High Bandwidth Scanner Based on Spatial-Spectral Holograms

Jingyi Xiong
jingyixiong@hotmail.com

Follow this and additional works at: https://scholar.colorado.edu/ecen_gradetds

Recommended Citation
Xiong, Jingyi, "High Bandwidth Scanner Based on Spatial-Spectral Holograms" (2010). Electrical, Computer & Energy Engineering Graduate Theses & Dissertations. 11.
https://scholar.colorado.edu/ecen_gradetds/11
High bandwidth scanner based on spatial-spectral holograms

by

Jingyi Xiong

B.S., Zhejiang University, 1999
M.S., Zhejiang University, 2002
M.S., University of Colorado, 2006

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Department of Electrical, Computer & Energy Engineering

2010
This thesis entitled:
High bandwidth scanner based on spatial-spectral holograms
written by Jingyi Xiong
has been approved for the Department of Electrical, Computer & Energy Engineering

______________________________
Kelvin H. Wagner

______________________________
Robert R. McLeod

Date ________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
I experimentally demonstrated a high bandwidth spatial-spectral holographic (SSH) scanner. Scanners or true time delay lines find their applications in phased-array antennas, radar range-Doppler processing and time-frequency ambiguity function analysis. A typical example of such a device is an acousto-optic deflector (AOD), which has limited bandwidth due to Bragg match conditions, frequency dependent acoustic attenuation of available materials and limitations of piezoelectric transducer technologies.

The system proposed in this thesis breaks through the bandwidth limitation of acousto-optic technology, yet resembles the function of an AOD since both operates as a scrolling scanner. It uses a material with large inhomogeneous bandwidth to record space-dependent time-delays as spatial-spectral holograms. The recording of the spatial-spectral holograms utilizes a Galvo scanning (GS) mirror and a chirped laser.

In Chapter 2, I experimentally show that a GS mirror can be sufficiently stable for the holographic recording process. After reviewing the relevant physics of the spatial-temporal holographic recording medium, the cryogenically-cooled rare earth doped crystals, in Chapter 3, I give further derivations that are useful in explaining the subsequent experimental results. Chapter 4 describes an efficient and stable numerical scheme for simulating the coherent light-atom interaction in a two-dimensional inhomogeneously-broadened crystal, allowing a search for the optimum experimental geometry for the recording experiment. Chapter 5 integrates the Galvo scanning mirror with the Tm$^{3+}$:YAG crystal, and gives the experimental demonstration of the first high bandwidth (1.5GHz bandwidth with 20 resolvable spots) spatial-spectral holographic scanner. This system uses one laser for the proof-of concept experiment. Finally, in Chapter 6, I explore the prospect for the future development of the high bandwidth SSH scanner. This chapter also gives the design and
demonstration of a two-laser stabilization circuit, with which we can extend our ability to realize the full version of the high bandwidth SSH scanner system.
Dedication

To my family, for all their love and support.
Acknowledgements

First and foremost, I would like to thank my advisor, Kelvin Wagner. Over the years, I have learned a lot from his passions over research and his breadth of knowledge. I will benefit from what I learned from him for the rest of my life.

I would like to thank the former and current students in KAOS group for all their help: Ben Braker, Max Colice, Dan Feldkhun, Lu Gao, Sangtaek Kim and Friso Schlottau. I especially want to thank Max for teaching me tremendous experimental tricks in the cool down experiments. I also want to thank Dan for giving me many useful advices on designing/debugging circuits.

I would like to thank Thomas Schibli for attending my design review and giving me great advices and help on designing and debugging the laser locking circuits.

I would like to thank Bengt Fornberg for teaching me various useful simulation techniques and spending time discussing with me about the simulation results.

I would like to thank Tiejun Chang for sharing with me his simulation experience.

I would also like to thank my friends and my family for all their love and support.

I am so lucky to have these talented and generous people helping me. Without their help and support, I could never get to this point.
Contents

Chapter

1 Introduction
   1.1 AOD as a scanner .................................................. 5
   1.2 An alternative approach ........................................... 11
   1.3 Thesis Outline ................................................... 18

2 Doppler processing systems
   2.1 Introduction ....................................................... 20
   2.2 GS mirror based Doppler processor ............................... 21
      2.2.1 Understanding the Doppler generation of a GS mirror .... 21
      2.2.2 Verification of the Doppler channels ......................... 26
         2.2.2.1 CCD as a time-integrating detector ..................... 27
         2.2.2.2 Photorefractive crystal as an integration medium .... 29
         2.2.2.3 Theoretical analysis of using SBN as an integration medium .... 31
         2.2.2.4 Experimental results of using SBN as an integration medium .... 43
   2.3 AOD as a Doppler Generator .................................... 49
      2.3.1 TeO$_2$ as an AOD material .................................. 52
         2.3.1.1 Phase matching of an AOD ............................... 55
         2.3.1.2 Acoustic rotation and optic rotation .................... 56
         2.3.1.3 Eigen polarizations of TeO$_2$ ......................... 61
2.3.2 Two cascaded AODs ................................................................. 62
  2.3.2.1 Topologies for in-plane AODs .............................................. 63
  2.3.2.2 Band shape of in-plane counter-propagating AODs ..................... 65
  2.3.2.3 Band shape of crossed AODs ................................................ 66
2.3.3 AOD Doppler verification experiment ....................................... 69
  2.3.3.1 Doppler slice verification experiment ..................................... 69
  2.3.3.2 Doppler bin verification experiment ...................................... 71

3 Two-level atoms ........................................................................... 75
  3.1 Two-level systems .................................................................. 76
  3.2 Rotating wave approximations .................................................. 78
    3.2.1 Rotating frame at laser frequency $\omega$ ................................. 79
    3.2.2 Rotating frame at atom resonant frequency $\omega_0$ - perturbation theory ................................................... 80
      3.2.2.1 Frequency domain perturbation theory - the full version ........ 83
      3.2.2.2 Frequency domain perturbation theory - with further approximations .................................................. 85
  3.3 Pulse vs chirp photon echo ...................................................... 89
    3.3.1 Pulse echo ........................................................................ 90
    3.3.2 Chirp echo ....................................................................... 93

4 Simulating light propagation in inhomogeneously broadened crystals .... 96
  4.1 Introduction ........................................................................... 96
  4.2 Numerical schemes to the Maxwell-Bloch equations ....................... 97
    4.2.1 The approximations we make ................................................. 99
    4.2.2 The Bloch equations ............................................................. 100
      4.2.2.1 Stability Analysis for ODEs .......................................... 103
      4.2.2.2 Numerical Schemes for Bloch Equations ......................... 107
      4.2.2.3 Numerical solutions to the Bloch equations ....................... 112
    4.2.3 2D Maxwell’s equations ...................................................... 113
4.2.3.1 The standard FFT-BPM .......................... 115
4.2.3.2 The modified FFT-BPM .......................... 116
4.2.3.3 Modified FFT-BPM using the paraxial approximation term .......................... 118

4.3 Carrying out the numerical schemes .......................... 119
4.3.1 Choosing the right parameters .......................... 121
4.3.2 Maxwell’s equation simulation .......................... 122
4.3.3 Bloch equation simulation .......................... 124
4.3.4 Maxwell-Bloch simulation for pulse photon echo .......................... 127

4.4 Simulation results .......................... 130
4.4.1 Rabi Oscillation .......................... 130
4.4.2 Two-pulse and three-pulse photon echo .......................... 132
4.4.3 Chirp generated photon echo .......................... 134

4.5 Experimental results .......................... 138
4.6 Discussions .......................... 141

5 High Bandwidth SSH scanner 143
5.1 Introduction .......................... 143
5.2 The general picture .......................... 146
5.3 Theoretical analysis of SSH scanner .......................... 151
5.3.1 Diffraction off the causal time edge .......................... 154
5.3.2 Finite recording bandwidth in the spectral domain .......................... 156
5.3.3 Nonlinearity of the “ramp” .......................... 157

5.4 Experimental setup for 1-laser high bandwidth scanner .......................... 160
5.4.1 Diagnostic experiments .......................... 168
5.4.1.1 Single spot spectral grating .......................... 171
5.4.1.2 Koheras chirping and characterization .......................... 172
5.4.2 Spectral grating read out experiment .......................... 173
A.0.1.3 Wavefront analyzer ................................................. 237
A.0.2 Arbitrary waveform generator (arb) programming ............ 238

B Simulation codes .......................................................... 242

C Circuit diagrams (schematic, PCB layout) .......................... 248

D Publications ................................................................. 254
Tables

Table

4.1 Numerical simulations schemes for the Maxwell-Bloch equations . . . . . . . . . . . . 98
## Figures

### Figure

1.1 Range-Doppler radar. ................................................................. 2  
1.2 Correlation of the transmitted signal and the sum of the transmitted and returned signals. ......................................................... 3  
1.3 AOD as a scanner. ................................................................. 5  
1.4 2-D uncertainty box for an AOD. ..................................................... 6  
1.5 Birefringent tangential phase matching condition. ......................... 10  
1.6 Two level atoms ................................................................. 11  
1.7 Photon echo correlator. ............................................................. 12  
1.8 S2CHIP correlator. ................................................................. 14  
1.9 SSH scanner correlator. .............................................................. 15  
1.10 SSH RF spectrum analyzer box geometry. ..................................... 15  
1.11 Copying AOD time delays to SSH crystal. .................................... 16  
1.12 Time shift and frequency shift are inter-changeable for a linear frequency chirp. ........................................................... 16  
1.13 Two chirps with spatial dependent time delays record sinusoids with changing periods along the spatial dimension. ........................................... 17  
2.1 Mirror Analysis Without Approximations ........................................ 21  
2.2 Normal incidence to the GS mirror. ............................................... 24  
2.3 Doppler verification setup using CCD as a time-integrating medium. ........................................................... 28
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.4</td>
<td>Doppler verification result - using CCD as a time-integrating medium.</td>
<td>30</td>
</tr>
<tr>
<td>2.5</td>
<td>Doppler verification setup - using SBN as a bias-free time-integrating correlator.</td>
<td>38</td>
</tr>
<tr>
<td>2.6</td>
<td>Two situations when SBN integration time $\tau &lt; 2T$.</td>
<td>39</td>
</tr>
<tr>
<td>2.7</td>
<td>In-plane hologram recording.</td>
<td>40</td>
</tr>
<tr>
<td>2.8</td>
<td>Out-of-plane hologram recording.</td>
<td>42</td>
</tr>
<tr>
<td>2.9</td>
<td>Doppler slices results.</td>
<td>44</td>
</tr>
<tr>
<td>2.10</td>
<td>GS mirror generated Doppler spots verification.</td>
<td>47</td>
</tr>
<tr>
<td>2.11</td>
<td>Doppler spot experimental results.</td>
<td>48</td>
</tr>
<tr>
<td>2.12</td>
<td>Multiple Doppler channel results.</td>
<td>49</td>
</tr>
<tr>
<td>2.13</td>
<td>Particle picture of acousto-optic interactions.</td>
<td>49</td>
</tr>
<tr>
<td>2.14</td>
<td>2-D uncertainty box for an AOD.</td>
<td>51</td>
</tr>
<tr>
<td>2.15</td>
<td>TeO$_2$ acoustic slowness surface.</td>
<td>53</td>
</tr>
<tr>
<td>2.16</td>
<td>TeO$_2$ optical momentum surface and acoustic momentum surface.</td>
<td>54</td>
</tr>
<tr>
<td>2.17</td>
<td>Tangential phase matching of AOD.</td>
<td>56</td>
</tr>
<tr>
<td>2.18</td>
<td>Acoustically unrotated and rotated AOD.</td>
<td>57</td>
</tr>
<tr>
<td>2.19</td>
<td>Conoscopic Pattern Setup.</td>
<td>58</td>
</tr>
<tr>
<td>2.20</td>
<td>Conoscopic Pattern for Unrotated AODs.</td>
<td>59</td>
</tr>
<tr>
<td>2.21</td>
<td>Conoscopic pattern for acoustically rotated AODs.</td>
<td>59</td>
</tr>
<tr>
<td>2.22</td>
<td>Eigen polarizations of TeO$_2$.</td>
<td>60</td>
</tr>
<tr>
<td>2.23</td>
<td>Topologies for cascaded in-plane Bragg cells.</td>
<td>62</td>
</tr>
<tr>
<td>2.24</td>
<td>Band shapes for single AODs.</td>
<td>65</td>
</tr>
<tr>
<td>2.25</td>
<td>Cascaded in-plane counter-propagating AODs.</td>
<td>66</td>
</tr>
<tr>
<td>2.26</td>
<td>k-space of TeO$_2$ AODs.</td>
<td>67</td>
</tr>
<tr>
<td>2.27</td>
<td>Band shape for the cascaded counter-propagating AODs.</td>
<td>67</td>
</tr>
<tr>
<td>2.28</td>
<td>Cascaded crossed AODs.</td>
<td>68</td>
</tr>
<tr>
<td>2.29</td>
<td>Band shape for the cascaded counter-propagating AODs.</td>
<td>69</td>
</tr>
<tr>
<td>2.30</td>
<td>AOD Doppler slice verification setup.</td>
<td>70</td>
</tr>
</tbody>
</table>
4.15 Schematic setup of the 3-pulse off-axis photon echo simulation. .................. 128
4.16 Maxwell-Bloch simulation for 3-pulse photon echo. ............................... 129
4.17 Integrated photon echo amplitude strength, $\int |E_{\text{echo}}| dt$, for 3-pulse photon echo with varying spatial steps. .......................................................... 130
4.18 The simulation results of Rabi oscillations of a Gaussian spatial shape long pulse. 131
4.19 Bloch vectors’ moving traces on a Bloch sphere. ..................................... 132
4.20 Spatial spectral gratings recorded by the time delayed angularly interfered Gaussian beams. .......................................................... 133
4.21 Angular output of the crystal. ................................................................. 134
4.22 Schematic setup of the chirp photon echo. ............................................ 135
4.23 Off-axis chirp photon echo simulation results. ....................................... 136
4.24 Interaction angle vs. echo efficiency. .................................................... 137
4.25 Photon echo experimental setup. .......................................................... 138
4.26 Experimental results of integrated photon echo intensity vs. interaction angle for different focal spot sizes at the center of the SSH crystal. .................... 140
4.27 Interaction angle vs. beam intensity weighted overlapped area. ............... 141
5.1 Two level atoms ........................................................... 144
5.2 The basic idea of the SSH scanner. ....................................................... 146
5.3 Time shift and frequency shift are inter-changeable for a linear frequency chirp. 147
5.4 Galvo scanning mirror generates linear frequency shift along the rotation direction. 149
5.5 A simple example of the high bandwidth SSH scanner. .......................... 151
5.6 Plot of the simulated diffraction edge kernel with finite bandwidth. .......... 157
5.7 Ramp approximations and error ......................................................... 158
5.8 Sine gratings written by nonlinear ramps ............................................. 159
5.9 Effects onto the gratings due to nonlinear ramp, nonlinear time delay, nonlinear readout, and nonlinear mirror response. ................................. 160
5.10 Experimental setup of 1-laser high bandwidth SSH scanner.  161
5.11 Top view and side view of beam 1 and beam 2 before the SSH crystal.  164
5.12 Echo and reference beam combined by an interferometer that consists of a PBS and polarizer.  165
5.13 High bandwidth SSH scanner electronic setup.  167
5.14 Time domain sequences of the high bandwidth SSH scanner electronic boxes.  169
5.15 Single spot spectral hole burning experiment time-frequency waveforms.  170
5.16 Single spot spectral hole burning readout experimental result.  171
5.17 Single spot spectral grating readout experiment frequency vs time waveforms.  172
5.18 Single spot spectral grating readout experimental results.  173
5.19 RSA waveform of Koheras-Velocity beat.  174
5.20 The power spectral density of spot-spot, spot-line and line-line experiments.  175
5.21 An example of the readout grating for a spot-line experiment.  176
5.22 Using 1-8 cosine order(s) to approximate the linear ramp.  177
5.23 Spot-line experiment.  178
5.24 Time delays generated at each spatial positions.  179
5.25 CCD images of the diffracted echo signal.  180
5.26 Profiles of the diffracted causal edge.  181
5.27 Time integrating correlation result for different GS mirror driving voltages and different time delays.  181
5.28 GS mirror driving voltage changes the number of resolvable of the system.  186

6.1 Two-laser high-bandwidth "Bragg cell" system.  189
6.2 Spectral feature with complex Lorentzian lineshape.  195
6.3 \(\Delta \delta\) and \(\Delta^2 \phi\) when \(\omega_m = 4\Delta \Omega\).  195
6.4 \(\Delta \delta\) and \(\Delta^2 \phi\) when \(\omega_m = 5\Delta \nu_{\text{fsr}}\) and cavity finesse is 500.  197
6.5 The basic idea of phase-lock loop.  198
6.6 Two-laser locking system. .................................................. 199
6.7 Close loop system response of a step function response when there is P, PI and PID
gain. .................................................................................. 202
6.8 Detailed block diagram of the 2-laser locking system. ....................... 202
6.9 Functional block diagram of AD9901. The figure is modified from AD9901 data
sheet from Analog Devices. .................................................. 205
6.10 AD9901 timing waveforms when two inputs are at the same frequency. ... 206
6.11 Phase gain plot of AD9901. .................................................. 206
6.12 AD9901 outputs when the two input frequencies are substantially different. ... 207
6.13 PCB block diagram - arranging the components according to signal flow. ... 210
6.14 Use the sines at the same frequency generated by a two-channel arb for RF and LO
input ports and probe the outputs of each of the chips for debugging the circuit. ... 215
6.15 AD9901 outputs when $f_{LO} \neq f_{RF}$. ........................................ 216
6.16 RSA3800 time-frequency diagram of the unstable two-laser beat-note with only pro-
portional gain. ........................................................................ 217
6.17 Time-frequency waterfall display of the two-laser beat-note with proportional gain
and differential gain. ................................................................. 217
6.18 Time-frequency waterfall display of the two-laser beat-note with proportional gain,
differential gain and integral gain. ............................................ 218
6.19 Locking spectrum of the Koheras-Velocity beat-note. ......................... 219
C.1 Schematic diagram of the locking circuit, part I - frequency to voltage conversion. .. 249
C.2 Schematic diagram of the locking circuit, part II - PID. ............................ 250
C.3 Schematic diagram of the locking circuit, part III - the voltage source. .......... 251
C.4 PCB board diagram power plane. ............................................. 252
C.5 PCB layout for manufacturing. ................................................ 253
Chapter 1

Introduction

Optical signal processing, due to its wide-bandwidth, parallelism, and physical implementation of many of the required transformations, plays an important role in RF signal processing. One of the problems in RF signal processing is range-Doppler processing, i.e. to pinpoint the time delay and velocity of a remote target or to image its internal structure. Fig. 1.1 shows the concept of range processing. The radar sends out RF signals $s_t(t)$, in this case random binary signals, to a remote target. The signals scattered back from the target, is attenuated to and time delayed to be

$$s_r(t) = \sum \sigma_n s_t[t - 2R_n(t)/c],$$

(1.1)

where $c$ is the speed of light in air, $R_n(t)$ is the distance of the multiple reflection points on the target to the radar. The time delay $t_f$ between the transmitted signals $s_t(t)$ and the returned signal $s_r(t)$ will help us figure out the distance of the remote target,

$$L = \frac{t_f c}{2}.$$  

(1.2)

To find out the time delay $t_f$, one can correlate the transmitted signal $s_t(t)$ with the returned signals $s_r(t)$

$$s_t(t) \star s_r(t) = \int s_t^*(\tau - t) s_r(\tau) d\tau,$$

(1.3)

where $\star$ is correlation, find out the peak of the correlation, and then measure the distance between the correlation peak and the center of the transmitted signal. An example of the correlation is shown in Fig. 1.2. To calculate the correlation function, one wants to get time-delayed transmitted
signals $s_t(t - \tau)$ with various time delay $\tau$. Getting these time-delayed transmitted signals is the essence of calculating the correlation function.

We can calculate the correlation function using digital methods, which involves discretizing the correlation Eq. (1.3) as

$$\text{Corr}[n] = \sum_{k=1-N}^{N-1} s_t^*[k - n]s_r[k],$$

(1.4)

where $N$ is the number of sampling data points in the signal. Calculating this equation involves calculating $N^2 - N + 1$ multiplications and $(2N - 1)^2$ summations. The computational complexity is $O(N^2)$.

Fast calculations use Fast Fourier Transform (FFT), a procedure discovered by Cooley and Tukey\cite{2} (it goes back at least to unpublished work of Carl F. Gauss around 1805). We first Fourier transform $s_t(t)$ and $s_r(t)$, calculate the multiplication of $S_t^*(f)$ and $S_r(f)$, then inverse Fourier transform to get $s_t(t) * s_r(t)$. The process can be mathematically described as following:

$$s_t(t) * s_r(t) = \mathcal{F}^{-1}\{S_t^*(f)S_r(f)\},$$

(1.5)

where $S_t(f) = \mathcal{F}\{s_t(t)\}$, $S_r(f) = \mathcal{F}\{s_r(t)\}$, and $\mathcal{F}\{\cdot\}$ is the Fourier transform. So now the correlation calculation involves two Fourier transformations, multiplications in the spectral domain and two inverse Fourier transformations. Using the recursive and periodic property of the discrete Fourier transforms, the number of operations using the FFT procedure Cooley and Tukey\cite{2}
Figure 1.2: Correlation of the transmitted signal and the sum of the transmitted and returned signals gives two correlation peaks, the separation $t_f$ of which indicates the flying time of the transmitted pulse.

It is suggested is $\frac{N}{2}\log_2 N$. To calculate one data point of the spectrum, we need to calculate $N$ multiplications,

$$S_r[k] = \sum_{n=0}^{N-1} s_r[n] e^{-i2\pi nk/N}.$$  \hspace{1cm} (1.6)

To calculate one data point of the spectrum $S_r[k]$, the FFT algorithm repeatedly subdivides the data into sublists that are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, \ldots of the original data size $N$. The total number of operations can be analyzed as the following[3]. The list sizes for each of the iteration are:

$$N, N/2^1, N/2^2, N/2^3, \ldots, N/2^m.$$
Eventually there will be an integer $m$ such that

$$N/2^m < 2.$$  

$m$ is the first integer for $N/2^m < 2$, thus

$$N/2^{m-1} \geq 2.$$  

Therefore we have

$$2^m \leq N < 2^{m+1}.$$  

Take the log of each term and the number of operations $m$ has the following property:

$$m \leq \log_2 N < m + 1.$$  

To calculate each of the spectral coefficient $S_r[n]$ ($n = 0, 1, 2, \ldots, N - 1$), it takes $N\log_2 N$ operations. Considering the fact that the Fourier spectrum is symmetric, the total number of operations to calculate one Fourier transform is $\frac{N}{2}\log_2 N$. Therefore, the total number of operations to calculate the correlation is on the order of $2N\log_2 N$, much less intensive than the straight forward $N^2$ calculations.

Optical solutions use physics properties, such as Fourier transform property of a lens and spectral analysis property of a spectral-hole burning crystal [4], to perform the correlation calculations and the complexity of calculations does not scale with the number of elements involving in the calculations. Although the final processing in optical solutions usually requires digitizing the data, the final processing result does not need to be high bandwidth [5] even with high bandwidth signal inputs.

Optical solutions to the correlation problem utilize the spatial dimensions to perform parallel processing. A typical transformation in optical processing is to map the time domain signal into space,

$$s_t(t - \tau) \leftrightarrow s_t(t - \alpha x),$$  

(1.7)
where $\alpha$ is the mapping coefficient and $x$ is the spatial dimension the time domain signal is mapped into. A scanner that spatially displays the time-delayed transmitted signals across a transverse spatial dimension can help us calculate the correlations.

1.1 AOD as a scanner

People use acousto-optic deflectors (AODs) as a scanner for correlation function calculation [6–10]. In Fig. 1.3, an RF pulse $g(t)$ is sent to the AOD. The vibrations of the AOD transducer broadcast acoustic waves across the AOD aperture with velocity $v$. Thus when the pulse propagates to position $x$, it experiences a time delay $\frac{x}{v}$. With a Gaussian shaped incident beam, the diffracted optical beam is modulated with the traveling pulse, giving

$$s(t,x) = g(t - \frac{x}{v}) \Pi \left( \frac{x}{A} \right) \exp(-\alpha_0 x) \exp \left[ - \left( \frac{x - A/2}{W} \right)^2 \right],$$

(1.8)
Figure 1.4: An AOD with aperture size $A$ and transducer length $L$ is driven by a single tone RF frequency. The region that the acoustic beam is overlapped with the optical beam is approximately a rectangular. The Fourier transform of the overlapped region is a 2-D sinc function with size $\frac{2\pi}{A} \times \frac{2\pi}{L}$.

where $A$ is the aperture of the AOD, $\Pi(t)$ is a rectangular function, $\alpha_0$ is the acoustic attenuation coefficient, and $W$ is the width of the incident Gaussian shaped optical beam. The diffracted beam $s(t, x)$ is a pulse with time delay $\frac{x}{v}$. Later in chapter 2, this property is also used to generate linear Doppler frequencies.

The AODs, however, have bandwidth limitations. Use isotropic diffraction as an example, shown in Fig. 2.14. The 1st order diffraction occurs when the incident beam angle equals the diffracted beam angle and the angle defined by

$$\sin \theta_B = \frac{\lambda_0}{2n\Lambda},$$

where $n$ is the refractive index, $\lambda_0$ is the free-space optical wavelength, and $\Lambda$ is the acoustic bandwidth.

---

1 The rectangular function $\Pi(t)$ is defined as

$$\Pi(t) = \begin{cases} 
1 & : |x| < \frac{1}{2} \\
\frac{1}{2} & : |x| = \frac{1}{2} \\
0 & : \text{otherwise.}
\end{cases}$$
wavelength. In practice, both the optical wave and the acoustic wave have finite beam widths \( A \) and \( L \). We call the Fourier transform of the overlapped region of the optical beam and the acoustic beam an uncertainty box. In the case shown in Fig. 2.14 (neglecting Gaussian laser profile and exponential acoustic attenuation), the uncertainty box is a 2D sinc function with the width to the first zero a size of \( \frac{2\pi}{A} \times \frac{2\pi}{L} \). The uncertainty box indicates the area that the incident optical k-vector, \( \mathbf{k}_{\text{in}} \), can be kicked into in momentum space. The overlap between the uncertainty box and the momentum circle of the optical beam with a radius of \( |\mathbf{k}| = \frac{2\pi}{\lambda} n \) giving the allowed propagation modes of the optical beam. The uncertainty box is frequently used to help us analyze the acousto-optical interaction in momentum k-space [11–13].

The momentum mismatch for the \( m \)th order diffraction, \( \Delta k_m \), is given by [6]

\[
\Delta k_m = \frac{\pi \lambda_0 m}{n \Lambda^2 \cos \theta} \left( m - \frac{2n \Lambda}{\lambda_0} \sin \theta \right), \tag{1.10}
\]

At \( z = L \), the normalized intensity for the first order diffraction is approximately [6]

\[
I_1 \cong I_{1p} \text{sinc}^2 \left( \frac{\Delta k_1 L}{2\pi} \right) \tag{1.11}
\]

where \( \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \) and

\[
I_{1p} = \sin^2 \eta^{1/2} \tag{1.12}
\]

is the peak intensity of the diffracted light beam under exact momentum matching condition (\( \Delta k_1 = 0 \)) and \( \eta \) is given by

\[
\eta = \frac{\pi^2}{2\lambda_0^2} \left( \frac{n^6 p^2}{\rho v^3} \right) \frac{P_a L}{H \cos^2 \theta}, \tag{1.13}
\]

where \( p \) is the effective elastooptic coefficient for the particular mode of acoustooptic interaction, \( \rho \) is the mass density, \( v \) is the acoustic velocity, \( P_a \) is the acoustic power, \( L \) is the transducer length, \( H \) is the transducer height. An approximate formula for the 3-dB bandwidth of the diffracted light intensity (i.e., for \( I_1 = 0.5 \)) is given by

\[
\Delta f = 1.8 \frac{f_0 L_0}{L} = 1.8 \frac{nv^2 \cos \theta}{\lambda_0 f_0 L}, \tag{1.14}
\]
where \( f_0 \) is the center acoustic frequency, \( L_0 = n\lambda_0 \tan \theta \) is a characteristic length and \( L/L_0 \) is the number of acoustic fringes that the Bragg matched incident beam crosses. For \( L >> L_0 \), the acousto-optic device operates at the Bragg regime, while for \( L << L_0 \), the device operates at the Raman-Nath regime. With a chosen acoustic material, the smaller the transducer length \( L \), the bigger the bandwidth. However, when \( L \) is too small, the device may not be operating in Bragg regime and the diffraction efficiency will be low.

Another factor that limits the bandwidth of AODs is the acoustic attenuation. To look at this effect, let’s first take a look at the resolution of the AODs. The resolution, or the maximum number of resolvable angular positions of an AOD is defined as the range of deflection angles \( \Delta \theta_d \), divided by the angular spread of the diffracted beam \( \delta \theta_d \):

\[
N = \frac{\Delta \theta_d}{\delta \theta_d}. \tag{1.15}
\]

The deflection angle is \( \theta_d = 2\theta_B \), where \( \theta_B \) is given by Eq. (1.9). Therefore, the range of deflection angle for a frequency range \( \Delta f \) is

\[
\Delta \theta_d = \frac{\lambda_0 \Delta f}{nv \cos \theta}. \tag{1.16}
\]

For AODs, the divergence of the diffracted beam is equal to that of the incident beam,

\[
\delta \theta_d = \frac{\lambda_0}{nA}, \tag{1.17}
\]

where a near unity constant factor is ignored. From Eq. (1.16) and Eq. (1.17), Eq. (1.15) becomes,

\[
N = \tau \Delta f, \tag{1.18}
\]

where \( \tau = A/(v \cos \theta) \) is the acoustic transit time across the optical beam.

For most crystalline solids, the acoustic attenuation is proportional to \( f^2 \) and is given by

\[
\mathcal{L}[\text{dB}] = \alpha_0 \tau f^2 \tag{1.19}
\]

where \( \alpha_0 [\text{dB/\mu sec\cdot GHz}^2] \) is the acoustic attenuation coefficient. For the maximum allow frequency \( f_{\text{max}} \) and octave bandwidth \( (f_{\text{max}} = 2\Delta f) \), the attenuation \( \mathcal{L} \) is proportional to the bandwidth \( \Delta f \)

\[
\mathcal{L} = 4 \cdot 10^3 \cdot N\alpha_0 \Delta f, \tag{1.20}
\]
which says that for an allowed attenuation of, say 4dB, the bandwidth of the AOD is limited to be

\[ \Delta f = \frac{1}{10^3 N \alpha_0}. \]  

(1.21)

For the typically reported attenuation of \( \alpha_0 = 1 \, [\text{dB/\mu sec-GHz}^2] \) for LiNbO\(_3\) and with \( N = 200 \) resolvable spots, the bandwidth \( \Delta f \) is limited to 5GHz.

One might want to ask: all your bandwidth limitation is based on the assumption that we can not bear less diffraction efficiencies. What if we are able to tolerate, say 10dB of acoustic attenuation, does that mean we can get 10GHz of bandwidth? Unfortunately, there are other hard limitations to the bandwidth. The geometric length constrain for the acoustic \( \mathbf{K} \) vector and the transducer bandwidth limitation are two of them. Eq. (1.14) is the approximate formula for 3-dB bandwidth of the isotropic diffraction, where the acoustooptic diffraction does not change the polarization of the incident light. In birefringent diffractions where the incident and diffracted polarizations are different, the acoustic \( \mathbf{K} \) vector can satisfy the tangential phase matching condition, as shown in Fig. 1.5, and therefore the momentum condition can be satisfied over a wider acoustic bandwidth than the isotropic diffraction case [6],

\[ \left( \frac{\Delta f}{f_0} \right)_\text{bir} = \sqrt{2} \left( \frac{\Delta f}{f_0} \right)_\text{iso}^{1/2}, \]  

(1.22)

we can use the low attenuation LiNbO\(_3\) crystal as an example to look at the bandwidth limitation due to the birefringence separation of the momentum of LiNbO\(_3\). Fig. 1.5 shows the tangential phase matching of a birefringent crystal. At 532nm, the refractive index of LiNbO\(_3\) are \( n_o = 2.3231 \) and \( n_e = 2.2342 \). The acoustic \( \mathbf{K} \) can be calculated by

\[ |\mathbf{K}| = k_0 \sqrt{n_o^2 - n_e^2}, \]  

(1.23)

and the center acoustic frequency is

\[ f_0 = \frac{v}{\lambda} = \frac{v}{\lambda_0} \sqrt{n_o^2 - n_e^2} = \frac{6.57 \times 10^5 \text{cm/sec}}{532\text{nm}} \sqrt{2.3231^2 - 2.2342^2} = 7.9\text{GHz}. \]  

(1.24)

With octave bandwidth, the bandwidth of the AOD in this case is \( \frac{2}{\pi} f_0 \approx 5.2\text{GHz}. \)

\(^2\) The refractive indices are calculated by \( n = (A + B/(\lambda^2 + C) + D\lambda^2)^{1/2}, \lambda; \mu m. \) For \( n_o, A = 4.9048, B = 0.11768, C = -0.0475, D = -0.027169; \) for \( n_e, A = 4.582, B = 0.099169, C = -0.044432, D = -0.02195 \) [14, 15].
Another bandwidth limiting factor is due to the transducer technology. The transducer bonding material (the material that bonds the transducer to an acoustooptic medium) needs to have low acoustic attenuation, such as gold (it is highly mismatched with most acoustooptic materials), and have matched mechanical impedance (otherwise acoustic waves will be reflected back to the source) to the acoustooptic materials, such as indium (it has high acoustic attenuation). Chang used aluminum, which has mechanical impedance close to that of indium and acoustic attenuation only slightly higher than that of gold, to achieve transducer conversion loss of less than 1dB at 350MHz [6]. The physical thickness of the transducer also needs to be brought down to fraction of the acoustic wavelength to minimize the acoustic reflection. Huang et al. [16] have reported to achieve a transducer of less than 20dB one-way-loss operation up to 11GHz.

With all the constrains that the acousto-optic technologies have, the bandwidth of the acousto-optic deflectors could be at most several GHz. Of the reported AODs, the highest bandwidth of a practical high performance AOD had been fabricated by Thomson in Nice, France to have 2GHz of bandwidth (from 3 to 5 GHz).
Figure 1.6: The rare earth doped crystals can be modeled by two-level atoms. The doped Tm$^{3+}$ ions experience different local environment depending on the positions they are at. And therefore the resonant frequency of each of the atoms will be shifted with respect to each other, resulting in inhomogeneously broadened bandwidth. When light at the resonant wavelength, 793.3 nm for Tm$^{3+}$:YAG, impinges on the crystal, ions are excited from the ground state to the excited state, changing the population inversion at that resonant frequency and leaving behind a 'spectral hole' in the absorption profile of the crystal. The ions stay at the excited state for a lifetime of 10msec and then relax back to the ground state.

1.2 An alternative approach

This thesis provides an alternative way of realizing the scanner without being limited by the intrinsic bandwidth limitations of the AODs. As is mentioned in the last section, our goal is to generate time-delayed versions of the input signal $g(t)$ along one spatial dimension, say $x$, giving an output $g(t - \alpha x)$ ($\alpha$ is a coefficient depending upon the chosen method). This scanner consists of three important parts: a chirped laser, a Galvo-controlled mechanically scanning mirror and a spatial-spectral hole-burning (SSH) crystal as a recording medium. Spectral hole burning crystals are crystals, such as YAG, LiNbO$_3$, etc, with dopant rare-earth ions (Tm$^{3+}$, Er$^{3+}$, etc.), which
exhibit a large inhomogeneous bandwidth. In this thesis, we use a Tm$^{3+}$:YAG crystal. When cryogenically cooled to about 4K, the homogeneous linewidth of Tm$^{3+}$:YAG crystal can be under 25kHz while the inhomogeneous bandwidth can exceed 20GHz, yielding nearly $10^6$ resolvable spectral channels. The inhomogeneously broadened 2-level atoms are resonant to different frequencies and change their population inversion accordingly at that specific incident frequency, resulting in spectral hole features in the absorption band as shown in Fig. 1.6. As a consequence, the power spectrum of a modulated incident light beam can be recorded in the inhomogeneous band and this crystal is a natural spectrum analyzer. It records the power spectrum of the incident beam in parallel, avoiding the typical sequential scanning process in conventional spectrum analyzers[4]. This property has many applications. The inhomogeneous band of the crystal can be employed for variable true-time delay programming[17], spectrum analysis[18–20], and data storage [21, 22]. When adding the spatial dimension of the crystal, we record spatial-spectral holograms into the crystal and will have more computational power and processing flexibility: Colice et. al. [23] uses the spatial dimension for pulse repetition frequency cuing RF spectrum analysis; Schlottau et. al.[24] for time-integrating spectrum analysis; Braker et. al.[25] for squint-free RF beamforming.

In this thesis, we will use both the spatial and spectral dimension of the crystal to record different time delays for ultra-wide bandwidth signal processing. We call the scanner a “High bandwidth SSH scanner”.

We verify the function of the SSH scanner by building a time-integrating correlator with it.
Previously, people also use the SSH crystal to build correlators. Bai et al.[26] built a photon echo correlator. The idea is that first two time-delayed signals $E_1$ and $E_2$ write spectral gratings in the spectral domain of the SSH crystal and a brief pulse $E_3 = \delta(t)$ reads out the correlation result as an echo, as is shown in Fig. 1.7. The output after the readout pulse $E_3$ will be $E_1 \ast E_2$. Since the correlation result has the same bandwidth as the two input signals $E_1$ and $E_2$, the output needs high bandwidth digitizing. Merkel et al.[5] developed a spatial-spectral coherent holographic integrating processor (S2CHIP) correlator that does not need high bandwidth digitizing. Instead of reading out the spectral gratings written by the first two pulses with a brief pulse, S2CHIP correlator uses a chirped laser to tune through the spectral band and read out the spectral gratings directly and detect on a high dynamic range, low bandwidth photodetector. It then uses a digital Fourier transform of the readout data and locates the correlation peak. However, this correlator can only integrate up to the lifetime of the SSH crystal. Our SSH scanner correlator pre-records space dependent time delay gratings in the spectral domain of the SSH crystal and calculates the correlation in the spatial domain on a time-integrating CCD. The correlation result only needs low bandwidth digitizing and the correlator can be integrated up to the CCD integration time. Therefore it can have higher processing gain than the S2CHIP correlator.

The idea of the SSH scanner originates from an SSH RF spectrum analyzer[27]. The SSH RF spectrum analyzer uses a 3D box geometry, as shown in Fig. 1.10, to record angular-dependent spectral gratings in the spectral domain of the SSH crystal. As the beam $\vec{k}_1$ scans across different angles, the laser also linearly changes its frequency. Beams $\vec{k}_1$ and $\vec{k}_2$ can then record spectral gratings in the SSH crystal that act as a dispersive grating. The RF signal $S(t)$ is modulated onto the optical carrier by using a Mach-Zehnder modulator comes in the opposite direction ($\vec{k}_3$) of beams $\vec{k}_1$ and $\vec{k}_2$ to diffract off the recorded spectral gratings. The different RF frequency content in beam $\vec{k}_3$ will be diffracted to different directions.

Our scanner is a space domain Fourier-transformed version of the the SSH RF spectrum analyzer. Instead of recording frequencies that are linearly dependent on angles as spectral gratings in the SSH crystal, we record time delays that are linearly dependent on spatial positions in the
SSH crystal. Schlottau and Wagner[24] demonstrated a low bandwidth version of the SSH scanner that copies the AOD time delays to the SSH crystal. As shown in Fig. 1.11, a pulse is launched into both the AOD and the AOM. The AOD aperture is imaged to the SSH crystal by using a 4f imaging system. The diffracted beam from the AOM illuminates the whole SSH aperture. As the pulse travels across the AOD aperture, it experiences increasing time delays with increasing...
distance from the AOD transducer. The time-delayed pulses from the AOD and the pulse from the AOM interfere and record spectral gratings with different periodicities in the spectral domain of the SSH crystal. Later, the signal beam is modulated by the AOM for readout. Time-delayed versions of the signal beam are diffracted at different spatial position of the SSH aperture. Since an AOD and an AOM are used for the spectral-grating recording process, the bandwidth of the
Figure 1.11: Copying AOD time delays to SSH crystal [figure from [28]].

The system is limited by the bandwidth of the AOD and AOM.

Figure 1.12: Time vs. frequency plots of two identical chirps with (a) time shift and (b) frequency shift between them. (c) Ignoring the edge effect, time shift and frequency shift are inter-changeable for a linear frequency chirp. This is a good approximation when the frequency shift is small with respect to the chirping bandwidth.

To break through the bandwidth limitation, the SSH scanner uses a scanning mirror and a chirped laser for the recording process. The idea of the scanner is to first generate space-dependent frequency shifts and then turn these into space-dependent time delays by using the combination of a chirped laser and a Galvo scanning (GS) mirror. A GS mirror with constant angular velocity $\Omega$ modulates the normal incident light $\lambda$ with a Doppler shift of $\frac{2\Omega x}{\lambda}$ at spatial position $x$ (please refer to Chapter 2 for more details). But how could a GS mirror producing a few KHz of spatially varying Doppler shift produce any time delay? Well, it usually can not, at least not for a significant amount, but for one specific situation, it can. That is when the incident laser is linearly frequency...
chirped. As shown in Fig. 1.12, a time shift and a frequency shift of a linear chirp are similar, although not identical. For example, a linear chirp (chirp rate $b$) with frequency shift $\Delta f$ will be equivalent to the same chirp with time shift $\frac{\Delta f}{b}$ over the overlapping bandwidth, we often ignore the edge effect when the shift is small. Since the linearly ramped GS mirror generates space-dependent frequency shift $\frac{2\Omega x}{\lambda}$, it equivalently generates space-dependent time shift, $\frac{2\Omega x}{\lambda} b$, of the chirped laser.

Now we have two chirps, one is the original chirp, $\exp(i \pi b t^2)$, the other is the one with space-dependent time delays, $\exp[i \pi b(t - \frac{2\Omega x}{\lambda b})^2]$. The second step is to record the interference power spectrum,

$$P(f) = 2 + 2 \cos(2\pi f \frac{2\Omega x}{\lambda b}),$$  \hspace{1cm} (1.25)

of the two chirps at each spatially resolvable position $x$ in the inhomogeneously broadened spectral band of the Tm$^{3+}$:YAG crystal (refer to chapter 3 for more theory of the Tm$^{3+}$:YAG crystal and chapter 4 for the simulation of light-atom interaction in the crystal). Fig. 1.13 shows an example of the recorded spectrum in the spatial-spectral domain of the crystal and spatial-spectral diffraction of an incident plane wave signal produces the desired space-dependent time delays. At each spatial position $x$, there is a sinusoidal grating and the spectral pitch of the sinusoidal grating, $\frac{2\Omega x}{\lambda b}$, is linearly proportional to the spatial position $x$. This “magic” crystal has a property: an incident
signal $s(t)$ can read out the recorded spectrum $P(f)$ in Eq. (1.25), and gives

$$s_{\text{out}} = s(t) * F^{-1}\{P(f)\} = 2s(t) + s(t + \frac{2\Omega}{\lambda_b} x) + s(t - \frac{2\Omega}{\lambda_b} x),$$

(1.26)

where $*$ is convolution. The second term in Eq. (1.26) is an acausal term that “arrives” before the signal $s(t)$ comes, thus it does not exist. The third term, $s(t - \frac{2\Omega}{\lambda_b} x)$, is what we want as a delay line. The bandwidth of this scanner depends on the bandwidth of the two recording laser chirps and the inhomogeneous bandwidth of the medium, which for a Tm$^{3+}$:YAG crystal could be 20GHz.

1.3 Thesis Outline

Chapter 2 of this thesis describes the holographic Doppler generation and verification of the Galvo scanning (GS) mirror in a photorefractive crystal. The experimental results show that GS mirror can be a stable source of linear frequency shifts along the spatial dimension. A possible substitute for the GS mirror in the linear frequency generation perspective is to use two acousto-optic deflectors, which is described in the latter part of chapter 2. Chapter 3 review the relevant physics of the rare earth doped crystals. The rare earth doped crystals can be modeled as inhomogeneously broadened two-level atoms. The equations that will be applied to explain the latter experimental results in chapter 5 are also derived in this chapter. Chapter 4 describes a technique for numerically simulating the light atom interaction in a two-dimensional crystal by using the Maxwell-Bloch equations. These simulations are used to search for the optimum experimental geometry for the recording experiment in chapter 5. Chapter 5 integrates the GS mirror with the Tm$^{3+}$:YAG crystal, and gives the experimental demonstration of the high bandwidth (1.5GHz bandwidth with 20 resolvable spots) spatial-spectral holographic (SSH) scanner. This system uses one laser for the proof-of-concept experiment. Chapter 6 explores the capabilities and limitation of this approach to the high bandwidth SSH scanner. This chapter also gives the design and demonstration of a two-laser stabilization circuit, with which we can extend our ability to realize the full version of the high bandwidth SSH scanner system.

---

3 Define inverse Fourier transform of a function $P(f)$ as $F^{-1}\{P(f)\} = \int_{-\infty}^{\infty} P(f) \exp(i2\pi ft)df$.?
Chapter 2

Doppler processing systems

One of the key components of the SSH scanner is the GS mirror that generates frequency shifts to the incident beam. One might want to ask: is the mechanically rotating mirror linear and stable enough for our holographic recording process? This chapter answers this question by using time-integrating systems to experimentally verify the Doppler frequencies generated by the GS mirror. In addition, counter-propagating AODs driven by appropriate chirps could substitute for the GS mirror as a space-dependent Doppler frequency generator.

This chapter consists of two parts: GS mirrors and AODs. Section 2.2 investigates the Doppler frequencies generated by a rotating mirror. The rotating mirrors have been used in super-resolution for rotating objects [29], beam deflection to access memory or record holograms, and in laser shows. In the experiment, we use scanning mirror with servo controls to linearize the scan [30]. The mechanism of Doppler generation by a rotating mirror is discussed in Section 2.2.1. To verify the Doppler channels generated by the GS mirror, we use a time-integrating correlator with a photorefractive crystal as the time-integrating element. Photorefractive materials are widely used in dynamic holograms [31–34], image amplification [35, 36], optical signal processing [37–40], and optical phase conjugation [41, 42]. The advantage of using it in our case is to get bias-free detections [43] and make effective use of the system dynamic range. In Section 2.2.2 of this chapter, we first give the verification of the Doppler channels produced by the rotating mirror using a CCD detector array as a time-integrating medium. Then we analyze the dynamics and geometries of using a photorefractive crystal as a time-integrating medium and give the experimental results.
Section 2.3 talks about AODs as a Doppler processor. People use the big aperture of the AODs to display time domain signals in space for RF spectrum analysis [44] and ambiguity function processing [45]. AODs are also used in two-dimensional laser scanning [46, 47], three-dimensional scanning in multi-photon microscopes [48], laser scanning vibrometer for micro-components [49], and optical switching in optical interconnect [50] and optical delay lines [51]. Section 2.3.1 starts with talking about a high efficiency material, TeO$_2$ as an AOD material. Section 2.3.2 discusses the geometries and bandshapes for two cascaded AODs: in-plane AODs and crossed AODs. Section 2.3.3 gives the experimental results for verifying Doppler slices and Doppler spots.

2.1 Introduction

In a range-Doppler processing system [52], the transmitted RF signals are correlated with the reflected signals from the targets to determine the position of the target. The movement of the target relative to the RF signals will alter the waveform of the reflected signals by Doppler shifting the signals, making the correlation used for measuring the range less effective. Mathematically, the Doppler effect decreases the correlation peak and broadens the correlation width, leaving bigger uncertainties to pinpoint the target position. One solution to this problem is to simultaneously generate a bank of precompensated Doppler correlator channels and compare the reflected signal with each through a correlation [5, 53]. This solution implements the optimum linear detector for a Doppler shifted and time delayed waveform and takes advantage of the spatial parallelism of an optical system, but is limited by the spreading of the available signal energy among the parallel bank of correlators.

There are at least two ways to optically generate spatial dependent Doppler channels: one is by using a rotating mirror, the other by using acousto-optic devices [54]. The first approach is energy efficient, but can only generate tens of KHz of Doppler. The second approach is less efficient than the first one, due to the diffraction efficiencies of the AODs, but can cover more bandwidth.
Figure 2.1: Mirror analysis without approximations. The object plane is the plane of interest for later processing. The incident beam has an angle $\alpha$ to the mirror normal when the mirror is not scanning. The mirror scans with a constant angular velocity $\Omega$ so the angular position of the mirror is $\theta(t) = \Omega t$.

2.2 GS mirror based Doppler processor

2.2.1 Understanding the Doppler generation of a GS mirror

The device we use for Doppler generation is a servo-controlled mechanically scanning mirror. For Doppler generation, the mirror is driven by a periodic ramp voltage. The mirror rotates with a constant angular velocity $\Omega$ until the ramp resetting point $t_r$, when the mirror starts to rotate in the reverse direction. As a Doppler generator, we only illuminate the mirror during the time when the GS mirror rotates continuously in the same direction (either clockwise or counter clockwise but not both). At the mirror pivoting point (point $O$ in Fig. 2.1), there is no relative movement between the mirror and the incident light, therefore it is a zero Doppler point. We define the mirror pivoting point $O$ as $y = 0$. The mirror half plane that moves away from the incident light is defined as the $y$-positive direction.

We start with the simplest situation: we only consider during the up ramp $t_{up}$ in Fig. 2.1.
Suppose the mirror rotates with a constant angular velocity $\Omega$ about the axis point $O$ and the instantaneous mirror angle is

$$\theta = \Omega t. \quad (2.1)$$

To analyze the effect of angular scanning on Doppler generation, we shine a plane wave on to the mirror and observe what the mirror does to the reflected beam. We image the vertical plane, the “object plane” in Fig. 2.1, the plane that goes through the mirror rotation axis, to the center of a photorefractive crystal and then to a CCD. The geometry of beam reflected by a rotating mirror is shown in Fig. 2.1. For plane waves incident on the mirror, there are two ways to calculate the Doppler generated at the object plane: one is to look directly at the phase of the reflected plane wave, the other is to look at the phase delay of two rays going through a specific spot when the mirror is at the angle $\theta = 0$ (the object plane) and when the mirror is at an angle $\theta(t)$.

The first method is quite straightforward. The plane wave incident with an angle $\alpha$ to the normal of the image plane can be written as

$$E_i(y, z) = \exp(i\omega t - ik_y y - ik_z z), \quad (2.2)$$

where $\omega$ is the optical frequency, $k_y^2 + k_z^2 = \omega^2/c^2$ and $\tan \alpha = k_y/k_z$. The reflected plane wave from the mirror at time $t$ has an angle $\alpha + 2\theta$ with the $z$-axis and can be written as

$$E_r(y, z; t) = \exp[i\omega_0 t - i k_y \sin(\alpha + 2\theta) - i k_z \cos(\alpha + 2\theta)], \quad (2.3)$$

where $k = \omega/c = 2\pi/\lambda$. The phase of the reflected beam is

$$\phi(y, z; t) = \omega_0 t - k y \sin(\alpha + 2\Omega t) - k z \cos(\alpha + 2\Omega t), \quad (2.4)$$

where we use Eq. (2.1) for $\theta$. Therefore, the instantaneous frequency, which is defined by the time domain derivative of phase, at a spatial position $(y, 0)$ is given by

$$f_i(y, 0) = \frac{1}{2\pi} \left. \frac{d\phi}{dt} \right|_{z=0} = \frac{\omega_0}{2\pi} - \frac{2\Omega y}{\lambda} \cos(\alpha + 2\Omega t). \quad (2.5)$$

Thus we see that the Doppler shift (the second term) linearly depends on position $y$, goes to 0 at $y = 0$, is negative for $y > 0$ (when $\Omega$ is positive) and positive for $y < 0$, and has a slope $-\frac{2\Omega y}{\lambda} \cos \alpha$. 
when \( t = 0 \). Only for part of each cycle where the scan velocity is nearly constant is the spatially dependent Doppler constant and usable.

The second method considers the path length difference between two rays that pass through the same point. Showing in Fig. 2.1, the ray reflected by the mirror at the object plane position and the the ray reflected by the mirror position at angle \( \theta \) go through the same point, \( A \). Consider the rays that go through point \( A \). The optical path length difference (OPD) between the rays that are reflected by \( \theta = 0 \) plane and \( \theta(t) = \Omega t \) position is

\[
\text{OPD} = -AA'P = -AA' - AP = -AA'[1 + \cos(2\theta + 2\alpha)] = -2y \sin \theta \cos(\theta + \alpha),
\]

(2.6)

where \( y \) is the position of spot \( A \) on the image plane. The instantaneous Doppler frequency at the spot \( A, (y, 0) \), is given by

\[
f_d(y, 0) = \frac{1}{\lambda} \frac{d\text{OPD}}{dt} = -\frac{2\Omega y}{\lambda} \left[ \cos \Omega t \cos(\Omega t + \alpha) - \sin \Omega t \sin(\Omega t + \alpha) \right] = -\frac{2\Omega y}{\lambda} \cos(\alpha + 2\Omega t).
\]

(2.7)

If we simplify the result in Aleksoff and Christensen’s paper [29], we can achieve the same result as Eq. (2.7). The term \( \cos(\alpha + 2\Omega t) \) gives nonlinearities to the generated Dopplers by the mirror. For a specific setup, \( \alpha \) is fixed and \( \Omega t \) is typically very small (on the order of mili-radians). By Taylor expanding \( \cos(\alpha + 2\Omega t) \) around \( \alpha \) to the second order, the nonlinearity can be written as

\[
\cos(\alpha + 2\Omega t) - \cos \alpha \approx -2\Omega t \cdot \sin \alpha - 2(\Omega t)^2 \cdot \cos \alpha.
\]

(2.8)

As \( \alpha \) increases, the nonlinearity increases. When \( \alpha = 0 \), the nonlinearity is given by \( \cos(2\Omega t) - 1 \approx -2(\Omega t)^2 \), which is only proportional to \( (\Omega t)^2 \) and therefore gives the smallest nonlinearity for all the \( \alpha \) within \([0, 90^\circ]\). With very small \( \Omega t \), we can approximate the Doppler frequency generated by the mirror as

\[
f_d \approx -\frac{2\Omega y}{\lambda} \cos \alpha.
\]

(2.9)

For \( y > 0 \) half plane in Fig. 2.1, the mirror rotates away from the incident light and modulates the incident light with a negative Doppler shift. For \( y < 0 \) half plane, it modulates the light with a positive Doppler. The GS mirror transformed the frequency signal \( f_d \) into space \( y \).
Figure 2.2: Normal incidence to the GS mirror. Vertically polarized beam is reflected by the PBS and converted to circularly polarized light by a $\frac{\lambda}{4}$ before hitting the GS mirror. The reflected light from the GS mirror is converted into horizontally polarized light by the same $\frac{\lambda}{4}$ and transmits the PBS.

**Normal incidence to the GS mirror**

We can use a PBS and a quarter waveplate to make the beam normally incident onto the GS mirror, as shown in Fig. 2.2. We use a vertically polarized light before the PBS. The light is reflected by the PBS and is turned into circular polarization by a quarter waveplate. When reflected back by the GS mirror, the light passes through the quarter waveplate again and is turned into horizontally polarized light which transmits through the PBS for later experiment. This geometry can also gives the smallest nonlinearity for Doppler generation.

**Periodic ramp driving signal**

When we observe the Doppler generated by the GS mirror for one period $T$ (or less) of its rotation, the generated Doppler is continuous along the spatial dimension $y$. If we observe the generated Doppler for more than one period, we will get discrete Doppler channels on the image plane of the GS mirror. Suppose the periodic bipolar ramp driving signal is

$$r(t) = \frac{A}{t_{up}} t \Pi\left(\frac{t}{t_{up}}\right) * \left[ \frac{1}{T} \text{comb}\left(\frac{t}{T}\right) \right] - \frac{A}{t_{down}} t \Pi\left(\frac{t}{t_{down}}\right) * \left[ \frac{1}{T} \text{comb}\left(\frac{t - T/2}{T}\right) \right],$$

(2.10)

where $r(t)$ has a unit of Volt, $\Pi(t) = 1$ ($|t| < 1/2$), $\Pi(t) = 0$ ($|t| > 1/2$), $*$ is convolution and $\text{comb}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n)$. Suppose the angular scanning coefficient of the GS mirror is $c_m$ with a
unit of rad/V. Then we have $\theta(t) = c_m r(t)$. According to Eq. (2.4), the phase of the reflected beam at position $(y, 0)$ can be written as

$$\phi(y, 0; t) = \omega_0 t + ky \sin[\alpha + 2c_m r(t)]. \quad (2.11)$$

Thus, the instantaneous Doppler frequency at a spatial position $(y, 0)$ is given by

$$f_d(y, 0) = \frac{\omega_0}{2\pi} + \frac{2c_m y}{\lambda} \cos[\alpha + 2c_m r(t)] \cdot \frac{dr(t)}{dt}$$

$$\approx \frac{\omega_0}{2\pi} + \frac{2c_m y}{\lambda} \cos \alpha \cdot \left\{ \frac{A}{t_{up}} \Pi\left(\frac{t}{t_{up}}\right) \ast \left[ \frac{1}{T} \text{comb}\left(\frac{t}{T}\right) \right] - \frac{A}{t_{down}} \Pi\left(\frac{t}{t_{down}}\right) \ast \left[ \frac{1}{T} \text{comb}\left(\frac{t-T/2}{T}\right) \right] \right\}. \quad (2.12)$$

The down ramp gives the opposite modulations to the incident light than the up ramp. In experiments, one can make $t_{down} \ll t_{up}$, s.t. the number of photons devoted to the unwanted Dopplers is small. Or one can use a chopper or laser modulator that blocks the light during $t_{down}$. In the following derivations, we use

$$r(t) = \frac{A}{T} t \Pi\left(\frac{t}{T}\right) \ast \left[ \frac{1}{T} \text{comb}\left(\frac{t}{T}\right) \right] \quad (2.13)$$

as a simplified idealization of a ramp that neglects the down ramp or gap during the blanking intervals.

If we use a large area detector to observe all the Doppler frequencies at the image plane of the GS mirror for a time duration of $T_i$. Displaying the signal from the detector on a spectrum analyzer with resolution bandwidth less than $\frac{1}{2T_i}$, we get

$$\mathcal{F}_t \{ E_r(y, z = 0; t) \} = \mathcal{F}_t \{ \exp[i\omega_0 t + iky \sin[\alpha + 2c_m r(t)] \} \ast \text{sinc}(T_i f) \}
\approx \mathcal{F}_t \{ \exp[i\omega_0 t + iky \sin \alpha + 2ic_m k y \cos \alpha \cdot r(t)] \} \ast \text{sinc}(T_i f), \quad (2.14)$$

where the approximation uses the fact that the angular scanning range of the mirror during each ramp segment is very small ($2c_m r(t) \ll \pi/2$). Plugging Eq. (2.10) into Eq. (2.14) and using the
fact that we can move the term $\Pi(\frac{t}{T}) * \left[\frac{1}{T}\text{comb}(\frac{t}{T})\right]$ out of the exponential, we get

$$
\mathcal{F}_t \{E_r(y, z = 0; t)\} = \exp(iky \sin \alpha) \delta(f - \frac{\omega_0}{2\pi})
\cdot \mathcal{F}_t \left\{ \exp(i2kc_m y \cos \alpha \cdot At) \Pi(\frac{t}{T}) * \left[\frac{1}{T}\text{comb}(\frac{t}{T})\right] \Pi(\frac{t}{T_i}) \right\}

= TT_i \cdot \exp(iky \sin \alpha) \delta(f - \frac{\omega_0}{2\pi})
\ast \left\{ \left[ \text{sinc}[T(f - \frac{2Ac_m y \cos \alpha}{\lambda})] \text{comb}(Tf) \right] \ast \text{sinc}(T_i f) \right\}.
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
$$(2.15)$$

For $T_i \rightarrow \infty$, Eq. (2.15) is not zero only when $f = \frac{n}{T_i}$ ($n = 0, 1, 2, \ldots$). Eq. (2.15) tells us that the Dopplers generated by the period ramp driven GS mirror is discrete with frequency step size of $\frac{1}{T_i}$. Since the GS mirror transformed the frequency signal into space by $\text{sinc}[T(f - 2Ac_m y \cos \alpha/\lambda)]$, discrete frequency steps also mean discrete spatial spots. The spatial step size is $y = \lambda/(2Ac_m T \cos \alpha)$. The spectral width of the Doppler bins is the convolution of two sinc functions with their first zeros at frequency $\frac{1}{T}$ and $\frac{1}{T_i}$. Thus the spectral width of the Doppler bins is $\frac{1}{T} + \frac{1}{T_i}$.

### 2.2.2 Verification of the Doppler channels

To confirm that the GS mirror does what we expect it to do, we want to test the Doppler channels generated by it. So we want to have something to compare with. We need to create a “ruler”, beam with some known Doppler shifted frequencies. An acousto-optic modulator (AOM) is a good candidate to generate highly coherent, phase stable Doppler references. AOMs typically have modulation bands centered at tens to several hundred Megahertz. For the small Doppler frequency (KHz to tens of KHz) range the GS mirror can generate, it is hard to compare the GS generated Dopplers with the single modulated AOMs. However, we can combine two AOMs. One Doppler up shifts the light by $f_0$ and the other Doppler down shifts the light by a slightly shifted frequency $f_0 + \delta f$. The modulation frequencies of the AOMs can be big, but it is possible to make their difference frequency small and precisely controlled as long as the signal generators producing these offset frequencies are locked to the same reference (typically 10MHz). We then can compare
the Dopplers generated by a GS mirror with the known frequency offset of the AOMs by using a
time-integrating (TI) medium (a TI CCD, a photorefractive crystal, etc.) through interferometric
detection.

In interferometric detection, the reference beam $E_{\text{ref}}$ interferes with the signal beam $E_{\text{sig}}$
(here we use a Mach-Zehnder interferometer). The intensity of the interference is given by

$$I = |E_{\text{ref}} + E_{\text{sig}}|^2 = |E_{\text{ref}}|^2 + |E_{\text{sig}}|^2 + 2E_{\text{ref}}E_{\text{sig}}^*.$$  \hfill (2.16)

It is sensitive to the phase/frequency changes between the reference and the signal beams. For the
Doppler verification experiments, $E_{\text{ref}} \propto \exp(i2\pi f_0 t)$ and $E_{\text{sig}} \propto \exp[i2\pi(f_0 + \delta f)t]\exp[i2\pi f(x)t]$, where $f(x)$ is the frequency generated by the GS mirror, the interference term is given by

$$2E_{\text{ref}}E_{\text{sig}}^* \propto \exp\{-i2\pi(\delta f + f(x))t\}. \hfill (2.17)$$

It has a constant phase only when $\delta f = -f(x)$. In the following sections, I will talk about using
CCD and photorefractive crystal as a TI medium for verifying the Doppler frequencies generated
by the GS mirror.

### 2.2.2.1 CCD as a time-integrating detector

The experiment setup for verifying the spatially varying Doppler produced by the Galvo
mirror using a time-integrating CCD is shown in Fig. 2.3. A Mach-Zehnder interferometer is used
to beat the reference frequency with the GS mirror and thereby compare the Doppler channels
generated by the GS mirror with the reference known Dopplers. Two AOMs, AOM1 and AOM2,
are used to generate the known KHz reference Doppler channels. The two AOMs can be put in each
arm of the interferometer and both are up shifted or down shifted in frequency, or both can be put
in one arm with one up shifted and one down shifted. To balance the power of the two arms of the
interferometer, we choose to separately put one AOM on each arm. The +1 order of both AOMs
are used. AOM1 is driven with $f_1 = 100\text{MHz} + \delta f\text{Hz}$ sinusoidal signals, $\exp(i2\pi f_1 t)$, and AOM2 is
driven with $f_2 = 100\text{MHz}$ sinusoidal signals, $\exp(i2\pi f_2 t)$. There is a $\delta f\text{Hz}$ offset frequency between
the two AOMs. The beam in AOM1 is then modulated by the GS mirror, $\exp[i2\pi \frac{2 c_m y \cos \alpha}{\lambda} r(t)]$.

Both beams are combined onto a time-integrating CCD. The photo-generated charges on the CCD is proportional to

$$Q(x, y, t) = \int_{T_i} \left| \exp(i2\pi k x \sin \theta) \exp(i2\pi f_1 \tau) \cdot \exp[i2\pi \frac{2 c_m y \cos \alpha}{\lambda} r(\tau)] + \exp(i2\pi f_2 \tau) \right|^2 d\tau$$

$$= 2 \Re \left\{ \exp(i2\pi k x \sin \theta) \int_{T_i} \exp(i2\pi \delta f \tau) \exp[i2\pi \frac{2 c_m y \cos \alpha}{\lambda} r(\tau)] d\tau \right\} + \text{bias},$$

(2.18)

where $T_i$ is the integration time of the CCD ($T_i > T$), $\alpha = 45^\circ$ is the beam incident angle to the GS mirror, $\theta$ is the angle between the two beams on the CCD and $\Re \{ \cdot \}$ is the real part operator and $\delta f = f_1 - f_2$. Using a periodic ramp, Eq. (2.10) as the driving signal of the GS mirror, letting
29

t_{\text{down}} << t_{\text{up}}$, and considering Eq. (2.15), we get

\[
Q(x, y, t) = 2\Re \left\{ \exp(i2\pi kx \sin \theta) \int_{T_i} \exp(i2\pi \delta f \tau) \right.
\times \left[ \exp \left[ -i2\pi \cdot (2A_c y \cos \alpha /\lambda) \cdot \tau \right] \cdot \Pi \left( \frac{\tau}{T_i} \right) \ast \left[ \frac{1}{T_i} \text{comb} \left( \frac{\tau}{T_i} \right) \right] \right] d\tau \}
\left. + \text{bias} \right\}
\]

\[
= 2TT_i \cdot \Re \left\{ \exp(i2\pi kx \sin \theta) \text{sinc}[T(\delta f - 2A_c y \cos \alpha /\lambda)] \right.
\times \text{comb}(T\delta f) \ast \text{sinc}(T_i\delta f) \} + \text{bias}
\]

(2.19)

When \( \delta f = 2A_c y \cos \alpha /\lambda = n/T, n = 0, 1, 2, \ldots \), or at position \( y = \lambda \delta f / (2A_c \cos \alpha) \), there are integrated fringes

\[
Q(x, y, t) = 2TT_i \cos(2\pi kx \sin \theta) + \text{bias}
\]

(2.20)

on the CCD. The fringe has a width of \( 1/T + 1/T_i \). The Doppler free fringes are also discrete, \( y = \lambda n / (2A_c T \cos \alpha), n = 0, 1, 2, \ldots \), with a frequency step of \( 1/T \). At other positions, the integration washes out to zero. In other words, there are integrated fringes on the CCD only at the position when the Doppler generated by the GS mirror matches with the Doppler generated by the two AOMs.

The experimental results is shown in Fig. 2.4. When changing the offset frequencies between the AOMs, the fringes on the CCD move accordingly in the GS mirror Doppler generation \( y \)-direction. The Doppler bin of this experiment is about 6KHz wide, due to the fact that the GS mirror is not imaged to the CCD but Fresnel diffracted to the CCD plane, as shown in Fig. 2.3. This will be improved in the next section.

2.2.2.2 Photorefractive crystal as an integration medium

The CCD used as a time integration medium also integrates up the bias in addition to the signals. The photo-generated charge on a CCD is proportional to the intensity of the incident beams. For this reason, to represent fringes that have both positive and negative values, one needs to increase the bias such that all the values are positive. Define DR as the dynamic range of the
detector and SBR as the signal-to-bias ratio on the detector. The effective dynamic range is given by [43]

$$DR' = DR \frac{SBR}{1 + SBR}. \quad (2.21)$$

In most cases, $SBR << 1$ and therefore the effective dynamic range is going to be much smaller than the dynamic range of the detector itself. In this next section, we will use a photorefractive crystal to record a hologram, which can be read out to remove the bias and increase the dynamic range on a detector.

Alternatively, we can use a photorefractive Strontium Barium Niobate (SBN) crystal doped with Cr as a time-integrating medium. When a photorefractive crystal is illuminated by an intensity grating, electrons move from the high intensity region to the low intensity region, forming a space-charge field. This field modifies the refractive index in the crystal through the electrooptic effect and forms a volume holographic phase grating. In our application, we use SBN as a dynamic holographic recorder in place of the CCD. We can then use another beam to read out the recorded phase grating. Since fringes only form at matched Doppler positions, the reading beam will only be diffracted at the Doppler matched positions. Imaging this diffraction onto a CCD produces a
bias-free measurement of the Doppler frequencies due to the scanning Galvo mirror.

SBN has a point group symmetry of 4mm with large electrooptic coefficients of $r_{33} = 1340\text{pm/V}$. It gives the optimum diffraction efficiency when the two green beams that form the grating have an angle of about $20^\circ$, and when the bisector of the two beams is normal to the grating vector $K_G$.

### 2.2.2.3 Theoretical analysis of using SBN as an integration medium

The setup of using SBN as an integrator is shown in Fig. 2.5. We use two lasers for the holographic writing (at 532nm) and reading (at 633nm) process. Two AOMs with frequency offsets $f_1 - f_2 = \delta f$ generates verification signals. A GS mirror generates Doppler frequencies along its $y$-axis. The electric field at the surface of the GS mirror can be written as

$$E_1(x_o, y_o, z_o = 0; t) = A_1 \exp(i2\pi f_1 t) \cdot \exp(-inkx_o \sin \theta_y) \cdot g(x_o, y_o) \times \exp[i2\pi \frac{2cmy_o \cos \alpha}{\lambda_g} r(t)],$$

(2.22)

where

$$g(x, y) = \exp \left[-\frac{x^2 + y^2}{W_0^2}\right]$$

(2.23)

is the Gaussian beam shape and $W_0$ is the beam half width. Since light is normal incident onto the GS mirror, $\alpha = 0$. The GS mirror in beam 1 is 1:1 imaged to $z_c = d$ (shown in Fig. 2.5) inside the SBN crystal with a thickness $L_z$ and refractive index $n$. For simplicity, we ignore the fact that the SBN crystal coordinate is tilted for $10^\circ$. Later in the derivation, we can see that for the two-wavelength Bragg matched holographic readout, the SBN position is not critical in the perspective of imaging the GS mirror to the CCD camera. At the center of the SBN crystal, the electric field $E_1$ is

$$E_1(x_c, y_c, z_c = d; t) = A_1 \exp(i2\pi f_1 t) \cdot g(x_c, -y_c) \cdot \exp[-i2\pi \frac{2cmy_c}{\lambda_g} r(t)].$$

(2.24)
The electric field at the other depths of the SBN crystal is given by

\[ E_1(x_c, y_c, z_c; t) = A_1 \exp(i2\pi f_1 t) \cdot R_g[z_c - d] \{ g(x_c, -y_c) \times \exp(-ink_g x_c \sin \theta_g) \cdot \exp[-i2\pi \frac{2cm_y c}{\lambda_g} r(t)] \} \cdot \exp(-ink_g x_c \sin \theta_g) \cdot \exp[-i2\pi \frac{2cm_y c}{\lambda_g} r(t)] \] \hspace{1cm} (2.25)

where the propagation operator inside the SBN crystal at 532nm, \( R_g[z_c\{U(x_1, y_1)\}] \), is defined by

\[ R_g[z_c\{U(x_1, y_1)\}] = \frac{1}{i\lambda_g z_c} \int_{-\infty}^{+\infty} U(x_1, y_1) \exp \left\{ \frac{nk_g}{2z_c} \left[ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right] \right\} \, dx_1 \, dy_1, \] \hspace{1cm} (2.26)

where \( U(x_1, y_1) \) is an arbitrary electric field, \( z_c \in [-\frac{L_z}{2}, \frac{L_z}{2}] \) is the propagation distance and \((x_2, y_2)\) is the coordinate that applies after propagation. At the SBN crystal, the electric field of beam 2 inside the SBN crystal can be written as

\[ E_2(x_c, y_c, z_c = d; t) = A_2 \exp(i2\pi f_2 t) \exp(ink_g x_c \sin \theta_g) g(x_c, y_c), \] \hspace{1cm} (2.27)

where \( A_1 \) and \( A_2 \) are the amplitudes of the two beams and they are real. In experiment, we balanced the two beam amplitudes, \( A_1 = A_2 = A \). We use a red laser beam to read out the grating,

\[ E_r(x_c, y_c, z_c) = A_r \exp[i\nu_r x \sin \theta_r] g(x_c, y_c), \] \hspace{1cm} (2.28)

where \( A_r \) is the amplitude of the red readout laser beam, and \( \theta_r \) is the incident angle of the red laser beam to the SBN crystal. The interference grating \( G(\mathbf{r}) \) building up in the SBN crystal with time can be written as

\[ \frac{\partial G(\mathbf{r}, t)}{\partial t} = c_b \cdot 2\text{Re} \{ E_1(\mathbf{r}, t) E_2^*(\mathbf{r}, t) \} - c_e (I_1 + I_2 + I_r) G(\mathbf{r}, t), \] \hspace{1cm} (2.29)

where \( c_b \) and \( c_e \) are the grating building-up and grating erasure coefficients, \( \text{Re}\{\cdot\} \) is the real part operator and the intensity of the beams \( I_1 = A_1^2 \), \( I_2 = A_2^2 \) and \( I_r = A_r^2 \). At steady state,

\[ \frac{\partial G(\mathbf{r}, t)}{\partial t} = 0, \] \hspace{1cm} (2.30)
and we have
\begin{equation}
G(r, t) = \frac{2c_b}{c_e} Re\{E_1(r, t)E_2^*(r, t)\},
\end{equation}
where \( I_0 = I_1 + I_2 + I_r \). The diffracted red beam from the grating is \([43, 56]\),
\begin{equation}
E_d(x_c, y_c, z_c = d; t) = E_r(r) \int_0^t G(r, t') \exp[(t' - t)/\tau]\, dt',
\end{equation}
where \( \tau \) is the complex time constant of the space-charge field \([57]\), \( \tau = K_2/I_0 \). In Eq. (2.32), the term \( E_1E_2^*E_r \) will be Bragg matched and will give out a diffraction; the term \( E_1^*E_2E_r \) will not be Bragg matched and therefore will give no diffraction terms. In the later derivations, we write \( c_b/c_e \) as
\begin{equation}
c_b/c_e = \frac{K_1}{\tau},
\end{equation}
where \( K_1 \) is a complex constant involving the material parameters (doping density, crystal point group, electrooptic coefficients, etc.) of the photorefractive crystal, the grating frequency, and the applied electric field. The center of the SBN crystal is imaged to the CCD by another 1:1 imaging system. The blurred images at the other depths of the SBN crystal propagate to the CCD to be
\begin{equation}
E_d(x_i, y_i, z_i = 0; t) = \mathcal{R}_r[d - z_c] \{E_d(x_c, y_c, z_c; t)\},
\end{equation}
where \( z_c \in [-L_z/2, L_z/2] \) and the propagation operator inside the SBN crystal at 633nm, \( \mathcal{R}_r[z]U(x_1, y_1) \), is defined by the equation\([55]\]
\begin{equation}
\mathcal{R}_r[z]U(x_1, y_1) = \frac{1}{i\lambda_r z} \int_{-\infty}^{+\infty} U(x_1, y_1) \exp\left\{i\frac{nk_r}{2z} [(x_2 - x_1)^2 + (y_2 - y_1)^2]\right\} \, dx_1 \, dy_1,
\end{equation}
where \( k_r = \frac{2\pi}{\lambda_r} \). All the images are integrated by the CCD, therefore the image on the CCD is a sum of the images from all the depths,
\begin{align*}
& E_d(x_i, -y_i, z_i = 0; t) \\
& = \int_{-L_z/2}^{L_z/2} \mathcal{R}_r[d - z_c] \{E_d(x_c, y_c, z_c; t)\} \, dz_c \\
& = \frac{K_1}{\tau I_0} \int_0^t \exp\left(\frac{t' - t}{\tau}\right) \, dt' \int_{-L_z/2}^{L_z/2} \mathcal{R}_r[d - z_c] \{\mathcal{R}_g[z_c - d]U(x_c, y_c, z_c = d; t')\} \, dz_c,
\end{align*}
(2.36)
where the electric field at $z_c = d$ of the SBN crystal

$$U(x_c, y_c, z_c = d; t')$$

$$= E_1(x_c, y_c, z_c = d; t') E_2^\dagger(x_c, y_c, z_c = d) E_r(x_c, y_c, z_c = d)$$

$$= A^2 A_r g(x_c, y_c) \exp(2\pi \delta f t' + i k_r x_c \sin \theta_r) \exp[-i 2\pi \frac{2\epsilon_m y_c}{\lambda_g} r(t')]$$

which uses the Bragg match condition

$$k_g \sin \theta_g = k_r \sin \theta_r.$$

We will talk about the Bragg match in two different geometries later. When the propagation operators at two different wavelengths, $R_r[-z]$ and $R_g[z]$ operates on an arbitrary electric-field $U(x_1)$, we have

$$R_r[-z] \{ R_g[z] \{ U(x_1) \} \} = \left( \frac{1}{\lambda_g \lambda_r z^2} \right)^{1/2} \exp \left( -i \frac{k_r}{2 z x_3^3} \right) \int_{-\infty}^{+\infty} U(x_1) \exp \left( \frac{k_g}{2 z x_1^2} \right) dx_1 \times \int_{-\infty}^{+\infty} \exp \left\{ \frac{i}{2 z} \left[ (k_g - k_r)x_2^2 + 2x_2(k_r x_3 - k_g x_1) \right] \right\} dx_2.$$

The integration over $x_2$ is

$$\int_{-\infty}^{+\infty} \exp \left\{ \frac{i}{2 z} \left[ (k_g - k_r)x_2^2 + 2x_2(k_r x_3 - k_g x_1) \right] \right\} dx_2 = \left( \frac{i z \lambda_r \lambda_g}{\lambda_r - \lambda_g} \right)^{1/2} \exp \left[ -i \pi \frac{\lambda_r \lambda_g}{z(\lambda_r - \lambda_g)} \left( \frac{x_1}{\lambda_g} - \frac{x_3}{\lambda_r} \right)^2 \right],$$

and Eq. (2.39) can be simplified as

$$U_d(x_3) = R_r[-z] \{ R_g[z] \{ U(x_1) \} \}$$

$$= \left[ \frac{i}{(\lambda_r - \lambda_g)} \right]^{1/2} \int_{-\infty}^{+\infty} U(x_1) \exp \left[ -i \frac{\bar{k}}{2 z_{eq}} (x_1 - x_3)^2 \right] dx_1,$$
as an indicator to how similar the diffracted beam is to the original beam. For green writing beam and red read beam holographic readout, the image on the CCD is the image of the GS mirror Fresnel propagating for a small distance of \( z_{eq} \). Using Eq. (2.41), Eq. (2.36) can be simplified as

\[
E_d(x_i, -y_i, z_i = 0; t) = \frac{i}{\lambda_r - \lambda_g} \cdot \frac{K_1}{T_0} \int_{0}^{t} \exp\left(\frac{t' - t}{\tau}\right) \int_{L/2}^{L/2} (z_c - d)^{-1} \, dz_c \\
\times \int_{-\infty}^{+\infty} U(x_i, y_i, z_c, t) \exp\left\{-\frac{i}{2z_{eq}}[(x_c - x_i)^2 + (y_c - y_i)^2]\right\} \, dx_c \, dy_c,
\]

(2.43)

where \( z_{eq} = (z_c - d) \cdot [(\lambda_r - \lambda_g)/2\lambda] \). For a Gaussian aperture of beam width \( W_0 = 2.5\text{mm} \), the confocal distance \( z_0 = \frac{\pi W_0^2}{\lambda} \approx 33.7\text{m} \). The diffraction of the Gaussian aperture remains a Gaussian beam shape. In fact, if the crystal thickness was 16.7\text{mm}, the indicator defined in Eq. (2.42) is only about 1%. For a 5mm-thick crystal, \( \max|z_{eq}| \approx 0.5\text{mm} \), \( p_d \approx 3.5 \cdot 10^{-6} \), indicating that the diffracted beam \( U_d \) is similar to the original beam \( U \). Thus, we can approximate the quadratic phase as a  \( \delta \)-function

\[
\exp\left\{-\frac{i}{2z_{eq}}[(x_c - x_i)^2 + (y_c - y_i)^2]\right\} \to -i(z_c - d)(\lambda_r - \lambda_g)\delta(x_c - x_i, y_c - y_i).
\]

Eq. (2.43) can be written as

\[
E_d(x_i, -y_i, z_i = 0; t) \approx \frac{K_1L}{\tau T_0} \int_{0}^{t} U(x_i, y_i, z_c, t') \exp[(t' - t)/\tau] \, dt.
\]

(2.44)

This result shows that the final image is not \( d \) dependent, which means no matter where the SBN crystal is, the GS mirror will be imaged to the CCD. If the read beam and write beam have the same wavelength, Eq. (2.39) can be simplified as

\[
\mathcal{R}[-z]\{\mathcal{R}[z]\{U(x_1)\}\} = \frac{1}{\lambda z} \exp\left(-\frac{i}{2z} \frac{k}{x_3^2}\right) \int_{-\infty}^{+\infty} U(x_1) \exp\left(\frac{i}{2z} \frac{k}{x_1^2}\right) \delta\left(x_3 - x_1, \frac{\lambda z}{2z}\right) \, dx_1
\]

\[
= U(x_3),
\]

(2.45)

which means whether or not if we image the GS mirror surface to the center of the SBN crystal, as long as there are two 4f systems between the GS mirror and the CCD, the surface of the GS mirror will be imaged onto the CCD [58].
Time integration of the SBN crystal

Using Eq. (2.44), the intensity of the diffracted beam at the CCD can be written as

\[
I_{\text{CCD}}(x_i, -y_i, z_i = 0; t) = \frac{K_1}{\tau I_0} \int_0^t \left| U(r_c, t') \exp\left(\frac{(t' - t)}{\tau}\right) dt' \right|^2,
\]

(2.46)

where \( r_c = (x_c, y_c, z_c) \). Eq. (2.46) shows that the strength of the grating in the photorefractive crystal decays with an intensity-dependent time constant \( \tau \), so the gratings that are currently in existence are the sums of all the decayed gratings from \( t' = 0 \). Write \( U(r_c, t) \) as the sum of its temporal Fourier components,

\[
U(r_c, t') = \int_{-\infty}^{+\infty} \tilde{U}(r_c, \omega) \exp(i\omega t') d\omega,
\]

(2.47)

then we can write Eq. (2.46) as

\[
I_{\text{CCD}}(x_i, -y_i, z_i = 0; t) = \frac{K_1}{\tau I_0} \int_{-\infty}^{+\infty} \int_0^t \left| \frac{\exp(-t/\tau)}{i\omega + 1/\tau} \tilde{U}(r_c, \omega) \right|^2 \exp\left(\frac{(i\omega + 1/\tau)t'}{\tau}\right) dt' d\omega.
\]

(2.48)

For long integration time \( t >> \tau, \exp(t/\tau) >> 1 \), thus \( \exp((i\omega + 1/\tau)t') - 1 \approx \exp((i\omega + 1/\tau)t) \).

Eq. (2.48) becomes

\[
I_{\text{CCD}}(x_i, -y_i, z_i = 0; t) = \frac{K_1}{\tau I_0} \int_{-\infty}^{+\infty} \exp(i\omega t) \tilde{U}(r_c, \omega) d\omega \right|^2.
\]

(2.49)

Eq. (2.49) can also be written as a convolution between \( U(r_c, t) \) and a time domain step function with exponential decay,

\[
I_{\text{CCD}}(x_i, -y_i, z_i = 0; t) = \left| \frac{K_1}{\tau I_0} \int_{-\infty}^{+\infty} \exp(-t/\tau) \tilde{U}(r_c, \omega) d\omega \right|^2.
\]

(2.50)

where \( H(t) = \begin{cases} 
1 & t > 0 \\
0 & t < 0
\end{cases} \) is a step function. We can treat the convolution as a sliding window integrator with integration time \( \tau \) [43]

\[
I_{\text{CCD}}(x_i, -y_i, z_i = 0; t) \approx \left| \frac{K_1}{\tau I_0} \int_{t-\tau}^t U(r_c, t') dt' \right|^2.
\]

(2.51)
Notice that the major difference between using photorefractive crystal as an integrator and using a CCD as an integrator is, there is no bias term in the photorefractive case. The readout beam is diffracted only at positions where gratings present and where the readout beam Bragg matches the gratings. We will discuss the Bragg match conditions in the next section. Here we want to find out the positions of the holographic gratings. As is analyzed in the last section, for integration time longer than the ramp period, there will only be gratings at Doppler matched positions. The following gives the detailed analysis.

In experiment, we make the two beam balanced in amplitudes, $A_1 = A_2$. Using Eq. (5.8) and Eq. (2.13), we can find out the diffracted beam intensity

$$I_{CCD}(x_i, -y_i, z_i = 0; t) \approx \left| \frac{K_1 A^2 A_r}{\tau I_0} g(x_c, y_c) \right|^2 \left| \int_{-\infty}^{+\infty} \exp(i2\pi \delta f t') \right|^2 \times \exp \left\{ -i2\pi \frac{2c_m y_c t'}{\lambda_g} \Pi \left( \frac{t'}{T} \right) * \left[ \frac{1}{T} \text{comb} \left( \frac{t'}{T} \right) \right] \right\} \Pi \left( \frac{t' - \tau/2}{\tau} \right) \mathrm{dt}'^2$$

For $\tau \geq 2T$, the crystal integrates multiple ramps and gives

$$I_{CCD}(x_i, -y_i, z_i = 0; t) \approx \left| \frac{K_1 A^2 A_r T}{I_0} g(x_c, y_c) \right|^2 \left| \text{sinc} \left[ T(\delta f - 2c_m y_c/\lambda_g) \right] \cdot \text{comb}(T \delta f) \right|^2 \left| \text{sinc}(\tau \delta f) \exp(i\pi \delta f \tau) \right|^2.$$  \hspace{1cm} (2.53)

Diffractions occur when $\delta f$ satisfies

$$\delta f = 2c_m y_c/\lambda_g = n/T, n = 0, 1, 2, \cdots,$$  \hspace{1cm} (2.54)

which means the Doppler channels are discrete with a frequency step size $1/T$. The width of the diffraction spot is determined by the convolution of the two sincs in Eq. (2.53), $1/T + 1/\tau$.

If $\tau < T$ and the crystal does not integrate the ramp reset, as shown in Fig. 2.6(a), then

$$I_{CCD}(x_i, -y_i, z_i = 0; t) \approx \left| \frac{K_1 A^2 A_r T}{I_0} g(x_c, y_c) \right|^2 \left| \text{sinc} \left[ T(\delta f - 2c_m y_c/\lambda_g) \right] \right| \left| \text{sinc}(\tau \delta f) \exp(i\pi \delta f \tau) \right|^2.$$  \hspace{1cm} (2.55)

Diffractions occur when $\delta f$ satisfies

$$\delta f = 2c_m y_c/\lambda_g,$$  \hspace{1cm} (2.56)
Figure 2.5: Doppler verification setup - using SBN as a bias-free time-integrating correlator. Two lasers with different wavelengths are used, the 532nm one is to write gratings in the SBN crystal, the 633nm one is for reading out the written gratings. Two AOMs with offset driven frequencies are used to generate the verification Doppler signals. A GS mirror that is driven by periodic ramp signals and rotates about its x-axis generates Doppler frequencies along the y-direction of the mirror. For maximum Doppler generation range, we use a PBS and a \( \frac{\lambda}{4} \) to achieve normal incidence to the GS mirror. The GS mirror is imaged to the SBN crystal by a 1:1 4-f system. The green beams interact in the SBN crystal with an angle of about 20°. Gratings are written at the matched Doppler positions. The red laser beam with an angle of about 12° with respect to the SBN normal diffracts off the gratings and forms Doppler slice on the CCD.

which means the Doppler channels are continuous. The width of the diffraction spot is again determined by the convolution of the two sincs in Eq. (2.57), \( 1/T + 1/\tau \).

If \( \tau < 2T \) and the crystal integrates the ramp reset, as shown in Fig. 2.6(b), then

\[
I_{CCD}(x_i, -y_i, z_i = 0; t) \approx \left| \frac{K_1 A^2 A_r T}{I_0} g(x_c, y_c) \right|^2 \left| \text{sinc} \left[ T(\delta f - 2c_{my_c}/\lambda_g) \right] \right| \text{sinc} \left( \tau_1 \delta f \right) \exp \left[i2\pi f(\tau - \tau_1/2) \right] + \left| \text{sinc} \left( \tau_2 \delta f \right) \exp \left[i\pi f \tau_2 \right] \right|^2, \tag{2.57}
\]
where $\tau_1$ and $\tau_2$ are shown in Fig. 2.6(b), $\tau_1 + \tau_2 = \tau$. Diffractions occur when $\delta f$ satisfies

$$\delta f = \frac{2c_m v_c}{\lambda_g}, \quad (2.58)$$

which again means the Doppler channels are continuous. However, the shape of the diffraction spot is now a more complicated function.

**In-plane vs. out-of-plane grating analysis**

To read out the gratings recorded by the green beams, the readout red beam should satisfy the Bragg match condition, Eq. (2.38). As we know from the last section, the Doppler channels generated by the GS mirror have finite frequency width of about $1/T + 1/\tau$, corresponding to a spatial width of $\lambda_g(1/T + 1/\tau)/(2c_m)$. Thus, there is an angular spread of $\Delta \theta \propto 2c_m/[\lambda_g(1/T + 1/\tau)]$ for the Doppler channels. The Bragg match condition will only be satisfied for one of the angles within $\Delta \theta$ range. In this section, we will analyze the momentum mismatch for all the angles within $\Delta \theta$. 
There are two recording geometries, in-plane and out-of-plane, depending on if the angular spread of the Doppler channels is within the plane that contains the two recording green beams. When we scan the GS mirror about its $y$-axis and make the two writing beams intersect in the $x-z$ plane, as shown in Fig. 2.7, we have an in-plane recording geometry. When we scan the GS mirror about its $x$-axis and make the two writing beams intersect in the $x-z$ plane, as shown in Fig. 2.8, we have an out-of-plane recording geometry. We will analyze the momentum mismatch in both situations below with K-space analysis similar to that in [13].

In the in-plane geometry, the Doppler channels are in the $x$-direction. Thus $\Delta \theta$ spreads out within the plane that contains $k_1$ and $k_2$. $k_1$ and $k_2$ write a grating $K_G$. The readout red beam $k_r$ reads out the grating. Since the diffracted light is only allowed to propagate on the red momentum circle, there is a momentum mismatch, $\delta k_d$. There is an angle of $\theta_r - \theta_g$ between the K-space circle of the green beam and that of the red beam. The momentum mismatch $\delta k_d$ of this case is

$$\delta k_d = k_r + K_G - k_d.$$  \hfill (2.59)
Since the recorded Doppler matched gratings is narrow along the $x$-axis, the spatial frequency of the grating spreads widely in $x$ direction. Thus, we want to consider the momentum mismatch along the $x$-axis. First we write the momentum of the diffracted beam along the $z$-direction,

$$k_d \cdot \hat{z} = (k_r + k_1 - k_2) \cdot \hat{z} = k_r \cos \theta_r + k_g [\cos(\theta_g - \frac{\Delta \theta}{2}) - \cos \theta_g]$$

$$\approx k_r \cos \theta_r + k_g \sin \theta_g \cdot \frac{\Delta \theta}{2},$$

(2.60)

where $k_1 = k_2 = k_g = \frac{2\pi}{\lambda_g}$. And the momentum of the diffracted beam along the $x$-direction is

$$k_d \cdot \hat{x} = -\sqrt{k_r^2 - (k_d \cdot \hat{z})^2} = -k_g \sin \theta_g \left(1 - \frac{1}{2} \frac{\cos \theta_r \Delta \theta}{\sin \theta_r}ight).$$

(2.61)

Then the amount of Bragg mismatch is

$$\delta k_d \cdot \hat{x} = -(k_1 - k_d) \cdot \hat{x}$$

$$= -k_g \left[\sin \theta_g - \frac{\Delta \theta}{2} \cos \theta_g\right] + k_g \sin \theta_g \left[1 - \frac{1}{2} \frac{\cos \theta_r \Delta \theta}{\sin \theta_r}\right]$$

$$\approx (k_g \cos \theta_g - k_r \cos \theta_r) \frac{\Delta \theta}{2}.$$

(2.62)

The momentum mismatch is on the order of $\Delta \theta$. The diffraction efficiency is given by

$$\eta \propto \sin^2 \left(\frac{L_x \Delta \theta}{2\pi}\right) = \sin^2 \left[\frac{L_x \Delta \theta}{2} \left(\frac{\cos \theta_g}{\lambda_g} - \frac{\cos \theta_r}{\lambda_r}\right)\right]$$

(2.63)

where $L_x$ is the dimension of the SBN crystal in the $x$-direction. The Bragg sensitivity is when $\Delta \theta$ reaches the first null of the sinc function,

$$\Delta \theta = 2 \left[\frac{L_x}{\lambda_g} - \frac{\cos \theta_r}{\lambda_r}\right].$$

(2.64)

In the out-of-plane geometry, as shown in Fig. 2.8, the Doppler channels are in the $y$-direction. $\Delta \theta$ spreads out orthogonal to the plane that contains $k_1$ and $k_2$.

The momentum of the diffracted beam along the $y$-direction can be written as

$$k_d' \cdot \hat{y} = |k_1'| \sin \frac{\Delta \theta}{2}$$

(2.65)
Figure 2.8: Out-of-plane hologram recording. The GS mirror rotates about the \( x \)-axis and thus generates Dopplers along the \( y \)-direction. The angular spread \( \Delta \theta \) of the Doppler images is along the \( y \)-direction. The two writing beams \( k_1 \) and \( k_2 \) write holographic gratings \( K_G \) in the \( x-z \) plane. The grating \( K_G \) is orthogonal to the \( \Delta \theta \) spread direction \( y \). Notice that \( k'_d = k_g \sin \theta_g = k_r \sin \theta_r \).

where

\[
|k'_1| = \sqrt{\left(k_g \cos \frac{\Delta \theta}{2} \sin \theta_g\right)^2 + \left(k_g \sin \frac{\Delta \theta}{2}\right)^2} = k_g \sqrt{\sin^2 \theta_g + \sin^2 \frac{\Delta \theta}{2} \cos^2 \theta_g} \quad (2.66)
\]

And the momentum of the diffracted beam along the \( x \)-direction is

\[
k'_d \cdot \hat{x} = -\sqrt{k'_d^2 - (k'_d \cdot \hat{y})^2} = -\sqrt{(k_g \sin \theta_g)^2 - \left(k_g \sqrt{\sin^2 \theta_g + \sin^2 \frac{\Delta \theta}{2} \cos^2 \theta_g} \sin \frac{\Delta \theta}{2}\right)^2} = -k_g \sqrt{\sin^2 \theta_g \cos^2 \frac{\Delta \theta}{2} - \sin^4 \frac{\Delta \theta}{2} \cos^2 \theta_g}. \quad (2.67)
\]
Then the amount of Bragg mismatch is
\[
\delta \mathbf{k}_d \cdot \hat{x} = (\mathbf{k}_1' - \mathbf{k}_d') \cdot \hat{x} \\
= -k_g \cos \frac{\Delta \theta}{2} \sqrt{\sin^2 \theta_g + \sin^2 \frac{\Delta \theta}{2} \cos^2 \theta_g} \\
+ k_g \sqrt{\sin^2 \theta_g \cos^2 \frac{\Delta \theta}{2} - \sin^4 \frac{\Delta \theta}{2} \cos^2 \theta_g} \\
\approx -k_g \cos^2 \theta_g \sin^2 \frac{\Delta \theta}{2} \\
\frac{2}{\sin \theta_g \cos \frac{\Delta \theta}{2}} (2.68)
\]

For small $\Delta \theta << \frac{\pi}{2}$, Eq. (2.68) can be further simplified as
\[
\delta \mathbf{k}_d \cdot \hat{x} \approx -\frac{k_g \cos^2 \theta_g}{2 \sin \theta_g} \Delta \theta^2. (2.69)
\]

The Bragg mismatch is on the order of $\Delta \theta^2$. The diffraction efficiency in this case is given by
\[
\eta \propto \text{sinc}^2 \left( \frac{L_x \delta \mathbf{k}_d \cdot \hat{x}}{2\pi} \right) = \text{sinc}^2 \left[ \frac{L_x \cos^2 \theta_g \Delta \theta}{2 \lambda_g \sin \theta_g} \right] (2.70)
\]
where $L_x$ is the dimension of the SBN crystal in the $x$-direction. The Bragg sensitivity is when $\Delta \theta$ reaches the first null of the sinc function,
\[
\Delta \theta = \left( \frac{2 \lambda_g \sin \theta_g}{L_x \cos^2 \theta_g} \right)^{1/2}. (2.71)
\]

Compared to the in-plane geometry, due to the tangential relationship of the momentum surface, the orthogonal geometry has 2nd order Bragg mismatch while the in-plane case has 1st order of mismatch, thus the Bragg match is less sensitive in the out-of-plane geometry. Therefore in the experiments, we choose to use the out-of-plane geometry.

**2.2.2.4 Experimental results of using SBN as an integration medium**

This section gives the experimental results of using SBN as a time-integrating medium. We first verify with the single Doppler channels, then we use signals with multiple Doppler channels as targets. In single Doppler channel experiments, we start with the Doppler slice experiment, where the GS mirror is imaged to the SBN crystal to form a line at equal Doppler positions. To increase the SBN integration speed, we sum up the beam along the equal Doppler direction along the GS mirror and achieve Doppler spots.
Doppler slices

Figure 2.9: Doppler slices results. (a) Doppler slice images. (b) Doppler slice profiles. To find out the position of the Doppler slices center, we fit the single slice experiment data with Gaussian function. We then use the first order polynomial fit to find out the relationship between the Doppler frequency and the pixel position on the CCD. The minimum resolution of the Doppler slice is determined by the driving frequency of the GS mirror. There are about 100 resolvable Doppler channels.

Fig. 2.5 shows the Doppler slice verification setup. A Coherent solid state 532nm laser is
used to write holographic gratings in the SBN crystal and a HeNe laser at 633nm is used to read out the gratings. The raw laser beam from the 532nm laser is split into two beams. One beam is modulated by AOM1 at 100MHz. The +1 order of diffraction from the AOM1 is spatial filtered. After passing through a $\frac{\lambda}{2}$ that changes the polarization from vertical to horizontal, the beam is collimated by a $F = 200$mm lens. The beam transmits a PBS and a $\frac{\lambda}{4}$ that rotates its polarization to circular. Then the beam hits the GS mirror that is driven by periodic ramp signals and rotates about its $x$-axis, generating a bank of Doppler filters along the $y$-direction. The reflected beam from the GS mirror passes the $\frac{\lambda}{4}$ again which changes the polarization into vertical, so that it reflects off the PBS. A 4-f system that consists of two $F = 200$mm lenses images the GS mirror onto a 10mm×10mm×5mm Cr:SBN (10.03% Cr) photorefractive crystal. The other beam is modulated by AOM2 at 100MHz+$\delta f$Hz. The +1 order of diffraction from AOM2 is spatial filtered and then collimated by a $F = 200$mm lens. Both beams are overlapped in the center of the SBN crystal with an angle of $\theta_y \approx 20^\circ$, chosen to optimize diffraction efficiency. The two green laser beams write volume holographic gratings in the SBN crystal at the matched Doppler positions. A red HeNe laser at 633nm is spatial filtered and collimated by a $F = 100$mm lens and aligned at the Bragg matching angle of about $10^\circ \cdot \frac{633}{532} \approx 12^\circ$ to diffract off this grating. The diffracted beam is filtered by a red filter and imaged onto a CCD by a 4-f system consisting of two $F = 150$mm lenses.

The image of Doppler slices are shown in Fig. 2.9(a1)-(a3), where the GS mirror is driven by 100Hz/200mV$_{pp}$, 300Hz/100mV$_{pp}$, 400Hz/100mV$_{pp}$ ramp voltage signals. Since the crystal integrates multiple periods of the signal, the Doppler bins that the crystal “sees” are discrete. And the minimum resolvable Doppler slice step size is determined by the driving frequency of the GS mirror. We locate the position of the Doppler matched slices with Gaussian fit. We then linear fit the positions of the Doppler slices with the known Doppler frequency of each slice. In the cases of 100Hz/200mV$_{pp}$, 300Hz/100mV$_{pp}$, 400Hz/100mV$_{pp}$, the relationship between the Doppler slice frequency and its spatial position on CCD is given by $f_d \approx 37.5y$, $f_d \approx 56.0y$, and $f_d \approx 74.7y$ respectively, where $f_d$ has a unit of Hz and $y$ has a unit of pixel on the CCD. The Doppler coefficients $c_d = 37.5, 56.0, 74.7$ of the GS mirror is proportional to both the driving voltage and the driving
frequency of the GS mirror. $c_d$ has a unit of Hz/pixel. Using the Doppler frequency equation, Eq. (2.9) and

$$\Omega = \frac{\theta_m}{t_{up}},$$  \hspace{1cm} (2.72)

where $\theta_m$ is the maximum mirror rotation angle (proportional to the maximum driving voltage, for normal incident light, we have

$$f_d = \frac{2\theta_m y}{t_{up} \lambda).}$$  \hspace{1cm} (2.73)

Therefore, the Doppler coefficient $c_d$ can be written as

$$c_d = \frac{2\theta_m}{t_{up} \lambda}.$$  \hspace{1cm} (2.74)

c_d \approx 18.7$ for a GS mirror driven by 100Hz/100mV$_{pp}$ ramp.

**Doppler spots**

To increase the reaction speed of the SBN crystal, we focus the Doppler slice down to be a spot. By doing this, we increase the reaction speed of SBN from seconds to subseconds. Fig. 2.10 shows the Doppler slice verification setup. A 532nm laser is used to write holographic gratings in the SBN crystal and a 633nm HeNe laser is used to read out the gratings. A GS mirror driven by periodic ramp signals rotates about its $x$-axis, generating Dopplers along the $y$-direction. Two AOMs with offset driven frequencies are used to generate the verification Doppler signals. For maximum Doppler generation range, we use a PBS and a $\frac{\lambda}{4}$ to achieve normal incidence to the GS mirror. The GS mirror is imaged to the SBN crystal by a 1:1 4-f system in the vertical $y$ direction and focused in the horizontal $x$ direction. The green beams interact in the SBN crystal with an angle of about 20°. Gratings are written at the matched Doppler positions. The red laser beam with an angle of about 12° with respect to the SBN normal diffracts off the gratings and forms Doppler spot on the CCD. The Doppler spot results are shown in Fig. 2.11. In this case, the GS mirror is driven by a 100Hz/100mV$_{pp}$ ramp. The the relationship between the Doppler slice frequency and its spatial position on CCD is given by $f_d \approx 19.8y$. The Doppler coefficient $c_d = 19.8$ of the GS
Figure 2.10: GS mirror generated Doppler spots verification - using SBN as a time-integrating medium. The raw laser beam from a Coherent solid state 532nm laser is split into two beams. One beam is modulated by AOM1 at 100MHz. The +1 order of diffraction from the AOM1 is spatial filtered. After passing through a $\lambda/2$ that changes the polarization from vertical to horizontal, the beam is collimated by a $F = 200\text{mm}$ lens. The beam transmits a PBS and a $\lambda/4$ that rotates its polarization to circular. Then the beam hits the GS mirror that tilts about the $x$-axis. The reflected beam from the GS mirror passes the $\lambda/4$ again which changes the polarization into vertical. The GS mirror is imaged in the vertical $y$ dimension and Fourier transformed in the horizontal $x$ dimension to a 10mm$\times$10mm$\times$5mm Cr:SBN (10.03% Cr) photorefractive crystal. The diffracted beam is filtered by a red filter and imaged to a CCD by a 4-f system consisting of two $F = 150\text{mm}$ lenses.

Multiple Doppler targets

If there are multiple Doppler signals presents simultaneously, gratings will be written at

mirror is slightly different from the that of the Doppler slice experiment. This discrepancy could be due to the alignment of the anamorphic imaging system between the GS mirror and the SBN crystal.
Figure 2.11: Doppler spot experimental results. The GS mirror is driven by 100Hz/100mV\textsubscript{pp} signals. (a) Images of the Doppler spots. To find out the position of the Doppler slices center, we fit the single spot experiment data with Gaussian function. We then use the first order polynomial fit to find out the relationship between the Doppler frequency and the pixel position on the CCD. The minimum resolution of the Doppler slice is determined by the driving frequency of the GS mirror. There are about 100 resolvable Doppler channels.

different spatial positions and diffractions will occur correspondingly. To demonstrate multiple Doppler channels, we drive one of the AOMs with frequency modulated (FM) signals,

\[ s(t) = \sin \left[ \omega_0 t + \frac{f_{FM}}{f_{INT}} \sin(2\pi f_{INT} t) \right], \quad (2.75) \]

where \( f_{FM} \) is the frequency modulation and \( f_{INT} \) is the internal modulation. \( s(t) \) can be Fourier expanded as

\[ s(t) = \exp(i\omega_0 t) \sum_{n=-\infty}^{\infty} J_n \left( \frac{f_{FM}}{f_{INT}} \right) \exp(i2\pi n f_{INT} t) + \text{c.c.}, \quad (2.76) \]

where \( J_n \) is the Bessel function of the first kind, order \( n \) and c.c. is the complex conjugate of the first term. The results of the multiple-Doppler experiment are shown in Fig. 2.12. To make the spot with small Bessel coefficients visible, we increase the total power level of the light and saturate the spots with high amplitudes. For the case of \( f_{INT} = 400\text{Hz} \), when \( f_{FM} = 1\text{kHz} \), \( J_0(2.5) \cong 0 \), the Doppler free spot is missing; when \( f_{FM} = 2\text{kHz} \), \( J_2(5) \cong 0 \), the 2nd order 800Hz Doppler spot is missing; when \( f_{FM} = 4\text{kHz} \), \( J_1(10) \cong 0 \), the 1st order 400Hz Doppler spot is missing.
Figure 2.12: Multiple Doppler channel results. The GS mirror is driven by 50Hz/100mV pp signals. (a) \( f_{\text{INT}} = 400\text{Hz} \). For \( f_{\text{FM}} = 1\text{kHz} \), \( J_0(2.5) \cong 0 \), the Doppler free spot is missing. For \( f_{\text{FM}} = 2\text{kHz} \), \( J_2(5) \cong 0 \), the 2nd order 800Hz Doppler spot is missing. For \( f_{\text{FM}} = 4\text{kHz} \), \( J_1(10) \cong 0 \), the 1st order 400Hz Doppler spot is missing. (b) \( f_{\text{INT}} = 1\text{kHz} \). For \( f_{\text{FM}} = 4\text{kHz} \), \( J_1(4) \cong 0 \), the 1st order 1kHz Doppler spot is missing.

Figure 2.13: Particle picture of acousto-optic interactions. (a) For the diffracted beam to be added up in phase in the direction \( \theta \), the optical path difference \( AO + OB = m\lambda \). (b) Momentum-conservation relation. [Figure from Yariv [59].]

2.3 AOD as a Doppler Generator

Another way of generating Doppler bins is to use acousto-optic deflectors (AODs). AODs can be used to shift the optical frequency by scattering an optical photon off an acoustic photon via the elasto-optical effect and yielding a frequency \( \omega' = \omega + \Omega \) that conserves energy [59], where \( \omega \), \( \Omega \) and \( \omega' \) are the frequencies of the incident optical wave, acoustic wave and the diffracted optical
wave, illustrated in Fig. 2.13(a). When acoustic waves are propagating through the transparent material, moving index gratings are induced. The incident optical beams can be deflected by this moving grating, just as it is deflected by a static grating and the Doppler shift due to the scattering off these moving gratings yields a frequency shift. The Doppler frequency of a wave reflected from a moving object is

$$f_d = 2\omega \frac{v}{c/n}. \quad (2.77)$$

where $n$ is the refractive index of the acoustic material and $v$ is the projection of the sound wave velocity vector along the optical wave propagation direction. From Fig. 2.13, we have $v = v \sin \theta$, and thus

$$f_d = 2\omega \frac{v \sin \theta}{c/n}. \quad (2.78)$$

The conservation of momentum requires that the momentum of the colliding particles, $\hbar (k_{in} + K)$ be equal to the momentum of the scattered photon $\hbar k_{out}$,

$$k_{out} = k_{in} + K. \quad (2.79)$$

Since the sound frequencies of interest are usually around 100MHz-1GHz and these of the optical beam are usually on the order of Tera-Hertz, we have

$$\omega' \approx \omega, \text{ so } |k_{in}| \approx |k_{out}| = |k|$$

and the magnitude of the sound wave vector is thus

$$K = 2k \sin \theta, \quad (2.80)$$

where $K = \frac{2\pi}{\lambda}$ and $k = \frac{2\pi n}{\lambda}$. Using this Bragg matching condition Eq. (2.80), the Doppler shift of the scattered light Eq. (2.78) can be written as

$$f_d = \frac{2\pi v}{\lambda} = \Omega, \quad (2.81)$$
Figure 2.14: An AOD with aperture size $A$ and transducer length $L$ is driven by a single tone RF frequency. The region that the acoustic beam is overlapped with the optical beam is approximately a rectangular. The Fourier transform of the overlapped region is a 2-D sinc function with size $\frac{2\pi}{A} \times \frac{2\pi}{L}$.

where $\Lambda$ is the wavelength of the acoustic wave and therefore the frequency of the reflected light is upshifted by the acoustic frequency $\Omega$. Due to this effect, the RF signals can be modulated onto the optical carrier.

In the momentum-space figure, Fig. 2.13(b), both the optical wave and the acoustic wave are plane waves. In practice, both of them have finite beam widths, as is shown in Fig. 2.14. The Fourier transform of the overlapped region of the optical beam and the acoustic beam is often called an uncertainty box. In the case shown in Fig. 2.14, the uncertainty box is a 2D sinc function with the first zero size of $\frac{2\pi}{A} \times \frac{2\pi}{L}$. The uncertainty box indicates the area that the incident optical $k$-vector, $k_{in}$, can be kicked into in the momentum space. The overlap between the uncertainty box and the momentum circle of the optical beam with a radius of approximately $k = \frac{2\pi}{A}$ is the allowed propagation modes of diffracted optical beam. The uncertainty box is frequently used to help us analyze the acousto-optical interaction in the momentum $k$-space.
AODs operate in the Bragg region where the $Q$ parameter

$$Q = \frac{2\pi \lambda L}{n\Lambda^2} = \frac{4\pi \theta_B}{\Phi}$$

(2.82)

is much greater than 1 [59], where $L$ is the beam width of the acoustic wave, $\theta_B$ is the approximated Bragg angle solved from Eq. (2.80),

$$\theta_B \simeq \frac{\lambda}{2n\Lambda},$$

(2.83)

and $\Phi$ is the angle of the acoustic beam spreading in the far field

$$\Phi \simeq \frac{\Lambda}{L}.$$  

(2.84)

$\Phi$ can also be understood as the angular uncertainty of the acoustic wave in the $z$-direction in Fig. 2.14 ($\Phi \propto \frac{2\pi}{L}$). For $Q >> 1$, the acoustic $K$ vector is well confined with respect to the Bragg angle, so multiple scatterings are forbidden.

Ideal AOD materials need to have high optical quality, low optical and acoustic attenuation, and high acousto-optic figures of merit [6]. Figures of merits are defined to determine the properties (such as diffraction efficiency and bandwidth) of the AODs [6]. Another important factor is the acoustic velocity. Given the same length of AO material, the slower the acoustic velocity, the longer the access time, which is good for a high resolution deflector but undesirable for a modulator. The AODs used in the following experiments in this chapter are made of TeO$_2$ crystals, which has anomalously slow velocity along the [110] crystal direction.

### 2.3.1 TeO$_2$ as an AOD material

TeO$_2$ is a non-centrosymmetric optically active, positive uniaxial crystal belonging to the 422 symmetry class, which is transparent throughout the visible and IR to 5$\mu$m, and is the most common material for high efficiency acousto-optic deflectors at low frequencies. This is because of the favorable acoustical properties of its anomalously slow shear mode, and a fortuitous diffraction geometry enable by the splitting of the optical surfaces near the $z$-axis due to the presence of optical activity. The most important reason for using TeO$_2$ devices is its large figure of merit M2.
Figure 2.15: TeO$_2$ acoustic slowness surface.  (a) TeO$_2$ slowness surface in 3D with 3 different acoustic propagation modes. (b) X-Y cross section of slowness surface showing three acoustic modes, at 45° angle the acoustic velocity of the in-plane shear mode is anomalously slow compared with all other acoustic modes. [Figure from Mcleod [60]]

The higher the figure of merit $M_2$ is, the higher the diffraction efficiency will be for a given device geometry and acoustic power [6]. Since

$$M_2 = \frac{n^6 p^2}{\rho v^5},$$

(2.85)

where $n$ is the refractive index, $p$ is the effective photo-elastic coefficient, $\rho$ is the mass density, and $v$ is the acoustic velocity, the very slow velocity (0.614mm/µsec) for the slow shear mode of TeO$_2$ at 45° allows very large $M_2 = 795.$

The acoustic properties of TeO$_2$ are best illustrated on the acoustic slowness surface shown in Fig. 2.15 [60, 61], which is analogous to the optical index surface, except that in acoustics there are always 3 modes (2 shear or quasi-shear, and 1 longitudinal or quasi-longitudinal) instead of only 2 in optics. The 3D TeO$_2$ acoustic slowness (inverse of the velocity) surface is shown in Fig. 2.15(a), and is more useful than a velocity surface since slowness is proportional to acoustic momentum and can thus be directly utilized in momentum matching diagrams. Fig. 2.15(b) shows the cross section of the slowness surfaces in X-Y plane with the polarizations indicated. There are three acoustic modes, two shear modes (one is a quasi-shear slow mode and one is fast pure-shear mode) and one longitudinal mode. Along ±45° direction in this X-Y plane, the slowness of TeO$_2$ is huge,
Figure 2.16: (a) TeO$_2$ optical momentum surface with exaggerated optical activity splitting with acoustic slowness surface at an exaggerated frequency to illustrate 3D momentum matching. (b) Y-Z plane of the momentum surfaces. The optical momentum surfaces of TeO$_2$ are separated near the z-axis due to optical activity. The slowness surface on top of the optical momentum surface illustrates the tangential birefringent phase matching for TeO$_2$ acousto optic diffraction. The center of the slowness surface is on the extraordinary momentum surface and the tip of the slowness surface is on the ordinary momentum surface. [Figure from Mcleod [60]]

A nearly unique feature of TeO$_2$ and a few other materials with such soft modes, such as HgCl and HgBr that unfortunately are also hygroscopic and toxic, a bad combination [62]. The large slowness corresponds to a very small acoustic velocity of 0.62 mm/$\mu$sec, which allows a longer access time in a reasonably sized crystal. We can also think of acoustic waves as traveling gratings. With the same aperture size, if the acoustic wave has slower velocity, the acoustic wavelength is smaller for a certain frequency, corresponding to more gratings within the aperture, a bigger diffraction angle, and a higher RF frequency resolution. The flip side of these slow modes is they have a large acoustic attenuation, with about $\alpha_0 = 18$ dB/$\mu$sec/GHz$^2$ in TeO$_2$ along [110]. For a maximally tolerable $N$ dB attenuation across the device aperture, the attenuation coefficient $\alpha_0$[dB/$\mu$sec/GHz$^2$], upper frequency $f_u$ and the maximum access time $T_m$ have the following relationship:

$$\alpha_0 f_u^2 T_m = N\text{dB}. \tag{2.86}$$

For an octave bandwidth device, $f_u = 2f_l = \frac{2}{3} f_0 = 2B$, where $f_l$ is the lower operating frequency, $f_0$
is the center frequency and $B$ is the bandwidth of the device. Therefore the time-bandwidth (TB) product of a device with attenuation coefficient $\alpha_0$, upper frequency $f_u$ and a maximally tolerable NdB attenuation across the device aperture can be written as

$$T_mB = \frac{N}{2\alpha_0 f_u},$$

(2.87)

and the maximum device length is

$$L = vT_m = \frac{vN}{\alpha_0 f_u^2},$$

(2.88)

where $v$ is the acoustic velocity in the material. So with the other conditions the same, $TB$ is inversely proportional to the upper frequency $f_u$. For example, for TeO$_2$, at 1GHz upper frequency the maximum device length would be about 0.1mm for an octave bandwidth limited $T_mB \approx 83$. Instead at 100MHz upper frequency (e.g. a 50-100MHz bandwidth) the attenuation limit is at 0.167µsec and the TB is 833, which shows that these devices will work best at very low frequencies.

### 2.3.1.1 Phase matching of an AOD

The optical momentum surfaces with acoustic slowness surface are shown in Fig. 2.16(a), where the two big balls are the extra-ordinary and ordinary optical momentum surfaces of TeO$_2$ and the little cross sitting on the top is the acoustic slowness surface (shown in Fig. 2.15). Circular birefringence in TeO$_2$ allows tangential birefringent diffraction at a low RF frequency, where the extraordinary incident light is diffracted by the acoustic wave from the outer optical momentum surface to the inner surface as ordinary light, shown in Fig. 2.16(b). Since the separation of the optical momentum surfaces due to optical activities is only about 0.1% of the optical birefringence, tangential phase matching could be achieved in the 20-100 MHz range. The uncertainty box of the acoustic wave is a sinc function for a rectangular shape transducer. As the acoustic frequency changes from low to high, different part of the sinc intersects the optical inner momentum surface. For tangential phase matching, the acoustic K-vector skims tangentially along the inner optical momentum surface as the RF frequency tunes across the band. For a low RF frequency, the left
Figure 2.17: Tangential phase matching of AOD. (a) Optical momentum surfaces with exaggerated optical activity splitting. The acoustic wave has a sinc uncertainty box due to the aperture shape of the rectangular transducer. The center frequency of the acoustic wave is tangentially incident on the inner ordinary optical momentum surface. As the RF frequency varies, the length of the acoustic $K$ varies and the sinc uncertainty box moves with the acoustic $K$. Therefore, different part of the uncertainty box will intersect with the inner optic momentum surface. So the incident optical momentum will be deflected with different efficiencies due to the different intersection points on the sinc function. (b) Acousto-optic band shape of the AOD. The band shape of AOD resembles the shape of the transducer uncertainty box. The diffraction efficiency (DE) varies with the applied RF frequency. [After S. Kim’s thesis [63].]

part of the sinc intersects with the inner momentum surface, where the diffracted acoustic amplitude at that point is low. As the frequency goes higher, the intersection point moves to the center, where the acoustic amplitude is the highest, and it then moves to the left of the sinc again, where the acoustic amplitude becomes lower again. In this way, the coupling efficiency can be calculated and the resulting AO bandshape is given in Fig. 2.17(b).

### 2.3.1.2 Acoustic rotation and optic rotation

A problem arises for the geometry of Fig. 2.17 as shown in Fig. 2.18(a). Because of the symmetry, the beam diffracted by the middle frequency beam from the outer momentum to the inner momentum can be subsequently diffracted by the same acoustic momentum to the outer optical momentum surface. The diffraction efficiency would be decreased at the center of the bandshape. This phenomenon is called mid-band degeneracy, as is illustrated in Fig. 2.18(a) and increases in
severity as the diffraction efficiency gets longer. This problem can be solved by acoustically rotating the AO to break the symmetry, as is shown in Fig. 2.18(b) in which the transducer face is cut at a small angle [110]. The unrotated AOD is cut such that the normal direction of the transducer is along [110], perpendicular to the optical axis. Alternatively, the rotated AOD is cut normal to the acoustic momentum \( k \) direction. However, the large acoustic power walkoff (in the direction of the normal of the momentum surface) can make the size of the acoustically rotated AODs several times bigger than the unrotated ones for the same size of time aperture.

Conoscopic pattern of the AOD can help us find out if the AOD is acoustically rotated or not. As is shown in Fig. 2.19, a 633nm horizontally polarized laser beam is focused by a 10\( \times \) 0.25 NA objective lens and is incident on the tested AOD. A vertical polarizer is used after the AOD and a conoscopic pattern can be seen on the screen. By examining the conoscopic pattern, we are able to find the direction of the optical axis for the TeO\(_2\) AODs. Since the center of the conoscopic pattern indicates the crystal axis and it will be shifted for acoustically or optically rotated TeO\(_2\) crystals. A uniaxial crystal will have a crossed pattern and the position of the cross indicates the direction of the optical axis of the crystal. A biaxial crystal has two “cat eye” patterns and the centers of those eyes indicate the directions of the two optical axes of the crystal.
Figure 2.19: Conoscopic Pattern Setup. A tightly focused polarized laser beam at 633nm impinged on the tested AOD. A polarizer is used after the AOD and interference pattern, which is called conoscopic pattern is obtained on the screen. The center of the conoscopic pattern indicates the optical axis direction of the AOD.

**Conventional Unrotated AOD**

Conoscopic patterns of the conventional unrotated AODs fabricated by Optech are shown in Fig. 2.20. Although TeO$_2$ is a uniaxial crystal, the cat eye patterns show that the material has some biaxial characteristic which might come from strain from the crystal mounting or from low quality crystals. The center of the eyes is in the center of the conoscopic pattern indicating that the AODs are not rotated.

**Acoustically rotated AOD**

Conoscopic patterns of two acoustically rotated AODs are shown in Fig. 2.21. The center of the rings is shifted to the left. By measuring the distance from the ring center to the center of the straight through beam and the distance from the objective lens focal point to the screen, we can verify that the optical axis is rotated away from the surface normal by about 6°. In addition, the region around the optical axis is circular without the elliptical figure that indicates biaxiality due to crystal strain.
Figure 2.20: Conoscopic Pattern for Unrotated AODs. The cat eye pattern indicates that the tested AODs are biaxial. (a) Conoscopic pattern for tested AOD1. (b) Conoscopic pattern for tested AOD2. Both patterns vary in space from $+45^\circ$ to $-45^\circ$ as the illuminating spot moves along the direction of propagation.

Figure 2.21: Conoscopic pattern for acoustically rotated AODs. (a) Conoscopic pattern for tested AOD1. (b) Conoscopic pattern for tested AOD2.

In contrast to the acoustic rotation, optical rotation can be achieved by rotating the AOD
Figure 2.22: Eigen polarizations of TeO$_2$. Top view of Fig. 2.16. Solid lines represents extraordinarily polarized light. Dashes lines represents ordinarily polarized light. the center is the $z$-axis. Along the $z$-axis, circular polarizations are the eigen modes. The eigen modes change from circular to elliptical and then linear as the direction of propagation is tilted away from the $z$-axis at the center. [Figure from S. Kim’s thesis [63]]

about the acoustic wave propagation direction so that the plane of incidence is tilted up from the optical axis which increases the spacing between the inner and outer momentum surface, allowing a tuning of the center RF frequency.
2.3.1.3 Eigen polarizations of TeO$_2$

The polarization change during the diffraction is complicated due to the optical activity of the TeO$_2$ material. Optical activity splits the optical surfaces near the z-axis and the eigen modes near the optical axis are circular. Away from the optical axis, the eigen polarizations change gradually from circular to elliptical and then to linear. For visible light, when the internal propagation angle is about 3° from the optical axis, the eigen-polarizations are almost pure linear. Viewed from near the optical axis, extraordinary polarization is radial and ordinary is circumferential as is illustrated in Fig. 2.22.

Referring to Fig. 2.22, in either an optically or acoustically rotated device, the input extraordinary light several degrees away from the optical axis should be mostly horizontally polarized at an orientation angle to match the eigen mode. However, for an optically un-rotated AOD, elliptical polarization is the required incident light, and the diffracted light is orthogonal circular at midband. We also need to make sure that the input optical k vector is several degrees away from the optical axis for an un-rotated AOD. In addition, the state of polarization for the diffracted light varies with acoustic frequency, as shown in Fig. 2.22. In an unrotated device, the center acoustic frequency diffracts light to be along the optical axis so that it becomes circularly polarized. At higher or lower acoustic frequencies the diffracted light changes from circular to elliptical. When we cascade two AODs in the later sections, the output polarization from the first AOD needs to be changed from circular to linear to accommodate the input polarization requirement for the second AOD. Although a polarizer can be used for this case, it will waste more than half the light for in-plane AODs (and exactly half on axis) and a little less than half of the light for crossed AODs. A $\frac{\lambda}{4}$ waveplate can change circular polarization to linear losslessly, but the polarization will vary somewhat with frequency, so the coupling into the correct optical mode will decrease on the edges of the RF bands. In fact, for a diffraction range from $-1$ to $+1$ degree, we could exactly match the eigen polarization into the second AOD at two points, say $-0.5^\circ$ and $+0.5^\circ$ giving a double peaked polarization coupling efficiency, at the expense of slightly less efficiency at midband.
Figure 2.23: Topologies for cascaded in-plane AODs. All imaging between the AODs is 1:1. All AODs are launched with up-chirps. (a) ++ Doppler shifted counter-propagating AODs. +1 order diffractions from both AODs are used. The first AOD is inversely imaged onto the second AOD, therefore the acoustic waves are propagating oppositely. (b) +− Doppler compensating co-propagating AODs. +1 order diffraction of the first AOD and −1 order of the second AOD are used. The first AOD is imaged to the second AOD in a way that the acoustic waves of both AODs propagate in the same direction. (c) +− Doppler compensating counter-propagating case. +1 order for the first and −1 order for the second AOD are used. The first AOD is inversely imaged onto the second AOD, therefore the acoustic waves are propagating oppositely. (d) ++ Doppler shifted co-propagating case. +1 order diffraction for both AODs are used. The first AOD is imaged to the second AOD in a way that the acoustic waves of both AODs propagate in the same direction.

2.3.2 Two cascaded AODs

A single AOD can only generate Dopplers around its center frequency $f_2$, as is shown in Fig. 2.17, and nothing around 0 Doppler can be produced. It will generate either positive or
negative Doppler shifts but not both. To generate both signs of Dopplers, we need cascade two AODs that generate opposite Doppler shifts.

### 2.3.2.1 Topologies for in-plane AODs

As shown in Fig. 2.23, there are four possible topologies for the cascaded in-plane Bragg cells fed with up-chirps. If the AODs are all fed with down chirps, the result will be the same except that the positive phase shift will become negative phase shift. The mixed up/down chirp situations will be equivalent to one of the following four listed situations. Since they give the same results as one of the listed cases in Fig. 2.23, I do not draw each of them in the figure. Of the 4 listed figures, only one is appropriate for generating the linear array of Doppler shifts. Suppose all the AODs are fed with low bandwidth repetitive up-chirp signals,

$$s_c(t) = \exp(i\pi bt^2) \cdot \Pi \left(\frac{t}{T}\right) \ast \text{comb} \left(\frac{t}{T}\right) + \text{c.c.},$$

(2.89)

where $b$ is the chirp rate. $\Pi$ is a rectangular function $\Pi(t) = 1$ ($|t| < 1/2$), $\Pi(t) = 0$ (otherwise). $T$ is the periodicity of the chirp. The diffraction from the AOD displays the RF driving signals along its aperture,

$$s(x, t) = s_c(t - x/v)\Pi\left(\frac{x}{X} - \frac{1}{2}\right),$$

(2.90)

where $x$ is the transverse coordinate of the AOD, $v$ the acoustic velocity and $X$ is the AOD width. In our experiment, the period of the chirp is much larger than the time the acoustic waves take to pass through the AOD aperture $A$, $T \gg \frac{A}{v}$, which enables us to neglect the situation when the acoustic waves from successive chirp repetitions are partly overlapped. Then we can find out the output for each of the cases in Fig. 2.23.

For case (a), the output doubles the single AOD Doppler shifts, since $+1$ orders for both AODs are used and the first AOD is inversely imaged onto the second AOD. In contrast to the position dependent Doppler output for a single AOD, the Doppler output of this case is position
independent,

\[ s_{\text{co}++}(x, t) = \exp[i\pi b(t + x/v)^2] \exp[i\pi b(t - x/v)^2] \Pi(\frac{x}{X} - \frac{1}{2}) \]

\[ = \exp \left[ i2\pi b(t^2 + \frac{x^2}{v^2}) \right] \Pi(\frac{x}{X} - \frac{1}{2}). \quad (2.91) \]

It is a chirp signal with doubled chirp rate $2b$. It also includes a quadratic phase factor $\exp \left( i2\pi b \frac{x^2}{v^2} \right)$ in the $x$ direction. This case is equivalent to mixed up/down chirps ++ Doppler shift counter-propagating AODs.

For case (b), +1 order for the first AOD and $-1$ order for the second AOD are used, and the first AOD is imaged onto the second one. The result is the second AOD will undo exactly what the first AOD does. We then have $+$ and $-$ signs for the first and second phase factor, and the output can be written as

\[ s_{\text{co}+}(x, t) = \exp[+i\pi b(t + x/v)^2] \exp[-i\pi b(t + x/v)^2] \Pi(\frac{x}{X} - \frac{1}{2}) \]

\[ = \Pi(\frac{x}{X} - \frac{1}{2}). \quad (2.92) \]

This geometry enables a dispersion free high contrast modulation. But it won’t generate position dependent Dopplers. This case is equivalent to mixed up/down chirps ++ Doppler shift co-propagating AODs.

For case (c), again, +1 order for the first AOD and $-1$ order for the second AOD are used, but the first AOD is inversely imaged onto the second one. The output can be written as

\[ s_{\text{co}+}(x, t) = \exp[+i\pi b(t + x/v)^2] \exp[-i\pi b(t - x/v)^2] \Pi(\frac{x}{X} - \frac{1}{2}) \]

\[ = \exp(i4\pi b \frac{x^2}{v^2}) \Pi(\frac{x}{X} - \frac{1}{2}). \quad (2.93) \]

Notice that the quadratic phase factors are all canceled out and only the linear frequency shift terms are left. The linear frequency shift, $f_d = \frac{2b}{v} x$, is position dependent and it is programmable by changing the chirp rate $b$. This is the topology for generating position dependent Doppler bins. This case is equivalent to mixed up/down chirps ++ Doppler shift counter-propagating AODs.
Figure 2.24: Band shape for (a) AOD1 and (b) AOD2.

For case (d), the +1 orders for both AODs are used, and the first AOD is imaged onto the second one. The output can be written as

\[
\begin{align*}
    s_{co+}(x,t) &= \exp[+i\pi b(t + x/v)^2]\exp[+i\pi b(t + x/v)^2]\Pi\left(\frac{x}{X} - \frac{1}{2}\right) \\
    &= \exp[+2\pi b(t + x/v)^2]\Pi\left(\frac{x}{X} - \frac{1}{2}\right). \\
\end{align*}
\]

This case just doubles the operating frequency and the chirp rate of a single AOD. For all the in-plane AOD cases, case (c) is the desired geometry for generating spatial position dependent Doppler shifts. This case is equivalent to mixed up/down chirps – Doppler shift co-propagating AODs.

### 2.3.2.2 Band shape of in-plane counter-propagating AODs

To investigate the effect of cascading AODs to the output signal bandwidth, we first measure the band shape of the two single AODs. We drive the AOD with slow chirps sweeping from 0 to 80MHz and use an oscilloscope to observe the diffraction of the AOD detected by a power meter, as shown in Fig. 2.24(a) and (b). These bandshapes vary with the incidence angle, but these plots represent the best compromise of bandwidth, efficiency and uniformity. We then cascade these two AODs by using a 1:1 4f imaging system, shown in Fig. 2.25. Both AODs are fed with the same 80MHz bandwidth repetitive chirp with chirp rate \( b = 80\text{MHz}/\text{ms} \). The +1 order of AOD1 and \(-1\)
Figure 2.25: Cascaded in-plane counter-propagating TeO$_2$ AODs. A horizontally polarized collimated laser beam illuminates AOD1. AOD1 is imaged onto AOD2 by a 4f system. The +1 diffraction order of AOD1 and −1 diffraction order of AOD2 are used. A $\frac{\lambda}{4}$ waveplate is used to turn the circularly polarized light into linearly polarized (horizontally polarized) light.

order of AOD2 are used. A horizontally polarized laser beam is used to match the input eigen mode for tangential phase matching in AOD1. Since in this condition, the output for AOD1 is nominally circularly polarized (although the polarization does vary with RF frequency), a $\frac{\lambda}{4}$ waveplate is used to turn the circularly polarized light into linearly polarized light such that the eigen mode for the tangential phase matching condition can be satisfied for AOD2. The AODs we use for the experiment are designed to tangentially phase match a single input plane wave. However, the input angle to AOD2 varies with the frequency driving AOD1. The bandwidth for the cascaded counter-propagating AOD system is much smaller than either of the single AODs, because most of the diffracted beam from AOD1 Bragg mismatches for AOD2, as is illustrated by Fig. 2.26. The cascaded AOD band shape is shown in Fig. 2.27. The bandwidth reduces from about 20MHz to less than 5MHz.

2.3.2.3 Band shape of crossed AODs

To access more of the available bandwidth the single AODs have, we can use two crossed AODs to avoid in-plane Bragg mismatch limitations in the second AOD, as shown in Fig. 2.28. We use the same AODs that we use for the counter-propagating AOD case. The AOD1 is imaged to AOD2 with a 4f system consisting of two identical lenses. The +1 order of AOD1 and −1 order of AOD2 are used. Since the transducer height of these AODs is a diamond of 8mm height, in
Figure 2.26: k-space of TeO$_2$ AODs. (a) k-space of TeO$_2$ AOD1 shows that the input optical $k$ is diffracted by the acoustic $k$ vectors into different directions. (b) k-space of AOD2 shows that only the center frequency can be tangentially phase matched.

Figure 2.27: Band shape for the cascaded counter-propagating AODs demonstrating bandwidth narrowing in cascaded operation, achieving only 4MHz of 3dB bandwidth.

this crossed geometry, up to an $4 \times 4$mm doubly diffracted beam can be generated. A horizontally
Figure 2.28: Cascaded crossed TeO$_2$ AODs. A horizontally polarized collimated laser beam illuminates AOD1. AOD1 is imaged onto AOD2 by a 4f system. The +1 diffraction order of AOD1 and −1 diffraction order of AOD2 are used. A $\lambda/4$ waveplate is used to turn the circularly polarized light into linearly polarized (horizontally polarized with respect to AOD2) light.

So places where $x + y = \text{constant}$ experience the same Doppler shift and the Doppler shift varies linearly in $x + y$. However, there is now an unwanted hyperbolic phase factor associated with the generated Doppler shift. We can compensate the phase factor by using a positive lens with $F = \frac{2v^2}{M}$ and a negative cylinder with $F_y = -\frac{x^2}{R}$. In this case, the light diffracted by both AODs scans in the Fourier plane along a 45° line. The cascaded band shape is given in Fig. 2.29. Because all
the diffracted light from AOD1 is in the orthogonal direction of the phase matching direction for AOD2, we can fine tune AOD2 to satisfy the phase matching condition for all the diffracted beams, resulting in wider bandwidth, achieving about 13MHz in this case, a 3-fold improvement.

2.3.3 AOD Doppler verification experiment

2.3.3.1 Doppler slice verification experiment

In the last section, we generated array of Doppler shifts along 45° angle and the Doppler shift linearly varies with the spatial location. To verify the Doppler shift for each spatial location, we interfere the array with a known single Doppler shifted reference beam. Only the positions where both arms have the same Doppler shift can interfere with each other and produce stationary time-integrated fringes on the CCD. So the locations where the fringe occurs has the same frequency of Doppler shift as the reference arm.

The experimental setup is shown in Fig. 2.30. A Mach-Zehnder interferometer is built to
Figure 2.30: AOD Doppler slice verification setup. A collimated HeNe beam is split by a beam splitter (BS) into two arms. The upper arm is the Doppler array generation arm, where two crossed AODs driven by the same chirp generate Doppler arrays. AOD1 is imaged onto AOD2 by a 4f system. The +1 order of AOD1 and the −1 order of AOD2 are selected with a 45° rotated slit spatial filter. The diffracted beam after AOD2 is then imaged by a tele-centric imaging system with a magnification of 2. The lower arm is the reference beam arm, where two counter propagating AOMs are used and driven by single frequency sines with a small frequency offset \( f \) to produce a small Doppler difference frequency. The −1 order of AOM1 and +1 order of AOM2 are used. The diffracted beam after AOM2 is collimated by a \( F = 200\text{mm} \) lens. Another BS combines these two beams which are detected on a time-integrating CCD.

Interfere the Doppler shifted arrays generated by two conjugate order crossed AODs with the single frequency Doppler shift beam generated by two conjugate order counter propagating AOMs. For the AOD arm, AOD1 is imaged onto the crossed AOD2 by a one-to-one 4f imaging system. The doubly diffracted light is then imaged by a 1:2 tele-centric imaging system onto the detector passing through a 45° rotated slit in the Fourier plane to block the undiffracted and the singly diffracted orders. Since the AODs are crossed and both AODs are fed with the same chirp, along the 45° lines of the illuminated aperture, light experiences equal Doppler shift. For the AOM arm, we apply single tones to both AOMs, but the RF frequencies are offset by small frequency differences on the order of kHz. AOM1 is imaged onto AOM2 by a one-to-one imaging system, with conjugate diffraction orders so that again Doppler compensation is used to produce a modulation at the frequency difference. The diffracted light after AOM2 is collimated by a \( F = 200\text{mm} \) lens and imaged onto the detector. So by varying the frequency difference between the two AOMs, we are able to generate known Doppler shifts in the AOM arm. By interfering both arms, we are able to
know the distribution of the Dopplers generated by the crossed AODs.

The experimental results are shown in Fig. 2.31 (a)-(e), each having 5KHz, 1KHz, 0KHz, -1kHz, -5kHz of Doppler shifts. We can see that the fringes along the $-45^\circ$ tilted line are moving from the upper right to lower left. At 0kHz, the extraneous fringes appear and this is because chirps are generated by an Agilent 33250A 80MHz AWG that in the time between the chirps produces an unwanted CW tone. This can be eliminated by using our DDS chirper or a more programmable AWG.

2.3.3.2 Doppler bin verification experiment

In the last section, we verified that an array of Doppler slices was generated by crossed AODs driven by the appropriate chirps. As a further step, we focus the beam along the $x + y$ direction to produce a 1-D array of Doppler shifted spots. An unwanted hyperbolic phase factor is associated with the crossed AOD geometry. But with $45^\circ$ focusing, the hyperbolic phase factor will not affect the linearity of the Doppler bins, which is also confirmed by the experimental results. Fig. 2.32 is the Doppler bins generation and verification setup, which is similar to the Doppler slice verification setup. Again, a Mach-Zehnder interferometer is built to interfere the array of Doppler bins generated by two conjugate order crossed AODs with the single-frequency Doppler shifted beam generated by two conjugate order counter propagating AOMs. The upper arm is the Doppler generation arm, where AOD1 is imaged onto AOD2 by a one-to-one 4f imaging system. Both AODs are driven by the same RF chirped signal. In AOD1, the chirp signal is propagating in the $x$ direction, which means across the beam aperture different position of the beam along
Figure 2.32: Doppler verification setup. A collimated beam is split by a BS into two arms. The upper arm is the Doppler array generation arm, where two crossed AODs driven by the same chirp are used. AOD1 is imaged onto AOD2 by a 4f system. The +1 order of AOD1 and -1 order of AOD2 are selected with a 45° rotated slit spatial filter. The diffracted beam after AOD2 is then imaged by a tele-centric imaging system with a magnification of 2 along the Doppler array direction and focused by a cylinder along the equal Doppler shift direction. The lower arm is the reference arm, where two counter propagating AOMs are used and driven by the same CW tones. The -1 order of AOM1 and +1 order of AOM2 are used. And the diffracted beam after AOM2 is expanded by a beam expander and focused by a 45° rotated cylinder. BS2 brings light from both arms to be overlapped with each other as a 45° rotated slice of light. A multi-mode fiber moving along a 45° rail is used to direct the beat signal to a photo-detector, which is connected to a RF spectrum analyzer to observe the beat signal.

$x$ is modulated by different Doppler up shifts. At the same time, in AOD2 the chirp signal is propagating in the $y$ direction, which modulates the beam in the $y$ direction with different Doppler down shifts. The one-to-one imaging condition and the conjugate order geometry help to cancel out the quadratic Doppler shifts and leave only the linear phase shifts across the aperture. 45° linear Doppler slices are thus generated by these two crossed AODs. Instead of imaging AOD2 onto the CCD detector with a 4f imaging system as in previous setups, we instead use a cylinder to focus the beam down along the $-45°$ equal Doppler slice direction and in the mean time still image along $+45°$ Doppler array direction with a tele-centric imaging system with a magnification of two. This gives the appropriate 1D array of Doppler shifts. The lower arm is the Doppler verification arm with AOMs driven by the same CW tones. Since the conjugate orders of AOMs are used, the beam after AOM2 is at a low frequency offset given by the difference of AOM1 and AOM2 drive frequencies across the whole aperture. Another cylinder is used to focus the beam along the same 45° line such that the focused beams from both arms are exactly overlapped with
each other. A multi-mode fiber mounted on a 45° tilted rail right at the focal plane of the 45° focused beams from both arms directs the light to a photo-detector. An RF spectrum analyzer is employed to analyze the frequency beat between both arms. As the fiber moves along the 45° tilted rail, the beat frequency peak will move with it, with zero beat frequency at the center and positive or negative beat frequencies on either side. By finely adjust the position of the fiber on the rail, we can find out the maximum beat frequency peak for regular spaced intervals in space. The recorded position versus frequency chart is shown in Fig. 2.33. By varying the chirp rate of the chirp signal, different Doppler spans are obtained. From Eq. 2.95, neglecting the spatial quadratic factor, the Doppler shift given by the crossed AODs is \( f_d = \frac{2b}{Mv}x \), where \( M \) is the magnification factor given by the tele-centric system and \( x \) is along the Doppler direction as defined in Fig. 2.33. The least square fit of the experimental data gives the relationship between the Doppler shifts and positions. The variance of the fits in Fig. 2.33 shows that the generated Doppler shifts are highly linear with spatial position. For a chirp rate of \( b = 0.5\text{MHz/msec} \), \( M = 2 \) and TeO\(_2\) acoustic velocity \( v = 0.614\text{mm/\mu sec} \), the theoretical Doppler shift is \( f_d = 0.8143x \) and the experimental result, \( f_d = (0.8195 \pm 0.0023)x \) has 0.64% error. For a chirp rate of \( b = 2\text{MHz/msec} \), the theoretical Doppler shift is \( f_d = 3.2573x \) and the experimental result, \( f_d = (3.2757 \pm 0.0028)x \) has 0.56% error. This probably indicates that the magnification of the imaging system is more like 1.99, which is within the range spanned by the lens focal length tolerance.

In conclusion, we have experimentally verified that the linear array of Doppler shifts produced by crossed AODs driven by chirps is highly linear and scalable and zoomable by varying the chirp rates as required for range-Doppler processing.
Figure 2.33: Doppler bin verification result. The Doppler shift linearly varies with position. The fiber position along the 45° tilted axis as given by a motorized stage is shown along the horizontal axis while the peak beat frequency in kHz measured by a dynamic signal analyzer is shown along the vertical axis for two different chirp rates. Least square fit of the data gives the slopes of the lines. The variance of the fit shows that the Doppler shift \( f \) is highly linear with the spatial position \( x \).

Acousto-optic Doppler Array Generation

\[ f = 0.8195x \]
\[ (\sigma = 2.32e^{-3}) \]
\[ b = 0.5 \text{MHz/msec} \]

\[ f = 3.2757x \]
\[ (\sigma = 2.79e^{-3}) \]
\[ b = 2 \text{MHz/msec} \]
Chapter 3

Two-level atoms

In the next few chapters, we will employ a rare earth doped crystal (Tm$^{3+}$ doped YAG) for signal processing. For simplicity the atoms doped into the crystal lattice can be modeled as two-level atoms, even though a bottleneck state in the relaxation path makes them more appropriately described as a three-level system, and the interaction of light with the atoms can be described by the optical Bloch equations. For further understanding of the light interaction with matter in two-level atoms, refer to Allen and Eberly’s book, *Optical Resonance and Two-Level Atoms*. Ken Anderson [64] and Friso Schlottau [28] also perform very clear derivations of the optical Bloch equations and perturbation theory in their theses. Max Colice [4] further derives the spectral-domain perturbation theory and applies the results to spectral-hole burning experiments. Benjamin Braker [65] discusses the assumptions of the perturbation analysis. The following derivations include the major results from the above references that are relevant to this thesis. Further, I analyze some of the special cases of the spectral-domain perturbation theory for the photon echo experiments (such as diffraction-off-the-causal-time-edge experiment) discussed in later chapters.

The derivations start with using quantum theory to formally describe the light-atom interactions. Then under the rotating wave approximations, when choosing different rotating frames, one can get the Bloch equations that we use in Chapter 4 for Maxwell-Bloch simulations; or one can get the time domain perturbation theory. Fourier transforming the time domain perturbation theory, one can get the spectral domain perturbation theory. With a few further approximations, the spectral domain perturbation theory can lead to explicit results for later experiments in Chapter
5. Finally, Section 3.3 compare the echo amplitude for using pulses or chirps to record the spectral domain gratings.

3.1 Two-level systems

Consider a two-level quantum system, upper level $|1\rangle$ and lower level $|0\rangle$, interacting with a laser field

$$E(r, t) = \left\{ E(t) \exp \left[ i\omega(t - n(\kappa) \kappa(t) \cdot r/c) \right] + \text{c.c.} \right\} \hat{E}, \quad (3.1)$$

where $\omega$ is the laser frequency, $E(t)$ is the envelope of $E(t)$ and $\kappa$ is the direction of propagation (it is a unit vector). We are concerned with electric dipole transitions and can write

$$\hat{H} = \hat{H}_0 + \hat{H}_1 \quad (3.2)$$

for the Hamiltonian (the system’s energy operator) of the interacting atoms. $\hat{H}_0$ is the unperturbed Hamiltonian. Suppose the energies at the eigenstates of $\hat{H}_0$, $|1\rangle$ and $|0\rangle$, are $W_1$ and $W_0$ respectively. Only transitions between $|0\rangle$ and $|1\rangle$ will be of interest in the discussion, so the matrix form of the stationary Hamiltonian is

$$\hat{H}_0 = \begin{bmatrix} W_1 & 0 \\ 0 & W_0 \end{bmatrix}. \quad (3.3)$$

$\hat{H}_1$ is the field dependent Hamiltonian,

$$\hat{H}_1(t) = -\hat{d} \cdot E(r, t), \quad (3.4)$$

where

$$\hat{d} = \begin{bmatrix} 0 & d_{21} \\ d_{12} & 0 \end{bmatrix} \quad (3.5)$$

is the electric dipole moment operator, $d_{12} = d_{21}^*$. Writing in matrix form, $\hat{H}_1$ becomes,

$$\hat{H}_1 = \begin{bmatrix} 0 & -\mu_{21}E(r, t) \\ -\mu_{12}E(r, t) & 0 \end{bmatrix}, \quad (3.6)$$
where \( \mu_{12} E(r, t) = d_{12} E(r, t) \) is the magnitude of the vector projection in Eq. (3.4). Therefore the total Hamiltonian is

\[
\hat{H} = \begin{bmatrix}
W_1 & -\mu_{21} E(r, t) \\
-\mu_{12} E(r, t) & W_0
\end{bmatrix}.
\]  

(3.7)

The density matrix,

\[
\rho = \begin{bmatrix}
\rho_{22} & \rho_{21} \\
\rho_{12} & \rho_{11}
\end{bmatrix},
\]

(3.8)

that describes the probability of wave distribution over the two eigenstates satisfies the Liouville equation

\[
\dot{\rho} = \frac{i}{\hbar} [\rho, \hat{H}],
\]

(3.9)

where the dot over \( \rho \) indicates a time derivative. Define \( W_1 - W_0 \equiv \hbar \omega_0 \) as the level-splitting angular frequency and we can write the equations of motion for the density matrix elements as

\[
\dot{\rho}_{11} = \frac{i}{\hbar} \mu_{12} E(r, t)(\rho_{12} - \rho_{12}^*),
\]

(3.10)

\[
\dot{\rho}_{22} = -\frac{i}{\hbar} \mu_{12} E(r, t)(\rho_{12} - \rho_{12}^*)
\]

(3.11)

\[
\dot{\rho}_{12} = -\frac{i}{\hbar} [\mu_{12} E(r, t)(\rho_{22} - \rho_{11}) + \hbar \omega_0 \rho_{12}],
\]

(3.12)

where using the relationship \( \rho_{12} = \rho_{21}^* \) simplifies the resulting expressions. Notice that the ground-state populations and the excited-state populations are equal and oppositely varying, i.e., \( \rho_{22} = -\rho_{11} \), because the total population is conserved. By defining \( \rho_{22} - \rho_{11} = \omega \), and writing the off-diagonal matrix component \( \rho_{12} \) as the sum of in-phase and quadrature components, \( \rho_{12} = (u + iv)/2 \), we can get the collision-less Bloch equations

\[
\dot{u} = \omega_0 v
\]

(3.13)

\[
\dot{v} = -\omega_0 u - \frac{2}{\hbar} \mu_{12} E(r, t)w
\]

(3.14)

\[
\dot{w} = \frac{2}{\hbar} \mu_{12} E(r, t)v.
\]

(3.15)
The collision-less Bloch equations do not consider the dephasing or the population decay from the excited state to the ground state. But in the real world, the phonon scattering in a solid can randomly alter the phase of dipole oscillation without changing its energy ($u$ and $v$ decay, but $w$ does not change). In addition, the population decays to the ground state through spontaneous emission. Incorporating the decoherence time $T_2$ and excited-state life time $T_1$ allows us to write the Bloch equations as,

$$
\dot{u} = -\frac{u}{T_2} + \omega_0 v \tag{3.16}
$$

$$
\dot{v} = -\omega_0 u - \frac{v}{T_2} - \kappa E(r, t)w \tag{3.17}
$$

$$
\dot{w} = \kappa E(r, t)v - \frac{w - w_{eq}}{T_1}, \tag{3.18}
$$

where $\kappa \equiv \frac{2}{\hbar} \mu_{12}$ and $w_{eq}$ is the equilibrium population inversion when $E(r, t) = 0$. Without incoherent external energy input, all the ions are at the ground state, i.e. $w_{eq} = -1$.

### 3.2 Rotating wave approximations

Notice that the off-diagonal term of the Hamiltonian $\hat{H}$,

$$
|0\rangle\langle 1| = -\mu_{12}E(r, t) = -\mu_{12}\left\{\mathcal{E}(t)\exp\left[i\omega[t - n(\hat{k})\hat{k}(t) \cdot r/c]\right] + c.c.\right\} \tag{3.19}
$$

contains two terms, one with frequency $\exp(i\omega t)$, the other with frequency $\exp(-i\omega t)$. Allen and Eberly [66] choose rotating frame at the optical frequency $\omega$. Under such a rotating frame,

$$
|0\rangle\langle 1| = -\mu_{12}E(r, t)
$$

$$
= -\mu_{12}\left\{\mathcal{E}(t)\exp\left[-i\omega n(\hat{k})\hat{k}(t) \cdot r/c\right] + \exp\left[-i2\omega t + i\omega n(\hat{k})\hat{k}(t) \cdot r/c\right]\right\} \tag{3.20}
$$

The first is stationary, while the second term counter-rotates at $2\omega$. Under the rotating wave approximation (RWA), we neglect the second term since its rapid variations average out, and thus we neglect the c.c. term of $E(t)$ in Eq. (3.1). The following discussions are all under the RWA.
3.2.1 Rotating frame at laser frequency $\omega$

If we choose a reference frame rotating at the laser frequency $\omega$, the rotation matrix can be written as

$$U = \exp(i\omega \sigma_z t/2),$$  \hspace{1cm} (3.21)

where $\sigma_z$ is the Pauli spin matrix,

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  \hspace{1cm} (3.22)

The rotation matrix in Eq. (3.21) is a simplified representation of

$$U = \begin{bmatrix} \exp(i\omega t/2) & 0 \\ 0 & \exp(-i\omega t/2) \end{bmatrix}.$$  \hspace{1cm} (3.23)

The transpose of the matrix $U^\dagger = \exp(-i\omega_0 \sigma_z t/2)$ or

$$U^\dagger = \begin{bmatrix} \exp(-i\omega t/2) & 0 \\ 0 & \exp(i\omega t/2) \end{bmatrix}.$$  \hspace{1cm} (3.24)

It follows that $UU^\dagger = 1$, where $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unitary matrix. With this coordinate transformation, the Liouville equation, Eq. (3.9), becomes,

$$U |\bar{\rho}| U^\dagger = \frac{i}{\hbar} U[\rho, \hat{\mathcal{H}}] U^\dagger.$$  \hspace{1cm} (3.25)

Define

$$\rho'(t) = U |\bar{\rho}| U^\dagger $$  \hspace{1cm} (3.26)

$$\hat{\mathcal{H}}' = U \hat{\mathcal{H}} U^\dagger.$$  \hspace{1cm} (3.27)

We can get

$$\rho' = \frac{i}{\hbar} [\rho', \hat{\mathcal{H}}'].$$  \hspace{1cm} (3.28)
where the transformed density matrix is
\[
\rho' = \begin{bmatrix}
\rho_{22} & \rho_{21} \exp(i\omega t) \\
\rho_{12} \exp(-i\omega t) & \rho_{11}
\end{bmatrix} = \begin{bmatrix}
\rho_{22}' & \rho_{21}' \\
\rho_{12}' & \rho_{11}'
\end{bmatrix}
\tag{3.29}
\]
and the transformed Hamiltonian is
\[
\hat{\mathbf{H}}' = \begin{bmatrix}
\hbar \Delta / 2 & s^*(r, t) \\
s(r, t) & -\hbar \Delta / 2
\end{bmatrix},
\tag{3.30}
\]
where
\[
s(r, t) = -\mu_{12} E(r, t) \exp \left[ -i \omega m (\hat{\mathbf{k}}) \cdot \mathbf{r} / c \right],
\tag{3.31}
\]
\[
\Delta = \omega_0 - \omega.
\tag{3.32}
\]
Again, by defining \( \rho_{22} - \rho_{11} \equiv w \) and \( \rho_{12} = (u + iv)/2 \), we can get the Bloch equations under RWA
\[
\dot{u} = -\frac{u}{T_2} + (\omega_0 - \omega) v
\tag{3.33}
\]
\[
\dot{v} = -(\omega_0 - \omega) u - \frac{v}{T_2} - \kappa s(r, t) w
\tag{3.34}
\]
\[
\dot{w} = \kappa s(r, t) v - \frac{w - w_{eq}}{T_1}.
\tag{3.35}
\]
In Chapter 4, we use this equation to simulate the Maxwell-Bloch equations.

### 3.2.2 Rotating frame at atom resonant frequency \( \omega_0 \) - perturbation theory

Mitsunaga and Brewer [67] choose a reference frame rotating at the level splitting frequency \( \omega_0 \), the rotation matrix can be written as
\[
U = \exp(i\omega_0 \sigma_z t / 2),
\tag{3.36}
\]
where \( \sigma_z \) is defined in Eq. (3.22). Then the transformed density matrix is
\[
\rho'' = \begin{bmatrix}
\rho_{22} & \rho_{21} \exp(i\omega_0 t) \\
\rho_{12} \exp(-i\omega_0 t) & \rho_{11}
\end{bmatrix} = \begin{bmatrix}
\rho_{22}'' & \rho_{21}'' \\
\rho_{12}'' & \rho_{11}''
\end{bmatrix}
\tag{3.37}
\]
and the transformed Hamiltonian is
\[ \hat{\mathcal{H}}'' = \begin{bmatrix} 0 & s^*(r, t) \\ s(r, t) & 0 \end{bmatrix}, \]
where
\[ s(r, t) = -\mu_{12} E(r, t) \exp \left[ -i \Delta t - i \omega n \left( \mathbf{k} \cdot \mathbf{r} \right) / c \right], \]
\[ \Delta = \omega_0 - \omega. \]

By defining \( \rho_{22} - \rho_{11} \equiv w \), we can get the Bloch-like equations
\[ \rho''_{12} = -i w s(r, t) - \frac{\rho''_{12}}{T_2}, \]
\[ \dot{w} = 2i(\rho''_{21} s(r, t) - \rho''_{12} s^*(r, t)) - \frac{w - w_0}{T_1}. \]

The difference between Eq. (3.33) and Eq. (3.41) is: in the laser frame, the density matrix \( \rho' \) in Eq. (3.33) has a free precession term with angular frequency \( \Delta \). In the atomic frequency rotating frame, the density matrix \( \rho'' \) in Eq. (3.41) does not contain the precession term, but the driving term \( s(r, t) \) contains a term, \( \exp(-i\Delta t) \) (\( s \) slowly rotates at the frequency \( \Delta \)). Integrating the equations in Eq. (3.41) yields
\[ \rho''_{12}(t) = -i \int_{-\infty}^{t} w(t') s(r, t') \exp[-(t - t')/T_2] \, dt', \]
\[ w(t) = w_0 + 2i \int_{-\infty}^{t} [\rho''_{21}(t') s(r, t') - \rho''_{12} s^*(r, t')] \exp[-(t - t')/T_1] \, dt'. \]

With the initial conditions \( \rho''_{12}(-\infty) = 0 \) and \( w(-\infty) = w_0 \), perturbative solutions of Eqs. (3.43) up to third order can be written as [67]
\[ w^{(0)} = w_0, \]
\[ \rho''_{12}^{(1)}(t) = -i w_0 \int_{-\infty}^{t} s(t_1) \exp[-(t - t_1)/T_2] \, dt_1, \]
\[ w^{(2)}(t) = 2i^2 w_0 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 [s^*(t_1) s(t_2) + s(t_1) s^*(t_2)] \times \exp[-(t_2 - t_1)/T_2 - (t - t_2)/T_1], \]
\[ \rho''_{12}^{(3)}(t) = -2i^3 w_0 \int_{-\infty}^{t} dt_3 \int_{-\infty}^{t_3} dt_2 \int_{-\infty}^{t_2} dt_1 s(t_3) [s^*(t_1) s(t_2) + s(t_1) s^*(t_2)] \times \exp[-(t - t_3 + t_2 - t_1)/T_2 - (t_3 - t_2)/T_1], \]
where the time ordering is
\[ t_1 < t_2 < t_3 < t. \]

In Eq. (3.43), the off-diagonal terms are the coherence related terms and they are odd order while the diagonal terms are population related and are even order. In addition, the off-diagonal terms decay as decoherence time \( T_2 \) while the diagonal terms decay with the population lifetime \( T_1 \).

Continuing the integration of Eq. (3.43) and Eq. (3.44), the \( n \)-th order solution is given by
\[
\rho_{12}^{(n)}(t) = -2^{l} \ii^{n} \omega_{0} \int_{-\infty}^{t} dt_{n} \int_{-\infty}^{t_{n}} dt_{n-1} \cdots \int_{-\infty}^{t_{2}} dt_{1} \times \exp[-(t-t_{n} + t_{n-1} - t_{n-2} + \cdots + t_{2} - t_{1})/T_{2}] \times \exp[-(t_{n} - t_{n-1} + \cdots + t_{3} - t_{2})/T_{1}] \times \sum_{i_{k}=\pm 1} s^{[i_{1}]}(t_{1}) s^{[-i_{1}]}(t_{2}) s^{[i_{2}]}(t_{3}) s^{[-i_{2}]}(t_{4}) \cdots s^{[i_{l}]}(t_{n-2}) s^{[-i_{l}]}(t_{n-1}) s(t_{n}),
\]
where \( i_{k} (k = 1, 2, 3, \ldots, l) \) is either \(-1\) or \(1\), the index \( n = 2l + 1 \) is an odd integer, and the bracketed function is defined as
\[
s^{[1]}(t) \equiv s(t), \quad s^{[-1]}(t) \equiv s^{*}(t). \]

Now by using the matrix,
\[
U' = \exp(-i \Delta \sigma_{z} t/2),
\]
we can transform the perturbation result to the conventional rotating frame at the laser frequency \( \omega \),
\[
\tilde{\rho}(t) = U \rho''(t) U^{T}.
\]

Up to this point, we have only considered a species of atoms with exactly the same resonance frequency \( \omega_{0} \). We now introduce an inhomogeneous broadening of the resonance frequency. We perform an average of \( \tilde{\rho}_{12}^{(n)}(t) \) over a Gaussian inhomogeneous line shape,
\[
g(\Delta) = \frac{1}{\sqrt{\pi} \sigma} \exp[-(\Delta/\sigma)^{2}],
\]
where
\[
t_1 < t_2 < t_3 < t.
\]
where $\sigma = 1/T_2^*$ is the inhomogeneous linewidth, $T_2^*$ is the inhomogeneous lifetime, giving
\[
\langle \tilde{\rho}_{12}^{(n)}(t) \rangle \equiv \int_{-\infty}^{\infty} \tilde{\rho}_{12}^{(n)} g(\Delta) d\Delta.
\]

In the cases we consider, we only consider a small part of the inhomogeneous bandwidth. Thus, the Gaussian inhomogeneous line shape becomes flat, $\exp[-(\Delta/\sigma)^2] \to 1$. We can obtain the general time-domain third-order solution for the coherent response of an inhomogeneously broadened medium,
\[
\langle \tilde{\rho}_{12}(t) \rangle = \mathcal{D} \int_{-\infty}^{t} dt_3 \int_{-\infty}^{t_3} dt_2 \int_{-\infty}^{t_2} dt_1 \mathcal{E}_3(t_3) \mathcal{E}_2(t_2) \mathcal{E}_1^*(t_1) \delta(t_2 - t_1 + t_3 - t)
\times \exp[-(t - t_3 + t_2 - t_1)/T_2] \exp[-(t_3 - t_2)/T_1],
\]

where $\mathcal{D} = -4i\sqrt{\pi}w_0\mu_{12}|\mu_{12}|^2T_2^*/\hbar^3 \exp(-i\omega \mathbf{r} \cdot \mathbf{n}/c)$. We neglect the acausal term, $\mathcal{E}_1 \mathcal{E}_2^*$, in the integration. In reference [67], Mitsunaga and Brewer also point out that for the perturbation to converge, the applied electric field must be low enough to satisfy
\[
\frac{|p_{12} \mathcal{E}(t)|^2}{\hbar^2} T_1 T_2 < 1.
\]

### 3.2.2.1 Frequency domain perturbation theory - the full version

Mitsunaga’s integral is in the time domain. It does not give much insight into the frequency domain operations. Colice [4] derives the frequency domain correspondence of this integral, Eq. (3.56). By writing the time domain electric field $\mathcal{E}(t)$ as $\mathcal{E}(t) = \mathcal{F}^{-1}\left\{ \tilde{\mathcal{E}}(f) \right\}$, he rewrite the

1. We define the Fourier transom pairs as follows:
\[
\tilde{\mathcal{E}}(f) = \mathcal{F}\{\mathcal{E}(t)\} = \int_{-\infty}^{+\infty} \mathcal{E}(t) \exp(-i2\pi ft) dt,
\]
\[
\mathcal{E}(t) = \mathcal{F}^{-1}\left\{ \tilde{\mathcal{E}}(f) \right\} = \int_{-\infty}^{+\infty} \tilde{\mathcal{E}}(f) \exp(i2\pi ft) df.
\]
time domain result in Eq. (3.56) as

\[
\langle \hat{\rho}_{12}(t) \rangle = \mathcal{D} \int_{-\infty}^{t} dt_3 \int_{-\infty}^{t_3} dt_2 \int_{-\infty}^{t_2} dt_1 \int_{-\infty}^{+\infty} df_3 \hat{E}_3(f_3) \exp(i2\pi f_3t_3) \\
\times \int_{-\infty}^{+\infty} df_2 \hat{E}_2(f_2) \exp(i2\pi f_2t_2) \int_{-\infty}^{+\infty} df_1 \hat{E}_1^*(f_1) \exp(-i2\pi f_1t_1) \\
\times \exp[-(t - t_3 + t_2 - t_1)/T_2] \exp[-(t_3 - t_2)/T_1] \delta(t_2 - t_1 + t_3 - t),
\]

(3.58)

where the time ordering is taken as

\[ t_1 < t_2 < t_3 < t \]  

(3.59)

Simplifying Eq. (3.58) by integrating over \( t_1, t_2, \) and \( t_3 \) one at a time, one can get

\[
\langle \hat{\rho}_{12}(t) \rangle = \mathcal{D} \int_{-\infty}^{+\infty} df_3 \hat{E}_3(f_3) \int_{-\infty}^{+\infty} df_2 \hat{E}_2(f_2) \int_{-\infty}^{+\infty} df_1 \hat{E}_1^*(f_1) \\
\gamma_1(f_2 - f_1) \gamma_2(f_3 + f_2 - 2f_1) \exp[i2\pi(f_3 + f_2 - f_1)t] 
\]

(3.60)

where

\[
\gamma_1(f) = \frac{1}{1/T_1 + i2\pi f},
\]

(3.61)

\[
\gamma_2(f) = \frac{1}{2/T_2 + i2\pi f},
\]

(3.62)

are two complex Lorentzians. The integrals are convolutions of the spectral domain products with the two Lorentzians. Eq. (3.60) contains the causality of Eq. (3.56), which can be seen by looking at the inverse Fourier transform of \( \gamma_1(f) \) and \( \gamma_2(f) \),

\[
\mathcal{F}^{-1} \{ \gamma_1(f) \} = H(t) \exp(-t/T_1),
\]

(3.63)

\[
\mathcal{F}^{-1} \{ \gamma_2(f) \} = H(t) \exp(-2t/T_2),
\]

(3.64)

where

\[
H(t) = \begin{cases} 
1 & t \geq 0 \\
0 & t < 0
\end{cases}
\]

(3.65)

is a Heaviside function that accounts for causality and time ordering.
3.2.2.2 Frequency domain perturbation theory - with further approximations

Eq. (3.60) of the last section considers both causality and the effects of population decay and coherence decay. The result is quite complicated. When applying it to analyzing the experimental problems, such as photon echoes and spectral analysis, we often make simplifying approximations, such as $T_1 \to \infty$.

To derive the perturbation equation with these approximations, I find it much simpler to re-derive the equation and make the approximations during the derivation rather than making approximations in the general form of the spectral domain result, i.e. Eq. (3.60). After all, the complex Lorentzian $\gamma_m(f) = (m/T_m + i2\pi f)^{-1}, m = 1, 2$, from Colice’s derivations are due to the combined effect of causality and lifetime/coherence time decay. The effect of not considering either of them will then be equivalent to substituting $\gamma_{1,2}(f)$ with $\delta(f)$. In other words, without causality, we mathematically substitute the Heaviside function $H(t)$ with “1” and in the spectral-domain, the effect is to substitute the Fourier transform of $H(t)$ with a $\delta$-function. The derivations here just follow Colice’s derivations. Derivations in Effect of population lifetime $T_1$ and coherence time $T_2$ neglect the $T_1, T_2$ population lifetime and coherence time. On top of that, derivations in Neglect the causality between $E_3(t)$ and $E_1(t)$, $E_2(t)$ neglect the $\gamma(f_2 - f_1)$ effect, or the causality between the reading pulse $E_3$ and the two writing pulses $E_1$ and $E_2$.

Effect of population lifetime $T_1$ and coherence time $T_2$

Start with the time-domain Mitsunaga’s integral, Eq. (3.56), when $T_{1,2} \to \infty$, we have

$$\langle \hat{\rho}_{12}(t) \rangle = \mathcal{D} \int_{-\infty}^{t} dt_3 \int_{-\infty}^{t_3} dt_2 \int_{-\infty}^{t_2} dt_1 E_3(t_3) E_2(t_2) E_1^*(t_1) \delta(t_2 - t_1 + t_3 - t).$$  

$$\left(3.66\right)$$

Again, by writing the time domain electric field $\mathcal{E}(t)$ as $\mathcal{E}(t) = \mathcal{F}^{-1}\left\{\hat{\mathcal{E}}(f)\right\}$, we can rewrite
Eq. (3.56) as

\[
\langle \tilde{\rho}_{12}(t) \rangle = \mathcal{D} \int_{-\infty}^{+\infty} df_3 \tilde{\xi}_3(f_3) \int_{-\infty}^{+\infty} df_2 \tilde{\xi}_2(f_2) \int_{-\infty}^{+\infty} df_1 \tilde{\xi}_1^*(f_1) \\
\times \int_{-\infty}^{t} dt_3 \exp(i2\pi f_3 t_3) \int_{-\infty}^{t_3} dt_2 \exp(i2\pi f_2 t_2) \\
\int_{-\infty}^{t_2} dt_1 \exp(-i2\pi f_1 t_1) \delta(t_2 - t_1 + t_3 - t), \tag{3.67}
\]

The integration over \( t_1 \) in Eq. (3.67) can be written as

\[
\int_{-\infty}^{t_2} \exp(-i2\pi f_1 t_1) \delta(t_2 - t_1 + t_3 - t) dt_1 = \int_{-\infty}^{+\infty} H(t_2 - t_1) \exp(-i2\pi f_1 t_1) \delta(t_2 - t_1 + t_3 - t) dt_1 \\
= H(t - t_3) \exp[-i2\pi f_1(t_2 + t_3 - t)]. \tag{3.68}
\]

The term \( H(t - t_3) \) indicates that considering the time ordering of \( t_1 \) and \( t_2 \) is equivalent to considering that of \( t \) and \( t_3 \). Eq. (3.67) then becomes

\[
\langle \tilde{\rho}_{12}(t) \rangle = \mathcal{D} \int_{-\infty}^{+\infty} df_3 \tilde{\xi}_3(f_3) \int_{-\infty}^{+\infty} df_2 \tilde{\xi}_2(f_2) \int_{-\infty}^{+\infty} df_1 \tilde{\xi}_1^*(f_1) \exp(i2\pi f_1 t) \\
\times \int_{-\infty}^{+\infty} dt_3 H(t - t_3) \exp[i2\pi (f_3 - f_1)t_3] \\
\times \int_{-\infty}^{+\infty} dt_2 H(t_3 - t_2) \exp[i2\pi (f_2 - f_1)t_2], \tag{3.69}
\]

where we use the Heaviside function \( H(t_3 - t_2) \) \((H(t) \text{ is defined in Eq. (3.65)) \) and \( H(t - t_3) \) to represent the causal integrals \( \int_{-\infty}^{t_3} \) and \( \int_{-\infty}^{t} \) respectively. Notice that the Fourier transform of a Heaviside function \( H(t) \) is given by

\[
\int_{-\infty}^{\infty} H(t) \exp(-i\omega t) dt = \mathcal{F}\{H(t)\} = -\frac{1}{i2\pi f} + \frac{1}{2}\delta(f). \tag{3.70}
\]

This Fourier transform equation can be proved by residue integrals. In the time domain, the first term in Eq. (3.70) gives an odd function that is non-zero everywhere from \(-\infty\) to \(+\infty\) (either \(-\frac{1}{2}\) or \(+\frac{1}{2}\)). The second term in Eq. (3.70) is due to the fact that the average of the Heaviside function is not zero. Causality is the combined effect of both terms. To make the later derivation easier, we write \( H(t) \) as

\[
H(t) = [\text{sgn}(t) + 1]/2 \tag{3.71}
\]
where \( \text{sgn}(t) = 1 : t > 0, 0 : t = 0, -1 : t < 0 \). It is easy to know that \( \text{sgn}(-t) = -\text{sgn}(t) \). Using this property, we know

\[
H(-t) = \left[-\text{sgn}(t) + 1\right]/2 = -H(t) + 1.
\]

It follows that

\[
\mathcal{F}\{H(-t)\} = -\mathcal{F}\{H(t)\} + \delta(f). \tag{3.72}
\]

The integration over \( t_2 \) in Eq. (3.69) can be rewritten as

\[
\int_{-\infty}^{t_3} dt_2 \exp[i2\pi(f_2 - f_1)t_2] = \int_{-\infty}^{+\infty} H(t_3 - t_2) \exp[i2\pi(f_2 - f_1)t_2] dt_2 \\
= \{-\mathcal{F}\{H(t_2)\} + \delta(f)\} \exp(i2\pi ft_3) |_{f = f_2 - f_1} \\
= \left[ \frac{1}{i2\pi f} + \frac{1}{2} \delta(f) \right] \exp(i2\pi ft_3) |_{f = f_2 - f_1} \\
= h(f_2 - f_1) \exp[i2\pi(f_2 - f_1)t_3], \tag{3.73}
\]

where

\[
h(f) = \frac{1}{i2\pi f} + \frac{1}{2} \delta(f) = \mathcal{F}\{H(-t)\}. \tag{3.74}
\]

Substituting the result from Eq. (3.73) into Eq. (3.69) yields,

\[
\langle \hat{\rho}_{12}(t) \rangle = \mathcal{D} \int_{-\infty}^{+\infty} df_3 \hat{\mathcal{E}}_3(f_3) \int_{-\infty}^{+\infty} df_2 \hat{\mathcal{E}}_2(f_2) \int_{-\infty}^{+\infty} df_1 \hat{\mathcal{E}}_1^*(f_1) h(f_2 - f_1) \exp(i2\pi ft) \\
\times \int_{-\infty}^{t} dt_3 \exp[i2\pi(f_3 + f_2 - 2f_1)t_3]. \tag{3.75}
\]

Similarly, we can calculate the integration over \( t_3 \),

\[
\int_{-\infty}^{t} dt_3 \exp[i2\pi(f_3 + f_2 - 2f_1)t_3] = h(f_3 + f_2 - 2f_1) \exp[i2\pi(f_3 + f_2 - 2f_1)t] \tag{3.76}
\]

And the complete spectral-domain result without considering the effects of limited population lifetime, \( T_1 \to \infty \) and coherence time, \( T_2 \to \infty \), but including causal time ordering in the perturbation expansion is

\[
\langle \hat{\rho}_{12}(t) \rangle = \mathcal{D} \int_{-\infty}^{+\infty} df_3 \hat{\mathcal{E}}_3(f_3) \int_{-\infty}^{+\infty} df_2 \hat{\mathcal{E}}_2(f_2) \exp[i2\pi(f_3 + f_2 - f_1)t] \\
\int_{-\infty}^{+\infty} \hat{\mathcal{E}}_1^*(f_1) h(f_2 - f_1) h(f_3 + f_2 - 2f_1) df_1. \tag{3.77}
\]
Although $h(f_2 - f_1)$ is a function of $f_1$ and $f_2$, it is the Fourier transform of $H(t_3 - t_2)$ (see Eq. (3.73)). Thus it enforces the timing between pulses $\mathcal{E}_3(t)$ and $\mathcal{E}_2(t)$. Whereas the term $h(f_3 + f_2 - 2f_1)$ is the Fourier transform of $H(t - t_3)$, it enforces the timing between the echo and the pulse $\mathcal{E}_3(t)$, which indirectly selects the causal part of the spectral grating $\tilde{\mathcal{E}}_1^*(f)\tilde{\mathcal{E}}_2(f)$.

**Neglect the causality between $\mathcal{E}_3(t)$ and $\mathcal{E}_1(t), \mathcal{E}_2(t)$**

The term $h(f_2 - f_1)$ in Eq. (3.77) is to enforce the time order of the reading pulse $\mathcal{E}_3(t)$ and the two writing pulses $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$. In our experiments, the reading pulse $\mathcal{E}_3(t)$ always happens after the writing pulses, $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$, are finished, but the mathematics imposes this in order to correctly represent all possible overlap of reading and writing pulses. What is the effect of neglecting the read-write pulses causality or the term $h(f_2 - f_1)$? A simple way to answer this question is to re-evaluate the integration over $t_2$ in Eq. (3.73). This time, instead of integrating from $-\infty$ to $t_3$, which gives a Heaviside function $H(t_3 - t_2)$, we integrate from $-\infty$ to $+\infty$,

$$\int_{-\infty}^{+\infty} dt_2 \exp[i2\pi(f_2 - f_1)t_2] = \delta(f_2 - f_1). \quad (3.78)$$

This integration can be thought as the Fourier transform of “1” evaluated at frequency $f_2 - f_1$ and the result is simply a $\delta$-function centers at $f_2 - f_1$. Finish the integration, we can get

$$\langle \tilde{\rho}_{12}(t) \rangle = D \int_{-\infty}^{+\infty} \tilde{\mathcal{E}}_3(f) \{ \left[ \tilde{\mathcal{E}}_1^*(f)\tilde{\mathcal{E}}_2(f) \right] * h(f) \} \exp(i2\pi ft) df, \quad (3.79)$$

where the convolution of $h(f)$ selects the causal part of the spectral grating $\tilde{\mathcal{E}}_1^*(f)\tilde{\mathcal{E}}_2(f)$. This equation is applied later to the high bandwidth scanner experiment. The same result can be achieved by approximating the lifetime limited Lorentzian as a delta function

$$\gamma_1(f) = \frac{1}{(1/T_1) + i2\pi f} \approx \delta(f).$$
In Eq. (3.60), and doing the integral by sifting the $\delta(f)$, we get

$$
\langle \hat{\rho}_{12}(t) \rangle = D \int_{-\infty}^{+\infty} df_3 \tilde{E}_3(f_3) \int_{-\infty}^{+\infty} df_2 \tilde{E}_2(f_2) \int_{-\infty}^{+\infty} df_1 \tilde{E}_1^*(f_1) \\
\delta(f_2 - f_1) \gamma_2(f_3 + f_2 - 2f_1) \exp[i2\pi(f_3 + f_2 - f_1)t] \\
= D \int_{-\infty}^{+\infty} df_3 \exp(i2\pi f_3 t) \tilde{E}_3(f) \int_{-\infty}^{+\infty} \tilde{E}_1^*(f) \tilde{E}_2(f) \gamma_2(f_3 - f_2) df_2 \\
= D \int_{-\infty}^{+\infty} \tilde{E}_3(f_3) \left\{ \left[ \tilde{E}_1^*(f_3) \tilde{E}_2(f_3) \right] \ast \gamma_2(f_3) \right\} \exp(i2\pi f_3 t) df_3.
$$

(3.80)

Notice that if we also neglect the effect of $h(f)$ in Eq. (3.79), we can get the Mosesberg’s integral [21],

$$
\langle \hat{\rho}_{12}(t) \rangle = D \int_{-\infty}^{+\infty} \tilde{E}_3(f) \tilde{E}_1^*(f) \tilde{E}_2(f) \exp(i2\pi f t) df,
$$

(3.81)

which neglects all aspects of causality and resolution limits in the spectral grating. Eq. (3.81) states that the echo is proportional to the Fourier transform of the spectral grating $\tilde{E}_3(f) \tilde{E}_1^*(f) \tilde{E}_2(f)$, or it is the convolution of $E_3(t)$ with the Fourier transform of the interference grating between $\tilde{E}_1^*(f) \tilde{E}_2(f)$, which is the correlation $E_1^*(f) \ast E_2(f)$.

**Consider $T_2$, but not $T_1$ nor the causality between $E_3(t)$ and $E_1(t), E_2(t)$**

Start with the time-domain Mitsunaga’s integral, Eq. (3.56), when $T_1 \to \infty$ and ignore the causality between $E_1$ and $E_2$, we have

$$
\langle \hat{\rho}_{12}(t) \rangle = D \int_{-\infty}^{+\infty} \tilde{E}_3(f) \left\{ \left[ \tilde{E}_1^*(f) \tilde{E}_2(f) \right] \ast \gamma_2(f) \right\} \exp(i2\pi f t) df \\
= DE_3(t) \ast \mathcal{F} \left\{ \tilde{E}_1^*(f) \tilde{E}_2(f) \right\} H(t) \exp(-2t/T_2). 
$$

(3.82)

where $\gamma_2$ is defined in Eq. (3.62). Decoherence attenuates the echo signal strength by the term $\exp(-2t/T_2)$. The Fourier transform of the spectral grating $\tilde{E}_1^*(f) \tilde{E}_2(f)$ is attenuated and convolved with beam 3, $E_3(t)$, giving the echo.

### 3.3 Pulse vs chirp photon echo

In photon echo experiments, we can choose to use different waveforms for the writing beams $E_1$ and $E_2$, and the reading beam $E_3$, as long as the power spectrum of the three pulses overlap. To
write population inversion gratings in the inhomogeneous band in an effective way, we want the first
two beams $\mathcal{E}_1$ and $\mathcal{E}_2$ to cover the same spectral range since they only interfere at the frequencies
that their spectra overlap. Interestingly, they do not need to overlap in time as with most other nonlinear processes. Of course, they must overlap in space. Two kinds of waveforms are the most
common choices: pulses and chirps. The bandwidth that the pulse waveform covers is inversely
proportional to the duration time of the pulse. While the bandwidth of the chirps is proportional
to their duration time when the chirp rate is constant. For light with a limited constant amplitude,
such as that produced by electro-optic modulating a CW laser, longer duration time means more
photons. Therefore, with the other conditions the same, chirp photon echoes could have higher
output echo strengths than pulse photon echoes. Of course mode locked pulsed lasers would change
this observation, since the laser is concentrating all of its power in a pulse, not wasting the power
in between pulses. To discuss their strength quantitatively, we define the following parameters: the
time domain amplitude of $\mathcal{E}_1(t)$, $\mathcal{E}_2(t)$, and $\mathcal{E}_3(t)$ are $e_1$, $e_2$ and $e_3$. The duration time of the beams
are $t_{p1}$, $t_{p2}$ and $t_{p3}$ for pulses and $t_{c1}$, $t_{c2}$ and $t_{c3}$ for chirps. $\Delta_1$, $\Delta_2$ and $\Delta_3$ are the bandwidths
(angular frequency) of the signals. To have the highest energy efficiency, we let all the signals cover
the same bandwidth, i.e. $\Delta_1 = \Delta_2 = \Delta_3 = \Delta$. In our experiments, the total power available for a
certain laser is usually fixed. We want to cover as much bandwidth as possible without losing too
much of the echo signal. We use the Mossberg’s integral, Eq. (3.81), with finite integration bounds
\[
\langle \tilde{\rho}_{12} (t) \rangle = D \int_{f_0-\Delta/4\pi}^{f_0+\Delta/4\pi} \tilde{\mathcal{E}}_3 (f) \tilde{\mathcal{E}}_1^* (f) \tilde{\mathcal{E}}_2 (f) \exp (i2\pi ft) df.
\] (3.83)
to find out the relationship between the bandwidth of the grating that the two writing beams cover
and the amplitude of the echo signal with fixed beam power $P$ and fixed beam sizes.

### 3.3.1 Pulse echo

We use time-domain pulses with width $t_{p_i}$ ($i = 1, 2, 3$) for writing and reading,
\[
E_i(t) = e_i \Pi \left( \frac{t}{t_{p_i}} \right) \ast \delta (t - t_i), i = 1, 2, 3,
\] (3.84)
where \( t_i \) is the arriving time of different pulses. To have the highest efficiency, the pulses should have the same bandwidth, thus we make \( t_{p1} = t_{p2} = t_{p3} = t_p \). The amplitude of the beams can be written in terms of the beam power \( P = \frac{|e_i|^2}{2\eta} A_i = \frac{\epsilon_0 cn}{2} |e_i|^2 A_i \)

\[
e_i = \sqrt{I_i} = \sqrt{\frac{2P}{\eta A_i}} \tag{3.85}
\]

where \( I_i \) is the intensity of beam \( i \), \( \epsilon_0 = \epsilon_0 cn \), \( \epsilon_0 \) is the electric permittivity of free space, \( c \) is the speed of light, \( n \) the material refractive index and \( A_i \) is the cross sectional area of beam \( i \) (again to maximize the echo amplitude, we make all the beams have the same beam size, \( A_1 = A_2 = A_3 = A \)). The spectrum of a brief rectangular pulse is a sinc function. The power spectrum of the pulse is on average proportional to \( 1/\Delta \). Notice that for pulses, we have \( t_p = 1/\Delta \).

In the spectral frequency domain, we can think that the photons of the beam is spreading to the bandwidth of pulse, so the amplitude of the spectrum can be written as

\[
\tilde{e}_i = \sqrt{\frac{2\eta P t_d}{A \Delta}} = \sqrt{\frac{2\eta P}{A \Delta^2}}. \tag{3.86}
\]

Now \( \tilde{E}^*_1(f) \tilde{E}_2(f) \) can be written as

\[
\tilde{E}^*_1(f) \tilde{E}_2(f) = \tilde{e}_1^* \tilde{e}_2 [\text{sinc}(t_p f) \exp(+i2\pi t_1 f)] \cdot [\text{sinc}(t_p f) \exp(-i2\pi t_2 f)]
\]

\[
= \tilde{e}_1^* \tilde{e}_2 \text{sinc}^2(t_p f) \exp[i2\pi(t_1 - t_2)f], \tag{3.87}
\]

where the amplitude of the Fourier transform of sinc\( (t_p f) \) is considered in \( \tilde{e}_1^* \) and \( \tilde{e}_2 \). To record resolvable sinusoidal gratings in the exposed portion of the spectrum, we want to have \( \frac{1}{t_2 - t_1} < \frac{1}{t_p} \), thus \( t_p < t_2 - t_1 \). If we keep the power of the beams the same (using the maximum power that the laser offers), using Eq. (3.86), we have

\[
\tilde{e}_1^* \tilde{e}_2 = \frac{2\eta P}{A \Delta^2}. \tag{3.88}
\]

Fig. 3.1 shows the power spectrum of brief pulses as the widths of the pulses increase by \( 10 \times \), \( 100 \times \) and \( 1000 \times \).
Plug $\tilde{\xi}_1^*(f)\tilde{\xi}_2(f)$ into Eq. (3.83), we get

$$
\langle \hat{\rho}_{12}(t) \rangle = D \int_{f_0-\Delta/4\pi}^{f_0+\Delta/4\pi} \tilde{\xi}_3(f)\tilde{\xi}_1^*\tilde{\xi}_2\text{sinc}^2(t_p,f) \exp[i2\pi(t-t_2+t_1)f] df
$$

$$
= D\tilde{\xi}_1^*\tilde{\xi}_2 E_3(t') [\frac{1}{t_p} \Pi \left( \frac{t'}{t_p} \right)] [\frac{1}{t_p} \Pi \left( \frac{t'}{t_p} \right)] \left[ \frac{\Delta}{2\pi} \text{sinc} \left( \frac{\Delta}{2\pi} t' \right) \right]_{t'=t+t_1-t_2},
$$

(3.89)

where the convolution of $\frac{1}{t_p} \Pi \left( \frac{t'}{t_p} \right)$ will not affect the amplitude of the result because the coefficient $1/t_p$ presents both in front of and inside the rectangular function (so is the case of $\frac{\Delta}{2\pi} \text{sinc} \left( \frac{\Delta}{2\pi} t' \right)$).

The echo is proportional to $\langle \hat{\rho}_{12}(t) \rangle$, and therefore will be proportional to the spectral grating amplitude $\tilde{\xi}_1^*\tilde{\xi}_2$ and the amplitude of the readout beam $e_3$,

$$
|\langle \hat{\rho}_{12}(t) \rangle| \propto \left( \frac{P}{A} \right)^\frac{3}{2} \frac{1}{\Delta^2}.
$$

(3.90)

In addition, we can analyze the strength of the echo in an approximate way by using the amplitude of the spectrum relationship given above in Eq. (3.86). The echo amplitude is proportional to

$$
|\langle \hat{\rho}_{12}(t) \rangle| \propto |\tilde{\xi}_1(\omega)\tilde{\xi}_2(\omega)\tilde{\xi}_3(\omega)| \cdot \Delta = \left( \frac{P}{A} \right)^\frac{3}{2} \frac{1}{\Delta^2}.
$$

(3.91)
If we keep the power of the beams the same, the echo amplitude is inversely proportional to the square of the pulse bandwidth, $\Delta^2$. To have stronger echo signal, we want to have pulses with longer durations, which means the system will access lower bandwidth.

### 3.3.2 Chirp echo

When the two recording beams are chirps with duration $t_{ci}$,

$$E_i(t) = e_i \exp(i\pi bt^2) \Pi \left( \frac{t}{t_{ci}} \right) * \delta(t - t_i), \quad i = 1, 2,$$

(3.92)

where $t_i$ is the central time of the chirps, $e_i$ are the amplitude of the chirps and $b = \Delta/t_c$ is the chirp rate with $\Delta$, the bandwidth of the chirp. The amplitude of the beams can also be written in terms of the beam power $P$

$$|e_i| = \sqrt{I_i} = \sqrt{\frac{2\eta P}{A_i}}$$

(3.93)

where $I_i$ is the intensity of beam $i$ and $A_i$ is the cross sectional area of beam $i$ (again to maximize the echo amplitude, we make all the beams have the same beam size, $A_1 = A_2 = A_3 = A$). The total energy in the chirp is $Pt_{ci}$, which will be the same either in the time domain or in the spectral domain. The intensity of the chirp is $I = \frac{P}{A}$. In spectral domain, the energy of the chirp is spreading over the bandwidth $\Delta$, so the power spectrum density $P_\omega$ is given by

$$P_\omega = \frac{\tilde{e}_i}{2\eta} A_i,$$

(3.94)

where the Fourier amplitude is

$$|	ilde{e}_i| = \sqrt{\frac{P t_{ci}}{A\Delta}}.$$ 

(3.95)

This can also be achieved by using the Parseval’s theorem [55], which says the power in one domain (it is time here) will be the same as the power in its Fourier transformed domain (frequency), since the chirp is nominally constant in amplitude in both domains gives

$$|e_i|^2 \cdot t_{ci} = |	ilde{e}_i|^2 \cdot \Delta.$$

(3.96)
Figure 3.2: Chirps power spectrum scaling with the chirp bandwidth. (a) Chirps with bandwidths increasing by $10\times$, $100\times$ and $1000\times$. (b) Spectral frequency domain power spectrum of chirps scales with $1/\Delta$.

So the energy in the chirp is $P t c_i = P_\omega \Delta$, $P_\omega$ is the power in spectral domain, and the total number of photons is

$$N_{\text{ph}} = P t c_i / h\nu. \quad (3.97)$$

In the high bandwidth scanner experiment, to have as many photons as possible, we keep $t c_i$ as long as we could have and keep it a constant, typically half of the lifetime of the crystal. We can vary the chirp rate $b$ to change the bandwidth of the chirps. Now $\tilde{\mathcal{E}}_1^*(f)\tilde{\mathcal{E}}_2(f)$ can be written as

$$\tilde{\mathcal{E}}_1^*(f)\tilde{\mathcal{E}}_2(f) = \tilde{c}_1^*\tilde{c}_2 \exp\left(-i\pi f^2/b\right) \ast \left[t c_i \text{sinc}(t c_i f)\right] \times \exp\left(i\pi f^2/b\right) \ast \left[t c_i \text{sinc}(t c_i f)\right] \exp\left[-i2\pi(t_2 - t_1)f\right] \quad (3.98)$$

$$= \tilde{c}_1^*\tilde{c}_2 \exp\left[-i2\pi(t_2 - t_1)f\right]. \quad (3.99)$$

Fig. 3.2 shows the power spectrum of chirps scaling with $1/\Delta$ as the bandwidths of the chirps increase by $10\times$, $100\times$ and $1000\times$. 
Plug $\tilde{\xi}_1(f)\tilde{\xi}_2(f)$ into Eq. (3.83), we get

$$\langle \hat{\rho}_{12}(t) \rangle = D \int_{f_0-\Delta/4\pi}^{f_0+\Delta/4\pi} \tilde{\xi}_3(f)\tilde{\xi}_1^*\tilde{\xi}_2 \exp[i2\pi(t - t_2 + t_1)f]df$$

$$= D \tilde{\xi}_1^*\tilde{\xi}_2 E_3(t') \left[ \frac{\Delta}{2\pi} \text{sinc}(\frac{\Delta}{2\pi}t) \right] \bigg|_{t' = t + t_1 - t_2}. \quad (3.100)$$

Again the convolution of $\frac{\Delta}{2\pi} \text{sinc}(\frac{\Delta}{2\pi}t)$ will not affect the amplitude of the echo since $\frac{\Delta}{2\pi}$ presents both in front of the sinc function and inside the sinc function. The echo is proportional to $\langle \hat{\rho}_{12}(t) \rangle$, and therefore will be proportional to the spectral grating amplitude $\tilde{\xi}_1^*\tilde{\xi}_2$ and the amplitude of the readout beam $e_3$,

$$|\langle \hat{\rho}_{12}(t) \rangle| \propto \tilde{\xi}_1^*\tilde{\xi}_2 e_3 = \left( \frac{P}{A} \right)^{3/2} (t_{c1} t_{c2})^{1/2} \frac{1}{\Delta} \propto \frac{1}{\Delta}. \quad (3.101)$$

Eq. (3.101) says with fixed beam power $P$, if we keep the chirp duration time and the reading beam the same, the amplitude (or efficiency) of the echo will be inversely proportional to the chirp bandwidth. In this case, the bandwidth of the chirp will not be inversely proportional to the duration time of the chirp. In fact, the longer the chirps, more photons will be deposited into the population inversion grating and the echo will be stronger. However, all the chirps must finish within the lifetime of the material to avoid big lifetime decay. In our experiments, we keep the chirp duration time the same, half of the lifetime. We can then vary the chirp rate to change the bandwidth it covers.
Chapter 4

Simulating light propagation in inhomogeneously broadened crystals

4.1 Introduction

Cryogenically cooled rare-earth-doped crystals, such as Tm$^{3+}$:YAG, contain two-level atoms with $T_1 = 10$ msec lifetimes, decoherence time less than $T_2 = 40 \mu$sec, and inhomogeneously broadened bandwidth larger than $1/T_2^* = 20$ GHz, with $1/\nu_0 = 2$ fsec optical cycle times. These very different time scales make numerical modeling challenging. These crystals can be used in time domain optical signal processing [5, 68], broadband RF spectrum analysis [20, 69], as well as for spatial holographic processing [70] and for coupled spatial-spectral holographic processing [23, 24]. These effects can be modeled by the Maxwell-Bloch equations [66] with an inhomogeneously broadened spectrum of resonances.

Previously, the finite difference time domain (FDTD) method was used to simulate the Maxwell-Bloch equations rigorously in 1-D [71, 72]. However, FDTD requires sampling with space steps one-tenth of the carrier wavelength, which could be meters in RF simulations. For simulations in optical wavelengths, however, this means space steps of less than $\lambda/10 \approx 0.1 \mu$m, and time steps of less than 0.3 fsec. For a typical millimeter-thick crystal and 10 $\mu$sec simulation duration, there would be $10^{11}$ time steps and $10^4 - 10^5$ space steps which would take months for even a 1-D simulation, much less a 2-D or 3-D simulation.

The slowly varying envelope approximation (SVEA) is introduced for the situations where the change of the electric field is much slower than that of the carrier. Using the SVEA can circumvent the requirement for sampling of the optical wavelength and the simulation can be many orders of
magnitude faster. Cornish derived and simulated 1D beam propagation in inhomogeneously broadened materials [73]. As a further step, Chang numerically solved for two symmetric spatial frequency components [74]. In this thesis, we discuss simulations of arbitrary 2D beam-propagation in inhomogeneously broadened absorber crystals by using a modified FFT beam propagation method (FFT-BPM) [75–77]. By employing the new method, we can simulate with nano-second of time steps and tens micro-meters of space steps, therefore being 10^9 times more efficient than the FDTD method for certain problems. The comparison of different numerical schemes for Maxwell-Bloch equations is summarized in Table 4.1. In addition, we discuss the stability of our numerical schemes and give comparisons with experiments.

Section 4.2 discusses the derivations of the spatial-temporal-spectral Maxwell-Bloch equations and gives numerical schemes to solve them. Section 4.3 discusses the implementation of the schemes, the simulation error and stability of the schemes. Section 4.4 gives the simulation results for the off-axis 3-pulse photon echoes. Section 4.5 gives experimental results of the off-axis 3-pulse photon echoes. Section 4.6 compares and discusses the simulation results and their experimental counterparts.

### 4.2 Numerical schemes to the Maxwell-Bloch equations

Although real atoms or ions doped into crystals are far more complex, as an idealization, we will model them as ensembles of two-level atoms. Each two-level atom has a specific energy level separation and corresponding optical resonant frequency. Atoms with different spatial locations see different local environment either due to local variations in the crystal or just due to the neighborhood doping of other ions, which can shift the resonant frequencies of the atoms, resulting in inhomogeneously broadened material. The coherent effect of light on these kind of inhomogeneously broadened two-level atoms is governed by the Bloch equations [66], where the incident E-field acts as a driving source for the atomic dipoles. The Bloch equations use the Bloch vector, a fictitious electric spin vector (or a pseudospin vector), to describe the evolution of the population inversion and complex polarization coherence of two-level atoms with different resonant frequencies. The
Table 4.1: Numerical simulations schemes for the Maxwell-Bloch equations

<table>
<thead>
<tr>
<th>Author</th>
<th>Numerical method</th>
<th>( \Delta z )</th>
<th>( \Delta t )</th>
<th>Spatial grid</th>
<th>Feature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ziolkowski [71]</td>
<td>FDTD and predictor-corrector</td>
<td>&lt; 0.1( \mu )m</td>
<td>&lt; 0.3fsec</td>
<td>1D</td>
<td>1 spectral line</td>
</tr>
<tr>
<td>Schlottau [72]</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>inhomogeneous band</td>
</tr>
<tr>
<td>Cornish [73]</td>
<td>Center difference &amp; Forward difference</td>
<td>100( \mu )m</td>
<td>0.2nsec</td>
<td>1D</td>
<td>ignore ( T_1 ) and ( T_2 ), dealing w/ absorption</td>
</tr>
<tr>
<td>Chang [74]</td>
<td>Forward difference</td>
<td>unknown</td>
<td>unknown</td>
<td>2D</td>
<td>Two symmetric spatial frequency components</td>
</tr>
<tr>
<td>Xiong [78]</td>
<td>FFT-BPM + trapezoidal rule &amp; RK-4</td>
<td>tens of ( \mu )m</td>
<td>nsec</td>
<td>2D</td>
<td>Arbitrary beam shape/ beam interaction angle</td>
</tr>
</tbody>
</table>


components of the Bloch vector are related to the expectation value of each resonant species’ atomic dipole moment and population inversion as determined from the density matrix. The radiation from these electric dipoles propagates through the medium and can change the attenuation of light as it propagates as well as induce coherent effect such as optical mutation, photon echoes, etc [66]. The propagation of light through the medium can be described by the Maxwell’s wave equation where the atomic dipole moment (the polarization) acts as a driving source for the E-field. The coupled Maxwell-Bloch equations govern the coherent interaction between the spatially propagating light and the temporally evolving atomic dipoles.

4.2.1 The approximations we make

Solving for the propagation of the envelope using the SVEA can simplify the optical evolution equation dramatically decreasing the complexity of solving the Maxwell’s equations using full wave techniques such as FDTD which require time steps less than 0.3fsec and space steps less than 0.1µm. To properly simulate the spatial property due to interference of angled beam, spatial steps on the order of 100µm is an upper limit due to Nyquist sampling limits. This corresponds to about 1 psec of time step size. Although simulations with SVEA are orders of magnitude more efficient than FDTD, it can still take 5 · 10^5 spatial steps×2 · 10^6 time steps×250 lines for a typical 2D simulation grid size of 200µm(x)×5mm(z)×2µsec(time)×250 (spectral lines). The inefficiency lies in the fact that the Maxwell-Bloch equations are simulated in a coupled way where one spatial step corresponds to one time step. To further improve the simulation efficiency, we break the dependence of the spatial and time step sizes and choose them independently. Mathematically, this means instead of simulating partial differential equations (PDEs), we simulate two ordinary differential equations (ODEs). The advantage of doing this is that we can now advance with nanoseconds of time steps while still keeping the spatial resolution.

Why can we break the the dependence of the spatial and time step sizes? The rare-earth-doped crystals are usually several millimeters thick, corresponding to a light propagation time of tens of picoseconds through the crystal. The typical applications [5, 20, 23, 24, 69, 70] of the
crystals involve CW lasers with less than 100mW of power, corresponding to a few MegaHertz of Rabi oscillation (the dipole oscillation under a constant E-field). A few megahertz of Rabi oscillation means it takes microseconds to go around the Bloch sphere. Thus nanoseconds time steps can more than adequately sample the Bloch sphere dynamics even for detuned atoms. Or in other words, there is not much change for the Bloch equation on a picosecond time scale and we do not need to consider the Bloch equation as we propagate the beam by using the Maxwell’s equation. On the other hand, if the incident pulses are much longer than the beam propagation time in the crystal (tens of picoseconds), say on the order of a nanosecond corresponding to a GHz of bandwidth, then we can simulate the beam propagation process as essentially instantaneous. Or in other words, there is not much change for the E-field during a nanosecond time span and we do not need to re-propagate the beam (the Maxwell’s equation) as we evolve the Bloch equation. Basically, we advance all the Bloch equation with nanosecond time steps. Then we beam propagate the E-fields to propagate and accumulate the dipole source terms and these accumulated total E-fields then drive the Bloch equations.

We simulate the Maxwell’s equations under the SVEA. We consider only the low spatial frequency components propagating at less than ±3° of angles. Under paraxial approximation, we use a modified FFT-BPM [75].

4.2.2 The Bloch equations

The propagation effect guided by the Maxwell’s equations distribute the incident E-field throughout the crystal. At a specific spatial position, the local E-field induces polarizations. This process is described by the Bloch equations. On the other hand, the induced polarization can in turn radiate and affect the propagation of the E-field. This is again governed by the Maxwell’s equations. In the last section, we discuss the Maxwell’s equations which govern the E-field propagation through an inhomogeneous material. In this section, we are going to discuss the Bloch equations which describe the interaction of an atom with a classical E-field. The solution of the Bloch equation, the Bloch vector, rotates at an optical frequency. If we observe the motion in a reference frame
Figure 4.1: Bloch sphere. $u$ and $v$ axes are the in-phase and in-quadrature components of the polarization. $w$ axis is the population inversion. The south pole of the sphere is the ground state $0>$. The north pole of the sphere is the upper state $1>$. That is itself rotating, the vector then moves much more slowly. This transformation into the rotating frame is called the rotating wave approximation (RWA). This approximation is similar to the SVEA, which separates the carrier and the envelope in the condition that the envelope is not varying faster than the carrier. Under the RWA, the Bloch equations can be written as [66]

$$
\frac{d}{dt} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & -\Delta & -\kappa E_i \\ \Delta & -\frac{1}{T_2} & \kappa E_r \\ \kappa E_i & -\kappa E_r & -\frac{1}{\tau_1} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\tau_1} \end{bmatrix} \tag{4.1}
$$

where $T_1$ is the population decay time of the excited state to the ground state $w(\infty) = -1$, $T_2$ is the decoherence time, $\Delta = \omega - \omega_0$ is the detuning of the dipole oscillation frequency $\omega$ from the optical frequency $\omega_0$, $u$ and $v$ are the in-phase and in-quadrature components of the polarization, $w$ is the population inversion of the two level atoms and $\mathcal{E}_r$ and $\mathcal{E}_i$ are the real and imaginary parts of the E-field $\mathcal{E}(x, z; t)$. Eq. (4.1) describes the light’s interaction with an ensemble of two-level atoms.
with a frequency detuning $\Delta$. On the Bloch sphere, as is shown in Fig. 4.1, $E_i$ drives the Bloch vector to rotate about the $v$-axis, while $E_r$ drives the Bloch vector to rotate about the $u$-axis. $\Delta$ makes the rotation axis slightly tilted towards the south pole of the sphere ($w = -1$). Considering inhomogeneously broadened two-level atoms, the total macroscopic polarization is the complex sum of dipole radiation contributions throughout the inhomogeneous band

$$
P(t) = \alpha \int_{-\infty}^{+\infty} g(\omega)[u(t, \omega) + iv(t, \omega)]d\omega, \quad (4.2)
$$

where $g(\omega)$ describes the inhomogeneous band shape and $\alpha$ is a real coefficient that includes the atom density and the expectation value of the dipole moment. Define $y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, $A = \begin{bmatrix} -\frac{1}{T_2} & -\Delta & -\kappa\Re\{E\} \\ \Delta & -\frac{1}{T_2} & \kappa\Im\{E\} \\ -\kappa\Im\{E\} & -\kappa\Re\{E\} & -\frac{1}{T_1} \end{bmatrix}$, and $b = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{bmatrix}$. Then Eq. (4.1) simply becomes

$$
y' = Ay + b. \quad (4.3)
$$

The solutions of the Bloch equations are on the surface or inside the Bloch sphere, $u^2 + v^2 + w^2 = 1$. The dipole transitions are described by the Bloch vector $y = (u, v, w)^T$ rotating around the Bloch sphere. With no decoherence nor population decay, $T_1 \to \infty$ and $T_2 \to \infty$, i.e., the on-diagonal components of matrix $A$ are 0, the length of the Bloch vector remains $|y| = 1$ and the solutions to the Bloch equations lie on the surface of the Bloch sphere. With decoherence, the Bloch vector spirals in towards the center of the Bloch sphere. As the Bloch vectors of different detuning frequencies get decohered, they are also dephasing. In the dephasing process, dipoles with different detuning frequencies are oscillating out of phase. The result of dephasing is a decreased ensemble sum of dipole moment $u + iv$ and the damping of polarization $P$.

Before we talk about the numerical schemes to the Bloch equations, let’s briefly review the importance of the eigenvalues for Ordinary Differential Equations (ODEs) and introduce the reasons for investigating the stabilities of numerical schemes for ODEs. A two-variable ODE which is a simplified case of the Bloch equations in the rotating wave approximation (RWA) is employed as an example to illustrate the necessity of stability analysis. In section. , several numerical schemes for simulating Bloch equations under RWA are explored and the stabilities of those schemes are
analyzed. Our purpose is to find out an efficient way to simulate the Bloch evolution dynamics.

### 4.2.2.1 Stability Analysis for ODEs

The Bloch equation in the RWA is a first-order linear ODE. A general first-order linear differential equation can be written as $y' = A(t)y + b(t)$ where $y$ and $b(t)$ are vectors and $A(t)$ is a matrix. The behavior of the ODE is governed by the eigenvalues of the matrix $A(t)$. By definition [79], the number $\lambda$ is an *eigenvalue* of the square matrix $A$ if there is a vector $v \in \mathbb{C}^n$, $v \neq 0$, such that $Av = \lambda v$. And the vector $v$ is an eigenvector of matrix $A$. In other words, the matrix $A$ can map its eigenvectors to multiples of its eigenvectors. The eigenvalues of a matrix are like the spectrum of a signal and play an important role in analyzing and simulating the ODEs. In order to be numerically simulated, the ODEs need to be discretized and advanced in discrete time steps. For ODEs with large eigenvalues, the step size needs to be very small to keep a certain numerical accuracy for the solution. For example, the equation

$$y'(t) = -100y(t), \quad y(0) = 1, \quad t > 0$$

(4.4)
has the exact solution

\[ Y(t) = e^{-100t}. \]  

(4.5)

If we solve this equation by using a forward difference scheme, we have

\[ y_{n+1} = y_n - 100hy_n = (1 - 100h)y_n. \]  

(4.6)

where \( h \) is the step size, \( y_n \) is the simulated value of \( y \) at step \( n \). By elementary induction,

\[ y_n = (1 - 100h)^ny_0. \]  

(4.7)

The exact solution

\[ Y(t) = e^{-100t}, \quad t > 0 \]  

(4.8)

decays exceedingly fast and will be almost zero for even a small \( t \). However, the simulated result given by the forward difference behaves differently for \( h > \frac{1}{100} \), where \(|1 - 100h| > 1\) and the iteration grows geometrically in magnitude. For an equation with smaller eigenvalue

\[ y'(t) = -y(t), \quad y(0) = 1, \quad t > 0 \]  

(4.9)

we can increase the step size a hundred times bigger before it blows up. For yet another problem

\[ y'(t) = iy(t), \quad y(0) = 1 \]  

(4.10)

\(|1 + ih| \) is always greater than 1 (time step \( h \) is real) and the simulation always blows up no matter what time step it has. In numerical theory, the stability region \( h\lambda \) is introduced to quantitatively analyze the behaviors of the numerical schemes and thus help choose proper numerical schemes for different problems.

Since it is convenient to illustrate the behavior of a two-variable linear ODE graphically, in the following discussions, we will use ODEs

\[
\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \quad \text{with initial condition} \quad \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]  

(4.11)
as a simple example reminiscent of the 3D Bloch equations. Now matrix \( A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) and vector \( \underline{y} = [u \, v] \). Eigenvalues of the matrix \( A \) are \( \lambda = \pm i \). The exact solutions are
\[
\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix},
\]
which is a constant tangential velocity trajectory on the circumference of a unit circle. Two illustrative schemes are employed here to solve this ODE numerically. The first and simplest is the forward difference scheme based on
\[
\frac{dy}{dt} \approx \frac{y_n - y_{n-1}}{\Delta t},
\]
which gives
\[
Y_{n+1} = Y_n + h\lambda Y_n.
\]
This is an explicit scheme in which the next value \( y \) is given by a function of the previous value. A
Figure 4.4: Stability region $h\lambda$ of (a) the forward difference scheme, which is inside the unit circle centered at $(-1,0)$ on the complex plane and (b) the central difference scheme, which is between $-1$ and $+1$ on the imaginary axis.

A slightly more symmetric scheme is based on the central difference approximation

\[
\frac{dy}{dt} \approx \frac{y_{n+1} - y_{n-1}}{2\Delta t},
\]

which gives

\[
Y_{n+1} = Y_{n-1} + 2h\lambda Y_n.
\]

For the forward difference scheme, with time step $h = 0.01$, after 10 rotations on the circle, there is about 37% error, as shown in Fig. 4.4(a). For the central difference scheme, at time step ten times bigger $h = 0.1$, after 100 rotations on the circle, the error is bounded under 0.5%. And the 2D simulation result of the traces are shown in Fig. 4.3.

Examination of the stability regions of both schemes helps to explain the behavior for both schemes. The scalar equation

\[
Y' = \lambda Y,
\]

where $Y$ is one component of the vector $y$, is a decoupled equation for the eigenvector subspace of the vector matrix equation $y' = Ay$. Substitute $Y_n = r^n$, $n \geq 0$ into both schemes (Eq. (4.14) and Eq. (4.16)), we are able to find out the region where $Y_n$ is bounded, i.e., $|r| \leq 1$. For the forward
difference scheme, we have

\[ r^{n+1} = (1 + h\lambda)r^n = (1 + h\lambda)^{n+1}, \quad (4.18) \]

and

\[ |r| = |1 + h\lambda| \leq 1. \quad (4.19) \]

\( h\lambda \) is bounded in a unit circle centered at \(-1\), as shown in Fig. 4.4(a). Since the time step is a real number, for any time steps which is greater than zero, the eigenvalues of the ODEs will never be inside the stability region of the forward difference scheme and this scheme will always blow up when being applied to the problem. On the other hand, for the central difference scheme, we have

\[ r^2 - 2h\lambda r - 1 = 0. \quad (4.20) \]

Here, it is easier to find out the stability region by letting

\[ r = e^{i\theta}, \quad (4.21) \]

where \( \theta \in [-\pi, \pi) \) (means \( r \leq 1 \)), and solving

\[ h\lambda = \frac{r^2 - 1}{r} = \frac{e^{2i\theta} - 1}{e^{i\theta}} = i\sin \theta \in [-i, i]. \quad (4.22) \]

So the stability region for the central difference scheme is a line between \(-1\) and \(+1\) on the imaginary axis, as is shown in Fig. 4.4(b). Because the eigenvalues of matrix \( A \) \( \pm i \) fall into this region for \( h \leq 1 \), this scheme is stable for \( h \leq 1 \). This stability region also tells us that if our matrix \( A \) has an eigenvalue away from the imaginary axis, the central difference scheme is not going to be stable no matter how small the step size \( h \) is. In the next section, I will apply the stability theory to our three-variable Bloch equations and find stable numerical schemes for simulating the equations.

4.2.2.2 Numerical Schemes for Bloch Equations

Typical inhomogenously broadened absorbers (IBAs) such as Tm\(^{3+}\):YAG or Er\(^{3+}\):EuYSO have about 20GHz of Gaussian shaped inhomogeneous line width consisting of 10\(^5\)
resolvable sub MegaHertz homogeneous Lorentzian lines. In stimulated photon echoes a spectral interference pattern is burned into this as a spectral grating. For the purposes of numerical simulations, it is not practical to have a million randomly spaced and overlapping Lorentzians, so instead we will use a regularly spaced array of Lorentzians across a portion of the inhomogeneous band that is sufficient to Nyquist sample the spectral variations engraved into the medium’s absorption profile. For example, using 500 lines spaced by 10MHz across a 5GHz bandwidth is sufficient for simulating several GHz of RF bandwidth and time delay up to 50nsec (e.g., $\frac{1}{20\text{MHz}}$) width time-bandwidth products of a few hundred. Three-pulse photon echoes with 500 inhomogenously broaden lines is simulated here. With rotating wave approximation, it is just barely possible to simulate a two-dimensional $100 \times 100$ array with each spot having more than 500 spectrum lines. Two stable schemes are implemented, predictor-corrector and Fourth order Runge-Kutta.

The Bloch equation (4.3) is a reasonably simple ODE and there are many finite difference schemes to simulate such an equation. For near resonance or on resonance spectrum lines, the frequency difference $\Delta$ is very small and the Bloch vector will be rotating at the frequency $\kappa E$, which is on the order of tens of MHz range. So if the simulation is advancing in nano-second time steps, the Bloch vector will be rotating for $\frac{1}{100}$ of circles at each time step. If we are off the resonance, the Bloch vector is rotating as a faster speed and the vector rotates more circles for the same time step size, which implies that this equation is easier to blow up when it is off resonant with the same time step size. In section 2.1, we know that different numerical schemes have different behaviours for even the same numerical problems and stability analysis can help to choose a stable numerical scheme.

To find out if the selected schemes are stable for the Bloch equations, we need to firstly find out the eigenvalues of the Bloch equations. Then proper times step size needs to be chosen to make all the eigenvalues inside the stability region of the chosen scheme. In some situation, the stability region is such that it is impossible to put the eigenvalues into the stability region.

For a given set of material parameters $T_1 = 10\text{msec}$, $T_2 = 5\mu\text{sec}$, $\Delta = 12\text{MHz}$, a sech $\pi/2$ pulse with duration $T_p = 0.314\mu\text{sec}$ and time step $dt = 5\text{nsec}$, we calculate the eigenvalues $\lambda$ of
Figure 4.5: $h\lambda$ plane (complex plane) of the eigenvalues of matrix $A$. $T_1 = 10$ msec, $T_2 = 5$ $\mu$sec, $T_p = 0.314$ $\mu$sec, $dt = 5$ nsec. Since matrix $A$ is a function of the electric field $\mathcal{E}$, for a sech pulse, the eigenvalues of matrix $A$ are different as the pulse steps in. Therefore there are more than 3 eigenvalues for the sech pulse photon echo simulation.

matrix $A$. The $h\lambda$ ($h$ is the time step $dt$) plane of matrix $A$ when the pulse comes in is shown in Fig. 4.5. We need to fit those values into the stability region of the numerical schemes we are going to choose.

**Predictor-corrector** [71] The predictor-corrector method is a second order method. There are two formulas in the method, an explicit predictor formula in which the next value $y$ is given by a function of only the previous value, and an implicit corrector formula, in which the next value $y$ is given by a function of the next unknown value as well as the previous value. The corrector formula is the critical formula which determines the accuracy and stability of this scheme. The corrector formula is an implicit formula based on integrating the differential equation $Y'(x) = f(x, Y(x))$ over $[t_n, t_{n+1}]$ to obtain

$$Y(t_{n+1}) = Y(t_n) + \int_{t_n}^{t_{n+1}} f(x, Y(x))dx.$$ (4.23)
Applying the simple trapezoidal rule, we have

\[ Y(t_{n+1}) \approx Y(t_n) + \frac{h}{2} [f(t_n, Y(t_n)) + f(t_{n+1}, Y(t_{n+1}))]. \tag{4.24} \]

Let \( Y(t_n) = y_n \), we have the corrector formula,

\[ y_{n+1}^{(j+1)} = y_n + \frac{h}{2} \left[ f(t_n, y_n) + f(t_{n+1}, y_{n+1}^{(j)}) \right], \quad j = 0, 1, 2, \ldots \quad \text{Corrector.} \tag{4.25} \]

where \( h \) is the time step and in our case, \( f(t, Y(t)) = A(t)y + b \). This corrector formula requires the value \( y \) at time step \( n + 1 \) in order to calculate a value at time step \( n + 1 \). We can use an explicit method to predict the value \( y \) at time step \( n + 1 \). Here we use a second order Runge-Kutta method as \( y_n^{(0)} \),

\[ y_{n+1}^{(0)} = y_n + \frac{h}{2} \left[ f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n)) \right], \quad \text{Predictor.} \tag{4.26} \]

This predictor equation is used only once to provide the first guess of \( y_{n+1} \), i.e., \( y_n^{(0)} \). Then we use the corrector equation iteratively to predict the more accurate \( y_n^{(j)} \), \( j = 1, 2, \ldots \) until the last predicted value converge to the exact value \( y_{n+1} \) (within certain tolerance). The reasons to choose the second order Runge-Kutta are: the corrector is of 2nd order; it is accurate enough to keep the required number of iterations to be less than 3; it is less time-consuming than the higher order Runge-Kutta method because it requires to evaluate fewer values.

The stability region of the predictor-corrector is derived from the corrector formula. Assume that the eigenvalues do not change much at each time step \( \lambda_n \approx \lambda_{n+1} \) and again we consider the scalar homogeneous equation \( Y' = \lambda Y \), where \( \lambda \) is one of the eigenvalues of matrix \( A \). Substitute \( Y_n = r^n, \ n \geq 0 \) into the corrector formula Eq. (4.25) and use \( f(t_n, y_n) = \lambda_n Y_n \) and the approximation \( \lambda_n \approx \lambda_{n+1} \), we can get

\[ r^{n+1} - r^n - \frac{h\lambda_n}{2} (r^n + r^{n+1}) = 0. \tag{4.27} \]

Doing some algebra, we can get

\[ r = \frac{1 + \frac{h\lambda_n}{2}}{1 - \frac{h\lambda_n}{2}}. \tag{4.28} \]
$|r| \leq 1$ is always true for $Re\{h\lambda_n\} \leq 0$, which means the stability region for this chosen corrector is the whole left side of the complex plane. This also implies that we can take infinite large time steps, which is not really practical because of the approximations we make. By the way, even if there is no assumption about the eigenvalues, we still cannot take infinite timesteps, since it means we need to iterate our corrector formula for huge amount of times to make it converge. At least, we can say that this method is very stable.

**Fourth order Runge-Kutta** A whole category of numerical methods for solving ODE, $Y'(x) = f(x, Y(x))$ uses approximations for numerical differentiation. The simplest case is the forward difference scheme, i.e., $y_{n+1} \approx y_n + hf(x_n, y_n)$, which uses the slope at $(x_n, y_n)$ to approximate the slope of $Y(x)$ on $[x, x + h]$. More precise higher order Runge-Kutta methods use averaged slopes. Illustrated in Fig. 4.6, the Fourth order Runge-Kutta uses the average of slope values at four different locations, $\bar{f}$ to approximate the slope of $Y(x)$ and then calculate the value at the next time step by using $y_{n+1} \approx y_n + h\bar{f}$. The first slope value $f_1$ is evaluated at $(x_n, y_n)$. Using $f_1$, we can find a position $(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf_1)$ and evaluate the second slope, $f_2$, at that position. We are calling it 'slope', but it is not really the slope of function $Y(x)$. It is actually the evaluation of function $f(x, y)$, which always has a value at the whole function space, at the position $(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf_1)$. Similarly, the third slope $f_3$ is evaluated at the position found by the second
slope. And the fourth one is evaluated at the position calculated from the third one. The average of the slopes are given by the equation

$$\bar{f} = \frac{1}{6}(f_1 + 2f_2 + 2f_3 + f_4).$$

(4.29)

This average slope $\bar{f}$ gives more accurate prediction of $y_{n+1}$ than the prediction that only uses slope $f_1$. The following is the general form of the equations:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

(4.30)

$$k_1 = hf(x_n, y_n), \quad k_2 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1),$$

$$k_3 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2), \quad k_4 = hf(x_n + h, y_n + k_3).$$

The stability region of the 4th order Runge-Kutta is shown in Fig. 4.7[1]. This region is shown on the complex $h\lambda$-plane. The vertical axis is the imaginary axis and the horizontal axis is the real axis. The region inside the shaded curve is the stability region. If $h\lambda$ of our problem is bounded by that shaded curve, the 4th order Runge-Kutta will be stable.

### 4.2.2.3 Numerical solutions to the Bloch equations

Considering simulation efficiency and simulation error, in this thesis, we use the fourth order Runge-Kutta [79] (RK-4) method to solve the Bloch equation (4.3) By applying RK-4, we have

$$y_{l+1,m,q,s} = y_{l,m,q,s} + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

(4.31)

$$P_{l,m,q} = \alpha \sum_s (u_{l,m,q,s} + iv_{l,m,q,s})g_s\Delta\omega,$$

(4.32)

where

$$k_1 = A(y_{l,m,q,s} + b), \quad k_2 = A(y_{l,m,q,s} + \frac{k_1}{2} + b),$$

$$k_3 = A(y_{l,m,q,s} + \frac{k_2}{2} + b), \quad k_4 = A(y_{l,m,q,s} + k_3 + b).$$

the inhomogeneous band shape $g_s = 1$ in the following simulations, the index $l$ is the time step, $m$ is the transverse space $x$ sampling index, $q$ is the propagation direction $z$ index, and $s$ indexes the inhomogeneous spectral lines, but only occurs in the Bloch sphere inhomogeneous band that is averaged to get the complex polarization in equations.
4.2.3 2D Maxwell’s equations

The rare-earth doped crystal we use in the experiment is a Tm$^{3+}$:Y$_3$Al$_5$O$_{12}$ (YAG) crystal. The Y$_3$Al$_5$O$_{12}$ crystal has cubic symmetry [80]. The Y$^{3+}$ ions occupy six sites and have dodecahedral point symmetry ($D_2$ point group). The doped Tm$^{3+}$ ions normally substitute the Y$^{3+}$ ions and experience the same $D_2$ symmetry. Since the six sites oriented differently from each other, the
six sets of dipoles will also be oriented differently. If the polarization of the E-field makes unequal angles with each of the six dipoles, different Rabi frequencies will be generated. When the polarization is along [111], [001], and their equivalents, the E-field will have the same angles relative to the transition dipoles, resulting in identical Rabi frequencies, while the remaining dipoles that are perpendicular to the E-field will not be excited. In experiment, we always make the E-field along the [111] direction of the crystal. In our simulation, we make the y-axis along the [111] direction and only consider the E-field that is polarized along the y-direction, assuming x is the transverse direction and z is the beam propagation direction. For an arbitrary incident polarization, one could simulate the polarization dependent effect by projecting the incident E-field onto the general solutions (Table I of [80]), simulating the polarization effect on each of the ion groups independently and then summing up to get the output E-field.

Let the traveling wave
$$\tilde{E}(x, z; t) = \mathcal{E}(x, z; t) e^{i(\omega_0 t - kz)} + c.c. \quad (4.33)$$

be a single sideband complex solution of the 2D wave equation
$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial^2 P}{\partial t^2}, \quad (4.34)$$
where $\mathcal{E}(x, z; t)$ is the slowly varying envelope of the incident E-field $E(x, z; t)$ expanded about a propagation with wave vector $K$ along $z$, $v = \frac{c}{n}$ is the velocity of light in the medium, $n$ is the refractive index of the host material, $\omega_0$ is the carrier frequency of the incident light and $k = \frac{\omega_0}{v}$.

The resonance induced single sideband complex polarization $P(x, z; t)$ can also be written as the product of the envelope and the modulation at the carrier frequency,
$$P(x, z; t) = \mathcal{P}(x, z; t) e^{i(\omega_0 t - kz)} \quad (4.35)$$

Plug Eq.(4.33) and Eq.(4.35) into the wave equation Eq.(4.34), for the situation when $\frac{\partial^2 \mathcal{E}}{\partial z^2} << 2k \frac{\partial \mathcal{E}}{\partial z}$, $\frac{\partial^2 \mathcal{P}}{\partial t^2} << 2\omega_0 \frac{\partial \mathcal{P}}{\partial t}$, $\frac{\partial^2 \mathcal{P}}{\partial x^2} << 2\omega_0^2 \mathcal{P}$, we can neglect the second order derivatives and make the SVEA, which results in
$$\frac{\partial^2 \mathcal{E}}{\partial x^2} - 2i k \frac{\partial \mathcal{E}}{\partial z} = \frac{2i \omega_0}{v^2} \frac{\partial \mathcal{E}}{\partial t} - \mu_0 \omega_0^2 \mathcal{P}. \quad (4.36)$$
Ignoring the time domain dispersion effect (the dispersion due to the dopant is considered in the polarization term $\mathcal{P}$) and writing Eq. (4.36) under moving coordinates (transform to a moving coordinate system $t' = t - \frac{z}{v}$ or $z' = z - vt$), we can get rid of the time derivative of the equation and simplify the equation as

$$2ik \frac{\partial \mathcal{E}}{\partial z} - \frac{\partial^2 \mathcal{E}}{\partial x^2} = \mu_0 \omega_0^2 \mathcal{P}.$$  \hspace{1cm} (4.37)

This partial differential equation (PDE) can be solved by complicated but general PDE schemes, like finite difference [81] or finite element methods [81]. An easier solution to this problem, however, is to use a spatial frequency domain method, usually referred to as the FFT-BPM, to transform this PDE into an ordinary differential equation (ODE) and use the ODE schemes [79] to solve this equation. This solution assumes a rectangular grid with periodic boundary conditions. The E-field and polarization can be expanded into sum of spatial frequency components,

$$\mathcal{E}(x, z; t) = \int_{-\infty}^{+\infty} \hat{\mathcal{E}}(k_x, z; t) \exp(-ik_x x) dk_x,$$  \hspace{1cm} (4.38)

$$\mathcal{P}(x, z; t) = \int_{-\infty}^{+\infty} \hat{\mathcal{P}}(k_x, z; t) \exp(-ik_x x) dk_x,$$  \hspace{1cm} (4.39)

where $\hat{\mathcal{E}}(k_x, z; t)$ and $\hat{\mathcal{P}}(k_x, z; t)$ are the spatial frequency components of $\mathcal{E}(x, z; t)$ and $\mathcal{P}(x, z; t)$.

Using the Fourier expansion Eqs. (4.38) and (4.39), we have

$$\frac{\partial^2 \mathcal{E}}{\partial x^2} = -\int k_x^2 \hat{\mathcal{E}}(k_x) \exp(-ik_x x) dk_x.$$  \hspace{1cm} (4.40)

Therefore, for each spatial frequency component, we have an evolution equation with coupling through the polarization source term

$$\frac{\partial \hat{\mathcal{E}}}{\partial z} = \frac{i}{2k_x} \left( k_x^2 \hat{\mathcal{E}} - \mu_0 \omega_0^2 \hat{\mathcal{P}} \right).$$  \hspace{1cm} (4.41)

The space domain polarization term $\mathcal{P}(x, z; t)$ can be calculated by using Eq.(4.2).

4.2.3.1 The standard FFT-BPM

The standard FFT-BPM is usually formulated with the polarization in the form of the E-field times a coefficient, the susceptibility (which can vary in space and time) although the coefficient
can be E-field dependent. Specifically, for an E-field propagating in an inhomogeneous material, the FFT-BPM solves the wave equation in the following form,

\[ \frac{\partial \hat{E}}{\partial z} = (\hat{D} + \hat{S})\hat{E}, \tag{4.42} \]

where \( \hat{D} \) and \( \hat{S} \) are the diffraction/dispersion linear operator and inhomogeneity nonlinear operator respectively. This equation is a formally solvable equation. The solution is given by

\[ \hat{E}(x, y, z + \Delta z) = \exp[(\hat{D} + \hat{S})\Delta z]\hat{E}(x, y, z). \tag{4.43} \]

People then calculate the diffraction effect in the spatial frequency domain and the refraction effect in the space domain. In case of Eq. (4.41), the paraxial diffraction operator \( \hat{D} = i\frac{k^2}{2k} \), which is a quadratic phase factor. Numerically, we consider this part of the solution in the spatial frequency domain by simply multiplying the E-field by a quadratic phase factor \( \exp(i\frac{k^2}{2k} \Delta z) \). Since our modal ODE is not in the form of typical BPM, we write Eq. (4.41) into a conventional format to see if a conventional BPM can be employed. In case of Eq. (4.41), \( \hat{S} = i\frac{\mu_0\omega_0^2 \hat{P}}{2k} \), which could amplify, attenuate or refract the beam. Numerically, we consider this part in space domain by multiplying the E-field by the phase factor \( \exp(i\frac{\mu_0\omega_0^2 \hat{P}}{2k} \Delta z) \) in the space domain. For the photon echo simulation we are going to discuss in section 4.3, the term \( \frac{\hat{P}}{\hat{E}} \) can give us trouble. Since in the photon echo simulation, the inhomogeneity gives a delayed amplification effect. At the time the echo comes out, the incident E-field is zero. The phase factor \( \exp(i\frac{\mu_0\omega_0^2 \hat{P}}{2k} \Delta z) \) goes to infinity (\( \lim_{\hat{E} \to 0} \frac{\hat{P}}{\hat{E}} = -i \cdot \infty \)). Then mathematically we have an echo strength of \( 0 \cdot \exp(+\infty) \). Depending on the order of the 0 and the order of the +\( \infty \), mathematically this product could be a finite number. However, numerically this is imprecise, inaccurate and unsatisfactory as a numerical technique. Therefore, we will introduce a new numerical approach in the next section.

4.2.3.2 The modified FFT-BPM

There is an easy way to fix this problem. Since after the spatial domain Fourier transformation, the wave equation becomes an uncoupled set of ODEs for the Fourier components driven
by the source terms for each Fourier component, we can simply use numerical schemes to solve the ODE directly. To observe the behavior of the Maxwell-Bloch equations, we first use forward difference to simulate both the wave equation and the Bloch equations. By decreasing time and spatial steps, the simulation results stabilize. This kind of diagnostic simulations give us ideas about our simulation problem. Since forward difference is a first order technique, we then switch to a more efficient second order accurate, unconditionally-stable scheme, the trapezoidal rule [79] to solve the wave equation Eq. (4.41). Comparing to the first order accurate forward difference scheme,

$$\hat{E}_{l,m,q+1} = \hat{E}_{l,m,q} + \frac{i}{2k} dz \left( k_x^2 \hat{E}_{l,m,q} - \mu_0 \omega_0^2 \hat{P}_{l,m,q} \right),$$

(4.44)

the trapezoidal rule gives us higher computation accuracy and thus allows bigger step sizes in the evolution direction $z$, resulting in higher computation efficiency. For a general ODE $\dot{y} = f(z, \dot{y})$, the trapezoidal rule is a numerical integration of $\dot{y}$ over $[z_q, z_{q+1}]$,

$$\hat{y}_{q+1} = \hat{y}_q + \frac{\Delta z}{2} \left[ f(z_q, \hat{y}_q) + f(z_q + \Delta z, \hat{y}_{q+1}) \right].$$

(4.45)

The trapezoidal rule is an implicit scheme, which usually requires root finding schemes, like Newton’s method [79], to solve for $y_{q+1}$. However, if $f(x,y)$ is a linear function of $y$, we can solve $y_{q+1}$ from Eq. (4.45) directly. So in our case of solving the wave equation (4.41), by applying the trapezoidal rule, we have

$$\hat{E}_{l,m,q+1} = \hat{E}_{l,m,q} + \frac{i}{4k} dz \left( k_x^2 \hat{E}_{l,m,q} - \mu_0 \omega_0^2 \hat{P}_{l,m,q} + k_x^2 \hat{E}_{l,m,q+1} - \mu_0 \omega_0^2 \hat{P}_{l,m,q} \right).$$

(4.46)

Since this is a linear equation, we can solve for $\hat{E}_{q+1}$,

$$\hat{E}_{l,m,q+1} = \frac{1 + \frac{i}{4k} k_x^2 dz}{1 - \frac{i}{4k} k_x^2 dz} \hat{E}_{l,m,q} - \frac{\frac{i}{4k} dz}{1 - \frac{i}{4k} k_x^2 dz} \cdot 2\mu_0 \omega_0^2 \hat{P}_{l,m,q}.$$ 

(4.47)

Notice that since we are not simulating the time evolution (dispersion) and propagation of a brief pulse, we assume that there is no variation with $l$ as we simulate the beam propagation through
the grid. Define \( \psi = \frac{k_x^2}{4k} dz \), the first term of Eq.(4.47) can be written as

\[
1 + i \frac{k_x^2}{4k} dz \hat{\chi}_{l,m,q} = \exp[i \cdot 2 \tan^{-1}(\psi)] \hat{\chi}_{l,m,q}.
\]

(4.48)

Compared with the analytical solution of the homogeneous wave equation,

\[
\hat{\xi}_{l,m,q+1} = \exp(i \cdot 2\psi) \hat{\xi}_{l,m,q},
\]

(4.49)

the phase error is given by

\[
\psi - \tan^{-1}(\psi) = \frac{\psi^3}{3} - \frac{\psi^5}{5} + \cdots,
\]

(4.50)

Since the error is proportional to \( \psi^3 \), the simulation error gets bigger for higher spatial frequency \( k_x \).

In this perspective, we can only simulate smaller spatial frequencies in comparison to the simulation ability of the FFT-BPM and especially compared to the wide-angle [82, 83] or non-paraxial FFT-BPM techniques [84–86]. For high interaction angles, low overlap area of the interacting beams leads to low photon echo efficiency. In our Maxwell-Bloch simulations, we only simulate beams with incident angles within \( \pm 3^\circ \), so a paraxial small angle approach is perfectly satisfactory.

### 4.2.3.3 Modified FFT-BPM using the paraxial approximation term

Since we know that the phase term \( \exp(i2\psi) \) in Eq. (4.51) is more accurate than the phase term \( \exp[i \cdot 2 \tan^{-1}(\psi)] \) in Eq. (4.48), can we instead use the phase term \( \exp(i2\psi) \) to advance the wave equation? With some algebra, it is practical to use the paraxial term \( \exp(i2\psi) \) instead.

First, let \( \hat{F} \) be the solution of the homogeneous wave equation,

\[
\frac{\partial \hat{F}}{\partial z} = i \frac{k_x^2}{2k} \hat{F}.
\]

(4.51)

The solution to the equation can be written as

\[
\hat{F}_{l,m,n+1} = \exp \left( \frac{k_x^2}{2k} \Delta z \right) \hat{F}_{l,m,n}.
\]

(4.52)

\( \hat{E} \) is the solution of the inhomogeneous wave equation, Eq. (4.41). We can write \( \hat{E} \) in terms of \( \hat{F} \) as

\[
\hat{E} = \hat{F} + \hat{G},
\]

(4.53)
where $\hat{G}$ is the difference term. Plug Eq. (4.53) into Eq. (4.41) and use Eq. (4.51), we can get

$$\frac{\partial \hat{G}}{\partial z} = \frac{i}{2k} k_z^2 \hat{G} - \frac{i \mu_0 \omega_0^2}{2k} \hat{P}.$$  \hspace{1cm} (4.54)

Now if we apply the trapezoidal rule to Eq. (4.54) and solve for $\hat{G}_{q+1}$, we get

$$\hat{G}_{t,m,q+1} = \frac{1 + \frac{i}{4k} k_x^2 dz}{1 - \frac{i}{4k} k_x^2 dz} \hat{G}_{t,m,q} - \frac{\frac{i}{4k} dz}{1 - \frac{i}{4k} k_x^2 dz} \cdot 2 \mu_0 \omega_0^2 \hat{P}_{t,m,q}. \hspace{1cm} (4.55)$$

Therefore we can calculate $\hat{E}_{t,m,q+1}$ by

$$\hat{E}_{t,m,q+1} = \hat{G}_{t,m,q+1} + \hat{F}_{t,m,q+1} \hspace{1cm} \Rightarrow \hspace{1cm} \hat{E}_{t,m,q+1} = \exp \left( \frac{i k_z^2 dz}{2k} \right) \hat{F}_{t,m,q} + \frac{1 + \frac{i}{4k} k_x^2 dz}{1 - \frac{i}{4k} k_x^2 dz} \hat{G}_{t,m,q} - \frac{\frac{i}{4k} dz}{1 - \frac{i}{4k} k_x^2 dz} \cdot 2 \mu_0 \omega_0^2 \hat{P}_{t,m,q}. \hspace{1cm} (4.56)$$

Comparing with the solution given in the last section, Eq. (4.47), the difference term is

$$\exp \left( \frac{i k_z^2 dz}{2k} \right) \hat{F}_{t,m,q} + \frac{1 + \frac{i}{4k} k_x^2 dz}{1 - \frac{i}{4k} k_x^2 dz} \hat{G}_{t,m,q} - \frac{1 + \frac{i}{4k} k_x^2 dz}{1 - \frac{i}{4k} k_x^2 dz} \hat{E}_{t,m,q} \hspace{1cm} \Rightarrow \hspace{1cm} \exp \left( \frac{i k_z^2 dz}{2k} \right) \hat{F}_{t,m,q} - \frac{1 + \frac{i}{4k} k_x^2 dz}{1 - \frac{i}{4k} k_x^2 dz} \hat{F}_{t,m,q}, \hspace{1cm} (4.57)$$

which is the difference between the analytical solution of the homogeneous wave and the trapezoidal rule solution of homogeneous equation. Eq. (4.55) will have better angular performance than Eq. (4.47).

### 4.3 Carrying out the numerical schemes

We repeat the following procedure, also shown in Fig. 4.8, to calculate the E-field and polarization at all the spatial-time steps:
Figure 4.8: Use Maxwell-Bloch schemes to calculate the E-field and the polarization at all the spatial-time steps. We update the E-field by using Eq.(4.47) in the spatial frequency domain and update the polarization by using Eq.(4.31) in the space domain. The transverse dimension $x$ is not shown in this figure. When updating the E-field and the polarization, we update all the transverse components simultaneously.

Algorithm 4.3.1: MAXWELL-BLOCH($l, q, s$)

**comment:** Maxwell-Bloch numerical schemes

for TimeStep ← 1 to $l$

\[
\begin{align*}
\mathcal{E}_{l,q=1} &= \text{Input E-field at time } l \\
\hat{\mathcal{E}}_{l,q=1} &= \text{FFT}\{\mathcal{E}_{l,q=1}\}
\end{align*}
\]

for SpatialStep ← 2 to $q$

\[
\begin{align*}
\text{do } \hat{\mathcal{E}}_{l,q+1} &= \left(1 + \frac{i}{4} k^2 dz \right) \hat{\mathcal{E}}_{l,q} - \frac{i}{8} \frac{dz}{k^2 dz} \cdot 2\mu_0\omega_0^2 \hat{P}_{l,q} \\
\end{align*}
\]

for SpectralLine ← 1 to $s$

\[
\begin{align*}
y_{l+1,q,s} &= y_{l,q,s} + \frac{\Delta t}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\
\text{do } k_1 &= A y_{l,q,s} + \frac{b}{2}, \\
k_2 &= A (y_{l,q,s} + \frac{k_1}{2}) + b, \\
k_3 &= A (y_{l,q,s} + \frac{k_2}{2}) + b, \\
k_4 &= A (y_{l,q,s} + k_3) + b \\
P_{l,q} &= \alpha \sum_s (u_{l,q,s} + iv_{l,q,s}) g_s \Delta \omega
\end{align*}
\]

- In spatial frequency domain, propagate the beam by employing the wave equation, Eq. (4.47), with about 40 $\mu$m of spatial steps;

- In space domain, evolve the dipoles across the inhomogeneous band by employing the Bloch equations, Eq. (4.31), with nanosecond time of steps.
The two processes are coupled by the induced polarization term, \( P_{l,m,q} \). The polarization induced by the dipole radiation drives the E-field, affecting the propagation of the beam. The E-field at each spatial location drives the Bloch equations. One thing to notice is that we alternate between the space and spatial frequency domain when doing the calculation. We use the Maxwell’s equation to propagate each spatial frequency components independently in spatial frequency domain. While in the space domain, we use the Bloch equation to calculate the induced polarization by using the E-field at each spatial position.

### 4.3.1 Choosing the right parameters

Simulation parameters need to be properly chosen for different simulation problems. For the Bloch equation simulation, several time/temporal frequency parameters are important. The one with the most importance is the time step size. We need to choose the time step such that it can at least Nyquist sample the Rabi oscillation for any detuning within our band, and in reality a factor of 10 times faster sampling than Nyquist is almost certainly required for accurate numerical evolution on the sphere. The Rabi frequency is determined by both the incident field intensity and the detuning frequency \( \Delta \),

\[
\Omega = \sqrt{\Delta^2 + \left(\kappa E\right)^2}.
\]  

Either higher detuning frequency or higher incident light intensity gives higher Rabi frequency. Therefore, both the incident beam intensity and the simulating inhomogeneous bandwidth affect the choice of the minimum allowable time step for the simulation. For example, if we have a maximum detuning frequency of \( f_m \), our time step size could be at most \( 1/2 \sqrt{f_m^2 + (\kappa E)^2} \). In simulations, it is convenient to validate the simulation time step size by decreasing the step sizes until the simulation result converges.

Another important parameter is the spectral domain resolution of the inhomogeneous band, which is the inhomogeneous bandwidth divided by the number of sampling spectral lines. Again, we need to at least Nyquist sample the smallest spectral features we want to resolve. For example,
for the pulse photon echo case, the spectrum of two identical pulses with time delay of $\tau$ has a periodicity of $1/\tau$ in the temporal frequency domain. For a given simulation bandwidth, we need to choose the number of spectral lines such that the spectral resolution $\Delta f < \frac{1}{2\pi}$. Because of this spectral domain periodic sampling, there is going to be an artificial recurrence of the echo and the two incoming signals every time interval of $1/\Delta f$. To eliminate the artificial recurrence of these signals, we can choose the number of sampling spectral lines such that $1/\Delta f$ is greater than the total simulation time. By doing this, the recurrence will happen after the simulation ends so we will not be able to observe it. Another way to eliminate the artifacts is to use random spectral domain sampling.

For the Maxwell’s equation simulation, the transverse sampling step $\Delta x$ is chosen such that $2\pi/\Delta x$ is bigger than the highest spatial frequency or the biggest angle of propagation we are interested in. We determine the longitudinal spatial step $\Delta z$ by decreasing the step sizes until the change of simulation results is small enough. When trying to find out the proper time and spatial step sizes, to minimize the dangers of numerical divergence, we fix the spatial domain step size and decrease the time domain step size until the change of simulation result is small enough. And we then fix the time domain step size and decrease the spatial domain step size to find ideal spatial step sizes for the simulation.

In the following subsections, we first verify the simulation results for the Maxwell’s equation and the Bloch equation separately. Then we discuss the issue that arise from combining the two equations for the case of pulsed photon-echo simulation.

4.3.2 Maxwell’s equation simulation

In order to find out the simulation error of our trapezoidal rule scheme for solving the Maxwell’s equations, we compare our numerical solution to the analytical solution of the homogeneous wave equation (free space beam propagation). At 793nm, we first simulate the diffraction pattern of a slit with a width of about 70 $\mu$m and a propagation distance of 1mm. As is shown in
Fig. 4.9, as we increase the number of spatial steps, the relative simulation error,
\[
\text{Error } \% = \frac{\int ||\mathcal{E}_s(x)| - |\mathcal{E}_t(x)|| \, dx}{\int |\mathcal{E}_t(x)| \, dx},
\]
where $\mathcal{E}_s$ and $\mathcal{E}_t$ are the simulated and the non-paraxial E-field respectively, decreases. The simulation scheme stays 2nd order accurate until at about $10^{-1}$m of step size where the accuracy starts to roll over due to the paraxial approximations we make in the Maxwell’s equation simulation. This is because a slit with sharp edges has numerous high spatial frequency components. For smaller spatial frequencies, only the 3rd order error in Eq. (4.50) is dominant, indicating the scheme is 2nd order accurate. As the spatial frequency gets higher, the 5th order error also comes into play, making the accuracy less than 2nd order.

Since in our photon echo simulation, limited by the simulation grid size and echo efficiency, we care about the spatial frequencies that are less than $3.0^\circ$, we also calculate the simulation errors for Gaussian beams with incident angles of $0^\circ$, $1.0^\circ$, $2.0^\circ$ and $3.0^\circ$ at different spatial step sizes, as is shown in Fig. 4.10. The relative simulation error, increases as we increase the angle of incidence for the Gaussian beam. The simulation scheme keeps to be 2nd order for all the spatial step sizes since the Gaussian beam contains mostly low spatial frequency components.
4.3.3 Bloch equation simulation

In order to verify our RK-4 scheme, we compare the well known theoretical solution of the Bloch equation [66] with our RK-4 numerical solution. For a steady incident E-field $E_0$, when the population decay time $T_1$ and decoherence time $T_2$ go to infinity, the analytical solution to Eq.(4.1) is given by [66]

$$
\begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix}
= 
\begin{bmatrix}
  \frac{(\kappa E_0)^2 + \Delta^2 \cos \Omega t}{\Omega^2} & -\frac{\Delta}{\Omega} \sin \Omega t & -\frac{\Delta \kappa E_0}{\Omega^2} (1 - \cos \Omega t) \\
  \frac{\Delta}{\Omega} \sin \Omega t & \cos \Omega t & \frac{\kappa E_0}{\Omega} \sin \Omega t \\
  -\frac{\Delta \kappa E_0}{\Omega^2} & -\frac{\kappa E_0}{\Omega} & \frac{\Delta^2 + (\kappa E_0)^2 \cos \Omega t}{\Omega^2}
\end{bmatrix}
\begin{bmatrix}
  u_0 \\
  v_0 \\
  w_0
\end{bmatrix},
$$

(4.60)

With an initial condition of $[u_0, v_0, w_0]^T = [0, 0, -1]^T$, we calculate the solutions to the Bloch equations both analytically and numerically over 7.5$\mu$s of time span. Comparing the numerical solution with the analytical solution, the maximum simulation error over the 7.5$\mu$s for population inversion $w$ is shown in Fig. 4.11. As we can see, the simulation error increases as we increase either the time step sizes or the detuning frequencies. The scheme is fourth order accurate since decreasing the time step by 10 decreases the error by $10^4$. The maximum Bloch vector simulation error over

![Figure 4.10: Relative simulation errors for Gaussian beams with a width of 35 $\mu$m (at the beam waist) and with incident angles of 0°, 1.0°, 2.0° and 3.0°.](image)
Figure 4.11: Maximum $w_0$ simulation error vs. different time step sizes for $w$ with different detuning frequency $\Delta$. We use RK-4 to simulate the Bloch equation driven by a constant amplitude E-field and without population decay or decoherence. The total simulation time is 7.5 $\mu$sec. The simulation error increases as we increase either the simulation time step size or the detuning frequencies. The detuning frequencies are from 0 to 10MHz, with 2MHz of increasing step. $\kappa E_0 = 5\pi$MHz.

A simulation time of 7.5$\mu$sec defined by

$$\text{Bloch vector error} = \text{Maximum} \left\{ [u(t) - u_a(t)]^2 + [v(t) - v_a(t)]^2 + [w(t) - w_a(t)]^2 \right\} \left| t \in (0, 7.5 \mu s) \right.$$ (4.61)

is shown in Fig. 4.12, where the subscript $s$ indicates the simulated result and the subscript $a$ indicates the analytical result. Since $u_a^2 + v_a^2 + w_a^2 = 1$, the "Bloch vector error" is actually a normalized error, the square root of which gives a length on the Bloch sphere, which is 4th order accurate. As we can see, the simulation error increases as we increase either the time step sizes or the detuning frequencies.

Since in the photon echo simulations that we care about, $T_1$ and $T_2$ are not infinitely long, we would like to also find out the behavior of the RK-4 scheme in this condition where there is no analytical solution available except in the simplest cases. In this case, we simulate the time-domain 3-pulse photon echo with decreasing sizes of time steps and see if the result converges. The pure time domain simulation only considers the Bloch equation, or it simulates a single spatial spot. To keep the simulation time reasonably short for future simulations, we decrease the decoherence time
to $T_2 = 700$ nsec corresponding to the coherence time in Eu$^{3+}$:YSO in 1 Tesla fields, instead of Tm$^{3+}$:YAG which has $T_2 \cong 10\mu$sec. The population decay time is $T_1 = 10$ msec. In the simulation, three identical time domain Gaussian shape pulses with widths of 50 nsec occur sequentially, with the first two time delayed by $\tau_{12} = 300$ nsec, and the third pulse appearing $\tau_{23} = 1.8\mu$sec after the first pulse. The 3-pulse photon echo is expected to appear at $2.1\mu$sec after the first pulse. The dipoles at each resonance frequencies inside the inhomogeneous band gain coherence from the incoming pulses at the time a pulse comes in. The interference spectrum of the first two time delayed pulses is recorded to the population inversion $w$ of the inhomogeneous band, as is shown in Fig. 4.13. The time delay between the first two pulses determines the periodicity of the spectral gratings and the shape of the pulses defines the Gaussian envelope of the spectral gratings. After the pulses are gone, we let the crystal decohere for $2T_2$ time. During the time, the dipoles with frequencies difference of $\delta$ then evolve out of phase with each other with a rate of $2\pi\delta$. After $2T_2$ of time, the dipole components $u$ and $v$ are shortened by $e^{-2}$. Yet the time delay between the first two pulses is still encoded in $w$ as a frequency domain periodic grating. The dipoles gain coherence from the third pulse upon its arrival at time $t_3$. The dipoles are nominally all in-phase at time $t_3$. Two dipoles with resonant frequency difference of $\delta$ have phase difference $2\pi\delta t$ at time $t_3 + t$. After every time interval of $1/\delta$, the phase difference will be $2\pi$, or in other words, these two
dipoles are in-phase after every time interval of $1/\delta_t$. Therefore, dipoles with frequency difference of $\delta \approx 3.3\text{MHz}$ will rephase after $1/\delta = 300\text{nsec}$, resulting in a 3-pulse photon echo.

With different simulation time steps, we get different integrated echo strengths, as is shown in Fig. 4.14. For time steps smaller than 5nsec, the difference of the echo strength between successive simulations decrease as time step size gets small. For time steps less than 5nsec, the simulation error of the RK-4 scheme dominants. The simulation error gets smaller as the time step size decreases and the simulation converges. For time steps bigger than 5nsec, the echo strength goes up rapidly (not shown in the figure). The insufficient sampling of high detuning frequencies causes the simulation to blow up. Therefore, we want to use time steps that are less than 5ns.

4.3.4 Maxwell-Bloch simulation for pulse photon echo

Now we know the behavior of the numerical schemes for the Maxwell’s equations and the Bloch equations separately. We can choose the time and spatial step sizes according to our desired accuracy. However, since the combination of those two schemes is nonlinear, convergence for either of them separately does not necessarily guarantee the convergence of the combination. Therefore, we continue our convergence and simulation error tests for the combination of the Maxwell’s equation simulation and the array of Bloch equations simulation. Changing both the spatial step size and
the time step size together could incur numerical instability, so we keep one thing fixed and change only the other. Since we do not have analytical solution to compare with in this case, we compare the integrated echo strength, $\int |E_{echo}|dt$, for each simulation. The setup for an off-axis 3-pulse photon echo is shown in Fig. 4.15, where the first two Gaussian pulses have widths of 50 nsec and a time delay of 300 nsec between them. The first two pulses have $35 \mu m$ beam waists and arrive with incident angles of $-0.8^\circ$ and $1.0^\circ$. Both beams are focused into the center of the 5mm long crystal. Those two tilted and time delayed pulses interfere in both the spatial and the spectral domains. We will discuss this spatial-spectral grating diffraction later. To find out the appropriate spatial and temporal step sizes, we first keep the spatial step size fixed and change only the time
Figure 4.16: Maxwell-Bloch simulation for 3-pulse photon echo. Integrated photon echo amplitude strength, \( \int |E_{\text{echo}}|dt \) for 3-pulse photon echo with varying time steps.

We compare the integrated 3-pulse photon echo strengths for different time step sizes in Fig. 4.16. As the time step gets smaller, the simulation result converges. The echo strength gradually grows until about 3nsec where it reaches its maximum, then it drops with a much higher rate. For the gradually growing part (time step less than 3nsec), the numerical error of the schemes dominates; while for time step greater than 3nsec, the sampling related error dominates. To keep the simulation result change for adjacent time steps under 5%, we need to keep the simulation step sizes under 1.5nsec for our Maxwell-Bloch simulation when the bandwidth is 20MHz and for \( \kappa E_0 = 5\pi \)MHz, giving a maximum Rabi frequency of about 25.4MHz. So we should sample about 10 times faster than Nyquist, giving about 4nsec of time step.

Secondly, we fix the time step size to 1nsec and observe the integrated echo strength with varying spatial step size, as is shown in Fig. 4.17. The result changes are all within 1% for successive simulation spatial steps, indicating that the simulation error is less sensitive with the spatial step size change.

In the following simulations, to keep the simulation reasonably stable we choose spatial step size to be about 50\( \mu \)m and time step size to be 2nsec.
The numerical schemes are carried out using MATLAB. The simulation results are given for the Rabi oscillation, pulse generated photon echoes and chirp generated photon echoes cases. Unless specified otherwise, the following are the simulation parameters. We choose the absorption coefficient corresponding to our experimental crystals such that $\alpha L = 1$ for the crystal. The simulation dimension of the hole burning crystal is $200\mu m(x) \times 5 mm(z)$. In the $x-z$ spatial plane, our grid size is $N_x \times N_z 128 \times 128$. Our time step size is $\Delta t = 1.5 nsec$. The simulated inhomogeneous bandwidth of the hole burning material is 100MHz (uniform band shape), sampled by $N_s = 250$ equally spaced spectral lines, $\Delta s = 100MHz/250 = 400KHz$. The simulated population life time $T_1 = 10msec$, and the decoherence time $T_2 = 0.7 \mu sec$.

### 4.4.1 Rabi Oscillation

The E-field is turned on at $t = 0.2 \mu sec$ and turned off at $t = 2.2 \mu sec$, as shown by the continuous line in Fig. 4.18(a). For better observation of the single frequency oscillations, in this simulation, we choose $\alpha L = 0.4$. The time it takes to turn on and turn off the pulses is 50nsec. The simulation time step size is 5nsec. Spatially, the Gaussian shaped beam with beam size of 35\mu m
is normally incident into the crystal. At the output, we use a detector that spatially integrates the output from the crystal to observe the output intensity change with time. For a Gaussian spatial shaped pulse, the E-field distribution is different across the transverse spatial dimension. Since Rabi frequency increases with the higher the E-field amplitude, different spatial location have different Rabi oscillations. The Rabi oscillation observed by the detector is thus an averaged effect of different spatial points. For a center E-field amplitude of $\kappa E = 5\pi \text{MHz}$, the Rabi frequency is about 2.3MHz, as is shown in Fig. 4.18(a). The oscillation is also a damped oscillation as the Rabi oscillations due to different detuning frequencies are getting out of phase. The Rabi oscillation in Fig. 4.18(a) is also an average of all the dipoles with different detuning frequencies. Fig. 4.18(b) plots the output intensity oscillations for different detuning frequencies. When simulating each of these frequencies, we assume that the material is homogeneously broadened such that there is only one spectral bin. As we can see, dipoles with different detuning frequencies start with different phase of the Rabi oscillation, indicating that different detunings have different population inversion at the abrupt E-field change point. After that, dipoles with different detuning frequencies oscillate with different frequencies. When considering all the intensity oscillations with detuning frequencies
Figure 4.19: Bloch vectors’ moving traces on a Bloch sphere. Different color represents Bloch vectors of different detuning frequencies. The length of the Bloch vector along the transverse $u-v$ plane indicates coherence. 0: ground state, 1: excited state. (a) The first pulse excites the Bloch vectors from the ground state towards the excited state. (b) After the first two pulses, the Bloch vectors decohere and shrink towards the center of the sphere. (c) The Bloch vectors re-gain coherence from the third pulse.

from 0 to 50MHz, we get the Rabi oscillation in Fig. 4.18(a). Notice that Fig. 4.18(a) is not a simple sum of all the single frequency simulations in Fig. 4.18(b) since dipoles with different frequencies are not independent to each other, instead their oscillations will couple back and forth between each other through the polarization term (sum of all the dipole oscillations).

4.4.2 Two-pulse and three-pulse photon echo

With the same angular and time setup in Fig. 4.15, we simulate an off-axis 3-pulse photon echo, where the first two Gaussian pulses have widths of 50 nsec and a time delay of 300 nsec and arrive with incident angles of $-1.0^\circ$ and $0.8^\circ$ and 35 $\mu$m beam waist at the center of the crystal, as shown in Fig. 4.15. At the positions and frequencies corresponding to constructive interference, more atoms are excited to the upper state while at the destructive interference locations fewer atoms are excited, resulting in periodic modulations of the population inversion in both the spatial and spectral dimensions. The Bloch vectors evolution of the whole process is described on a Bloch sphere as is shown in Fig. 4.19. In Fig. 4.19(a), when the first pulse just arrives, the Bloch vectors
Figure 4.20: Spatial spectral gratings recorded by the time delayed angularly interfered Gaussian beams. (a) 3D view showing $x - z$ space and spectrum (b) slice view at $z = 2.5\text{mm}$, of the spatial-spectral population inversion gratings. Different colors are coded to represent the frequency detunings of the dipoles.

are excited towards the excited state while spreading out in the transverse plane due to the effect of detuning. The cross spectral interference is recorded in the inhomogeneous band of two-level atoms as sinusoidal gratings of period $1/300\text{nsec} \approx 3.3\text{MHz}$ with a Gaussian envelope of 20 MHz width. In addition to this spectral domain grating, we also record spatial domain gratings along the $x$ direction with the two inclined beams. The spatial-spectral population inversion grating is shown in Fig. 4.20. The direction of the grating vector points towards the time (giving a time delay) and angle of the photon echo. We then let the medium decohere for $2T_2$ time before the third pulse arrives. During the decoherence time, the Bloch vectors decrease in their length in the transverse plane, as shown in Fig. 4.19(b), indicating a loss of the phase information from the previous process. By the time the third pulse comes in, the dipoles have “forgotten” the phase information of the first two pulses. But since the population grating is still present, the time delay information of the first two pulses remains in the spectral periodicity of the inversion. The dipoles gain coherence from the third pulse, rotating them back towards the equatorial plane of the Bloch sphere. At that time, they all lie in the same vertical plane. Then they freely precess from there with their own frequencies, as shown on the Bloch sphere in Fig. 4.19(c). Once more, because the population grating periodicity is 3.3MHz, 300 nsec after the third pulse, the dipoles rephase again,
resulting in a 3-pulse photon echo. Because the population grating is also in the space domain, the phase coherent superposition of all the radiating dipoles causes the spatial grating to diffract the third beam into the angular direction of pulse 2, thereby conserving momentum. The angular output of the crystal is shown in Fig. 4.21 as a function of time.

In addition to the 3-pulse echo, there is also a 2-pulse echo following the second pulse. The 2-pulse echo can be understood as the following [87]: the first two incoming pulses record a spectral grating with period \( \frac{1}{300\text{ nsec}} \approx 3.3\text{ MHz} \); the second half of pulse two reads out this grating, giving a 2-pulse grating at a time 300 nsec after the second pulse.

### 4.4.3 Chirp generated photon echo

In experiments, we usually use chirps instead of brief pulses for photon echo generation, since chirps are more photon efficient without losing bandwidth. Chirps can last up to the lifetime of the crystal and in the mean time keep their high bandwidth. While for brief pulses, the uncertainty principle dictates that it lasts for only a short time if we want to maintain a large bandwidth. Therefore, with the same beam intensity, beam size and bandwidth, chirps with longer duration can deposit much more energy into the crystal than brief pulses. We simulate the chirp photon echo with the setup shown in Fig. 4.22. Since our simulation time is on the order of several micro-seconds,
there is essentially no lifetime decay during the simulation although the coherence decay dictates the delay between corresponding frequency components must be much less than $T_2 = 700\text{nsec}$. We simulate a spatial grid size of $128(x) \times 128(z)$ with spatial dimension $250\mu m(x) \times 5\text{mm}(z)$ and a time step size of $3\text{nsec}$. We use 250 spectral bins to sample the $150\text{MHz}$ inhomogeneous bandwidth. Therefore the maximum time delay we can simulate without aliasing is $1.67\mu\text{sec}$. Chirp 1 and 3 hits the crystal with an angle $\theta = 1.8^\circ$, with duration times $\tau_1 = 1\mu\text{sec}$ and $\tau_3 = 0.2\mu\text{sec}$. Chirp 2 has a normal incidence into the crystal with a duration time $\tau_2 = 0.8\mu\text{sec}$. Chirp 2 arrives $t_{d1} = 0.3\mu\text{sec}$ after chirp 1. Chirp 3 arrives $1.8\mu\text{sec}$ after the ending time of chirp 2. All the chirps have $100\text{MHz}$ bandwidth, ranging from $f_{\text{str}} = 10\text{MHz}$ to $f_{\text{end}} = 110\text{MHz}$. The frequency components of chirp 1 interfere with the corresponding frequencies of chirp 2 and write chirped gratings in the inhomogeneous band. Since the time delays of the frequency components in chirp 1 and chirp 2 vary from $0.1\mu\text{sec}$ to $0.3\mu\text{sec}$, the grating periods in the spectral domain vary from about $3.3\text{MHz}$ to $10\text{MHz}$. When chirp 3 reads out the chirped spectral grating, it reads out the time delays from $0.3\mu\text{sec}$ (for $10\text{MHz}$) to $0.1\mu\text{sec}$ (for $110\text{MHz}$). Because chirp 3 has a duration of $0.3 - 0.1 = 0.2\mu\text{sec}$, echoes for all the frequency components collapse at the same time, i.e., $0.3\mu\text{sec}$
Figure 4.23: Off-axis chirp photon echo simulation results. (a) Angular output of the crystal vs time. Chirp 1 starts 300ns before chirp 2 and ends 100ns before chirp 2. 200ns after the read chirp, the echo comes out in the direction of chip 2, 0°. The echoes right after chirp 1 and chirp 2 are simulation artifacts due to the discrete sampling in the inhomogeneous band of the crystal. (b) Intensity of the output along the direction of chirp 1, 1.8° and the direction of chirp 2, 0°. Both figures share the same time axis.

after the starting time of chirp 3 and form an echo in the direction of chirp 2, as shown in Fig. 4.23. In Fig. 4.23(a), the simulation artifacts are due to the fact that the sampling in the spectral domain is discrete which incurs a recurrence of the chirp inputs. In Fig. 4.23(b), the intensity of the chirps varies across the bandwidth since different detuning frequencies have different population inversion with the same beam intensity.

With the same simulation setup in Fig. 4.22, if we keep the other parameters the same while varying the incident angle of beam 1, θ, and the delay time t_{d1} between chirp 1 and chirp 2, (since chirp duration times τ_1 and τ_2 are not changed, the duration of chirp 3, t_{d1} - t_{d2}, is a constant), we get Fig. 4.24(a). The integrated echo intensity decreases as the time delay t_{d1} gets bigger. The T_2 decoherence time is responsible for this effect. As the time delay between the chirps increases, the dipoles excited by the two chirps are less coherent and therefore they record population fringes
with smaller fringe visibility. For all the time delays $t_{d1}$, the echo intensity first slightly increases as the beam interaction angle gets bigger and then decreases. Thus, to maximize the echo intensity in experiment, we want to have small beam interaction angle around $0.5^\circ$.

Similarly, with the same simulation setup in Fig. 4.22, if we keep the other parameters the same and change the focused spot sizes in the center of the crystal $W_0$, and the incident angle of beam 1 $\theta$, we get Fig. 4.24(b). Since we keep the power of the beams constant, the amplitude of the beams is inversely proportional to the beam size $W_0$ in this 1D simulation, while in 2D the amplitude would fall as $1/W_0^2$. The results show that with the experimental beam size (from about $35\mu$m to $100\mu$m), the smaller the beam size, the higher the echo efficiency. For the $W_0 = 35\mu$m case, the echo efficiency decreases as the interaction angle increases. For the $W_0 = 60\mu$m and $100\mu$m cases, as the interaction angle increases, the echo intensity first increases slightly and peaks at about $0.2^\circ$, then decreases afterward. The results indicate that we need to keep the interaction angle small and beam size small for high echo efficiency.
4.5 Experimental results

We did on-axis (1D) and off-axis (2D) photon echo experiments to compare with the simulation results. The setup is shown in Fig. 4.25. A New focus Velocity laser that is stabilized to the spectral hole of a Tm$^{3+}$:YAG crystal is power amplified with a tapered traveling wave semiconductor amplifier and spatially filtered with a fiber coupler (not shown in Fig. 4.25) before it is used by our system. The beam is collimated by a fiber collimator into $d = 2.5$ mm width spot. To control the power of the two arms, we use a $\frac{\lambda}{2}$ and a PBS. For the on-axis photon echo experiment, we rotate the $\frac{\lambda}{2}$ to maximize the power of beam 2. For the off-axis photon echo experiment, we balance the power of beam 1 and beam 2. Another $\frac{\lambda}{2}$ is used in beam 1 to change the polarization back to vertical. Each beam is focused by an $f = 150$ mm lens into a AOM (ISOMET-SLM-16-835). The AOMs are PhMoO$_4$ crystals with 700 µm of transducer height and 70 MHz of modulation band-
width and are illuminated with a 60μm focal spot. The AOMs are fed with the chirps generated by an AWG520. The signals with 3dBm of power from the AWG520 are amplified by 24dBm RF amplifiers giving 27dBm sent into the AOMs. At this applied power, the modulation efficiency of the AOMs is about 50%. After the AOMs, the beams are collimated by an $f = 250$mm lens, DC blocked, and focused to the center of the Tm$^{3+}$:YAG crystal by a lens. To get different focus spot sizes, we use $f = 150$mm, 200mm, 250mm and 300mm lenses in the experiments. To vary the interaction angle of beam 1 and beam 2 in the crystal, we mount M3 on a translation stage that changes the beam separation before the lens in front of the crystal and therefore the beam interaction angle. We use a Thorlabs WM-100 beam profiler to measure the beam separation distance before the lens and use the separation divided by the focal length of the lens to calculate the interaction angle. A flipping mirror M4 in front of the cryo and a knife edge beam profiler (Thorlabs WM-100) at the same virtual position of the crystal center help to align the beam focus and beams overlap to the crystal center. Beam 1 is blocked after the cryo. Beam 2 (the echo also appears in this direction) is first collimated by an $f = 250$mm lens then focused into the gate AOM that turn on only after the read chirp to let the echo diffract. The sinusoidal gating signal for the gate AOM is a bandpass filtered square wave burst from the marker output of AWG520 amplified by 24dB to 27dBm. The diffracted signal from the gate AOM is focused by an $f = 50$mm lens into an APD. The output of the APD is sent to an oscilloscope. In order to automate the data-taking process, the AWG520 and the oscilloscope are both controlled by LabVIEW programs.

With four different focal length lenses, we take the 3-pulse echo data for various interaction angles and different time delays ($t_{d1}$ in Fig. 4.25) from 20μsec to 140μsec. The integrated echo strengths with different interaction angle and time delays are plotted in Fig. 4.26. Since we use a mirror (M3) to control the distance (thus the interaction angle) of the two beams, we can not access the beam separations of less than the beam size and with Gaussian beams and mirror edges this results in clipping of the weak beam edges as the beams get close. With the available data, we can see that as time delay $t_{23}$ increases, the echo strength decreases as the population decays to the ground state. Interestingly, except for Fig. 4.26(a), the case with $f = 150$mm, some of the off-axis
Figure 4.26: Experimental results of integrated photon echo intensity vs. interaction angle for different focal spot sizes at the center of the SSH crystal. Time delay: $t_{23}$.

Photon echo measurements are higher than the on-axis echo measurements, especially for the small angle, small $t_{23}$ time delay cases. However, as angle increases or as $t_{23}$ increases, the on-axis echo strength tends to be stronger than the off-axis echo strength. The reason the $f = 150$mm case is different from the other cases is probably because we can not measure echo strengths for interaction angles less than 2.5°.
4.6 Discussions

Comparing the simulation results with the experiment results, we can see that the experiment results resemble the feature of the simulation results: as the interaction angle increases, the echo strength first increases then decreases. What could be the possible reason for this phenomenon? The overlapped intensity weighted area of the two beams plays an important role. Shown in Fig. 4.27 is the product of beam 1 and beam 2 in the simulation setup Fig. 4.22 changing with different beam interaction angle. This, however, does not explain why some of the off-axis echo strengths are stronger than the on-axis echo strength.

For the experiment results, the on-axis (when interaction angle is $0^\circ$) interaction case is done by using a half waveplate and a PBS. Due to the imperfectness of the waveplate and the PBS, not all the light will be used for the on-axis 1D photon echo experiment. Therefore, the on-axis 1D photon echo experiment could have less photon echo strength than the off-axis 2D photon echo experiment. In addition, the mirror M3 in Fig. 4.25 could be blocking part of beam 2 when the two beams have small interaction angles. This could be another reason that the on-axis experiment has higher echo efficiency than the off-axis echo experiment. A better way to combine these two beams is to use a beam-splitter (and still use a mirror on a translation stage to change the separation of
the beams). Although a beam-splitter will throw away half of the power, it won’t have the beam blocking issues and will allow smaller beam interaction angles closer to 0°.
Chapter 5

High Bandwidth SSH scanner

5.1 Introduction

True time delays (TTDs) are useful in avoiding beam squint for phased-array antennas [88], radar return signal correlations to find the range of remote targets [89], and time-frequency analysis of time-varying spectra [28]. An example of a device that generates such time delays is an acousto-optic deflector (AOD) [44]. An AOD generates linear time delays across its aperture by the propagation delays of the acoustic waves. It works in the following way: the RF signal of interest drives the piezo-electric transducer of the AOD which vibrates and broadcasts acoustic waves traveling across the aperture of the AOD. It takes a certain time for the acoustic waves to travel to any specific position within the aperture. Depending on the distance away from the piezo-electric transducer, acoustic waves experience different time delays at different positions across the AOD aperture. The incident light that is deflected by the acoustic waves will be modulated by the frequency content of the acoustic waves and so time delayed in proportion to the position of the acoustic wave. Therefore, we get different time-delayed signals across the AOD aperture. However, several factors limit the bandwidth of the AODs: phase matching condition, acoustic attenuation, material birefringence, and transducer technology [6]. For a certain diffraction efficiency, only a range of the acoustic frequencies can satisfy the phase matching conditions. Acoustic attenuation which increases as the square of the RF frequency [6], can limit the AOD bandwidth to a few GHz. Birefringent AODs have higher bandwidth than their isotropic counterparts, but the bandwidth for birefringent AODs is limited by the refractive indices difference of the birefringent
Figure 5.1: The rare earth doped crystals can be modeled by two-level atoms. The doped Tm$^{3+}$ ions experience different local environment depending on the positions they are at. And therefore the resonant frequency of each of the atoms will be shifted with respect to each other, resulting in inhomogeneously broadened bandwidth. When light at the resonant wavelength, 793.3nm for Tm$^{3+}$:YAG, impinges on the crystal, ions are excited from the ground state to the excited state, changing the population inversion at that resonant frequency and leaving behind a 'spectral hole' in the absorption profile of the crystal. The ions stay at the excited state for a lifetime of 10msec and then relax back to the ground state.

crystal. For a LiNbO$_3$ crystal AOD with octave bandwidth, the bandwidth is limited to about 5GHz at 532nm. An AOD transducer needs to have low acoustic attenuation and have matched mechanical impedance to the acoustooptic materials. Of all the constrains that AODs have, the highest bandwidth an AOD can achieve could be about 5GHz.

In this chapter, we will introduce a new approach to building a traveling wave deflector for signals up to 20GHz of bandwidth by using a spatial-spectral hole-burning (SSH) crystal. Spectral hole burning crystals are crystals, such as YAG, LiNbO$_3$, etc, with dopant rare-earth ions (Tm$^{3+}$, Er$^{3+}$, etc.), which exhibit a large inhomogeneous bandwidth. In this chapter, we use a Tm$^{3+}$:YAG
crystal. When cryogenically cooled to about 4K, the homogeneous linewidth of Tm$^{3+}$:YAG crystal can be under 25kHz while the inhomogeneous bandwidth can exceed 20GHz, yielding nearly $10^6$ of resolvable spectral channels. The inhomogeneously broadened 2-level atoms are resonant to different frequencies and change their population inversion accordingly at that specific incident frequency, resulting in spectral hole features in the absorption band as shown in Fig. 5.1. As a consequence, the power spectrum of a modulated incident light beam can be recorded in the inhomogeneous band. This property has many applications. The inhomogeneous band of the crystal can be employed for variable true-time delay programming[17], spectrum analysis[18-20], and data storage [21, 22]. When adding the spatial dimension of the crystal, we record spatial-spectral holograms into the crystal and will have more computational power and processing flexibility: Colice et. al. [23] uses the spatial dimension for pulse repetition frequency cuing RF spectrum analysis; Schlottau et. al.[24] for time-integrating spectrum analysis; Braker et. al.[25] for squint-free RF beamforming. In this paper, we will use both the spatial and spectral dimension of the crystal to record different time delays for ultra-wide bandwidth signal processing. In the following text, we refer to this system as a “SSH scanner”.

This chapter is organized as follows. Section 5.2 introduces the basic idea of the SSH scanner. Section 5.3 analyzes the theory for programming and using the SSH scanner. In the subsections, we start with a general setup. By making some approximations and simplifications, we mathematically show the basic idea of the high bandwidth SSH scanner. Then by taking off the approximations one by one, we analyze the non-ideal situations that impact the performance of the experiments. Section 5.4 outlines the simplified version of the experiment by using one laser for the programming and the reading process and gives corresponding experimental results. Section 5.5 gives the full version of the experiment, identifying the sub-systems that have been realized in this paper and the parts that are yet to be done to realize an entire system for ultra-wideband arbitrary signal processing using a SSH scanner.
5.2 The general picture

The SSH scanner is a system that with any inputs within its bandwidth, its outputs are copies of the input signal with spatial-position-dependent time delays. To understand how it works, first consider how to generate a specific time delay in a SSH crystal. Since time delays and linear phase ramps (or complex exponentials) are Fourier duals of each other, generating linear phase ramps in the temporal frequency domain, or in this case the spectral frequency gratings in the SSH crystal, will be equivalent to generating time-delays directly (with causality coming into play). This is the basis of time-delayed photon echoes [67]. With sinusoidal gratings recorded in the inhomogeneous band of the SSH crystal, a readout signal that needs to be time-delayed can excite a time-delayed echo. As a further step, in order to generate time delays proportional to the spatial positions, in the temporal frequency domain, we need to generate linear phase ramps (or sinusoidal grating) that change the grating period with spatial position. The whole process is illustrated in Fig. 5.2. Suppose we have recorded sinusoidal gratings across a wide bandwidth in the spectral domain of the SSH crystal and these sinusoids have different periodicities at different spatial positions. When illuminating the SSH crystal with a time-domain brief pulse, at each spatial position of the crystal,
the input pulse will read out spectral domain gratings with different periodicities, generating echoes with time delays that are inversely proportional to the grating periodicities. By doing this, we can generate a whole bank of spatially dependent linear time delays. The inhomogeneous bandwidth of the SSH crystal will be an upper limit of the bandwidth of our SSH scanner, which is about 20GHz for Tm$^{3+}$:YAG crystal. So by using Tm$^{3+}$:YAG, we can generate time delays like an AOD, and further more with potentially 20GHz of bandwidth.

Schlottau [24] did a low bandwidth version of the experiment, in which he recorded the array of time delays generated by an AOD in the spatial-spectral domain of the Tm$^{3+}$:YAG with forms of sinusoidal spectral gratings varying in space. The bandwidth in his case, however, was limited by the AOD used for recording, so this demonstration does not have technological advantage over the AOD by itself in terms of bandwidth limitation. In contrast, in this paper, we present an SSH scanner that can scale to many tens of gigahertz of bandwidth without the fundamental limitations that constrain the AOD technology.

Now how can we record those spatial-spectral gratings with changing periods in the inhomogeneous band of the SSH crystal? This paper proposes a way of recording sinusoidal gratings in the inhomogeneous band of the SSH crystal without limiting the system bandwidth to that imposed by the AOD bandwidth. Recall that the Fourier transform of two identical chirps (chirp rate is $b$)
with a time delay $\tau$ between them, $\mathcal{F}\{\exp(i\pi b t^2) + \exp[i\pi b(t - \tau)^2]\}$, is a spectral chirp (Fourier transform of the original chirp) times a spectral sinusoid, $\exp(i\pi f^2) \cdot 2\exp(i\pi f\tau)\cos(\pi f\tau)$, the power spectrum of which, $|S(f)|^2 = 4\cos^2(\pi f\tau) = 2 + 2\cos(2\pi f\tau)$, can engrave sinusoidal spectral gratings that has a periodicity of $\frac{1}{\tau}$ in the inhomogeneous band of the SSH crystal. Actually, beside chirp, any waveform with a flat power spectrum spanning the required bandwidth can be used to produce sinusoidal spectral gratings, but a chirp is readily produced over an extremely wide bandwidth by tuning a laser so this is the implementation utilized here. In addition, we show in Fig. 5.3 that by frequency shifting such a chirp, we can produce an effective time shift (other waveforms do not have this property), which is the key to programming the SSH delay line. The spectral period of the spectral grating is inversely proportional to the time delay between the two identical chirps. One way of looking at the two chirps is: each instantaneous frequency of the chirp is time delayed by the same amount across the whole chirp bandwidth. Another way of seeing the two time delayed chirps is: at each point in time, there is a frequency shift between the two chirps. The time shift $\Delta t$ and frequency shift $\Delta f$ is related by the chirp rate $b$ with $\Delta f = b\Delta t$. Therefore, if we can generate frequency shifts between two identical time-domain chirps and the frequency shifts are linearly proportional to the spatial position, we will be able to record linear time delays along the spatial position and in the inhomogeneous band of the SSH crystal.

There are two ways of getting linear frequency shifts with respect to spatial position, i.e., $\exp(i\omega yt)$ [54]:

1. Use two oppositely shifted AODs, one is up-shifted, the other down-shifted, that are driven by identical chirps. The first AOD is imaged onto the second AOD. The advantage of this approach is its bandwidth. It can provide MHz of Doppler shifts, higher than the second approach. The drawback is that it has low efficiency due to double diffractions from the AODs.

2. Use a Galvo Scanning (GS) mirror that is driven by a linear ramp signal, as is shown in

---

1 Define Fourier transform of a function $g(t)$ as $G(\omega) = \mathcal{F}\{g(t)\} = \int_{-\infty}^{+\infty} g(t) \exp(-i\omega t)dt$
Figure 5.4: (a) Galvo scanning mirror that pivots about the center of the mirror. During the scan, the mirror rotates with a constant angular velocity of $\Omega$. Within a small angular rotation range, light near the mirror surface with $y$ coordinate experiences a frequency shift of $f_d \approx 2\nu \cos \alpha$. (b) At position $y_2$ on the surface of the mirror, the light is Doppler shifted by a frequency of $\Delta f_2 \approx -\frac{2\nu y_2}{\lambda} \cos \alpha$, $y_2 < 0$. Since the mirror is moving away from the light the Doppler shift is negative. (c) At time $t_1$, light along the $y$-direction of the mirror experiences Doppler shifts that linearly vary with the spatial position. (d) At position $y_1$ on the surface of the mirror, the light is Doppler shifted by a frequency of $\Delta f_1 \approx -\frac{2\nu y_1}{\lambda} \cos \alpha$. Since the mirror is moving towards the light, the Doppler shift is positive.

Fig. 5.4. When the GS mirror is driven by a linear voltage ramp, it rotates with a constant angular velocity until the ramp resets. We only use the parts when the GS mirror rotates in the same direction (either clockwise or counter clockwise but not both). For the $y > 0$ half plane in Fig. 5.4(a), the mirror rotates away from the incident light and gives the incident light a negative Doppler shift, $f_d \approx -\frac{2\nu y}{\lambda} \cos \alpha$ (assuming that the mirror rotation range is small)[29], where $\alpha$ is the light incident angle onto the mirror. While for the $y < 0$ half plane, a positive Doppler shift is produced. The incident light at each position $y$ is Doppler shifted for a frequency that is proportional to the position $y$. When the incident light is a linear chirp, at each spatial position, the frequency shift can be viewed as a time shift.
which is proportional to the frequency shift. Therefore, by using a GS mirror, we can get
a linear time shift that is proportional to the spatial position $y$,

$$\Delta t = \frac{\Delta f}{b} \approx \frac{2\Omega y}{b\lambda} \cos \alpha.$$ (5.1)

This approach is more photon efficient, but the bandwidth of the Doppler shifts is limited
to about 15KHz (with 100Hz of 100mVpp ramp, at 532nm normal incident light) by the
rotation speed of the GS mirror and the size of the mirror. 20KHz requires a velocity of
$2 \cdot 10^4 \lambda$ per second, so a 100Hz ramped mirror must have an edge that moves 200\lambda.

Since light efficiency is important to us, we choose the GS mirror approach for the following
experiments. Therefore, all of the theoretical analysis in this paper is also devoted to the GS mirror
approach. But the analysis for the dual AOD approach won’t be very different.

Let’s put the idea of the SSH scanner in a more specific experimental setup. A simplified
experimental layout of the high bandwidth SSH scanner is shown in Fig. 5.5. Two lasers are
used in the SSH scanner. The frequency scanned writing laser along with the GS mirror records
sinusoidal gratings in the SSH crystal. The high bandwidth signal that is needed to be time-delayed
is modulated onto the CW reading laser beam by using an electro-optic modulator (EOM). Two
choppers are used to control the on and off of each of the laser beams for the recording and reading
process. In the recording process, chopper 1 opens, chopper 2 closes. A linear ramp signal (control
1) chirps the writing laser and simultaneously angularly scans the GS mirror to write sinusoidal
gratings in the inhomogeneous band of the SSH crystal. In the reading process, chopper 2 opens
and chopper 1 closes. A high bandwidth signal is modulated onto the CW reading laser carrier
and arrives at the SSH crystal to read out the space dependent time delays, producing echoes with
different time delays across the SSH crystal aperture. To verify these time delays, we delay the
same signal that goes into the SSH crystal with a known time delay and observe the interferometric
time-integration on a CCD.
Figure 5.5: Experimental design of the high bandwidth SSH scanner. Writing process: Chopper 1 opens. Chopper 2 closes. The writing laser and the GS mirror are controlled by a linear ramp signal (control 1) to record sinusoidal gratings in the SSH crystal. Reading process: Chopper 2 opens. Chopper 1 closes. The reading laser is modulated by a signal of interest with an EOM. The modulated beam reads out the time delays recorded in the SSH crystal, resulting in time delayed echoes. To verify the time delays, a small portion of the modulated signal beam is time delayed with a known amount and interferes with the echoes on a CCD.

5.3 Theoretical analysis of SSH scanner

The recording process prepares the SSH crystal as an SSH scanner. A frequency tunable laser and a GS mirror are required for the recording process. The tunable laser, in our case a 4KHz linewidth 1586nm Koheras fiber laser which is frequency doubled to 793nm, and the GS mirror are driven by the same ramp signal

\[ r(t) = \frac{1}{4T} t \Pi \left( \frac{t}{T} \right) \ast \text{comb} \left( \frac{t}{4T} \right), \]

where \( \Pi(t) = 1 \) (\(|t| < \frac{1}{2}\)), \( \Pi(t) = 0 \) (\(|t| > \frac{1}{2}\)), * is convolution and \( \text{comb}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n) \). \( r(t) \) has a unit of V. The frequency tuning coefficient of the piezo transducer of the Koheras laser is \( c_l \) with a unit of Hz/V and the angular scanning coefficient of the GS mirror is \( c_m \) with a unit of rad/V. The instantaneous frequency of the Koheras laser is \( \omega_{k0} \) with \( \frac{1}{2} \omega_{k0} = \omega_0 \). The 1586nm beam
from the Koheras laser is frequency doubled in a temperature-controlled 40mm-long periodically-poled LiNbO\(_3\) (PPLN) crystal and divided into two beams that interfere with each other in the SSH crystal. The two beams interact with each other in the horizontal plane. One beam goes directly to the SSH crystal

\[
E_1(x, y, t) = \exp(i\omega_0 t) \cdot \exp[i\pi c l r(t) t] \cdot \exp(ik \sin \theta_1 x) \cdot g(x, My), \tag{5.3}
\]

where \(\theta_1\) is the beam 1 incident angle on the SSH crystal in the horizontal plane \(k = \frac{\omega_0}{c}\), \(\omega_0\) is the instantaneous angular frequency of the frequency doubled laser and

\[
g(x, My) = \exp \left[ -\frac{x^2 + (My)^2}{W_0^2} \right], \tag{5.4}
\]

where \(M < 1\) and \(W_0\) is the beam half width in the horizontal direction at the crystal. Since the laser with frequency \(f_0 = \frac{\omega_0}{2\pi}\) is driven by a linear ramp giving a linear chirp, the instantaneous frequency \(f = \frac{2\pi}{c}[f_0 + c l r(t)]\). The other beam is modulated by the GS mirror that scans in the vertical direction (rotates about a horizontal axis at the center of the beam)

\[
E_2(x, y, t) = \exp(i\omega_0 t) \cdot \exp[i\pi c l r(t) t] \cdot \exp(ik \sin \theta_2 x + ik \sin(\alpha + 2c_m r(t)) y) \cdot \exp\{i^2 \cdot \frac{2\pi}{c} \cdot 2c_m \cos \alpha \cdot [f_0 - c l r(t)] r(t) y\}, \tag{5.5}
\]

where \(\theta_2\) is the beam 2 incident angle on the SSH crystal in the horizontal plane and \(\alpha\) is the beam incident angle on the GS mirror in the vertical plane when the mirror is stationary. When \(2c_m r(t) \ll 1\), \(\sin(\alpha + 2c_m r(t)) \approx \sin \alpha + 2c_m r(t) \cdot \cos \alpha\) and the phase term due to the GS scanning mirror can be written as

\[
E_2(x, y, t) = \exp(i\omega_0 t) g(x, My) \exp[i(k(\sin \theta_2 x + \sin \alpha y))] \times \exp[i\pi c l r(t) t] \exp\left\{\frac{2\pi}{c} \cdot 2c_m \cos \alpha \cdot [f_0 - c l r(t)] r(t) y\right\}, \tag{5.6}
\]

where the second term in the last exponential \(i^2 \cdot \frac{2\pi}{c} \cdot 2c_m c l r^2(t) y\), is due to the combined effect of both GS mirror scanning and laser chirping. When \(c l r(t) \ll f_0\), i.e., when the chirping range of the laser is small in comparison to the central frequency of the laser which is almost always the case in
experiments, we can neglect this term (for a typical scanning range, \( \frac{cl_{\text{max}}}{f_0} \approx 10^{-6}, \frac{cm_{\text{max}}}{2\pi} \approx 10^{-3} \)). And after a little algebra,

\[
E_2(x, y, t) = \exp(i\omega_0 t)g(x, My) \exp[ik(\sin\theta_2 x + \sin\alpha y)]
\]

\[
\times \exp \left[ i\pi\beta \left( t - \frac{2}{\lambda_0 c_t} y \cos\alpha \right) \cdot \frac{1}{4T} \Pi \left( \frac{t}{T} \right) \ast \text{comb} \left( \frac{t}{T} \right) \right]
\]

\[
\times \exp \left\{ i\pi\beta \left( \frac{2}{\lambda_0 c_t} y \cos\alpha \right) \cdot \frac{1}{4T} \Pi \left( \frac{t}{T} \right) \ast \text{comb} \left( \frac{t}{T} \right) \right\},
\]

The last spatial quadratic phase term is negligible (\( \frac{cm_{\text{max}}}{\lambda_0^2 c_t} \approx 10^{-6} \)) since the chirping range of the laser is far more than the Doppler shifts generated by the mirror rotation. The position dependent time delay, \( \frac{2}{\lambda_0} \frac{cm}{c_t} y \cos\alpha \) in Eq. (5.7) is generated by the Doppler due to the mechanical scanning of the GS mirror. It is proportional to the spatial position \( y \) of the beam on the mirror. Since the GS mirror rotates about the central horizontal axis, it generates positive Dopplers in one half plane and negative Dopplers on the other half during a single-sided (upside or downside) ramp, as shown in Fig. 5.4. At each spatial position, the material sees two identical time-delayed chirps. Due to the linear Doppler shifts proportional to spatial positions, the time delays at each spatial position is also proportional to the spatial coordinates. The spectral grating written in the SSH crystal by these two beams can be written as \( E_1^*(\omega)E_2(\omega) + E_1(\omega)E_2^*(\omega) \). Considering only the grating that gives causal diffraction (the diffraction that happens at a later time instead of at an earlier time), the spectral grating is given by [21, 90],

\[
E_1^*(\omega)E_2(\omega) = \mathcal{F} \{ B(t) \cdot E_1(x, y, t) \}^\ast \mathcal{F} \{ B(t) \cdot E_2(x, y, t) \}
\]

\[
= \exp \left( -i \frac{2cm}{c_t \lambda_0} y \cos\alpha \cdot \omega \right) \exp \{ ik[\sin(\theta_2 - \theta_1)x + \sin\alpha y] \},
\]

where \( B(t) = \frac{1}{\pi T} \Pi \left( \frac{t}{T} \right) \ast \text{comb} \left( \frac{t}{m T} \right) \) is the gate that controls the recording pulses and is on a much slower time scale than the resolvable spectrum. To maximize the number of photons for the recording process, we make \( T \) about half the lifetime of the crystal. To minimize the interference from the adjacent recordings, we wait until the material relaxes by \( \exp(-4) \) before the next recording, i.e., \( n = 4 \). At a specific spatial position \( y = y_0 \), the two linear chirps with constant time delay \( \frac{2}{\lambda_0} \frac{cm}{c_t} y_0 \cos\alpha \) write sinusoidal gratings \( \cos \left( \frac{2cm}{c_t \lambda_0} y_0 \cos\alpha \cdot \omega \right) \) with constant spectral periodicity...
\frac{c \lambda_0}{2c_m y_0 \cos \alpha}$ in the spectral domain of the crystal. Notice that this result agrees with the result, Eq. (5.1), the result we get in section 2 \( \frac{\Gamma[\text{rad/sec}]}{\vartheta[\text{Hz/sec}]} = \frac{c_m[\text{rad/V}]}{c_l[\text{Hz/V}]} \).

In the reading process, the broadband signal beam \( s(t) \) is modulated on a stabilized velocity laser with frequency \( \omega_{v0} = 2 \omega_{k0} = \omega_0 \), where \( \omega_{v0} \) is the frequency of the velocity laser and \( \omega_{k0} \) is that of the Koheras laser. The reading beam comes at a latter time in the direction of beam 1

\[
E_3(x, y, t) = \exp(i \omega_0 t) \cdot s(t) \cdot \exp(i k \sin \theta_1 x) \cdot g(x, My)
\]

(5.9)

to read out the spectral grating, where \( s(t) \) is the time domain readout signal. We write \( s(t) \) as

\[
s(t) = s'(t) \cdot \frac{1}{4T} \Pi \left(\frac{t - T}{T}\right) \ast \text{comb} \left( \frac{t}{4T} \right),
\]

(5.10)

where \( s'(t) \) is an ideal infinitely long signal. \( s(t) \) consists of copies of windowed \( s'(t) \). So the echo is

\[
E_e(x, y, t) = \exp(i \omega_0 t) \int E_1^*(\omega) E_2(\omega) S(\omega) \exp(-i \omega t) d\omega
\]

(5.11)

\[
= \exp(i \omega_0 t) \exp[i \kappa (\sin \theta_2 x + \sin \alpha y)] g(x, My) s \left( t - \frac{2c_m}{c_l \lambda_0} y \cos \alpha \right),
\]

which means the echo is a time-delayed version of the read beam with time delays proportional to the spatial position \( y \) and in the direction of beam 2. Remember that the GS mirror could generate both positive and negative Dopplers, which means the time delay could be both positive (when \( y > 0 \)) and negative (when \( y < 0 \)) mathematically. However, due to causality, one can’t get the echo before the readout beam occurs. So diffraction happens only for \( y > 0 \) spatial positions. Notice that Mossberg’s integral [90] manually select causality of the writing and reading pulses by choosing the \( E_1^*(\omega) E_2(\omega) \) term. However, this manually selection process can not consider the fact that the GS mirror gives both positive and negative Dopplers and thus positive and “negative” time delays. For this reason, Eq. (5.11) does not show the partial diffraction effect. The next subsection talks about how to use the spectral-domain Mitsunaga’s integral to derive the echo equation.

### 5.3.1 Diffraction off the causal time edge

Mitsunaga’s perturbation theory [67] can describe the third order photon echo effect. Mitsunaga’s analysis is in the time domain. To better interpret the result, Colice [4] (in chapter 3)
converts the time domain perturbation theory into the frequency domain. Following Colice’s approach, if we consider only the causality of the two writing beams and neglect the effect of finite lifetime $T_1$ and decoherence time $T_2$ on the spectral gratings, the echo can be written as

\[ E_e(x, y, t) = \mathcal{D} \int d\omega_3 E_3(x, y, \omega_3) \exp(i\omega_3 t) \int E_2(x, y, \omega_2)E_1^*(x, y, \omega_2)\gamma(\omega_3 - \omega_2)d\omega_2 \]

\[ = \mathcal{D} \int E_3(x, y, \omega)\{[E_1^*(x, y, \omega)E_2(x, y, \omega)] \ast \gamma(\omega)\} \exp(i\omega t) d\omega \]

\[ = \mathcal{D}E_3(x, y, t) \ast \{\mathcal{F}^{-1}\{E_1^*(x, y, \omega)E_2(x, y, \omega)\}H(t)\}, \tag{5.12} \]

where

\[ \gamma(\omega) = \frac{1}{i\omega} + \frac{1}{2}\delta(\omega). \tag{5.13} \]

The inverse Fourier transform of $\gamma(\omega)$ is a Heaviside function $H(t)$,

\[ \mathcal{F}^{-1}\{\gamma(\omega)\} = H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}. \tag{5.14} \]

Substituting the spectral grating in Eq. (5.8) into Eq. (5.12), we get

\[ E_e(x, y, t) = \mathcal{D} \exp[ik \sin(\theta_2 - \theta_1)x] \mathcal{F}^{-1}\left\{E_3(x, y, \omega) \left[ \exp\left(- \frac{2c_m}{c_1 \lambda_0} y \cos \alpha \cdot \omega \right) \ast \gamma(\omega) \right] \right\} \]

\[ = \mathcal{D} \exp[ik \sin(\theta_2 - \theta_1)x]E_3(x, y, t) \ast \left\{ \delta \left( t - \frac{2c_m}{c_1 \lambda_0} y \cos \alpha \right) \cdot H(t) \right\} \]

\[ = \mathcal{D} \exp[ik \sin(\theta_2 - \theta_1)x] \int E_3(x, y, t - \tau)\delta \left( \tau - \frac{2c_m}{c_1 \lambda_0} y \cos \alpha \right) H(\tau) d\tau \]

\[ = \mathcal{D} \exp(i\omega_0 t) \exp[ik(\sin \theta_2 x + \sin \alpha y)]g(x, My)s \left( t - \frac{2c_m}{c_1 \lambda_0} y \cos \alpha \right) \cdot H(y). \tag{5.15} \]

Due to $H(y)$, there is only diffraction in half of the spatial plane where $y > 0$. Thus, the time domain edge function $H(t)$ is transformed into spatial domain. Notice that we always define the negative Doppler half plane of the mirror as the $y$-positive direction. Echoes will only diffract from the half plane that generates negative Dopplers and thus positive time delays, as is illustrated in Fig. 5.4. Rebane et. al [91, 92] performed similar experiments by using femtosecond laser pulses which they referred to as “diffraction off the causal edge”. When two pulses with a certain spatial
beam width and inclined at a small angle incident onto the SSH crystal simultaneously, part of the pulse (say pulse 1) arrives earlier than the other pulse (say pulse 2) at the same spatial position: let’s call this half plane A. The rest of pulse 1, however, arrives latter than that of pulse 2: let’s call this half plane B. If a third pulse arrives after the SSH material has decohered and in the direction of pulse 1, it reads out only the half plane A where pulse 1 arrives earlier than pulse 2. The theoretical analysis of [91]-[92] uses Mossberg’s [90] integral which first neglects the causality of Mitsunaga’s [67] integral. Then it uses causality to get rid of one of the diffraction orders in time domain and Fourier transform back into spectral domain to observe the spectrum. Instead of explaining the causal edge effect this way, we use the spectral domain representation of Mitsunaga’s integral to explain it. Since Mitsunaga’s integral considers the timing orders of the signals, its spectral domain representation derived by Colice[4] naturally contains the causality effect. Therefore, the above equation, Eq. (5.12), derived from Colice’s spectral domain perturbation theory considers the causality of the pulses. By using this equation, one does not need to manually select the grating diffraction order that is causal.

5.3.2 Finite recording bandwidth in the spectral domain

When the bandwidth of the recorded spectral gratings, Δ, is finite, say varying from (ω₀, ω₀ + Δ), Eq. (5.12) becomes

\[
E_e(x, y, t) = \mathcal{D} \exp[i k \sin (\theta_2 - \theta_1) x] \int E_3(x, y, \omega) \times \left\{ \exp \left( -i \frac{2 c_m}{c_l \lambda_0} y \cos \alpha \cdot \omega \right) \Pi \left[ \frac{\omega - (\omega_0 + \Delta/2)}{\Delta} \right] \right\} \ast \gamma(\omega) \exp(i \omega t) d\omega \\
= \mathcal{D} \exp(i \omega_0 t) \exp[i k (\sin \theta_2 x + \sin \alpha y)] g(x, My) \times \int s(t - \tau) \text{sinc} \left[ \Delta \left( \tau - \frac{2 c_m}{c_l \lambda_0} y \cos \alpha \right) \right] H(\tau) d\tau \\
= \mathcal{D} \exp(i \omega_0 t) \exp[i k (\sin \theta_2 x + \sin \alpha y)] g(x, My) \times s(t) \ast \left\{ \text{sinc} \left[ \Delta \left( t - \frac{2 c_m}{c_l \lambda_0} y \cos \alpha \right) \right] H(t) \right\},
\]

Since the sinc function is not a finite duration function, \(\text{sinc}[\Delta(t - \frac{2 c_m}{c_l \lambda_0} y \cos \alpha)]H(t) \neq 0\) no matter whether \(t > 0\) or \(t < 0\). In addition, the sinc function in Eq. (5.16) relates time to space. It means:
at a certain spatial position $y$, the sinc function is delayed by $\frac{2cm}{c/\lambda_0} y \cos \alpha$. And the sinc is truncated by $H(t)$, as is shown in Fig. 5.6. For spatial positions $y > 0$, the center of the sinc is not truncated, so the echo is big. For spatial positions $y < 0$, the center of the sinc is truncated and the echo is small, but it is not zero. Although causality introduces a clear-cut edge $H(t)$ in the time domain, in the spatial domain, there are still diffractions into the “forbidden” $y < 0$ half plane.

5.3.3 Nonlinearity of the “ramp”

Now we want to evaluate the consequence of an important but troublesome experimental detail: the nonlinearity of the ramp functions driving the laser piezo and GS mirror. Instead of assuming the input voltage to the GS mirror and the laser piezo is a linear function of time, suppose the GS mirror and the laser are driven by ramps approximated by $m$ orders of sines, as an approximation to a symmetric triangle function, as illustrated in Fig. 5.7(a) (the figure only shows half of the period) for $m = 1, 2, \cdots, 8$
Figure 5.7: (a) Including successively refined Fourier harmonics 1 – 8 for the bandwidth limit approximation of the ramp. Offsets are for clarity only. (b) Error $r(t) - r_m(t)$ between the ramp approximations $r_m(t)$ and the linear ramp $r(t)$.

\[ r_m(t) = \frac{AT}{\pi^2} \sum_{n=0}^{n=m} (-1)^{n+1} \frac{\sin(2n+1)\omega t}{(2n+1)^2}, \]  

where $A$ is the amplitude of the ramp and $T$ is the periodicity of the ramp. Now the two writing beams, Eq. (5.3) and Eq. (5.6), become

\[ E_1(x, y, t) = \exp(i\omega_0 t)g(x, My)\exp(ik\sin\theta_1x)\exp[i\pi cr_m(t)t], \]  

\[ E_2(x, y, t) = \exp(i\omega_0 t)g(x, My)\exp[ik(\sin\theta_2x + \sin\alpha y)] \times \exp[i\pi cr_m(t)t] \exp \left[ i\frac{2\pi}{\lambda_0} \cdot 2c_mr_m(t)y \cos\alpha \right], \]

Instead of having two linear chirps with constant time delays recording sinusoidal gratings with constant periodicities, the two recording beams will be two nonlinear chirps and the Doppler shifts picked up from the GS mirror will not be constant in time. The spectral domain gratings recorded by these two beams will be sinusoids but with varying periodicities, as shown in Fig. 5.8. When $m = 1$, the center of the grating has bigger periodicity than the edge of the grating. When $m = 2$,
the center has smaller periodicity. When $m = 3$, there are two symmetric denser grating periods areas and as $m$ increases, the denser grating areas move towards the edges of the gratings.

The nonlinearity of the response of the GS mirror or of the laser piezo also affects the linearity of the chirps for the recording process, which is similar to the effect of nonlinear ramp. In Fig. 5.9, we compare the various factors that affects the spectral domain grating. Sinusoidal writing chirps make the grating sparser in the center; Sinusoidal frequency shifts make the grating denser in the center; cubic mirror nonlinear term ($x^3$) makes the grating denser at the edge. In fact, with a known nonlinearity, we can precompensate the nonlinearities by appropriate Tayloring of the Fourier amplitude coefficients and eliminate this nonlinearity. More troublesome is noise, non-
 repeatability, hysteresis, and jitter in both the laser scanning piezo and GS scanner, since these can not be precompensated. In the experiments, the grating written in the spectral domain are combinations of all these effects. We will give the experimental results that show the different effect by taking different numbers of sinusoidal approximation orders.

5.4 Experimental setup for 1-laser high bandwidth scanner

To avoid the complications of two-laser locking, we use a frequency-doubled narrow-linewidth stable Koheras Boostik fiber laser for both the writing and reading process to demonstrate the concept for the high bandwidth SSH scanner system. The Koheras laser is an extremely stable (2KHz linewidth) yet rapidly sweepable with a piezo-electric Bragg grating stretcher operating at a wavelength of 1586nm. We use the same frequency doubling setup that Colice [4] used to double
Figure 5.10: High bandwidth SSH scanner experimental setup. The Koheras fiber laser is frequency doubled by a PPLN, amplified by a BoosTA amplifier, and spatial filtered by a PM single-mode fiber. The light is split into two arms. Two AOMs control the writing and reading process of the experiment. One of the writing arms is vertically scanned by a GS mirror. The two elliptical beams overlap at the SSH crystal in a cryo. These two beams record spatially varying spectral population gratings in the inhomogeneous band of the SSH crystal. After recording, beam 1 comes in to read the recorded gratings and is diffracted into the direction of beam 2. After the cryo, beam 1 is blocked. By flipping up/down the flipping mirrors M1 and M2, the latter part of the setup can be used to do two kinds of experiments: a spectral grating readout experiment that uses the photo-detector and a time-integrating correlator experiment that uses the time-delay bouncing mirrors and the CCD camera.

The output frequency of the Koheras from 1586nm to 793nm in order to access the inhomogeneous band of the Tm$^{3+}$:YAG crystal. This setup is shown in the rectangle on the right corner of Fig. 5.10. The output of the Koheras fiber laser is collimated to a 1mm diameter beam by a fiber collimator. The output polarization of the Koheras laser is elliptical. After the laser has warmed up for several hours, the ellipticity still varies with time, but the major axis of the elliptical polarization remains fixed. A $\lambda/4$ waveplate and a $\lambda/2$ waveplate can align the polarization to the linear state needed by the
cascaded pairs of OFR optical isolators at 1550nm used for preventing reflections back to the fiber laser. Each OFR isolator consists of two prism polarizers at both ends and a 45° Faraday rotator at the center. However, since the polarization is mostly linear \( I_\xi/I_\eta > 100 \), where \( I_\xi \) and \( I_\eta \) are the light intensity along the major and minor axes of the polarized light) and the ellipticity of the light is changing randomly on a time scale of minutes (means the \( \frac{\lambda}{4} \) waveplate needs to be rotated accordingly), we only use a tilted tunable Alphalas \( \frac{\lambda}{2} \) waveplate for preparing the polarization. The power of the light after the cascaded optical isolators is about 500 – 600mW.

**Frequency doubling**

We use a 40mm-long Stratophase periodically poled LiNbO₃ (PPLN) for frequency doubling. Boyd and Kleinman show that the maximum conversion efficiency occurs when the total length of the crystal for beams at normal incidence is about 2.84 times the confocal parameter of the input beam when there is no spatial walkoff [93]. This gives a confocal parameter of \( 2z_0 \approx 14\text{mm} \) in air. With the collimated beam of about 2.5mm width, a 154 mm focal-length lens is appropriate and the focused spot size is about \( 2 \left( \frac{z_0 \lambda_0}{\pi} \right)^{1/2} = 119\mu\text{m} \). Experimentally, we find that an \( f = 100\text{mm} \) lens focused into the PPLN grating with 19.5\( \mu \text{m} \) periodicity heated at 158°C gives the highest efficiency. Since the fundamental and second harmonic waves are both extraordinary polarized (horizontal) and colinear, to access the large \( d_{33} = -41.1\text{pm/V} [94] \) coefficient in LiNbO₃, we use mirrors coated with a narrow band high reflectivity mirror at 793nm to separate the second harmonic beam from the fundamental beam after the PPLN. Because of the coating, the second harmonic beam is reflected by the mirror; while the fundamental frequency beam transmits to the back surface of the mirror and gets partially reflected and scattered there. So there is a transverse spatial offset between the scattered fundamental and second harmonic beam (similar to the effect of parallel plate), making them spatially separable. The power of the frequency doubled output is 4.5 – 5.5mW, so the efficiency is unexpectedly low, only about 1% while theoretical calculations predict closer to 10% efficiency.

After the PPLN, the 793nm light goes through one stage of NIR OFR 800nm optical isolator
(it rotates the polarization by $45^\circ$) which prevents the formation of a cavity between the BoosTA amplifier we use later in the setup and the uncoated PPLN surface that can act as a reflective cat’s eye. After using a $\frac{\lambda}{2}$ waveplate to rotate the light back to vertical polarization, we use mirror pairs to couple the light into a Toptica BoosTA semiconductor tapered laser amplifier. The BoosTA amplifier contains a tapered semiconductor wave guide that amplifies the beam power by about 16dB to be about 180mW. Again, two more OFR optical isolators help prevent back reflections of the later components back into the BoosTA. Most amplified spontaneous emission (ASE) in other spatial modes propagates with an angle and is filtered out by the spatial aperture of the OFR isolator and only about 100mW of light is left after the cascaded OFRs. Two cylinders with focal lengths of $f_x = 250$mm and $f_x = 150$mm form a 4-f system to correct the elliptical beam shape and astigmatism due to the BoosTA. A single mode polarization maintaining (PM) fiber filters out the other ASE spatial modes and cleans the beam profile. Due to fiber coupling losses, the slightly elliptical beam shape and residual astigmatism reduce the achievable PM fiber output to about 40mW of power.

**Acousto-optic modulators (AOMs)**

The amplified and fiber coupled light is collimated by a fiber collimator into 2.5mm width spot and split into two beams. Each beam is focused by an $f = 150$mm lens to a gate AOM (ISOMET SLM-16-835). The AOMs are PbMoO$_4$ crystals with 700$\mu$m transducer height and 70MHz of modulation bandwidth and are illuminated with a 60$\mu$m focal spot. The AOMs control the timing for the grating writing and reading process. The AOMs are controlled by the output from channel 1 and channel 2 of an AWG520 which consist of gated 100MHz sine waves. The signals with 3dBm of power from the AWG520 are amplified by 24dBm RF amplifiers before sending into the AOMs. At this applied power, the modulation efficiency of the AOMS is about 50%.

**SSH**

After the AOM, a small portion of beam 1 is split out by using a $\frac{\lambda}{2}$ waveplate and a PBS to act as time-delayed reference for interfering with the delayed diffraction from the SSH crystal.
Figure 5.11: Top view of beam 2: the GS mirror that is scanning vertically is vertically imaged to the SSH crystal. The 2 reflecting mirrors are not shown. Side view of both beam 1 and beam 2: both beams are focused to the SSH crystal horizontally. The transverse offset of the beams is about $D = 7\text{mm}$ before the $F = 150\text{mm}$ lens while their Gaussian beam waists are $2W_0 = 3.2\text{mm}$, giving a ratio $D/W_0 = 4.4$. The length of the SSH crystal is $L = 5\text{mm}$. A flipping mirror is used with a WM-100 beam profiler placed at a plane equivalent to the crystal center for beam overlap alignment.

Beam 1 goes through a vertical 1 : 1 4-f imaging system consisting of a cylinder $f_y = 150\text{mm}$ and a lens $f = 150\text{mm}$. The spherical lens focuses in the horizontal direction and it is imaged in the vertical direction into a $100\mu\text{m}\times3.2\text{mm}$ vertical line in the SSH crystal. Beam 2 is collimated by an $f = 250\text{mm}$ lens to a size of $3.4\text{mm}$ that illuminates the GS mirror oriented to scan vertically. The mirror is vertically imaged and horizontally Fourier transformed to the SSH crystal to about the same $100\mu\text{m}\times3.2\text{mm}$ beam size (measured by a Thorlabs WM-100 beam profiler) as that of beam 1 by a cylinder $f_y = 150\text{mm}$ and the same shared lens ($f = 150\text{mm}$) in beam 1. Fig. 5.11
Figure 5.12: The vertically polarized echo is rotated into horizontal polarization by a half waveplate. The PBS transmits the horizontally polarized echo and reflects the vertically polarized reference beam, spatially combining these two beam. The two beams are focused to a vertical line at the chopper wheel. The chopper rotates at about 25Hz (40msec period) with the opening time window of 7msec. During the writing phase, it blocks beam 1 and beam 2, while during the readout phase it lets through the echo beam 4 and the reference beam. The readout beam 3 is spatially blocked by a razor blade (not shown in this figure) that also blocks beam 1. After the chopper, the echo and the reference beams are collimated by a $f = 200$mm lens. A polarizer projects the two orthogonally polarized beams into the same polarization and allows them to interfere.

shows the imaging-Fourier transform relationship of the beams before the cryo. The separation of the beams is about 2.2 times the beam width. The two beams intersect with an angle of 2.4°. Because of the holographic nature of the recording and matched readout, the imaging of the GS mirror onto the SSH crystal does not have to be accurate [58], but it is important to image the mirror to the CCD camera at the end of the system. This experiment is a nonlinear experiment. For the best recording and diffraction efficiency, it is important to make the two beams maximumly overlapped and focused in the SSH crystal to obtain the highest writing intensity. Since the SSH crystal sits in a cryo, we use a flipping mirror and a WM-100 beam profiler to aid the focusing and overlapping alignment. Another flipping mirror (not shown in Fig. 5.11) that reflects light to the opposite direction of the beam profiler and a CCD are used for helping the alignment of imaging the GS mirror to the SSH crystal. The positions of the CCD camera and the WM-100 beam profiler are found by using a small depth of field (high NA) camera.

PBS

After the cryo, the writing beam 1 (also the reading beam 3) is blocked by a razor blade.
With the edge of the razor blade at the center of beam 1 and beam 2, only $6 \cdot 10^{-5}$ of the peak intensity of beam 1 (also beam 3) leaked through. A 4-f system consisting of two $f = 200$mm lenses images the SSH crystal to a Palo Alto Research chopper wheel that rotates at about 25Hz (40msec period) with two overlapping chopper blades adjusted to give an opening time window of 7msec. The chopper is triggered by and synchronized to the Marker signal of the AWG520. It opens when beam 3, the readout beam that is incident collinear with beam 1 (well after the decoherence time of the SSH crystal), arrives and lets through the diffracted echo (beam 4). Notice that beam 3 is spatially blocked by the same razor blade that blocks beam 1 and beam 2 is temporally blocked by the chopper. Therefore, only the diffracted echo (beam 4) will go through the chopper and arrive at the CCD at the end of the setup. A PBS combines the echo (in the beam 2 direction) and the reference beam, forming an interferometer. A more detailed beam combining diagram is shown in Fig. 5.12. The echo and the reference beam have the same vertical polarization and are both vertical slits. After a half waveplate ($\frac{\lambda}{2}$), the echo polarization is rotated into horizontal. The PBS then spatially combines the horizontally polarized echo and vertically polarized reference beam. A polarizer projects the two beams that have orthogonal polarizations into the same polarization which allows them to interfere. By rotating the polarizer, one can balance the intensity between the echo and reference beam and optimize the interference fringe visibility. Finally, the chopper is imaged either onto a CCD or onto a photo-detector by another 4-f system.

**Electronic setup and timing of the waveforms**

The electronic setup of this high bandwidth SSH scanner system is shown in Fig. 5.13. We use two arbs, a 3-channel Tektronix AWG430 and a 2-channel Tektronix AWG520, to control the Koheras laser piezo, gate AOMs, GS mirror, chopper and scope. AWG430 works as the master of all the other electronic boxes. The Koheras laser piezo is driven by channel 1 of the AWG430. Since AWG430 has a maximum output voltage of $2V_{pp}$, we use a high voltage op-amp OPA445 to amplify the AWG430 output voltage up to $73V_{pp}$. We use the same band-limited triangle wave approximated by multiple orders of sinusoids to drive both the piezo of the Koheras laser (amplified
Figure 5.13: High bandwidth SSH scanner electronic setup. We use two arbs to control the Koheras laser piezo, gate AOMs, GS mirror, chopper and scope. CH1 and CH2 of AWG430 send out the same ramp waveforms with different amplitudes. AWG430 CH1 output is amplified by a high voltage op-amp OPA445 and fed to the Koheras laser piezo. AWG430 CH2 and CH2 controls the GS mirror + and − ports. AWG520 outputs are amplified by 24dB RF amplifiers and fed to gate AOMs. AWG520 triggers the chopper that lets through the echo signals to the photodetector. A scope that is triggered by AWG520 displays the signal of the photodetector.

by OPA445) and the GS mirror. During the up-side of the triangle, both gate AOMs are turned on. Beam 1 and beam 2 write gratings in the SSH crystal. During the downside of the triangle, gate AOM 1 is turned on to produce the readout beam, beam 3 (which has the identical spatial position as beam 1) which is perfectly Bragg matched to read out the spatial-spectral gratings in the SSH crystal and produce the diffracted echo, beam 4. In the mean time, the chopper opens, letting through the off-axis diffracted spectral readout signal or the photon echo signal. But the chopper is closed during the recording process, blocking the beam 2 light from scattering to the detector. The time domain sequences for AWG430, AWG520 and chopper are shown in Fig. 5.14. CH1 of AWG430 is amplified by a high voltage amplifier OPA445 and sent to the piezo of Koheras laser that has a piezo coefficient of 25MHz/V. CH2 and CH2 of AWG430 controls the command+ and command− ports of the GS mirror. CH1 and CH2 has the same waveform but with different
output voltages. CH3 of AWG430 triggers the AWG520 whenever the triangle wave of CH1 and CH2 starts every 40msec, chosen to be sufficiently long \((4T_1)\) that the previous exposures have decayed away \(\exp(-4)\). CH1 and CH2 of AWG520 produce gated RF tones at 100MHz to control gate AOM1 and gate AOM2 respectively. They are responsible for the timing of the writing and reading process. Marker 1 of CH1 triggers the chopper. The chopper has a timing uncertainty of about 100\(\mu\)sec and a rise time of about 30\(\mu\)sec due to the focal spot size of 100\(\mu\)m at the chopper wheel of 5cm radius. To read out the whole inhomogeneous bandwidth that has been written by the chirped lasers, the chopper needs to be opened slightly before the reading gate AOM 1 turns on, but not too early to keep the writing beam from sneaking through and scattering onto the detector. The timing of the chopper opening can be moved back and forth by changing the phase of the chopper. Marker 1 of CH2 triggers the scope for the spectral grating readout experiment.

With this setup, one can choose to do two different kinds of experiments:

(1) By flipping up mirrors M1 and M2, one can choose the undelayed reference beam and the photo-detector for a spectral grating readout experiment;

(2) By flipping down the mirrors M1 and M2, one can choose to interfere the echo with the delayed beam 3 and use the UNIQ CCD camera to do a time integrating correlator experiment.

In the subsections, we will describe each of these experiments respectively.

### 5.4.1 Diagnostic experiments

It is very important to test EACH part of the setup independently if it is at all possible, from optics alignments, to electronic boxes and circuits, to make sure they perform as they are supposed to at room temperature before attempting a cool down. In fact, almost all the tests could be done to a certain degree at room temperature. Before the cool down experiments, I make sure both spatial alignment and time domain “alignment” are right. Important spatial alignments include:
Figure 5.14: Time domain sequences of the high bandwidth SSH scanner electronic boxes. We use AWG430 to control the Koheras laser piezo and the GS mirror. We use AWG520 to control the gate AOMs. AWG430 triggers AWG520 and AWG520 triggers the chopper and the scope. The same ramp signal drives the GS mirror and the Koheras laser piezo. The AOMs are fed with pulsed sine signals at 100MHz. During the writing process, both gate AOMs are opened during the up chirp of CH1 and CH2. During the reading process, only gate AOM1 opens and in the mean time, the chopper transmits light.

(1) Imaging condition between the GS mirror through the crystal and onto the photo-detector (PD) or UNIQ CCD camera.

(2) The GS mirror's in-plane rotation about the z-axis (beam propagation direction) relative to the cylinders' in-plane rotation about the z-axis.
Figure 5.15: Single spot spectral hole burning experiment time-frequency waveforms. A CW beam 1 burns spectral hole at 105MHz for 1ms. A slow chirp beam 2 reads out the spectral hole. The actual waveform has $10^5$ oscillations during the CW tone and $10^4$ during the chirp.

(3) Transverse overlap between beam 1 and beam 2 at the center of the SSH crystal.

I talk about some of these alignments in Appendix A. One could check the time-domain signal alignment easily by bringing both beam 1/3 and beam 2 on the PD and observing the PD output on a scope. By changing the phase of the chopper, one can make beam 3 go through the chopper opening. Notice that due to hysteresis of the laser piezo during frequency up chirp and down chirp, in real experiments, beam 1 and beam 3 are not exactly symmetric about the peak of the ramp in the time domain. Unfortunately this can only be determined in cool down experiments.

In a cool down experiment, I start with simple verification experiments and then add one thing at a time to increase the complexity. By doing this, I am able to reduce the uncertainties of identifying the cause of any problems. Although it may sound like more work to do, it really saves a lot of time and effort during the cool down experiments. It helps to catch some small things that may have been missed and keeps me from touching the wrong knobs.

I typically start with a single spot spectral hole burning experiment by doing the following:

- Remove cylinder 1 and cylinder 2 (use kinematic alignment mounts allowing easy replacement of the cylinders on a rail) in order to produce small overlapped circular spots that image AOM1 and AOM2 onto the SSH.
- Use gate AOM 1 to modulate beam 1 with a CW tone at 105MHz.
Figure 5.16: Single spot spectral hole burning readout experimental result. Beam 1 has about 1mW of power before the cryo and beam 2 (readout beam) has about 0.3mW of power before the cryo. The ringing present in the spectral hole is due to the fast chirp readout. It can be removed by spectral recovery [95], i.e., deconvolving the raw signal from the read chirp.

- Then, after a short delay, modulate beam 2 with a slow chirp signal that sweeps from 100MHz to 110MHz within 100µsec to read out the engraved spectral hole.

The frequency vs time plot of this waveform is shown in Fig. 5.15. While observing the spectral hole, I align the overlap of beam 1 and beam 2 to maximize the spectral hole signal. Fig. 5.16 shows an example of the spectral hole readout. This confirms the wavelength and stability of the laser and optimizes the overlap between the two beams.

5.4.1.1 Single spot spectral grating

Then I do a single-spot spectral-grating read-out experiment by only using beam 1 and the chopper:

- Two identical 10MHz bandwidth chirps with 100µsec duration delayed by 0.5µsec, 1µsec and 2µsec are produced by the arb and modulated using AOM1 to write sinusoidal spectral gratings.
Figure 5.17: Single spot spectral grating readout experiment frequency vs time waveforms. Two identical chirps with a time delay of 1μsec between them sweep from 100MHz to 110MHz within 100μsec. They write sinusoidal gratings in the inhomogeneous band of the SSH crystal. 1 msec after the second chirp is gone, a 1 msec slow chirp that covers the same 10MHz bandwidth comes in to read out the spectral grating. A chopper opens about 300 – 400μsec before the readout chirp. The chopper rotates at 25Hz and opens for about 3.3msec.

- Let the material decohere for 1 msec, during which time the chopper opens. The 1 msec decoherence time accommodates the jittering of the chopper.

- Read out the spectral grating with a 1ms duration, 10MHz bandwidth chirp.

The frequency vs time plot of the waveforms is shown in Fig. 5.17. I did this experiment once to check if the chopper works properly as a read-out signal gate and if the spectral grating readout experiment works. In this experiment, one could even lock the chirp signals that are fed to the AOM to the chopper (acting as the master) if one can not afford the long decoherence time. An example of the spectral grating readout is shown in Fig. 5.18 (the data only show 0.5msec of the readout).

5.4.1.2 Koheras chirping and characterization

All of the above test experiments are done without chirping the laser and scanning the mirror. Unfortunately, the effect of scanning the GS mirror and chirping the Koheras fiber laser on spectral grating recording can not be tested separately with ease. Previous experiments in Chapter 2 (Doppler processor) used photorefractive holograms to verify the coherent processing ability of the GS mirror when it is driven within the appropriate limits on the drive voltage and repetition rate. By looking at the position output signal of the GS mirror, one can also get a rough idea if
the GS mirror is working properly. To verify the chirping range of the Koheras laser, I look at the frequency beat of the chirping Koheras with a Velocity laser on a real-time RF spectrum analyzer, Tektronix RSA3308A. When the piezo of the Koheras laser is driven by $0.9V_{pp}$ of 110Hz single sine burst with a repetition rate of 100Hz, the beat tone resets every 10msec and the reset frequency has a variation range of about 10MHz, as is shown in Fig. 5.19(a). Without the driving signal, the Koheras - Velocity beat is shown in Fig. 5.19(b).

5.4.2 Spectral grating read out experiment

By flipping up mirrors M1 and M2, one can choose the un-delayed reference beam and the photo-detector for a spectral grating readout experiment. For the spectral-grating readout experiment, we beat the un-delayed reference beam, which is the same as beam 3, with the echo (beam 4) that comes out in the beam 2 direction and has a spatial-dependent time delay. The beat is a sine wave in the time domain. By beating beam 3 with the echo, we map the spectral-domain sinusoidal gratings (population inversions) into the time domain (intensity changes). An easier
starting point is to do a spot-line experiment, where we don’t use cylinder 1 and only use cylinder 2 in the GS mirror arm. Therefore, at the SSH crystal, beam 1 is focused into a $75 \mu m(x) \times 55 \mu m(y)$ spot which overlaps as a small portion on beam 2 which is focused into a $75 \mu m(x) \times 3.5 mm(y)$ vertical line. Both beams have about 10mW of power right before the SSH crystal. The spot-line experiment avoids spreading the power out spatially in both beams since only the tilting beam is spread vertically resulting in a much higher exposure power spectral density ($W/cm^2/\text{Hz}$) and will thus have much higher diffraction efficiency than a line-line experiment where beam 1 and beam 2 are both vertically spread lines at the SSH crystal. The configuration of spot-spot, spot-line and line-line experiment is illustrated in Fig. 5.20. The spot-spot experiment has the highest power spectral density. However, to verify the high bandwidth SSH scanner with this setup, we need to focus a beam to the surface of the GS mirror and then image the spot to the SSH crystal. This experimental setup is not readily expandable to the ultimate line-line configuration. So instead, we start with the less efficient but more readily expandable spot-line configuration.

In the spot-line experiment, the spatial-spectral gratings are only written in the SSH crystal at the position where the spot (beam 1) and line (beam 2) are overlapped. Since the GS mirror scans vertically, although the crystal is in an image plane of the GS mirror, different vertical spatial
Spot-Spot Spot-Line Line-Line

1, 3 ●● 2

Writing \((P_0/A/B)^2\)

Reading \((P_0/A/B)\)

Spot-line experiment: beam 1 and beam 3 are focused to a spot, beam 2 is focused to a line. The spectral radiance of the echo is proportional to \(\frac{1}{N} \left(\frac{P_0}{AB}\right)^3\). Line-line: all beams are focused to a line in the SSH crystal. The spectral radiance of the echo is proportional to \(\frac{1}{N^2} \left(\frac{P_0}{AB}\right)^3\).

Figure 5.20: The power spectral density of spot-spot, spot-line and line-line experiments. Beam 1: writing beam. Beam 2: writing beam. Beam 3 (in the same spatial position of beam 1): reading beam. \(P_0\): power of one of the beams. \(A\): horizontal beam width. \(B\): vertical beam width. \(N\): number of resolvable spots in vertical direction. Spot-spot experiment: the writing beams and the reading beam are all focused to a spot in the SSH crystal. The spectral radiance of the echo is proportional to \(\left(\frac{P_0}{AB}\right)^3\). Spot-line experiment: beam 1 and beam 3 are focused to a spot, beam 2 is focused to a line. The spectral radiance of the echo is proportional to \(\frac{1}{N} \left(\frac{P_0}{AB}\right)^3\). Line-line: all beams are focused to a line in the SSH crystal. The spectral radiance of the echo is proportional to \(\frac{1}{N^2} \left(\frac{P_0}{AB}\right)^3\).
tilt knob of M3 changes the periodicities of the readout spectral grating.

5.4.2.1 Fourier series approximations to ramp

By using different orders of cosine that approximate the linear ramp, one can see the effect of nonlinear chirps and nonlinear GS mirror scans on recording the spectral gratings. As is shown in Fig. 5.22(a), as the number of orders increases, the periodicities of the sinusoidal gratings become more constant throughout the readout bandwidth. In addition to the recording process, the nonlinearity of the readout down chirp will make the readout gratings even more nonlinear in periodicities. For the grating recorded and read with one cosine order, the grating has more constant periods at the center where the cosine is a closer approximation to a linear ramp. Fig. 5.22(b) shows the Fourier transform of the grating readouts. As the approximation orders increase, the Fourier transform of the gratings becomes a narrower peak, indicating that the periodicity of the gratings is becoming more constant and the readout more faithfully produces the desired constant periodicity sinusoidal grating. This experiment shows a ratio of peak width to available delay of more than 20.

Figure 5.21: An example of the readout grating for a spot-line experiment.
Figure 5.22: Using 1-8 cosine order(s) to approximate the linear ramp. The Koheras piezo driving voltage is 37V_{pp} covering 925MHz in 5msec. The opening time window of the gate AOMs is 2.5msec. So the bandwidth for the grating recording is about 654MHz. (a) Spectral gratings readout for different sine orders approximations. As we increase the number of orders that approximate the ramp, the periodicities of the sinusoidal gratings recorded in the SSH crystal becomes more regular. (b) Fourier transform of the sinusoidal gratings gives time delayed pulses.

Another spot-line experiment we did is to read out the spectral gratings at different spatial position of the SSH crystal by translating the micrometer of the mirror M3. Fig. 5.23(a) shows the readout gratings with bandwidth of over 1.5GHz at different vertical spatial positions of the SSH crystal. The grating periodicities decrease as we move to the center of the GS mirror (image at the SSH crystal). This is due to the fact that the beam at the center of the GS mirror (where the mirror pivots) experiences less Doppler shifts and thus less time delay (larger period spectral gratings). After Fourier transform the readout gratings, we can get the time delays corresponding to each of the gratings, as is shown in Fig. 5.23(b). Notice that the upper half of the readout has much lower amplitudes than the lower half of the readout. This is the causal time edge discussed in Section 3. Plot Fig. 5.23(b) in the coordinates of spatial position vs. time delay, as is shown in Fig. 5.24,
Figure 5.23: Spot-line experiment. The Koheras piezo driving voltage is $73\text{V}_{\text{pp}}$. The opening time window of the gate AOMs is 4msec. So the grating recording bandwidth is 1556MHz. GS mirror driving voltage is 100mV. (a) Spectral gratings readout for different spatial positions. The spatial step size is $36\mu\text{m}$ at the SSH crystal. (b) Fourier transform of the sinusoidal gratings gives time delays linearly proportional to the spatial positions as desired. Again, this experiment shows a ratio of peak width to available delay of more than 20.

one can see the diffraction timing edge effect more clearly. As discussed in Section 5.3.2, due to the finiteness of the spectral grating bandwidth, there are still some diffractions on the acausal half plane.

5.4.2.2 Diffraction off the causal edge

By flipping down mirror M2, blocking the light from the bouncing mirrors that generate time delays, and use both cylinders, one can do a diffraction-off-the-causal-edge experiment. This has to be a line-line experiment, where both beams are focused into vertical lines in the SSH crystal. In this case, instead of looking at the spectral gratings recorded in the inhomogeneous band, we look
at the echoes of the readout chirp signal. The readout chirp acts as $s(t)$ of $E_3(x,y,t)$ in Eq. (5.9), it diffracts off the spatial-spectral gratings that are written in the SSH crystal. Since the GS mirror generates opposite Doppler shifts for the half planes above and below its pivot axis, the time delays that are recorded in the SSH crystal will be positive and negative respectively on opposite sides of this pivot axis. Negative time delays would give diffractions that violate causality and thus cannot produce an echo into this direction. Only positive time delays can actually be diffracted. So diffracted echoes will be generated only on one side of the pivot axis of the GS mirror, resulting in a sharp edge in space indicating the image of the pivot axis. Fig. 5.25(a) is an echo image with background that consists of the undiffracted DC signals of AOM 1 and AOM 2. When turning off gate AOM 1 or gate AOM 2, no spectral gratings will be recorded in the SSH crystal, so the diffracted echo signals disappear as shown in Fig. 5.25(b)-(c). The chopper lets through unwanted leakage from beam 3 that diffracts around the razor blade, the echo beam 4 and the DC leakage from the AOMs. When turning off AOM 1, beam 3 and the echo are both gone, so Fig. 5.25(b) is

Figure 5.24: Time delays generated at each spatial positions. On the left side of the diffraction edge, there are diffracted fringes; while on the right side, there is essentially no diffracted fringes due to the violation of causality.
the background that consists of the DC leakage of AOM 1 and AOM 2. When turning off AOM 2, only the echo is gone, Fig. 5.25(c) consists of the spatial leakage of beam 3 (from the “beam block for beam 1” after the cryo in Fig. 5.10) in addition to the DC leakages of AOM 1 and AOM 2. These DC leakages are the major problems preventing us from going to higher bandwidth with current power level. Better beam shaping in the AODs and spatial filtering of the diffractions could potentially overcome this limitation, but it may require further engineering, such as polarization switching AOMs [96]. Subtracting the background (Fig. 5.25(c) from Fig. 5.25(a), one can get the diffracted edge signal shown in Fig. 5.25(d). Translating the GS mirror vertically, the diffraction edge moves accordingly, as shown in Fig. 5.26. Notice that the Gaussian beam profile illuminating the crystal is not moving, it is only the GS mirror that is moving within this beam profile, so the resulting edge is a purely nonlinear SSH diffraction effect due to causality.
Figure 5.26: Profiles of the diffracted causal edge. As the GS mirror translates towards the acausal side, less of the light is diffracted. The fixed dip is a dust speck or other artifact on the SSH crystal.

Figure 5.27: Time integrating correlation result for different GS mirror driving voltages and different time delays. The number of resolvable time delays is about 20.

5.4.3 Time-integrating (TI) correlator experiment

To further verify that the SSH scanner generates space-dependent time delays like an AOD, we use a TI correlator where the time-delayed signals from the SSH scanner are correlated with the
signal with a known time delay on a TI CCD. By flipping down mirrors M1 and M2, letting the echo (beam 4) in direction of beam 2 interfere with the time-delayed reference beam (either tilted horizontally or vertically from the echo), and use both cylinders, the system can operate as a TI correlator by using a TI CCD detector array instead of a time domain point detector at the output [24]. Operating as a TI correlator is advantageous since the CCD pixels can accumulate the time delayed echoes for a long integration time, accumulating more photoelectrons and detecting very weak echoes. In fact TI can continue for many times the crystal lifetime \( T_1 \), if the diffracting SSH grating is refreshed and rewritten or multiple frames of CCD TI correlators can be arranged. The image accumulated on a CCD can be written as

\[
I(x, y, m) = \int_{(m-1)T_i}^{mT_i} |E_e(x, y, \tau) + E_{\text{ref}}(x, y, \tau)|^2 \, d\tau, \tag{5.20}
\]

where \( E_e(x, y, \tau) \) is the echo signal and \( E_{\text{ref}}(x, y, \tau) = E_3(x, y, \tau - t_d) \exp(i k_0 y \sin \theta_r) \) is the reference signal delayed by the bouncing mirrors for time \( t_d \) and incident with an angle \( \theta \) in the \( y \)-direction. Using Eq. (5.11), we can rewrite Eq. (5.20) as

\[
I(x, y, m) = g(x, M y) \int_{(m-1)T_i}^{mT_i} \left| s(\tau - \frac{2c_m}{c_i\lambda_0} y \cos \alpha) + s(\tau - t_d) \exp(i k_0 y \sin \theta_r) \right|^2 \, d\tau
\]

\[
= 2g(x, M y) \mathcal{R}e \left\{ \exp(i k_0 y \sin \theta_r) \int_{(m-1)T_i}^{mT_i} s^*(\tau - \frac{2c_m}{c_i\lambda_0} y \cos \alpha) s(\tau - t_d) \, d\tau \right\} + \text{bias}, \tag{5.21}
\]

where \( T_i \) is the integration time of the CCD camera, and \( \mathcal{R}e\{\cdot\} \) is the real part operator. The integral is a correlation between the reference beam and the echo. When the reference beam matches with the echo, i.e., \( t_d = \frac{2c_m}{c_i\lambda_0} y \cos \alpha \) or at position \( y = \frac{c_i\lambda_0}{2c_m \cos \alpha} \), the integral peaks; while at other positions, the integration over time washes out. The width of the correlation peak is determined by the bandwidth, \( \Delta \), of the spectral gratings written in the SSH crystal during the recording process. On the CCD the spatial width of the correlation peak is \( W_y = \frac{2c_i\lambda_0}{c_m \Delta \cos \alpha} \). The interaction angle \( \theta \) between the beams gives a sinusoidal spatial modulation, \( \sin(k_0 y \sin \theta) \), to the correlation peak. The modulation has a period of \( \frac{\lambda_0}{\sin \theta} \), which should be 2-3 times smaller than the peak width \( W_y \). To sample this, the CCD must have a pixel size of less than \( \frac{\lambda_0}{2 \sin \theta} \).
In the experiment, the GS mirror is driven by a 100Hz symmetric ramp approximated by 10 orders of sinusoids. The Koheras laser is driven by the same ramp signal but with about 74Vpp of amplitude. This chirps the laser over 1.5GHz of bandwidth over 4msec. When gate AOM 1 opens during the down chirp of the Koheras laser, beam 3 diffracts off the spectral gratings with different periodicities at each spatial position of the SSH crystal, resulting in beam 4 that contains various time-delayed beam 3 signals with a time delay linearly proportional to spatial position. The time-delayed echoes interfere with the beam 3 signal delayed by the bouncing mirrors at the CCD camera. Since in this case we use a downchirp for the readout, $s(t)$ can be written as

$$s(t) = \exp(-i\pi bt^2) \cdot \frac{1}{4T} \Pi \left( \frac{t}{T} \right) \ast \text{comb} \left( \frac{t}{4T} \right).$$  \hspace{1cm} (5.22)

For simplicity, we define $t_g(y) = \frac{2c_o}{\lambda N} y \cos \alpha$. The time delays that could be generated by this system and by the bouncing mirrors are on the order of nano-seconds, therefore are much less than the chirp duration time $T$, $t_d$, $t_g$, $|t_d - t_g| << T = 10\text{msec}$. Since for the camera we use, $T_i = \frac{1}{60}\text{sec}$, there is only one chirp integrated during the camera integration time. At the positions that the echo’s time delay matches with the time delay generated by the bouncing mirrors, i.e., $t_g(y_0) = t_d$, the interference signal integrates up at the CCD camera and there will be vertical spatial interference fringes,

$$I(x, y, m) = 2T g(x, My) \cos(k_0 y \sin \theta_r) : t_g(y_0) = t_d. \hspace{1cm} (5.23)$$

While at other positions where the delays are different, i.e., $t_g(y) \neq t_d$, at $y \neq y_0$

$$I(x, y, m) \cong 2g(x, My) \Re \{ \exp[i k_0 y \sin \theta_r + \pi b(t_g^2(y) - t_d^2)] \times \int_{-\infty}^{+\infty} \exp[i 2\pi \delta t(t_d - t_g(y))] \Pi \left( \frac{t - t_d}{T} \right) \text{d}t \} + \text{bias}$$

$$= 2T g(x, My) \text{sinc}[b T(t_d - t_g(y))] \cos[k_0 y \sin \theta_r + \pi b(t_g^2(y) - t_d^2)]$$

$$+ 2\pi b[t_d - t_g(y)] t_d \} + \text{bias}, \hspace{1cm} (5.24)$$

Using $bT = \Delta$, there are three different situations:
\[ t_d - t_g(y) << \frac{1}{\Delta}, \quad \text{sinc}\{bT[t_d - t_g(y)]\} \cong 1, \]

\[
I(x, y, m) \cong 2Tg(x, My) \cos\{k_0 y \sin \theta_r + \pi b[t_g^2(y_0) - t_d^2] + 2\pi b[t_d - t_g(y_0)]t_d\} + \text{bias.} \tag{5.25}
\]

\[ t_d - t_g(y) \approx \frac{1}{\Delta}, \quad \text{sinc}\{bT(t_d - t_g)\} \cong 0, \]

\[
I(x, y, m) \cong \text{bias.} \tag{5.26}
\]

\[ t_d - t_g(y) >> \frac{1}{\Delta}, \quad \text{sinc}\{bT[t_d - t_g(y)]\} \cong 0, \text{ therefore, we have} \]

\[
I(x, y, m) \cong \text{bias.} \tag{5.27}
\]

Thus fringes will accumulate when \(|t_d - t_g(y)| \approx \frac{1}{\Delta}\), with a full width of \(\frac{2}{\Delta}\) (width between zero points), while at other positions, there is only the DC background. Fig. 5.27 shows the time-integrating correlator results at different GS mirror driving voltages and with different time delays generated by the bouncing mirrors. Different time delayed signals match with the echo generated at different spatial positions and the time-integrating CCD integrates the interference up to generate spatial fringes. Therefore as the time delay varies, the fringe position moves with it. As we vary the mirror driving voltage, since time delays of the echoes are proportional to driving voltage of the GS mirror, the same time delay corresponds to different spatial positions at the SSH crystal and thus at different positions on the CCD which is at an image plane of the SSH crystal and GS mirror.

The vertical beam size in the SSH crystal is about 3.1mm with about 110\(\mu\)m of the smallest focal spot size, so the number of resolvable spots is about 30. However, the number of resolvable spots of the whole system is not necessarily determined by the spatial imaging system. Another possible factor is the GS mirror-Kohera laser recording system. If we write angular velocity \(\Omega\) of the GS mirror in Eq. (5.1) in terms of the GS mirror’s maximum rotation angle \(\theta_m\) and ramp signal periodicity \(T\), and write laser chirp rate \(b\) in terms of the chirp bandwidth \(\Delta\) and periodicity \(T\), we
\[
\Delta t = \frac{\theta_m / (T/2) y_{\text{max}}}{\Delta/(T/2) \lambda} = \frac{2\theta_m y_{\text{max}}}{\Delta \lambda}.
\]

(5.28)

Since the smallest resolvable time delay is \(1/\Delta\), the number of resolvable spots is determined by

\[
\text{# of resolvable spots} = \frac{2\theta_m y_{\text{max}}}{\lambda},
\]

(5.29)

which is the number of wavelengths that the edge of the mirror moves. So the number of resolvable spots is proportional to the GS mirror’s maximum rotation angle \(\theta_m\) and the spatial size of the beam \(y_{\text{max}}\), and is inversely proportional to the wavelength \(\lambda\). The maximum rotation angle of the GS mirror \(\theta_m\), is determined by the peak amplitude of its driving voltage which is determined by the GS mirror bandwidth. For 100mV_{pp} of driving voltage, \(\theta_m \approx 2.5\text{mrad}\). For beam size \(y = 3.1\text{mm}\), the number of resolvable spots determined by the recording process is thus about 19.

The resolvable spots of this whole system is the smaller of the number of spatial spots and the number of time domain spots. For 100mV_{pp} of GS mirror driving voltage, the number of time domain spots is the limiting factor. But with increasing GS mirror driving voltage and thus bigger \(\theta_m\), the spatial spots number could be the limiting factor. The increasing number of resolvable spots effect with the increasing driving voltage, as shown in Fig. 5.28.

In the experiment, we propagate the reference beam up to about 10 meters by multiple bouncing the beam between mirrors, which makes the reference beam expand substantially. This introduces a quadratic phase factor \(\exp(i k_0 z_0^2 / 2\sigma)\) (\(z_0\) is the propagation distance) to the reference term in Eq. (5.21) and distorts the integration fringes. To compensate the effect, one could use a lens after the bouncing mirrors. But for different time delays, one needs to use lenses with different focal lengths. Another solution is to use a 4-f system and by changing the distance between the lenses one could compensate the quadratic phase due to the long propagation distances required to induce the required time delay \(t_d\).

This correlation experiment further verifies that the SSH scanner can generate space dependent time delays. The SSH scanner was developed to be applied to situations where a high
Figure 5.28: GS mirror driving voltage changes the number of resolvable of the system. With higher driving voltage of the GS mirror and the beam size keeps the same, the number of resolvable spots gets more.

A bandwidth AOD is needed but unavailable for bandwidths beyond the limits of AO technology (a few GHz), e.g., a TI ambiguity function processor [24]. However, one major difference between the AOD and the SSH scanner is the AOD Doppler shifts the diffracted beam while the SSH scanner does not, so a SSH scanner can not cover all the functions of an AOD.

5.5 Conclusion

We theoretically analyzed and experimentally demonstrated a high bandwidth SSH scanner that can time delay high bandwidth signals with different amount along its spatial aperture. The SSH scanner resembles the function of an AOD and it has much higher (up to hundreds of GHz depending on the material) bandwidth potentials than an AOD. In the spectral grating readout experiment, we demonstrate 20 resolvable spectral grating periods with about 1.5 GHz of bandwidth. In the diffraction off the causal edge experiment, we show the causality of the time delays generated by the SSH scanner. In the TIC experiment, we demonstrate 20 time delays with about 1.5 GHz of bandwidth.
bandwidth.

The Fourier transform of the SSH scanner is an RF spectrum analyzer [27]. During the recording process, the laser changes its frequency and in the mean time the GS mirror scans the beam in angle. The result is to record frequency-selective holograms at each angle. When the signal of interest comes in to read out the hologram, different frequencies will be diffracted to different angles. In other words, the Fourier transform of the SSH scanner can be thought as a dispersive grating that has super-fine wavelength resolution. The recording requirement for the SSH RF spectrum analyzer is less stringent than that of the SSH scanner. The recording process of the SSH scanner requires that the GS mirror scans are coherent or that it scans with a constant angular velocity. This insures that the spectral gratings recorded at each spatial position has the same phase and therefore the Fourier transform of which will give a time delay. Therefore, a SSH scanner can be used as an RF spectrum analyzer, whereas a SSH RF spectrum analyzer might not be able to function as a SSH scanner unless it is recorded with a phase coherent scanning mechanism.
Chapter 6

Prospects for future SSH scanner

Chapter 5 discusses the proof-of-concept experiment of a high bandwidth SSH scanner. It uses one laser for both the programming and the reading process. The reading waveform is constrained in that experiment to be a chirp. However, this is not a hard limit for the system. To have an arbitrary reading waveform, we need to use two lasers for the experiment, one is a chirped laser for the writing process and the other is a stabilized CW laser for the reading process. The arbitrary rf reading waveform can be modulated onto the CW laser by using an electro-optic modulator (EOM) with a bandwidth only limited by the technological limitations of EOMs (20GHz to as much as 100GHz). The two lasers need to be locked to each other to avoid long-term frequency drifting.

Section 6.1 gives a general picture of how this future system will work and what performance it could achieve. Section 6.2 talks about the design of a two-laser locking circuit and gives the locking result.

6.1 General picture of the future system

A full version of the SSH scanner experiment requires two frequency locked lasers, as is shown in Fig. 6.1, one laser needs to be chirped for the recording process and the other laser needs to be stabilized for the reading process. We can use the narrow linewidth Koheras Boostik fiber laser for the recording process and a hole-locked Velocity laser for the reading process or vice versa, but using a narrow linewidth laser for the recording process avoids the stochastic linewidth broadening during chirping that is unavoidable with a stabilized laser when it chirps and is no longer stabilized.
Figure 6.1: Two-laser high-bandwidth "Bragg cell" system. The recording process uses Koheras laser and the reading process use the Velocity laser that locks to a spectral hole of the SSH crystal. During the recording process, the Koheras and GS mirror are controlled by the same ramp signal. The recording process happens at the up chirp. After sufficient decoherence time, the EOM modulated velocity laser light reads out the spectral gratings. The Koheras laser is locked to the velocity laser by using a PDH locking circuit when it is not chirped then the locking circuit is held during the chirp. Chopper 1 and chopper 2 can be replaced by AOMs if accurate timing control is necessary.

Since the Velocity laser drifts randomly and has an inherent linewidth of nearly 1MHz, one needs to stabilize it to a frequency reference such as an atomic line or a spectral hole of burned in the SSH crystal, by using a Pound-Drever-Hall locking box, such as the 2007 model from Scientific Material. The recording and reading process needs to be addressing the same chunk of spectrum and one way of ensuring this is to periodically lock the two lasers together when the recording Koheras laser is not chirped.

During the recording process, the Koheras fiber laser is independent from the Velocity laser. We use a slow sine wave signal to drive the piezo of the Koheras laser. This process is the same as the recording process of the one-laser experiment. Currently, the background due to the un-
diffracted beams of the gate AOMs is about at the same level as the photon echo signal. To explore more bandwidth, one needs to solve the diffraction leakage of the DC beams from the gate AOMs by using less tightly focused beams and devices with less surface roughness and scattering. Another way of avoiding the scattering is to use choppers instead of AOMs for timing control. The one-laser experiment requires complicated timing waveform sequences that is not easily implemented by using choppers. However, for the two-laser experiment, it is possible to use two synchronized choppers for the recording and reading process and therefore avoid the scattering problem.

The reading process is different from the one-laser experiment. Instead of using the downchirp of the Koheras laser as a signal that needs to be time delayed differently, we use an EOM to modulate an arbitrary high-bandwidth RF signal onto the optical carrier and read out the time delays pre-recorded in the SSH crystal. Again, we can use a CCD detector as an interferometric time-integrating correlator to test the recorded time delays. In the reading process, we only need to use the Velocity laser. However, we need to lock the Koheras laser to the stabilized Velocity laser during this reading stage so that the Koheras laser won’t slowly drift away from the Velocity laser in long term, especially when trying to reset back to the original frequency after each chirp excursion.

The one-laser experiment can be expanded to a two-laser experiment by including a periodic two-laser locking circuit. The locking circuit can be built on top of a two-laser locking circuit that is based on an electric phase/frequency discriminator[97],[98]. With a high bandwidth RF signal source that goes up to 20GHz or more bandwidth and can drive the EOM appropriately at these rates, one can again use an interferometric time-integrating correlator experiment to verify the operation of the wideband SSH scanner at bandwidths of 20GHz, or potentially even more. This should be compared to the few GHz bandwidth limitation of AO technology.

Notice that in the diffraction-off-the-causal-edge experiment and the TIC experiment in the last section, we used a ND3.5 filter after the SSH to prevent the echo from saturating the UNIQ CCD camera. Thus the echoes are 35dB larger than required for clear detection on the CCD, and another 20dB above the noise floor. This makes exploring much higher bandwidth than 1.5GHz
even at the currently available power level at the crystal promising. But the bandwidth dependence
of the required power needs to be clarified. The amplitude of the echo can be found by applying
Eq. (5.12). In Eq. (5.12), by neglecting the spectral resolution, coherence decay and causality
(substitute $\gamma(\omega)$ with $\delta(\omega)$), we get the Mossberg's integral [90],

$$E_e(x, y, t) = D \int_{\omega_0}^{\omega_0 + \Delta} E_3(x, y, \omega)[E_1^*(x, y, \omega)E_2(x, y, \omega)] \exp(-i\omega t) d\omega. \quad (6.1)$$

This equation says, for a TIC (photon echo) experiment, the electric field amplitude of the echo
is proportional to the square root of the power spectral density (power/bandwidth/area) of each
of the recording and reading beams. To maximize the echo efficiency, we want to maximize the
number of photons that write the spectral grating (population inversion), but in experiment, the
maximum power we can get from a continuous wave writing laser is fixed, $P$. So to maximize the
number of photons used to write a spectral grating, we want to maximize the writing chirp duration
time. The maximum chirp duration time, however, is constrained by the lifetime of the crystal.
In our experiments, we chose the chirp duration time to be half of the lifetime, leaving half of the
lifetime for reading out, and keep it fixed, $T_c = \frac{T_1}{2}$. The spectral area of the beams are the same,
$A$. The amplitude of the writing beams can be written in terms of the beam power $P$

$$|e| = \sqrt{I} = \left(\frac{P}{A}\right)^{1/2} \quad (6.2)$$

where $I$ is the intensity of beams. Using Parseval’s theorem [55], which says the energy in one
domain (it is time domain here) will be the same as the energy in its Fourier transformed domain
(frequency domain),

$$|e|^2 \cdot T_c = |\tilde{e}|^2 \cdot \Delta, \quad (6.3)$$

where $\tilde{e}$ is the amplitude of the beams in spectral domain, we have

$$|\tilde{e}| = \left(\frac{PT_c}{A\Delta}\right)^{1/2}. \quad (6.4)$$
Now Eq. (6.1) becomes

\[ E_e(x, y, t) = D \int_{\omega_0}^{\omega_0+\Delta} E_3(x, y, \omega)e^2 \exp \left[ -i\omega \left( t - \frac{2c_m}{c_i\lambda_0} y \cos \alpha \right) \right] d\omega \]

\[ = De^2 \exp(i\omega t) \exp(ik \sin \theta_2 x)g(x, My)s(t - \frac{2c_m}{c_i\lambda_0} y \cos \alpha) * [\Delta\text{sinc} (\Delta t)] \]

\[ \propto \frac{1}{\Delta}, \]  

(6.5)

where the \( \Delta \) in front of the sinc function should not be counted in the bandwidth calculation because \( \Delta\text{sinc}(\Delta t) \) is convolved with \( s(t - \frac{2c_m}{c_i\lambda_0} y \cos \alpha) \). The bigger the \( \Delta \) is, the narrower the sinc function will be. Therefore, keeping the other things the same and increasing the bandwidth of the writing/reading beams by 10 times will make the intensity of the echo only \( \frac{1}{10} \) of before. Eq. (6.5) tells us that without using the ND3.5 filter, we can go up to about 1500GHz of bandwidth using the same available laser power with the same SSH diffraction efficiency, indicating that the system bandwidth limited by the \( \text{Tm}^{3+}:\text{YAG} \) crystal bandwidth. In this case, we would have to use some other high inhomogeneous bandwidth material, such as \( \text{Er}^{3+}:\text{LiNbO}_3 \) (200 GHz of bandwidth, but with 50 times the dipole matrix element giving 7 times the absolute efficiency as \( \text{Tm}^{3+}:\text{YAG} \) [99].

In Eq. (6.5), we have assumed that we could eliminate the background noise caused by unwanted leakage, diffraction and scattering from the AOMs, spatial filters, cryostat windows and other components. With the current experimental setup, the background leakage due to the nominally blocked un-diffacted beams of the gate AOMs is about at the same level as the photon echo signal. To explore more bandwidth, one needs to solve the DC leakage of the gate AOMs due to diffraction and scattering around the Fourier plane filter. One solution is to use less tightly focused beams in the AOMs, thereby allowing the diffracted and un-diffracted spots to be better resolved. Another way of avoiding the scattering is to use choppers instead of AOMs for timing control. The one-laser experiment required complicated timing waveform sequences that were not easily implemented by using choppers. However, for a two-laser experiment, it is practical to use two synchronized choppers for the recording/reading control and therefore they will not have the AO scattering problem at all and will have higher throughput (100% vs at best 50% AO diffraction efficiency) at the expense of slight jitter of the timing. This makes it easier to explore the 20GHz
bandwidth of the Tm$^{3+}$:YAG crystal even with the current power level.

The following part of this chapter discusses the two-laser locking circuit for realizing the full version of the experiment.

6.2 Two-laser locking circuit

6.2.1 Introduction to laser locking techniques

Laser diodes are subject to noises that affect their instantaneous linewidth and long term stability. High frequency fluctuations broaden the linewidth of the laser diodes. Low frequency fluctuations, such as thermal drifts of the laser cavity, shift the center frequency of the laser. Laser locking techniques compare the laser frequency with a stable source and use feedback techniques to compensate the fluctuations, thus reducing the linewidth and stabilizing the central frequency of the laser. The stable source could be an optical cavity [100], a spectral hole [101], or another more stable laser [102]. All the locking systems contain an important part, a phase/frequency discriminator that measures the frequency error signal and turns it into voltage error. Both the cavity locking system and the hole locking system are developed from frequency modulation (FM) spectroscopy [103, 104]. The cavity locking system uses the narrow transmission features of a Fabry-Perot cavity as a phase/frequency discriminator. The spectral hole locking system uses the transmission of a spectral hole as the phase/frequency discriminator. And a two-laser locking system detects the interferometric beat between the two lasers and uses an electronic phase/frequency discriminator.

FM spectroscopy is a type of optical heterodyne spectroscopy [103, 104]. A laser beam with amplitude $E_0$ at frequency $\omega_c$ is represented as

$$E_1(t) = E_0 \exp(i\omega_c t)$$

(6.6)

is phase modulated at frequency $\omega_m$, producing two weak sidebands at frequencies $\omega_c \pm \omega_m$ to be

$$E_2(t) = E_0 \left\{ -\frac{M}{2} \exp[i(\omega_c - \omega_m)t] + \exp[i\omega_c t] + \frac{M}{2} \exp[i(\omega_c + \omega_m)t] \right\},$$

(6.7)

where $M << 1$ is the modulation index allowing us to neglect the higher order sideband set $\pm k\omega_m$ ($k = 2, 3, \cdots$). The modulated beam passes through an optical reference, such as a Fabry-Perot
cavity or atomic gas cell or hole-burning media that contains spectral features. In the case of an absorptive cell, the sample has a length of \(L\) with intensity absorption coefficient \(\alpha(\omega)\) and refractive index \(n\). The transmission of the sample can therefore be written as

\[
T_j = \exp(-\delta_j - i\phi_j),
\]

where \(j = 0, \pm 1\) denotes the components at \(\omega_c\) and \(\omega_c \pm \omega_m\) the real part \(\delta_j = \alpha_j L/2\) describes the amplitude attenuation and the imaginary part \(\phi_j = n_j L(\omega_c + j\omega_m)/c\) describes the optical phase shift experienced by each of the orders \(j = 0, \pm 1\). Then the transmitted field is

\[
E_3(t) = E_0 \left\{ |T_0|^2 - \frac{M}{2} \left[ T_{-1}^* T_0 \exp(i\omega_m t) + T_{-1} T_0^* \exp(-i\omega_m t) \right] 
+ \frac{M}{2} \left[ T_0^* T_1 \exp(i\omega_m t) + T_0 T_1^* \exp(-i\omega_m t) \right] \right\}. \tag{6.9}
\]

Dropping terms of order \(M^2\) and we get

\[
I_3(t) \approx E_0^2 \left\{ |T_0|^2 - \frac{M}{2} \left[ T_{-1}^* T_0 \exp(i\omega_m t) + T_{-1} T_0^* \exp(-i\omega_m t) \right] 
+ \frac{M}{2} \left[ T_0^* T_1 \exp(i\omega_m t) + T_0 T_1^* \exp(-i\omega_m t) \right] \right\}, \tag{6.10}
\]

assuming that \(\delta_0 - \delta_1\) and \(\delta_0 - \delta_{-1}\) are small compared with unity, the transmitted intensity can be written as

\[
I_3(t) \approx E_0^2 e^{-2\delta_0} \left[ 1 - M(\delta_0 - \delta_{-1}) \cos(\phi_{-1} - \phi_0 + \omega_m t) 
+ M(\delta_0 - \delta_1) \cos(\phi_0 - \phi_1 + \omega_m t) \right], \tag{6.11}
\]

assuming that \(|\phi_0 - \phi_1|\) and \(|\phi_0 - \phi_{-1}|\) are small compared with unity. The transmitted intensity can be written as

\[
I_3(t) \approx E_0^2 e^{-2\delta_0} \left[ 1 + \Delta \delta M \cos \omega_m t + \Delta^2 \phi M \sin \omega_m t \right]. \tag{6.12}
\]

where \(\Delta \delta = \delta_1 - \delta_{-1}\) and \(\Delta^2 \phi = (\phi_1 - \phi_0) - (\phi_0 - \phi_{-1})\). When \(\omega_m\) is small with respect to the spectral feature, \(\Delta \delta\) is proportional to the derivative of the absorption of the spectral feature and \(\Delta^2 \phi\) is proportional to the second derivative of the dispersion. For a Lorentzian spectral feature

\[
\mathcal{L}(\omega) = \frac{R(\omega)}{1 + R^2(\omega)} + i \frac{1}{1 + R^2(\omega)}, \tag{6.13}
\]
Figure 6.2: Spectral feature with complex Lorentzian lineshape.

where \( R(\omega) = \frac{\omega - \Omega}{\Delta \Omega/2} \), \( \Omega \) is the center of the Lorentzian line and \( \Delta \Omega \) is the full width at half maximum, \( \delta \) and \( \phi \) can be written as \( \delta(\omega) = \frac{1}{1 + R^2(\omega)} \) and \( \phi(\omega) = \frac{R(\omega)}{1 + R^2(\omega)} \). Define \( R_0 = R(\omega_c) \),

Figure 6.3: \( \Delta \delta \) and \( \Delta^2 \phi \) when \( \omega_m = 4\Delta \Omega \).
Fig. 6.2 shows the real and imaginary parts of the Lorentzian line with $\Omega = 0$. Scanning $\omega_c$ while keeping $\omega_m$ constant, one can trace $\Delta \delta$ and $\Delta^2 \phi$ as is shown in Fig. 6.3 where $\omega_m = 4\Delta \Omega$. Between frequency $R_L$ and $R_R$, $\Delta^2 \phi$ can be used as a frequency discriminator: when $\omega < \Omega < 0$, $\Delta^2 \phi > 0$; when $\omega > \Omega > 0$, $\Delta^2 \phi < 0$. When $\Delta^2 \phi$ is fed back to the laser frequency controller (piezo or current), the laser frequency can be locked to the center of a Lorentzian line. Most commonly, atomic vapors are used as a Lorentzian spectral reference, when an appropriate atom can be found at the desired frequency, but as an alternative for our SHB experiments, we utilize a spectral hole engraved by the laser itself. By locking to a Lorentzian line, the laser linewidth can be reduced to fraction of a spectral hole width. More detailed analysis of spectral hole locking to rare earth doped crystals can be found in [105].

A cavity locking system also uses FM spectroscopy. Instead of using the transmission of a spectral feature as a frequency discriminator, it uses the reflection of a stable high Finesse Fabry-Perot cavity. Suppose

$$E_{\text{inc}} = E_0 e^{i\omega t} \quad (6.14)$$

is the electric field of the incident beam, $r$ is the amplitude reflection coefficient of the two end mirrors and $\Delta \nu_{\text{fsr}} = \frac{c}{2L}$ is the free spectral range of the cavity of length $L$. Define phase $\psi = \frac{\omega}{\Delta \nu_{\text{fsr}}}$. Without losses, the reflected beam can be written as

$$E_{\text{ref}} = E_{\text{inc}} \left[ r - (1 - r^2)^{1/2} \cdot r \cdot (1 - r^2)^{1/2} \exp(i\psi) \right. \right.$

$$
\left. \left. - (1 - r^2)^{1/2} \cdot r^3 \cdot (1 - r^2)^{1/2} \exp(i2 \cdot \psi) \right. \right.$$

$$
\left. \left. \cdots \left. \left. \left. - r^{2n-1} \cdot (1 - r^2) \exp(1 - n \psi) \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r

And the complex reflection coefficient $F(\omega)$ of the cavity is

$$F(\omega) = \frac{E_{\text{ref}}}{E_{\text{inc}}} = r \frac{\exp(i\psi) - 1}{1 - r^2 \exp(i\psi)} \quad (6.16)$$
Now $F(\omega)$ can be viewed as the “spectral feature”. In this case,

$$
\begin{align*}
\delta(\omega) &= \text{Real}\{F(\omega)\} = \frac{r(1 + r^2)(\cos \psi(\omega) - 1)}{1 - 2r^2 \cos \psi(\omega) + r^4} \\
\phi(\omega) &= \text{Imag}\{F(\omega)\} = \frac{(1 - r^2) \sin \psi(\omega)}{1 - 2r^2 \cos \psi(\omega) + r^4}.
\end{align*}
$$

(6.17) (6.18)

Scanning $\omega_c$ while keeping $\omega_m$ constant, one can trace $\Delta \delta$ and $\Delta^2 \phi$ as is shown in Fig. 6.4 where $\omega_m = 5\Delta \nu_{\text{fsr}}$ with a cavity finesse of 500. Similarly, $\Delta^2 \phi$ can be used as feedback error signal locking the laser to the cavity. An alternative way of analyzing the cavity locking system can be found in [106].

A two-laser locking system uses an electronic phase/frequency discriminator [97, 98] that changes the output voltage linearly with the input phase/frequency difference. I will talk more about this in Section 6.2.2.2.

### 6.2.2 Design PDH two-laser locking circuit

The high bandwidth SSH scanner experiment in Section 6.1 requires two lasers, one (Koheras) for the recording process and the other (Velocity) for the reading process. To make sure that the...
recording and reading process happen at the same fraction of the spectral band, we must lock the two lasers together. The locking is not to reduce the linewidth, but is to make the central frequency of the lasers the same over the recording and reading process of the experiment. Since one of the lasers, the Koheras fiber laser, is chirped during the recording process, it cannot be locked to the other laser (Velocity ECDL) during this time. The two lasers need to be locked only during the reading time when the Koheras is operating CW. So the two-laser locking needs to be periodic and then held during the chirp of the Koheras fiber laser.

In the following subsections, I will start with the general idea of the two-laser locking circuits in subsection 6.2.2.1. Subsection 6.2.2.2 talks in details about the considerations of choosing the circuit components. And subsection 6.2.2.3 describes the PCB layout of the circuit.

6.2.2.1 General considerations

The two-laser locking circuit uses a phase-lock loop. The basic idea of a phase-lock loop is shown in Fig. 6.5. First, we compare the frequencies of the two lasers (\(f_{\text{master}}\) and \(f_{\text{slave}}\)) and convert the phase/frequency difference into a voltage signal that drives one of the lasers’ piezo or current port (in the phase-lock loop circuit, the piezo or current port that controls the lasers frequency can also be thought of as voltage controlled oscillator (VCO) since it changes the frequency of the laser). For example, when the master laser has a frequency of \(f_0\) and the slave laser has a frequency of \(f_0 + 25\text{KHz}\), the desired circuit will generate a 1mV signal that drives the piezo with
a coefficient of 25MHz/V and decreases the slave laser’s frequency by 25KHz. When this feedback process is dynamic, the frequency of the two lasers can be constantly locked together. There are two key parts of this locking circuit: a phase/frequency detector that converts the phase/frequency difference into a voltage signal; a feedback circuit that changes the output of the phase/frequency detector and shapes the signal’s bandwidth to accommodate the slave laser’s piezo or current port, thereby locking the slave laser frequency to be the same as the master laser frequency.

In practice, the phase/frequency detector can not detect the frequency difference of the two lasers directly. Instead we beat the two lasers in an interferometer and heterodyne-detect the beat-note with a photodetector as is shown in Fig. 6.6. Write the electric field of the Koheras laser and the Velocity laser as

\[ E_k(t) = \mathcal{E}_k(t) \sin[2\pi f_k t + \phi_k(t)] \]

and

\[ E_v(t) = \mathcal{E}_v(t) \sin[2\pi f_v t + \phi_v(t)], \]

where \( \mathcal{E}_k(t) \) and \( \mathcal{E}_v(t) \) are the slow varying amplitudes of the Koheras and Velocity laser, \( f_k \) and \( f_v \) are the center frequency and \( \phi_k(t) \) and \( \phi_v(t) \) are the time-varying phase of the two lasers and there is no linear frequency terms in either \( \phi_k(t) \) or \( \phi_v(t) \). The output voltage of the photo-detector
is proportional to the intensity of light incident onto the detector. The intensity of the detected signal at the photo-detector can be written as

\[
I \propto [E_k(t) + E_v(t)]^2
\]

\[
= \frac{1}{2} [E_k^2(t) + E_v^2(t)] + 2E_k(t)E_v(t) \exp \left[ i(f_k - f_v)t + \phi_k(t) - \phi_v(t) \right] + \text{c.c}
\]

\[
= I_{DC} + I_{\text{diff}}
\]  

(6.21)

where \( I_{DC} \) is responsible for the DC voltage and the sum frequency terms are dropped since the beat is well out of the detector’s detection range and can be neglected.

\[
I_{DC} = \frac{1}{2} [E_k^2(t) + E_v^2(t)],
\]  

(6.22)

and

\[
I_{\text{diff}} = E_k(t)E_v(t) \cos \left[ 2\pi(f_k - f_v)t + \phi_k(t) - \phi_v(t) \right].
\]  

(6.23)

The difference frequency term \( I_{\text{diff}} \) is the term we use for the laser locking circuit. Locking the two lasers at the same frequency \( f_k = f_v \) directly is undesirable since the difference frequency term will also appear at DC making the locking sensitive to amplitude noise of the lasers as well as technical \( 1/f \) noise. Modulating the laser frequency to a higher frequency enables detection at the region where the technical noise is near the shot-noise limit. In our case, instead of modulating one of the laser’s frequency, we lock the two lasers with a frequency offset of \( \delta f = f_k - f_v \). The detected voltage signal is compared with a CW reference signal produced by a stable RF oscillator using a phase detector. After a proportional-integral-derivative (PID) feedback control circuit, the signal is fed back to the Koheras laser piezo. The transfer function of this feedback loop can be written as

\[
H(s) = \frac{G_K(s)}{1 + G_K(s)G_H(s)G_{\text{phase}}(s)G_{\text{PID}}(s)}
\]  

(6.24)

where \( G_K(s) \) is the frequency dependent sensitivity of the piezo of the Koheras laser (Hz/V), \( G_H \) is the transfer function for the heterodyne detection, \( G_{\text{phase}} \) is the transfer function of the
phase/frequency discriminator that has a unit of V/Hz, and $G_{\text{PID}}$ is the transfer function of the PID circuit. Suppose the laser piezo has a second order response, 

$$G_K = \frac{1}{s^2 + a_1 s + a_2},$$

where $s = j \omega$ is the Laplace complex plane frequency variable. Neglect the amplitude variation effect (slowly varying envelop) on the output frequency of the heterodyne detection, so that $G_H = \alpha_H$. The phase/frequency detector is a linear system within certain frequency range with $G_{\text{phase}} = \alpha_{\text{phase}}$. For an input error signal of $e(t)$, the output $u(t)$ of a PID circuit can be written as [107]

$$u_{\text{PID}}(t) = K_p e(t) + \frac{K_p}{T_I} \int_0^t e(\tau) d\tau + K_p T_D \frac{de(t)}{dt},$$

where $K_p$ is the proportional gain, $T_I$ is the integral or reset time and $T_D$ is the derivative time. The three terms correspond to proportional, integral and differential gain. The proportional gain compensates the error signal that is currently present in the system. The integral gain takes care of the errors of the past. It makes it possible for the output to have nonzero outputs even when the input is zero. This is important for getting rid of a constant disturbance to the system. And the derivative gain anticipates the future since it leads the proportional gain. The transfer function of the PID circuit can be written as [107]

$$G_{\text{PID}}(s) = K_p + \frac{K_p}{T_I s} + K_p T_D s$$

(6.26)

So the close-loop characteristic equation of the feedback loop can be written as

$$s^3 + (a_1 + K_p T_D) s^2 + (a_2 + K_p) s + \frac{K_p}{T_I} = 0.$$  

(6.27)

The roots of the characteristic equation Eq. 6.27 determine the response of the system can be uniquely determined by controlling the parameters $K_p$, $T_I$ and $T_D$. Fig. 6.7 compares the response of the system with only P, PI, and PID when there is a unit disturbance. Bigger P gain gives bigger oscillations and destabilize the system. I gain can also destabilize the system, but it compensates the constant offset error. D gain will not decrease the error but it will affect the damping rate of the oscillations. It makes the system more stable. Detailed analysis can be found in [107].

A more detailed version of the feedback locking system is shown in Fig. 6.8. The light from the Koheras laser and the Velocity laser with a frequency offset of 40MHz is focused and
Figure 6.7: Close loop system response of a step function response when there is P, PI and PID gain.

Figure 6.8: Detailed block diagram of the 2-laser locking system.

interferometrically detected by two identical photo-detectors ET-2030A with 1.2GHz of detection bandwidth. The output of one of the detectors goes to a bandpass filter and the other goes to a spectrum analyzer for monitoring and tuning the feedback control behavior. Without the
frequency offset, one will detect the error signal at DC, where there can be significant amplitude noise. The frequency offset between the lasers allows us to use a bandpass filter to filter out the beat signal at the unwanted frequency band from the detector and increase the signal-to-noise ratio for later processing. This offset also allows for the disambiguation of positive vs negative frequency excursions of one laser with respect to the other (39MHz is different clearly different from 41MHz while +1MHz is not distinguishable from −1MHz). We want the offset frequency of the lasers to be sufficiently high so that the filtering process does not yield a significant phase shift within the desired feedback bandwidth but not too high to make the bandwidth of the post-processing circuits too demanding. In our experiment, we choose to offset the laser frequencies by 40MHz. After the bandpass filter, the beat signal of the two lasers is amplified and sent to the LO port of a phase/frequency detector where the frequency is compared with a CW tone at 40MHz generated by a signal generator. The output from the phase detector is lowpass filtered. Then for the purpose of periodic locking, the signal is sent to a sample/hold (S/H) chip that can be controlled by logic signals to choose between two states: a sample state and a hold state. In the sample state, the input signal goes through the chip without being modified. In the hold state, the input signal will not affect the output signal and the output will instead hold at the signal value right before the logic signal switches signs. A PID circuit that insures the stability of the feedback loop follows the S/H. The periodic chirp control signal that chirps the laser is summed with the error signal with a summing amplifier. The sum is then amplified by 40 and fed to the Koheras piezo to stretch the fiber and control the laser frequency.

The following sub-sections discuss the components contained in the circuit.

6.2.2.2 Components of the circuit

I learned lots of practical considerations from [108]. In this section, I will walk through the key components of the circuit schematic diagram in Fig. C.1- Fig. C.3 presented in appendix C. There are two input ports in Fig. C.1 of the circuit, a LO input port and a RF input port. The LO port is fed with the 40MHz offset CW tone from a signal generator. The RF port is fed with the
beat-note between the master and slave lasers. The beat-note signal of the two-lasers are detected by a 1.2GHz bandwidth photo-detector, ET-2030A. Most of the bandwidth is useless for the later processing and will hurt the SNR. A bandpass filter will help reduce the noise out of the bandwidth of interest. In practice, we use a simple 5MHz first order RC-highpass filter in the circuit combined with a 70MHz Mini-circuits lowpass filter in front of the circuit to let through the 40MHz offset frequency between the two lasers. An alternative solution is to design a bandpass filter centered at 40MHz by using a filter design software such as “AADE filter design”. However, a disadvantage of this software is that it uses capacitor and inductor values that in most cases are not commercially available. In some filter designs, the filter properties are not very sensitive to the slight changes of the capacitor/inductor values and we can use commercially available components to substitute the desired components. Unfortunately, this is not the case for the bandpass filter we want to design here. So we end up using a lowpass-highpass filter.

**RF amplifier**

I need an RF amplifier that amplifies the signal from the photo-detector to feed the analog-to-ECL converter required by the phase/frequency detector. The non-distorted highest output voltage of the photo-detector is $0.5V_{pp}$ and the input voltage range of the analog-to-ECL converter I use is from $-2V$ to $3V$. Therefore we need a gain of 10 or 20dB. In experiment, the beat signals without bias is typically smaller than $0.5V_{pp}$ and we found that using a Mini-circuits MAR-1SM+ (DC-1GHz, 16.5dB gain) works fine for the later circuit processing. For smaller input detection signal and higher requirements of gain, one could consider using higher gain amplifiers (ERA-8SM+ 19dB or MAR-6SM+ 20dB).

After an analog-to-ECL converter ADCMP564, the reference signal (CW tone at 40MHz from a signal generator) and the amplified local oscillator signal (beat-note of the master and slave lasers) are converted into ECL signals ready for processing by the phase/frequency detector.

**Phase/frequency discriminator**

There are two types of phase detectors [109]. Type I phase detector is simply an exclusive-OR
gate. Type II phase detector is sensitive only to the relative timing edges between the two input signals. I use an easy-to-use type I phase detector from Analog Devices AD9901 [97, 98]. It has higher linear phase detection range (360° up to 40KHz) than a typical mixer (90°).

Fig. 6.9 shows the functional block diagram of AD9901. It has two exchangeable input ports, reference input and oscillator input, both of which only accept ECL signals. It works in two distinct modes: as a phase detector and as a frequency discriminator. When the two inputs are close in frequency, only the phase detector circuit is active; when the two input frequencies are substantially different, the frequency discriminator circuit overrides the phase detector output. The phase detector contains two D-type flip-flops and an XOR gate. The D-type flip-flop switches the output upon input rising edges.

Fig. 6.10 shows the timing waveforms when the two inputs are at exactly the same frequencies but with different phases. Fig. 6.10(a) shows the situations when the inputs are exactly π out of phase. The flip-flop outputs halves the frequency of the inputs since it only switches upon rising edges. The flip-flop outputs are applied to an XOR-gate (1: different inputs; 0: same inputs), giving rectangular pulses with 50% duty cycle. Similarly, Fig. 6.10(b) shows the situations when the oscillator input lags behind the reference input, where the XOR-gate outputs have more than
Figure 6.10: AD9901 timing waveforms when two inputs are at the same frequency. The figures are extracted from AD9901 data sheet from Analog Devices.

Figure 6.11: Phase gain plot of AD9901. The figures are extracted from AD9901 data sheet from Analog Devices.

50% duty cycle. Fig. 6.10(c) shows the situations when the oscillator input leads the reference input, where the XOR-gate outputs have less than 50% duty cycle. After low-pass filtering the XOR-gate outputs, we can extract the DC information, which is proportional to the phase difference of the inputs, as is shown in Fig. 6.11. We can use the filtered XOR outputs to lock the oscillator inputs to the reference inputs at the \( \pi \) phase shift point. Locking at the \( \pi \) phase shift point instead of at the 0 phase shift point helps to use the linear region of the phase detector, avoiding the dead phase detection zone around 0 phase.
When the reference and oscillator frequencies are substantially different, the frequency high or low flip-flop will override the XOR gate output and hold the output voltage at appropriate level, either high or low, to pull the oscillator frequency toward the reference frequency. The outputs of the AD9901 for different input frequency situations are shown in Fig. 6.12(a)-(c).

After the AD9901, a lowpass filter with a 600KHz corner frequency is used to extract the DC mean value of the error signal and prevent high frequency oscillation in the feedback circuit from oscillating. The output of the phase/frequency detector are negative voltage signals from 0 to about $-60\text{mV}$. To make the error signal centered at 0, a differential amplifier (AD8132) is used to convert the differential output signals from the AD9901 into single-ended signals. An AD8132 is typically used as receivers for the transmission of high speed signals over twisted-pair cables. It has good common mode rejection (-70dB), which enables us to decrease the affects of external noise or crosstalk. One could also use a low offset voltage, low bias current op-amp and operate it as a subtractor to get the same result, although trimming would be required.

The PID circuit

So far I have talked about the front end of the circuit. I now want to get into the part that connects the error voltage signal and the laser piezo. I use a PID circuit for this part. Now that we have changed the frequency difference signal into an error voltage signal, we want to feed the error
signal to the VCO, in this case it is our laser piezo. But the voltage error might not be exactly what we want, so we need to build a stable feedback control system. This is the place where the proportional gain comes in. But the proportional gain itself might make the output overshoot, to avoid such oscillation, we can use a derivative of the voltage error (the rate of change of the voltage error), so that as the beat frequency approaches the 40MHz of a reference oscillator, the gain decreases its rate of change. To avoid large DC error, we can either tune up the proportional gain, but this may cause ringing, or we can integrate the DC of the error signal with an integrator. Then we use an adder to sum up all these error signals. In a typical PID circuit, it is often required to also use an inverter after the summer. But in our case here, it is not necessary because we could simply tune one of the laser frequencies to switch the sign of the error signal ($f_{\text{Koheras}} > f_{\text{Velocity}}$ or $f_{\text{Koheras}} < f_{\text{Velocity}}$ will make the error signal switch sign).

The above is the ideal situation. Real op-amps have bias currents and offset voltages. We need to slightly modify the circuit by adding some resistors and capacitors. Ideal op-amps have zero flow of current at the input ports and have equal voltage at the inverting and non-inverting port. But real op-amps have a small current into or out of the input ports, called the bias current. They also could have a voltage difference between the two ports, called the offset voltage. Those two things can change the integration range of the integrator and introduce voltage error to the output. Because the two ports have about the same bias current, adding a resistor to the + input port could get rid of the effect of bias current. To reduce the offset bias voltage, we could add a voltage divider at one of the input ports the op-amp. I use one voltage divider, $R_{65}$ for the whole circuit.

Now what kind of op-amps should I use for the PID circuit? I mainly looked at the following parameters: slew rate, input/output voltage range, noise density, bias current and offset voltage. Slew rate is the maximum rate of change of a signal. Going beyond it gives non-linear output for the op-amp. And the effect of this to the PDH circuit is to transform the higher frequency error signal to lower frequency so that we get a fake lower frequency error signal. Because I want both positive output and negative outputs, I need op-amps with dual rail supplies. Low bias current and
offset voltage are especially important for the integrator and differentiator. AD8620, AD8066 and AD8034 are all pretty good candidates with the same package footprints.

**Choosing the other parts**

A configurable sample/hold LF298 is used to make the periodic locking of the two lasers possible. It is controlled by logic input signals to switch between the sample mode where the error signal goes straight through without being affected and the hold mode where the output of LF298 holds at the last error signal value. In sample mode, the two lasers will be locked. In hold mode, the two lasers are running independently.

I use an analog switch U8 (ADG1212) to switch the output between chirp control signals (hold mode) and laser locking error signals (sample mode). Another analog switch U8 (ADG1212) is used to switch off the integrator during the hold mode to prevent the integrator from saturating. I also use several analog switches U9 and U10 as a gain selection knob for the integrator. When choosing the analog switches, I mainly looked at the leakage current and charge injection. Because the signal that is controlled by the analog switches could be both positive and negative, I choose switches with dual power supplies that can support both signs of signals. One thing I kept doing wrong and forgetting about is the input/output range of the switches and op-amps. It is a necessary condition for the circuit to work. The noise performance and leakage current characteristics are conditions for the circuit to work well.

I need 7 different power supplies. When choosing the voltage regulators, I first looked at the components that are driven by each of the power supplies and calculate the total current needed for that voltage source. Then I considered the noise characteristics of the regulators. I mainly used regulators from Linear Technology because its regulators generally have lower noise. Fig. C.3 is the schematic diagram of the voltage regulators. Each of the regulator has at least one input and one output capacitor for stability (big value capacitor) and for high frequency noise filtering (small value capacitor). For non-polarized ceramic capacitors, I use X5R or X7R dielectric, because of they have more stable temperature coefficient. For polarized capacitors, I use low ESR tantalum. Tantalum
capacitors are vulnerable to voltage overshoot, so it is important to have capacitor voltage ratings at least twice of the highest voltage of the circuit. Some of the regulators are adjustable and require resistors to set the output voltage. I use 1% accurate resistors for accurately setting the voltage outputs.

6.2.2.3 PCB board design

I use a four-layer board. The top and bottom layers are signal layers. The two internal planes are power plane and ground plane. I use the software program *Altium* to do the PCB layout. I first arrange the major components according to the signal flow, as is shown in Fig. 6.13. Signal flow always has the first priority when arranging the components because it is the major
factor that affects the length of the traces. The longer the trace length, the bigger inductance the trace has and the signal is also more subject to coupling noise. When roughly arranging the major components, I also take into consideration that the components using the same power supply stay together in a convenient way for the power plane divisions. Then I arrange the other components, i.e. resistors and capacitors around the major components. The power plane is divided into 7 parts for 7 different voltage sources: ±2V digital, ±5V digital, ±5V analog, ±15V analog. Fig. C.4 in appendix C shows the power planes. After rough arrangements of the components and division of power planes, I consider putting the voltage regulators in place. The voltage regulators are put in place according to the rough assignment of power planes. Since I use top and bottom layers for signal traces, traces could cross on those two layers. But power planes can not cross. So I crossed traces at the voltage sources to avoid problems due to crossed power plane. For PCB routing, the arrangement of the components takes most of the time. Good arrangements make the routing process almost effortless. The shaded parts within the blue box in Fig. 6.13 have digital signal flow and the rest have analog signal flow. The circuit before the differential amplifier has about 100MHz of bandwidth. In Fig. C.1, between the ADCMP564 and the integrating capacitors (C15 and C16), the signals are digital. I made an effort to make the traces short for the digital part of the circuit. The circuit after the low-pass filter has a bandwidth of less than 1MHz so it is not that critical to have short trace length.

Fig. C.5 in appendix C is the PCB diagram I sent to 4pcb.com for manufacturing. I used 8mil of width for regular traces and fatter traces, - 20mil, for the power traces. I also used fatter traces, 14mil, that connect the bypass capacitors to ground to allow higher current flow. I used 18/30mil for regular vias and 30/50mil for regulator related vias. Smaller value bypass capacitors are put to the power pin of the chips as close as possible. I used copper pour (ground plane) for the top and bottom of the PCB layers for better heat conduction for the voltage regulators. This also helps shield high frequency noise.
6.2.3 Test of the PDH circuit

Like aligning optical components one sub-system at a time, it is a good idea to solder one subsection of the circuit at a time and debug them one by one. To prevent damage to the chips, I start with only soldering the parts related to the voltage regulators. Unfortunately, the solder pad layouts recommended by the application notes of the voltage regulators are not for hand soldering, so I had big trouble soldering especially the big grounding pads for heat dissipation. I constantly had lots of loose connections. It would be much easier for soldering if one makes the pad layout slightly bigger than what is suggested on the application notes. After soldering the voltage regulators and their related components (capacitors and resistors), I make sure there were no short circuits between the voltage sources and the ground. Then I powered the circuit by four independent power supplies: ±17V and ±7V. After that, I confirm the output voltages for each of the voltage regulators (−2V, −5.2V digital, +5V digital, ±5V analog, ±15V).

I soldered the rest of the major chips one at a time, making sure that the outputs of these chips are what we expect. To debug the functioning of the comparator (ADCMP564) and phase/frequency discriminator (AD9901), I used sine waves at the same frequency generated by a two-channel arb for the LO input port and the RF input port of the circuit. Fig. 6.2.3(a) is the input sine wave for both ports. Fig. 6.2.3(b) shows outputs of amplified LO by the MAR - 1SM+ RF amplifier. Fig. 6.2.3(c) is the ECL output in the LO arm. Fig. 6.2.3(d) is the ECL output in the RF arm. When the LO input and RF input had the same phase and the same frequency, the ECL outputs of the LO and RF arms are the same. Notice that due to the bandwidth of the probe, the rectangular waveforms have rings at the corners. Fig. 6.2.3(e) shows the AD9901 outputs with different relative input phase between the LO and RF ports. The integrated outputs of AD9901 linearly vary with the phase difference of the two input ports. This property is why the AD9901 can be used to build a phase lock loop. The integrated output of AD9901 can be used to control a voltage controlled oscillator (VCO) that changes the frequency of the RF input.

When the LO and RF ports are fed with different frequencies, the AD9901 has “chirped”
square outputs, as is shown in Fig. 6.15(a)–6.15(b). When \( f_{LO} < f_{RF} \), the AD9901 output stays longer at high voltage (0V). When \( f_{LO} > f_{RF} \), the AD9901 output stays longer at low voltage (about −0.6V). Fig. ?? shows the Phase/frequency discriminator (AD9901) outputs with different relative phase differences from 0° to 360° between the LO and RF ports. The duty cycle is proportional to the phase differences of the LO and RF ports. The integrated output (DC mean value) is linearly proportional to the phase differences of the two ports. This is how AD9901 works as a phase discriminator. Notice that AD9901 output is not 0V at 180° phase since it gives all negative output voltages. This is the reason why we use a potentiometer \( R_{65} \) in Fig. C.2 to adjust the bias voltage. The integrated outputs of AD9901 can be used to bring the RF frequency to the LO frequency via laser piezo.

After debugging the open-loop circuit behavior, I debugged the close-loop two-laser locking circuit. Since the Velocity laser has a current modulation port that has higher bandwidth (1MHz) than the Koheras piezo port (100KHz) and at room temperature the free-running Koheras laser is more stable than the free-running Velocity laser, I started with locking the Velocity laser to the Koheras laser instead of the other way around. The current modulation port of the Velocity laser has an input voltage range of −10V to +10V. Before closing the loop, I measured the output voltage range of the circuit, making sure that it is within this range. Initially, I only used the proportional gain and made \( R_{40} = 100\text{k}\Omega \). By tuning \( R_{38} \), I can change the proportional gain. I found that without the other two gains, a small proportional gain (\( R_{38} = 0.5\text{k}\Omega \)) can make the laser frequency oscillate easily, as is shown in the RSA3800 time-frequency diagram in Fig. 6.16.

I gradually turned up the differential gain until the side band was about the same height as the main peak. After adding the differential gain (\( T_D = 3.3\mu\text{sec} \)), the oscillation problem is eliminated, but the lock is not very stable, as is shown in Fig. 6.17. The two lasers can only be locked together temporarily. The lock is not very robust. Once the two lasers are unlocked they will not re-lock automatically.

When the lasers are temporarily locked, one can gradually increase the integral gain (\( T_I = 78\mu\text{sec} \)). The lasers should stay locked. The lock stayed stable for about 1 hour and 40 minutes
until the laser frequency drifted out of the tuning range of the Velocity laser current modulation port. The time-frequency display of the two-laser beat-note is shown in Fig. 6.18. Limited by the size of the data, I did not take longer data set. The spectrum at one instant of time is shown in Fig. 6.19(a). The loop bandwidth is about 180KHz with about 18% of the power in the error spectrum shown in Fig. 6.19(b). Fine-tuning the gains can slightly decrease the amount of power in the error spectrum and make the lock more robust.

In Section 6.2, I demonstrated a two-laser locking circuit that locks a Velocity laser to a Koheras fiber laser constantly. By including the function of the sample/hold circuit, one can modify the constant two-laser locking circuit to a periodic locking circuit. Demonstrating the function of the periodic locking circuit requires cryogenically cooling down the Tm$^{3+}$:YAG crystal and stabilizing the Velocity laser to the spectral hole feature burned into the Tm$^{3+}$:YAG crystal, say using a Scientific Material locking box. The periodic locking circuit locks the Koheras laser to the stabilized Velocity laser only during the read-out process. During the recording process, the circuit is in the “hold” state and the Koheras laser will be free-running. The periodic locking circuit is an essential part of the two-laser SSH scanner. With the periodic locking circuit, one can realize the future version of the SSH scanner and explore more bandwidth.
(a) Input sine waves at 40MHz for RF and LO ports of the circuit.

(b) Amplified outputs of RF amplifier (MAR - 1SM+).

(c) ECL output of the comparator, AD-CMP564 for the RF port.

(d) ECL output of the comparator, AD-CMP564 for the LO port.

(e) AD9901 outputs with different relative phase differences from 0° to 360° between the LO and RF ports.

Figure 6.14: Use the sines at the same frequency generated by a two-channel arb for RF and LO input ports and probe the outputs of each of the chips for debugging the circuit.
Figure 6.15: AD9901 outputs when $f_{LO} \neq f_{RF}$. 

(a) $|f_{LO} - f_{RF}| = 10$ MHz

(b) $|f_{LO} - f_{RF}| = 0.1$ MHz
Figure 6.16: RSA3800 time-frequency diagram of the unstable two-laser beat-note with only proportional gain. The span of the displayed beat-note spectrum is 5MHz. The upper figure shows the beat-note power spectrum at an instant of time. The lower figure shows the time-frequency waterfall display of the beat-note with the vertical time axis and horizontal frequency axis.

Figure 6.17: Time-frequency waterfall display of the two-laser beat-note with proportional gain and differential gain. The vertical axis is time and the horizontal axis is frequency. The frequency span of the displayed beat-note spectrum is about 10MHz and the time span is 105msec.
Figure 6.18: Time-frequency waterfall display of the two-laser beat-note with proportional gain, differential gain and integral gain. The vertical is time axis and the horizontal is frequency axis. The frequency span of the displayed beat-note spectrum is about 0.6MHz and the time span is about 375msec.
(a) Beat-note of the two locked lasers. The loop bandwidth is about 180KHz.

(b) Power of the spectrum in linear scale. About 18% of the power is in the error spectrum.

Figure 6.19: Locking spectrum of the Koheras-Velocity beat-note.
Chapter 7

Concluding Remarks

Optical scanners have found a variety of applications varying from the mundane supermarket scanners to laser printers, optical data storage systems and optical signal processing. In advanced applications the acousto-optic Bragg cell has been the dominant technology, allowing the production of slowly scanned spots using a Fourier lens and changing the RF frequency, or rapidly scanning an arbitrary signal using the traveling-wave nature of acoustic propagation. But these devices typically operate with less than 100MHz bandwidth when using high efficiency photoelastic crystals and at most can achieve a few GHz bandwidth in wideband acousto-optic deflectors (AODs) with much lower efficiency. In this thesis, I have investigated an alternative technology for producing ultra-wideband scanners for applications in optical signal processing based on cryogenically cooled spatial-spectral holography. These devices have already been applied to the problem of RF spectrum analysis [27], and I wished to demonstrate here that they could also be utilized in a broader range of optical signal processors, so I chose to demonstrate a time-integrating correlator using our wideband SSH scanner.

To program the SSH scanner, we use a linearly chirped laser that chirps through the whole desired programming bandwidth as well as a synchronized Galvo scanning (GS) mirror that generates Doppler frequencies that linearly vary across its spatial aperture. The combination of the chirped laser and the GS mirror can generate Doppler, producing frequency shifted chirps that behave similar to time delays that linearly vary across one spatial dimension. A rare-earth-doped crystal is used to record the interference between a reference chirp and the linear spatially-dependent time delay
chirps to produce spectral gratings in the crystals inhomogeneous band whose spectral periodicity varies with position. The SSH scanner utilizes the physical nature of the rare-earth-doped crystal therefore avoids the computational complexities of digital processing techniques. In addition, it has the potential to break through the bandwidth limitations of AODs which are the dominant technology currently utilized.

There are three key components in the SSH scanner: a GS mirror, a SSH crystal and a chirped laser. To demonstrate this scanner, we first needed to understand the properties and behavior for each of them. I discussed the functions of the GS mirror and SSH crystal in Chapter 2, 3 and 4. In Chapter 5, I demonstrated the function of the SSH scanner with one laser. In Chapter 6, I discussed the future perspective of the full version SSH scanner that is capable of scanning a very broad band electro-optically modulated signal of interests.

In Chapter 2, I talked about two approaches to generating rocking plane waves that produce linear space dependent Doppler frequencies as required for programming the SSH delay line: a Galvo scanning mirror driven by a ramp wave form (GS mirror approach), or a pair of counter-propagating acousto-optic deflectors driven by slow chirps (AOD approach). In the GS mirror sections, I first reviewed some previous applications of GS mirrors and derived the frequency shifts the GS mirror driven by a ramp waveform generates. To experimentally determine the Doppler generation bandwidth and the stability of the GS mirror, I built a time-integrating interferometer using a photorefractive crystal as a recording medium. In the experiment, I used orthogonally polarized dual wavelength writing and reading of the photorefractive hologram in order to produce the highest fidelity, bias-free, correlations possible in order to precisely characterize the space dependent frequency shifts. I analyzed the Bragg match conditions for the recording and reading process of the two-wavelength holographic schemes and experimentally verified the Doppler channels generated by the GS mirror. In the AOD Doppler generation sections, two cascaded AODs driven by chirp waveforms were used for linear Doppler generation. I first summarized the material properties of the AODs we used for Doppler processing. Then I discussed the topologies and bandshapes of two cascaded AODs, investigating issues due to Bragg matching, polarization dependence of the
diffracted beam, and in-plane vs crossed architectures. Finally, I build an interferometer to verify the stability, accuracy, reproducibility, and spatial dependence of the Doppler frequencies generated by two crossed AODs.

To understand the physics of the cryogenically cooled SSH crystal as a recording medium, in Chapter 3, I reviewed the semi-classical quantum treatments of light-atom interactions for two-level atoms, and the major results of time and spectral-domain perturbation theory. With a few approximations to the spectral domain perturbation theory, I derived the results that are used to explain subsequent experimental observations. Finally, I compare the echo amplitude for using pulses or chirps to record the spectral domain gratings. These results are applied to predict the bandwidth potential of the SSH scanner based upon the available experimental results later in Chapter 6.

To further understand the dynamics of beam propagation and light atom interaction in the SSH crystal, in Chapter 4, I developed a novel approach to the simulation of light propagation and nonlinear interaction in SSH crystals. First, I reviewed and compared numerical techniques used previously for Maxwell-Bloch simulations and discussed the difficulties of such high dimensional simulations. Then I developed a new approach to robustly simulating time delayed photon echoes in both space and time and then numerically simulated the 2D Maxwell-Bloch equations and analyzed the simulation error and the stability of the algorithm. I gave simulation results for Rabi oscillations, two and three-pulse photon echoes and chirp photon echoes. Finally, I compared the simulation results with the experimental result for chirp photon echo experiments.

In Chapter 5, I put everything together and experimentally demonstrated a 1.5GHz bandwidth spatial-spectral holographic scanner with 20 resolvable spots using one laser for both writing the delay grating and reading them out. Using the results in the previous chapters, I analyzed the function of the SSH scanner. Then I did some diagnostic experiments to make sure each essential experimental component works well. Finally, I read out the spectral gratings recorded in the SSH crystal and also used a time-integrating correlator to verify the time delays recorded in the SSH crystal.
Using one laser for both recording and reading process, the scanner I built in Chapter 5 cannot process arbitrary signals of interests. In Chapter 6, I discuss the full version of the scanner that uses two lasers and can process an arbitrary signal modulated onto a CW laser using an EOM. The one-laser experiment can be extended to a two-laser experiment by including a two-laser periodic locking circuit. I then designed and demonstrated a two-laser locking circuit by using Pound-Dreaver-Hall locking techniques.

7.1 Novel contributions

The following are the major contributions I made in this thesis.

- In Chapter 2, I experimentally characterized the Doppler generation bandwidth and the linearity of the Doppler channels of a ramp-driven GS mirror and chirp-driven pair of counter-propagating AODs. The GS mirror can generate Doppler frequencies up to about 20KHz of bandwidth but is very power efficient. The AOD approach is less power efficient but could have many Megahertz of Doppler bandwidth.

- In Chapter 3, I applied the spectral domain perturbation theory to explain the effect of causality in the SSH scanner experiment. The result is used in Chapter 5 to explain the diffraction-off-the-causal-edge experiment. I also discussed the bandwidth dependence of the echo intensity.

- In Chapter 4, I developed a novel, stable, and efficient numerical simulator to solve the 2D Maxwell-Bloch equations, analyze the simulation error and the stability of the algorithm and compared the simulation results with the experimental result for chirp photon echo experiments.

- In Chapter 5, I experimentally demonstrated a 1.5GHz bandwidth SSH scanner with 20 resolvable spots. This is the first wideband demonstration of such a scanner that shows the coherence in the spatial domain.
7.2 Limitations of the SSH scanner

Like many other things in the world, the two biggest advantages of this SSH scanner do not come free. One needs to pay an engineering cost or trade off for both of them. The parallelism of this system comes at a price of power spreading. To make the processing parallel, the SSH scanner (as well as the Doppler processing system described in Chapter 2) divides the available power into multiple time-delay channels. Each channel corresponds to a distinct spatial position. Therefore, depending on the number of channels, the power available at each channel is only a fraction of the total power. A nonlinear process like photon echo requires sufficient power at each spatial spot to produce a diffraction larger than artifacts, scattering noise (e.g. from the gate AOMs) and leakage from the DC of the AOMs. Whereas a sequential solution like S$^2$CHIP [5], in which all the available power is concentrated for one channel each time and one needs to scan through all the time delays one at a time, is more power efficient than the power splitting parallel solutions.

Another advantage of this system is its high bandwidth. However, according to the analysis in Chapter 5, the highest time delay we can achieve is inversely proportional to the bandwidth. Thus, the higher the bandwidth, the smaller the total time delay we can get. The number of resolvable spots for this scanner is a better criteria for this system. The resolving power of this system is limited by the scanning range of the GS mirror we use. Switching the GS mirror to the two-counter-propagating-AOD solution mentioned in Chapter 2 can massively increase the number of resolvable spots of the system with a price of losing some of the power due to the cascaded AOD diffraction efficiencies.

7.3 Possible future improvements

To improve the performance of this SSH scanner, one could consider the following improvements:

- Use a PPLN waveguide to increase the frequency doubling efficiency [110, 111]. Currently our PPLN only gives us 1% of frequency doubling efficiency. With the 70% efficiency
reported in [110], we could get 70 times more power and therefore 70 times more bandwidth, or 70 times more scanning time, or 10 times more bandwidth and 7 times more scanning time. In that case, the inhomogeneous bandwidth of the Tm\(^3^+\):YAG crystal might become a limit to the bandwidth.

- To access bandwidths higher than 20GHz, we can switch to other high bandwidth materials, such as Er\(^{3^+}\):LiNbO\(_3\)[99, 112], or Tm\(^{3^+}\):LiNbO\(_3\)[113], which have several hundred GHz of available bandwidth. Thiel et al [113] report the studies of spectral hole burning in Tm\(^{3^+}\):LiNbO\(_3\) within at least a 30GHz region with up to minutes of lifetime when a magnetic field of a few hundred Gauss was applied. Thiel et al [112] also explore the decoherence and population relaxation dynamics of Er\(^{3^+}\):LiNbO\(_3\), which has an inhomogeneous linewidth of 180GHz and homogeneous linewidth of as narrow as 2KHz.

- Use a counter-propagating pair of AODs driven by slow chirps instead of the GS mirror to generate higher frequency Dopplers. It might be possible to achieve 50% combined diffraction efficiency in such a cascaded system, although less than 10% cascaded efficiency is more likely, which will make the whole system less power efficient, but potentially could increase the total time delay of the SSH scanner.

- I did a one-laser demonstration of the SSH scanner which does not have the versatility to process an arbitrary high bandwidth waveform. Using two lasers, one CW modulated by a wide bandwidth electro-optic modulator and the other repetitively chirping over the scanner bandwidth for writing, one can realize the full function of this scanner. In a two-laser experiment, one needs to keep the frequencies of the two lasers locked to each other by using a two-laser locking circuit. Since one of the lasers is chirped periodically for the grating-recording process, the laser need to be periodically chirped, and then relocked to the CW laser in between chirps using the circuit presented in Chapter 6.

- In the one-laser experiment, we use two acousto-optic modulators (AOMs) as gates to
control the timing for the grating writing and reading process. Even with a beam block, part of the undiffracted optic beam still leaks to the diffracted beam, leaving a background signal overlapping with the echo (refer to Fig. 5.25 in Chapter 5). To get rid of the background, we can use synchronized choppers instead of AOMs as gate signals to avoid the leakage of the undiffracted DC beam from the AOMs, or use improved AOMs with smaller scatterings or less tightly focused beam within the AOMs to reduce the leakage around the Fourier plane knife edge filter.

- In the one-laser experiment, I used periodic triangle waveforms to drive the piezo of the fiber laser and to scan the GS mirror. To avoid the high frequency components and mechanical ringing due to the sharp corners, instead of using a triangle waveform to chirp the laser and the GS mirror, I used multiple Fourier orders of sinusoids to approximate triangle waveforms. This gives nonlinear chirps that leads to nonlinearly spaced time delays. Instead, one could use self-heterodyne servo-loop [114–117] to linearize and stabilize the laser chirp.

- In the recording process of the one-laser experiment, I did a single scan recording. One could possibly do an accumulation experiment [118, 119], in which repeated recording singles are applied to the crystal, for higher grating modulation depth and therefore higher echo efficiency. In this experiment, it is important that the GS mirror is stable from scan to scan and the accumulation needs to finish within the lifetime of the crystal, so it might be necessary to switch to the more stable AOD approach to do accumulated recording. It would be interesting to see experimentally if this approach really wins.
Bibliography


Appendix A

Experimental techniques

Experimental wise, to make an optical system work, one needs to be familiar with the following issues:

- How to align the optical components?
- How to make the electronic signals that control the active optical components, i.e., acousto-optic devices, GS mirror, tunable laser sources?
- How to automatically control instruments, i.e., scopes, signal generators, etc?

In the following three sections, I will briefly describe the techniques I used in building the aforementioned optical systems.

A.0.1 Alignment tools

It is not hard to imagine that the most important techniques for building a successful optical system is the alignment techniques. In building any free-space optical systems, an iris is a must-have tool. In near IR optical systems, due to the invisibility of light, an IR card and a camera that response to 793nm are also must-have tools. In addition to these most obvious must-have tools, a very useful alignment tool is a beam profiler (Thorlabs WM100 Omega Meter). Beam profiler and oscilloscope are used for the physical alignment of the beams. They are especially useful when there are multiple anamorphic components, such as cylinders and GS mirror, in the system. Wavefront analyzers are used when the aberrations of the beam is important, e.g., when one wants to correct
the beam shape/astigmatism of the output from a laser diode or a BoosTA amplifier. Polarimeters are useful in aligning polarized components, such as waveplates and polarization maintaining fibers.

A.0.1.1 Beam profiler

A beam profiler measures the beam diameter based on the knife edge technique. A constantly revolving knife edge scans across the beam of interest and the photodiode right behind the knife edge integrates the unblocked beam. It calculates the beam size by electronically differentiating to synthesize a scanned slit from the detected scanned knife edge data. One could use it to find the focal plane of a lens. It can also be used to align a 4f imaging system by checking if the beam is collimated. One could also use it to align the overlap of several beams.

The output of a beam profiler could be displayed on an oscilloscope. The combination of the two is very useful for aligning the in-plane tilt of cylinders with the scanning of the GS mirror. The idea is to use the knife edge of the beam profiler as a reference and make everything parallel to it. Using the alignment in Chapter 5 as an example, I need to align the parallel of two vertical lines involving two cylinders and make the scanning of a GS in one beam parallel to the two lines. In other words, the GS mirror scanning direction needs to be orthogonal to the cylinders’ focusing direction. Before making the GS mirror scanning, I first align the rotation around the optical axis of the cylinders respectively by minimizing the beam size (or maximizing the beam amplitude on a scope). Then I intentionally separate the two beams. So on the scope, I can see two separated peaks. I use a 1Hz 100mV_{pp} sine wave to make the GS mirror scanning slowly and observe the two peaks on the scope. If the two peaks are static on the time axis with respect to each other, the scanning of the mirror is perfectly orthogonal to the cylinder focusing. Otherwise, one peak is moving back and forth with respect to the other. One could adjust the rotation around the optical axis of the GS mirror to make the two static to each other.
A.0.1.2 Oscilloscope

The combination of an oscilloscope in videoline trigger mode and a CCD camera is especially useful in the alignment related to lines focused by cylinders. Bringing the CCD camera image on a scope could make the cylinder-GS mirror alignment mentioned in the last section accurate to a pixel of the CCD camera. The procedure is quite similar to the procedure in the last section. In the Doppler processor experiment in Chapter 2, I used this trick to align the cylinder focusing and the scanning direction of the GS mirror. In that case, I firstly align the GS mirror to one line of the CCD camera. I focus the beam into a spot on the CCD camera and look at the line with the highest intensity on the scope. I then send a DC voltage to the GS mirror, rotating the GS mirror in-plane to make the highest intensity still at the same video line. Turning off the DC voltage, I can translate the GS mirror to bring the highest intensity line to the current line. By alternating between these two adjustments, I can make the GS mirror scanning along the video line of the CCD camera. I can then make the focusing of the beam along the same video line of the camera.

A.0.1.3 Wavefront analyzer

I used wavefront analyzers when I wanted to couple the output beam from a laser diode and a BoosTA to a single mode PM fiber. The beams in both cases has an elliptical beam shape and suffers from a little bit of astigmatism. Coupling the beam into the fiber directly introduces big coupling loss. One could use cylinder pairs with proper amplification ratio and in 4f setup to correct the beam shape. By moving one of the cylinders and make the setup slightly off the 4f imaging system, one could compensate the astigmatism. The alignment of cylinders could be monitored by using a wavefront analyzer. I used BlueSky wavefront analyzer and Spiricon Hartman wavefront analyzer. The latter uses pinhole arrays and a CCD camera behind it to measure the projection of the wavefront onto the CCD behind the pinholes. It infers the aberrations from the deviation of the beam with respect to the pinhole positions. Unfortunately, one needs to massively attenuate the beam to keep the camera from being saturated. The Blue sky wavefront analyzer is less sensitive
to strong beam intensity than the HWA we have.

A.0.2 Arbitrary waveform generator (arb) programming

In optical signal processing experiments, one needs to control the active optical elements, such as AOs and tunable lasers. Finding an efficient way of programming the control waveforms is important. In cool-down experiments, sometimes it is also very important to have the ability to modify the waveforms quickly to compare the experimental results. Arbs provide the advantage of programming versatile waveforms. My favorite way of generating arbitrary waveforms is to use the Waveform Programming Language of the Tektronix arb to generate sequence files. Sequence files contain blocks of multiple waveform files. One could change the repetition of each waveform file easily. The two arbs I used, Tektronix AWG520 and AWG430, have almost the same programming languages. Following the format instructions provided in the Tektronix arb Programmer Manual, one can make the program in a .txt format and compile it in the arb to generate the waveform files and sequence files. This is much easier than loading the waveform files directly in the sequence file. In direct loading, when the waveform file names are longer than 5 characters (which is typically the case), only part of the file name is shown, making it hard to identify the file when debugging the sequence.

The following is the sequence file I used for the AWG520 in the high BW Bragg cell experiment (chapter 5). Channel 1 & 2 of AWG520 control the gate AOs. Channel 1/2 marker 1 sends out signals that trigger the chopper and the scope when the read-out gate AO turns on.

```plaintext
'sequence for AWG520
delete("hbwb\_laser.seq")
size=1000
clock=1e9
num=8

'write sequence file header
write("hbwb\_laser.seq","MAGIC 3002\n")
write("hbwb\_laser.seq","LINES ":num:"
")

'create a 1us DC waveform file
```
"dead-1us.wfm"=0

' create a 10us DC waveform file with mk1=1
"dead-1us-mk1.wfm"="dead-1us.wfm"
"dead-1us-mk1.wfm".marker1=1

' create a 1us sine waveform file
freq=100e6
"sin100-1us.wfm"=sin(2*pi*freq*time)

' generate sequence file
' line 1: time delay
rep=1250
write("hbwb_laser.seq","\"dead-1us.wfm\",  \"dead-1us.wfm\",":rep:\"n")

' line 2: making ch1 ahead of ch2
' the time of line 2 should equal that of line 4
' the total of line2 and line3 should be 4ms
rep=1
write("hbwb_laser.seq","\"sin100-1us.wfm\",  \"dead-1us.wfm\",":rep:\"n")

' line 3
rep=4000
write("hbwb_laser.seq","\"sin100-1us.wfm\",  \"sin100-1us.wfm\",":rep:\"n")

' line 4: making ch1 ahead of ch2, ch1 ends before ch2
rep=1
write("hbwb_laser.seq","\"dead-1us.wfm\",  \"sin100-1us.wfm\",":rep:\"n")

' line 5: time delay before readout
rep=1000
write("hbwb_laser.seq","\"dead-1us.wfm\",  \"dead-1us.wfm\",":rep:\"n")

' line 6: trigger for the chopper
rep=100
write("hbwb_laser.seq","\"dead-1us-mk1.wfm\",  \"dead-1us-mk1.wfm\",":rep:\"n")

' line 7: readout gate is open
rep=4000
write("hbwb_laser.seq","\"sin100-1us.wfm\", 

The .txt file I used for the AWG430 in the high BW Bragg cell experiment (Chapter 5) is as follows. The compiling time for this sequence file is about an hour. To compare the effect of including different cosine orders, one can open the sequence in sequence editor and change the cosine file through "Data Entry -> Enter Filename...". For sequences that include long waveform files like this one, it takes some time to save the file.

\begin{verbatim}
%\begin{quotation}
' sequence for the Koheras laser and GS mirror
 delete("hbwb.seq")

time=10e-3
clock=2e8
size=time*clock
num=2

' write sequence file header
write("hbwb.seq","MAGIC 3003\n")
write("hbwb.seq","LINES ":num:"\n")

' create a 10ms 100Hz cosine waveform file
freq=100
A=2

' generate "ramps" that include 1-10 cosine orders
for i=1 to 10
  "cos100Hz-10ms-":i:.wfm"=0
  for j=1 to i
    "cos100Hz-10ms-":i:.wfm"=4*A/pi\^2/(2*j-1)\^2 *cos(2*pi*freq*time*(2*j-1)-pi) +"cos100Hz-10ms-":i:.wfm"
  next
next

' create a trigger waveform
"DC-10ms.wfm"=1

' create a 10ms 0 waveform file
"dead-1ms.wfm"=0

' line 1
rep=1
write("hbwb.seq","\textbackslash cos100Hz-10ms-1.wfm\")
\end{verbatim}
\"cos100Hz\-10ms\-10.wfm\",\"DC\-10ms.wfm\",\"rep:\n\")

'line 2
rep=3
write ("hbwb.seq",\"cos100Hz\-10ms\-1.wfm\",\"cos100Hz\-10ms\-10.wfm\",\"dead\-10ms.wfm\",\"rep:\n\")
Appendix B

Simulation codes

The following is a sample of the Matlab simulation codes using trapezoidal rule and the 4th order Runge-Kutta method.

```matlab
%------------------------------------------- Main -------------------------------------------
clear;
clc;
disp('bp_bloch_trapezoidal_rk4.m');

% Define Constants
cl = 2.99792458e8;
lambda = 1.0546e-34;
kappa = 1.9e-32/hbar; % kappa = 1.9 e-32/hbar; from paper
mu0 = 4.0*pi*1.0e-7; % permeability of free space

% Define the material parameters
nlines = 250;
T1 = 10.0e-3; % excited state lifetime (sec)
T2 = .7e-6; % dephasing time (sec)
Nat = 1.0e21; % number of atoms (m^-3)
gamma = 1.0e-32; % default = 1.0e-29
nr = 1.5;

% wavelength related beam parameters
lambda = 793.0e-9; % m
f0 = c / lambda;

% Define simulation grid parameters
x_points = 128.;
z_points = 128;
z_dim = 5.0e-3; % m
dz = z_dim / z_points;
```
% Simulation time
dt = 4e-9;  % time step (sec)
% 10us decay time for debugging, 50us for the final simulation
temporal_length = 2.4e-6;  % simulation length (sec)
n_timestep = round(temporal_length/dt);  % # of time steps

% Time/frequency domain beam parameters
Delta = 2*pi*((0:(nlines-1))-round(((nlines)/2))/nlines*100e6);

Tp = pi*1.0e-7;  % Pulse duration (def=20.0/f0)
E0 = 4.2186e2*1.0/4.0*1.59151;% Pulse peak intensity
tau1 = 0.3e-6;  % Time separation between 1st and 2nd pulses
tau2 = 1.8e-6;  % Time separation between 1st and 3rd pulses
t_p1 = Tp/2.;
t_p2 = t_p1 + tau1;
t_p3 = t_p1 + tau2;
% t_start = 2*dt;
% t_end = 100*dt;
t_start = .2e-6;
t_end = 2.2e-6;

% Beam prop parameters
% spatial domain
dx = 2.5;
deltax = dx*lambda;  % sampling period
x_dim = dx*x_points*lambda;
N21 = round(x_points/2);

k0=2*pi/lambda;
k= nr*k0;
x=((1:x_points)-x_points/2)*deltax;
% x=linspace(-1,1,x_points)*x_points*deltax/2;
kx=((((1:x_points)-round(x_points/2)-1)*2*pi/x_dim)');

% DEFINE BEAM PARAMETERS
% Spatial Domain
w0 = 35.0*lambda;
q = i*pi*w0^2/lambda;
% beam tilted angle
% theta1 = 1.0*pi/180.;
theta1 = 0;
theta2 = 0.8*pi/180.;
rayleigh = pi*w0^2/lambda;
focus = 15.e3*lambda;
% focusing factor
quad = -i * k0*(x.^2)/(2.*focus);
% t=0
% Beam prop (with only the source E field)
% ************************************************************

% SPATIAL PROFILE
% Gaussian beam linear phase factor
% beam1 = (exp(-i*k0*(x+0.8*w0).^2/(2*q)) .* ... 
% exp(+i*k0*sin(theta1)*x) .* exp(quad))';
beam1 = (exp(-i*k0*(x+0.0*w0).^2/(2*q)) .* ... 
       exp(+i*k0*sin(theta1)*x) .* exp(quad))';

% beam2 = (exp(-i*k0*(x-0.8*w0).^2/(2*q)) .* ... 
% exp(-i*k0*sin(theta2)*x) .* exp(quad))';

% amp spatial time domain signal
s = zeros(n_timestep,1);

E2d0(:,1) = E0 * beam1 * (source(0,t_p1,dt,Tp) + ... 
                        source(0,t_p3,dt,Tp)) + ... 
            E0 * beam2 * source(0,t_p2,dt,Tp);

% **************************************** Beam Propagation ****************************************
for z=2:z_points
   Efft = fftshift(fft(fftshift(E2d0(:,z-1))));
   Efft = (1+i/(4*k)*kx.^2*dz)./(1-i/(4*k)*kx.^2*dz).*Efft;
   E2d0(:,z) = ifftshift(ifft(ifftshift(Efft)));
end

% clear quad x;
G = [0.0 0.0 -1.0/T1]';
u0 = zeros(x_points,z_points,nlines);
v0 = zeros(x_points,z_points,nlines);
w0 = zeros(x_points,z_points,nlines);
u0(:,:,:) = 0.0;
v0(:,:,:) = 0.0;
w0(:,:,:) = -1.0;

% BEGIN OF THE TIME STEP
coef = Nat*gamma*mu0*c^2/nlines*1.0e7/dz*2;
tic;
filename='data/bp_bloch_trpzl_rk4_rabi_4ns_5mm_p1MHz.dat';
file_write_flag = 0;
if file_write_flag == 1
   fidElg = fopen(filename,'w');
end
EI = zeros(n_timestep,1);
for n = 2: n_timestep,
kEr = real(kappa*E2d0);
kEi = imag(kappa*E2d0);

E1 = E2d0(:,128);

if file_write_flag == 1
    fwrite(fidElg, real(E1), 'double');
    fwrite(fidElg, imag(E1), 'double');
end

% update u, v, w by RK4
for line = 1:nlines,

k1u = -(1/T2*u0(:,line)+Delta(line)*v0(:,line)+... 
    kEi.*w0(:,line))*dt;

k1v = (Delta(line)*u0(:,line)-1/T2*v0(:,line)+... 
    kEr.*w0(:,line))*dt;

k1w = (kEi.*u0(:,line)-kEr.*v0(:,line)-... 
    1/T1*w0(:,line)-1/T1)*dt;

k2u = k1u-1/2*dt*(1/T2*k1u+Delta(line)*k1v+kEi.*k1w);

k2v = k1v+1/2*dt*(Delta(line)*k1u-1/T2*k1v+kEr.*k1w);

k2w = k1w+1/2*dt*(kEi.*k1u-kEr.*k1v-1/T1*k1w);

k3u = k1u-1/2*dt*(1/T2*k2u+Delta(line)*k2v+kEi.*k2w);

k3v = k1v+1/2*dt*(Delta(line)*k2u-1/T2*k2v+kEr.*k2w);

k3w = k1w+1/2*dt*(kEi.*k2u-kEr.*k2v-1/T1*k2w);

k2u = k1u+k2u+k3u;

k2v = k1v+k2v+k3v;

k2w = k1w+k2w+k3w;

clear k3u k3v k3w;

u0(:,line) = u0(:,line) +k1u-1/6*dt*(1/T2*k2u+... 
    Delta(line)*k2v+kEi.*k2w);

v0(:,line) = v0(:,line) +k1v+... 
    1/6*dt*(Delta(line)*k2u-... 
    1/T2*k2v+kEi.*k2w);

w0(:,line) = w0(:,line) +k1w+1/6*dt*(kEi.*k2u-... 
    kEr.*k2v-1/T1*k2w);

clear k1u k1v k1w k2u k2v k2w;

end % line
\[ P = \text{coef} \ast (\text{sum}(u0,3)+i\text{sum}(v0,3)); \]

% Update the E-field by trapezoidal rule
\[ E2d0(:,1) = E0 \ast \text{beam1} \ast (\text{source}(n,t_{p1},dt,Tp) + \ldots \text{source}(n,t_{p3},dt,Tp)) + \ldots \]
\[ E0 \ast \text{beam2} \ast \text{source}(n,t_{p2},dt,Tp); \]

for \( z=2:z_{\text{points}} \)
\[ \text{Efft} = \text{fftshift(fft(fftshift(E2d0(:,z-1))))}; \]
\[ \text{Pfft} = \text{fftshift(fft(fftshift(P(:,z-1))))}; \]

% Use trapezoidal rule for the beam prop
% \[ k1 = i/(2*k)*(kx.^2.*\text{Efft}-\text{Pfft}); \]
% \[ k2 = i/(2*k)*(kx.^2.*(\text{Efft}+k1*dz)-\text{Pfft}); \]
% \[ \text{Efft} = \text{Efft} + 1/2*dz*(k1+k2); \]
\[ E2d0(:,z) = \text{ifftshift(ifft(ifftshift(Efft))}); \]
end % \( z \)
\[ E1(n) = \text{sum(abs(E2d0(:,z_{\text{points}})).^2)}-\text{sum(abs(E2d0(:,1)).^2)}; \]
if \( (n-2)/10 == \text{round}((n-2)/10) \)
\[ \text{disp('simulating to')}; \]
\[ \text{disp('Has been simulating for (hours)');} \]
\[ \text{time=toc/3600} \]
% if \( \text{file_write_flag} == 1 \)
% \[ \text{fwrite(fidE2d0, real(E2d0), 'double');} \]
% \[ \text{fwrite(fidE2d0, imag(E2d0), 'double');} \]
% end
\[ \text{energyIncrease=sum(abs(E2d0(:,z_{\text{points}})).^2)}-\text{sum(abs(E2d0(:,1)).^2)}; \]
end % time step
\[ \text{time = toc/60} \]
\[ \text{close_status=fclose('all')}; \]

\[ \text{fidElg2=fopen(filename,'r');} \]
\[ \text{detector=zeros(n_{\text{timestep}},1)}; \]
% Plot the detector output
for \( l=1:n-1 \),
\[ \text{E2r=fread(fidElg2, 128, 'double');} \]
\[ \text{E2i=fread(fidElg2, 128, 'double');} \]
\[ \text{E2=E2r+i*E2i}; \]
\[ \text{detector(1)=sum(abs(E2(:,1)).^2)}; \]
end
\[ \text{figure(1)}; \]
plot((1:n_timestep)*dt*1e6,abs(detector),'k-.','LineWidth',2);
xlabel('time(\musec)','FontSize',18);
ylabel('Intensity (a.u.)','FontSize',18);
set(gca,'FontSize',14);
hold on;

%Plot the input power
Esource=zeros(n_timestep,1);
for ind=1:n,
    Esource(ind)=sum(abs(E0*beam1*s(ind)).^2);
end
plot((1:n_timestep)*dt*1e6,abs(Esource),'r-','LineWidth',2);
h=legend('Output','Incident E-field');
figure(2);
imagesc(abs(E2d0));
title('2d E field')
Appendix  C

Circuit diagrams (schematic, PCB layout)
Figure C.1: Schematic diagram of the locking circuit, part I - frequency to voltage conversion.
Figure C.2: Schematic diagram of the locking circuit, part II - PID.
Figure C.3: Schematic diagram of the locking circuit, part III - the voltage source.
Figure C.4: PCB board diagram power plane.
Figure C.5: PCB layout for manufacturing.
Appendix D

Publications


