \cf \kappa \rangle \text{ of distinct members of } I \text{ such that } \mu_\alpha \in PT(A_{i\alpha}) \text{ and } \sup_{\alpha < \cf \kappa} \mu_\alpha = \kappa \text{ and there exists a set } P \in [B]^{\leq \kappa} \text{ such that } P \text{ is a partition of } I \setminus \bigcup_{\alpha < \cf(\kappa)} \{ i_\alpha \} \}$. 

Proof. $(\exists)$ Suppose that $X \subseteq A_i$ is a partition of 1. Then for any $x \in X$ the set
\[
\{ \chi_{I \setminus \{i\}} + \chi_{x}^i \} \cup \{ \chi_{y}^i : y \in X \setminus \{x\} \}
\]
is a partition of 1 in $A$. If $X \subseteq B$ is a partition of 1 in $B$ then $\{\chi_x : x \in X\}$ is a partition of 1 in $A$.

Now suppose that $\kappa \in S$. For each $\alpha < \text{cf}(\kappa)$, let $X_\alpha \in A_{i_\alpha}$ witness $\mu_\alpha \in \text{PT}(A_{i_\alpha})$. Let $P \subseteq B$ be the partition of $I \setminus \{i_\alpha : \alpha < \text{cf}(\kappa)\}$ as in the definition of $S$. Then
\[
\bigcup_{\alpha < \text{cf}(\kappa)} \{ \chi_{x}^i : x \in X_\alpha \} \cup \{ \chi_p : p \in P \}
\]
is a partition of 1 of size $\kappa$ in $A$.

$(\subseteq)$. Suppose that $X \subseteq A$ is a partition of 1. Say $|X| = \kappa$. Then $J := \bigcup_{x \in X} m(x)$ has size at most $\kappa$. Now define
\[
Q := \{ 1(x) : x \in X \} \text{ and } M := \bigcup_{x \in X} \{ \chi_{x(i)}^i : i \in m(x) \}.
\]

If $|Q| = \kappa$ or $|J| = \kappa$, then the set $Q \cup \{\{j\} : j \in J\}$ is a partition of 1 of size $\kappa$ in $B$, as desired. Suppose then that both $|Q|$ and $|J|$ are strictly smaller than $\kappa$. We claim $|M \cup Q| = |X| = \kappa$. Consider the map $f : X \to [M]^{<\omega} \times Q$ defined by $f(x) = \left( \{ \chi_{x(i)}^i : i \in m(x) \}, 1(x) \right)$. Clearly this map is 1-1, and $|[M]^{<\omega} \times Q| = \max \{ \aleph_0, |Q|, |M| \}$. Since $X$ has size $\kappa$, it follows that $|M \times Q| = \kappa$, so $|Q| < \kappa$ implies that $|M| = \kappa$. Now let $X(J) := \bigcup_{j \in J} \{ x(j) : x \in X \setminus \{1, 0\} \}$. We claim that $|X(J)| = |M| = \kappa$. We will define a map $\varphi : X(J) \to M$. If $a \in X(J)$ then we may choose some $x \in X$ and some $j$ such that $x(j) = a$. We may assume that $A_i \cap A_j = \emptyset$ for $i \neq j$, so that the choice of $j$ is not arbitrary. Since $a \neq 0, 1$, we have $j \in m(x)$ and we may set $\varphi(a) = \chi_{x(j)}^j \in M$. Also, for such $j$, since $X$ is a partition, we cannot have distinct $x, y \in X$ such that $x(j) = y(j)$, so the map is well-defined. We claim that $\varphi$ is a bijection. Suppose that $a$ and $b$ are distinct members of $X(J)$. Say $a = x(j)$ and $b = y(i)$. Either $j \neq i$ or $j = i$ but $x(j) \neq y(j)$. In either case we have $\varphi(a) = \varphi(x(j)) \neq \varphi(y(i)) = \varphi(b)$, so $\varphi$ is 1-1. Next, if $m \in M$, there is an $x \in X$ and an $i \in J$ such that $\chi_{x(i)}^i = m$. Then $x(i) \in X(J)$, and $\varphi(x(i)) = m$, so $\varphi$ is onto. We claim that
\[
|X(J)| = \sum_{j \in J} |\{ x(j) : x \in X \}|.
\] (3.1)
In fact $\leq$ is clear since $X(J) = \bigcup_{j \in J} \{x(j) : x \in X\} \setminus \{0, 1\}$. On the other hand we have $|X(J)| = \kappa$ and by assumption $|J| < \kappa$, and clearly $|\{x(j) : x \in X\}| \leq \kappa$ for each $j \in J$. So equality holds in equation 3.1. Notice that for each $j \in J$, the set $\{x(j) : x \in X\}$ is a partition of 1 in $A_j$. Thus if $|x(j) : x \in X| = \kappa$ for some $j \in J$, then $\kappa \in \text{PT}(A_j)$. Suppose then that $|x(j) : x \in X| < \kappa$ for all $j \in J$. By equation (3.1), this implies that $\kappa$ is singular, and there is a sequence $\{i_\alpha : \alpha < \text{cf} \kappa\} \subseteq J$ such that $\sup_{\alpha < \text{cf} \kappa} |\{x(i_\alpha) : x \in X\}| = \kappa$. Then $(i_\alpha)$ and $(\mu_\alpha) := \langle |\{x(i_\alpha) : x \in X\}| \rangle$ are as in the definition of $S$. The set

$$P := Q \cup \left\{ \{j\} : j \in J \setminus \bigcup_{\alpha < \text{cf} \kappa} \{i_\alpha\} \right\}$$

satisfies the definition of $S$ as well. \hfill \Box

The small version of cellularity for $A$ denoted $a(A)$ and is equal to $\min \text{PT}(A)$. As a consequence of the above we have

**Corollary 3.2.8.** $a(A) = \min \{\min_{i \in I} a(A_i), a(B)\}$.

3.2.4 Pseudo-Intersection Spectra

For any BA $A$, call $X \subseteq A$ a **p-set** (for $A$) if $X$ is infinite, $\sum X = 1$ but $1$ is not the least upper bound of $X'$ for all $X' \in [X]^{<|X|}$.

For any Boolean algebra $A$ define

$$\mathfrak{p}_{\text{spect}}(A) = \{|X| : X \text{ is a p-set for } A\}.$$  

**Lemma 3.2.9.** For any BA $A$, $\mathfrak{p}(A) = \min \mathfrak{p}_{\text{spect}}(A)$.

**Proof.** Recall that $\mathfrak{p}(A)$ can be defined as $\min \{|X| : X \in S\}$ where

$$S = \left\{ X \subseteq A : \sum X = 1 \text{ but } \sum X' \neq 1 \text{ for every } X' \in [X]^{<\omega} \right\}.$$  

Note that $S$ consists only of infinite sets, so every p-set is a member of $S$. Thus the inequality $\leq$ is clear. Now suppose that $X \in S$ is of minimal size. Then, by minimality, we have additionally that $1$ is not the least upper bound of $X'$ for every $X' \in [X]^{<|X|}$, so $|X| \in \mathfrak{p}_{\text{spect}}(A)$, which shows the other inequality. \hfill \Box
In what follows we use $\sum X \neq 1$ to abbreviate the statement: “1 is not the least upper bound of $X$.”

**Proposition 3.2.10.** $p_{\text{spect}}(A) = S \cup p_{\text{spect}}(B) \cup \bigcup_{i \in I} p_{\text{spect}}(A_i)$, where $S = \{\kappa : \kappa \text{ is singular and there are sequences } \langle \mu_\alpha : \alpha < \text{cf}(\kappa) \rangle \text{ and } J = \{i_\alpha : \alpha < \text{cf}(\kappa)\} \subseteq I \text{ such that for all } \alpha, \mu_\alpha \in p_{\text{spect}}(A_{i_\alpha}), \sup_{\alpha < \text{cf}(\kappa)} \mu_\alpha = \kappa, \text{ and there is a } Z \in [\mathcal{P}(I \setminus J) \cap B]^<\kappa \text{ such that }$

$\forall i \in I \setminus J \exists z \in Z(i \in z)$.

**Proof.** Let $X \subseteq B$ be a $p$ set. We claim that $Y = \{\chi_x : x \in X\}$ is a $p$ set in $A$. Suppose that $\exists a \in A$ with $1_A > a \geq \chi_x$ for all $x \in X$. Then $1(a) \geq x$ for all $x \in X$, a contradiction, so $\sum Y = 1_A$. Suppose that for some $Y' \in [Y]^{|Y|}$ we have $\sum Y' = 1_A$. Say $Y' = \{\chi_x : x \in X'\}$ for some $X' \in [X]^{|X|}$. Then $\sum_{x \in X'} x = 1_B$, a contradiction.

Now if $X \subseteq A_i$ is a $p$ set then we claim that $Y := \{\chi^i_x + \chi_{I \setminus \{i\}} : x \in X\}$ is a $p$ set in $A$. Clearly $\sum Y = 1$, and if $Y' \in [Y]^{|Y|}$, then $(\sum Y')(i) \neq 1$. So $Y$ is as claimed.

Finally suppose that $\kappa \in S$. Let $\langle \mu_\alpha \rangle$, $J$ and $Z$ be as in the definition of $S$. For $i = i_\alpha \in J$, let $X_i \subseteq A_i$ be a $p$ set of size $\mu_\alpha$. We define a $p$ set of size $\kappa$ in $A$ as follows. For each $i = i_\alpha \in J$, let $Y_i = \{\chi^i_x : x \in X_i\}$, and let $Y = \{\chi_z : z \in Z\} \cup \bigcup_{i \in J} Y_i$. First we check that $\sum Y = 1$. Letting $a \in A \setminus \{1\}$ be arbitrary it suffices to check that for some $i \in I$, there is a $y \in Y$ such that $y(i) \not\leq a(i)$. Let $i \in I$ be such that $a(i) \neq 1$. If $i \in I \setminus J$ then there is a $z \in Z$ with $i \in z$, thus $a(i) < \chi_z(i)$, and $\chi_z \in Y$. If $i \in J$ then we use the fact that $\sum X_i = 1_A$ to find $x \in X_i$ such that $x \not\leq a(i)$. Then $\chi^i_x \in Y$ is as needed. Next we check that the minimality condition holds, so suppose that $Y' \in [Y]^{|Y|}$. Then for some (in fact infinitely many) $i \in J$ we must have $|Y_i \cap Y'| < \mu_i$. Now note that for all $y \in Y \setminus Y_i$, $y(i) = 0$. Thus $\pi_i"(Y') = \pi_i"(Y_i \cap Y')$, which does not have least upper bound $1_{A_i}$ because $X_i$ was a $p$ set. Thus also $\sum Y' \neq 1$.

Now suppose that $X \subseteq A$ is a $p$ set, with $\kappa := |X| \not\in \bigcup_{i \in I} p_{\text{spect}}(A_i)$. Let $Y = \{p(x) : x \in X\}$. Then $\sum Y = 1$. If $|Y| = \kappa$ and $Y$ is a $p$ set in $B$ then $\kappa \in p_{\text{spect}}(B)$, so suppose that one of those fail, hence for some $Y' \in [Y]^<\kappa$ we have $\sum Y' = 1$. Say $|Y'| = \lambda < \kappa$. Write $Y' = \{p(x) : x \in X'\}$,
with \(X' \in [X]^\lambda\). Now \(m(x)\) is finite for all \(x \in X\), so \(|\bigcup_{x \in X'} m(x)| \leq |X'|\). Thus if we define
\[
I' = \left( \bigcup_{x \in X'} m(x) \right) \setminus \left( \bigcup_{x \in X'} 1(x) \right),
\]
also \(|I'| \leq |X'| < \kappa\). Now note that for each \(i \in I\) we have \(\sum \pi_i '' X = 1\) (in \(A_i\)), and \(|\pi_i '' X| \leq |X|\), so the assumption that \(\kappa \not\in \bigcup_{i \in I} \mathcal{P}(A_i)\) implies that for each \(i\) there is some \(X'_i \subseteq X\) with \(|X'_i| < \kappa\) and \(\sum \pi_i '' X'_i = 1\). Now for each \(i \in I'\) let \(X_i \subseteq X'_i\) be of minimal cardinality subject to the condition that \(\sum \pi_i '' X_i = 1\). Thus for each \(i \in I'\), if \(X_i\) is infinite, then \(\pi_i '' X_i\) is a \(\mathbb{p}\) set in \(A_i\).

Letting \(X'' = X' \cup \bigcup_{i \in I'} X_i\) we claim that \(\sum X'' = 1\). Let \(a \in A \setminus \{1\}\) be arbitrary, we will show that \(a\) does not bound every member of \(X''\). Fix \(i \in I\) such that \(a(i) < 1\). In case \(i \in I'\) we use the fact that \(\sum \pi_i '' X_i = 1\) to find an \(x \in X_i\) such that \(x(i) \leq a(i)\). Thus as \(X_i \subseteq X''\), this is the desired \(x\). Now suppose that \(i \in I \setminus I'\). Recall that \(1 = \sum Y' = \sum \{ p(x) : x \in X' \}\), so for each \(i \in I\) there is an \(x \in X'\) with \(x(i) > 0\). Thus \(I \setminus I' = \bigcup_{x \in X'} 1(x)\). Hence there is an \(x \in X'\) with \(x(i) = 1\), so clearly also \(x \leq a\) in this case.

Now \(|X''| \leq \max \{ \lambda, |I'|, \sup_{i \in I'} |X_i| \}\), and \(|X''| = \kappa\) because \(X\) was a \(\mathbb{p}\) set, so it must be the case that \(\sup_{i \in I'} |X_i| = \kappa\). Thus \(J \overset{\text{def}}{=} \{ i \in I' : |X_i| \geq \omega \}\) has size at least \(\text{cf}(\kappa)\), so for some subset \(J' \subseteq J\) we have that \(|J'| = \text{cf}(\kappa)\) and \(\{|X_i| : i \in J'\}\) is in the definition of \(S\). We claim that also \(Z \overset{\text{def}}{=} \{ 1(x) : x \in X' \} \cup \{ \{ i \} : i \in I' \setminus J' \}\) is as in the definition of \(S\). Clearly \(Z \subseteq B\) as \(1(x) \in B\) for all \(x \in A\) and all finite subsets of \(I\) are also members of \(B\). Next we check that \(Z \subseteq \mathcal{P}(I \setminus J)\).

Clearly \(\{ \{ i \} : i \in I' \setminus J'\}\) is disjoint from \(I'\), and \(J' \subseteq I'\), so \(1(x) \subseteq I \setminus J'\) as claimed. To check the final condition, let \(i \in I \setminus I'\). If \(i \in I'\) then \(\{ i \} \in Z\) by definition, so suppose that \(i \in I \setminus I'\). Recall from above that \(I \setminus I' = \bigcup_{x \in X'} 1(x)\), therefore for this \(i\) there is an \(x \in X'\) with \(i \in 1(x)\). Thus \(Z\) is as claimed and so \(\kappa \in S\) in this case.

\[\square\]

**Corollary 3.2.11.** \(p(A) = \min \{ \min_{i \in I} p(A_i), p(B) \}\).

One could make an alternative definition for \(p_{\text{spect}}\) as follows. Define a \(\mathbb{p}^+\) set as any \(X \subseteq A\) such that \(\sum X = 1\) but \(\sum X' \neq 1\) for all \(X' \in [X]^{<\omega}\). Then let \(p_{\text{spect}}^+(A) = \{|X| : X\text{ is a }\mathbb{p}^+\text{ set}\}\).
Every \( p \) set is also a \( p^+ \) set so \( \text{p}_\text{spect}(A) \subseteq \text{p}^\text{+}_\text{spect}(A) \) and the inclusion can be proper. In fact also \( \text{u}_\text{spect}(A) \subseteq \text{p}^\text{+}_\text{spect}(A) \), while this is not true in general for \( \text{p}_\text{spect} \) as we will show in an example below. First, to see that \( \text{u}_\text{spect} \subseteq \text{p}^\text{+}_\text{spect} \) consider any base \( X \) for a nonprincipal ultrafilter \( U \) on a BA \( A \). Now clearly for every \( F \in [X]<\omega \) we have \( \prod F \neq 0 \), and we claim that \( \prod X = 0 \) so \( X \) is dual to a \( p^+ \) set. In fact we will show that any lower bound of \( X \) is 0. So let \( a \in A \) with \( a \leq x \) for all \( x \in X \). If \( a \in U \) then clearly \( U \) is principally generated by \( a \), contradiction. So \( -a \in U \). Then there is an \( x \in X \) such that \( x \leq -a \). So \( a \leq -x \) and \( a \leq x \), hence \( a = 0 \).

To illustrate that \( u_{\text{spect}}, p^+_{\text{spect}}, \) and \( p_{\text{spect}} \) can all be different, consider \( A = \text{Fr}(\kappa) \), the free algebra on \( \kappa \) generators for \( \kappa \) uncountable. Then we have

(i) \( u_{\text{spect}}(A) = \{\kappa\} \),

(ii) \( p_{\text{spect}}(A) = \{\omega\} \), and

(iii) \( p^+_{\text{spect}}(A) = [\omega, \kappa] \),

where the interval above is meant to consist only of cardinals.

Part (i) is proved as part of the proof of Corollary 3.2.2.

For (ii), consider any subset \( Y \subseteq A \) such that \( \sum Y = 1 \). Since \( \text{Fr}(\kappa) \) has ccc, using Lemma 10.2 of [KMB89] there is a countable \( X \subseteq Y \) such that \( \sum X = 1 \) (and clearly no finite subset has sum 1).

For (iii), let \( X \) be a countable \( p \) set as in (ii). For each \( x \in X \), there is a finite set of free generators \( Y_x \) such that \( x \in \langle Y_x \rangle \). Let \( Y = \text{Fr}(\kappa) \setminus \bigcup_{x \in X} Y_x \). So \( Y \) has size \( \kappa \), hence there is an independent subset \( Y' \in [Y]^\kappa \). Let \( \lambda \in [\omega, \kappa] \), and let \( Z \in [Y']^\lambda \). Then \( X \cup Z \) is a \( p^+ \) set of size \( \lambda \), proving (iii).
The intervals in what follows are intended to denote only cardinals.

Lemma 3.2.12. Let $A$ and $B$ be arbitrary Boolean algebras such that at least one of $p(A)$ or $p(B)$ is defined (i.e. at least one of $A, B$ is infinite). Let $\kappa = \min\{p(A), p(B)\}$ and $\lambda = |A| + |B|$. Then $p^+_{\text{spect}}(A \times B) = [\kappa, \lambda]$.

Proof. First note that the inclusion $p^+_{\text{spect}}(A \times B) \subseteq [\kappa, \lambda]$ is clear. Now we claim that both $[p(A), |B|]$, and $[p(B), |A|]$ are contained in $p^+_{\text{spect}}(A \times B)$, from which the other inclusion follows. By symmetry it suffices to show containment for the first interval, which we may assume wlog is nonempty. Let $\delta \in [p(A), |B|]$. Let $X \subseteq A$ be a $p^+$-set of size $p(A) \leq \delta$. Let $Y$ be any subset of $B$ of size $\delta$. Then let $Z = (X \times \{1\}) \cup (\{0\} \times Y)$. Clearly $Z$ is a $p^+$-set in $A \times B$ of the correct size. \qed

Proposition 3.2.13. Assume again that $B$ is infinite. Let $A = \prod_{i \in I} A_i$. Then

$$p^+_{\text{spect}}(A) = \left[\min\\{\{p(A_i) : i \in I\} \cup \{p(B)\}\}, |B| + \sum_{i \in I} |A_i|\right].$$

By Corollary 3.2.11, we can simplify the right side, and write

$$p^+_{\text{spect}}(A) = [p(A), |A|].$$

Proof. Choose $j \in I$ and let $J$ be the principal ideal in $A$ generated by $\chi_{\{j\}}$. We will show that both $[p(A_j), |A|] \subseteq p^+_{\text{spect}}(A)$ and $[p(B), |A|] \subseteq p^+_{\text{spect}}(A)$. Note that $A \cong A_j \times A/J$. So the first inclusion holds by Lemma 3.2.12. Next, let $X$ be a $p$-set in $B$ of size $p(B)$. Clearly then $X$ is also a $p^+$-set. Let $Y = \{[\chi_x] : x \in X\}$, where $[\chi_x]$ represents the equivalence class of $x$ modulo $J$. We claim that $Y$ is also a $p^+$-set for $A/J$. First $Y$ has sum 1, for, suppose that $[\chi_x] \leq [y] \leq 1$ for all $x \in X$. Then $\chi_x \cdot y \leq \chi_{\{j\}}$ for all $x \in X$. Take any $k \in I \setminus \{j\}$. Since $\sum X = 1$ there is an $x \in X$ such that $k \in x$. Since $\chi_x \cdot y \leq \chi_{\{j\}}$, it follows that $-y(k) = 0$. Thus $-y \leq \chi_{\{j\}}$, so $[y] = 1$, as needed.

Now suppose that $F \in [X]^{<\omega}$ and $\sum_{y \in F} [\chi_y] = 1$. Thus $\prod_{y \in F} [-\chi_y] = 0$, so $\prod_{y \in F} -\chi_y \leq \chi_{\{j\}}$, and hence $\prod_{y \in F} -\chi_y \cdot -\chi_{\{j\}} = 0$. Now $\sum X = 1$, so there is a $z \in X$ such that $j \in z$. Also,
\[ \sum F + z \neq 1, \text{ so there is a } k \in I \setminus (\bigcup F \cup z). \] Then \( \chi\{k\} \neq 0 \) and \( \chi\{k\} \leq \prod_{y \in F} -\chi y \cdot -\chi\{j\}, \) contradiction.

So again applying Lemma 3.2.12 we see that \( |p(B), |A|] \subseteq p^+_{\text{spect}}(A). \) Since \( \min p^+_{\text{spect}}(A) = \min p_{\text{spect}}(A), \) Proposition 3.2.10 shows that \( p^+_{\text{spect}} \) cannot contain smaller cardinals. As the upper bound is the size of the algebra, \( p^+_{\text{spect}} \) cannot contain larger cardinals either, so we are done. \( \Box \)

### 3.2.5 Reaping numbers

The reaping number \( r(A) \) is defined as follows:

\[
    r(A) = \min \{|X| : \forall a \in A\exists x \in X [x \leq a \text{ or } x \leq -a]\}.
\]

A generalized version which has been studied for example in [BS92] and [DSW96] is as follows:

\[
    r_n = \min \{|X| : \forall \text{ partitions } P \text{ of } 1 \text{ with } |P| \leq n (\exists x \in X \exists p \in P [x \leq p])\}
\]

**Proposition 3.2.14.** If the \( A_i \) are atomless and \( n \in \omega \setminus 2 \) then \( r_n(A) = \omega. \)

**Proof.** Let \( J \in [I]^{\omega} \) and \( X = \{\chi\{j\} : j \in J\}. \) Suppose that \( \{y_0, \ldots, y_{n-1}\} \) is a partition of 1 in \( A. \) Then \( F = \bigcup_{k=0}^n m(y_k) \) is finite, so pick \( j \in J \setminus F. \) Then for some \( k \in \{0, \ldots, n\}, y_k(j) = 1. \) So \( \chi\{j\} \leq y_k, \) as needed. \( \Box \)

### 3.2.6 Splitting number

Call a set \( S \subseteq A \) a splitting set iff for all \( a \in A \setminus \{0\} \) there is an \( s \in S \) such that \( s \cdot a \neq 0 \neq -s \cdot a. \)

Define \( s(A) \) to be the minimum size of any splitting subset of \( A. \)

Suppose that the \( A_i \) are atomless, then \( s(A) = \max (\sup_{i \in I} \{s(A_i)\}, |I|) \). The same proof as for weak product in [Mon01a] works.
3.2.7 Distributivity number

A is \((\kappa, \infty)\)-distributive iff every family \(\langle P_\alpha : \alpha < \kappa \rangle\) of partitions of unity of \(A\) has a common refinement to another partition of unity of \(A\). Let

\[ h(A) = \min \{ \kappa : A \text{ is not } (\kappa, \infty)\text{-distributive} \}. \]

Suppose that the \(A_i\) are atomless, then \(h(A) = \min_{i \in I} h(A_i)\). The same proof as for weak product in [Mon01a] works.

3.2.8 Tower number

In this section, a subset, \(T\), of a BA is a \textbf{tower}, if \((T, \leq)\) is a well-order, \(1 \notin T\), and \(\sum T = 1\). Then we define \(t(A) = \min \{|T| : T \text{ is a tower in } A\}\).

**Proposition 3.2.15.** \(t(A) = \min \{\min_{i \in I} t(A_i), t(B)\}\)

**Proof.** Again \(t(A) \leq \min_{i \in I} t(A_i)\) is clear. Now suppose that \(T\) is a tower in \(B\). Then clearly \(\{\chi_t : t \in T\}\) gives a tower of the same sort in \(A\), so also \(t(A) \leq t(B)\).

Now suppose that \(T \subseteq A\) with \(|T| < \min_{i \in I} t(A_i)\), \(T\) is well-ordered by \(\leq\), \(1_A \notin T\), and \(\sum T = 1_A\). We claim that \(S := \{1(t) : t \in T\}\) is a tower in \(B\). Clearly \(1_B \notin S\) because \(1_A \notin T\). Also, if \(t, t' \in T\) with \(t \leq t'\) then \(1(t) \subseteq 1(t')\), so \(S\) is well-ordered because \(T\) is. We claim that \(\forall i \in I \exists t \in T\) with \(t(i) = 1_{A_i}\). Suppose not, so fix \(i \in I\) with \(t(i) < 1_{A_i}\) for all \(t \in T\). Then \(\{\pi_i(t) : t \in T\}\) is a tower in \(A_i\), a contradiction, so the claim is proved. Thus \(\sum S = 1_B\). So \(S\) is a tower in \(B\) as claimed, and \(|S| \leq |T|\), so also \(t(A) \leq t(B)\). \(\square\)

3.2.9 Independence

For any BA \(A\), \(X \subseteq A\) is \textbf{independent} if \(X\) is a free set of generators for \(\langle X \rangle\). Then the \textbf{independence} of \(A\) is

\[ \sup \{|X| : X \text{ is an independent subset of } A\}. \]

**Proposition 3.2.16.** \(\text{Ind}(A) = \max(\sup_{i \in I} \text{Ind}(A_i), \text{Ind}(B))\)
Proof. Given an independent family $X$ in one of the $A_i$, the set $\{x^i_x : x \in X\}$ is independent in $A$. Given an independent family $X$ in $B$, the set $\{x^i_x : x \in X\}$ is independent in $A$. Together, these two observations give $\geq$.

Now suppose that $X \subseteq A$ is independent and $|X| > \sup_{i \in I}(\text{Ind}(A_i))$. Let $Y = \{p(x) : x \in X\}$. We claim that $|Y| = |X|$. Suppose that $p(x) = p(y)$ for $x \neq y$. Thus $1(x) \cap 0(y) = \emptyset$. Set $z = x \cdot -y$. Then
\[
p(z) \subseteq (1(x) \cup m(x)) \cap (0(y) \cup m(y)) \subseteq m(x) \cup m(y) \in [I]^{<\omega}.
\]
Now, for any finite, disjoint $F,G \subseteq X \setminus \{x,y\}$ we have $z \cdot (\prod_{a \in F} a) \cdot (\prod_{b \in G} -b) \neq 0$, but $z(i) = 0$ for every $i \notin p(z)$. So, for each such $F,G$, $(\prod_{a \in F} a) \cdot (\prod_{b \in G} -b) (i)$ is not identically 0 for all $i \in p(z)$. Thus the restriction of $X \setminus \{x,y\}$ onto the factor $\prod_{i \in p(z)} A_i$ is independent, contradicting $|X| > \sup_{i \in p(z)}(\text{Ind}(A_i))$ since by Corollary 10.4 of [Mon96] we have $\text{Ind} \left( \prod_{i \in p(z)} A_i \right) = \sup_{i \in p(z)} \text{Ind}(A_i)$. Thus for distinct $x,y \in X$ we have $p(x) \neq p(y)$, so $|Y| = |X|$ as claimed. Now we show that $Y$ is independent in $B$. Suppose not, so take finite disjoint $F,G \subseteq X$ such that
\[
\left( \prod_{a \in F} p(a) \right) \cdot \left( \prod_{b \in G} I \setminus p(b) \right) = \emptyset.
\]
Now for each $b \in G$, put $J_b = p(-b) \cap p(b)$, and note that $J_b$ is finite for each $b$. Note that if $i \in \prod_{b \in G} p(-b) \setminus \left( \prod_{b \in G} I \setminus P(b) \right)$, then for some $b \in G, i \in J_b$. Thus, letting $J = \bigcup_{b \in G} J_b$, we have
\[
p \left( \left( \prod_{a \in F} a \right) \cdot \left( \prod_{b \in G} -b \right) \right) \subseteq J \in [I]^{<\omega}.
\]
Then the projection of $X \setminus (F \cup G)$ on $\prod_{i \in J} A_i$ is independent, a contradiction. So $>$ does not hold, thus the equality is proved. \qed

We also consider the small version of independence. For a BA $A$ let
\[
i(A) = \min \{|X| : X \text{ is an infinite maximal independent subset of } A\}.
\]

**Proposition 3.2.17.** Suppose that the $A_i$ are atomless, then $\iota(A) = \omega$.

**Proof.** We show that the construction in [Mon01a] works; the construction is repeated here. Assume for convenience that $I = \kappa$ is an infinite cardinal. For each $n < \omega$ let $\{x_{n,i} : i \in \omega\}$ be an independent
family in $A_n$. We define a system of elements $\{y_n : n \in \omega\}$ of $A$ by specifying the values at each factor:

$$y_n(\alpha) = \begin{cases} 
  x_{\alpha, n-\alpha} & \text{if } \alpha < n \\
  1 & \text{if } \alpha = n \\
  0 & \text{if } \alpha > n.
\end{cases}$$

We claim that $\{y_n : n \in \omega\}$ is independent. Let $F$ and $G$ be finite, disjoint subsets of $\omega$. Define $a = \prod_{n \in F} y_n$ and $b = \prod_{m \in G} -y_m$, so we must show that $a \cdot b \neq 0$. Note that if $F \neq \emptyset$ then for $n = \min F$, $a(n) \geq \prod_{j \in F} x_{n,n-j}$. If $F$ is empty then choose $n \in \omega \setminus G$. Then in either case, $b(n) \geq \prod_{k \in G, k > n} -x_{n,n-k}$, so $(a \cdot b)(n) \geq 0$ by the independence of the family $\{x_{n,i} : i \in \omega\}$ and the fact that $F \cap G = \emptyset$.

Now let $a \in A$; we will show that $\{y_n : n \in \omega\} \cup \{a\}$ is no longer independent. Since $m(a)$ is finite, there is some $n < \omega$ such that $a(n) \in 2$. Wlog $a(n) = 0$. Then we claim

$$z := \left( \prod_{k < n} -y_k \right) \cdot y_n \cdot a = 0.$$

If $\alpha > n$ then $y_n(\alpha) = 0$. If $\alpha = n$ then $a(\alpha) = 0$. If $\alpha < n$ then $-y_\alpha(\alpha) = 0$, so $z$ is as claimed. \qed

**Proposition 3.2.18.** Suppose that $\kappa \in \text{i-spect}(B)$ and that for all $i$, $\kappa \geq \text{Ind}(A_i)$. Then $\kappa \in \text{i-spect}(A)$.

**Proof.** Let $X \subseteq B$ be a maximal independent set of size $\kappa$. Then $Y := \{\chi_x : x \in X\}$ is independent in $A$. We claim that for all $Z$ extending $Y$ with $Z$ independent, still $|Z| = |Y| = \kappa$. In fact, suppose that $Z \supseteq Y$ and $|Z| > \kappa$.

We claim that

$$\forall z \in Z \setminus Y \text{ there is a } Y\text{-monomial } w_z \text{ such that } p(w_z \cdot z) \text{ is finite.} \quad (3.2)$$

To see that (3.2) holds, let $z \in Z \setminus Y$ and define $s := \{i : z(i) < 1\}$. We claim $s \not\subseteq X$. In fact,
suppose that \( s \in X \). Then \( \chi_s \in Y \), hence also \( \chi_s \in Z \). However, for any \( i \in I \),

\[
(-z \cdot -\chi_s)(i) = \begin{cases} 
-z(i) & \text{if } i \not\in s \text{ (in which case } z(i) = 1) \\
0 & \text{if } i \in s 
\end{cases}
\]

\( = 0. \)

This contradicts the independence of \( Z \), so \( s \not\in X \), as claimed. So by maximality of \( X \), \( X \cup \{ s \} \) is not independent, hence there are finite, disjoint \( F, G \subseteq X \) such that for some \( \varepsilon \in 2 \), \( \prod_{x \in F} x \cdot \prod_{y \in G} -y \cdot s^\varepsilon = \emptyset \). Define \( w_z = \prod_{x \in F} \chi_x \cdot \prod_{y \in G} -\chi_y \). Since \( w_z \) is a product of characteristic functions, we have \( 1(w_z) = p(w_z) \). So \( p(w_z) \cdot s^\varepsilon = 1(w_z) \cdot s^\varepsilon = 0 \). If \( p(w_z) \cdot s = 0 \) then \( p(w_z) \subseteq -s \). But since \( s = \{ i : z(i) < 1 \} = \{ i : -z(i) > 0 \} \) we have \(-s = \{ i : -z(i) = 0 \} \), so \(-z \cdot w_z = 0 \), a contradiction of the independence of \( Z \). Thus it must be the case that \( p(w_z) \cdot -s = 0 \). Hence \( p(w_z) \subseteq s \), so \( p(w_z \cdot z) \subseteq m(z) \), which is finite. Thus (3.2) is proved.

So let \( z, w_z \) be as in (3.2) and let \( Z' = (Z \setminus Y) \setminus \{ z \} \). We claim that \( Z' \upharpoonright m(z) \) is an independent set of size \( |Z'| \) in \( \prod_{i \in m(z)} A_i \). First we check independence. Supposing to the contrary let \( u \) be a \( Z' \)-monomial such that \( u \upharpoonright m(z) = 0 \). Then \( u \cdot w_z \cdot z = 0 \), contradicting the independence of \( Z \). For the cardinality claim, suppose to the contrary that for some distinct \( z_0, z_1 \in Z' \) we have \( z_0 \upharpoonright m(z) = z_1 \upharpoonright m(z) \). Then \( z_0 \cdot -z_1 \cdot w_z \cdot z = 0 \), again a contradiction.

Thus \( \prod_{i \in m(z)} A_i \) has an independent set of size \( |Z'| > \kappa \). By Theorem 2 of [MM04] this implies that for some \( i \in m(z) \), \( |Z'| \in i_{\text{spec}}(A_i) \). This is a contradiction since for each \( i \in I \), \( \kappa \geq \text{Ind}(A_i) \).

3.2.10 Free Caliber

We say that \( A \) has free caliber \( \kappa \) if \( \kappa \leq |A| \) and \( \forall X \in [A]^\kappa \exists Y \in [X]^\kappa \) (\( Y \) is independent).

**Proposition 3.2.19.** Suppose that \( \text{cf}(\kappa) > |I| \). Let \( A = \prod_{i \in I} A_i \). Then the following are equivalent:

(1) \( \kappa \in \text{Freecal}(A) \)
\((2)\) \(\kappa \in \bigcap \{\text{Freecal}(A_i) : \kappa \leq |A_i|\}\) and if \(|B| \geq \kappa\) then also \(\kappa \in \text{Freecal}(B)\).

**Proof.** \((1) \Rightarrow (2)\) is clear.

Suppose \(\kappa\) is as in \((2)\) and let \(X \in [A]^\kappa\). We have

\[X = \bigcup_{F \in [I]^{<\omega}} \{x : m(x) = F\}.
\]

As \(\text{cf}(\kappa) > |I|\), there is some \(Y \in [X]^\kappa\), and some \(F \in [I]^{<\omega}\) such that \(m(y) = F\) for all \(y \in Y\). We consider two cases.

- **Case 1.** There is some \(i \in F\) such that \(|\{x(i) : x \in Y\}| = \kappa\). Restricting \(Y\) to such an \(i\) gives a \(\kappa\)-sized family in \(A_i\), so there is a \(Z \in [Y]^\kappa\) such that \(\{z(i) : z \in Z\}\) is independent in \(A_i\). Clearly this \(Z\) is independent in \(A\).

- **Case 2.** For all \(i \in F\) we have \(|\{x(i) : x \in Y\}| < \kappa\).

  * Subcase 2.1. There is an \(a \in \prod_{i \in F} A_i\) such that \(|\{y \in Y : y|F = a\}| = \kappa\). So let \(S \in [Y]^\kappa\) be such that \(s|F = a\) for each \(s \in S\). Recall that for all \(s \in S\) we have \(m(s) = F\). Since all the members of \(S\) are distinct, if we define \(Z = \{1(s) : s \in S\}\) we must have \(|Z| = \kappa\). Now \(Z \subseteq B\) and \(\kappa \in \text{Freecal}(B)\) so let \(Z' \in [Z]^\kappa\) be independent. Then for each \(z \in Z'\) choose \(y_z \in Y\) such that \(1(y_z) = z\). Then \(Y' = \{y_z : z \in Z'\}\) is independent in \(A\) as desired.

  * Subcase 2.2. \(|\{1(y) : y \in Y\}| = \kappa\). Note that in particular this happens if \(F = \emptyset\). Then we proceed as in the above subcase, with \(S = Y\), since \(Z \overset{\text{def}}{=} \{1(y) : y \in Y\}\) already has size \(\kappa\).

  * Subcase 2.3. The above cases don’t hold. So for all \(a \in \prod_{i \in F} A_i\) we have \(|\{y \in Y : y|F = a\}| < \kappa\), and \(|\{1(y) : y \in Y\}| < \kappa\). Recall that also \(|\{x(i) : x \in Y\}| < \kappa\) for each \(i \in F\), so \(|Y| = \kappa\) implies that there must be some sequence \(\langle a_\alpha : \alpha < \text{cf}(\kappa)\rangle\) of elements of \(\prod_{i \in F} A_i\) such that \(\sup \{|y \in Y : y|F = a_\alpha| = \kappa\}\). For each \(\alpha < \text{cf}(\kappa)\) let \(Y_\alpha = \{y \in Y : y|F = a_\alpha\}\). Let
\[ Z = \{1(y) : y \in Y\} \]. Now for each \( \alpha < \text{cf}(\kappa) \) we have that \( |Y_\alpha| = |\{1(y) : y \in Y_\alpha\}| \leq |Z| \), so \( \sup |Y_\alpha| \leq |Z| \), hence \( |Z| = \kappa \), which contradicts the last part of the subcase assumption.

**Proposition 3.2.20.** Suppose that \( \text{cf}(\kappa) \leq |I| \). Let \( A = \prod_{i \in I} A_i \). Then the following are equivalent:

(1) \( \kappa \in \text{Freecal}(A) \)

(2) The following conditions hold:

(a) \( \kappa \in \bigcap \{ \text{Freecal}(A_i) : \kappa \leq |A_i| \} \) and if \( |B| \geq \kappa \) then also \( \kappa \in \text{Freecal}(B) \).

(b) \( \sup \{ |A_i| : i \in I, |A_i| < \kappa \} < \kappa \)

(c) \( |\{ i \in I : \kappa \leq |A_i| \}| < \text{cf}(\kappa) \).

(d) \( |I| < \kappa \) (so \( \kappa \) is singular).

**Proof.** (\( \Rightarrow \)). Assume (1). Then 2(a) is clear.

Suppose that 2(b) does not hold. Then letting \( I' = \{ i \in I : |A_i| < \kappa \} \) we have \( \sup_{i \in I'} |A_i| \geq \kappa \). Then define \( X = \bigcup_{i \in I'} \{ \chi^i_a : a \in A_i \} \). So \( X \in [A]^\geq \kappa \). Now \( |A_i| < \kappa \) for each \( i \in I' \), so for any \( Y \in [X]^\kappa \), there are distinct \( i, j \in I' \) and \( a \in A_i, b \in A_j \) such that \( \chi^i_a \cdot \chi^j_b \in Y \). Since \( \chi^i_a \cdot \chi^j_b = 0 \), this contradicts (1). So 2(b) holds.

Next suppose that 2(c) does not hold. Then let \( I' \subseteq I \) be such that \( |A_i| \geq \kappa \) for all \( i \in I' \), and \( |I'| = \text{cf}(\kappa) \). For \( i \in I' \) choose a subset \( S_i \in [A_i]^{< \kappa} \) such that \( \sup_{i \in I'} |S_i| = \kappa \). Define \( X = \{ \chi^i_a : a \in S_i \text{ for some } i \in I' \} \) Then we reach a contradiction as in the above paragraph. So 2(c) also holds.

Toward 2(d) consider the set \( X \stackrel{\text{def}}{=} \{ \chi_{\{i\}} : i \in I \} \), which has size \( |I| \), and is pairwise disjoint, so has no independent subset. So (1) implies \( |I| < \kappa \).

(\( \Leftarrow \)). Assume (2). Let \( X \in [A]^\kappa \). First consider the set \( \{1(x) : x \in X\} \). In case this set has size \( \kappa \) then by 2(a) there is some \( Z \in [\{1(x) : x \in X\}]^\kappa \) such that \( Z \) is an independent subset of \( B \).
For each \( z \in Z \) choose some \( x = f(z) \in X \) such that \( 1(x) = z \). Define \( X' = \{ f(z) : z \in Z \} \). First note that if \( x, y \in X' \) are distinct, then \( 1(x) \neq 1(y) \). Say \( x = f(z) \) and \( y = f(z') \) for \( z, z' \in Z \). Then \( 1(x) = z \neq z' = 1(y) \). Next we claim that \( X' \) is an independent subset of \( X \) (it has size \( \kappa \) since \( Z \) does). In fact suppose that for some finite, disjoint \( F, G \subseteq X' \) we have

\[
\left( \prod_{x \in F} x \right) \cdot \left( \prod_{y \in G} y \right) = 0.
\]

In particular, equation 3.3 implies that \( (\prod_{x \in F} 1(x)) \cdot (\prod_{y \in G} 0(y)) = 0 \), since \( 0(y) = 1(\bar{y}) \). Now \( -1(y) = 0(y) \cup m(y) \), so

\[
\left( \prod_{x \in F} 1(x) \right) \cdot \left( \prod_{y \in G} -1(y) \right) = \left( \prod_{x \in F} 1(x) \right) \cdot \left( \prod_{y \in G} (0(y) \cup m(y)) \right) \subseteq \bigcap_{y \in G} m(y).
\]

Thus the left hand side of equation 3.4 is finite. But since \( Z \) is independent and infinite, there are infinitely many subsets of \( (\prod_{x \in F} 1(x)) \cdot (\prod_{y \in G} -1(y)) \) (a contradiction). To see this, note that for each finite disjoint \( H, K \subseteq (Z \setminus (F \cup G)) \) we have

\[
\left( \prod_{z \in H} z \right) \cdot \left( \prod_{w \in K} -w \right) \cdot \left( \prod_{x \in F} 1(x) \right) \cdot \left( \prod_{y \in G} -1(y) \right) \neq 0.
\]

So \( X' \) is indeed independent of size \( \kappa \).

So we proceed, assuming henceforth that \( |\{1(x) : x \in X\}| \) is strictly smaller than \( \kappa \). We have

\[
X = \bigcup_{F \in |I|^{<\omega}} \{ x \in X : m(x) = F \}.
\]

- Case 1. For some \( F \in |I|^{<\omega} \) we have \( |\{ x \in X : m(x) = F \}| = \kappa \). We can proceed as in the above proposition.

- Case 2. For each \( F \in |I|^{<\omega} \) we have \( |\{ x \in X : m(x) = F \}| < \kappa \). Let \( \mathcal{F} = \bigcup_{x \in X} m(x) \). So

\[
X = \bigcup_{F \in |\mathcal{F}|^{<\omega}} \{ x \in X : m(x) = F \}.
\]

By the case assumption, each set in that union is smaller than \( \kappa \), so \( |\mathcal{F}| = ||\mathcal{F}|^{<\omega}| \) must be at least \( \text{cf}(\kappa) \). On the other hand, since \( |I| < \kappa \) we must have \( |\mathcal{F}| < \kappa \), so we must have

\[
\sup_{F \in \mathcal{F}} |\{ x \in X : m(x) = F \}| = \kappa.
\]
Let $I_1 = \{ i \in I : |A_i| \geq \kappa \}$. We claim

$$\text{There is an } i \in I_1 \text{ and a sequence } \langle F_\alpha : \alpha < \cf(\kappa) \rangle \text{ of members of } |\mathcal{F}|^\omega$$

such that $i \in F_\alpha$ for every $\alpha$ and $\sup_{\alpha < \cf(\kappa)} |\{ x \in X : m(x) = F_\alpha \}| = \kappa$ \hspace{1cm} (3.6)

Suppose not. For each $i \in I$ let $\mathcal{F}_i = \{ F \in \mathcal{F} : i \in F \}$. So for each $i \in I_1$ have that

$$\sup_{F \in \mathcal{F}_i} |\{ x \in X : m(x) = F \}| < \kappa. \hspace{1cm} (3.7)$$

By assumption 2(c) we have that $|I_1| < \cf(\kappa)$, so letting $\mathcal{F}' = \bigcup_{i \in I_1} \mathcal{F}_i$ we still have

$$\sup_{F \in \mathcal{F}'} |\{ x \in X : m(x) = F \}| < \kappa. \hspace{1cm} \text{Thus by equation } 3.5 \text{ we have}$$

$$\sup_{F \in (\mathcal{F} \setminus \mathcal{F}') } |\{ x \in X : m(x) = F \}| = \kappa \hspace{1cm} (3.8)$$

Let $\langle F_\alpha : \alpha < \cf(\kappa) \rangle$ enumerate some subset of $\mathcal{F} \setminus \mathcal{F}'$ such that with

$$\lambda_\alpha \overset{\text{def}}{=} |\{ x \in X : m(x) = F_\alpha \}|, \text{ the sequence } \langle \lambda_\alpha \rangle \text{ has supremum } \kappa. \hspace{0.5cm} \text{Say } |\{ 1(x) : x \in X \}| = \lambda,$$

and recall that $\lambda < \kappa$. Let $\alpha$ be such that $\lambda_\alpha > \lambda$, and write

$$\{ x \in X : m(x) = F_\alpha \} = \bigcup_{y \in \{ 1(x) : x \in X \}} \{ x \in X : m(x) = F_\alpha \text{ and } 1(x) = y \}. \hspace{1cm} (3.9)$$

Note that for each $y \in \{ 1(x) : x \in X \}$ if $|\{ x \in X : m(x) = F_\alpha \text{ and } 1(x) = y \}| = \eta$ then $|\prod_{i \in F_\alpha} A_i| \geq \eta$. The index set on the right side of (3.9) has size $\lambda$, and the union has size $\lambda_\alpha > \lambda$. If $\lambda_\alpha$ is regular, there is some $y$ such that the set $\{ x \in X : m(x) = F_\alpha \text{ and } 1(x) = y \}$ has size $\lambda_\alpha$, hence $|\prod_{i \in F_\alpha} A_i| \geq \lambda_\alpha$. Otherwise $\lambda_\alpha$ is singular, but still the supremum of the sizes of sets in the union is $\lambda_\alpha$. So by the observation immediately below (3.9), $|\prod_{i \in F_\alpha} A_i| \geq \eta$ for each $\eta < \lambda_\alpha$, (and $\lambda_\alpha$ is a limit cardinal) so again in this case $|\prod_{i \in F_\alpha} A_i| \geq \lambda_\alpha$. Then for some $i \in F_\alpha$, $|A_i| \geq \lambda_\alpha$. This contradicts 2(b). So we have proved claim (3.6).

So fix a sequence $\langle F_\alpha : \alpha < \cf(\kappa) \rangle$ as in (3.6). Now for each $\alpha$ let $G_\alpha = I_1 \cap F_\alpha$. As each $G_\alpha$ is a finite subset of $I_1$, and $|I_1| < \cf(\kappa)$, we must have that $\cf(\kappa)$-many of the sets $G_\alpha$ are the same. By removing some of the members sequence $F_\alpha$ if necessary, we may assume there is a $\xi < \cf(\kappa)$ and a $G$ such that $G_\alpha = G$ for all $\alpha > \xi$. We claim that there is an $i \in G$ such that
\(|\{x(i) : x \in X\}| = \kappa\). In fact, say \(\sup \{|A_i| : i \in I \setminus I_1\} = \mu\), and recall that \(\mu < \kappa\) and also \(\lambda = |\{1(x) : x \in X\}| < \kappa\). Let \(\langle \lambda_\alpha : \alpha < \text{cf}(\kappa) \rangle\) be a sequence of cardinals with supremum \(\kappa\). By renumbering the sequence \(F_\alpha\), we may assume that \(|\{x \in X : m(x) = F_\alpha\}| = \lambda_\alpha\).

Then for each \(\alpha > \xi\) with \(\lambda_\alpha > \max \{\lambda, \mu\}\), the fact that \(|\{x \in X : m(x) = F_\alpha\}| = \lambda_\alpha\) implies that \(|\{x \in G : x \in X\}| = \lambda_\alpha\). To see this, consider the function

\[ f : \{x \in X : m(x) = F_\alpha\} \to \left(\bigcup_{x \in X} \{x \mid F_\alpha\}\right) \times \{1(x) : x \in X\}, \]

defined by \(f : x \mapsto (x \mid F_\alpha, 1(x))\).

The above \(f\) is clearly injective, so the range has size at least \(\lambda_\alpha\). Since \(|\{1(x) : x \in X\}| < \lambda_\alpha\), the set \(\bigcup_{x \in X} \{x \mid F_\alpha\}\) has size at least \(\lambda_\alpha\). Since \(|\prod_{i \in F_\alpha \setminus G} A_i| < \lambda_\alpha\), we must have \(|\{x \in G : x \in X\}| = \lambda_\alpha\).

Since this holds for all \(\alpha\) sufficiently large, we conclude that \(|\{x \in G : x \in X\}| = \kappa\), so for some \(i \in G\), \(\{x(i) : x \in X\}\) has size \(\kappa\). So, by choosing an independent subset of \(\{x(i) : x \in X\}\), we obtain the desired independent subset of \(X\).

\[ \square \]

3.2.11  Ideal-Independence

First, we define \(S(B) = (\{|b| : b \in B\} \setminus \omega)\).

**Lemma 3.2.21.** For any \(B \subseteq \mathcal{P}(\kappa)\) containing the singletons, we have \(S(B) \subseteq s_{\text{spect}}(B)\). In particular \(\kappa \in s_{\text{spect}}(B)\).

**Proof.** Suppose that \(b \in B\) is infinite. Fix an arbitrary \(i_0 \in b\). We define a maximal ideal-independent family of size \(|b|\) as follows:

\[ Y = \{\{i\} : i \in b \setminus \{i_0\}\} \cup \{-b \cup \{i_0\}\}. \]

We claim that \(Y\) is ideal-independent. For any \(y \in Y\), \(b \cap y\) is a singleton, say \(\{j\}\). Then \(\{j\} \in \langle Y \rangle^{\text{id}}\) but \(\{j\} \notin \langle Y \setminus y \rangle^{\text{id}}\). Now suppose that \(a \in B \setminus Y\); we will show that \(Y \cup \{a\}\) is not ideal-independent. This is clear if \(a \subseteq -b \cup \{i_0\}\). So assume there is some \(i \in a \cap (b \setminus \{i_0\})\). Then \(\{i\} \subseteq a\).  

\[ \square \]
Proposition 3.2.22. Assume that \( \{ A_i : i \in I \} \) is a family of atomless BAs. Then \( \text{spect}(B) \cup \bigcup_{i \in I} \{ \text{spect}(A_i) \} \subseteq \text{spect}(A) \).

Proof. Suppose that \( Y \) is a maximal ideal-independent subset of \( A_i \). Then for each \( y \in Y \) define

\[
f_y(j) = \begin{cases} y & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}
\]

Let \( X = \{ f_y : y \in Y \} \). We claim that \( X \) is maximal ideal-independent. In fact, ideal-independence is clear because \( Y \) is ideal-independent. Suppose that \( a \in A \), we will show that \( X \cup \{ a \} \) fails to be ideal independent. By maximality of \( Y \) we have that either

(I) \( \exists F \in [Y]^{<\omega} \left[ a(i) \leq \sum F \right] \), or

(II) \( \exists F \in [Y]^{<\omega} \exists b \in Y \setminus F \left[ \sum F + a(i) \geq b \right] \).

If (I) is the case then also \( a \leq \sum_{y \in F} f_y \). Suppose that (II) holds then. If \( F \) is nonempty then also

\( a + \sum_{y \in F} f_y \geq f_b \).

Otherwise, let \( y \in Y \setminus \{ b \} \), and then the above holds with \( F = \{ y \} \).

Now suppose that \( \lambda \in \text{spect}(B) \). Let \( Y \subseteq B \) be maximal ideal-independent such that \( |Y| = \lambda \). Let \( X \subseteq A \) be the family \( \{ \chi_y : y \in Y \} \). Clearly \( X \) is an ideal-independent family of size \( \lambda \). Letting \( a \in A \), we will show that \( X \cup \{ a \} \) fails to be ideal independent. Now by maximality of \( Y \) we have that either

(I) \( \exists F \in [Y]^{<\omega} \left[ 1(a) \subseteq \sum F \right] \), or

(II) \( \exists F \in [Y]^{<\omega} \exists b \in Y \setminus F \left[ \sum F \cup 1(a) \supseteq b \right] \).

If item (II) is the case then we have that \( \chi_b \subseteq a + \sum_{y \in F} \chi_y \), so we are done. So we assume that (I) holds. Now we claim that for each \( i \in \kappa \) we must have some \( y \in Y \) such that \( \{ i \} \subseteq y \). Suppose not; consider the set \( \{ i \} \cup Y \). Because \( Y \) is maximal and \( \{ i \} \not\subseteq y \) for any \( y \in Y \) then we
must have some \( y \in Y \) and some \( F \in [Y]^\omega \) with \( y \subseteq \sum F \cup \{i\} \). But \( i \notin y \) so also \( y \subseteq \sum F \), a contradiction of the ideal-independence of \( Y \). So the claim holds, and we can find, for each \( i \in \kappa \), a set \( y_i \in Y \) such that \( i \in y_i \). Now the set \( G := \{y_i \in Y : i \in m(a)\} \) is finite, and we have

\[
a \subseteq \sum_{y \in F \cup G} \chi_y
\]

which completes the proof.

**Corollary 3.2.23.** \( s_{mm}(A) \leq \min \{s_{mm}(A_i) : i \in I\}, s_{mm}(B) \)

**3.2.12 \( \pi \)-Weight**

**Definition 3.2.1.** The \( \pi \)-weight of \( A \), denoted \( \pi(A) \), is the least size of a dense subset of \( A \).

**Proposition 3.2.24.** \( \pi(A) = \max \{|I|, \sup_{i \in I} \pi(A_i)\} \).

**Proof.** The inequality \((\geq)\) follows in from the following two facts:

1. \( \pi(A) \geq \pi(A') \) whenever \( A \geq_{\text{reg}} A' \) (c.f. [Mon10]), and
2. \( \pi(B) = |I| \) since any dense subset of \( B \) must contain the singletons, and the singletons are dense.

For the other inequality, we will exhibit a dense subset of size \( \max \{|I|, \sup_{i \in I} \pi(A_i)\} \). Let \( D_i \subseteq A_i \) for each \( i \in I \) be a dense of minimal size. Next define

\[
D = \bigcup_{i \in I} \{\chi_d : d \in D_i\}.
\]

Now if \( a \in A^+ \) then there is some \( i \in I \) such that \( a(i) \neq 0 \), so for some \( d \in D_i \), \( \chi_d \leq a \).
Bibliography


