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Development and Applications of Soliton Perturbation Theory

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Development and Applications of Soliton Perturbation Theory

by

S. D. Nixon

B.A., University of Central Florida, 2005

A thesis submitted to the
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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
This thesis examines the effects of small perturbation to soliton solutions of the nonlinear Schrödinger (NLS) equation on two fronts: the development of a direct perturbation method for dark solitons, and the application of perturbation theory to the study of nonlinear optical systems including the dynamics of ultra-short pulses in mode-locked lasers.

For dark soliton solutions of the NLS equation a direct perturbation method for approximating the influence of perturbations is presented. The problem is broken into an inner region, where core of the soliton resides, and an outer region, which evolves independently of the soliton. It is shown that a shelf develops around the soliton which propagates with speed determined by the background intensity. Integral relations obtained from the conservation laws of the NLS equation are used to determine the properties of the shelf. The analysis is developed for both constant and slowly evolving backgrounds. A number of problems are investigated including linear and nonlinear dissipative perturbations.

In the study of mode-locking lasers the power-energy saturation (PES) equation is a variant of the nonlinear NLS equation, which incorporates gain and filtering saturated with energy, and loss saturated with power (intensity). Solutions of the PES equation are studied using adiabatic perturbation theory. In the anomalous regime individual soliton pulses are found to be well approximated by soliton solutions of the unperturbed NLS equation with the key parameters of the soliton changing slowly as they evolve. Evolution equations are found for the pulses’ amplitude, velocity, position, and phase using integral relations derived from the PES equation. It is shown that the single soliton case exhibits mode-locking behavior for a wide range of parameters. The results from the integral relations are shown to agree with the secularity conditions found in multi-scale perturbation theory.
In the normal regime both bright and dark pulses are found. Here the NLS equation does not have bright soliton solutions, and the mode-locked pulse are wide and strongly chirped. For dark pulses there are two interpretations of the PES equation. The existence and stability of mode-locked dark pulses are studied for both cases.

Soliton strings are found in both the constant dispersion and dispersion-managed systems in the (net) anomalous and normal regimes. Analysis of soliton interactions show that soliton strings can form when pulses are a certain distance apart relative to their width. Anti-symmetric bi-soliton states are also obtained. Initial states mode-lock to these states under evolution.
Dedication

To Jenny Negin, Ann Singleton and Gregg Bantz who gave me something to do with my time.
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Chapter 1

Introduction

An important problem in nonlinear waves is to find and understand the effect of small perturbation on soliton solutions. The term soliton was originally coined by Zabusky and Kruskal in 1965 [72] to describe elastic solitary-wave pulse interactions in the Korteweg-deVries (KdV) equation. In physics and engineering the term soliton has been broadened to mean most localized waves; i.e. solitary waves. We will use the latter terminology in this thesis.

In optics, a central equation which describes the envelope of a quasi-monochromatic wave-train is the nonlinear Schrödinger (NLS) equation, which in normalized form is given by

\[ iU_z + \frac{d_0}{2} U_{tt} + n |U|^2 U = 0 \]  

(1.0.1)

where \(d_0, n\) are constant. We will consider the NLS equation in a typical nonlinear optics context where \(d_0\) corresponds to the group-velocity dispersion (GVD), \(n > 0\) is related to the the nonlinear index of refraction, \(z\) is the direction of propagation and \(t\) corresponds to the retarded time. In this form, the sign of \(d_0\) determines whether the light focuses or defocuses. In the anomalous GVD regime \((d_0 > 0)\) the NLS equation exhibits so-called ‘bright’ solitons which are pulses which decay rapidly at infinity. In this case the solitons are formed due to a balance between dispersion and self-focusing cubic nonlinearity. In the normal GVD regime \((d_0 < 0)\) decaying pulses broaden and bright solitons of the NLS equation do not exist. Instead solitons can be found as localized dips in intensity which decay off of a continuous-wave (cw) background. These dark solitons, which are termed black when the intensity of the dip goes to zero and grey otherwise, are also associated with a rapid change in phase across the pulse.
A particular perturbation of interest is the Power Energy Saturation (PES) equation

\[ iU_z + \frac{d_0}{2} U_{tt} + n_i|U|^2 U = i \left( g \frac{U}{1 + E/E_{sat}} + \tau \frac{U_{tt}}{1 + E/E_{sat}} + l \frac{U}{1 + P/P_{sat}} \right) \] (1.0.2)

where \( E \) is the pulse energy, \( E_{sat} \) is the saturation energy, \( P \) is the pulse power, \( P_{sat} \) is the saturation power, and \( g, \tau \) and \( l \) are positive real constants. Typical physical parameters can be found in [7]. The first two terms model energy saturated gain and filtering, and the last term models a power saturated loss. These are the three main components of any mode-locking laser. We refer the reader to Haus [39] for more background on the derivation of models like the PES equation. For bright solitons this type of rational function model has worked remarkably well when compared to experimental results regarding Ti: sapphire lasers [62]. Recent experimental [25] and theoretical [7] results in ML lasers in the anomalous dispersion regime indicate that the normalized intensity of a pulse in a ML laser can be described by the bright solitons of the unperturbed NLS equation. Different types of models have been considered in the past for both bright [64, 67] and dark [15] pulses.

1.1 History

1.1.1 Soliton Perturbation Theory

Perturbation theory as applied to solitons which decay at infinity, i.e. so-called bright solitons, has been developed over many years; it has been used to investigate a number of equations including the KdV equation [47, 54, 43], the NLS equation [49, 36], and the sine-Gordon equation [31]. The analytical work employs a diverse set of methods including perturbations of the inverse scattering transform (IST), direct multi-scale perturbation analysis, perturbations of conserved quantities, etc; the analysis applies to a wide range of physical problems. Using these analytical methods, the propagation of bright solitons under perturbation is described by the adiabatic evolution of the soliton parameters; i.e. the soliton’s height, velocity, position shift and phase shift. An early and extensive list of reference may be found in the review article by Kivshar and Malomed [52].
The experimental observations of dark solitons in both fiber optics [27] and planar waveguides [45] sparked significant interest in the asymptotic analysis of their propagation dating back two decades. However, the non-vanishing boundary of dark solitons introduces serious complications when simply applying the perturbative methods developed for bright solitons. In early work, the particular case of added linear loss was studied both numerically [75] and analytically [33]. The analysis was originally developed for black solitons and solved explicitly for higher order correction terms. These results were re-derived [58] with a more straightforward method.

The method was extended to grey solitons and general perturbations but only for the two of the four main soliton parameters; background height and soliton depth, were determined. The evolution of the background was shown to be independent of the soliton by Kivshar [51] where the asymptotic behavior at infinity was used to separate the propagation of the background magnitude from the rest of the soliton. The amplitude/width of the soliton ‘core’ was determined via a perturbed Hamiltonian. Of the methods proposed many have employed perturbation theory based on IST theory. In [55] orthogonality conditions are derived from a set of squared Jost functions (eigenfunctions of the linearized NLS operator) [49]; from these conditions the soliton parameters are, in principle, determined; over the years various corrections/modifications have been made to the details [23, 20].

There is a central issue systemic through all these methods. For dark solitons finding the adiabatic evolution of the soliton depth and background height alone is insufficient to fully characterize the evolution of a dark soliton. In fact there is a generation of a small but broad shelf that is important and needed to determine all leading order parameters of the soliton. Numerically speaking, generation of shelves around dark solitons under external perturbation [22] was suggested as an explanation regarding discrepancies in the perturbed conservation laws (Hamiltonian and Energy). Moreover shelves in soliton perturbation theory had been found earlier in a different, but as it turns out much easier, class of problems. They were needed to effectively understand perturbations of the decaying soliton solution of the KdV equation under perturbation cf. [4, 53]. In the KdV equation there is a small and broad shelf produced in the wake of the soliton.
The height/speed of the soliton, shelf and the additional soliton parameter which determines the center of the soliton are all determined by perturbation theory [54]. The importance of radiative effects (like shelves) for dark soliton perturbations was not considered again until [57], in which the continuous spectrum of the IST is analyzed. However, the soliton parameter for the position and phase shift are not considered and they do not find the shelf structure. The complete picture and details associated with dark solitons are presented in this thesis for the first time.

1.1.2 Soliton Mode-Locking

Ultra-short pulses in mode-locked (ML) lasers are the topic of extensive research due to their wide range of applications ranging from communications [39], to optical clock technology [71] and even to measurements of the fundamental constants of nature [30]. Although ML lasers have been studied for many years [68, 38], it is only recently that researchers have begun to better understand and explore their complicated dynamics.

The first mathematical model for mode-locked lasers was set down in the seminal paper by Haus and Meccozzi [42] (the culmination of work in [40, 41]). Here the assumption is made that the change of the pulse per round trip is small and the effects of the discrete laser components may be replaced by a continuous approximation. This approximation is termed the "master" equation. It is essentially the PES model with, however, the power term expanded to two terms in its Taylor series; i.e. assuming small amplitude. Unfortunately, the master equation has only a small parameter regime where mode-locking to stable soliton states occurs [56]; it has not been shown to support higher-order states which have been observed [35, 34] nor does it have bright solitons in the normal ($d_0 < 0$ regime (also found in experiments [29]). For certain values of the parameters this equation exhibits a range of phenomena including: mode-locking evolution, pulses which disperse into radiation, and some whose amplitude grows rapidly [46]. In the latter case, if the nonlinear gain is too high, the linear attenuation terms are unable to prevent the pulse from blowing up, suggesting the breakdown of the master mode-locking model [56].

Other models used to describe this propagation include Ginzburg-Landau (GL) type equa-
tions [18, 17] which may be found in the constant energy limit of the “master” equation. Such GL systems in the anomalous regime have been found to support steady high-order soliton solutions [16]; they also contain a wide range of solutions including unstable, chaotic and quasi-periodic states and even blow-up can occur. Furthermore, soliton interactions within the framework of such GL systems can be complicated. It is found [16] that both in-phase and anti-phase high-order soliton states are either unstable or weakly stable. In either case these states are not attractors [19]. Hence they do not correspond to observations of higher order soliton states in a ML laser system which have discrete, nearly fixed separations [65].

The PES model was proposed in [7] as an alternative model for mode-locking. As well as exhibiting mode-locking in the anomalous dispersion regime, the PES equation has stable modes in the normal dispersion regime [9] and for dispersion managed systems [10] and yields mode-locking of dark soliton states [11] (a phenomena recently studied [28, 74]). Furthermore, the PES model reduces to the “master” equation in the small power (or large power saturation) limit. In fact this reduction is what was originally used [38] to arrive at the loss term for the “master” equation. At the time the “master” equation was deemed preferable since it had an explicit solution,

1.2 Outline of the Thesis

This thesis examines the effects of small perturbation to soliton solutions of the NLS equation on two fronts: the development of a direct perturbation method for dark solitons, and the application of perturbation theory to the study of mode-locking in a variety of optical systems.

In Chapter 2, we analyze the effect perturbations to the NLS equation have on dark solitons. The problem is divided into an inner “soliton” region consisting of the soliton and radiative effects and an outer “background” region which is unaffected by the local disturbance. A moving boundary layer bridges the difference between these two regions. The soliton develops a shelf on either side which expands at a rate proportional to the background magnitude. The basic ideas are first laid out for an example problem with a black soliton and a dissipative perturbation and then extended to grey solitons and general perturbations. The method uses a multiple-scales expansion
with perturbed conservation laws and asymptotic information about the first correction term to find the evolution of soliton depth and phase as well as a first approximation for the magnitude and phase of the shelf. To find the evolution of the center of the soliton we must explicitly solve for the first correction term. We also show that the condition found from the perturbed conservation laws is consistent with Fredholm Alternative type arguments for the solvability of the equation for the first correction term.

In Chapter 3, we apply perturbation theory to the study of the PES equation in the anomalous dispersion regime. We use the method of multiple-scales and perturbed conservation laws to find the evolution of the amplitude, velocity, position and phase of a bright soliton. Mode-locked solutions are found to be solitons of the NLS equation. For high energy initial conditions single pulses break into multiple pulses which appear to mode-lock. In the case of weak interaction the effect of neighboring solitons may be viewed as a perturbation and this effect is studied in detail for case of two solitons interacting. The apparent mode-locking is found to have variations which decay slowly (logarithmically). We also show that the evolution equation found from perturbed conservation laws are equivalent to the secularity condition derived from perturbation theory based on the inverse-scattering transform.

In Chapter 4, we extend our analysis of the PES equation to the normal dispersion regime. In the normal dispersion regime with dark solitons there are two interpretations of the PES model. One alternative is define the energy as the dark energy, i.e. the difference in energy from a continuous wave of the same amplitude. In this case mode-locking is found, but the equilibrium solutions are unstable. A second alternative is to restrict the problem to a finite domain with periodic boundary conditions with the energy defined as the average power. Here pulsed tend toward black soliton solutions, but the shelves interacting with each other cause increasing fluctuations in the background. Both cases are discussed in the thesis.

In Chapter 5, we study the PES model numerically for the case of dispersion management (DM). Similarly to constant dispersion mode-locked solutions are found to be solitons of the dispersion managed NLS equation (DMNLS). The interaction of solitons here is dependent on
the map strength (a measure of the variance in dispersion). As well as single solitons bi-soliton solutions which have the structure of two joined solitons for small map strength and one large soliton with a pi phase jump at the peak for large map strength. Such anti-symmetric DM solitons have been observed [61, 60].

This thesis is based on work found done by the author in the following publications: [3, 2, 1, 11, 13].
Chapter 2

Dark Soliton Perturbation Theory

In this chapter we develop a direct perturbation theory for dark solitons in the NLS equation. In Section 2.1 we pose the problem and illustrate how the background evolves under perturbation independent of any localized solitary wave disturbances. Sections 2.2 - 2.5 set up the basic analysis and a prototypical problem is discussed which helps describe the ideas. The method of multiple scales is employed to find the first order approximation for a black soliton under the action of a dissipative perturbation which decays to zero well away from the soliton core. The concept of a moving boundary layer is used to bridge the differences between the inner soliton solution and the outer background. This discrepancy between the approximate soliton solution and the background manifests itself as a shelf developing on either side of the soliton. Perturbed conservation laws are used to find the growth of the shelf in both magnitude and phase. The analytic results are shown to be in agreement with numerical simulations of the perturbed NLS equation. In Sections 2.6 - 2.8 the method is extended to grey solitons under general perturbations. Asymptotic information about the shelf is obtained from the linear first order perturbation equation; to determine the asymptotic states and ‘center of phase’, the complete solution of the linear problem is not required. However, to find the soliton center we find the first order correction term. In Sections 2.9 - 2.11 the perturbation method is applied to some physically relevant perturbations: dissipation and two photon absorption. We find that the spatial frequency of the soliton differs from that of the background that it resides on. All of the adiabatically varying core soliton parameters and the shelf have not been obtained in the many previous studies of perturbed dark solitons. In Section
2.12 we derive two secularity conditions from Fredholm alternative type arguments which agree with the results found from the perturbed conservation laws.

2.1 The Boundary at Infinity

Let us consider the dimensionless NLS equation with normal dispersion: $D = -1$, $n = 1$ (we can always rescale NLS to get these normalized values) and with an additional small forcing perturbation

$$i U_z - \frac{1}{2} U_{tt} + |U|^2 U = \epsilon F[U]$$

(2.1.1)

where $|\epsilon| \ll 1$. Further we will assume a non-vanishing boundary value at infinity; i.e., $|U| \to 0$ as $t \to \pm \infty$. The effect the perturbation has on the behavior of the solution at infinity is independent of any local phenomena such as pulses which do not decay at infinity; i.e. dark solitons. In the case of a continuous wave background, which is relevant to perturbation problems with dark solitons as well as in applications to lasers, we have $U_{tt} \to 0$ as $t \to \pm \infty$ and the evolution of the background at either end $U \to U^\pm(z)$ is given by the equation

$$i \frac{d}{dz} U^\pm + |U^\pm|^2 U^\pm = \epsilon F[U^\pm]$$

(2.1.2)

We write $U^\pm(z) = u^\pm(z)e^{i\phi^\pm(z)}$ where $u^\pm(z) > 0$ and $\phi^\pm(z)$ are both real functions of $z$. Then the imaginary/real parts of the above equations are

$$\frac{d}{dz} u^\pm = \epsilon \text{Im} \left[ F[u^\pm e^{i\phi^\pm}] e^{-i\phi^\pm} \right]$$

(2.1.3a)

$$\frac{d}{dz} \phi^\pm = (u^\pm)^2 - \epsilon \text{Re} \left[ F[u^\pm e^{i\phi^\pm}] e^{-i\phi^\pm} \right]/u^\pm$$

(2.1.3b)

The above equations completely describe the adiabatic evolution of the background under the influence of the perturbation $F[U]$. Although this is true for all choices of perturbation, we will further restrict ourselves to perturbations which maintain the phase symmetry of equation (2.1.1); i.e., $F[U(z, t)e^{i\theta}] = F[U(z, t)]e^{i\theta}$. As we show next, this is a sufficient condition to keep the magnitude of the background equal on either side and a property of most commonly considered perturbations. We assume that at $z = 0$, $u^+(0) = u^-(0)$, then, since $u^\pm(z)$ satisfy the same
equation, the evolution is the same for all \(z\). Hence \(u^+(z) = u^-(z) \equiv u_\infty(z)\). While this restriction is convenient the essentials of the method presented here apply in general. The equations for the background evolution (2.1.3) can now be further reduced by considering the phase difference \(\Delta \phi_\infty(z) = \phi^+(z) - \phi^-(z)\) which is the parameter related to the depth of a dark soliton (see below); here \(\phi^\pm(z)\) represents the phase as \(t \to \pm \infty\) respectively

\[
\frac{d}{dz}u_\infty = \epsilon \text{Im} \left[ F[u_\infty] \right] \tag{2.1.4a}
\]

\[
\frac{d}{dz} \Delta \phi_\infty = 0 \tag{2.1.4b}
\]

Thus, while the magnitude of the background evolves adiabatically the phase difference remains unaffected by the perturbation.

Let us now focus on the evolution of a dark soliton under perturbation. To simplify our calculations we take out the fast evolution of the background phase

\[
U = u_0 e^{\int_0^z u_\infty(s)^2 ds} \tag{2.1.5}
\]

so equation (2.1.1) becomes

\[
iu_z - \frac{1}{2} u_{tt} + (|u|^2 - u_\infty^2)u = \epsilon F[u] \tag{2.1.6}
\]

The dark soliton solution to the unperturbed equation is given by

\[
u_s(t, z) = (A + iB \tanh (B(t - Az - t_0))) e^{i\sigma_0} \tag{2.1.7}
\]

where the core parameters of the soliton: \(A, B, t_0, \sigma_0\) are all real, the magnitude of the background is \((A^2 + B^2)^{1/2} = u_\infty\) and the phase difference across the soliton is \(2\tan^{-1}\left( \frac{B}{A} \right)\), \(A \neq 0\). When \(A = 0\) equation (2.1.7) describes a black soliton, which has a phase difference of \(\pi\).

Below, we employ the method of multiple scales by introducing a slow scale variable \(Z = \epsilon z\) with the parameters \(A, B, t_0\) and \(\sigma_0\) being functions of \(Z\). A perturbation series solution for equation (2.1.6) is assumed

\[
u = u_0(Z, z, t) + \epsilon u_1(Z, z, t) + O(\epsilon^2) \tag{2.1.8}
\]
The order 1 approximation \( u_0(Z, z, t) \) should satisfy the slowly varying boundaries from equations (2.1.4), which means two of the parameters are already pinned down \( A(Z) = u_\infty(Z) \cos \frac{\Delta \phi_\infty}{2} \), \( B(Z) = u_\infty(Z) \sin \frac{\Delta \phi_\infty}{2} \) and we take \( \sigma_0(0) = 0 \).

### 2.2 The First Order Correction

We write the solution in terms of the amplitude and phase: \( u = q e^{i \phi} \) where \( q \) and \( \phi \) are both real functions of \( z \) and \( t \) so equation (2.1.6) becomes

\[
 iq_z - \phi_z q - \frac{1}{2} (q_{tt} + i2\phi_t q_t + q(i\phi_{tt} - \phi^2_t)) + (|q|^2 - u^2_\infty)q = \varepsilon F[u]
\]

Once we introduce the additional multiple scale variable \( Z = \varepsilon z \); the real and imaginary parts of the above equations are:

\[
 q_z = \frac{1}{2} (2\phi_t q_t + q q_{tt}) + \varepsilon (\text{Im}[F[u]] - q_Z)
\]

\[
 \phi_z q = -\frac{1}{2} (q_{tt} - \phi^2_t q) + (|q|^2 - u^2_\infty)q + \varepsilon (\text{Re}[F[u]] - \phi Z q)
\]

Expanding \( q \) and \( \phi \) as series in \( \varepsilon \): \( q = q_0 + \varepsilon q_1 + O(\varepsilon^2) \) and \( \phi = \phi_0 + \varepsilon \phi_1 + O(\varepsilon^2) \), we have at \( O(1) \)

\[
 q_{0z} = \frac{1}{2} (2\phi_{0t} q_0 t + q_0 \phi_{0tt}) \quad (2.2.10a)
\]

\[
 \phi_{0z} q_0 = -\frac{1}{2} (q_{0tt} - \phi^2_{0t} q_0) + (|q_0|^2 - u^2_\infty)q_0 \quad (2.2.10b)
\]

with the general dark soliton solution

\[
 q_0 = \left( A(Z)^2 + B(Z)^2 \tanh^2(x) \right)^{1/2} \quad (2.2.11a)
\]

\[
 \phi_0 = \tan^{-1} \left[ \frac{B(Z)}{A(Z)} \tanh(x) \right] + \sigma_0(Z) \quad (2.2.11b)
\]

\[
 x = B \left( t - \int_0^z A(\varepsilon s)ds - t_0(Z) \right) \quad (2.2.11c)
\]

For a black soliton the form of solution (2.2.11) is taken to be

\[
 q_0(Z, z, t) = u_\infty \tanh \left[u_\infty (t - t_0(Z)) \right] \quad (2.2.12a)
\]

\[
 \phi_0(Z, z, t) = \sigma_0(Z) \quad (2.2.12b)
\]
where we note that in this representation \( q_0 \) is allowed to be negative. At \( \mathcal{O}(\varepsilon) \) we have

\[
\begin{align*}
q_{1z} &= \frac{1}{2} [2(q_0 t_1 t + q_0 t_1 t) + q_0 t_1 t] + q_0 t_1 t + q_1 \phi_{0tt} + 1 \text{Im}[F] - q_0 Z \quad (2.2.13a) \\
\phi_{1z} q_0 &= -q_1 \phi_0 Z - \frac{1}{2} [q_1 t + (2q_0 t_1 t)q_0 - q_0^2 q_1] + 3q_0^2 q_1 - u_\infty^2 q_1 + \text{Re}[F] - \phi_{0Z} q_0 \quad (2.2.13b)
\end{align*}
\]

We begin with a somewhat simpler problem than others we deal with later; i.e. consider the linear dissipative filter perturbation

\[ F[u] = i\gamma u_{tt}, \quad \gamma > 0 \quad (2.2.14) \]

and for concreteness, at leading order we assume a black pulse, (2.2.12), which satisfies the boundary conditions, (2.1.4); i.e. \( u_\infty(Z) = \text{constant} \). This leaves us with slow evolution terms

\[
\begin{align*}
q_0 Z &= -t_0 Z q_0 t \quad (2.2.15a) \\
\phi_0 Z &= \sigma_0 Z \quad (2.2.15b)
\end{align*}
\]

If we look for a stationary solution, \( q_{1z} = \phi_{1z} = 0 \), and note that \( \phi_{0t} = \phi_{0tt} = 0 \) then equations (2.2.13) reduce to

\[
\begin{align*}
0 &= \frac{1}{2} [2(q_0 t_1 t) + q_0 t_1 t] + \text{Im}[F] + t_0 Z q_0 t \quad (2.2.16a) \\
0 &= -\frac{1}{2} [q_1 t + (2q_0 t_1 t)q_0 - q_0^2 q_1] + 3q_0^2 q_1 - u_\infty^2 q_1 + \text{Re}[F] - \sigma_0 Z q_0 \quad (2.2.16b)
\end{align*}
\]

where \( \text{Im}[F] = \gamma q_{0tt} \) and \( \text{Re}[F] = 0 \).

First we look at equation (2.2.16a)

\[
q_0 t_1 t + \frac{1}{2} q_0 t_1 t = -\gamma q_{0tt} - t_0 Z q_0 t \quad (2.2.17)
\]

which after multiplying by \( q_0 \), using properties of the leading order solution and integrating, yields

\[
\phi_{1t} = \frac{4}{3} \gamma q_0 - t_0 Z + c_1 q_0^{-2} \quad (2.2.18)
\]
Since $q_0^{-2}$ has a singularity at $t_0$ we set $c_1 = 0$ to have a bounded solution. After integrating we are left with

$$
\phi_1 = \frac{4}{3} \gamma \ln \left[ \cosh(u_\infty(t - t_0)) \right] - t_0Z t + c_2
$$

(2.2.19)

Asymptotically for large $t$ we have

$$
\phi_1^+ = \frac{4}{3} \gamma u_\infty - t_0Z, \quad \phi_1^- = -\frac{4}{3} \gamma u_\infty - t_0Z
$$

(2.2.20)

where the superscript $\pm$ indicates the value of a function as $t \to \pm \infty$ respectively.

We can also solve explicitly for $q_1$. After a change of variables $x = u_\infty(t - t_0)$ and substituting in for $q_0$ equation (2.2.16b) becomes

$$
q_{1xx} + (6\tanh^2(x) - 4)q_1 = -2 \frac{\sigma_0Z}{u_\infty} \tanh(x)
$$

(2.2.21)

The homogenous problem is now a special form of

$$
Q_{xx} + (n(n + 1) \text{sech}^2x - n^2)Q = 0
$$

(2.2.22)

which has the bounded solution

$$
Q = \text{sech}^n x
$$

(2.2.23)

With this solution we can use reduction of order to solve equation (2.2.21)

$$
q_1 = \left[ c_1 + c_2 \left( \frac{1}{4} \sinh(4x) + 2 \sinh(2x) + 3x \right) + \frac{\sigma_0Z}{8u_\infty} \left( x - \frac{1}{4} \sinh(4x) \right) \right] \text{sech}^2(x)
$$

(2.2.24)

By looking at the asymptotic behavior at $x \to \pm \infty$

$$
q_1 \sim \frac{1}{16} \left( c_2 - \frac{\sigma_0Z}{u_\infty} \right) e^{\pm 2x}
$$

(2.2.25)

we see that to avoid exponential growth we must take $c_2 = \frac{\sigma_0Z}{u_\infty}$. Furthermore, it is convenient to require that the full solution $q_1$ vanish at $t = t_0$ so that the $u$ remains anti-symmetric. If $u$ must be anti-symmetric, then it also follows that $t_0Z$ must be 0 in equation (2.2.18). We corroborate this result in Section 2.5. Now the unique solution to (2.2.16b) is

$$
q_1 = \frac{\sigma_0Z}{4u_\infty} \left[ \sinh(2u_\infty(t - t_0)) + 2u_\infty(t - t_0) \right] \text{sech}^2(u_\infty(t - t_0))
$$

(2.2.26)
Looking at the asymptotic behavior as $t \to \pm \infty$ we have

$$q_1 \to (2u_\infty^2)^{-1} \sigma_0 z q_0^\pm = \pm \frac{\sigma_0 z}{2u_\infty}$$

(2.2.27)

### 2.3 Boundary Layer

Notice that $q_1 \to 0$ and $\phi_1 \to 0$ as $t \to \pm \infty$. As a result, the solution $u \approx (q_0 + \epsilon q_1) e^{i(\phi_0 + \epsilon \phi_1)}$ to order $\epsilon$ does not match the boundary conditions at infinity. Thus, our problem is now broken into two regions: the region which matches imposed decaying boundary condition behavior at infinity which is unaffected by the soliton, and the region in which the $O(\epsilon)$ correction term is valid and where the solution is quasi-stationary. We introduce a boundary layer in which there is a transition from a nonzero value in the perturbation term to zero (see also [54, 53]). Note in this section we will consider the more general case when $u_\infty$ is a function of $Z = \epsilon z$. We find the behavior of this boundary layer, where the regions are matched. For this we return to equation (2.1.6) and seek a solution perturbed around the solution at infinity, say $u \approx (u_\infty + \epsilon w) e^{i(\phi_0 + \epsilon \theta)}$ where $w$ and $\theta$ are real functions of $z$ and $t$; the equation is automatically satisfied at $O(1)$ and we have at $O(\epsilon)$

$$-\theta_z u_\infty + iw_z - \frac{1}{2} [iu_\infty \theta_{tt} + w_{tt}] + 2u_\infty^2 w = F[u_\infty + \epsilon w] - \left( \frac{iu_\infty}{iZ} - u_\infty \frac{d\phi^\pm}{dZ} \right)$$

(2.3.28)

After substituting equations (2.1.3) and (2.1.4) and noting $F[u_\infty + \epsilon w] - F[u_\infty] \approx \epsilon F'[u_\infty] w$ it follows that the right hand side is actually a higher order term and may be dropped. As a corollary, to leading order, the boundary layer is independent of perturbation. We now break (2.3.28) into real and imaginary parts

$$\theta_z u_\infty = 2u_\infty^2 w - \frac{1}{2} w_{tt}$$

(2.3.29a)

$$w_z = \frac{1}{2} u_\infty \theta_{tt}$$

(2.3.29b)

Taking a derivative with respect to $z$ of equation (2.3.29b) and then substituting in for $\theta_z$ and, similarly, taking a derivative with respect to $z$ of equation (2.3.29a) and then substituting in for $w_z$
we get

\[ w_{zz} = u_\infty^2 w_{tt} - \frac{1}{4} w_{tttt} \] (2.3.30a)

\[ \theta_{zz} = u_\infty^2 \theta_{tt} - \frac{1}{4} \theta_{tttt} \] (2.3.30b)

which is the same equation for both functions, though we will need a different solution for each. This is because of the differing boundary conditions to correctly match the inner region to the outer region. On the left side of the shelf we match the nonzero quasi-stationary shelf to the equilibrium state at infinity (which is on the left of the soliton); the boundary conditions are

\[ w(-\infty) = 0 \quad w(\infty) = q^-_1 \] (2.3.31a)

\[ \theta(-\infty) = 0 \quad \theta_t(\infty) = \phi^-_{1t} \] (2.3.31b)

On the right side of the shelf (on the right of the inner soliton) the boundary conditions are

\[ w(-\infty) = q^+_1 \quad w(\infty) = 0 \] (2.3.32a)

\[ \theta_t(-\infty) = \phi^+_1 \quad \theta(\infty) = 0 \] (2.3.32b)

If we let \( w = e^{i(kt + \frac{1}{2} \int_0^z \omega(s,k) ds)} \), then the 'dispersion' relation for equation (2.3.30a) is found to be

\[ \omega^2 = u_\infty^2 (Z)|k|^2 + \frac{1}{4} k^4 \] (2.3.33)

For long waves (\( k \ll 1 \)) we approximately have \( \omega(Z,k) \approx \pm u_\infty(Z)|k| \) or \( w = e^{i(kz + \frac{1}{2} \int_0^z u_\infty(z) \, dz)} \).

Thus, we see that long wave solutions (i.e. \( |k| \ll 1 \)) move with instantaneous velocity \( V(z) = \pm u_\infty(z) \). This is also true for equation (2.3.30b).

With this in mind, we look for solutions to equations (2.3.30) in a moving frame of reference:

\[ x = t - \int_0^\zeta V \, dz \quad \text{and} \quad \zeta = z. \]

\[ w_{\zeta\zeta} = 2Vw_{\zeta x} + (u_\infty^2 - V^2)w_{xx} - \frac{1}{4} w_{xxxx} \]

\[ \theta_{\zeta\zeta} = 2V\theta_{\zeta x} + (u_\infty^2 - V^2)\theta_{xx} - \frac{1}{4} \theta_{xxxx} \]
And, for $V = \pm u_\infty$

\[ w_{\xi\xi} = 2Vw_{\xi x} - \frac{1}{4}w_{xxxx} \quad (2.3.34a) \]
\[ \theta_{\xi\xi} = 2V\theta_{\xi x} - \frac{1}{4}\theta_{xxxx} \quad (2.3.34b) \]

We assume that derivatives with respect to $x$ are small; i.e. long waves. There are several ways to balance the terms in equations (2.3.34) (see section 2.12), the optimal one being

\[ \partial_{\xi\xi} \ll \partial_{\xi x} \sim \partial_{xxxx} \ll 1 \]

leaving us with

\[ 0 = 2Vw_{\xi x} - \frac{1}{4}w_{xxxx} \quad (2.3.35a) \]
\[ 0 = 2V\theta_{\xi x} - \frac{1}{4}\theta_{xxxx} \quad (2.3.35b) \]

There are now two similarity solutions which we find to satisfy the boundary conditions (2.3.31) and (2.3.32) derived from matching the two regions. First, by making the transformation

\[ \theta_{x} = \tilde{f}(\tilde{\xi}) \text{ and } \tilde{\xi} = x/\zeta^{1/3} \text{ in equation (2.3.35b)} \]

we get

\[ 0 = \frac{2}{3}V\tilde{\xi}\tilde{f}' - \frac{1}{4}\tilde{f}''' \]

which can then be further reduced by the transformation $f = \tilde{f}'$ and $\xi = -2 \left(\frac{V}{3}\right)^{1/3} \tilde{\xi}$ to get

\[ 0 = f'' - \xi f \quad (2.3.36) \]

Equation (2.3.36) is the well known Airy equation [14] with general solution $f(\xi) = c_1 \text{Ai}(\xi) + c_2 \text{Bi}(\xi)$ where $\text{Ai}(\xi)$ and $\text{Bi}(\xi)$ are special functions defined in terms of infinite series or improper integrals. Since, $\text{Bi}(\xi)$ grows exponentially we take $c_2$ to be 0. For $V = -u_\infty$ we are looking for a solution $\theta$ which goes to zero as $x \to -\infty$ and as a direct result $\tilde{f} = \theta_{x} \to 0$ as $\tilde{\xi} = x/\zeta^{1/3} \to -\infty$. With this we can now unwrap the transformations made earlier. If we consider the boundary conditions on the left of the soliton

\[ \theta(\zeta, x) = c_1 \int_{-\infty}^{x} \int_{-\infty}^{a\tilde{\xi}/\zeta^{1/3}} \text{Ai}(s) \text{d}s \text{d}\tilde{\xi} \quad (2.3.37a) \]
where \( a = -2 \left( \frac{V}{3} \right)^{1/3} \) and \( c_1 = \phi_{0t}^- \). Note that the sign of \( a \) depends on the sign on \( V \). In the same way on the right of the soliton we find that when \( V = u_\infty \) the solution is

\[
\theta(\zeta, x) = c_2 \int_{-\infty}^{x} \frac{\alpha x}{\zeta^{1/3}} \int_{-\infty}^{\alpha x/\zeta^{1/3}} \text{Ai}(s) ds \text{d}x
\]

(2.3.37b)

This solution matches the boundary conditions for the phase \( \theta \) with \( c_4 = \phi_{0t}^+ \).

To get the other solution we begin by factoring out a derivative with respect to \( x \) in equation (2.3.35a)

\[
0 = \left( 2Vw_\zeta - \frac{1}{4}w_{xxx} \right)_x
\]

\[
c_3 = 2Vw_\zeta - \frac{1}{4}w_{xxx}
\]

For this to satisfy the zero boundary condition (on either side) it must be \( c_3 = 0 \) leaving us with

\[
0 = 2Vw_\zeta - \frac{1}{4}w_{xxx}
\]

Which under the same procedure used above has solution

\[
w(x) = c_4 \int_{-\infty}^{\alpha x/\zeta^{1/3}} \text{Ai}(s) ds
\]

(2.3.38)

for both \( V = -u_\infty \) and \( V = u_\infty \). An important point is that in the case of a black soliton there are two boundary layers moving away from the soliton solution with speed \( u_\infty \) generating a shelf. The shelf must be carefully taken into consideration when dealing with the integrals employed in soliton perturbation theory, be it in the conservation laws that we will be employing or in integral secularity conditions.

### 2.4 Perturbed Conservation Laws

We still need to solve for the slowly evolving parameters \( \sigma_0(Z) \) and \( t_0(Z) \) for the black soliton. This can be done by deriving equations for the growth of the shelf from the perturbed conservation laws associated with the perturbed NLS (2.1.6) equation from only the leading order solution and asymptotic information about the perturbed solution. The shelf is associated with
the asymptotic parameters $q_1^\pm$ and $\phi_1^\pm$, which are in turn expressed in terms of $\sigma_{0Z}$ and $t_{0Z}$. We
use the Hamiltonian $H$, the energy $E$, the momentum $I$, and the center of energy $R$ defined below.
For grey solitons, $t_0$ proves to be more difficult to obtain than $\sigma_0$. To find $t_0$ we employ $u_1$ in the
expansion $u = u_0 + \epsilon u_1 + \ldots$. To find $\sigma_0$ only asymptotic information is needed.

\[
H = \int_{-\infty}^{\infty} \left[ \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 + \frac{1}{2} (u_{\infty}^2 - |u|^2)^2 \right] dt \tag{2.4.39a}
\]

\[
E = \int_{-\infty}^{\infty} [u_{\infty}^2 - |u|^2] dt \tag{2.4.39b}
\]

\[
I = \int_{-\infty}^{\infty} \text{Im} [u u^*_t] dt \tag{2.4.39c}
\]

\[
R = \int_{-\infty}^{\infty} t (u_{\infty}^2 - |u|^2) dt \tag{2.4.39d}
\]

where $u^*$ denotes complex conjugate.

Note that since the standard total energy ($E_{\text{Total}} = \int |u|^2 dt$) would be infinite, we define
the energy of a dark pulse to be the difference of the total energy and the energy of a continuous
wave of corresponding magnitude. For the unperturbed NLS equation the first three integrals
are conserved quantities while the last can be written in term of the momentum, i.e.; $\frac{dR}{dz} = -I$.

Evolution equations for these integrals may be easily obtained from equations (2.1.1) and (2.1.4)

\[
\frac{dH}{dz} = \epsilon \left( E \frac{d}{dz} u_{\infty}^2 + 2 \text{Re} \int_{-\infty}^{\infty} F[u] u_{\infty}^* dt \right) \tag{2.4.40a}
\]

\[
\frac{dE}{dz} = 2 \epsilon \text{Im} \int_{-\infty}^{\infty} F[u_{\infty}] u_{\infty} - F[u] u^* dt \tag{2.4.40b}
\]

\[
\frac{dI}{dz} = 2 \epsilon \text{Re} \int_{-\infty}^{\infty} F[u] u^*_t dt \tag{2.4.40c}
\]

\[
\frac{dR}{dz} = -I + 2 \epsilon \text{Im} \int_{-\infty}^{\infty} t (F[u_{\infty}] u_{\infty} - F[u] u^*) dt \tag{2.4.40d}
\]

2.5 The Black Soliton

We begin with the perturbed conservation of energy

\[
\frac{d}{dz} \int_{-\infty}^{\infty} [u_{\infty}^2 - |u|^2] dt = 2 \epsilon \text{Im} \int_{-\infty}^{\infty} (F[u_{\infty}] u_{\infty} - F[u] u^*) dt \tag{2.5.41}
\]
Substituting in \( u = qe^{i\phi} \), \( F[u] = i\gamma u_{tt} \), \( T = t - t_0 \) expanding \( q = q_0 + \epsilon q_1 + \ldots \) and taking the terms up to \( O(\epsilon) \) we have

\[
\frac{d}{dz} \int_{-\infty}^{\infty} [q_0^2 - u_\infty^2 + \epsilon 2q_0 q_1] \, dT = 2\epsilon \int_{-\infty}^{\infty} \gamma q_0 TT_0 dT \tag{2.5.42a}
\]

At \( O(1) \) equation (2.5.42a) is satisfied: \( \frac{d}{dz} \int [q_0^2 - u_\infty^2] \, dT = 0 \); At \( O(\epsilon) \) we have

\[
\frac{d}{dz} \int_{-\infty}^{\infty} q_0 q_1 dT = -\gamma \int_{-\infty}^{\infty} q_0^3 dT \tag{2.5.42b}
\]

This is an equation for the change in energy caused by the propagation of the shelf. Notice that on the left hand side of equation (2.5.42b) we are only integrating over \( T \in [-u_\infty z, u_\infty z] \), the inner region around the soliton defined by the boundary layers found in the last section. Since \( q_0 \) and \( q_1 \) are only functions of \( T \) and \( Z \), we can apply the fundamental theorem of calculus to arrive at

\[
u_\infty [q_1(u_\infty z)q_0(u_\infty z) + q_1(-u_\infty z)q_0(-u_\infty z)] = -\gamma u_\infty^3 \frac{4}{3} \tag{2.5.42c}
\]

And, for large \( z \) (although in practice \( u_\infty z \) only needs to be modestly larger than the the full-width-half-max), we take \( q_0 \to \pm u_\infty \) and \( q_1 \to q_1^\pm \) leaving us with

\[
q_1^+ - q_1^- = -\frac{4}{3} u_\infty \gamma \tag{2.5.42d}
\]

By substituting in the asymptotic approximation (2.2.27) found early for \( q_1^\pm \), we arrive at an expression for \( \sigma_0 \)

\[
\sigma_0 Z = -\gamma \frac{4}{3} u_\infty^2 \tag{2.5.43}
\]

Next, we consider the modified conservation of momentum

\[
\frac{d}{dz} \Im \int_{-\infty}^{\infty} uu_t^* \, dt = 2\epsilon \Re \int_{-\infty}^{\infty} F[u]u_t^* \, dt \tag{2.5.44}
\]

Again, we let \( u = qe^{i\phi} \), \( F[u] = i\gamma u_{tt} \), \( T = t - t_0 \) and use the perturbation expansion for \( u \) up to \( O(\epsilon) \) so that equation (2.5.44) becomes

\[
-\frac{d}{dz} \int_{-\infty}^{\infty} [\phi_{0T} q_0^2 + \epsilon(2\phi_{0T} q_0 q_1 + \phi_{1T} q_0^2)] \, dT = \epsilon 2\Re \int_{-\infty}^{\infty} i\gamma q_{0TT} q_0 dT \tag{2.5.45a}
\]
which, using $\phi_{0T} = 0$, reduces in the same way as the conservation of energy to

$$\phi_{1t}^+ + \phi_{1t}^- = 0 \quad (2.5.45b)$$

By substituting in the asymptotic approximations (2.2.20) found early for $\phi_{1t}^\pm$, we arrive at an expression for $t_0$

$$t_0 \tau = 0 \quad (2.5.46)$$

Later we will see that the above result agrees with the more general grey soliton case.

This can now be compared with direct numerical simulations. The magnitude and phase are depicted in figure 2.1 and figure 2.2 respectively for $z = 30$ and $u_\infty = 1$. Here we see the inner region, discussed earlier is $t \in (-30, 30)$ where the asymptotic solution agrees well with the numerical solution. The remainder of the domain constitutes the outer region where the inner asymptotic solution matches to the exterior rest state; this is discussed next. The boundary layer shown in figure 2.3 compares the solutions (2.3.38) and (2.3.37a) to numerics and illustrates how the inner and outer solutions are connected. The propagation of this boundary layer can be seen in figure 2.4 where the contour plot illustrates the soliton down the middle and the shelf extending out from it. The speed of the boundary layer matches the speed predicted by the long wave approximation in Section 2.3.
Figure 2.1: Numerical results plotted against the asymptotic approximation for the magnitude $|u|$ up to $O(\epsilon)$. Here $u_\infty = 1$, $z = 30$ and $\epsilon \gamma = 0.05$.

Figure 2.2: Numerical results plotted against the asymptotic approximation for the phase $\phi$ up to $O(\epsilon)$. Here $u_\infty = 1$, $z = 30$ and $\epsilon \gamma = 0.05$. 
Figure 2.3: Asymptotic approximation $|u| \approx u_\infty + \epsilon w$ and $\phi \approx -\frac{\Delta \phi_\infty}{2t} + \epsilon \theta$ for the boundary layer compared to numerics. Here $u_\infty = 1$, $z = 30$ and $\epsilon \gamma = 0.05$.

Figure 2.4: The predicted location of the boundary layer $t_{BL} = \pm u_\infty z$ displayed over a contour plot of $|u|$. Here $u_\infty = 1$ and $\epsilon \gamma = 0.05$
2.6 The Grey Soliton

Now we consider the general case of a grey soliton with velocity \( A(Z) \); we also recall \( (A^2 + B^2)(Z) = u_\infty^2(Z) \). Let \( u = q e^{i\phi} \) where \( q > 0 \) and \( \phi \) are real functions of \( z \) and \( t \) and introduce moving frame of reference \( T = t - \int_0^z A(\epsilon s) ds - t_0 \) and \( \zeta = z \), so that with \( u = u(\zeta, T, Z) \) equation (2.1.6) becomes

\[
iu_\zeta - i\Lambda u_T - \frac{1}{2}u_{TT} + (|u|^2 - u_\infty^2)u = \epsilon F[u] \tag{2.6.47}
\]

And, then using \( u = q e^{i\phi} \)

\[
i(q_\zeta + i\phi_\zeta q) - i\Lambda (q_T + i\phi_T q) - \frac{1}{2} [q_{TT} + i2\phi_T q_T + (i\phi_{TT} - \phi_T^2)q] + q^3 - u_\infty^2 q = \epsilon F[q, \phi] \tag{2.6.48}
\]

This is now broken into imaginary and real parts respectively

\[
q_\zeta = \Lambda q_T + \frac{1}{2}(2\phi_T q_T + q\phi_T) + \epsilon \text{Im}[F[q, \phi]] \tag{2.6.49a}
\]

\[
\phi_\zeta q = \Lambda \phi_T q - \frac{1}{2}(q_{TT} - \phi_T^2 q) + (|q|^2 - u_\infty^2)q - \epsilon \text{Re}[F[q, \phi]] \tag{2.6.49b}
\]

We now write equations (2.6.49b) in terms of the slow evolution variable \( \zeta = \epsilon Z \) and series expansions \( q = q_0 + \epsilon q_1 + O(\epsilon^2) \) and \( \phi = \phi_0 + \epsilon \phi_1 + O(\epsilon^2) \). At \( O(1) \) the equations are satisfied by the soliton solution (2.2.11).

At \( O(\epsilon) \) we have

\[
q_1\zeta = \Lambda q_{1T} + \frac{1}{2} [2(\phi_{0T} q_{1T} + \phi_{1T} q_{0T}) + \phi_{1TT} q_0 + \phi_{0TT} q_1] + \text{Im}[F[u_0]] - q_0 Z
\]

\[
\phi_1 q_0 = -\phi_{0\zeta} q_1 + \Lambda (\phi_{0T} q_1 + \phi_{1T} q_0) - \frac{1}{2} (q_{1TT} - \phi_{0T}^2 q_1 - 2\phi_{0T} q_0 \phi_{1T}) + 3q_0^2 q_1 - u_\infty^2 q_1
\]

\[- \text{Re}[F[u_0]] - \phi_0 Z q_0
\]

where \( u_0 = q_0 e^{i\phi_0} \). We look for stationary solutions at \( O(\epsilon) \)

\[
0 = \Lambda q_{1T} + \frac{1}{2} [2(\phi_{0T} q_{1T} + \phi_{1T} q_{0T}) + \phi_{1TT} q_0 + \phi_{0TT} q_1] + \text{Im}[F[u_0]] - q_0 Z \tag{2.6.50a}
\]

\[
0 = \Lambda (\phi_{0T} q_1 + \phi_{1T} q_0) - \frac{1}{2} (q_{1TT} - \phi_{0T}^2 q_1 - 2\phi_{0T} q_0 \phi_{1T}) + 3q_0^2 q_1 - u_\infty^2 q_1 \tag{2.6.50b}
\]

\[- \text{Re}[F[u_0]] - \phi_0 Z q_0
\]
where

\[ q_{0Z} = \frac{1}{2} \left( AA + BB \tanh^2(x) \right) q_0^{-1} + q_{0T} \left( \frac{BZ}{B} - t_{0Z} \right) \] (2.6.51a)

\[ \phi_{0Z} = (AB - BA) \tanh(x) q_0^{-2} + \phi_{0T} \left( \frac{BZ}{B} - t_{0Z} \right) + \sigma_{0Z} \] (2.6.51b)

Next we assume a shelf structure similar to the one found in our example problem will develop; this is supported by numerical computations. Consider equation (2.6.50a) in the limit \( T \to \pm \infty \) using \( q_0 \to u_\infty \) and

\[ 0 = A q_{1T}^\pm + \frac{u_\infty}{2} \phi_{1TT}^\pm \] (2.6.52)

We assume \( q_1 \) tends to a constant with respect to \( t \); i.e. \( q_{1T} \to 0 \) as \( t \to \pm \infty \). As a result \( \phi_{1TT} \to 0 \). Then \( q_1 \) and \( \phi_{1T} \) both tend asymptotically to constants as \( t \to \pm \infty \) which corresponds to a shelf developing around the soliton. Substituting \( \phi_{0T} \) into equation (2.6.50b), in the limit \( T \to \pm \infty \) we get

\[ A \phi_{1T}^\pm + 2u_\infty q_1^\pm = -\Re \left[ F[u_\infty] / u_\infty \right] + \frac{(AB - BA)}{u_\infty^2} + \sigma_{0Z} \] (2.6.53)

We define \( \Delta \phi_0 \) by

\[ \Delta \phi_0 = 2 \tan^{-1} \left( \frac{B}{A} \right) \] (2.6.54a)

the phase change across the core soliton. This is consistent with the soliton parameters \( A \) and \( B \) being expressed in terms of background magnitude, \( u_\infty \), and phase change, \( \Delta \phi_0 \),

\[ A = u_\infty \cos \left( \frac{\Delta \phi_0}{2} \right) \quad B = u_\infty \sin \left( \frac{\Delta \phi_0}{2} \right) \] (2.6.54b)

Using \( \phi_Z^\pm = -\Re \left[ F[u_\infty] \right] / u_\infty \), from Section 2.1 and substituting (2.6.54b) into equation (2.6.53) we find

\[ A \phi_{1T}^\pm + 2u_\infty q_1^\pm = \phi_Z^\pm \pm \frac{\Delta \phi_0 Z}{2} + \sigma_{0Z} \] (2.6.55)

### 2.7 Conservation Laws for the Grey Soliton

Next we use the modified conservation equations (2.4.40) to solve for the shelf parameters \( q_1^\pm \) and \( \phi_{1T}^\pm \) as well as the slow evolution variables \( A, \sigma_{0Z} \). More work is required in order to find
t₀. Note that if we find A, then B = \( (u_\infty^2 - A^2)^{1/2} \). The edge of the shelf still propagates with velocity \( V(Z) = u_\infty(Z) \), however the speed may now vary in \( z \). In terms of the moving frame of reference the boundaries of the shelf are

\[
S_L(\zeta) = -\int_0^\zeta [u_\infty(\epsilon s) + A(\epsilon s)] \, ds \\
S_R(\zeta) = \int_0^\zeta [u_\infty(\epsilon s) - A(\epsilon s)] \, ds
\]

where \( S_L \) and \( S_R \) give the position in \( T \) of the left and right boundaries of the shelf respectively at \( \zeta \). Note that \( A \leq u_\infty \) for all \( Z \), thus the soliton cannot overtake the shelf. In fig 2.5 we illustrate the general structure of a perturbed dark soliton with moving shelf. The inner region consists of the core soliton and the shelf expanding around it, while the outer region consists of the infinite boundary conditions characterized by equations (2.1.4). The boundaries between these regions are delineated by dotted red lines at \( t = S_L \) and \( t = S_R \).

We begin with the evolution equation for the Hamiltonian (2.4.40a).

\[
\frac{d}{d\zeta} \int_{-\infty}^{\infty} \left[ \frac{1}{2} |u_t|^2 + \frac{1}{2} (u_\infty^2 - |u|^2)^2 \right] \, dt = \epsilon (u_\infty^2) Z \int_{-\infty}^{\infty} [u_\infty^2 - |u|^2] \, dt + 2 \epsilon \text{Re} \int_{-\infty}^{\infty} F[u] u^*_0 \, dt \quad (2.7.57)
\]

Substituting in \( u = (q_0 + \epsilon q_1)e^{i(\phi_0 + \epsilon \psi)} \) and changing variables to the moving frame of reference, we have up to \( O(\epsilon) \)

\[
\frac{d}{d\zeta} \int_{-\infty}^{\infty} \left[ (q_0^2 + q_0^2 q_0^2) + (u_\infty^2 - q_0^2)^2 \right] \, dT = 2 \epsilon (u_\infty^2) Z \int_{-\infty}^{\infty} [u_\infty^2 - q_0^2] \, dT - 4 \epsilon \text{Re} \int_{-\infty}^{\infty} F[u_0] u^*_0 \, dT
\]

where both here and later on \( u_0 = q_0 e^{i\phi_0} \). The Hamiltonian is unique among the evolution equations (2.4.40) in that the contribution of the shelf appears only at \( O(\epsilon^2) \) or higher and to \( O(\epsilon) \) may be ignored. We now put in the soliton form (2.2.11) to get

\[
2B^2 B_Z = (u_\infty^2) Z B - A \text{Re} \int_{-\infty}^{\infty} F[u_0] u^*_0 \, dT
\]

(2.7.59)

Taking a derivative with respect to \( Z \) of the equation \( u_\infty^2 = A^2 + B^2 \) we get

\[
(u_\infty^2)_Z = 2AA_Z + 2BB_Z
\]

(2.7.60)
which can be used consolidate equations (2.7.59) down to

\[ 2BA_Z = \text{Re} \int_{-\infty}^{\infty} F[u_0]u_0^* dT, \quad (2.7.61) \]

The evolution equations for energy (2.5.41) and momentum (2.5.44) both remain the same after transforming to the moving frame of reference

\[ \frac{d}{d\zeta} \int_{-\infty}^{\infty} [u_\infty^2 - |u|^2] dT = 2\varepsilon \text{Im} \int_{-\infty}^{\infty} [F[u_\infty]u_\infty - F[u^*]u^*_0] dT \quad (2.7.62) \]

\[ \frac{d}{d\zeta} \text{Im} \int_{-\infty}^{\infty} uu^*_T dT = 2\varepsilon \text{Re} \int_{-\infty}^{\infty} F[u]u^*_T dT \quad (2.7.63) \]

The inner region over which \( q_1 \) and \( \phi_1 \) are relevant is \( T \in [S_L(\zeta), S_R(\zeta)] \), and outside this region \( q_1 = \phi_{1T} = 0 \). At \( O(1) \) the equations are satisfied and at \( O(\varepsilon) \) we have

\[ B_Z - \frac{d}{d\zeta} \int_{S_L(\zeta)}^{S_R(\zeta)} q_0 q_1 dT = \text{Im} \int_{-\infty}^{\infty} [F[u_\infty]u_\infty - F[u^*_0]u^*_0] dT \quad (2.7.64a) \]

\[ -2(AB)_Z - \frac{d}{d\zeta} \int_{S_L(\zeta)}^{S_R(\zeta)} [2\phi_0 q_0 q_1 + \phi_{1T} q_0^2] dT = 2\varepsilon \text{Re} \int_{-\infty}^{\infty} F[u^*_0]u^*_T dT \quad (2.7.64b) \]

Since the integrands on the left hand side are not functions of \( \zeta \), we can apply the fundamental theorem of calculus to arrive at

\[ B_Z - u_\infty \left[ (u_\infty - A)q_1^+ + (u_\infty + A)q_1^- \right] = \text{Im} \int_{-\infty}^{\infty} [F[u_\infty]u_\infty - F[u^*_0]u^*_0] dT \quad (2.7.65a) \]

\[ 2(AB)_Z + u_\infty^2 \left[ (u_\infty - A)\phi_{1T}^+ + (u_\infty + A)\phi_{1T}^- \right] = -2\varepsilon \text{Re} \int_{-\infty}^{\infty} F[u^*_0]u^*_T dT \quad (2.7.65b) \]

We are left now with the evolution of the center of energy

\[ \frac{d}{d\zeta} \int_{-\infty}^{\infty} t(u_\infty^2 - |u|^2) dt = -\text{Im} \int_{-\infty}^{\infty} uu^*_T dt + 2\varepsilon \text{Im} \int_{-\infty}^{\infty} t(F[u_\infty]u_\infty - F[u^*_0]u^*_0) dt \quad (2.7.66) \]

which after transforming to the moving frame of reference is now

\[ \frac{d}{d\zeta} \int_{-\infty}^{\infty} (T + \int_{0}^{\zeta} A + t_0)(u_\infty^2 - |u|^2) dt = -\text{Im} \int_{-\infty}^{\infty} uu^*_T dT \]

\[ + 2\varepsilon \text{Im} \int_{-\infty}^{\infty} (T + \int_{0}^{\zeta} A + t_0)(F[u_\infty]u_\infty - F[u^*_0]u^*_0) dT \]
After rearranging some terms we have

\[
\frac{d}{d\zeta} \int_{-\infty}^{\infty} T(u_{\infty}^2 - |u|^2) dT + \left( \int_{0}^{\zeta} A + t_0 \right) \left[ \frac{d}{d\zeta} \int_{-\infty}^{\infty} [u_{\infty}^2 - |u|^2] dT - e2 \text{Im} \int_{-\infty}^{\infty} (F[u_{\infty}]u_{\infty} - F[u]u^*) dT \right] \\
+ A \int_{-\infty}^{\infty} [u_{\infty}^2 - |u|^2] dT + \text{Im} \int_{-\infty}^{\infty} uu^*_T dT \\
= -e t_0 Z \int_{-\infty}^{\infty} [u_{\infty}^2 - |u|^2] dT + 2e \text{Im} \int_{-\infty}^{\infty} T (F[u_{\infty}]u_{\infty} - F[u]u^*) dT 
\]  

(2.7.67a)

The terms on line (2.7.67b) reproduce the energy equation (2.7.62) and cancel out. The terms on line (2.7.67c) are calculated up to \(O(\epsilon)\) using the previous results by integrating the energy and momentum equation (2.7.65)

\[
E(Z) = 2B - 2 \left[ S_R(Z)q_1^+ - S_L(Z)q_1^- \right] u_{\infty} + eE^{(1)}(Z) + O(\epsilon^2) \\
I(Z) = -2AB - u_{\infty}^2 \left[ S_R(Z)\phi_{1t}^+ - S_L(Z)\phi_{1t}^- \right] + eI^{(1)}(Z) + O(\epsilon^2) 
\]  

(2.7.69)

(2.7.70)

Noting that \(d/d\zeta = \epsilon d/dZ\) and \(S_R\) and \(S_L\) are \(O(1/\epsilon)\) in terms of \(Z\).

When we put everything together in terms of slow evolution variable \(Z = \epsilon \zeta\), we get from (2.7.67)

\[
e^{2B} t_{0Z} = 2e \text{Im} \int_{-\infty}^{\infty} T (F[u_{\infty}]u_{\infty} - F[u_0]u_0^*) dT \epsilon + AE_1(Z) + eI_1(Z) \\
+ 2u_{\infty} \left[ S_R(u_{\infty} - A)q_1^+ + S_L(u_{\infty} + A)q_1^- \right] + 2u_{\infty} A \left[ S_R q_1^+ - S_L q_1^- \right] \\
+ u_{\infty}^2 \left[ S_R \phi_{1t}^+ - S_L \phi_{1t}^- \right] 
\]  

(2.7.71)

After some cancelations, this breaks into \(O(1)\) terms

\[
2 \left[ S_R q_1^+ + S_L q_1^- \right] + \left[ S_R \phi_{1t}^+ - S_L \phi_{1t}^- \right] = 0 
\]  

(2.7.72)

and \(O(\epsilon)\) terms which include \(t_{0Z}\) and higher order energy and momentum terms have not been determined. The six equations (2.6.55), (2.7.61), (2.7.65a), (2.7.65b) and (2.7.72) can now be used to
solve for the set of six parameters $q^+_1$, $\phi^+_1 (= \phi^+_1 T)$, $\Lambda$ and $\sigma_0$. To find $t_0 Z$ we need to employ more information—see next section.

\[
\frac{d}{dZ} u_\infty = \text{Im} [F[u_\infty]] \tag{2.7.73a}
\]
\[
2B \frac{d}{dZ} A = \text{Re} \int_{-\infty}^{\infty} F[u_0] u_0^* dT \tag{2.7.73b}
\]
\[
u_\infty \frac{d}{dZ} \sigma_0 = B_Z - \text{Im} \int_{-\infty}^{\infty} F[u_\infty] u_\infty - F[u_0] u_0^* dT + \text{Re}[F[u_\infty]] \tag{2.7.73c}
\]
\[
q^+_1 = \frac{1}{2} \left( \sigma_0 Z + \Delta \phi_0 Z \right) / \left( u_\infty - \Lambda \right) \tag{2.7.73d}
\]
\[
q^-_1 = \frac{1}{2} \left( \sigma_0 Z - \Delta \phi_0 Z \right) / \left( u_\infty + \Lambda \right) \tag{2.7.73e}
\]
\[
\phi^+_1 T = -2q^+_1 \tag{2.7.73f}
\]
\[
\phi^-_1 T = 2q^-_1 \tag{2.7.73g}
\]

\[
B_Z = (u_\infty u_\infty Z - \Lambda A_Z) / B \tag{2.7.73h}
\]
\[
\Delta \phi_0 Z = (2AB Z - 2BA Z) / u_\infty^2 \tag{2.7.73i}
\]

Thee equation may now be solved from top to bottom. We have added equations (2.7.73h) and (2.7.73i) to the list since it is often easier to use these formulations for $B_Z$ and $\Delta \phi_0 Z$ rather than working out $B$ and $\Delta \phi_0$ explicitly and then taking derivatives.

By combining equations (2.7.61) and (2.7.65b) we arrive at

\[
2(AB)_Z + u_\infty^2 \left[ (u_\infty - \Lambda) \phi^+_1 T + (u_\infty + \Lambda) \phi^-_1 T \right] = 4BA Z \tag{2.7.74}
\]

which may be rewritten as

\[
2AB Z - 2BA Z + u_\infty^2 \frac{d}{dC} [\phi_1(S_R) - \phi_1(S_L)] = 0 \tag{2.7.75}
\]

If we define $\phi_1$ as follows

\[
\Delta \phi_1 = \phi_1(S_R) - \phi_1(S_L) \tag{2.7.76}
\]
then $\epsilon \phi_1$ is the phase change across the shelf. Substituting this definition along with (2.6.54a) into equation (2.7.75) we arrive at

$$
\frac{d}{dZ} \Delta \phi_0 + \epsilon \frac{d}{dZ} \Delta \phi_1 = 0
$$

(2.7.77)

Thus, the total phase change across the inner region remains constant, which is consistent with our earlier result that $\Delta \phi_\infty$ (the phase change from $-\infty$ to $\infty$) remains constant for all perturbations. As an example, figure 2.5 shows that the entire phase change remains consistent with the given boundary condition.
Figure 2.5: Typical numerical simulation of a perturbed soliton for both the Magnitude and Phase. Here $F[u] = i\gamma u_{tt}$, $z = 30$, $\epsilon\gamma = 0.05$ and $\Delta\phi_0 = 4\pi/5 = \Delta\phi_0 + \epsilon\Delta\phi_1$. 
2.8 $t_0z$ and Higher Order Terms

To find the final parameter $t_0$ we employ the first order correction term. We look for series solution to equation (2.1.1) of the form $u = u_0 + \epsilon u_1 + O(\epsilon^2)$, and at $O(\epsilon)$ we have

$$iu_{1z} + \left(-\frac{1}{2}\partial_z^2 + 2|u_0|^2 - u_{\infty}^2\right)u_1 + (u_0^2)u_1^* = F[u_0] - iu_0z$$  \hspace{1cm} (2.8.78)

After changing variables to moving frame of reference $T = t - \int_0^z A(\epsilon s)ds - t_0$, $z = \zeta$ we have

$$iu_{1\zeta} + \left(-iA\partial_T - \frac{1}{2}\partial_T^2 + 2|u_0|^2 - u_{\infty}^2\right)u_1 + (u_0^2)u_1^* = F[u_0] - iu_0z$$  \hspace{1cm} (2.8.79)

Here

$$u_{0Z} = A_Z e^{i\sigma} + \frac{B_Z}{B} (u_0 - Ae^{i\sigma}) + u_{0T} \left(-t_0Z + \frac{B_Z}{B}T\right) + i\sigma_0Z u_0$$  \hspace{1cm} (2.8.80)

If we look for stationary solutions $(\frac{\partial}{\partial z} = 0)$, this can be written as a system of coupled second order differential equations

$$LU_1 = G[u_0]$$  \hspace{1cm} (2.8.81a)

where

$$U_1 = \begin{pmatrix} \text{Re}[u_1] \\ \text{Im}[u_1] \end{pmatrix} \quad \text{and} \quad G[u_0] = \begin{pmatrix} \text{Re}[F[u_0] - iu_{0Z}] \\ \text{Im}[F[u_0] - iu_{0Z}] \end{pmatrix}$$  \hspace{1cm} (2.8.81b)

and

$$L = \begin{bmatrix} -\frac{1}{2}\partial_T^2 + (3A^2 + B^2\tanh(BT) - u_{\infty}^2) & \Lambda\partial_T + 2AB\tanh(BT) \\ -A\partial_T + 2AB\tanh(BT) & -\frac{1}{2}\partial_T^2 + (A^2 + 3B^2\tanh(BT) - u_{\infty}^2) \end{bmatrix}$$  \hspace{1cm} (2.8.81c)

In the limit $A \to 0$ this system decouples into two second order differential equations which are not difficult to solve and give two strictly real solutions and two strictly imaginary solutions. For each solution we found for $A = 0$ we assume there exists a solution for $A \neq 0$ that differs only in the perpendicular direction; e.g. if $U_H = \begin{pmatrix} u_R \\ 0 \end{pmatrix}$ satisfies (2.8.81) with $A = 0$, then there exists $u_I$ such that $U_H = \begin{pmatrix} u_R \\ u_I \end{pmatrix}$ satisfies (2.8.81) with $A \neq 0$; namely only the second component
changes and hence the system reduces to first order order equation which is consistent with the remaining equations. Under this assumption we find a complete set of homogeneous solution

\[
U_{11} = \begin{pmatrix} 0 \\ \text{sech}^2(BT) \end{pmatrix}
\]  

(2.8.82a)

\[
U_{12} = \begin{pmatrix} B\tanh(BT) \\ -A \end{pmatrix}
\]  

(2.8.82b)

\[
U_{13} = \begin{pmatrix} B(B\tanh(BT) - 1) \\ A \left( -BT + \frac{3}{2}BT\text{sech}^2(BT) + \frac{3}{2}\tanh(BT) \right) \end{pmatrix}
\]  

(2.8.82c)

\[
U_{14} = \begin{pmatrix} -\frac{4AB}{A^2-B^2} \cosh^2(BT) \\ 3B\text{sech}^2(BT) + 4\tanh(BT) + \tanh(BT)\cosh(2BT) \end{pmatrix}
\]  

(2.8.82d)

and using variation of parameters we can obtain a particular solution, \( U_{1p} \), for the forcing \( G[u_0] \).

After combining real and imaginary parts, the full solution to equation (2.8.79) is given by

\[
u_1 = c_1u_{11} + c_2u_{12} + c_3u_{13} + c_4u_{14} + u_{1p}
\]  

(2.8.83)

Where \( c_1, c_2, c_3 \) and \( c_4 \) are functions of \( Z \) and \( u_{1p} \) is dependent on the yet to be determined \( t_0 \). We take \( c_4 = 0 \) to remove the exponential growth in \( u_{14} \) and we separate out the contribution of \( t_0Z \) which appears linearly in the particular solution \( u_{1p} \)

\[
u_1 = c_1u_{11} + c_2u_{12} + c_3u_{13} + t_0Zu_{1p}^{(1)} + u_{1p}^{(2)}
\]  

(2.8.84)

where

\[
u_{1p}^{(1)} = 1 - i \left[ BT\text{sech}^2(BT) + \tanh(BT) \right] \frac{A}{B}
\]  

(2.8.85)

so that \( u_{1p}^{(2)} \) has no unknowns left in it.

To put \( u_1 \) in terms of the magnitude and phase functions \( q_0, q_1, \phi_0 \) and \( \phi_1 \), we expand our previous approximation for \( u \)

\[
u = (q_0 + \epsilon q_1)e^{i(\phi_0 + \epsilon \phi_1)} = q_0e^{i\phi_0} + \epsilon (q_1 + i\phi_1 q_0) e^{i\phi_0} + O(\epsilon^2)
\]  

(2.8.86)
so that

\[ u_0 = q_0 e^{i\phi_0} \quad (2.8.87) \]

\[ u_1 = (q_1 + i\phi_1 q_0) e^{i\phi_0} \quad (2.8.88) \]

\[ = [q_1 \cos(\phi_0) - \phi_1 q_0 \sin(\phi_0)] + i [q_1 \sin(\phi_0) + \phi_1 q_0 \cos(\phi_0)] \quad (2.8.89) \]

By looking at the asymptotic behavior of the solution \( u_1 \) as \( t \to \pm \infty \) we find the equation

\[ u_{1T}^\pm = -\phi_{1T}^\pm (\pm B) + i\phi_{1T}^\pm (A) \quad (2.8.90) \]

Since \( u_{11T}, u_{12T} \) and \( u_{1pT} \) all go to zero in the limit \( t \to \pm \infty \) the above equation can be used to find \( c_3 \).

With this, we are now able to find a second order differential equation for \( t_0 \) from the Hamiltonian at \( O(\epsilon^2) \)

\[
\frac{d}{d\zeta} H_1 + \frac{d}{d\epsilon} H_2 = \\
- 4u_\infty \text{Im}[F[u_\infty]] \text{Re} \int_{-\infty}^{\infty} u_0 u_1^* d\tau + 2 \text{Re} \int_{-\infty}^{\infty} F[u_0] u_0^* d\tau \quad (2.8.91a) \\
- 2A \text{Re} \int_{-\infty}^{\infty} (F[u_0] u_1^* + F'[u_0] [u_1] u_0^*) d\tau
\]

where

\[ F'[u_0][u_1] = \frac{d}{d\epsilon} F[u_0 + \epsilon u_1] \quad (2.8.92) \]

On the left hand side we have the slow evolution of the \( O(\epsilon) \) terms and the fast evolution of the \( O(\epsilon^2) \) terms. \( H_1 \) is dependent on \( u_0 \) and \( u_1 \) and is given by

\[ H_1 = \int_{-\infty}^{\infty} \text{Re}(u_0^* u_1^* d\tau) + (u_2^2 - |u_0|^2) \text{Re}(u_0 u_1^*) d\tau \quad (2.8.93) \]

\( H_2 \) is dependent on \( u_0, u_1 \) and the order \( \epsilon^2 \) correction \( u_2 \). However, as before we assume a stationary (in the moving frame of reference) solution \( u_2 \) (as was done for \( u_0 \) and \( u_1 \)) then the derivative of \( H_2 \) with respect to the fast evolution variable \( \zeta \) only depends on the asymptotic behavior of \( u_0 \) and \( u_1 \) and is given by

\[
\frac{d}{d\zeta} H_2 = 4u_\infty^2 [u_\infty (q_1^{+2} + q_1^{-2}) - A (q_1^{+2} - q_1^{-2})] 
\]

(2.8.94)
Though it is not immediately obvious, we find that $c_1$ and $c_2$ do not contribute to the Hamiltonian in equation 2.8.91, so $t_0$ is the only unknown. We take $t_0(0) = 0$ and to find a suitable initial condition $t_{0Z}(0)$ we require that the Hamiltonian be accounted for by $H_0$ at $z = 0$; i.e., the higher order terms are initially zero

$$H_1(0) = 0 \quad (2.8.95)$$

Our prediction for $t_0$ differs greatly from that given by methods based on a ‘so called’ complete set of squared Jost function. This discrepancy may be partially explained by the assumption that the squared Jost function form a basis for the solution space of equation (2.8.79). The eigenfunctions found in the inverse scattering theory for the defocusing NLS equation with non-vanishing boundary values [73] and as a direct result the acquired basis functions associated with the soliton are localized and bounded. However, we have solved explicitly for the first correction term and we find that the solution has an expanding shelf. From this we deduce that the squared Jost functions associated with the soliton are an insufficient basis.

### 2.9 Dissipative Perturbation

Let us return to the perturbation $F[u] = i\gamma u_{tt}$, however we now consider the evolution of a general dark soliton with $u_{\infty}(0) = 1$. As was the case for black solitons, the background height $u_{\infty}$ is found to be constant from equations (2.1.4); i.e. $u_{\infty}(Z) = 1$. In figure 2.6 we see that the velocity of the soliton does not effect the velocity of shelf which still moves with velocity $V = \pm u_{\infty}$. Using the equations derived in Section 2.7 and Section 2.8 we can now solve for all relevant parameters

$$A_Z = 0 \quad (2.9.96a)$$

$$\sigma_{0Z} = -\frac{4}{3} \gamma B^3 / u_{\infty} \quad (2.9.96b)$$

$$q_{1t}^{\pm} = -\frac{2}{3} \gamma B (u_{\infty} \pm A) / u_{\infty} \quad (2.9.96c)$$

$$\phi_{1t}^{\pm} = \pm \frac{4}{3} \gamma B (u_{\infty} \pm A) / u_{\infty} \quad (2.9.96d)$$

$$t_{0Z}Z = -\frac{16}{9} \gamma^2 B^3 A / u_{\infty}, \quad t_{0Z}(0) = -\frac{2}{3} \gamma AB / u_{\infty} \quad (2.9.96e)$$
We also note that these results agree with the black soliton when \( A = 0 \). In the limit \( A \to 0 \) we have \( q_0 \to u_\infty |\tanh(u_\infty T)| \), however, this has a discontinuity in its derivative so instead we used \( q_0 = u_\infty \tanh(u_\infty T) \) for our black soliton calculations. As a result there is a sign difference in \( q_{1-} \) from (2.2.27).

Unlike the speed of the shelf at the edges, the magnitude and phase, \( q_{1+}, \phi_{1+} \), of the shelf does depend on the soliton’s velocity (which is in turn related to the soliton’s depth and the phase across the soliton). As illustrated in figure 2.7 the shelf is shallower behind the soliton for larger speeds (or smaller phase change \( \Delta \phi_0 \)). The extra phase \( \sigma_0(z) = -\epsilon z^4 \gamma^3 B^3 / u_\infty \) induced by the perturbation means that the spatial frequency of the soliton is different then the frequency of the cw background that it lies on. Though \( \sigma_0 \) evolves adiabatically the soliton eventually becomes noticeably out of phase from the background as shown in figure 2.8. Here the background phase (\( \phi_+ \) and \( \phi_- \)) is constant since the fast evolution of the background phase was taken out in equation (2.1.1). Finally, in figure 2.9, we show the improvement finding \( t_0(Z) \) makes on predicting the center of the soliton over just using the velocity \( A \).

### 2.10 Linear Damping

We now apply our results to the case of linear damping

\[
F[u] = -i\gamma u
\]  

which was both the first [33] and a commonly used example used in the study of perturbed dark solitons.

In this example we now have a moving background found by solving equation (2.1.4)

\[
\frac{d}{dZ} u_\infty = -\gamma u_\infty
\]  

(2.10.98)
Figure 2.6: The predicted shelf edge overlaid on the contour plot of numerical results. Here $\epsilon \gamma = .05$, and $\Delta \phi_0 = 4\pi/5$.

Figure 2.7: The numerical shelf height for various values of $\Delta \phi_0$ is plotted as are the asymptotic approximations at $z = 30$. Here $F[u] = i\gamma u_{tt}$, $u_\infty = 1$, $\epsilon \gamma = .05$. 
Figure 2.8: \( \sigma_0(Z) \) plotted against the phase at plus and minus infinity along with the phase at the center of the soliton. Here \( u_\infty = 1, \epsilon \gamma = .05, \) and \( \Delta \phi_0 = 4\pi/5. \)

Figure 2.9: Two approximations for the soliton center are shown. One using just the velocity \( \lambda \) and one taking into account \( t_0(Z). \) A comparison of numerics to asymptotics for \( t_0 \) is also given. Here \( \epsilon \gamma = .05, \) and \( \Delta \phi_0 = 4\pi/5. \)
From equations (2.7.73) we can determine the slowly varying soliton parameters

\[ A_Z = -\gamma A \]  \hspace{1cm} (2.10.99a)

\[ \sigma_0Z = \frac{B}{u_\infty} \]  \hspace{1cm} (2.10.99b)

\[ q_1^\pm = \gamma \frac{(u_\infty \pm A)}{2Bu_\infty} \]  \hspace{1cm} (2.10.99c)

\[ \phi_{1T}^+ = \gamma \frac{(u_\infty + A)}{Bu_\infty}, \quad \phi_{1T}^- = \gamma \frac{(u_\infty - A)}{Bu_\infty} \]  \hspace{1cm} (2.10.99d)

\[ t_0ZZ = -\gamma t_0Z + \gamma^2 \frac{3A}{2Bu_\infty}, \quad t_0Z(0) = \gamma \frac{A(0)}{2B(0)u_\infty(0)} \]  \hspace{1cm} (2.10.99e)

Figure 2.10 shows that the existence of a raised shelf and dynamics of the shelf edge is well predicted by the asymptotic theory. The background height and trough of the soliton \((A(Z))\) are accurately approximated by our method (see figure 2.11); this agrees with previously found approximations [51]. Our results for \(t_0\) and \(\sigma_0\) are plotted in figure 2.12. As mentioned earlier, previous attempts using IST are not adequate [23, 57, 20]. \(t_0\) was not obtained in [51]; furthermore, no previous work has considered the evolution of the parameter \(\sigma_0\).

### 2.11 Two Photon Absorption

Dark solitons have been proposed in the development of optical switching devices [59]. Here, materials with high nonlinearities are used to reduce the power for soliton formation and the switching threshold, however, an enhanced two-photon absorption (TPA) coefficient also accompany these materials. An example of TPA with strong defocusing nonlinearity is the semiconductor ZnSe [63]. This is represented by the perturbation term.

\[ F[u] = -i\gamma |u|^2 u \]  \hspace{1cm} (2.11.100)

From equations (2.7.73) we can find all parameters other than \(t_0\); they are given below. This yields the evolution of the background height and trough height (see figure 2.14); we also find that the phase change across the core soliton does not remain constant (as had been the case for our
Figure 2.10: The predicted shelf edge from the asymptotic theory overlaid on the contour plot of numerical results. Here $u_\infty(0) = 1.5$, $\epsilon \gamma = 0.03$ and $\Delta \phi_0 = 4\pi/5$.

Figure 2.11: Numerical compared to analytic results for both $u_\infty$ and $A$. Here $\epsilon \gamma = 0.03$ and $\Delta \phi_0 = 4\pi/5$. 
Figure 2.12: Numerical compared to analytic results for both $\sigma_0$ and $t_0$. Here $u_{\infty}(0) = 1.5$, $\epsilon\gamma = 0.03$ and $\Delta\phi_0 = 4\pi/5$.

Figure 2.13: The predicted shelf edge from the asymptotic theory overlaid on the contour plot of numerical results. Here $\epsilon\gamma = 0.02$, $u_{\infty}(0) = 2$ and $\Delta\phi_0(0) = 7\pi/10$. 
Figure 2.14: The background height and magnitude of the soliton trough found both numerically and asymptotically. Here $\epsilon \gamma = 0.02$, $u_\infty(0) = 2$ and $\Delta \phi_0(0) = 7\pi/10$.

Figure 2.15: Numerical results for the phase of $u$ both initially and at $z = 50$. Here $\epsilon \gamma = 0.02$, $u_\infty(0) = 2$ and $\Delta \phi_0(0) = 7\pi/10$. 
previous examples).

\[
\frac{d}{dZ} u_\infty = -\gamma u_\infty^3 \tag{2.11.101a}
\]

\[
\frac{d}{dZ} A = -\gamma \left( \frac{2}{3} A^2 + \frac{1}{3} u_\infty^2 \right) A \tag{2.11.101b}
\]

\[
\frac{d}{dZ} \Delta \phi_0 = -\frac{4}{3} \gamma AB \tag{2.11.101c}
\]

\[
\sigma_{0Z} = \gamma \frac{B}{u_\infty} \left( 2A^2 + \frac{1}{3} u_\infty^2 \right) \tag{2.11.101d}
\]

\[
q_{i\pm} = \gamma \frac{(u_\infty \pm A)}{Bu_\infty} \left( 2A^2 + \frac{1}{3} u_\infty^2 \pm \frac{4}{3} A u_\infty \right) \tag{2.11.101e}
\]

\[
\phi_{1T}^+ = -2\gamma \frac{(u_\infty + A)}{Bu_\infty} \left( 2A^2 + \frac{1}{3} u_\infty^2 + \frac{4}{3} A u_\infty \right) \tag{2.11.101f}
\]

\[
\phi_{1T}^- = 2\gamma \frac{(u_\infty - A)}{Bu_\infty} \left( 2A^2 + \frac{1}{3} u_\infty^2 - \frac{4}{3} A u_\infty \right) \tag{2.11.101g}
\]

As in the linear damping example, a raised shelf develops around the core soliton as seen in figure 2.13. Again we see remarkable correlation with our predicted shelf velocity. Figure 2.14 shows strong agreement between numerics and asymptotic analysis for the evolution of both \(u_{\text{inf}}(Z), A(Z)\). Figure 2.15 shows that while the phase change across the core soliton decrease by half this is compensated for by the change in phase over the shelf and the total phase change from negative infinity to positive infinity remains constant which agrees with (2.7.77).

### 2.12 Generalized Fredholm Alternatives

In bright soliton perturbation theory Fredholm alternatives are used to derive equations for the slowly varying soliton parameters cf. [54]. These conditions are identical to those derived from the perturbed conservation laws for energy and momentum cf. [3]. A similar connection can be drawn for dark solitons for some of the parameters (not \(t_0, \sigma_0\)). Even here, however, the existence of non-decaying eigenfunctions means that normal Fredholm alternative methods are insufficient to generate the related equations and a generalized method must be employed.

If we return to our notation from Section 2.8, the linearized NLS operator \(L\) is self-adjoint for
the inner product
\[
<f, g> = \int_{-\infty}^{\infty} [f(s)^*]^T g(s) ds
\] (2.12.102)

and the homogeneous problem \(LU_h = 0\) has two bounded solutions
\[
U_{h1} = \begin{pmatrix} \text{Re}[u_0] \\ \text{Im}[u_0] \end{pmatrix} \quad U_{h2} = \begin{pmatrix} \text{Im}[u_0] \\ -\text{Re}[u_0] \end{pmatrix}
\] (2.12.103)

We now develop a Fredholm alternative type argument for \(U_{h1}\), which is in the function space of our norm. The result follows from eq. (2.8.81a)
\[
<U_{h1}, G[u_0]> = 0
\] (2.12.104)

which reduces to
\[
-2BAZ = e \text{Re} \int_{-\infty}^{\infty} F[u_0]u_0^* dT
\] (2.12.105)

the same equation that was derived from the evolution of the leading order Hamiltonian (2.7.61).

Since \(U_{h2}\) does not decay to zero as \(T \to \pm \infty\) there is not an immediately available associated Fredholm alternative. However by using the same type of argument we can derive a generalized Fredholm alternative. Let us define the set of bilinear operators
\[
<f, g>_M = \int_{-M}^{M} [f(s)^*]^T g(s) ds
\] (2.12.106)

so that, for any \(M\), both sides of
\[
<U_{h2}, LU_1>_M = <U_{h2}, G[u_0]>_M
\] (2.12.107)

are finite. Here \(U_1 = [u_{1R} \quad u_{1I}]^T\) is a quasi-stationary solution of the linear problem (2.8.81). With integration by parts the operator \(L\) now gives
\[
\text{IBP} = <LU_{h2}, U_1>_M = <U_{h2}, G[u_0]>_M
\] (2.12.108a)

where
\[
\text{IBP} = \left( A (\text{Im}[u_0]u_{11} - \text{Re}[u_0]u_{1R} ) \right. + \left. \frac{1}{2} (\text{Im}[u_0]|_{-M}^M \partial_T u_{1R} + \text{Re}[u_0]|_{-M}^M \partial_T u_{11} - \text{Im}[u_0]|_{-M}^M u_{1R} ) \right)
\] (2.12.108b)
are the terms which arise from the integration by parts. Putting $u_1$ in terms of the magnitude and phase functions $q_0, q_1, \phi_0$ and $\phi_1$, we expand our previous approximation for $u$

$$u = (q_0 + \epsilon q_1)e^{i(\phi_0 + \epsilon \phi_1)} = q_0 e^{i\phi_0} + \epsilon (q_1 + i\phi_1 q_0) e^{i\phi_0} + O(\epsilon^2)$$  \hspace{1cm} (2.12.109)$$

so that

$$u_0 = q_0 e^{i\phi_0}$$  \hspace{1cm} (2.12.110)$$

$$u_1 = (q_1 + i\phi_1 q_0) e^{i\phi_0}$$  \hspace{1cm} (2.12.111)$$

Now as $M \to \infty$ equation (2.12.108) is

$$Au_\infty (q_1^+ - q_1^-) + \frac{1}{2} u_\infty^2 (\phi_1^+ - \phi_1^-) = -B_Z + \text{Im} \int_{-\infty}^{\infty} (F[u_\infty] u_\infty - F[u_0] u_0^*) \, dt$$  \hspace{1cm} (2.12.112)$$

This is consistent with equations (2.7.73).

### 2.13 Conclusions

In conclusion, we develop a novel approach to dark soliton perturbation theory which breaks the problem into an inner region around the soliton and a shelf which matches to the boundary at infinity. Under perturbation a dark soliton develops a shelf the edge of which propagates out at speed proportional to the magnitude of the cw background. In analytical terms the shelf arises due to properties of the perturbation which serve to drive mean contributions in the amplitude and growing terms in the phase. It is also found that the soliton can have a different frequency than the cw background. The method can be applied to general perturbations which can have both moving and constant backgrounds. For typical perturbations the asymptotic approximations were calculated and were compared favorably with direct numerical results. These comparisons confirmed the existence of the analytically predicted shelf and support our results. The non-vanishing background and soliton are treated separately from the core soliton; in this way we obtain a consistent perturbation theory for dark solitons.
Chapter 3

The PES equation

In this chapter the Power-Energy Saturation (PES) equation is considered. This is a variant of the classical nonlinear Schrödinger (NLS) equation which is used to model the effects of dispersion and self-phase modulation. The effects of gain, loss and filtering are taken to be perturbations to the NLS (not necessarily small!). The gain and filtering terms are saturated with energy and the loss term is saturated with power. Other models used to describe this propagation include Ginzburg-Landau (GL) type equations [18, 17] and the master equation [38, 42]. In Section 3.1 perturbation theory is employed to find mode-locking behavior in the PES equation for single pulses. Section 3.2 introduces the phenomena of pulse splitting and proposes the possibility of higher-order bound states in the PES equation. In sections 3.3-3.5 we investigate weakly interacting solitons by considering the effect of a neighboring soliton as a new perturbation. Finally, in section 3.6 we demonstrate the equivalence of using secularity conditions to using integral relations in soliton perturbation theory.

The PES equation is written in non-dimensional form as

\[ \text{i}\psi_z + \frac{d_0}{2}\psi_{tt} + |\psi|^2 \psi = \frac{ig}{1 + E/E_{\text{sat}}}\psi + \frac{i\tau}{1 + E/E_{\text{sat}}}\psi_{tt} - \frac{il}{1 + P/P_{\text{sat}}}\psi \]  

(3.0.1)

where \( E = \int_{-\infty}^{\infty} |\psi|^2 \text{d}t \) is the pulse energy, \( E_{\text{sat}} \) is the saturation energy, \( P = |\psi|^2 \) is the pulse power, \( P_{\text{sat}} \) is the saturation power. The dispersion is in the anomalous regime: \( d_0 > 0 \); \( g, \tau \) and \( l \) are positive real constants and here we take \( d_0 = 1, E_{\text{sat}} = 1, P_{\text{sat}} = 1, \tau = l = 0.1 \) and vary \( g \). Typical physical parameters can be found in [7]. This equation has been shown numerically to
exhibit mode-locking behavior for a large range of parameters with solutions that are found to be well approximated by NLS solitons [8]. Mode-locking associated with the other models is much more limited. Like the master-laser equation, the PES model has gain and filtering saturated by energy. Unlike the master-laser equation in which cubic nonlinearity models additional loss-gain, the PES equation has loss modeled by the power saturation term, which reflects the “iris” effect in the laser media (e.g Ti:sapphire crystal) where only a portion of the field power is transmitted. For small $P/P_{sat}$ the PES equation reduces to the master-laser equation by taking the first order Taylor series expansion of the loss term. The model can then be further reduced by taking pulse energy to be constant, in which case the “master-equation” reduces to a GL type system.

3.1 Single soliton evolution

We begin by studying the effects of the right hand side of the PES equation as perturbations on a soliton solution and use the following form for the four parameter soliton family of solutions to NLS

$$\psi = u e^{i\phi}$$

$$\xi = \int_0^z Vdz + t_0, \quad \theta = t - \xi, \quad \sigma = \int_0^z [\mu + \frac{V^2}{2}]dz + \sigma_0, \quad \phi = V\theta + \sigma$$

with

$$u = \eta \text{sech}(\eta \theta)$$

where $\mu, V, t_0, \sigma_0$ are arbitrary constants defining the the height/width, speed, temporal shift and phase shift of the soliton respectively. We also note that $\eta$ is directly related to $\mu$ by $\mu = \eta^2/2$.

The gain, filtering and loss terms are considered perturbations to the NLS

$$i\psi_z + \frac{1}{2}\psi_{tt} + |\psi|^2 \psi = F[\psi]$$

where $F$ is assumed small. Remarkably, the contribution of the individual terms in $F$ may not be small, but the combined terms are and thus only provides the mode-locking mechanism. We
account for this perturbation in our solution by letting the four free parameters of the soliton solution to vary slowly in \( z \). Rather than introducing a new slow space variable related to a small term \( \mathcal{O}(\epsilon) \) and then factoring an \( \epsilon \) out of \( F \), we directly consider \( \eta, V, t_0, \sigma_0 \) as functions of \( z \). The assumption that they vary \textit{adiabatically}, i.e slowly, in \( z \) is due to the smallness of \( F \).

The analysis uses integrated conservation laws modified by the RHS terms and integral identities derived from equation (3.1.4) given by

\[
\begin{align*}
\frac{d}{dz} \int |\psi|^2 & = 2 \text{Im} \int F |\psi| \psi^* \quad (3.1.5a) \\
\frac{d}{dz} \text{Im} \int \psi \psi^*_t & = 2 \text{Re} \int F |\psi| \psi^*_t \quad (3.1.5b) \\
\frac{d}{dz} \int t |\psi|^2 & = -d_0 \text{Im} \int \psi \psi^*_t + 2 \text{Im} \int t F |\psi| \psi^* \quad (3.1.5c) \\
\text{Im} \int \psi_z \psi^*_\mu & = -\frac{d_0}{2} \text{Re} \int \psi_t \psi^*_\mu + \text{Re} \int |\psi|^2 \psi \psi^*_\mu - \text{Re} \int F |\psi| \psi^*_\mu \quad (3.1.5d)
\end{align*}
\]

where all integrals are taken over \(-\infty < t < \infty\).

The virtue of using conservation laws rather than multi-scale perturbation theory is that allows one to readily obtain a convenient system of ordinary differential equations for \( \eta, V, t_0, \sigma_0 \) without formulating detailed perturbation theory and secularity conditions. Substituting the soliton solution into the above integral relations and allowing the functions \( \eta, V, t_0, \sigma_0 \) to vary slowly in \( z \) satisfies the following list of conservation laws derived from equation (3.1.4).

\[
\begin{align*}
\frac{dE}{dz} & = \int_{-\infty}^{\infty} u(Fe^{-i\Phi} - F^* e^{i\Phi}) d\theta/i \quad (3.1.6a) \\
E \frac{dV}{dz} & = \int_{-\infty}^{\infty} u_\theta(Fe^{-i\Phi} + F^* e^{i\Phi}) d\theta \quad (3.1.6b) \\
E \frac{dt_0}{dz} & = \int_{-\infty}^{\infty} u_\theta(Fe^{-i\Phi} - F^* e^{i\Phi}) d\theta/i \quad (3.1.6c) \\
E_\mu V \frac{dt_0}{dz} + \frac{d\sigma_0}{dz} & = \int_{-\infty}^{\infty} u_\mu(Fe^{-i\Phi} + F^* e^{i\Phi}) d\theta \quad (3.1.6d)
\end{align*}
\]

The above equations are in fact the same secularity equations found from the method of multi-scale perturbation theory [70, 69], and are valid for the generalized NLS with a nonlinear term of the form \( N(|\psi|^2)\psi \)–see section 3.6.

Substituting equation (3.1.3) into equations (3.1.6) yields the following set of differential
equations governing $\eta$, $V$, $t_0$, $\sigma_0$ respectively

\[
\frac{d\eta}{dz} = g \left[ \frac{2\eta}{2\eta + 1} \right] - \tau \frac{1}{2\eta + 1} \left[ \frac{2}{3} \eta^3 + 2V^2 \eta \right] + l \left[ 2\eta - \frac{1}{a - b} \log \left( \frac{a}{b} \right) \right] \quad (3.1.7a)
\]

\[
\frac{dV}{dz} = -\tau V \left[ \frac{4\eta^2}{3(2\eta + 1)} \right] \quad (3.1.7b)
\]

\[
\frac{dt_0}{dz} = 0 \quad (3.1.7c)
\]

\[
\frac{d\sigma_0}{dz} = 0 \quad (3.1.7d)
\]

where here $a$ and $b$ are the roots of the polynomial $x^2 + 2(1 + 2\eta^2)x + 1 = 0$ Note $a$ and $b$ can be chosen to be either root of the quadratic equation. Also, $t_0$ and $\sigma_0$ are constant and the first two equations are independent of $t_0$, $\sigma_0$.

Consider first the case $V(z) = 0$; in fact this turns out to be a stable equilibrium for any $\eta > 0$ ($\eta < 0$ is only a phase shift from its positive counterpart). This results in the following equation for $\eta$

\[
\frac{d\eta}{dz} = g \left[ \frac{2\eta}{2\eta + 1} \right] - \tau \frac{1}{2\eta + 1} \left[ \frac{2}{3} \eta^3 \right] + l \left[ 2\eta - \frac{1}{a - b} \log \left( \frac{a}{b} \right) \right]
\]

Clearly, $\eta$ has at least one equilibrium at 0. To classify its stability we consider the first derivative of $\frac{dn}{dz}$ with respect to $\eta$ evaluated at $\eta = 0$.

\[
\frac{d}{d\eta} \left( \frac{dn}{dz} \right) = g \left[ \frac{2}{(2\eta + 1)^2} \right] - \tau \eta^2 \left[ \frac{4\eta + 1}{3(2\eta + 1)^2} \right] - 2l \left[ \log \left( \frac{a}{b} \right) \frac{1}{b - a} + \eta \frac{b}{a} \right] \left[ \frac{a}{b} \right] \eta + \eta \log \left( \frac{a}{b} \right) \frac{a\eta - b\eta}{(b - a)^2}
\]

\[
\frac{d}{d\eta} \left( \frac{dn}{dz} \right)_{\eta=0} = 2 \left[ g - l \right]
\]

When $g > l, \eta = 0$ is an unstable equilibrium while for $g < l, \eta = 0$ is stable. The phase portrait for various values of $g < l$ indicates that $\frac{dn}{dz}$ is a monotonic decreasing function in $\eta$ and so $\eta \to 0$ as $z \to \infty$ for any initial condition $\eta > 0$. Physically this corresponds to loss overtaking gain and the pulse decays. For $g > l$, $\frac{dn}{dz}$ is initially positive and as $\eta$ increases the filtering term, $\tau \frac{1}{2\eta + 1} \left[ \frac{2}{3} \eta^3 \right]$ becomes dominant and $\frac{dn}{dz}$ takes on negative values. Thus there must exist another equilibrium. Plotting the phase portrait for various values of $g$ for $g > l$ shows there is a single stable equilibrium for $\eta > 0$, as shown in figure 3.1.
Figure 3.1: Phase portrait of the amplitude equation for several values of $g$ illustrating the single stable equilibrium.
As \( \eta \) approaches its stable equilibrium \( \eta_* \neq 0 \) and \( V \) approaches its stable equilibrium \( V_* = 0 \) the system of equations (3.1.7) tend to

\[
\frac{d\eta}{dz} = \frac{dV}{dz} = 0
\]

which means the soliton tends to a particular NLS soliton of speed zero and height/width determined by \( \eta_* \), thus suggesting the mode-locking capabilities of equation (3.0.1). In fact, \( \eta_* \) is an attractor that any arbitrary initial condition will eventually converge to, even when it is far way from its soliton solution [7, 10].

This agrees with the results found numerically for the PES equation. An example of the final state found numerically compared to what is predicted is given below in figure 3.2 (top). Numerically mode-locked solutions are observed for \( g > l = 0.1 \); for \( g < l = 0.1 \) initial pulses decay to zero which agrees with the turn over point \( g = l \) found above regarding the stability of the zero solution for \( \eta \). For soliton initial conditions the evolution of the height matches very well (see figure 3.2 (bottom)).

### 3.2 Higher-order solitons

As seen in section 3.1 solitons are obtained when the gain is above a certain critical value, \( g > l \), otherwise pulses dissipate and eventually vanish. For high energy initial conditions the PES equation exhibits pulse splitting. An example case is shown in figure 3.3. As gain becomes stronger additional soliton states are possible and 2, 3, 4 or more coupled pulses are found to be supported. The value of \( \Delta \xi/\alpha \), where \( \Delta \xi \) and \( \alpha \) are the pulse separation and pulse width respectively, is an important parameter. The full width of half maximum (FWHM) is used for pulse width, \( \Delta \xi \), is measured between peak values of two neighboring pulses and \( \Delta \phi \) is the phase difference between the peak amplitudes.

The resulting individual pulses are similar to the single soliton mode-locking case, i.e. individual pulses are approximately solutions of the unperturbed NLS equation, namely hyperbolic secants. The pulses differ from a single soliton in that the individual pulse energy is smaller then
Figure 3.2: Comparison of asymptotic results to numerical simulations for the peak values of the soliton $\eta$ (top) and for the resulting profile (bottom).
Figure 3.3: Mode-splitting of the anomalously dispersive PES equation (top) and (bottom) at $z = 200$. Here $g = 0.5$. 
that observed for the single soliton mode-locking case for the same choice of $g$, while the total energy of the two soliton state is higher. This is due to the non-locality of energy saturation in the gain and filtering terms.

To investigate the minimum distance, $d^*$, between the solitons in order for no interactions to occur we evolve the PES equation starting with two solitons. If the initial two pulses are sufficiently far apart then the propagation evolves to a two soliton state and the resulting pulses have a constant phase difference. If the distance between them is less than a critical value then the two pulses interact in a way characterized by the difference in phase between the peaks amplitudes: $\Delta \phi$. When initial conditions are symmetric (in phase) two pulses are found to merge into a single soliton of equation (3.0.1). When the initial conditions are anti-symmetric (out of phase by $\pi$) then they repel each other until their separation is above this critical distance while retaining the difference in phase, resulting in an effective two pulse high-order soliton state. This does not occur in the classical (unperturbed) NLS equation as shown in section 3.4 below.

In the constant dispersion case this critical distance is found to be $\Delta \xi = d^* \approx 9\alpha$ (see figure 3.3) corresponding to soliton initial conditions. Interestingly, this is consistent with the experimental observations of [65]. To further illustrate, we plot the evolution of these cases in figure 3.4. At $z = 500$ for the repelling solitons $\Delta \xi/\alpha = 8.9$.

### 3.3 Weakly Interacting Solitons

Let us begin with the unperturbed NLS equation; i.e $F = 0$. When solitons are widely separated to leading order the full solution may be viewed as the superposition of two single solitons $\psi = \psi_1 + \psi_2$; we take $\psi_1$ to be the soliton on the left and $\psi_2$ to be the soliton on the right. Without loss of generality we analyze the small effect (due to the wide separation) of soliton 2 on soliton 1. By taking $\psi_2$ to be small locally and expanding the nonlinear term around $\psi_1$ the evolution of $\psi$ where $\psi_1$ is the dominant term is well approximated by

$$i\psi_{1z} + \frac{1}{2}\psi_{1tt} + |\psi_1|^2 \psi_1 = -2|\psi_1|^2 \psi_2 - \psi_1^2 \psi_2^* = G[u_1,u_2]$$

(3.3.8)
Figure 3.4: Two pulse interaction when $\Delta \xi < d^*$. Initial pulses ($z = 0$) in phase: $\Delta \phi = 0, \Delta \xi/\alpha \approx 7$ (top) merge while those out of phase by $\Delta \phi = \pi$ with $\Delta \xi/\alpha \approx 6$ (bottom) repel. Here $g = 0.5$. 
and similarly an equation for when $\psi_2$ dominates is found to satisfy
\[
\imath \psi_{2z} + \frac{1}{2} \psi_{2tt} + |\psi_2|^2 \psi_2 = -2 |\psi_2|^2 \psi_1 - \psi_2^* \psi_1^* = G[u_2, u_1]
\] (3.3.9)

Notice that if $\psi_1$ and $\psi_2$ satisfy this system of coupled PDEs then the sum $\psi$ satisfies NLS. This is not a requirement for the method to work, but an added feature of having a cubic nonlinearity. See [48] for a more general perturbative treatment of such interactions.

We can now use the same procedure from the one soliton case to derive equations for the evolution of the eight parameters which define our two solitons. Substituting the perturbations in equations (3.3.8)-(3.3.9) for $F$ into the integral relations (3.1.5) we arrive at the following set of differential equations with right hand sides as integrals which must be computed
\[
\frac{d\eta_k}{dz} = (-1)^k \int_{-\infty}^{\infty} u_k^2 u_{3-k} \sin(\Delta \phi) d\theta
\] (3.3.10a)
\[
\eta_k \frac{dV_k}{dz} = -\int_{-\infty}^{\infty} 3u_k \theta u_{k z}^2 u_{3-k} \cos(\Delta \phi) d\theta
\] (3.3.10b)
\[
\eta_k \frac{d\sigma_k}{dz} = (-1)^k \int_{-\infty}^{\infty} \theta u_{k \theta}^3 u_{3-k} \sin(\Delta \phi) d\theta
\] (3.3.10c)
\[
V_k \frac{dt_0}{dz} + \frac{d\sigma_0}{dz} = -\int_{-\infty}^{\infty} 3u_k \eta u_{k z}^2 u_{3-k} \cos(\Delta \phi) d\theta
\] (3.3.10d)

where $\Delta \phi = \phi_2 - \phi_1$ for $k = 1, 2$. Further simplifications are needed to get more explicit results. We assume that $\Delta V = V_2 - V_1$ is small and then from the definition $\Delta \phi \approx -\bar{V} \Delta \xi + \Delta \sigma$ which importantly is independent $\theta$ so those terms may be taken outside the integral. We also assume $\Delta \eta = \eta_2 - \eta_1$ is small and approximate $\eta_1$ and $\eta_2$ as $\bar{\eta}$. For all variables the bar denotes the mean over the two solitons and the difference of the right minus the left. We finally approximate $u_{3-k} \approx 2\bar{\eta} e^{(-1)^k \eta \theta_{3-k}} \approx 2\bar{\eta} e^{(-1)^k \eta \theta_{k} e^{-\eta \Delta \xi}}$ using the tail which interacts with $u_k$. Then equations (3.3.10) may now be reduced to
\[
\frac{d\eta_k}{dz} = (-1)^k \int_{-\infty}^{\infty} u_k^2 2\bar{\eta} e^{(-1)^k \eta \theta} \sin(\Delta \phi) e^{-\eta \Delta \xi} d\theta
\]
and since we know the form of $u_k$ the integral can now be evaluated to give

\[
\frac{d\eta_k}{dz} = (-1)^k 4\eta^3 \sin(\Delta \phi) e^{-\eta \Delta \xi} \quad (3.3.11a)
\]

\[
\frac{dV_k}{dz} = (-1)^{k+1} 4\eta^3 \cos(\Delta \phi) e^{-\eta \Delta \xi} \quad (3.3.11b)
\]

\[
\frac{dt_0}{dz} = 2\eta^2 \sin(\Delta \phi) e^{-\eta \Delta \xi} \quad (3.3.11c)
\]

\[
\frac{d\sigma_0}{dz} = -6\eta^2 \cos(\Delta \phi) e^{-\eta \Delta \xi} - V_k 2\eta^2 \sin(\Delta \phi) e^{-\eta \Delta \xi} \quad (3.3.11d)
\]

Since these equations are in terms of $\Delta \phi$ and $\Delta \xi$, it is useful to derive evolution equations directly for these variables as well. First we combine equations (3.3.11) (sums and differences) to get the following stationary variables

\[
\frac{d\bar{\eta}}{dz} = 0, \quad \frac{d\bar{V}}{dz} = 0, \quad \frac{d\Delta t_0}{dz} = 0, \quad \frac{d\Delta \sigma_0}{dz} = 0 \quad (3.3.12)
\]

And now we may use the definitions of $\Delta \phi = -\bar{V}\Delta \xi + \Delta \sigma$ and $\Delta \xi = \int_0^z (\Delta V) \, dz - \Delta t_0$ and take the derivative to arrive at

\[
\frac{d\Delta \xi}{dz} = \Delta V + \Delta t_{0z} = \Delta V
\]

\[
\frac{d\Delta \phi}{dz} = -\bar{V}_z \Delta \xi - \bar{V} \Delta \xi_z + \Delta \sigma_z
\]

\[
= -\bar{V} \Delta V + (\bar{\eta} \Delta \eta + \bar{V} \Delta V) + \Delta \sigma_{0z} = \bar{\eta} \Delta \eta
\]

Now consider the combined effect of the perturbation associated with soliton interactions and the perturbation from the PES equation

\[
i\psi_{1z} + \frac{1}{2}i\psi_{1tt} + |\psi_1|^2 \psi_1 = G[u_1, u_2] + F[u_1]
\]

\[
i\psi_{2z} + \frac{1}{2}i\psi_{2tt} + |\psi_2|^2 \psi_2 = G[u_2, u_1] + F[u_2]
\]

The effect of interaction in the $F$ terms is a higher order term and omitted for our first order approximation. This method has also been successfully applied to the case of unperturbed generalized NLS [76]. Since the integral relations used to derive evolution equations for the soliton parameters are linear in the perturbation we may find a set of equations for the evolution of two weakly
interacting solitons in the PES equation by taking the superposition of our previous results. For this, it is convenient to define the following functions

\[
S_1(\eta, V, \bar{\eta}) := g \left[ \frac{2\eta}{E_0 + 1} - \tau \frac{1}{E_0 + 1} \left( \frac{2}{3} \eta^3 + 2V^2\eta \right) + \frac{1}{2\eta} \frac{1}{a-b} \log \left( \frac{a}{b} \right) \right]
\]
\[
S_2(\eta, V, \bar{\eta}) := -\tau V \left[ \frac{4}{3} \frac{\eta^2}{E_0 + 1} \right]
\]

Here the energy is computed to be \( E_0 = 4\eta \). Since the energy saturation is a nonlocal effect this adds yet another interaction between the solitons. The final set of evolution equations governing the weak interaction is

\[
\frac{d\eta_k}{dz} = S_1(\eta_k, V_k) + (-1)^k 4\eta^3 \sin(\Delta\phi) e^{-\eta\Delta\xi}
\]
\[
\frac{dV_k}{dz} = S_2(\eta_k, V_k) + (-1)^k 1 4\eta^3 \cos(\Delta\phi) e^{-\eta\Delta\xi}
\]
\[
\frac{d\Delta\xi}{dz} = \Delta V
\]
\[
\frac{d\Delta\phi}{dz} = \eta \Delta\eta
\]

For two solitons whose respective peaks differ in phase by \( \Delta\phi \) we find that for \( 0 \leq \Delta\phi < \pi/2 \) the solitons are attracted to each other, while for \( \pi/2 < \Delta\phi \leq \pi \) they are repelled. In figure 3.5 and 3.6 we show some comparisons to numerical results. Here initial conditions are NLS solitons of height \( \eta_{10} = \eta_{20} = 2.67 \) and peak separation \( \Delta\xi_0 = 4 \) with an initial phase difference \( \Delta\phi_0 \). In certain cases the two solitons collide. Prior to collision, perturbation theory gives excellent agreement with direct numerical simulation as in left- figure 3.6. Near and after collision the effects can no longer be modeled by weak interaction. In these cases the perturbation equations experience blow up.

In equations (3.3) the terms \( S_1 \) and \( S_2 \) are the contributions from gain, loss and filtering, while the other terms come from the tail interactions. Considering just the effects of gain, loss and filtering, the velocities have a single stable equilibrium \( V_1 = V_2 = 0 \) for any positive value of \( \eta_1 \) and \( \eta_2 \) (negative values of \( \eta_1 \) or \( \eta_2 \) are only a phase shift from their positive counterparts), so we look at the dynamics of \( \eta_1 \) and \( \eta_2 \) for \( V_1 \) and \( V_2 \) at this equilibrium. Figure 3.7 shows the phase
Figure 3.5: Comparison of the asymptotic results for the soliton centers to the contour plots found numerically for $\Delta \phi_0 = \pi$ (top) and comparison of the final profile (bottom).
Figure 3.6: Comparison of the asymptotic results for the soliton centers to the contour plot found numerically for $\Delta \phi_0 = 0$ (top) and for $\Delta \phi_0 = \pi/4$ (bottom).
plane for gain values above and below the threshold for stable two pulse solutions. In both cases there exists a stable equilibrium at \( \eta_k = 0 \) for \( k = 1 \) or \( 2 \) which amounts to a reduction to the single soliton case and an unstable equilibrium when both \( \eta_1 = \eta_2 = 0 \) for any \( g > 1 \). Also, in both cases we see an equilibrium at \( \eta_1 = \eta_2 = \eta^* \), however, for \( g < g_c \) it is found to be unstable and for \( g > g_c \), it is stable; \( g_c \approx 0.45 \).

In figure 3.7 typical cases are shown. Here \( g = 0.4 \) (top) is unstable and \( g = 0.6 \) (bottom) is stable. The \( \eta^* \) found from perturbation theory agrees with the height found numerically to several decimal places. We also note that there are two other two equilibria found for \( g = 0.6 \) which are unstable.

When the soliton interaction terms are taken into consideration, \( S_1 \) and \( S_2 \) are found to be the dominant terms for sufficiently large \( \Delta \xi \), hence \( \eta_k \to \eta^* \) and \( V_k \to 0 \) for \( k = 1, 2 \). However, the small contributions from the interaction terms can cause small decaying oscillations for large \( z \). Solving the perturbation theory derived system, we find a stable equilibrium at \( \Delta \phi = \pi \) for any \( \eta \) as well as an unstable equilibrium for \( \Delta \phi = 0 \) and \( \Delta \eta = 0 \). See typical cases in figure 3.8. The peak separation is further characterized by the phase difference; the pulses attract each other when \(-\pi/2 \leq \Delta \phi < \pi/2\) and repel when \(\pi/2 < \Delta \phi \leq 3\pi/2\). This is seen from inspecting the sign of \( \Delta V_t = -8\overline{\eta}^3 \cos(\Delta \phi) \exp(\overline{\eta} \xi) \) for two solitons with zero velocity. Thus, for initial phase difference \( \Delta \phi_0 = 0 \) the pulses will eventually collide and combine into a single pulse, while for \( \Delta \phi_0 \neq 0 \) the phase difference will evolve to \( \pi \) and after some initial oscillations the pulses will repel.

### 3.4 Comparison to classic NLS

To illustrate the significance of the results in section 3.3 we compare the results for the classical NLS to what is found for the PES equation. When the interactions result in the solitons eventually moving away from each other out to infinity we find the rate at which the solitons move apart is an order of magnitude different, see figure 3.9. Indeed, direct numerical evaluation indicates that the departure rate of the PES equation is even slower than that predicted from perturbation
Figure 3.7: Typical phase planes for \( \frac{d\eta_1}{dz} \) and \( \frac{d\eta_2}{dz} \) with \( V_1 = V_2 = 0 \) for the unstable two soliton case, \( g = 0.4 \) (top) and the stable two soliton case, \( g = 0.6 \) (bottom).
Figure 3.8: Evolution of the phase for $\eta_{10} = \eta_{20} = 3.3$, $\Delta \xi_D = 3.0$ ($\Delta \xi_D/\alpha \approx 6$) and several choices of initial phase difference. Here $g = 0.6$. $\Delta \phi = \pi$ is shown with a dashed line.
theory. Effectively we have bound states in the PES equation. This also agrees with experimental observations [65]. The difference between the equations can be explained by the strong damping effect the filtering term has on the velocity of the solitons. This will be shown analytically below.

![Graph comparing NLS and PES equations](image)

**Figure 3.9:** Comparison of the interaction behavior for the classical NLS equation and the PES equation, for the initial conditions used in figure 3.5.

Similar to the one soliton case, the heights of the two solitons in the PES equation both tend towards a fixed height $\eta^*$ defined by root of $S_1(\eta^*, 0, \eta^*) = 0$. As $\eta_1$ and $\eta_2$ tend to equilibrium the phase difference tends to $\pi$. In contrast, the classical NLS equation with $\eta_1 \neq \eta_2$ has (small decaying) oscillations in the heights as illustrated in figure 3.10; generally the distance between solitons increases linearly for classical NLS while it grows logarithmically for the PES equation. In some special cases in the classical NLS equation the distance between solitons only oscillates. This is found for initial conditions $\eta_1 \neq \eta_2$ and $\Delta \phi_0 = 0$ or $\pi$. For these oscillating bound states the oscillations in the heights do not decay as opposed to the case of separating solitons; an example is given below—see figure (3.11).
Figure 3.10: Comparison of classical NLS equation and the PES equation for the heights and centers of two solitons. Here $\Delta \phi_0 = \pi/4$ and $\Delta \xi_0 = 4$. 
Figure 3.11: An example of the oscillating bound states found for classical NLS, but not apparent for the PES equation. Here $\Delta \phi_0 = \pi$ and $\Delta \xi_0 = 4$. 
3.5 Asymptotic results for $\Delta \xi$

Using the fact that $\eta_1 = \eta_2 \to \eta^*$ and $\Delta \phi \to \pi$ as $z \to \infty$ we can derive the asymptotic behavior of $\Delta \xi$. First we take the difference of $dV_2/dz$ and $dV_1/dz$ and the derivative of $d\Delta \xi/dz$ giving us

$$\frac{d\Delta V}{dz} = -\tau \left[ \frac{4}{3} \frac{\eta^{*2}}{4\eta^* + 1} \right] \Delta V + 8\eta^{*3} e^{-\eta^* \Delta \xi}$$

$$\frac{d^2}{dz^2} \Delta \xi = \frac{d\Delta V}{dz}$$

which may now be combined to form a second order differential equation for $\Delta \xi$,

$$\frac{d^2}{dz^2} \Delta \xi = -\tau \left[ \frac{4}{3} \frac{\eta^{*2}}{4\eta^* + 1} \right] \frac{d}{dz} \Delta \xi + 8\eta^{*3} e^{-\eta^* \Delta \xi}$$

Since the evolution of $\Delta \xi$ is seen from numerical simulations to be evolving slowly in $z$ we expect that $\frac{d^2}{dz^2} \Delta \xi \ll \frac{d}{dz} \Delta \xi$ and so $\frac{d^2}{dz^2} \Delta \xi$ is dropped as being a higher order term leaving

$$\tau \left[ \frac{4}{3} \frac{\eta^{*2}}{4\eta^* + 1} \right] \frac{d}{dz} \Delta \xi = 8\eta^{*3} e^{-\eta^* \Delta \xi}$$

which we solve to get the long term behavior of $\Delta \xi$

$$\Delta \xi \to \frac{1}{\eta^* \log \left( \frac{\eta^* (1 + 4\eta^*)}{\tau} \right) 6z}$$

This logarithmic growth in soliton separation differs significantly from the linear separation growth in the classical NLS. In fact, after an initial interaction period, during which there is transient motion the soliton centers tend essentially to a fixed distance; the motion is hardly noticeable numerically. We call this an effective bound state since the small variations would be too small and take too long to seen experimentally. For example, when the initial $\Delta \xi$ is taken to be 9 times the full-width-half-max the the change in separation over a thousand units in $z$ is approximately 0.1.

3.6 Secularity Conditions

Here we will show the equivalence of using conservation laws and secularity conditions in multiple-scale perturbation theory for the perturbed generalized NLS equation

$$i\psi_z + \frac{d_0}{2} \psi_{tt} + N(|\psi|^2)\psi = F[\psi] = e\hat{F}[\psi]$$

(3.6.13)
In both cases we assume a solution of the form (3.1.2) where \( u \) satisfies

\[-\mu u + \frac{1}{2} u_{\theta \theta} + N(u^2)u = 0\]

and \( \mu, V, t_0, \sigma_0 \) are all function of \( Z = \epsilon z \). For conservation laws the first three integral identities (3.1.5) are the same for generalized NLS as for classical NLS just replacing \( F \) with \( \epsilon \hat{F} \) while the fourth (3.1.5d) is replaced by

\[
\text{Im} \int \psi_z \psi_{\mu}^* = - \frac{d_0}{2} \text{Re} \int \psi_t \psi_{\mu}^* + \text{Re} \int N(|\psi|^2) \psi \psi_{\mu}^* - \text{Re} \int \epsilon \hat{F}[\psi] \psi_{\mu}^*
\]

In either case the equations (3.1.6) will result from substituting in (3.1.2) into the integral identities. For the multi-scale method we begin by substituting (3.1.2) into (3.6.13) to arrive at

\[
i u_z - \mu u + \frac{1}{2} u_{\theta \theta} + N(u^2)u = G
\]  
(3.6.14)

where

\[
G = \epsilon \hat{F}[\psi] e^{-i \phi} - \epsilon [iu_{\mu} u_z - i u_0 t_0 Z] - \epsilon [V t_0 Z - V Z \theta + \sigma_0 Z] u
\]

Expanding \( u \)

\[
u = u_0 + \epsilon u_1 + O(\epsilon^2)
\]

we see that the leading order is satisfied immediately since \( u_0 \) satisfies (3.6.14) and at order \( \epsilon \) the equation for \( u_1 \) can be written as

\[
i U_z + LU = \tilde{G}
\]  
(3.6.15)

where

\[
U = \begin{pmatrix} \text{Re}(u_1) \\ -i \text{Im}(u_1) \end{pmatrix}, \quad L = \begin{pmatrix} 0 & L_0 \\ L_1 & 0 \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} -i \text{Im}(G) \\ -\text{Re}(G) \end{pmatrix}
\]

and

\[
L_0 = -\frac{1}{2} \partial_{\theta \theta} + \mu - N(u^2)
\]

\[
L_1 = -\frac{1}{2} \partial_{\theta \theta} + \mu - N(u^2) - 2u^2 N'(u^2)
\]
The operator $\mathbb{L}$ has two pairs of solutions (eigenfunctions and generalized eigenfunctions) corresponding to the zero eigenvalue:

\[ V_1 = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}, \quad \hat{V}_1 = \begin{pmatrix} 0 \\ -\theta u/2 \end{pmatrix} \]

and

\[ V_2 = \begin{pmatrix} 0 \\ u \end{pmatrix}, \quad \hat{V}_2 = \begin{pmatrix} -u_\mu \\ 0 \end{pmatrix} \]

For the solution $U$ of (3.6.15) to be non-secular for large $z$, the inhomogeneous term $\tilde{G}$ must be the orthogonal to the eigenfunction and generalized eigenfunctions given above; i.e.

\[ \langle \tilde{G}, V_k \rangle = \langle \tilde{G}, \hat{V}_k \rangle = 0, \quad k = 1, 2 \] (3.6.16)

where the inner product is defined by

\[ \langle A, B \rangle = \int_{-\infty}^{\infty} A^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B d\theta \]

Evaluating the four integrals defined by (3.6.16), the evolution equations for $\mu$, $V$, $t_0$, $\sigma_0$ are found to be

\[
\frac{dE}{dZ} = \int_{-\infty}^{\infty} u(\hat{F} e^{-i\phi} - \hat{F}^* e^{i\phi}) d\theta / i \] (3.6.17)

\[
\frac{dV}{dZ} = \int_{-\infty}^{\infty} u\theta(\hat{F} e^{-i\phi} + \hat{F}^* e^{i\phi}) d\theta \] (3.6.18)

\[
\frac{dt_0}{dZ} = \int_{-\infty}^{\infty} u\theta(\hat{F} e^{-i\phi} - \hat{F}^* e^{i\phi}) d\theta / i \] (3.6.19)

\[
E \mu \frac{dt_0}{dZ} + \frac{d\sigma_0}{dz} = \int_{-\infty}^{\infty} u\mu(\hat{F} e^{-i\phi} + \hat{F}^* e^{i\phi}) d\theta \] (3.6.20)

which are exactly equations (3.1.5) after the simple change of variables $Z = \varepsilon z$ and $\hat{F} = \hat{F} / \varepsilon$.

### 3.7 Conclusions

We presented an analytic theory based on asymptotic analysis to describe solitons and soliton interactions in mode-locked lasers. The power-saturation equation was used. This is a
generalization of the classical NLS, appropriately modified to model gain, loss and filtering in the laser. It has been shown, that solitons of the PES system are essentially attractors that even arbitrary initial condition will tend to after evolution. The mode-locking capabilities of the model distinguishes it from others, such as the master mode-laser equation. Effective multi-soliton bound states are found.
In this chapter we take the PES equation (3.0.1) with normal dispersion $d_0 = -1 < 0$

\[
\begin{align*}
&i\psi_z - \frac{1}{2}i\psi_{tt} + |\psi|^2\psi = \frac{ig}{1 + E/E_{\text{sat}}}\psi + \frac{i\tau}{1 + E/E_{\text{sat}}}\psi_{tt} - \frac{i\ell}{1 + P/P_{\text{sat}}}\psi,
\end{align*}
\]

where the complex electric field envelope $\psi(z, t)$ is subject to then boundary conditions $|\psi(z, t)| \to \psi_\infty$ as $|t| \to \infty$. Since the system energy $E_s = \int_{-\infty}^{\infty} |\psi|^2 \, dt$ is not finite here, we require a new definition for $E$ in the energy saturated gain and filtering (and a complimentary definition for $P$ in the power saturated loss). We investigate two different alternatives. Section 4.2 introduces a model where the energy is take as the drop in energy associated with the dip in the background as was the case in Chapter 2. Section 4.3 introduces a model where the system energy is used but the domain is taken to finite. In section 4.4 we investigate each model on a finite domain with periodic boundary conditions, and in section 4.5 we verify our numerical results with the perturbation methods developed in Chapter 2.

### 4.1 Development of mode-locking for dark solitons

Laser frequency stabilization via mode-locked (ML) lasers has become an indispensable tool in many research activities. Advances in optical frequency standards have resulted in the development of precise frequency measurement capability in the visible and near-infrared spectral regions. Although the potential for using ML lasers in optical frequency synthesis was recognized early, the available lasers did not provide the properties necessary for fulfilling this potential until
recently [26]. The recent explosion of measurements based on ML lasers has been largely due
to the development of the Kerr-lens mode-locked Ti:Sapphire laser and its capability to generate
sufficiently short pulses. Alternatively, amplified pulses of a similar duration are created through
the compression of pulses via self-phase modulation in a hollow core fiber [74].

Dark solitons are intensity dips on a constant background with a phase jump across their
intensity minimum. Since their discovery [37], dark solitons have attracted considerable attention,
especially in the fields of nonlinear optics [50] and Bose-Einstein condensates [32]. In addition,
the development of dark pulse laser open the door to new applications in the fields of optical
communications [50] and femtosecond lasers [71]. In the non-mode-locking regime, the first
train of dark solitons was successfully achieved in an all normal dispersion Erbium-doped fiber
laser with an in-cavity polarizer. In ML lasers they have been difficult to generate with the first
experimental observation only recently reported [74, 28]. Apart from the bright pulse emission,
it was observed that, under strong continuous wave emission, appropriate pump strength and
negative cavity feedback, a fiber laser can also emit single or multiple dark pulses [74].

4.2 The dark energy (DE) model

The first model is based on dark energy, i.e. the drop in energy associated with the dip in the
background where $E(z) = \int_{-\infty}^{\infty} (|\psi_\infty|^2 - |\psi|^2) \, dt$ is the dark energy and $P(z, t) = |\psi_\infty|^2 - |\psi|^2$ is the
instantaneous power. This definition of the pulse energy, although unconventional, corresponds to
the physical properties of the system. Indeed, as the peak-to-background intensity ratio can achieve
large values in bright pulses, so can the focused-to-unfocused vacuum ratio in dark pulses. As
such, dark pulses can be thought of as focusing the vacuum [66]. The definition of power follows
consistently from $E = \int_{-\infty}^{\infty} P \, dt$. This expression for the energy also agrees with that given in
equation (2.4.39b). We refer to this as the dark energy (DE) model.

To study the dynamics and mode-locking capabilities of our model, we integrate equation
(4) with a fourth-order Runge-Kutta method, with initial profile $\psi(0, t) = \tanh t$, though others,
with a $\pi$-phase jump, can be used as well. The gain parameter $g$ is varied, while $E_{\text{sat}} = P_{\text{sat}} = 1\,$
and \( \tau = l = 0.1 \). In the top panel of figure 4.1 we show the evolution of the background amplitude, which sets the dark soliton amplitude, for different values of the gain parameter \( g \). Locking onto

![Graph of |\psi| vs z for different values of g](image)

Figure 4.1: Top panel: evolution of the pulse background of an arbitrary initial profile under the PES equation with different values of gain. Bottom panel: complete evolution of the initial profile for \( g = 0.5 \).

stable dark solitons is only achieved when the gain term is sufficiently strong, i.e. the parameter \( g \)
is large enough to counterbalance the losses. Thus, for \( g = 0.1 \) (see dashed line in the top panel of figure 4.1), the dark soliton decays quickly due to excessive loss. On the other hand, for \( g = 0.2 \), the pulse initially decreases, but then locks to a specific amplitude and width. Finally, for \( g = 0.5, 0.7, 1.0 \), a stable evolution characterized by an increase of the soliton amplitude is obtained almost instantaneously; see bottom panel of figure 4.1.

To determine the evolution of the background we use equations (2.1.4) to get

\[
\frac{d\psi_\infty}{dz} = \frac{g}{1 + 2|\psi_\infty|/E_{\text{sat}}} \psi_\infty - l\psi_\infty. \tag{4.2.1}
\]

Here, an approximate solution of equation (4) in the form \(|\psi(z, t)| = |\psi_\infty \tanh(\psi_\infty t)|\) is assumed, which gives \( E = 2|\psi_\infty| \). Equation (4.2.1) also describes the evolution of \( \psi_\infty \), as depicted in the top panel of figure 4.1, and illustrates the importance of energy saturation. Indeed, if we consider the so-called similariton supporting equation (NLS with linear gain) corresponding to \( \tau = l = 0 \) [21], with energy saturation absent \((2|\psi_\infty|/E_{\text{sat}} = 0)\), the pulse amplitude grows exponentially at a rate defined by the gain \( g \). In some models, loss is introduced in the form of fast saturable power absorbers placed periodically [44]. It has been shown that the lumped and distributive models yield similar results [8].

A stable equilibrium exists for equation (4.2.1) and can be found setting \( d\psi_\infty/dz = 0 \), namely,

\[
|\psi_\infty| = \frac{E_{\text{sat}}}{2} \left( \frac{g}{l} - 1 \right). \tag{4.2.2}
\]

This is the resulting background amplitude of the dark soliton and agrees with direct numerical simulation, as seen in figure 4.2. Thus, dark solitons tend to an equilibrium (mode-lock) with constant energy and background.

The above findings agree with the experiment of Ref. [74], where the mode-locked dark pulses were identified as NLS dark solitons. In particular, they are black solitons (stationary kinks) characterized by a \( \pi \)-phase jump across the soliton notch and they can be described analytically by a hyperbolic tangent profile, i.e., \( \psi(z, t) = \psi_\infty \tanh(\psi_\infty t) \). In figure 4.2 we plot the black solitons of the DE model and NLS equations (for the same value of \( \psi_\infty \), as determined from equation
Figure 4.2: Black solitons of the PES and NLS equations for different values of $g$. 
for different values of $g$. The amplitudes match so closely that they are indistinguishable in the figure, meaning that the combined perturbation of gain, filtering and loss in the DE model underlies the mode-locking mechanism: its effect is to mode-lock to a black soliton of the pure NLS equation with the appropriate background, equation (4.2.2). On the other hand, mode-locking to grey solitons, namely moving dark pulses with a phase-jump less than $\pi$ at their (non-zero) intensity minima, are not found in this model as they have insufficient dark-pulse energy. This too agrees with reference [74] where grey solitons were not observed.

### 4.3 The average power model

Our second model is based on averaged power, i.e. $E = \frac{1}{T} \int_{-T/2}^{T/2} |u(z, t)|^2 \, dt$ and $T$ is the averaged time through the cavity. We term this the average power (AP) model. In this case, looking for stationary solutions of equation (4.2.2) yields

$$|\psi_{\infty}|^2 = \frac{g - l}{\frac{1}{T} \int_{-T/2}^{T/2} |\psi|^2 \, dt} \approx \frac{1}{T} |\psi_{\infty}|^2 T = |\psi_{\infty}|^2,$$

for the background where, for sufficiently large $T$, $E = \frac{1}{T} \int_{-T/2}^{T/2} |\psi|^2 \, dt \approx \frac{1}{T} |\psi_{\infty}|^2 T = |\psi_{\infty}|^2$. Equation (4.3.3) is a more restrictive condition for mode-locking which can also be attributed to the experimental difficulties of the problem. Not only must the gain be greater than the loss as in equation (4.2.2), but the gain must saturate faster with respect to increases in background intensity. This definition also results in more complicated dynamics and the development of a shelf on the soliton background, which is more pronounced as compared to the previous case. In fact, as shown earlier in Chapter 2, shelves occur naturally in the perturbation theory for dark solitons. We illustrate the resulting shelf in figure 4.3, where now $g = 0.2$, $l = 0.1$, $E_{\text{sat}} = 1$ and $P_{\text{sat}} = 5$. Due to the noisy background this feature maybe difficult to observe in an experiment. However, the shelf also affects the phase of the resulting pulse (see inset in figure 4.3).
Figure 4.3: The development of a shelf in the solution to the PES equation (solid line) as compared to the solution of the NLS equation (dashed line).
4.4 Periodic Domains

Though it is far less prominent in the DE model (see figure 4.11), both the DE model and the AP model exhibit shelves that expand from the dark pulse. In a physical system, where the domain is both finite and periodic, this means that the shelves will begin interacting. In this section we consider multiple dark solitons on a finite domain, $[-T/2, T/2]$, with periodic boundary conditions. Here a single soliton is insufficient to satisfy the periodic boundary conditions, since the phase change across the pulse is between 0 (no soliton) and $\pi$ (a black soliton) and the total phase change must be a multiple of $2\pi$. Initial conditions for a string of $N$ dark solitons can be approximated by

$$\psi(z, t) \approx u_\infty \prod_{k=1}^{N} \psi_k(z, t)$$

where $u_\infty$ is the background height, $-2\pi \leq x_k \leq 2\pi$ is the phase change across the $k^{th}$ soliton and $t_k$ is the center of the $k^{th}$ soliton. To satisfy the periodic boundary conditions we require

$$\sum_{k=1}^{N} \alpha_k = m\pi \text{ for some } m \in \mathbb{Z}. \quad (4.4.6)$$

The simplest ways this condition may be satisfied are two solitons with opposite phase change (including the degenerate case of two black solitons) and a string of three solitons whose phase changes have the same sign and add up to $2\pi$. This is illustrated in figures 4.4 and 4.5. We begin by considering initial conditions consisting of two black solitons.

In the DE model ($E = \int_{-\infty}^{\infty} (u_\infty^2 - |u|^2)dt$) the evolution settles on an equilibrium solution which is close to two black solitons. A slight difference can be seen in the phase which exhibits curvature between the solitons. A slight difference can be seen in the phase which exhibits curvature between the solitons. A slight difference can be seen in the phase which exhibits curvature between the solitons (for unperturbed solitons the phase would be constant as seen in figure 4.4). An example is shown in figure 4.6 for parameters $g = 0.5$, $l = \tau = 0.1$ and $E_{sat} = P_{sat} = 1$. This equilibrium is not found under slight variations in the $t_k$'s so that the solitons are not equally spaced, and slight variations in the phase change over the solitons (giving grey solitons for initial conditions instead of black). When this equilibrium solution is slightly perturbed we find numerically that the solution is unstable and that the pulses eventually vanish.
Figure 4.4: Two solitons with opposite phase change.

Figure 4.5: A string of solitons whose phase changes add up to $2\pi$. 
Figure 4.6: The evolution of black solitons in the dark energy model with periodic boundary conditions and the phase at $z = 300$.

Figure 4.7: Blow up occurs when the equilibrium of the dark energy model is perturbed.
and the background magnitude grows exponentially. In figure 4.7 the equilibrium was perturbed by random noise on the order of $10^{-4}$; by $z = 150$ the solution has moved noticeably away from the black soliton equilibrium and by $z = 300$ the background is growing exponentially.

In the AP model no equilibrium emerges, the moving shelves emanating from the two solitons continue to interact with each other resulting in continually increasing fluctuations in the background height. The shelf fronts move at a constant speed and form a diamond-like grid structure; e.g. see the contour plot in figure 4.8. Though fluctuations in the background increase as $z$ increases, the average height remains close to 1 and the black solitons persist. (In the numerics: $g = 0.18$, $l = \tau = 0.1$, $E_{\text{sat}} = 1$, and $P_{\text{sat}} = 10$.) In figures 4.9 and 4.10 we consider an example consisting of three initial solitons with $\alpha_i = \pi/3$ for $i = 1, 2, 3$, the same grid structure appears in the contour plot. While the background height oscillates around 1 the soliton troughs decrease and the phase changes across the solitons increase as $z$ increases. This illustrates the general trend for grey solitons to become black solitons in the averaged power model. The grid structure still appears in the contour plot of figure 4.9, furthermore the insert shows that the phase change over the solitons now longer needs to sum to a multiple of $2\pi$ since there are now variations in the phase between the solitons.

### 4.5 Dark Mode-locking

We now use tools for analyzing the effect of small perturbations on dark soliton solutions of the NLS equation with normal dispersion developed in chapter 2 to discuss out two models for the study of dark pulses in mode-locked lasers. To put this problem in the notation of chapter 2 let $\psi = ue^{i \int_0^z \psi_\infty ^2 ds}$ and

$$
\epsilon F[u] = i \left[ \frac{g}{1 + E/E_{\text{sat}}} + \frac{\tau u_{tt}}{1 + E/E_{\text{sat}}} - \frac{lu}{1 + P/P_{\text{sat}}} \right].
$$

(4.5.7)

where $E = \int_{-\infty}^{\infty} P(t,z)dt$ and $P$ is the power which is linearly related to $|u|^2$. Note that the small parameter $\epsilon$ is implicitly contained in the right hand side of equation (4.5.7); the term $\epsilon F[u]$ is small.
Figure 4.8: The evolution of black solitons in the averaged power model with periodic boundary conditions and the magnitude at $z = 300$.

Figure 4.9: The evolution of three grey solitons in the averaged power model with periodic boundary conditions and the phase at $z = 300$. 
Figure 4.10: The background height (measured at the boundary) and soliton trough height for evolution of three grey solitons. The dips in the background occur as the solitons pass across the boundary.
Here the energy appears explicitly in the perturbation term, so it is convenient to treat the energy as its own parameter. From equations (2.7.73c) and (2.5.41) we have

\[
\frac{d}{dz} E^{(0)} = 2\epsilon \text{Im} \int_{-\infty}^{\infty} (F[u_\infty]u_\infty - F[u_0^*]u_0^*) \, dt \tag{4.5.8}
\]

\[
\frac{d}{dz} \sigma_0 = \left( B \frac{e^{E^{(0)}}}{2} \right) / u_\infty \tag{4.5.9}
\]

where \( E^{(0)} \) is the first order approximation for the energy, i.e. \( E = E^{(0)} + \epsilon E^{(1)} + O(\epsilon^2) \). This is consistent with the expression for \( E^{(0)} \) in equation (2.7.69); the energy at \( O(1) \) has contributions from both the core soliton and the shelf. We have also used the fact that \( \text{Re}[F[u_\infty]] = 0 \) for this perturbation.

We first analyze the DE model. From equations (2.7.73), the evolution of the key parameters is given by

\[
\frac{d}{dz} u_\infty = \frac{g}{1 + E^{(0)}/E_{\text{sat}}} u_\infty - l u_\infty \tag{4.5.10a}
\]

\[
\frac{d}{dz} A = \frac{g}{1 + E^{(0)}/E_{\text{sat}}} A - \frac{l P_{\text{sat}}}{B \sqrt{B^2 + P_{\text{sat}}}} \text{ArcTanh} \left( \frac{B}{\sqrt{B^2 + P_{\text{sat}}}} \right) A \tag{4.5.10b}
\]

\[
\frac{d}{dz} E^{(0)} = \frac{4g}{1 + E^{(0)}/E_{\text{sat}}} B + \frac{2\tau/3}{1 + E^{(0)}/E_{\text{sat}}} B^3 \tag{4.5.10c}
\]

\[
-4l \frac{u_\infty^2 + P_{\text{sat}}}{\sqrt{B^2 + P_{\text{sat}}}} \text{ArcTanh} \left( \frac{B}{\sqrt{B^2 + P_{\text{sat}}}} \right)
\]

expressed here in terms of \( z \) since the small parameter \( \epsilon \) is implicitly contained in the RHS. From \( u_\infty, A, E^{(0)} \) then \( \sigma_0 \) follows from (4.5.9) and then the other soliton and shelf parameters may be calculated from equations (2.7.73).

It follows that the equilibrium background for black solitons \( (B = u_\infty, A = 0) \) using the assumption \( E = 2u_\infty \)

\[
u_\infty = \frac{E_{\text{sat}}}{2} \left( \frac{g}{l} - 1 \right) \tag{4.5.11}
\]

does not satisfy equation (4.5.10c) in general. This leads to a discrepancy between the soliton energy and the total dark energy and indicates that a shelf will develop around the soliton. The shelf height can be calculated as part of the perturbation analysis. With this perturbation the shelf generally has small size. Corresponding to typical parameters in figure 4.11 we see the shelf is
O\(10^{-3}\), too small to be seen on a plot of the soliton. So we zoom in to make our comparison between numerics and asymptotic prediction. Though the shelf may seem small, when considered on a periodic computational domain the interaction of shelves can eventually have a noticeable effect on the dark energy. In figure 4.13 a comparison of numerics to the asymptotic results is given, as well as the effect of shelf interactions.

Equilibrium solutions of equations 4.5.10 with \(A = 0\) can be shown to be unstable. In fact any small variation from a black pulse eventually degenerates into a continuous wave with dark energy \(E = 0\), at which point the equation for the background becomes

\[
\frac{d}{dz}u_\infty = (g - 1)u_\infty
\]  

(4.5.12)

which implies exponential growth. This is observed in figure 4.12. In the case of periodic boundary conditions this variation may come from shelf interaction.

In the AP model where \(E = \int_{-T/2}^{T/2} P(z, t)dt/T\) where \(P = |u(z, t)|^2\) we define the dark energy as \(E_D(z) = T(u_\infty^2 - E)\). For \(T \gg 1\) we may approximate \(E = u_\infty^2 - \frac{1}{4}E_D \approx u_\infty^2\) with small error for large \(T\). The evolution equations the key parameters (from (2.7.73)) are

\[
\frac{d}{dz}u_\infty = \frac{g}{1 + u_\infty^2/E_{sat}}u_\infty - \frac{l}{1 + u_\infty^2/P_{sat}}u_\infty
\]  

(4.5.13a)

\[
\frac{d}{dz}A = \frac{g}{1 + u_\infty^2/E_{sat}}A - \frac{lP_{sat}}{\sqrt{A^2 + P_{sat}}} \text{Arctan} \left( \frac{B}{\sqrt{A^2 + P_{sat}}} \right) A
\]  

(4.5.13b)

\[
\frac{d}{dz}E_{D(0)} = \frac{4g}{1 + u_\infty^2/E_{sat}}B + \frac{8\tau/3}{1 + u_\infty^2/E_{sat}}B^3
\]  

(4.5.13c)

\[-4l \left( \frac{P_{sat}}{u_\infty^2} + 1 \right) \frac{1}{\sqrt{A^2 + P_{sat}}} \text{Arctan} \left( \frac{B}{\sqrt{A^2 + P_{sat}}} \right).
\]

These equations are compared to numerical results in figure 4.14.

Equations (4.5.13) have an equilibrium at

\[
u_\infty = \left( \frac{g - l}{l} \frac{1}{E_{sat} - P_{sat}} \right)^{1/2}
\]  

(4.5.14a)

\[A = 0\]

(4.5.14b)
Figure 4.11: A black soliton with predicted equilibrium background is given as initial conditions and a small shelf develops around the soliton. The inset shows the magnitude of the soliton; the scale is too large for the shelf to be seen. Here $g = 0.3$, $\tau = 0.05$, $l = .1$ and $E_{\text{sat}} = P_{\text{sat}} = 1$.

Figure 4.12: A grey soliton evolving in the under the dark energy model. Here $g = 0.3$, $\tau = 0.05$, $l = .1$ and $E_{\text{sat}} = P_{\text{sat}} = 1$. 
Figure 4.13: Dark energy and background height evolution compared to asymptotic results for the dark energy model. The vertical line indicates when the shelves begin interacting. Here $\eta = 0.5$, $\tau = 0.1$, $l = .1$ and $E_{\text{sat}} = p_{\text{sat}} = 1$. 
when \( \text{sign}(g - l) = \text{sign} \left( \frac{l}{E_{\text{sat}}} - \frac{g}{P_{\text{sat}}} \right) \). This equilibrium is stable for a wide range of parameters.

Figure 4.15 shows a typical phase portrait with a stable black soliton solution. Since the perturbation is over a finite domain it is natural to consider the problem on a periodic domain as in the case of a fiber ring laser. In this case the shelves continue to interact with each other. In figure 4.16 this interaction is shown to create a diamond-like grid structure in the contour plot. Despite the interaction of the shelves we can still see evolution towards black type solitons. Once the shelves begin interacting the perturbation theory is no longer valid, however the pulses still maintain the characteristics of a black soliton with a trough near near zero and a phase change near \( \pi \). This is illustrated in figure 4.17 where the value at the boundary and soliton trough are plotted for the evolution depicted in figure 4.16.

### 4.6 Conclusions

In conclusion, we presented two versions of the PES equation for describing mode-locking of dark solitons in lasers. Much like their bright counterparts, these pulses (as well as multiple pulses) exist when sufficient gain is present in the system, which agrees with the experimental observations. The specific energy saturated gain and filtering, and power saturated loss, are crucial to the mode-locking mechanism. The resulting pulses are essentially modes of the unperturbed NLS equation with background amplitudes appropriately defined by the gain and loss parameters. The dark energy model exhibits mode-locking to an equilibrium solutions consisting of black solitons but these equilibria are weakly unstable; the average power model exhibits increasing fluctuations in the background but this does not disrupt black soliton pulses.
Figure 4.14: Background height and soliton trough compared to asymptotic results for the averaged power model. Here $g = 0.3$, $\tau = 0.05$, $l = 0.1$, $E_{sat} = 1$ and $P_{sat} = 10$.

Figure 4.15: The phase plane for $u_\infty$ and $A$ with a dotted red line to indicate the equilibrium for $u_\infty$. Here $g = 0.3$, $\tau = 0.05$, $l = 0.1$, $E_{sat} = 1$ and $P_{sat} = 10$. 
Figure 4.16: Two grey solitons with opposite phase change evolve in the averaged power model. Here \( g = 0.18, \tau = 0.1, l = 0.1, E_{\text{sat}} = 1 \) and \( P_{\text{sat}} = 10 \).

Figure 4.17: The value at the boundary and minimum value associated with figure 4.16. Here \( g = 0.18, \tau = 0.1, l = 0.1, E_{\text{sat}} = 1 \) and \( P_{\text{sat}} = 10 \).
Chapter 5

Bi-Solitons and Dispersion Management

In this chapter we remark on some numerical results for the PES equation in the normal
dispersion and dispersion managed regimes including bi-soliton solutions. In section 5.1 we give
results for the normal regime. In sections 5.2 and 5.3 we derive the dispersion managed PES
equation and give some numerical results.

5.1 Normal higher order solitons

Next we briefly mention some results for the constant normal dispersive case: $d_0 = -1 < 0$.
As mentioned above, individual pulses of equation (3.0.1) in the normal regime exhibit strong chirp
and cannot be identified as the solutions of the unperturbed NLS equation. Indeed, the classical
NLS equation does not exhibit decaying bright soliton solutions in the normal regime. If we
begin with an initial gaussian, $\psi(0, t) = \exp(-t^2)$, the evolution mode-locks into a fundamental
soliton state; see figure 5.1 bottom two figures. These figures clearly exhibit the mode-locking
evolution and the significant chirp of the pulse. On the other hand we can obtain a higher-order
anti-symmetric soliton, i.e. an anti-symmetric bi-soliton, one which has its peaks amplitudes
differing in phase by $\pi$. Such a state can be obtained if we start with an initial state of the form
$\psi(0, t) = t \exp(-t^2)$ (i.e. a Gauss-Hermite polynomial). The evolution results in an anti-symmetric
bi-soliton and is shown in figure 5.2 along with a comparison to the profile of the single soliton.
This is a true bound state. Furthermore, the results of our study finding anti-symmetric solitons
in the normal regime are consistent with recent experimental observations [24].
Figure 5.1: Mode-locking evolution for a single soliton evolving from a unit gaussian initial state in normal regimes; the state at $z = 300$ is depicted below. The phase-chirp is shown in the inset.
Figure 5.2: Evolution of the anti-symmetric soliton (top) and the anti-symmetric (bi-soliton) superimposed with the relative single soliton (bottom) at $z = 1000$. Here $g = 1.5$. 
It is interesting that the normal regime also exhibits higher soliton states when two general initial pulses (e.g. unit gaussians) are taken sufficiently far apart. The resulting pulses, shown in figure 5.3, individually have a similar shape to the single soliton of the normal regime with lower individual energies. These pulses if initially far enough apart can have independent chirps which may be out of phase by an arbitrary constant. We again find that \( d^* \approx 9\alpha \) is a good estimate for the required pulse separation just as in the constant anomalous dispersive case. These pulses are “effective bound states” in that after a long distance they separate very slowly (too slowly to measure).

![Figure 5.3: Symmetric two soliton state of the normal regime with in phase pulses. The phase structure is depicted in the inset. Here \( g = 1.5 \).](image)

Additional higher order 3, 4, ... soliton states can also be found. We will not go into further details here.
5.2 Dispersion managed systems

Standard dispersion managed (DM) solitons, like their constant dispersive counterparts, are obtained from the PES equation over a wide range of anomalous dispersion [10]. This is important since recent mode-locked laser experiments have been conducted in dispersion-managed regimes [44, 24, 62]. In correspondence with Ti:Sapphire DM laser systems we allow the dispersion to vary in z as well as introducing nonlinear management in the form of the function \( n(z) \) multiplying the nonlinear term. Equation (3.0.1), is used but we change the symbol of the envelope from \( \psi = \psi(z, t) \) to \( u = u(z, t) \) to distinguish the two cases (constant dispersion and DM, respectively). The effect of dispersion management is obtained by splitting the dispersion \( d(z) \) into two components \( d(z) = d_0 + \Delta(z/z_\alpha)/z_\alpha \) where \( 0 < z_\alpha << 1 \) is the dispersion-map period. Hence \( d(z) \) is large and periodic. If \( d(z) \) was only \( O(1) \) then the multi-scale averaging method employed below would result in the constant dispersive case discussed earlier. The path averaged dispersion is \( d_0 \) and \( \Delta(z/z_\alpha) \) is a rapidly varying function with zero average which we define as follows: \( \Delta(\zeta) = \{ -\Delta_1, 0 < \zeta < 1/2; \Delta_2, 1/2 < \zeta < 1 \} \). Here we consider the case of positive average dispersion \( (d_0 > 0) \). We define the map strength \( s = \Delta_1/2 \) which gives a measure of the variability of dispersion around the average. We take \( d_0 = 1, z_\alpha = 0.1 \) and vary \( s \). The effect of the nonlinear management is to turn the nonlinearity “off and on”, i.e. \( n(\zeta) = \{ 0, 0 < \zeta < 1/2; 1, 1/2 < \zeta < 1 \} \).

For example, in a Ti:Sapphire laser the nonlinearity is negligible in the anomalous regime where one has a prism pair that compensates for the normal dispersion in the crystal. The averaged equation is derived using the method of multiple scales and perturbation theory [5]. The variation in dispersion occurs on the short distance scale \( \zeta = z/z_\alpha \) and the pulse envelope evolves on the long scale \( Z = z \).

The method proceeds by expanding \( u \) in powers of \( z_\alpha \):

\[
u(\zeta, Z, t) = u^{(0)}(\zeta, Z, t) + z_\alpha u^{(1)}(\zeta, Z, t) + O(z_\alpha^2)
\]

and substituting this into equation (3.0.1) to obtain a series of equations by relating terms by
powers of $z^\alpha$. At $O(z^{-1})$

$$\imath \frac{\partial u^{(0)}}{\partial \zeta} + \frac{\Delta(\zeta)}{2} \frac{\partial^2 u^{(0)}}{\partial t^2} = 0$$

which can be solved using Fourier transforms to arrive at

$$\hat{u}^{(0)}(\zeta, Z, \omega) = \hat{U}_0(Z, \omega) \exp \left[-\frac{\imath \omega}{2} C(\zeta) \right]$$

where $C(\zeta) = \int_0^\zeta \Delta(\zeta') \, d\zeta'$, $\hat{U}_0(Z, \omega) = \hat{u}^{(0)}(\zeta = 0, Z, \omega)$, and the Fourier transform pair is defined as

$$\hat{u}(\omega) = \mathcal{F}\{u(t)\} = \int_{-\infty}^{\infty} u(t) e^{\imath \omega t} \, dt$$

$$u(t) = \mathcal{F}^{-1}\{\hat{u}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\omega) e^{-\imath \omega t} \, d\omega$$

Thus $\hat{u}_0$ separates into a slowly evolving envelope $\hat{U}_0$ and fast oscillations due to changes in the local dispersion. The equation for $\hat{U}_0$ is obtained by imposing secularity conditions on the $O(1)$ terms,

$$\imath \frac{\partial \hat{U}_0}{\partial Z} - \frac{\omega_0^2}{2} \hat{U}_0 + \int_0^1 \exp \left[-\frac{\imath \omega}{2} C(\zeta) \right] \left(\mathcal{F}\{|u^{(0)}|^2 u(0)\} - \mathcal{F}[F[u^{(0)\, 2} u(0)]]\right) d\zeta = 0, \quad (5.2.1)$$

This is the averaged, or mean-field equation which we solve to find the pulse dynamics. In the case where $F = 0$ this is known as the dispersion managed NLS or DMNLS equation [5].

The method of spectral renormalization [12] can be employed to find single mode-locked DM solitons for equation (5.2.1). Here we initially superimpose two such DM soliton pulses at varying peak separations with phase difference $\Delta \phi = 0$ and let them evolve to find two soliton “effective” bound states. As a criterion for an “effective” bound state, we require the the peak separation differ less than 0.05$\alpha$ after evolving 500 units in $z$. Typical examples with comparison of initial vs. final states are depicted in Figs. 5.4 and 5.5. Much like the single soliton case the individual pulses are well approximated by the solutions of the unperturbed DMNLS equation. We also note that due to the nonlocality of the equation, the individual pulses have a smaller energy than the single pulse for the same parameters, as was in the case of constant dispersion.
Figure 5.4: “Effective” bound state for two solitons in DM system for $s = 0.1$ (top) and $s = 1.0$ (bottom). Here $g = 0.6$. 
As is indicated in the figure, the minimum initial distance $d^* \approx 9\alpha$ no longer holds for the dispersion (and nonlinearly) managed PES equation (DMPES). In the nonlinear managed system we find that the critical distance $d^* \approx 7\alpha$ for $s = 0$, i.e. constant dispersion, and $d^*$ depends on the map strength $s$ for $s > 0$. Since this change is much more dramatic between $s = 0$ and $s = 1$ than $s = 1$ and $s = 10$ we investigate this region more thoroughly. In figure 5.5 the value of $\Delta \xi / \alpha$ found for the pulses to be “effectively” bound are plotted for varying map strengths. The general trend is for the needed separation to decrease as $s$ increases, however, as can be seen between $s = 0.1$ and $s = 1.0$ where more $s$ values were tested this is not a monotonically decreasing process.

For more information on the normal ($d_0 < 0$) DM case we refer the reader to reference [9].

5.3 Anti-symmetric bi-solitons for DM system

We now superimpose two net anomalous DM solitons with a $\pi$ phase difference. For $s < s_\ast \approx 0.25$ pulses repel each other as was the case for constant anomalous dispersion; we note that for the above values of the map strength both of the local dispersions being used ($d_0 + \Delta_1/z_\alpha$ and $d_0 + \Delta_2/z_\alpha$) are in the anomalous regime. For $s > s_\ast$, pulses which are taken close enough together, i.e. the distance between peak values, $d < d_\ast \approx 2.5$, are found to lock into a bi-soliton state. Examples are given in figure 5.6. Pulses taken further apart: $d > d_\ast$ are found to repel. These anti-symmetric bi-solitons are the mode-locked (due to gain-loss) analog of what was obtained in the case of pure DM systems without gain loss [60, 6].

5.4 Conclusions

To conclude, we investigated the PES equation numerically for the normal dispersion and dispersion management and found a large class of localized solutions including: mode-locked solitons in both the constant anomalous and normal regimes, high-order solitons in the constant anomalous regime and anti-symmetric bi-solitons in the constant normal regime. These results are consistent with experimental observations of higher-order solitons in the anomalous and bi-
Figure 5.5: “Effective” bound state for two solitons in DM system for $s = 10$ (top). Numerically found relation between the map strength $s$ and the minimum initial distance $\Delta \xi_0/\alpha$ for no interactions to be seen (bottom) after $z = 500$ units. Here $g = 0.6$. 
Figure 5.6: Bi-Soltion states found for $s = 0.3, 0.7$ and 5.0. Here $g = 0.6$. The relative phases and the $\pi$ jumps at the origin are shown in the insets.
solitons in the normal dispersive regimes. The dispersion and nonlinear managed system was also investigated. Here in the averaged anomalous regime single and higher-order soliton pulses were obtained, including anti-symmetric bi-solitons in the net anomalous regime. For the constant dispersion case, it is found that when individual pulses are initially separated by $d^* \approx 9\alpha$ where $\alpha$ is the width of the individual pulse the result is a soliton string. For the DM system the results indicate that the high-order soliton strings in the DM case can exist in much closer proximity to each other.


