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MODAL AND COUPLING CHARACTERISTICS OF INHOMOGENEOUS DIELECTRIC SLAB WAVEGUIDES - Part I: LOSSLESS SLABS

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ABSTRACT

A method for the calculation of the propagation constants for the surface wave modes of an inhomogeneous slab waveguide is presented. The method is based upon the invariant imbedding concept, and utilizes the resonances of an equivalent transverse non-uniform transmission line representation to find the modes. A Ricatti equation for the transverse wave impedance (or admittance) is obtained and numerically integrated across the slab, and the resonance is found by a modification of Newton's method. In this way, convergence to any desired accuracy can be obtained. Results are presented for several different permittivity profiles.

It is found that the cutoff frequency for the lowest order modes, which appears in an asymmetric structure, depends quite sensitively upon the asymmetry when the structure is nearly symmetric. In addition, as a waveguide structure is varied from an abrupt junction profile to a gradual permittivity change, intersections of the group velocity curves for different modes at relatively low frequencies disappear. Finally, the well-known property of the so-called "lens-like medium" of having nearly equal group velocities in all modes is verified for a more realistic, truncated version at higher frequencies, when this version is a good approximation to the ideal medium.
Chapter 1: Introduction

Recent advances in laser technology have aroused interest in the use of optical frequencies for communication systems. The prospect of greatly increased bandwidths combined with the advantage of small physical size for the components makes this frequency range attractive for future use in high-data-rate transmission. The present capability in optical fiber fabrication technology is such that loss figures of 4 dB/km are obtainable, with theoretical limits that are much lower [1]. This has made feasible long-distance communication over optical fibers, and the consequent need for circuit-type components to process and direct the signals has led to considerable research and growth in the field of integrated optics.

In the areas both of fiber optics and of integrated optics, some interest has been shown in the use of various inhomogeneous dielectric surface wave structures. Specifically, it has been recognized that certain continuous transverse permittivity distributions can significantly reduce delay distortion below what is attainable in abrupt junction profiles [2]. In curved guides with inhomogeneous cross-section, the radiation loss due to the bending of the guide depends rather sensitively upon the dielectric profile; the attenuation seems to be increased over the abrupt-junction value when the profile is made gradual within the guide [3], while considerable reduction is possible if the gradation is made outside
the waveguide [4]. An ability to control this radiation would be highly desirable for such applications as coupling between guides. In any case, inhomogeneous profiles are almost inevitable as a result of most fabrication techniques [5,6]; in a diffusion process, for example, the permittivity profile is approximately in the form of an error function, the exact shape of which is determined by factors such as temperature, electron mobility, etc.

In the case of a single dielectric waveguide, analytical solutions are possible only for a few specific permittivity profiles in simple geometries; for example, the parabolic and hyperbolic secant profiles [7,8]. Asymptotic methods such as the WKB approximation have been applied to the high order modes in a weakly inhomogeneous multimode guide [9], and to the case where the permittivity variation is small for distances on the order of a wavelength [10, pp. 193-215]. In general, however, the problem can only be solved numerically.

In this work, the properties of an inhomogeneous dielectric slab will be studied as a simple model of more realistic inhomogeneous structures. The slab considered is diagrammed in Fig. 1.1. The boundary regions 1 and 2 are homogeneous and characterized by the constant permittivities $\varepsilon_1$ (for $x < 0$) and $\varepsilon_2(x > d)$ respectively. The slab, which lies between $0 \leq x \leq d$, has a dielectric constant which is an arbitrary function, $\varepsilon(x)$, of $x$ on this interval, subject only to the restrictions that $\varepsilon(x) \geq \varepsilon_0$ everywhere, and that only a finite number of finite discontinuities be allowed, so that no
Figure 1.1: Geometry of the Problem
Pathological situations will arise. All media are assumed
to have permeability $\mu_0$, and to be uniform and of infinite
extent in the plane perpendicular to the x-axis. The direction
of propagation is identified as the z-axis, so that $\frac{\partial}{\partial y} = 0$
for all quantities involved.

In a sourceless region where $\varepsilon(x)$ is continuous, the
fields must satisfy Maxwell's equation; therefore, assuming
and suppressing a factor $\exp[j(\omega t - \beta z)]$ for field dependence
on $z$ and $t$, we have two independent sets of equations.

For a TE-wave,

$$\frac{d^2}{dx^2} E_y + (k^2 - \beta^2) E_y = 0 \quad (1.1)$$

$$H_x = -\frac{\beta}{\omega \mu_0} E_y \quad (1.2)$$

$$H_z = -\frac{1}{j\omega \mu_0} \frac{d}{dx} E_y \quad (1.3)$$

and for a TM wave,

$$\frac{d}{dx}[\frac{1}{\varepsilon(x)} \frac{d}{dx} H_y] + \frac{k^2 - \beta^2}{\varepsilon(x)} H_y = 0 \quad (1.4)$$

$$E_x = \frac{\beta}{\omega \varepsilon(x)} H_y \quad (1.5)$$

$$E_z = \frac{1}{j\omega \varepsilon(x)} \frac{d}{dx} H_y \quad (1.6)$$

where $k^2 = \omega^2 \mu_0 \varepsilon$. At points where $\varepsilon(x)$ is discontinuous,
the tangential fields ($E_y$ and $H_z$ or $H_y$ and $E_z$) are required
to be continuous.

The approach most often taken to this problem [11-13] is
Figure 1.2: Piecewise Constant Approximation to a Continuous Profile.
number of terms retained in the infinite determinant; these approaches become cumbersome at best. Indeed, there does not exist a satisfactory criterion for determining the number or size of layers, or the number of retained determinant entries, required to obtain a specified accuracy; convergence may be tested only by trial and error with a consequent increase in computer time requirements.

In this work, a different method, based upon the invariant imbedding principle and utilizing the transverse impedance concept, is developed to analyze the modal characteristics of dielectric slabs with arbitrary permittivity profiles. A first-order Ricatti differential equation for the impedance (or admittance) is formulated and numerically integrated across the slab using a fourth-order Runge-Kutta method with error estimation. The value of $\beta$ is then adjusted to satisfy the transverse resonance condition for any specific mode. It appears that this approach not only is rapidly convergent and highly accurate but can also provide useful insight into the design of dielectric waveguides. In Chapter 2, the method will be derived for a lossless slab, the equivalent transverse representation given, and the numerical results discussed.
Chapter 2: Lossless Inhomogeneous Slabs

2.1 Formulation of the modal characteristic equation

The problem outlined in Chapter 1 will now be studied in the case of a lossless slab. We shall be interested only in those modes which may exist independently and carry only a finite amount of power near or within the surface of the slab, i.e., the surface wave modes. Although the method to be developed may, with certain variations, be applied to the continuous mode spectrum of the guide, where the problem is not an eigenvalue problem but simply a two-point boundary problem, this will not be done here. A full discussion of the discrete and continuous mode spectra of an inhomogeneous guide is beyond the scope of this work and may be found in [15].

We consider the surface wave modes, therefore, as part of a complete set of mode functions over which we may expand an arbitrary field. We thus consider that the amplitude of the mode is arbitrary, though non-zero, and normalized when used in such an expansion. In addition, only forward traveling modes will be considered. As mentioned in Chapter 1, the modes fall into either the TE or TM classification since $E_y$ or $H_y$ may be set equal to zero independently. Because of the formal similarity of equations (1.1) - (1.3) and equations (1.4) - (1.6), a single notation will be introduced to cover both cases. We first introduce the normalized quantities

$$\alpha = \beta / k_0 \quad \quad u = k_0 x$$
where \( k_0 = \omega \sqrt{\mu_0 \varepsilon_0} \), and let \( \varepsilon_r = \varepsilon(x)/\varepsilon_0 \) to get

\[
\frac{d}{du} \left( \frac{1}{K(u)} \right) \frac{d}{du} f(u) + \frac{\gamma^2(u; \alpha)}{K(u)} f(u) = 0
\]

(2.1.1)

where \( \gamma^2(u; \alpha) = \varepsilon_r - \alpha^2 \) and \( K(u) \) and \( f(u) \) are given in Table 2.1 for both TE and TM modes.

Now calling the boundary points \( u_1 = 0 \) and \( u_2 = k_0 d \), we can express the solutions of (2.1.1) in regions (1) and (2) as

\[
f_1 = A_1 \exp[\Gamma_1(u-u_1)] \quad u < u_1
\]

\[
f_2 = A_2 \exp[-\Gamma_2(u-u_2)] \quad u > u_2
\]

where

\[
\Gamma_i = +\sqrt{\alpha^2 - \varepsilon_r^i} > 0 \quad i = 1, 2
\]

and the plus sign is used in order to satisfy the finite energy condition as \( |u| \to \infty \). \( \Gamma_i \) must always be real, otherwise there is a component of travelling wave in the \( u \)-direction and the energy condition would again be violated. At \( u_1 \) and \( u_2 \) (and for that matter at any point where \( \varepsilon_r \) is discontinuous) the appropriate pair of tangential field components must be continuous. For either type of mode, this is easily seen to be equivalent to requiring the continuity of \( f(u) \) and \( f'(u)/K(u) \) at such points (see Table 2-1); here the prime denotes differentiation with respect to \( u \). At the boundary points \( u_1 \) and \( u_2 \), in particular, these requirements give

\[
f'(u_i) = r_i \Gamma_i f(u_i); \quad i = 1, 2
\]

(2.1.2)
Table 2-1
Parameter Definitions for TE and TM Waves

<table>
<thead>
<tr>
<th>Symbol</th>
<th>TE</th>
<th>TM</th>
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<tr>
<td>( f(u) )</td>
<td>( E_y )</td>
<td>( H_y )</td>
</tr>
<tr>
<td>( f'(u) )</td>
<td>(-j\eta_0 H_z)</td>
<td>( j\varepsilon_r E_z/\eta_0 )</td>
</tr>
<tr>
<td>( K(u) )</td>
<td>( 1 )</td>
<td>( \varepsilon_r )</td>
</tr>
<tr>
<td>( r_1 )</td>
<td>( 1 )</td>
<td>( \varepsilon_r(u_1)/\varepsilon_r )</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>(-1 )</td>
<td>(-\varepsilon_r(u_2)/\varepsilon_r )</td>
</tr>
<tr>
<td>( R(u) )</td>
<td>( jE_y/(\eta_0 H_z) )</td>
<td>(-j\eta_0 H_y/E_z )</td>
</tr>
<tr>
<td>( S(u) )</td>
<td>(-j\eta_0 H_z/E_y )</td>
<td>( jE_z/(\eta_0 H_y) )</td>
</tr>
</tbody>
</table>

as boundary conditions for equation (2.1.1), where the \( r_i \) are given in Table 2-1. The pair of equations (2.1.1)-(2.1.2) constitutes a two-point boundary/eigenvalue problem in the normalized propagation constant \( \alpha \). Any discontinuities of \( \varepsilon_r \) within the interval \([u_1, u_2]\) may be thought of as imposing additional, internal boundary conditions; however, in the method to be developed, these conditions will be shown to be satisfied in a trivial manner.

We now make use of a polar coordinate transformation due to Prüfer (see [16] for example) by defining two new functions: \( \theta(u) \), the phase function; and \( \rho(u) \), the amplitude function, where

\[
\begin{align*}
  f(u) &= \rho(u) \sin \theta(u) \\
  f'(u)/K(u) &= \rho(u) \cos \theta(u)
\end{align*}
\]  

(2.1.3)
If we require $\rho > 0$, then it can be shown [16] that $\rho$ is uniquely defined as

$$\rho(u) = \left\{ [f(u)]^2 + \left[ \frac{f'(u)}{K(u)} \right]^2 \right\}^{\frac{1}{2}} \quad (2.1.4)$$

and does not vanish for any non-trivial solution $f(u)$ of (2.1.1). Similarly, $\theta$ is uniquely defined by

$$\tan \theta(u) = \frac{K(u)f(u)}{f'(u)} \quad (2.1.5)$$

if we require that $0 \leq \theta(u) \leq \pi$ at the boundary point $u = u_1$ and that $\theta(u)$ be continuous wherever $f(u)$ and $f'(u)/K(u)$ are continuous (in this case, everywhere). It is thus seen that the phase function contains all the information about the zeroes of $f(u)$ and $f'(u)/K(u)$ since $\rho$ never vanishes.

Substituting (2.1.3) into (2.1.1) and making use of (2.1.5) we obtain

$$\rho'(u) = \rho(u) \left\{ \frac{\sin 2\theta(u)}{2} \left[ K(u) - \frac{\gamma^2(u;\alpha)}{K(u)} \right] \right\} \quad (2.1.6)$$

$$\theta'(u) = K(u) \cos^2 \theta(u) + \frac{\gamma^2(u;\alpha)}{K(u)} \sin^2 \theta(u) \quad (2.1.7)$$

and the only boundary conditions we obtain from (2.1.2) are

$$\theta(u_1) = \arccot(q_1 \Gamma_1); \quad 0 < \theta(u_1) < \frac{\pi}{2} \quad (2.1.8a)$$

$$\theta(u_2) = \arccot(q_2 \Gamma_2) + p\pi; \quad \frac{\pi}{2} \leq \arccot(q_2 \Gamma_2) < \pi \quad (2.1.8b)$$

where $q_i = r_i/K(u_i); \quad i = 1, 2$, and $p$ is (as yet) an arbitrary integer. The restrictions on the ranges of the inverse cotangents, in addition to what was assumed for the uniqueness of $\theta$, come from the requirements that $\Gamma_1$ and $\Gamma_2$ be positive.
If we now denote by \( \theta(u;\alpha) \) the particular solution of (2.1.7) subject to the initial condition (2.1.8a), the boundary condition (2.1.8b) enables us to construct the modal characteristic equation

\[
P_p(\alpha) = 0 \tag{2.1.9}
\]

where the "characteristic function", \( P_p(\alpha) \), is given by:

\[
P_p(\alpha) = \theta(u_2;\alpha) - \arccot(q_2 \Gamma_2) - p\pi \tag{2.1.10}
\]

such that the \( p^{th} \) root, \( \alpha \), of (2.1.9) is the propagation coefficient of the surface wave mode corresponding to the integer \( p \).

2.2 Properties of the characteristic function

In order to examine the behaviour of \( P_p(\alpha) \), it is convenient to define a function \( \chi(u;\alpha) \) [17]

\[
\chi(u;\alpha) = \frac{\partial}{\partial \alpha} \theta(u;\alpha) \tag{2.2.1}
\]

which can be shown, by differentiating (2.1.7) with respect to \( \alpha \), to satisfy the following equation:

\[
\chi'(u;\alpha) = \left[ \gamma^2 \frac{\theta(u;\alpha)}{K(u)} - K(u) \right] \sin 2\theta(u;\alpha) \cdot \chi(u;\alpha) - \frac{2\alpha}{K(u)} \sin^2 \theta(u;\alpha) \tag{2.2.2}
\]

\[
= -2\left\{ \frac{\rho'(u;\alpha)}{\rho(u;\alpha)} \chi(u;\alpha) + \frac{\alpha}{K(u)} \sin^2 \theta(u;\alpha) \right\}
\]

In the last expression we have denoted by \( \rho(u;\alpha) \) the solution of (2.1.6) whose initial condition is
\[ \rho(u_1; \alpha) = 1. \]

This value was chosen arbitrarily since \( \rho \) is the amplitude function and has no boundary condition specified from the original problem. This is merely a reflection of the fact that no specific excitation is being considered in this treatment. The initial condition for \( \chi \) is obtained by differentiating (2.1.8a) with respect to \( \alpha \):

\[ \chi(u_1; \alpha) = -\frac{\alpha}{\Gamma_1} \frac{q_1}{1 + q_1^2 \Gamma_1^2} \]  

(2.2.3)

Since (2.2.2) is a first-order linear inhomogeneous equation in \( \chi \), an unique solution can be obtained from (2.2.3), together with the use of the following transformation

\[ \chi(u; \alpha) = \psi(u, \alpha) / \rho^2(u, \alpha) \]

as

\[ \chi(u; \alpha) = -\frac{\alpha}{\rho^2(u; \alpha)} \left\{ \frac{q_1}{\Gamma_1} \frac{q_1^2}{1 + q_1^2 \Gamma_1^2} + 2 \int_{u_1}^{u} \frac{\rho^2(\xi; \alpha)}{K(\xi)} \sin^2\theta(\xi; \alpha) d\xi \right\} \]  

(2.2.4)

Provided that the condition \( \alpha > \alpha_{\text{min}} \), where

\[ \alpha_{\text{min}} = \max\{\sqrt{\varepsilon R_1}, \sqrt{\varepsilon R_2}\}, \]

is satisfied, we have right away that both \( \Gamma_1 \) and \( \Gamma_2 \) are positive and real. Thus, since \( q_1, K(u), \) and \( \Gamma_1 \) are all positive, the value of \( \chi(u; \alpha) \) must be negative on the
interval \([u_1, u_2]\) whenever \(\alpha > \alpha_{\text{min}}\). This implies that for any fixed \(u\), \(\theta\) is a decreasing function of \(\alpha\), and in particular,

\[
\theta(u_2; \alpha_{\text{min}}) \geq \theta(u_2; \alpha) \quad \forall \alpha > \alpha_{\text{min}} \quad (2.2.5)
\]

Furthermore, since for each \(p\) the boundary condition (2.1.8b) is an increasing function of \(\alpha\), the characteristic function \(P_p(u)\) is therefore strictly decreasing for \(\alpha > \alpha_{\text{min}}\). As a consequence, the integer

\[
P_{\text{max}} = \text{int}[P_0(\alpha_{\text{min}})/\pi], \quad (2.2.6)
\]

where "int" denotes the truncation or "greatest integer" function, gives the largest value of \(p\) for which (2.1.8b) can be satisfied (see Fig. 2-1). Although it is not evident at first glance that \(p_{\text{max}}\) must be the same for non-symmetric structures no matter whether the original or "mirror image" structure is considered, it is necessary on physical grounds, and as is shown analytically in Appendix A.

If we now denote

\[
\alpha_{\text{max}} = \sqrt{\varepsilon_{\text{r}}^{\text{max}}} = \max_{0 < x < d} \sqrt{\varepsilon_{\text{r}}(x)}
\]

and suppose that \(\alpha > \alpha_{\text{max}}\), then \(\gamma^2(u; \alpha)\) is negative throughout the slab. Then if \(\theta\) is to have a trajectory which crosses any of the lines \(\theta = (n + \frac{1}{2})\pi\), where \(n\) is an integer, the slope at the point of crossing must be, from (2.1.7):

\[
\theta' = \frac{\gamma^2(u; \alpha)}{K(u)} < 0;
\]
Figure 2.1: Dependence of $\theta(u)$ and the Boundary Conditions on the Parameter $\alpha$. All arrows denote the direction of increasing $\alpha$. 
but since $\theta(u_1)$ is less than $\frac{\pi}{2}$ to begin with, all possible trajectories of $\theta$ are bounded from above by $\frac{\pi}{2}$. In a similar manner if a $\theta$-trajectory is to cross one of the lines $\theta = m\pi$ we must, at the crossing point, have

$$\theta' = K(u) > 0;$$

but since $\theta(u_1) > 0$, all trajectories of $\theta$ are bounded from below by 0, because $\theta$ would have to be decreasing to cross the line $\theta = 0$. Thus we have shown that

$$0 < \theta(u_2; \alpha) < \frac{\pi}{2}$$

for all $\alpha > \alpha_{\text{max}}$, and since no value assumed by the boundary condition (2.1.8b) falls in this range, no eigenvalues may exist for $\alpha > \alpha_{\text{max}}$.

The characteristic equation (2.1.9) has, therefore only a finite number of discrete solutions corresponding to the integers $p = 0, 1, 2, \ldots, p_{\text{max}}$; the possession of only a finite number of surface-wave modes is a well-known phenomenon associated with open guided-wave structures [19, Chapter 11]. For each such mode, the normalized propagation constant is greater than the normalized wavenumber of either of the bounding media ($\alpha > \alpha_{\text{min}}$); this implies that all surface-wave modes are slow wave in nature. For each mode, since the $y$-component of the appropriate field vector has modes when $\theta$ assumes the values of integer multiples of $\pi$, an examination of Fig. 2-1 shows that the mode corresponding to the integer $p$ possesses exactly $p$ of these nodes; consequently we label the mode
$TE_p$ or $TM_p$ as is appropriate. If it happens that the normalized propagation constant is equal to $\alpha_{\min}$ for some mode, this mode is said to be at cutoff, which in this case means that in at least one of regions (1) and (2) the fields no longer decay exponentially away from the slab. Since, as $\alpha$ decreases from $\alpha_{\max}$, $P_0(\alpha)$ increases from some negative value, it is clear that the roots of (2.1.9) are encountered in the sequence $p = 0, 1, 2, \ldots, p_{\max}$ as indicated in Fig. 2.1. From this we see that no mode "crossover" can occur, that is, as mode number increases, $\alpha$ always decreases:

$$\alpha_{\min} < \alpha_p < \cdots < \alpha_1 < \alpha_0 < \alpha_{\max}.$$

Another very significant feature of the characteristic equation is that in the case of asymmetrical waveguide, i.e. $\varepsilon_2 \neq \varepsilon_1$, there exists a cut-off frequency for each surface-wave mode. However, the cut-off frequency of $TE_0$ and $TM_0$ are identically zero for a symmetrical structure. (Appendix A).

2.3 Equivalent transverse impedance representation

It is seen that for a numerical solution of (2.1.7), trigonometric functions must be performed during each step of the process. This leads us to consider another transformation [20]:

$$R(u) = \tan \theta(u) = K(u)f(u)/f'(u)$$

$$S(u) = \cot \theta(u) = f'(u)/[K(u)f(u)] \quad (2.3.1)$$
R and S are evidently a normalized wave impedance and its reciprocal, a normalized wave admittance. The precise correspondence depends upon whether TE or TM modes are being considered, and may be found in Table 2-1. Using (2.1.7) it is easy to show that these functions satisfy

\[ S'(u) = -K(u)S^2(u) - \gamma^2(u; \alpha)/K(u) \]  
(2.3.2)

\[ R'(u) = K(u) + \gamma^2(u; \alpha)R^2(u; l)/K(u) \]  
(2.3.3)

and the initial condition (2.1.8a) is now

\[ S(u_1) = [R(u_1)]^{-1} = q_1 \Gamma_1 \]  
(2.3.4)

Equations (2.3.2) and (2.3.3) are first-order differential equations of the Ricatti type, involving no trigonometric functions explicitly. Ricatti equations have been used previously to study scattering problems [21], propagation problems in tropospheric layers [22], and propagation modes in coaxial waveguides [23]. Ricatti equations also arise in the determination of reflection coefficients in inhomogeneous media [10, pp. 215-233]. A full discussion of the relationships between invariant imbedding, wave propagation in inhomogeneous media, the Ricatti equation and the non-uniform transmission line can be found in [24].

The Ricatti equations may be thought of as the limiting case of the equations for the transverse wave impedance in a system of piecewise constant-permittivity layers. Based upon the appropriate field equation it is easy to show that (within such a constant layer)
\[ S(u;\alpha) = \frac{\gamma_i K_i S_i \cos \gamma_i(u-u_i) - \gamma_i \sin \gamma_i(u-u_i)}{K_i S_i \sin \gamma_i(u-u_i) + \gamma_i \cos \gamma_i(u-u_i)} \]  

(2.3.5)

where

\[ \gamma_i = \sqrt{\gamma^2(u;\alpha) - \alpha^2} \]

\[ K_i = K(u) \]

\[ S_i = S(u_i;\alpha) \]

and \( \varepsilon_{ri} \) is the (constant) relative permittivity of the layer. The point \( u_i \) may be any place inside the layer, but is usually taken to be one of its boundary points. Equation (2.3.5) is recognized as the law of impedance transformation for transmission lines [25], and so we identify the characteristic impedance, \( Z_0 \), with \(-j\gamma/K_i\), the input impedance with \( S_i \), and the phase constant with \( \gamma_i \) (the same statements hold true for admittance when \( S \) is an admittance). In the limit as the layers become infinitesimal, (2.3.5) is easily shown to approach (2.3.2) which may thus be regarded as describing a non-uniform transmission line which is the limit of a sequence of piecewise uniform transmission-line representations of the inhomogeneous slab [26,27]. The non-uniform line has parameters which are functions of \( u \); the characteristic impedance (admittance) is given by

\[ Z_0(u) = -j\sqrt{\gamma^2(u;\alpha)/K(u)} \]

and the phase constant by

\[ h(u) = \sqrt{\gamma^2(u;\alpha)} \]
Figure 2.2: Equivalent Transverse Transmission Line Representation
Alternatively, the line can be characterized by the series impedance \( z(u) \) and shunt admittance \( y(u) \) per unit length [24]:

\[
z(u) = -\gamma^2(u;\alpha)/K(u)
\]

\[
y(u) = K(u)
\]

In any case, when expressed in terms of \( R \) and \( S \), the boundary conditions at \( u_1 \) and \( u_2 \) amount to the presence of lumped terminating impedances (admittances) at these points:

\[
S(u) = q_1\Gamma_1 = S_1 \quad \text{for} \quad u < u_1
\]

\[
S(u) = q_2\Gamma_2 = S_2 \quad \text{for} \quad u > u_2
\]

(2.3.6)

and the requirement that these boundary conditions be satisfied is simply a statement that the nonuniform line must be in resonance with its terminating loads. The equivalent transverse representation is shown in Fig. 2-2, and the transverse resonance condition is a generalization of that used for piecewise uniform problems as outlined in [19, Chapter 6].

2.4 The numerical solution of the characteristic equation

For an arbitrary permittivity profile, neither \( \theta(u;\alpha) \) nor \( R \) and \( S \) are available in closed form, and therefore a solution of the characteristic equation (2.1.9) must be obtained approximately via some numerical methods: first for evaluating \( \theta \) (or \( R \) and \( S \)) and \( \chi(u;\alpha) \), and second for actually finding the zero of \( P_p(\alpha) \). If we assume for the moment that the first task has already been accomplished, we can attack the second in some detail.
For each value of $p$ between 0 and $p_{\text{max}}$, $P_p(\alpha)$ has exactly one zero on the interval $[\alpha_{\text{min}}, \alpha_{\text{max}}]$, and the corresponding value of $\alpha$ is the desired solution. In addition, as we saw in Section 2.2, the derivative

$$P_p'(\alpha) = \chi(u_2; \alpha) + \frac{\alpha}{T_2} \frac{q_2}{1 + q_2 T_2^2}$$

(2.4.1)

of $P_p(\alpha)$ is always negative, so that with this much information localizing the root, we look to Newton's method as a method of solving the characteristic equation which will converge rapidly enough so as not to involve an excessive amount of computing time. Newton's method consists in successive approximations to the actual root $\alpha^{(p)}$ obtained by the formula

$$\alpha_{n+1} = \alpha_n - \frac{P_p(\alpha_n)/P_p'(\alpha_n)}{P_p'(\alpha_n)}$$

(2.4.2)

and ultimately from some initial estimate, $\alpha_0$.

It is well-known that Newton's method may not converge if the function whose roots are to be found possesses certain characteristics or if the initial estimate $\alpha_0$ falls outside a certain "interval of convergence." To avoid this problem, we utilize our knowledge of $P_p(\alpha)$.

In the first place, $p_{\text{max}}$ may be calculated from (2.2.6) by our assumed method of calculating $\theta$, thus assuring us of the existence of a root for properly chosen $p$. Now consider any estimate $\alpha_n$ to the root $\alpha^{(p)}$. Since $P_p'(\alpha)$ is negative throughout the interval $[\alpha_{\text{min}}, \alpha_{\text{max}}]$, the correction term in
(2.4.2) will always be of the correct sign; that is, if \(-P'(\alpha_n)/p'(\alpha_n)\) is positive, then \(\alpha_n\) is less than \(\alpha^{(p)}\), and conversely. Therefore, at any stage in the procedure, an upper bound (at most, \(\alpha_{\text{max}}\)) and a lower bound (at least, \(\alpha_{\text{min}}\)) on the root \(\alpha^{(p)}\) are available [17].

As an example, consider the situation shown in Figure 2-3a. The initial estimate \(\alpha_0\) is such that \(\alpha_1\) (the intersection of the tangent line at \(P_0(\alpha_0)\) with the \(\alpha\)-axis) is much larger than \(\alpha_{\text{max}}\). Obviously \(\alpha_1\) is not a better approximation than \(\alpha_0\) to the root \(\alpha^{(0)}\). Now, however, instead of the value recommended by (2.4.2), suppose we select

\[
\alpha_1 = \frac{\alpha_0 + \alpha_{\text{max}}}{2}
\]

as our next approximation as in Figure 2-4a. This is done on the basis that \(\alpha_0\) is certainly a lower bound to \(\alpha^{(0)}\) and \(\alpha_{\text{max}}\) certainly an upper bound, and amounts to a step in the so-called bisection method. Now, if \(\alpha_2\) is evaluated from \(\alpha_1\) by Newton's formula, quick convergence may be expected. In a similar fashion, if, as in Fig. 2-3b, a lower bound on \(\alpha^{(0)}\) is undershot by Newton's formula, we again resort to bisection as shown in Figure 2-4b, and would continue this, as necessary, until the approximations to the root came sufficiently close to \(\alpha^{(0)}\) that Newton's method "took over" and converged to the desired result. If this rule is applied, we are assured convergence by the bisection formula, and achieve the quadratic convergence of Newton's method eventually, but without the possibility of its divergence.
Figure 2.3: Divergence Tendencies in the Execution of Newton's Method
Figure 2.4: Modified Newton's Method Approximations
We now return to the first task; that of evaluating $\theta(u_2; \alpha)$. We have already noted that the Ricatti equations may be evaluated in a much smaller amount of time than (2.1.7); however, some minor difficulties present themselves. It is evident that $R$ and $S$ exhibit poles (corresponding to $\theta = (n + \frac{1}{2}) \pi$ and $\theta = n \pi$, respectively) which pose obstacles to a numerical solution of either (2.3.2) or (2.3.3) separately. This problem is avoided [20] by noting that when $R$ has a pole, $S$ has a zero, and vice-versa. Therefore, one may simply alternate between equations (2.3.2) and (2.3.3) near a pole of $S$, and in this way the solution may be carried across the slab without numerical difficulty. A convenient criterion for determining when to switch between $R$ and $S$ is to do so whenever either one exceeds 1 in absolute value.

Now, even though $R$ (or $S$) is available at $u = u_2$, $\theta$ is still defined only within an arbitrary multiple of $\pi$. This ambiguity is resolved by noting the number of poles of $S$ which occur during the integration across the slab. We note that $S$ has only simple, non-repeated poles (that is, $\theta$ attains the value $n \pi$ at most once, and passes through this value with positive slope - see Section 2.2) so that if $m$ poles of $S$ (that is, zeroes of $R$) are encountered during the integration of (2.3.2) - (2.3.3) between $u_1$ and $u_2$, we may reconstruct $\theta$ as follows:

$$\theta(u_2) = m \pi + \arccot S(u_2) \quad 0 \leq \arccot S < \pi$$

$$= (m + \frac{1}{2}) \pi - \arccot R(u_2) \quad 0 \leq \arccot R \leq \frac{\pi}{2}$$

$$= (m - \frac{1}{2}) \pi + \arccot R(u_2) \quad \frac{\pi}{2} \leq \arccot R \leq \pi$$
Finally, it is also necessary to evaluate $\chi(u_2;\alpha)$. Although the differential equation for $\chi$, (2.2.2), contains trigonometric functions, these disappear when the equation is written in terms of $R$ and $S$:

$$\chi'(u;\alpha) = \left[ \gamma^2(u;\alpha)/K(u) - K(u) \right] \cdot 2S(u;\alpha)\chi(u;\alpha) - 2\alpha/K(u) \right] / \left[ 1 + S^2(u;\alpha) \right]$$

$$= \left[ \alpha^2(u;\alpha)K(u) - K(u) \right] \cdot 2R(u;\alpha)\chi(u;\alpha) - 2\alpha R^2(u;\alpha)/K(u) \right] / \left[ 1 + R^2(u;\alpha) \right]$$

(2.4.3)

In view of this, there is no reason to introduce functions such as

$$\frac{\partial R}{\partial \alpha} \quad \text{or} \quad \frac{\partial S}{\partial \alpha}$$

which have double poles where $R$ and $S$ have only simple poles; whereas $\chi(u;\alpha)$ remains always negative and finite. Thus, using (2.4.3) along with (2.3.2) and (2.3.3), $R$ and $S$ (hence $\theta$) and $\chi$ may be evaluated using a Runge-Kutta scheme.

In order to obtain an error estimate on this evaluation (hence, on the calculated eigenvalue $\alpha$), a fourth-order Runge-Kutta method with a good error estimator was chosen [28]. The error estimate, in turn, allows the step size to be adjusted automatically to maintain the truncation error for each step at an approximately constant value. Such a scheme is superior to any of the conventional methods where the inhomogeneity is approximated by piecewise uniform slabs since, even if some method of varying the width of these layers is used, there is still no way of determining what errors are involved and how
Figure 2.5: \( \theta(u) \) vs. \( u \) - TE\(_0\) mode.

Uniform Symmetric Profile with \( \varepsilon_{r_1} = \varepsilon_{r_2} = 1.50, \varepsilon_{r_{\text{max}}} = 1.53, d = 4 \times 10^{-6}, f = 10^{-15}\text{Hz}. \)
Figure 2.6: $\theta(u)$ vs. $u$ - TE$_4$ mode.
Uniform Symmetric Profile with $\varepsilon_r = 1.50$,
$\frac{r_1}{r_2}$

$\varepsilon_{r_{\text{max}}} = 1.53$, $d = 4 \times 10^{-6}$ m, $f = 10^{15}$ Hz.
Figure 2.7: $P_0(\alpha)$ vs. $\alpha$. Uniform Symmetric profile with $\varepsilon_r = \varepsilon = 1.50$, $\varepsilon_{\text{max}} = 1.53$, $r_1$, $r_2$, $r_{\text{max}}$, $d = 4 \times 10^{-6}$ m, and $f = f_{\text{max}} = 10^{15}$ Hz.
this variation affects them. The method used here, however, is able to change the step length locally as needed, to satisfy any desired error bound.

In Figs. 2.5 through 2.7, some sample curves are given to demonstrate the behavior of the functions \( \theta(u) \) and \( P_0(\alpha) \). Figures 2.5 and 2.6 are plots of \( \theta \) vs. \( u \) for a TE\(_0\) and a TE\(_4\) mode of a uniform guide. This behavior can be expected, in general, since \(|\gamma^2(u;\alpha)| << 1\) for the guides we are considering, while \(K(u) \gg 1\). Thus we expect \( \theta \) to vary quite slowly around \( \theta = (n+\frac{1}{2})\pi \), when \( \cos \theta \approx 0 \) in (2.1.7) but relatively rapidly elsewhere. In Fig. 2.7 a plot of \( P_0(\alpha) \) vs. \( \alpha \) at a frequency where 5 modes of each kind exist is given (the graph shown is for TE modes). The modes are the roots of \( P_0(\alpha) - p\pi = 0 \) for \( p = 0,1,2,3,4 \). The bunching of the modes near \( \alpha_{\text{max}} \) is evident in this graph, and this motivated selection of \( \alpha_{\text{max}} \) as the initial guess in Newton's method. Once a lower order mode was calculated, this value of \( \alpha \) was used as an initial value for the next higher order mode, since it evidently is an upper bound (See Section 2.2).

2.5 Results for various permittivity profiles

This method, as outlined in Section 2.4, was implemented on a CDC 6400 computer, and used to calculate dispersion curves and group velocity curves for several different permittivity profiles. Print-out of the FORTRAN source programs will be found in Appendix B. All computations used an error
Figure 2.8: Five types of permittivity profile.
bound of $10^{-7}$ on an individual Runge-Kutta step, and an error of $10^{-5}$ was allowed in calculating the eigenvalues $\alpha$. Total error estimates were made for each data point; these averaged about $6 \times 10^{-6}$, and were never significantly larger than $10^{-5}$.

An average dispersion curve, with data calculated at 30 points, required about 30 seconds of computer central processor time to calculate, or about one second per data point. The time to compute one data point varied from a high of 2.4 seconds to a low of 0.3 seconds. The main factor influencing the computing speed seemed to be the amount of time spent in evaluating the dielectric constant $\varepsilon_r(x)$, since this must be done for every step of a Runge-Kutta process. Uniform profiles ($\varepsilon_r(x) =$ constant) proved to be the fastest, while profiles involving both long arithmetic expression evaluations and complicated logical statements tended to take the longest. No attempt was made, however, to reduce any of these times by an optimization of the subroutine which evaluated $\varepsilon_r(x)$. To a lesser degree, the rate of convergence of Newton's method for asymmetrical profiles (i.e., where $\varepsilon_{r_1} \neq \varepsilon_{r_2}$) had some dependence on the permittivity of the region of lower relative dielectric constant (in this case, $\varepsilon_{r_2}$). In one set of computations with $\varepsilon_{r_1} = 1.50$, varying $\varepsilon_{r_2}$ from 1.00 to 1.50 was found to change the computing time per data point from 0.83 seconds to 0.48 seconds. Again, no attempt was made to study or optimize these figures.

The types of profiles for which dispersion curves were calculated are shown in Figs. 2.8 a-e. Figure 2.8b is an idealization of a guide formed by placing a layer of dielectric
on a substrate of slightly lower permittivity, and allowing the other side of the dielectric an air interface. Figure 2.8d is a slightly more realistic model of such a guide, with a finite transition region representing a diffusion region resulting from the fabrication process. Figures 2.8a and 2.8c may be regarded as similar models of guides embedded within a substrate with no air interface. Finally, Fig. 2.8e is the truncated parabolic profile, an approximation to the so-called "lenslike medium" idealization studied by many researchers [2].

Dispersion and group velocity curves for various profiles are shown in Figs. 2.9 - 2.20. Figures 2.9 and 2.10 show the results for a simple, symmetric uniform slab for both TE and TM modes. In all profiles for which curves were computed, the results for TE and TM modes, though different, were almost identical (in fact, differences were rarely greater than $10^{-3}$, where $\alpha$ was on the order of 1), so as to be indistinguishable from each other on a graph. Therefore, we omit the TM curves from here on.

In Fig. 2.11, the results for an asymmetric uniform guide are given, and the non-zero cutoff for the lowest order mode, as predicted in Appendix A, is evident. A study of the dependence of this cutoff frequency on the relative permittivity was made, and the results plotted in Fig. 2.12. The extreme sensitivity of this dependence very close to symmetry (i.e., for $\varepsilon_{r2}$ close to 1.50) suggests that in actual waveguides, the symmetry should be closely controlled; otherwise, when
Figure 2.9: TE Modes for a Uniform Symmetric Profile.
\( \varepsilon_{r_1} = 1.50, \varepsilon_{r_{\text{max}}} = 1.53, \ d = 4 \times 10^{-6} \text{ m}, \ \beta_{\text{min}}/k_0 = 1.225. \)
Figure 2.10: TM Modes for a Uniform Symmetric Profile. 
$\varepsilon_{r_1} = 1.50, \ \varepsilon_{r_m} = 1.53, \ d = 4\times10^{-6} \text{m}, \ \beta_{\text{min}}/k_0 = 1.225.$
Figure 2.11: TE Modes for a Uniform Asymmetric Profile.

\( \varepsilon_{r_1} = 1.50, \ \varepsilon_{r_2} = 1.00, \ \varepsilon_{r_{\text{max}}} = 1.53, \ d = 4 \times 10^{-6} \text{m}, \ \beta_{\text{min}}/k_0 = 1.225. \)
Figure 2.12: TE₀ Mode Cutoff Value of $k_0d$ vs. $\varepsilon_{r2}$ for a uniform asymmetric-profile. $\varepsilon_{r1} = 1.50$, $\varepsilon_{r_{\text{max}}} = 1.53$. 
Figure 2.13: TE Modes for a Symmetric Trapezoidal Profile.
\(\varepsilon_{r1} = 1.50, \varepsilon_{r_{\text{max}}} = 1.53, \ d = 2 \times 10^{-6} \text{ m}, \ w = 1 \times 10^{-6} \text{ m}, \ \beta_{\text{min}}/k_0 = 1.225.\)
(a) Propagation Constant

(b) Group Velocity

Figure 2/14: TE Modes for an Asymmetric profile with Cosinusoidal Transition. $\varepsilon_{r_1} = 1.50$, $\varepsilon_{r_2} = 1.00$, $\varepsilon_{r_{\text{max}}} = 1.53$, $d = 4 \times 10^{-6} \text{m}$, $w/d = 0.0$, $\beta_{\text{min}}/k_0 = 1.225$. 
Figure 2.15: TE Modes for an Asymmetric Profile with Cosinusoidal Transition. \( \varepsilon_r = 1.50, \quad \varepsilon_r = 1.00, \quad \varepsilon_r = 1.53, \quad \varepsilon_{r_{\text{max}}} = \varepsilon_{r_{\text{min}}}, \quad d = 4 \times 10^{-6}, \quad w/d = 0.2, \quad \beta_{\text{min}}/k_0 = 1.225. \)
Figure 2.16: TE Modes for an Asymmetric Profile with Cosinusoidal Transition. $\varepsilon_r = 1.50$, $\varepsilon_{r_2} = 1.00$, $\varepsilon_{r_{\text{max}}} = 1.53$, $d = 4 \times 10^{-6}$, $w/d = 0.4$, $\beta_{\text{min}}/k_0 = 1.225$. 

(a) Propagation Constant

(b) Group Velocity
Figure 2.17: TE Modes for an Asymmetric Profile with Cosinusoidal Transition. \( \epsilon_r = 1.50, \epsilon_r = 1.00, \epsilon_{r_{max}} = 153, \)
\( \frac{d}{w/d} = 0.6d, \quad \frac{\beta_{min}}{k_0} = 1.225. \)
(a) Propagation Constant

(b) Group Velocity

Figure 2.18: TE Modes for an Asymmetric Profile with Cosinusoidal Transition. $\varepsilon_r = 1.50$, $\varepsilon_{r_1} = 1.00$, $\varepsilon_{r_2} = 1.53$, $r_{\text{max}}$, $d = 4 \times 10^{-6} \text{m}$, $w/d = 0.8$, $\beta_{\text{min}}/k_0 = 1.225$. 
Figure 2.19: TE Modes for an Asymmetric Profile with Cosinusoidal Transition. $\varepsilon_r = 1.50$, $\varepsilon_{r_2} = 1.00$, $\varepsilon_{r_{max}} = 1.53$, $d = 4 \times 10^{-6}$, w/d = 1.0, $\beta_{min}/k_0 = 1.225$. 

(a) Propagation Constant

(b) Group Velocity
(a) Propagation Constant

(b) Group Velocity

Figure 2.20: TE Modes for a Truncated Parabolic Profile.
\( \varepsilon_r = 1.50, \quad \varepsilon_{r_{max}} = 1.53, \quad d = 4 \times 10^{-6} \text{ m}, \quad \beta_{\text{min}}/k_0 = 1.225. \)
the frequency of operation is such that the guide is in single-mode operation, a slight asymmetry may produce an "effective cutoff frequency" near the frequency of operation, so that the guided fields may radiate into the surrounding medium in the vicinity of the asymmetry.

In Fig. 2.13, a symmetric trapezoidal profile is studied. The result of the finite transition region is to make the dispersion curves slightly more gentle, and separate somewhat the group velocity curves for the first and second modes.

Figures 2.14 through 2.19 consider the asymmetric guide with cosinusoidal transition regions of widths varying between 0 and the full guide width, d. Between \( w = 0.2d \) and \( w = 0.4d \), the intersection of the group velocity curves for the first and second modes disappears, and as \( w \) increases further, the separation between these two curves increases somewhat, then becomes fairly constant. As will be shown in Part II of this report, the signal distortion arising from the frequency dependence of the group velocity for a single mode is generally much smaller than the possible effects of differences in group delay when the signal propagates in two different modes. This suggests that the point of intersection of the group velocity curves, when it exists (and, indeed, if it exists in more realistic 3-dimensional structures) would be an optimum operating point for the guide in terms of signal distortion.

Finally, in Fig. 2.20, the results for a truncated parabolic profile are presented. This profile is frequently credited [2] with having better group delay characteristics for multimode operation than any other profile. Although this is
strictly true only for the idealized "lenslike medium", it is an excellent approximation for modes sufficiently far from cutoff that little field penetration into the surrounding medium occurs. Indeed, the group velocities of the first two modes become very nearly equal at only moderately high frequencies, which is not true for any other profile studied here.
Chapter 3: Conclusions

This part of the report has developed a method for the solution of inhomogeneous lossless slab waveguides based upon invariant imbedding and the transverse resonance of an equivalent non-uniform transmission line representation for the guide. A modification of Newton's method has been used to solve for the resonance, such that convergence to any desired accuracy is assured for an arbitrary permittivity profile. Results have been presented for various permittivity profiles, and the effect of a continuous transition region, in particular, has been noted.

In the second part of this report, the method outlined here will be extended to study lossy structures and coupling between parallel slabs. In addition, a study of pulse distortion for slightly more general structures will be made.
BIBLIOGRAPHY


Appendix A: Non-Symmetrical Structures

In cases where the slab structure is not symmetric, i.e., when \( \varepsilon_{r_1} \neq \varepsilon_{r_2} \) and/or the profile within the slab itself does not possess reflection symmetry about the midpoint of the slab, it is not immediately obvious that the same solutions for the surface wave modes are obtained regardless of whether the original or "mirror-image" problem is studied (although on physical grounds it may be argued they must be identical). We shall demonstrate here that, although \( P_0(\alpha) \) is, in general, different for the two cases, the roots of the two characteristic equations (corresponding to \( P_0(\alpha) = \frac{1}{\varepsilon_{r}} \)) are identical. In addition, for that class of non-symmetric structures with \( \varepsilon_{r_1} \neq \varepsilon_{r_2} \), it will be shown that all modes must exhibit a non-zero cutoff frequency.

We consider the original wave equation for \( f(u) \) (the \( y \)-component of the electric (TE) or magnetic (TM) field):

\[
\frac{d}{du} \left( \frac{1}{K(u)} \frac{d}{du} f(u) \right) + \frac{\gamma^2(u;\alpha)}{K(u)} f(u) = 0 \quad (21.1)
\]

Since this is a linear second-order equation, it has two independent solutions, \( F(u;\alpha) \) and \( G(u;\alpha) \), say; and a general solution

\[ f(u;\alpha) = AF(u;\alpha) + BG(u;\alpha) \]

where \( A \) and \( B \) are constants. For the particular solution obtained by applying the boundary condition at \( u = u_1 \) in (2.1.2), we obtain an expression relating \( A \) and \( B \):
A[F'(u_1; \alpha) - r_1 \Gamma_1 F(u_1; \alpha)] + B(G'(u_1; \alpha) - r_1 \Gamma_1 G(u_1; \alpha)] = 0

Thus, after some manipulation, we obtain an expression for

\[ P_0(\alpha) = \arccot \left[ \frac{f'(u_2; \alpha)}{K(u_2)f(u_2; \alpha)} \right] - \arccot (q_2 \Gamma_2) \]

\[ = \arccot \left[ \frac{H(\alpha) + r_1 \Gamma_1 J(\alpha)}{K(u_2)[L(\alpha) + r_1 \Gamma_1 M(\alpha)]} \right] - \arccot (q_2 \Gamma_2) \]

where

\[ H(\alpha) = G'(u_1; \alpha) F'(u_2; \alpha) - F'(u_1; \alpha) G'(u_2; \alpha) \],

\[ J(\alpha) = F(u_1; \alpha) G'(u_2; \alpha) - G(u_1; \alpha) F'(u_2; \alpha) \],

\[ L(\alpha) = G'(u_1; \alpha) F(u_2; \alpha) - F'(u_1; \alpha) G(u_2; \alpha) \],

and

\[ M(\alpha) = F(u_1; \alpha) G(u_2; \alpha) - G(u_1; \alpha) F(u_2; \alpha) \].

Finally, using a trigonometric identity, we have

\[ \cot P_0(\alpha) = \frac{q_2 \Gamma_2 [H(\alpha) + r_1 \Gamma_1 J(\alpha)] + K(u_2)[L(\alpha) + r_1 \Gamma_1 M(\alpha)]}{q_2 \Gamma_2 K(u_2)[L(\alpha) + r_1 \Gamma_1 M(\alpha)] - [H(\alpha) + r_1 \Gamma_1 J(\alpha)]} \]

(A.1)

Now, denoting quantities of the "mirror-image" structure with a superscript "T", we have

\[ K^T(u) = K(u_1 + u_2 - u) \]

\[ \gamma^T(u; \alpha) = \gamma^2(u_1 + u_2 - u; \alpha) \]

\[ \epsilon^T_{r_1} = \epsilon_{r_2} ; \epsilon^T_{r_2} = \epsilon_{r_1} \]

\[ q_1^T = -q_2 ; q_2^T = -q_1 \]
so that, calling \( v = u_1 + u_2 - u \), the boundary points \( v = u_1 \)
and \( u_2 \) correspond to \( u = u_2 \) and \( u_1 \), respectively, and \( f^T(u) \)
satisfies
\[
\frac{d}{dv} \left( \frac{1}{K(v)} \frac{d}{dv} f(v) \right) + \gamma^2(v;\alpha) \frac{1}{K(v)} f(v) = 0
\]
if \( f^T(u) = f(v) \). We now proceed exactly as before, noting
that the general solution \( f^T(u) \) has the form
\[
f^T(u;\alpha) = AF(v;\alpha) + BG(v;\alpha)
\]
and finally obtaining
\[
cot P_0^T(\alpha) = \frac{q_1 \Gamma_1^{-1}[H(\alpha)+r_2 \Gamma_2 L(\alpha)]-K(u_1)}{q_1 \Gamma_1 K(u_1)[-J(\alpha)+r_2 \Gamma_2 M(\alpha)]+[-H(\alpha)+r_2 \Gamma_2 L(\alpha)]} \quad (A.2)
\]
where use has been made of the relations
\[
H^T(\alpha) = -H(\alpha) , \\
J^T(\alpha) = -L(\alpha) , \\
L^T(\alpha) = -J(\alpha) , \\
M^T(\alpha) = -M(\alpha) .
\]

Although the expressions (A.1) and (A.2) are not identi-
cal, the surface wave modes, which correspond to the zeroes
of the denominators of these expressions, are identical,
since \( q_i K(u_i) = r_i \) for \( i = 1,2 \). As a corollary to this,
\( P_{\text{max}}^T \) as obtained from (2.2.6), must be the same in either
case, although \( P_0^T(\alpha_{\text{min}}) \) will not, in general, be equal to
\( P_0^T(\alpha_{\text{min}}) \).
Because of the invariance of the solutions of a problem under mirror reflection, we consider a guide structure for which \( \varepsilon_{r_1} \neq \varepsilon_{r_2} \), and assume without loss of generality that \( \varepsilon_{r_1} > \varepsilon_{r_2} \), so that \( \alpha_{\text{min}} = \sqrt{\varepsilon_{r_1}} \). If the lowest order \((p = 0)\) mode is to exist, we must have

\[
\theta(u_2; \sqrt{\varepsilon_{r_1}}) > \arccot \left( q_2 \sqrt{\varepsilon_{r_1} - \varepsilon_{r_2}} \right) \quad (A.3)
\]

However, from (2.1.7) we can show that

\[
|\theta(u_2; \alpha_{\text{min}}) - \theta(u_1; \alpha_{\text{min}})| = \left| \int_{u_1}^{u_2} \left[ K(u) \cos^2 \theta(u) + \gamma^2(u; \alpha_{\text{min}}) \sin^2 \theta(u) \right] \frac{du}{K(u)} \right|
\]

\[
\leq \left| \int_{u_1}^{u_2} \left[ K(u) + \frac{\gamma^2(u; \alpha_{\text{min}})}{K(u)} \right] \frac{du}{K(u)} \right|
\]

A little inspection of the integrand of the last term makes it apparent that it has an upper bound of \((\varepsilon_{r_{\text{max}}} + 1)\), so that

\[
|\theta(u_2; \alpha_{\text{min}}) - \theta(u_1; \alpha_{\text{min}})| \leq k_0 d(\varepsilon_{r_{\text{max}}} + 1) \quad (A.4)
\]

Since \( \theta(u_1; \alpha_{\text{min}}) = \frac{\pi}{2} \), we obtain the following inequality from (2.2.5), (A.3), and (A.4):

\[
\arccot \left( q_2 \sqrt{\varepsilon_{r_1} - \varepsilon_{r_2}} \right) - \frac{\pi}{2} \leq k_0 d(\varepsilon_{r_{\text{max}}} + 1) \quad (A.5)
\]

This is a necessary condition for the existence of the lowest-order mode. Obviously (A.5) can be satisfied only for \( k_0 \) above some certain positive value, since the left-hand side is by assumption a positive number. We have thus shown that
all waveguides possessing this particular type of asymmetry exhibit a nonzero cutoff frequency, below which no surface-wave mode of any kind can exist.
Appendix B:

The program implementing the numerical procedure for calculating the waveguide propagation constants is given in this Appendix.

The main program, SLAB, is mainly organizational and serves to read in data, define the frequency interval for which modes are to be calculated, to collect the data calculated in the subroutines and arrange it for output. Excluding the function subprogram ER(x), which serves to provide the required information about the permittivity profile, the subprograms fall into four groups. In each of these groups are routines written for TE modes and corresponding routines for TM modes; the last letter in the subprogram name is E or M, respectively. The first group, NROOT-, contains NROOTE and NROOTM. These contain the modified Newton's method for calculating the propagation constants at a particular frequency. Since this requires an evaluation of \( \theta(u_2;\alpha) \) and \( \chi(u_2;\alpha) \), reference is made to a routine of the second group, PHASB-, made up of PHASBE and PHASBM, which calculate these values numerically, using the R and S functions as described in Section 2.4. As this is accomplished using the Runge-Kutta-England method referenced in that section, routines in the third group, RK-, consisting of RKSBE, RKRBE, RKSBM, and RKRBM, are referenced. These routines execute single Runge-Kutta steps on either R or S. Finally, these single steps require derivative evaluations provided by the D-group of functions (DSE, DRE, DCHBSE, DCHBRE,
DSM, DRM, DCHBSM, DCHBRM) which reference the profile function, ER(X). Thus the subroutine groups are arranged in a calling hierarchy which is broken only on two minor occasions:

```
  SLAB
   ↓
  NROOT-
   ↓
 PHASB-
   ↓
  RK-
   ↓
   D-
   ↓
 ER(X)
```

The exceptions to this hierarchy are: (1) when SLAB calls the entry point EMAX in ER(X) to perform the necessary initializations and communication of parameters to the main program, and (2) when SLAB calls the PHASB- group directly in order to calculate the number of modes which exist at FMAX.

The roles of each of these routines in detail is best described by the source listings found on the following pages. Since only calculational details differ between the TE routines and TM routines, only the former will be presented here.
PROGRAM SLAB(INPUT,OUTPUT,PUNCH)

C
C THIS MAIN PROGRAM CONSISTS OF SEVERAL PARTS--
C THE FIRST PART READS IN PARAMETERS FOR THE SLAB GUIDE.
C FMAX IS THE LARGEST FREQUENCY CONSIDERED, IN HZ
C ESTEP IS THE ERROR BOUND ON AN INDIVIDUAL R-K STEP
C EPS IS THE ERROR BOUND ON THE CALCULATION OF B
C D IS THE WIDTH OF THE SLAB
C IMOS IS THE APPROXIMATE NUMBER OF MODES DESIRED FOR THE OUTPUT
C
REAL KO
COMMON D,KO,B,BMAX,PI,G1,G2,ER1,ER2,BMIN
COMMON ERR/EPS/EBD/ESTEP,EACT/ETOT/ET
DIMENSION Y(10)
DIMENSION XP(100)
PI=3.14159265359 $ CLT=2.997925E8
READ 199,IMOS
199 FORMAT(13)
   READ 100,D,EPS,ESTEP,FMAX
100 FORMAT(3F10.5,G14.5)
   PUNCH 170,D,EPS,ESTEP
170 FORMAT(3E18.10)

C THE CALL TO THE ENTRY POINT EMAX OF THE FUNCTION ER(X) (WHICH SEE)
C INITIALIZES THE QUANTITIES ER1, ER2, AND EMAX, AS WELL AS OTHERS
C WHICH MAY BE NECESSARY
C
KD=2*PI*FMAX/CLT $ BMAX=SQR(FMAX/ER1*ER2)
BMXN=SQR(E/R1)
PRINT 101,D,FMAX,EPS,ESTEP
101 FORMAT(* THE SLAB WIDTH, D, IS *,E10.3, * METERS*/
     * THE MAXIMUM FREQUENCY, FMAX, IS *,E10.3, * HERTZ*/
     * THE ERROR BOUND ON BETA IS *,E10.3/
     * THE ERROR BOUND ON A SINGLE R-K STEP IS *,E10.3/)
C
C THE SECOND PART CALCULATES THE NUMBER OF MODES OF EACH TYPE--
C --TE OR TM----SUPPORTED BY THE GUIDE AT FMAX
C
B=BMXN $ G2=SQR(ER1-ER2) $ G1=0
CALL PHASEB(THETA, CHI)
N1=INT((THETA-ATAN2(-1.,SQR(ER1-ER2)))/PI)
IF(N1.EQ.1)GO TO 20
PRINT 102,N1
102 FORMAT(* THERE ARE *,I3, * TE MODES AT FMAX*/)
GO TO 21
20 PRINT 103,N1
103 FORMAT(* THERE IS *,I3, * TE MODE AT FMAX*/)
21 CALL PHASRM(THETA, CHI)
N2=INT((THETA-ATAN2(-ER2,SQR(ER1-ER2)))/PI)
C
C THIS PART OF THE PROGRAM CALCULATES BETA FOR VARIOUS FREQUENCIES
C UP TO FMAX( CORRESPONDING TO KD ) AND STORES THESE VALUES IN AN ARRAY
C TO BE PUNCHED ONTO DATA CARDS
C
MXM=MAX0(N1,N2)
IF(MXM.EQ.0)GO TO 75
IF(IMDS.GT.MXM)IMDS=MXM
KO=K0*IMDS/MXM
IPTS=8*IMDS
STPK=KO/IPTS
PUNCH 175,BMIN,BMAX,MXM,N1,N2,STPK,IPTS,K0
175 FORMAT(2I15,8,3I15,E15.7,I5,E15.7)
IF(N1.EQ.0)GO TO 65
DO 11 I=1,10
Y(I)=BMAX
11 CONTINUE
DO 40 IFR=1,IPTS
KO=STPK*IFR
XP(IFR)=KO*D
B=BMIN & G2=SQRTR(ER1-ER2) & G1=0
CALL PHASEDE(THETA,CHI)
NE=INT((THETA-DTAN2(-1.,SQRTR(ER1-ER2)))/PI)
PUNCH 600,NE
600 FORMAT(13)
IF(N1.EQ.0)GO TO 40
CALL VROOTE(Y,NE)
PRINT 500,XP(IFR),NE,(Y(I),I=1,NE)
500 FORMAT(1X,5HKO*D=,2X,F10.5,4X,I3,* MODES */
1* BETA/KO= *,10F10.5)
PUNCH 601,(Y(I),I=1,NE)
601 FORMAT(5F15.10)
40 CONTINUE
70 IF(N2.EQ.1)GO TO 30
   PRINT 104,N2
104 FORMAT(* THERE ARE * I3, * TM MODES AT FMAX*/*)
   GO TO 65
75 STOP 01
30 PRINT 105,N2
105 FORMAT(* THERE IS * I3, * TM MODE AT FMAX*/*)
65 IF(N2.EQ.0)GO TO 85
   DO 12 I=1,10
      Y(I)=BMAX
12 CONTINUE
   DO 50 IFR=1,IPTS
      KO=STPK+IFR
      B=BMN $ G2=SQR((ER1-ER2) $ G1=0
      CALL PHABM(THETA,CHI)
      NM=INT(((THETA-ATAN2(-ER2,SQR(ER1-ER2)))//PI)
      PUNCH 600,NM
      IF(NM.EQ.0)GO TO 50
      CALL NRDTM(Y,NM)
      PRINT 500,XP(IFR),NM,(Y(I),I=1,NM)
      PUNCH 601,(Y(I),I=1,NM)
50 CONTINUE
85 CONTINUE
END
SUBROUTINE NROOTE(Y,N1)

C THIS SUBROUTINE CALCULATES THE NORMALIZED PROPAGATION CONSTANT, B,
C AT THE CURRENT FREQUENCY CORRESPONDING TO KO USING A MODIFICATION OF
C NEWTON'S METHOD
C
REAL KO
DIMENSION Y(N1)
COMMON D,KO,B,BMAX,PI,G1,G2,ER1,ER2,BMIN/ERR/EPS
COMMON/ETOT/ET
DO 10 I=1,N1

C THE INITIAL ESTIMATES FOR THE VALUES OF B FOR THE VARIOUS MODES
C ARE PASSED FROM THE MAIN PROGRAM IN THE ARRAY Y
C
BUB=BMAX $ BLB=BMIN $ B=Y(I)
17 G1=SQRT(ABS(B*B-ER1)) $ G2=SQRT(ABS(B*B-ER2))
CALL PHASBE(THETA,CHI)
P2=THETA-ATAN(G2)-(1-0.5)*PI
D2=CHI-B/G2/(1+G2*G2)
DO=-P2/D2
IF(B+DO.GE.BUB)GO TO 11
IF(B+DO.LE.BLB)GO TO 12
  B=B+DO
  IF(DO)13,14,15
  13 HUB=B-DO $ GO TO 16
  15 BLB=B-DO
  16 IF(ABS(DO/BMAX).LT.EPS)GO TO 14
    GO TO 17
  11 BLB=B $ GO TO 16
  12 BUB=B
  18 B=(BUB+BLB)/2
    GO TO 16
  14 Y(I)=B+DO
C
C THE ERROR ESTIMATE FOR THE CALCULATION OF B IS TAKEN TO BE DO
C SINCE THE CONVERGENCE OF NEWTON'S METHOD IS QUADRATIC, THIS IS MUCH
C LARGER THAN THE ACTUAL ERROR.
C
  ET=T+ABS(DO) $ PRINT 444,ET
444 FORMAT(* MAX ERROR IN BETA IS *,E10.3)
  10 CONTINUE
    RETURN
  END
SUBROUTINE PHASBE(T,C)

C THIS SUBROUTINE PERFORMS ONE NUMERICAL SOLUTION OF THE INITIAL VALUE
C PROBLEM OBTAINED BY USING THE BOUNDARY CONDITION AT U=0 AS STARTING
C VALUE, FOR THE VALUE OF B SPECIFIED IN THE MAIN PROGRAM
C

REAL KO
COMMON D,KO,B,BM,PI,G1,G2,ER1,ERK2/STEP/N,H,NP,NDFG,HMAX
COMMON/SSTP,CONS/RSTP/CONR
COMMON/ETOT/ET/EBD/ESTEP,EACT
C=0 $ ET=0 $ NDFG=0 $ HMAX=KO*D/10 $ CONS=CONR=0.05
C
C IF B=BMIN, THEN G1=0 AND THE INITIAL CONDITION FOR CHI WOULD BE
C INFINITE. SINCE THE ONLY EVALUATIONS OF THESE FUNCTIONS WITH B=BMIN
C OCCUR WHEN THE NUMBER OF MODES IS BEING CALCULATED, CHI IS NOT
C NECESSARY AND SO IS SET EQUAL TO ZERO AND NOT COMPUTED
C
IF (G1.EQ.0.0) GO TO 100
C=-H/G1/(1+G1*G1)
100 S=G1 $ U=0. $ N=0 $ H=1/40/BM $ NP=1
20 IF (ABS(S).GT.1) GO TO 10
90 CALL RKSEBE(U,S,C)
   ET=ET+EACT
   IF (NDFG. NE.1) GO TO 20
IF(ABS(S).GT.1)GO TO 30
70 T=(N+0.5)*PI-ATAN(S)
RETURN
10 R=1/S
80 CALL RKRRE(U,R,C)
   ET=ET+EACT
   IF(NDFG.NE.1)GO TO 40
   IF(ABS(R).GT.1)GO TO 50
60 IF(R)1,2,3
   1 T=(N+1)*PI+ATAN(R)
   RETURN
   3 T=N*PI+ATAN(R)
   RETURN
   2 T=N*PI
   RETURN
30 R=1/S
GOTO 60
50 S=1/R
GOTO 70
40 IF(ABS(R).LE.1)GO TO 80
   S=1/R
   GO TO 90
END
SUBROUTINE RK5BE(U, S, C)

C THIS SUBROUTINE SELECTS AN APPROPRIATE STEP SIZE AND PERFORMS ONE STEP
C OF A RUNGE-KUTTA-ENGLAND 4TH ORDER PROCESS ON S AND C+I

REAL K0
REAL K1, K2, K3, K4, K5, K6, K7, K8, K9
REAL M1, M2, M3, M4, M5, M6, M7, M9
COMMON D, K0, B, BM/EBD/E, EA/DSCN/DSC(20)
COMMON STEP/N, H, NP, NDFG, HMAX/SSTP/CONH
TS=DFSE(U, S) & TR = ABS(TS)
IF (TR .LT. CONH/HMAX) GO TO 10
H=CONH/TR & GO TO 20

10 H=HMAX
20 IF (U+2*H .GE. DSC(NP)*K0) GO TO 60
40 K1=H*TS
   K2=H*DFSE(U+H/2, S+K1/2)
   K3=H*DFSE(U+H/2, S+(K1+K2)/4)
   K4=H*DFSE(U+H, S-K2+2*K3)
   S1=S+(K1+4*K3+K4)/6
   U1=U+H
   K5=H*DFSE(U1, S1)
   K6=H*DFSE(U1+H/2, S1+K5/2)
   K7=H*DFSE(U1+H/2, S1+(K5+K6)/4)
   K8=H*DFSE(U1+H, S+(-K1-96*K2+92*K3-121*K4+144*K5+6*K6-12*K7)/6)
EA IS THE ERROR ESTIMATE FOR THE R-K STEP. IF IT IS TOO LARGE,
THE STEP SIZE IS CUT IN HALF AND THE EVALUATION TRIED AGAIN.
WHEN THE ERROR IS FOUND ACCEPTABLE, THE STEP SIZE IS ALLOWED TO
CREEP UPWARD BY 10 PERCENT EACH STEP THE ERROR IS LESS THAN HALF
OF THE ALLOWABLE VALUE

\[
EA = \text{ABS}((-K1+4\times K3+17\times K4-23\times K5+4\times K7-K8)/90)
\]

IF (EA.GT.E) GO TO 30
K9 = H*DSE(U1+H*S1-K6+2*K7)
IF (C.EQ.0.0) GO TO 50
M1 = H*DCHBSE(U1, S1, C)
M2 = H*DCHBSE(U+H/2, S+K1/2, C+M1/2)
M3 = H*DCHBSE(U+H/2, S+(K1+K2)/4, C+(M1+M2)/4)
M4 = H*DCHBSE(U+H, S-K2+2*K3, C-M2+2*M3)
C = C+(M1+M3*4+M4)/6
M5 = H*DCHBSE(U1, S1, C)
M6 = H*DCHBSE(U1+H/2, S1+K5/2, C+M5/2)
M7 = H*DCHBSE(U1+H/2, S1+(K5+K6)/4, C+(M5+M6)/4)
M9 = H*DCHBSE(U1+H, S1-K6+2*K7, C-M6+2*M7)
C = C+(M5*4*M7*M9)/6

50 S = S1+(K5+4*K7+K9)/6
U = U1+H
IF (U.GE.K0*DSC(NP)) NP = NP+1
IF (U.GE.K0*NP) STOP
IF (EA.E.GT.0.5) RETURN
CONH = 1.1*CONH
RETURN
30 H = H/2 $ CONH=CONH/2
GO TO 40
60 H = (DSC(NP)*K0-U)/2*E
GO TO 40
END
SUBROUTINE RKRB(E(U,R,C)

C THIS SUBROUTINE SELECTS AN APPROPRIATE STEP SIZE AND PERFORMS ONE STEP
C OF A RUNGE-KUTTA-ENGLAND 4TH ORDER PROCESS ON R AND CHI

REAL K0
REAL K1,K2,K3,K4,K5,K6,K7,K8,K9
REAL M1,M2,M3,M4,M5,M6,M7,M9
COMMON O, K0, B, BM, EBD, E, EA, DSCN/DSC(20)
COMMON/STEP/N, H, NP, NDFG, HMAX/RSTP/CONH
TS=DRE(U,R) $ TR=ABS(TS)
IF (TR.LT. CONH/HMAX) GO TO 10
H=CONH/TR $ GO TO 20

10 H=HMAX
20 IF (U+2*H.GE. DSC(NP)*K0) GO TO 60
40 K1=H*TS
K2=H*DRE(U+H/2, K+K1/2)
K3=H*DRE(U+H/2, K+(K1+K2)/4)
K4=H*DRE(U+H, R-K2+2*K3)
R1=R+(K1+4*K3+K4)/6
U1=U+H
K5=H*DRE(U1, R1)
K6=H*DRE(U1+H/2, R1+K5/2)
K7=H*DRE(U1+H/2, R1+(K5+K6)/4)
K8=H*DRE(U1+H, R+(-K1+96*K2+92*K3-121*K4+144*K5+6*K6-12*K7)/6)
CA IS THE ERROR ESTIMATE FOR THE R-K STEP. IF IT IS TOO LARGE,
THE STEP SIZE IS CUT IN HALF AND THE EVALUATION TRIED AGAIN.
WHEN THE ERROR IS FOUND ACCEPTABLE, THE STEP SIZE IS ALLOWED TO
CREEP UPWARD BY 10 PERCENT EACH STEP THE ERROR IS LESS THAN HALF
OF THE ALLOWABLE VALUE

\[
\begin{align*}
EA &= \text{ABS}((-K1+4*K3+17*K4-23*K5+4*K7-K8)/90) \\
& \text{IF}(EA \cdot \text{GT} . E) \text{GO TO} 30 \\
K9 &= H \cdot \text{DER}(U1+H \cdot R1-K6+2*K7) \\
& \text{IF}(C \cdot \text{EQ} . 0.0) \text{GO TO} 50 \\
M1 &= H \cdot \text{DCHBRE}(U \cdot R \cdot C) \\
M2 &= H \cdot \text{DCHBRE}(U+H/2 \cdot R+K1/2 \cdot C+M1/2) \\
M3 &= H \cdot \text{DCHBRE}(U+H/2 \cdot R+(K1+K2)/4 \cdot C+(M1+M2)/4) \\
M4 &= H \cdot \text{DCHBRE}(U+H, R-K2+2*K3, C-M2+2*M3) \\
C &= C+(M1+M3+4*M4)/6 \\
M5 &= H \cdot \text{DCHBRE}(U1 \cdot R1 \cdot C) \\
M6 &= H \cdot \text{DCHBRE}(U1+H/2 \cdot R1+K5/2 \cdot C+M5/2) \\
M7 &= H \cdot \text{DCHBRE}(U1+H/2 \cdot R1+(K5+K6)/4 \cdot C+(M5+M6)/4) \\
M9 &= H \cdot \text{DCHBRE}(U1+H \cdot R1-K6+2*K7, C-M6+2*M7) \\
C &= C+(M5+4*M7+M9)/6
\end{align*}
\]
C AT THIS POINT THE CHANGE IN R IS EXAMINED TO CHECK FOR ZERO CROSSINGS
C THE RUNNING TOTAL OF THESE CROSSINGS IS N
C

50 DELR=(K1+4*K3+K4+K5+4*K7+K9)/6
   IF(R*(R+DELR))1,2,3
1   N=N+1
    GO TO 3
2   IF(R+DELR.EQ.0)N=N+1
3   R=R+DELR
    U=U+H
    IF(U.GE.K0*DSC(NP))NP=NP+1
    IF(U.GE.K0*D)NDFG=1
    IF(EA/E.GT.0.5)RETURN
    CONH=1.1*CONH
    RETURN
30 H=H/2  CONH=CONH/2
    GO TO 40
60 H=(DSC(NP)*K0-U)/2+E
    GO TO 40
END
FUNCTION DSE(U,S)
C
C THIS FUNCTION COMPUTES THE DERIVATIVE OF S
C
REAL KO
COMMON D,KO,B
DSE=-S*S-(ER(U/KO)-B*B)
RETURN
END

FUNCTION DRE(U,R)
C
C THIS FUNCTION COMPUTES THE DERIVATIVE OF R
C
REAL KO
COMMON D,KO,B
DRE=1+(ER(U/KO)-B*B)*R*R
RETURN
END
FUNCTION DCHBSE(U,S,C)
C
C THIS FUNCTION COMPUTES THE DERIVATIVE OF CHI USING S
C
REAL KO
COMMON D,KO,B
DCHBSE=(2*S*C*((ER(U/KO)-B*B)-1)-2*B)/(1+S*S)
RETURN
END

FUNCTION DCHBRE(U,R,C)
C
C THIS FUNCTION COMPUTES THE DERIVATIVE OF CHI USING R
C
REAL KO
COMMON D,KO,B
DCHBRE=(2*R*C*((ER(U/KO)-B*B)-1)-2*B*R*R)/(1+R*R)
RETURN
END
FUNCTION ER(X,ER1,ER2)
C
THIS FUNCTION DESCRIBES THE ASYMMETRIC PROFILE WITH COSINUSOIDAL TRANSITION. THIS COMMON STATEMENT MUST APPEAR IN ALL ER(X) FUNCTIONS.
C
COMMON C/DSCN/DSC(20)
C
THE FOLLOWING STATEMENTS MUST CALCULATE THE VALUE OF ER AT THE POINT GIVEN BY X, ON THE INTERVAL FROM 0 TO D, AND ASSIGN THAT VALUE TO ER. THE LAST STATEMENT IN THIS SECTION MUST BE A RETURN.
C
IF(ABS(X-D/2).LT.W)GO TO 1
IF(X-D/2)2,2,3
1 ER=1.515+0.015*SIN((X-D/2)*PI/2/W) $ RETURN
2 ER=1.50 $ RETURN
3 ER=1.53 $ RETURN
C
THIS SECTION WHOSE ENTRY POINT MUST BE LABELED EMAX SERVES TO PROVIDE THE MAIN PROGRAM WITH VARIOUS CONSTANTS WHICH IT MAY REQUIRE. THESE MUST INCLUDE ER1, ER2, EMAX, AND AT LEAST THE FIRST MEMBER OF THE ARRAY DSC. CSC IS TO CONTAIN, IN INCREASING ORDER, THE POINTS OF DISCONTINUITY OF THE PROFILE BETWEEN 0 AND D, INCLUDING D BUT NOT INCLUDING 0. THUS IF THERE ARE OTHERWISE NO POINTS OF DISCONTINUITY, THEN DSC(1)=D, MUST AT LEAST APPEAR.
ENTRY EMAX
ER1=1.50
ER2=1.00
EMAX=1.53

C NOTICE THESE PARAMETERS WHICH ARE USED IN THE FIRST PART OF THIS
C FUNCTION ARE INITIALIZED HERE.

PI=3.14159265359
A=0.2
W=A*D
CSC(1)=(D-W)/2 $ CSC(2)=(C+W)/2 $ CSC(3)=D

C ADDITIONALLY, ANY PRINT STATEMENTS DESCRIBING THE PROFILE MAY BE
C INSERTED HERE IF DESIRED.

PRINT 100, ER1, ER2
100 FORMAT(*1FOR X .LT. O ER = *,E10.3/
1* FOR X .GT. D ER = *,E10.3)
PRINT 101,W
101 FORMAT(* THE GUIDE PROFILE VARIES FROM 1.50 TO 1.53 SINUSOIDALLY*/
1* OVER A TRANSITION WIDTH OF *,E10.3,* METERS*)
RETURN
END
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