Giving Spitzer’s Zero Range Process a Positive Range

by

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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
We study a generalization, which allows interaction between two sites at a time, of Spitzer’s zero range interaction process. We first identify the stationary measure for finitely many interaction sites. Then we study the process for infinitely many interaction sites and by using free energy consideration we identify all of the translation invariant stationary measures satisfying a certain mixing condition.
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A lot of things had happened to me in the past few years, and none of which was not so pleasantly memorable. As I am finishing this, I hope that this will be a some kind of turning point of my life where I can leave all the bad memories behind and move on.

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## Contents

### Chapter

1 Introduction
   1.1 Model ......................................................... 1

2 Finitely many cities
   2.1 Stationary measures ........................................ 4
   2.2 Free energy ................................................... 6
   2.3 Microcanonical ensemble ..................................... 8
   2.4 Fourier analysis ............................................... 12

3 Infinitely many cities
   3.1 Translation invariance ....................................... 17
   3.2 Specific free energy .......................................... 23

4 Conclusion
   4.1 Return to finitely many cities .............................. 35
   4.2 Generalization ................................................ 37

Bibliography ......................................................... 40
Chapter 1

Introduction

The origin of this thesis was an attempt to understand the well known phenomena from linguistics and the social sciences known variously as the rank-size law or the rank-size rule. It says, in the case of cities, that the largest city in a country is twice as large as the second largest city, and three times as large as the third largest city, etc. see [5], [10]. These ratios are, of course, only approximate and the causes are no doubt many such as geographical, economical, political, social, etc. Our desire was to find a simple model that displayed the phenomena. Our model is too simple to be of interest to the social scientists; however, it is interesting mathematically and is a generalization of Spitzer’s zero range processes [6]. Thus, in spite of the original motivation for the model, we study the model for its own sake rather than for a contribution to the social scientists.

1.1 Model

Let \( \eta_t \) be a Markov process defined, for a finite set \( \Lambda = \{0,1,\ldots,n\} \subset \mathbb{Z} \) and \( b \in \mathbb{Z}^+ \) or \( b = \infty \), on a finite state space \( S^b_{\Lambda,N} = \{ \eta : \Lambda \to \{0,1,\ldots,b\} \mid \sum_{k \in \Lambda} \eta(k) = N \} \), where \( \eta(k) \) is the configuration (i.e., population of the city) at \( k \in \Lambda \). Suppose that the transition probability \( p(x,y) \) for \( x, y \in \mathbb{Z} \), has the following properties:

(i) \( p(x,y) = p(y,x) \);

(ii) \( p(x,y) = p(0,y-x) \);

(iii) \( p(0,1) > 0 \) and;
(iv) $\sum_y |y|p(0, y) < \infty$.

The assumption that $\sum_x p(0, x) = 1$ is not necessary. Any $p$ satisfying (i), (ii), (iii), and (iv) would work. However, by a change of time scale, we can always assume that $\sum_x p(0, x) = 1$.

For given $\lambda > 0$, define the rate

$$c(\eta; x, y) = \begin{cases} p(x, y) \frac{\eta(x)(\eta(y) + \lambda)}{\eta(x) + \eta(y) + 2\lambda} & \text{if } x, y \in \Lambda \text{ and } \eta(y) < b, \\ 0 & \text{otherwise} \end{cases}$$

Note if we let $\lambda \to \infty$, we get $c(\eta; x, y) = p(x, y) \frac{\eta(x)}{2}$. This is a special case of Spitzer’s zero range process [9].

We may explain $c(\eta; x, y)$ as follows:

Let $|\Lambda|$ be the number of cities. For each $x \in \Lambda$, let $\eta(x)$ be the number of people in the city $x$ that can not exceed $b \leq \infty$. Then $N$ would be the total number of people in cities in $\Lambda$ and $\eta_t$ would be the configuration of people at time $t$. Suppose, at each city $x$, that people independently of one another may move to any other cities in $\Lambda$. At rate $p(x, y)$ each individual at city $x$ considers moving to city $y$, and then the probability of that individual actually moving to city $y$ is $\frac{\eta(y) + \lambda}{\eta(x) + \eta(y) + 2\lambda}$ if $\eta(y) < b$. To understand where this comes from we assume that the attraction of a city is a linear function of its population size, say $\alpha + \beta \eta(y)$ where $\eta(y)$ is the population of the city $y$. Then when choosing between a city with population $\eta(x)$ and one with population size $\eta(y)$ we choose $y$ with probability $\frac{\alpha + \beta \eta(y)}{\alpha + \beta \eta(x) + \alpha + \beta \eta(y)} = \frac{(\alpha / \beta) + \eta(y)}{2((\alpha / \beta) + \eta(x) + \eta(y))}$. If we let $\frac{\alpha / \beta}{\beta} = \lambda$, we get $\frac{\eta(y) + \lambda}{\eta(x) + \eta(y) + 2\lambda}$. Then $c(\eta; x, y)\Delta t$ is the probability of a move from $x$ to $y$ during infinitesimally small time interval $[t, t + \Delta t]$. The corresponding infinitesimal generator $\mathcal{U}$ of this Markov process is given by $\mathcal{U} f(\eta) = \sum_{x \in \Lambda} \sum_{y \in \Lambda} c(\eta; x, y)(f(\eta^{x,y}) - f(\eta))$ for $\forall f : \mathcal{S}_N^b \to \mathbb{R}$, where

$$\eta^{x,y}(k) = \begin{cases} \eta(k) & \text{if } k \neq x, y, \\ \eta(k) - 1 & \text{if } k = x \\ \eta(k) + 1 & \text{if } k = y. \end{cases}$$
In chapter 2, we will show, under the assumption above, that

$$\Pi^b_{\Lambda,N} = \begin{cases} \frac{\prod_{x \in \Lambda} \frac{\Gamma(\eta(x)+1)}{\Gamma(\eta(x)+\lambda)}}{Z^b(\Lambda,N)} & \text{if } \eta(\cdot) \in \mathcal{S}^b_{\Lambda,N} \\ 0 & \text{otherwise,} \end{cases}$$

where $Z^b(\Lambda,N) = \sum_{\eta \in \mathcal{S}^b_{\Lambda,N}} \prod_{k \in \Lambda} \frac{\Gamma(\eta(k)+1)}{\Gamma(\eta(k)+\lambda)}$, is a stationary distribution for $\eta_t$. Then we will define and discuss the free energy for $\eta_t$. We will also discuss the stationary distribution for $\eta_t$ when the number of cities $|\Lambda|$ and the total number of people $N$ increase to $\infty$. We will prove, in this case, if $N \to \infty$ and $|\Lambda| \to \infty$ such that $\frac{N}{|\Lambda|} \to M$, then the stationary distribution has a marginal density

$$P_{m,b}(\eta(x) = k) = \begin{cases} \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} m^k \quad & \text{for } k = 0, 1, \ldots, b, \\ \sum_{i=0}^b \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i & \text{otherwise,} \end{cases}$$

where, for given $M > 0$, $m$ is such that $M = \mathbb{E}[\eta(x)]$ for $\forall x \in \Lambda$. In particular, $P_{m,\infty}(\eta(x) = k) = \left(\frac{\lambda}{M+\lambda}\right)^\lambda \left(\frac{M}{M+\lambda}\right)^k \frac{\Gamma(k+\lambda)}{\Gamma(k+1)}$.

In chapter 3, we will show that $\mathbb{E}_{\mu_0}[\eta(x)] < \infty$ and $\mathbb{E}_{\mu_t}[(\eta(x))^2] < \infty$ for $\forall t \geq 0$ if the initial probability measure $\mu_o$ is translation invariant and $M = \mathbb{E}_{\mu_0}[\eta(x)] < \infty$ and $\mathbb{E}_{\mu_0}[(\eta(x))^2] < \infty$ for $\forall x \in \Lambda$. In fact, $\mathbb{E}_{\mu_t}[\eta(x)] = M$ for $\forall t \geq 0$. Also, we will show that the specific free energy, the free energy per site as $\Lambda \to \mathbb{Z}$, is finite. Finally, under Assumption 3.2.8 we will show that if $\mu_o$ is ergodic for translation and a stationary distribution for $\eta_t$ with $\mathbb{E}_{\mu_0}[\eta(x)] = M$, then $\mu_o$ is the infinite product of $P_{m,b}$'s.
Chapter 2

Finitely many cities

2.1 Stationary measures

Let

\[ \Pi^b_{\Lambda,N} = \begin{cases} \frac{\Pi_{\tau \in \Lambda} \frac{\Gamma(\eta(x) + \lambda)}{\Gamma(\eta(x) + 1)}}{Z^b(\Lambda,N)} & \text{if } \eta(\cdot) \in S^b_{\Lambda,N} \\ 0 & \text{otherwise,} \end{cases} \]

where \( Z^b(\Lambda, N) = \sum_{\eta \in S^b_{\Lambda,N}} \Pi_{\tau \in \Lambda} \frac{\Gamma(\eta(x) + \lambda)}{\Gamma(\eta(x) + 1)} \). We prove that \( \Pi^b_{\Lambda,N} \) is a stationary distribution for \( \eta \).

Lemma 2.1.1. \( \Pi^b_{\Lambda,N}(\eta)c(\eta; x, y) = \Pi^b_{\Lambda,N}(\eta^{x,y})c(\eta^{x,y}; y, x) \) for \( \forall \eta, \eta^{x,y} \in S^b_{\Lambda,N} \).

Proof. Case 1) If \( \eta(x) = 0 \), then \( c(\eta; x, y) = 0 \) by definition of \( c(\eta; x, y) \). But \( \eta^{x,y} \notin S^b_{\Lambda,N} \) since \( \eta^{x,y}(x) = \eta(x) - 1 < 0 \), so \( \Pi^b_{\Lambda,N}(\eta^{x,y}) = 0 \).

Case 2) If \( \eta(y) = b \), then \( c(\eta; x, y) = 0 \) by definition of \( c(\eta; x, y) \). But \( \eta^{x,y} \notin S^b_{\Lambda,N} \) since \( \eta^{x,y}(y) = \eta(y) + 1 > b \), so \( \Pi^b_{\Lambda,N}(\eta^{x,y}) = 0 \).

Case 3) If \( \eta(x) > 0 \) and \( \eta(y) < b \), then \( \eta^{x,y} \in S^b_{\Lambda,N} \). Consider \( \frac{\Pi^b_{\Lambda,N}(\eta)}{\Pi^b_{\Lambda,N}(\eta^{x,y})} \). Since \( \eta(k) = \eta^{x,y}(k) \) if \( k \neq x, y \) and \( \Gamma(x + 1) = x\Gamma(x) \),

\[
\frac{\Pi^b_{\Lambda,N}(\eta)}{\Pi^b_{\Lambda,N}(\eta^{x,y})} = \frac{\Gamma(\eta(x) + \lambda)}{\Gamma(\eta(x) + 1)} \frac{\Gamma(\eta(y) + \lambda)}{\Gamma(\eta(y) + 1)} \frac{\Gamma(\eta(x) - 1 + \lambda)}{\Gamma(\eta(x) - 1 + 1)} \frac{\Gamma(\eta(y) + 1 + \lambda)}{\Gamma(\eta(y) + 1 + 1)} = \frac{(\eta(x) - 1 + \lambda)(\eta(y) + 1)}{\eta(x)(\eta(y) + \lambda)}.
\]

Then the ratio

\[
\frac{\Pi^b_{\Lambda,N}(\eta)c(\eta; x, y)}{\Pi^b_{\Lambda,N}(\eta^{x,y})c(\eta^{x,y}; y, x)} = \frac{(\eta(x) - 1 + \lambda)(\eta(y) + 1)}{\eta(x)(\eta(y) + \lambda)} \frac{c(\eta; x, y)}{c(\eta^{x,y}; y, x)} \]

\[
= \frac{(\eta(x) - 1 + \lambda)(\eta(y) + 1)}{\eta(x)(\eta(y) + \lambda)} \frac{p(x, y) \frac{\eta(x)(\eta(y) + \lambda)}{\eta(x) + \eta(y) + 2\lambda}}{p(y, x) \frac{(\eta(y) + 1)(\eta(x) - 1 + \lambda)}{\eta(x) + \eta(y) + 2\lambda}} = \frac{p(x, y)}{p(y, x)} = 1.
\]
since \( p(x, y) = p(y, x) \).

**Lemma 2.1.2.** For \( f, g : S_{A,N}^b \to \mathbb{R} \),
\[
\sum_{\eta \in S_{A,N}^b} f(\eta) \mathcal{W} g(\eta) \Pi_{A,N}^b(\eta) = \sum_{\eta \in S_{A,N}^b} g(\eta) \mathcal{W} f(\eta) \Pi_{A,N}^b(\eta)
\]
\[
= \frac{1}{2} \sum_{\eta \in S_{A,N}^b} \sum_{x \in A} \sum_{y \in A} (f(\eta x, y) - f(\eta y, x)) c(\eta; x, y) \Pi_{A,N}^b(\eta).
\]

**Proof.** First, look at \( \sum_{\eta \in S_{A,N}^b} f(\eta) \mathcal{W} g(\eta) \Pi_{A,N}^b(\eta) \)
\[
= \sum_{\eta \in S_{A,N}^b} \sum_{x \in A} \sum_{y \in A} f(\eta x, y)(g(\eta x, y) - g(\eta y, x)) \Pi_{A,N}^b(\eta)
\]
\[
= \sum_{\eta \in S_{A,N}^b} \sum_{x \in A} \sum_{y \in A} (f(\eta x, y) - f(\eta y, x)) c(\eta; x, y) \Pi_{A,N}^b(\eta)
\]
\[
= \frac{1}{2} \sum_{\eta \in S_{A,N}^b} \sum_{x \in A} \sum_{y \in A} (f(\eta x, y) - f(\eta y, x)) c(\eta; x, y) \Pi_{A,N}^b(\eta).
\]
by lemma 2.1.1
\[
= \frac{1}{2} \sum_{\eta \in S_{A,N}^b} \sum_{x \in A} \sum_{y \in A} (f(\eta x, y) - f(\eta y, x)) c(\eta; x, y) \Pi_{A,N}^b(\eta).
\]
By exchanging \( f \) and \( g \), we also have \( \sum_{\eta \in S_{A,N}^b} g(\eta) \mathcal{W} f(\eta) \Pi_{A,N}^b(\eta) \)
\[
= \frac{1}{2} \sum_{\eta \in S_{A,N}^b} \sum_{x \in A} \sum_{y \in A} (g(\eta x, y) - g(\eta y, x)) c(\eta; x, y) \Pi_{A,N}^b(\eta)
\]
\[
= \sum_{\eta \in S_{A,N}^b} f(\eta) \mathcal{W} g(\eta) \Pi_{A,N}^b(\eta).
\]

**Corollary 2.1.3.** For \( \forall g : S_{A,N}^b \to \mathbb{R} \), \( \sum_{\eta \in S_{A,N}^b} g(\eta) \Pi_{A,N}^b(\eta) = 0 \).

**Proof.** Set \( f(\eta) \equiv 1 \) on lemma 2.1.2.
Lemma 2.1.4. $\Pi_{\Lambda,N}^b$ is a stationary distribution for $\eta_t$.

Proof. Let $T(t)$ be the semigroup of Markov process $\eta_t$, i.e., $T(t) = e^{t\mathcal{L}}$ and $T(t)f(\eta) = \mathbb{E}_\eta[f(\eta)]$.

In the case $\Lambda$ is finite, $\mathcal{L}$ is a bounded operator on $C(S_{\Lambda,N}^b)$ and since we know that

$$T(t)f(\eta) - f(\eta) = \int_0^t \mathcal{L}T(s)f(\eta)ds,$$

we have

$$\sum_{\eta \in S_{\Lambda,N}^b} T(t)f(\eta)\Pi_{\Lambda,N}^b - \sum_{\eta \in S_{\Lambda,N}^b} f(\eta)\Pi_{\Lambda,N}^b = \sum_{\eta \in S_{\Lambda,N}^b} \int_0^t \mathcal{L}T(s)f(\eta)ds\Pi_{\Lambda,N}^b.$$

But $\sum_{\eta \in S_{\Lambda,N}^b} \int_0^t \mathcal{L}T(s)f(\eta)ds\Pi_{\Lambda,N}^b = \int_0^t \sum_{\eta \in S_{\Lambda,N}^b} \mathcal{L}T(s)f(\eta)ds\Pi_{\Lambda,N}^b = 0$ by Corollary 2.1.3 since $T(s)f(\eta)$ is a function from $S_{\Lambda,N}^b \to \mathbb{R}$.

Therefore, $\sum_{\eta \in S_{\Lambda,N}^b} T(t)f(\eta)\Pi_{\Lambda,N}^b - \sum_{\eta \in S_{\Lambda,N}^b} f(\eta)\Pi_{\Lambda,N}^b = 0$, i.e., $\mathbb{E}_{\Pi_{\Lambda,N}^b}(\eta_t) = \mathbb{E}_{\Pi_{\Lambda,N}^b}(\eta_0)$.

\[\square\]

2.2 Free energy

For the Markov process $\eta_t$ on $S_{\Lambda,N}^b$, we now think of the probability measure $\mu_t$ on the configuration space $S_{\Lambda,N}^b$ as the states of an isothermal system. We know that the probability measure converges to a stationary distribution $\Pi_{\Lambda,N}^b$ as the time $t$ goes to infinity for a finite state space $S_{\Lambda,N}^b$. However, it may not be so when the configuration space is infinite ($|\Lambda| \to \infty$). In this case, studying free energy of probability measure space with time will give us a better understanding of the behavior of $\mu_t$ as $t \to \infty$, which will be discussed later in chapter 3.

We define the free energy $H_\Lambda(\mu_t)$ as follows:

first, let us note that, at a given temperature $\frac{1}{\beta}$, Gibb’s density of a state is defined to be

$$g(\eta) = \frac{e^{-\beta E(\eta)}}{Z(\beta)},$$

where $E(\eta)$ is the energy of $\eta$ and $Z(\beta)$ is the partition function and that free energy $H_\Lambda(\mu_t) = U(\mu_t) - \frac{1}{\beta}S(\mu_t)$, where $U(\mu_t)$ is the internal energy and $S(\mu_t)$ is the entropy of the system of state $\mu_t$. Thus, the free energy of state $\mu_t$ is given by

$$H_\Lambda(\mu_t) = \sum_{\eta \in S_{\Lambda,N}^b} E(\eta)\mu_t(\eta) - \frac{1}{\beta}(\sum_{\eta \in S_{\Lambda,N}^b} \mu_t(\eta)\ln(\mu_t(\eta)))$$

$$= \sum_{\eta \in S_{\Lambda,N}^b} -\frac{1}{\beta}(\ln(g(\eta)) + \ln(Z(\beta)))\mu_t(\eta) + \frac{1}{\beta} \sum_{\eta \in S_{\Lambda,N}^b} \mu_t(\eta)\ln(\mu_t(\eta))$$

$$= \frac{1}{\beta} \sum_{\eta \in S_{\Lambda,N}^b} \ln\left(\frac{\mu_t(\eta)}{\Pi_{\Lambda,N}^b(\eta)}\right)\mu_t(\eta) - \frac{1}{\beta} \ln(Z(\beta)).$$

Because a Gibb’s state is an equilibrium state and $\Pi_{\Lambda,N}^b$ is a stationary distribution for the process we will take $g(\eta) = \Pi_{\Lambda,N}^b(\eta)$ for $\forall \eta \in S_{\Lambda,N}^b$. Noting also that $\ln(Z(\beta))$ is a constant at a
given temperature $\beta$ and does not depend on the state $\mu_t$, we may define the free energy to be

$$H_\Lambda(\mu_t) = \sum_{\eta \in S^b_{\Lambda,N}} \mu_t(\eta) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right).$$

Next two lemmas will show that free energy on $S^b_{\Lambda,N}$, $H_\Lambda(\mu_t)$, is a nonincreasing function of $t$, and is minimized as $\mu_t$ converges to $P^b_{\Lambda,N}$, the Gibbs’s distribution.

**Lemma 2.2.1.** For $\forall f : S^b_{\Lambda,N} \rightarrow \mathbb{R}$, $\sum_{\eta \in S^b_{\Lambda,N}} \frac{d}{dt} \mu_t(\eta) f(\eta) = \sum_{\eta \in S^b_{\Lambda,N}} \mu_t(\eta) \frac{d}{dt} f(\eta)$.

**Proof.**

$$\sum_{\eta \in S^b_{\Lambda,N}} \frac{d}{dt} \mu_t(\eta) f(\eta) = \frac{d}{dt} \sum_{\eta \in S^b_{\Lambda,N}} \mu_t(\eta) f(\eta) = \frac{d}{dt} \mathbb{E}_\mu_t[f(\eta)]$$

$$= \frac{d}{dt} \sum_{\eta \in S^b_{\Lambda,N}} \mathbb{E}_\eta[f(\eta)] \mu_\eta(\eta) = \frac{d}{dt} \sum_{\eta \in S^b_{\Lambda,N}} T(t) f(\eta) \mu_\eta(\eta)$$

$$= \sum_{\eta \in S^b_{\Lambda,N}} T(t) \frac{d}{dt} f(\eta) \mu_\eta(\eta) = \sum_{\eta \in S^b_{\Lambda,N}} \mathbb{E}_\eta[\frac{d}{dt} f(\eta)] \mu_\eta(\eta)$$

$$= \mathbb{E}_\mu_t[\frac{d}{dt} f(\eta)] = \sum_{\eta \in S^b_{\Lambda,N}} \mu_t(\eta) \frac{d}{dt} f(\eta).$$

□

**Lemma 2.2.2.** $\frac{d}{dt} H_\Lambda(\mu_t) \leq 0$, and $\frac{d}{dt} H_\Lambda(\mu_t) = 0$ implies $\mu_t = P^b_{\Lambda,N}$.

**Proof.**

$$\frac{d}{dt} H_\Lambda(\mu_t) = \frac{d}{dt} \sum_{\eta \in S^b_{\Lambda,N}} \mu_t(\eta) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) = \sum_{\eta \in S^b_{\Lambda,N}} \left( \frac{d}{dt} \mu_t(\eta) \right) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right)$$

$$= \sum_{\eta \in S^b_{\Lambda,N}} \left( \frac{d}{dt} \mu_t(\eta) \right) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) + \frac{d}{dt} \sum_{\eta \in S^b_{\Lambda,N}} \mu_t(\eta) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right)$$

$$= \sum_{\eta \in S^b_{\Lambda,N}} \frac{d}{dt} \mu_t(\eta) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right)$$

$$= \sum_{\eta \in S^b_{\Lambda,N}} \mu_t(\eta) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right).$$

By lemma 2.1.2,

$$\frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right)$$

$$= \frac{1}{2} \sum_{\eta \in S^b_{\Lambda,N}} \sum_{x \in \Lambda} \sum_{y \in \Lambda} \mu_t(\eta) \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right)$$

$$= \frac{1}{2} \sum_{\eta \in S^b_{\Lambda,N}} \sum_{x \in \Lambda} \sum_{y \in \Lambda} \mu_t(\eta) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right)$$

$$= \frac{1}{2} \sum_{\eta \in S^b_{\Lambda,N}} \sum_{x \in \Lambda} \sum_{y \in \Lambda} \mu_t(\eta) \ln \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right) \left( \frac{\mu_t(\eta)}{P^b_{\Lambda,N}(\eta)} \right).$$

This form of the derivative for the free energy is due to Ollagnier and Pinchon in the context of spin-flip processes [8].
Note that if \((\mu_t(\eta)c(\eta; x, y) - \mu_t(\eta^{x,y})c(\eta^{x,y}; y, x)) > 0\), then \(\ln \left( \frac{\mu_t(\eta^{x,y})c(\eta^{x,y}; y, x)}{\mu_t(\eta)c(\eta; x, y)} \right) < 0\); and
if \((\mu_t(\eta)c(\eta; x, y) - \mu_t(\eta^{x,y})c(\eta^{x,y}; y, x)) < 0\), then \(\ln \left( \frac{\mu_t(\eta^{x,y})c(\eta^{x,y}; y, x)}{\mu_t(\eta)c(\eta; x, y)} \right) > 0\). Thus, each term of \(\frac{d}{dt} H_\Lambda(\mu_t)\) is nonpositive, i.e., \(\frac{d}{dt} H_\Lambda(\mu_t) \leq 0\).

Suppose \(\frac{d}{dt} H_\Lambda(\mu_t) = 0\). Then we must have \(\mu_t(\eta^{x,y})c(\eta^{x,y}; y, x) = \mu_t(\eta)c(\eta; x, y)\) for \(\forall \eta, \eta^{x,y} \in S_{\Lambda,N}^b\) with \(c(\eta; x, y), c(\eta^{x,y}; y, x) > 0\). Recall that \(p(0,1) > 0\) and \(p(x,y) = p(y,x)\) for \(\forall x, y \in \Lambda\).

Thus, for \(\forall \eta, \zeta \in S_{\Lambda,N}^b\), there is a path \(\eta = \eta_0, \eta_1, \ldots, \eta_k = \zeta\) such that \(\eta_{i+1} = \eta^{x_i, y_i}\) for some \(x_i, y_i \in S_{\Lambda,N}^b\) with \(p(x_i, y_i) > 0\) and \(\mu_t(\eta_i)c(\eta_i; x_i, y_i) = \mu_t(\eta_{i+1})c(\eta_{i+1}; y_i, x_i)\). Thus \(\frac{\mu_t(\eta_i)}{\mu_t(\eta_{i+1})} = \frac{c(\eta_{i+1}; y_i, x_i)}{c(\eta_i; x_i, y_i)} = \Pi_{\Lambda, N}(\eta_i) / \Pi_{\Lambda, N}(\eta_{i+1})\) for \(i = 0, 1, \ldots, k - 1\). Hence \(\frac{\mu_t(\eta)}{\mu_t(\zeta)} = \Pi_{\Lambda, N}(\eta) / \Pi_{\Lambda, N}(\zeta)\) for \(\forall \eta, \zeta \in S_{\Lambda,N}^b\), which implies that \(\frac{\mu_t(\eta)}{\Pi_{\Lambda, N}(\eta)} = C\) for some constant \(C\). But \(\sum_{\eta \in S_{\Lambda,N}^b} \mu_t(\eta) = \sum_{\eta \in S_{\Lambda,N}^b} \Pi_{\Lambda, N}^b(\eta) = 1\), so \(C = 1\).

\(\square\)

### 2.3 Microcanonical ensemble

In this section, we will study what happens when we let \(|\Lambda| \to \infty\) and \(N \to \infty\) such that \(\frac{N-|\Lambda|M}{\sqrt{|\Lambda|}} \to 0\), where \(0 < M < b\).

Let \(\lambda > 0\) be fixed and we will suppress it from the notation. If \(b < \infty\) and \(m > 0\) or \(b = \infty\) and \(0 < m < 1\), define

\[
P_{m,b}(\eta(x)) = \begin{cases} \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} m^k \prod_{i=0}^{k} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i \quad &\text{for } k = 0, 1, \ldots, b, \\ 0 &\text{otherwise.} \end{cases}
\]

Clearly, \(\mathbb{E}_{m,b}[\eta(x)] \in [0,b]\). We will show that for every \(M \in [0,b]\), there is a unique solution \(m\) such that \(\mathbb{E}_{m,b}[\eta(x)] = M\). We will call such \(m\) \(m(M,b)\). Before we discuss the existence of such \(m\), we prove the following lemmas.

**Lemma 2.3.1.** For \(0 \leq m < 1\),

\[
\lim_{b \to \infty} \sum_{i=0}^{b} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i = (1 - m)^{-\lambda} \Gamma(\lambda).
\]

**Proof.** For \(0 \leq m < 1\), consider \((1 - m)^{-\lambda} = \sum_{n=0}^{\infty} (-m)^n\frac{\Gamma(n+1)}{n!} (-\lambda)^n = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+1)}{n!(\lambda+1)} m^n\).
Proof. If \( f \) is a strictly increasing function of \( m \) on \( [0, \infty) \), then for \( m, b \) such that \( b < 1 \), we have:
\[
\sum_{i=0}^{\infty} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i = \sum_{i=0}^{\infty} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i \Gamma(\lambda) = (1 - m)^{-\lambda} \Gamma(\lambda).
\]
Thus, \( \lim_{b \to \infty} \sum_{i=0}^{b} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i = \sum_{i=0}^{\infty} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i \Gamma(\lambda) = (1 - m)^{-\lambda} \Gamma(\lambda). \]

For \( 0 \leq m < \infty \) and \( b < \infty \), define \( F(m, b) = \sum_{k=0}^{b} k P_{m,b}(k) \).

**Lemma 2.3.2.** For a fixed \( b < \infty \), \( F(m, b) \) is a continuous and strictly increasing function of \( m \) on \( [0, \infty) \) with \( F(0, b) = 0 \) and \( \lim_{m \to \infty} F(m, b) = b \). Moreover, for a fixed \( m \), \( F(m, b) \) is also a strictly increasing function of \( b \) on \( \{0, 1, 2, \ldots\} \).

**Proof.** If \( m = 0 \), then \( m^k = 0 \) for \( \forall k \geq 1 \). So \( F(0, b) = 0 \).

Since \( b < \infty \),
\[
\lim_{m \to \infty} \frac{\sum_{k=0}^{b} \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} m^k}{\sum_{i=0}^{b} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i} = \frac{b \Gamma(b+\lambda)}{\Gamma(b+1)} = b.
\]
To show \( F(m, b) \) is a strictly increasing function of \( m \) on \( [0, \infty) \) for a fixed \( b \), we show that \( \frac{d}{dm} F(m, b) > 0 \) for \( \forall m \in [0, \infty) \). First, look at \( \frac{d}{dm} \sum_{k=0}^{b} \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} m^k \)
\[
= \sum_{k=0}^{b} k \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} m^{k-1} = \frac{1}{m} \sum_{k=0}^{b} k^2 \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} m^k
\]
Now let \( g(m, b) = \sum_{i=0}^{b} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i \).

Then \( \frac{d}{dm} g(m, b) = \frac{d}{dm} \sum_{i=0}^{b} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i = \sum_{i=0}^{b} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} i m^{i-1} = \frac{1}{m} \sum_{i=0}^{b} i^2 \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i \).

Then \( \frac{d}{dm} F(m, b) = \frac{d}{dm} \sum_{k=0}^{b} \frac{k \Gamma(k+\lambda)}{\Gamma(k+1)} m^k \bar{g}(m, b) = \sum_{k=0}^{b} \frac{k \Gamma(k+\lambda)}{\Gamma(k+1)} m^k \bar{g}(m, b) \)
\[
= \frac{1}{m} \sum_{k=0}^{b} k^2 \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} m^k \bar{g}(m, b) - \left[ \frac{\sum_{k=0}^{b} k \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} m^k \bar{g}(m, b)}{\bar{g}(m, b)} \right]^2
\]
\[
= \frac{1}{m} \sum_{k=0}^{b} k^2 P_{m,b}(k) - \left( \sum_{k=0}^{b} k P_{m,b}(k) \right)^2 > 0
\]
Therefore, \( F(m, b) \) is a strictly increasing function of \( m \) for a fixed \( b \). In fact, \( 0 \leq F(m, b) \leq b \) on \( m \in [0, \infty) \).

Finally, it remains to show that \( F(m, b) \) is a strictly increasing function of \( b \) for a fixed \( m \).
\[
F(m, b) = \sum_{k=0}^{b} k P_{m,b}(k) = \sum_{k=1}^{b} \sum_{j=k}^{b} P_{m,b}(j) = \sum_{k=1}^{b} (1 - \sum_{j=0}^{k-1} P_{m,b}(j)).
\]
For fixed \( j \) and \( m \), the numerator of \( P_{m,b}(j) \) does not change as \( b \) increases, but the denominator of \( P_{m,b}(j) \) increases as \( b \) increases. Thus, \( P_{m,b}(j) \) strictly decreases as \( b \) increases and \( (1 - P_{m,b}(j)) \)
strictly increases as $b$ increases. $F(m,b)$ is the sum of strictly increasing terms as $b$ increases. Moreover, as $b$ increases there are more positive terms in the sum. Therefore, $F(m,b)$ is strictly increasing as $b$ increases for a fixed $m$.

\[ F(m,b) = \sum_{k=0}^{\infty} k \frac{\Gamma(k+\lambda)}{\Gamma(\lambda) \Gamma(k+1)} m^k (1-m) = (1-m) \sum_{k=0}^{\infty} k \frac{\Gamma(k+1+\lambda)}{\Gamma(\lambda+1) \Gamma(k+1)} m^k \]

\[ = (1-m)^{\lambda} m \lambda (1-m)^{-(\lambda+1)} = \frac{m \lambda}{1-m}. \]

\[ \lim_{b \to \infty} F(m,b) = \frac{m \lambda}{1-m} = F(m,\infty). \]

**Proof.** It follows from lemma 2.3.1 that

\[ \lim_{b \to \infty} F(m,b) = \sum_{k=0}^{\infty} k \frac{\Gamma(k+\lambda)}{\Gamma(\lambda) \Gamma(k+1)} m^k (1-m)^{\lambda} = (1-m)^{\lambda} \sum_{k=0}^{\infty} k \frac{\Gamma(k+1+\lambda)}{\Gamma(\lambda+1) \Gamma(k+1)} m^k \]

\[ = (1-m)^{\lambda} m \lambda (1-m)^{-(\lambda+1)} = \frac{m \lambda}{1-m}. \]

\[ \lim_{b \to \infty} F(m,b) = \frac{m \lambda}{1-m} = \frac{M}{M+\lambda}. \]

**Proof.** First, note $F(m(M,b),b) = M$ and that $F(m,b)$ is a strictly increasing function of both variables $m$ and $b$. Then $m(M,b)$ must be strictly decreasing in $b$.

Suppose there exists $b_1$ such that $m(M,b_1) < \frac{M}{M+\lambda}$. Then $m(M,b_1) > m(M,b)$ for $\forall b > b_1$. By lemma 2.3.3, $M = \lim_{b \to \infty} F(m(M,b),b) \leq \lim_{b \to \infty} F(m(M,b_1),b) = \frac{m(M,b_1) \lambda}{1-m(M,b_1)} < \frac{M}{M+\lambda} = M$, which is contradiction. Thus, $m(M,b_1) \geq \frac{M}{M+\lambda}$ for $\forall b$.

Given $\epsilon > 0$, if $m(M,b) > \frac{(M+\epsilon)}{(M+\epsilon)+\lambda}$ for $\forall b$, then $F(m(M,b),b) > F\left(\frac{(M+\epsilon)}{(M+\epsilon)+\lambda},b\right)$ for $\forall b$. Then $M = \lim_{b \to \infty} F(m(M,b),b) \geq \lim_{b \to \infty} F\left(\frac{(M+\epsilon)}{(M+\epsilon)+\lambda},b\right) = M + \epsilon$, which is contradiction. Therefore, for $\forall \epsilon > 0$, $m(M,b) < \frac{(M+\epsilon)}{(M+\epsilon)+\lambda}$ for all sufficiently large $b$.
**Theorem 2.3.5.** Let \( b \leq \infty \) and \( 0 < M < b \) be fixed. If \( |\Lambda| \to \infty \) and \( N \to \infty \) such that

\[
\frac{N - |\Lambda| M}{\sqrt{|\Lambda|}} \to 0,
\]

then, for fixed \( x_1, x_2, \ldots, x_l \in \Lambda \) and \( k_1, k_2, \ldots, k_l \) such that \( k_i \leq b \) and \( l \geq 1 \),

\[
\Pi_{\Lambda,N}^b(\eta(x_1) = k_1, \eta(x_2) = k_2, \ldots, \eta(x_l) = k_l) \to \prod_{i=1}^l P_{m,b}(k_i),
\]

where \( m = m(M,b) \). If \( b = \infty \), \( P_{m,\infty}(k) = (\frac{\lambda}{M+\lambda})^\lambda (\frac{M}{M+\lambda})^k \frac{\Gamma(k+\lambda)}{\Gamma(\lambda+1)} \).

To prove the theorem, we need the following lemma.

**Lemma 2.3.6.** Let \( b(\leq \infty) \) be fixed. For each \( M \in (0,b) \), let \( X_1, X_2, \ldots, X_l \) be i.i.d. with distribution \( P_{m,b} \), where \( m = m(M,b) \), i.e., \( \mathbb{E}[X_i] = M \) for \( i = 1, 2, \ldots, l \). Then

\[
\sqrt{l} \sigma_{m,b} \mathbb{P}_{m,b} \left( \sum_{i=1}^l X_i = k \right) \to \frac{1}{\sqrt{2\pi}}
\]

as \( l \to \infty \) and \( k \to \infty \) such that \( \frac{k-lM}{\sqrt{l}} \to 0 \), where \( \text{Var}[X_i] = \sigma_{m,b}^2 \) for \( i = 1, 2, \ldots, l \).

**Proof.** Recall that, by Local Central Limit Theorem [1],

\[
\lim_{l \to \infty} \sup_k \left| \sqrt{l} \sigma_{m,b} \mathbb{P}_{m,b}(\sum_{i=1}^l X_i = k) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{(k-lM)^2}{\sigma_{m,b}^2}\right) \right| = 0.
\]

Given \( \epsilon > 0 \),

By LCLT, \( \left| \sqrt{l} \sigma_{m,b} \mathbb{P}_{m,b}(\sum_{i=1}^l X_i = k) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{(k-lM)^2}{\sigma_{m,b}^2}\right) \right| < \frac{\epsilon}{2} \) for large \( l \) uniformly on \( k \). And

\[
\left| \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \frac{(k-lM)^2}{\sigma_{m,b}^2}\right) - \frac{1}{\sqrt{2\pi}} \right| < \frac{\epsilon}{2} \text{ for large } l \text{ and } k \text{ such that } \frac{k-lM}{\sqrt{l}} \text{ is small enough.}
\]

So we have

\[
\left| \sqrt{l} \sigma_{m,b} \mathbb{P}_{m,b}(\sum_{i=1}^l X_i = k) - \frac{1}{\sqrt{2\pi}} \right| < \epsilon \text{ for large } l \text{ and } k \text{ such that } \frac{k-lM}{\sqrt{l}} \text{ is small enough.}
\]

\( \square \)

**Proof of theorem 2.3.5.** Let \( b(\leq \infty) \) be fixed. For fixed \( y_1, y_2, \ldots, y_l \in \Lambda \) and \( k_1, k_2, \ldots, k_l \) such that \( k_i \leq b \), \( \sum_{i=1}^l k_i = L \leq N \), and \( l \leq |\Lambda| \), let \( B_{\Lambda,N}^b = \{ \eta \in S_{\Lambda,N}^b : \eta(y_i) = k_i, i = 1, 2, \ldots, l \} \).

\[
\Pi_{\Lambda,N}^b(\eta(y_1) = k_1, \eta(y_2) = k_2, \ldots, \eta(y_l) = k_l) = \sum_{\eta \in B_{\Lambda,N}^b} \frac{\Gamma(\eta(y)+\lambda)}{\Gamma(\eta(y)+1)} \frac{\eta^y(y)}{Z^b(\Lambda, N)}
\]

\[
= \frac{\sum_{\eta \in B_{\Lambda,N}^b} \Pi_{\eta \in \Lambda} \frac{\Gamma(\eta(y)+\lambda)}{\Gamma(\eta(y)+1)} \eta^y(y)}{\sum_{\zeta \in S_{\Lambda,N}^b} \Pi_{\zeta \in \Lambda} \frac{\Gamma(\zeta(y)+\lambda)}{\Gamma(\zeta(y)+1)} \zeta^y(y)}
\]

\[
= \frac{\sum_{\eta \in B_{\Lambda,N}^b} \Pi_{\eta \in \Lambda} \frac{\Gamma(\eta(y)+\lambda)}{\Gamma(\eta(y)+1)} \eta^y(y)}{\sum_{\zeta \in S_{\Lambda,N}^b} \Pi_{\zeta \in \Lambda} \frac{\Gamma(\zeta(y)+\lambda)}{\Gamma(\zeta(y)+1)} \zeta^y(y)}
\]
since $\sum_{y \in \Lambda} \eta(y) = \sum_{y \in \Lambda} \zeta(y) = N$,

$$
\frac{\sum_{\eta \in B_{\Lambda,N}^b} \prod y \in \Lambda \left( \frac{\Gamma(\eta(y)+\lambda)}{\Gamma(\eta(y)+1)} m^\eta(y) / \sum_{j=0}^b \Gamma(j+\lambda) m^j \right)}{\sum_{\zeta \in S_{\Lambda,N}^b} \prod y \in \Lambda \left( \frac{\Gamma(\zeta(y)+\lambda)}{\Gamma(\zeta(y)+1)} m^\zeta(y) / \sum_{j=0}^b \Gamma(j+\lambda) m^j \right)}
= \frac{\sum_{\eta \in B_{\Lambda,N}^b} \prod y \in \Lambda P_{m,b}(\eta(y))}{\sum_{\zeta \in S_{\Lambda,N}^b} \prod y \in \Lambda P_{m,b}(\zeta(y))}.
$$

(2.1)

Note that (2.1) is equal to $P_{m,b}(X_1 = k_1, X_2 = k_2, \ldots, X_l = k_l \mid \sum_{i=1}^{\vert \Lambda \vert} X_i = N)$, where $X_i$s are defined as in lemma 2.3.6. It is also equal to

$$
(2.1) = \frac{\prod_{i=1}^l P_{m,b}(X_i = k_i) \sum_{\eta \in B_1} \prod y \in \Lambda P_{m,b}(\eta(y))}{\sum_{\zeta \in S_{\Lambda,N}^b} \prod y \in \Lambda P_{m,b}(\zeta(y))},
$$

where $\Lambda_1 = \Lambda \setminus \{y_1, y_2, \ldots, y_l\}$ and $B_1 = \{ \eta \in B_{\Lambda,N}^b \mid \sum_{y \in \Lambda_1} \eta(y) = N - L \}

= \prod_{i=1}^l P_{m,b}(X_i = k_i) \frac{P_{m,b}(\sum_{i=l+1}^{\vert \Lambda \vert} X_i = N - L)}{P_{m,b}(\sum_{i=1}^{\vert \Lambda \vert} X_i = N)}.$

If $\vert \Lambda \vert \to \infty$ and $N \to \infty$ such that $\frac{N-\vert \Lambda \vert M}{\sqrt{\vert \Lambda \vert}} \to 0$,
then $\sqrt{\vert \Lambda \vert} \sigma_{m,b}P_{m,b}(\sum_{i=1}^{\vert \Lambda \vert} X_i = N) \to \frac{1}{\sqrt{2\pi}}$ by lemma 2.3.6.

Since $\frac{N-\vert \Lambda \vert M}{\sqrt{\vert \Lambda \vert}} \to 0$ implies $\frac{(N-L)-(\vert \Lambda \vert - L)M}{\sqrt{(\vert \Lambda \vert - L)}} \to 0$,

$\sqrt{(\vert \Lambda \vert - L)} \sigma_{m,b}P_{m,b}(\sum_{i=l+1}^{\vert \Lambda \vert} X_i = N - L) \to \frac{1}{\sqrt{2\pi}}$.

Thus, $\Pi_{\Lambda,N}^b(\eta(y_1) = k_1, \eta(y_2) = k_2, \ldots, \eta(y_l) = k_l)$

$$
= \prod_{i=1}^l P_{m,b}(X_i = k_i) \frac{\sqrt{(\vert \Lambda \vert - L) \sigma_{m,b}P_{m,b}(\sum_{i=l+1}^{\vert \Lambda \vert} X_i = N - L)}}{\sqrt{\vert \Lambda \vert \sigma_{m,b}P_{m,b}(\sum_{i=1}^{\vert \Lambda \vert} X_i = N)}} \frac{\sqrt{\vert \Lambda \vert}}{\sqrt{(\vert \Lambda \vert - L)}}
$$

$\to \prod_{i=1}^l P_{m,b}(X_i = k_i)$ as $\vert \Lambda \vert \to \infty$ and $N \to \infty$ such that $\frac{N-\vert \Lambda \vert M}{\sqrt{\vert \Lambda \vert}} \to 0.$

2.4 Fourier analysis

In this section, we will show using Fourier analysis that the conclusion of theorem 2.3.5 is still true if $N \to \infty$ and $\vert \Lambda \vert \to \infty$ such that $\frac{N}{\vert \Lambda \vert} \to M$. To show this, it suffices to prove the following theorem. Then the rest follows similarly to the proof of theorem 2.3.5.
Theorem 2.4.1. For a fixed $b$ and $\lambda$, let $0 < M_n < b$ be a sequence that converges to $M \in (0, b)$ and let $X_1, X_2, \ldots$ be i.i.d. with distribution $P_{m,b}$, where $m = m(M_n, b)$, i.e., $E[X_1] = M_n$. Then, for all $k, l \in \mathbb{N}$, $\sqrt{2\pi}\sqrt{n} \sigma_{M_n,b} P_{m,b}(\sum_{i=1}^{n-k} X_i = [nM_n] - l) \to 1$ as $n \to \infty$, where $Var[X_i] = \sigma^2_{m,b}$ and $\lfloor a \rfloor$ is the largest integer that does not exceed $a$.

We will prove this theorem using the technique given in [3], pp.297–301.

Proof. Since $b$ and $\lambda$ are fixed, for the simplicity of notations we denote $P_{m_n}$ for $P_{m(M_n,b),b}$ and $\sigma_{M_n}$ for $\sigma_{m(M_n,b),b}$. Let $\sigma = \inf_n \sigma_{M_n} > 0$ and $\rho = \sup_n E[X_1^2]$. If $b < \infty$, then $\rho < b^3$. Since $\sup_n M_n < \infty$ if $b = \infty$, $\rho < \infty$. Now let $\varphi_{M_n}(\theta) = \sum_{j=0}^{b-1} e^{ij\theta} P_{M_n}(j)$, where $i^2 = -1$.

Then $P_{M_n}(\sum_{j=1}^{n-k} X_j = [nM_n] - l) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i([nM_n]-l)\theta} \varphi_{M_n}(\theta)^{(n-k)} d\theta$.

Note that $\varphi_{M_n}(\theta) \to \varphi(\theta)$ uniformly in $\theta$ as $n \to \infty$ since $P_{M_n} \to P_M$ and that $|\varphi_{M_n}(\theta)| < 1$ on $(0, 2\pi)$ since $P_{M_n}(0), P_{M_n}(1) > 0$. Since $\sqrt{n} \to 1$ and $\frac{\sigma_{M_n}}{\sup_n M_n} \to 1$ as $n \to \infty$, it is enough to show that $\sqrt{2\pi}\sqrt{n} - k\sigma_{M_n} P_{M_n}(\sum_{j=1}^{n-k} X_j = [nM_n] - l) \to 1$ as $n \to \infty$. Now consider

$$\sqrt{2\pi} \sqrt{n} - k\sigma_{M_n} P_{M_n}(\sum_{j=1}^{n-k} X_j = [nM_n] - l) = \frac{\sigma_{M_n}}{\sqrt{2\pi}} \left[ \int_{-\pi}^{\pi} e^{-i([nM_n]-l)\theta} \varphi_{M_n}(\theta)^{(n-k)} d\theta - \int_{-\pi}^{\pi} e^{-u^2\frac{\sigma_{M_n}^2}{2}} du \right]$$

by letting $\sqrt{n-k} \theta = u$,

$$= \frac{\sigma_{M_n}}{\sqrt{2\pi}} \left[ \int_{-\sqrt{n-k} \pi}^{\sqrt{n-k} \pi} e^{-i([nM_n]-l)\theta} \varphi_{M_n}(\frac{u}{\sqrt{n-k}})^{(n-k)} du - \int_{-\infty}^{\infty} e^{-u^2\frac{\sigma_{M_n}^2}{2}} du \right]$$

$$= \frac{\sigma_{M_n}}{\sqrt{2\pi}} \left[ \int_{-\sqrt{n-k} \pi}^{\sqrt{n-k} \pi} e^{-iM_n(\theta)} \varphi_{M_n}(\frac{u}{\sqrt{n-k}})^{(n-k)} e^{\frac{u^2}{\sigma_{M_n}^2}}((n-k)M_n + l - \lfloor nM_n \rfloor) du - \int_{-\infty}^{\infty} e^{-u^2\frac{\sigma_{M_n}^2}{2}} du \right]$$

(2.2)

Now write $\beta_n = (n-k)M_n + l - \lfloor nM_n \rfloor$; $\beta_n$ is bounded since $M_n$ is bounded, and $\gamma_n(\theta) = e^{-iM_n(\theta)} \varphi_{M_n}(\theta)$; $|\gamma_n(\theta)| \leq 1$.

(2.2) $= I_{n}^{(1)}(A) + I_{n}^{(2)}(A) + I_{n}^{(3)}(A, r) + I_{n}^{(4)}(r)$, where

$$I_{n}^{(1)}(A) = \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u| < A} \gamma_n(\frac{u}{\sqrt{n-k}})^{(n-k)} e^{\beta_n \frac{u}{\sqrt{n-k}} - e^{-u^2\frac{\sigma_{M_n}^2}{2}}} du;$$
\[ I_n^{(2)}(A) = \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u|>A} e^{-\frac{u^2+2r\pi}{2}} du; \]

\[ I_n^{(3)}(A, r) = \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{A<|u|<\sqrt{n-k\pi}} \gamma_n\left(\frac{u}{\sqrt{n-k}}\right)(n-k)e^{i\beta_n \frac{u}{\sqrt{n-k}}} du; \]

and

\[ I_n^{(4)}(r) = \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{\sqrt{n-k\pi}<|u|<\sqrt{n-k\pi}} \gamma_n\left(\frac{u}{\sqrt{n-k}}\right)(n-k)e^{i\beta_n \frac{u}{\sqrt{n-k}}} du. \]

By letting \( v = \sigma_{M_n}u \), we get

\[ I_n^{(2)}(A) = \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma_{M_n}A} e^{-v^2/2} dv \leq \frac{1}{\sqrt{2\pi}} \int_{|v|>\sigma_{M_n}A} e^{-v^2/2} dv \rightarrow 0 \]

as \( A \rightarrow \infty \).

\[ |I_n^{(4)}(r)| \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{\sqrt{n-k\pi}<|u|<\sqrt{n-k\pi}} \left| \gamma_n\left(\frac{u}{\sqrt{n-k}}\right)(n-k)e^{i\beta_n \frac{u}{\sqrt{n-k}}} \right| du \]

\[ = \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{\sqrt{n-k\pi}<|u|<\sqrt{n-k\pi}} \left| \varphi_{M_n}\left(\frac{u}{\sqrt{n-k}}\right) \right| (n-k) du \]

since \( |\gamma_n\left(\frac{u}{\sqrt{n-k}}\right)e^{i\beta_n \frac{u}{\sqrt{n-k}}}| = |\gamma_n\left(\frac{u}{\sqrt{n-k}}\right)| = |e^{-iM_n\left(\frac{u}{\sqrt{n-k}}\right)}\varphi_{M_n}\left(\frac{u}{\sqrt{n-k}}\right)| = |\varphi_{M_n}\left(\frac{u}{\sqrt{n-k}}\right)| \).

For \( |u| > \sqrt{n-k}\pi, \frac{u}{\sqrt{n-k}} > r \), \( \varphi_M(\theta) \) is continuous on \([-\pi, \pi]\) and \( |\varphi_M(\theta)| < 1 \) if \( \theta \neq 0 \).

So there exists \( \epsilon > 0 \) such that \( |\varphi_M(\theta)| < 1 - 2\epsilon \) if \( r \leq |\theta| \leq \pi \). Also, \( \varphi_{M_n}(\theta) \) converges to \( \varphi_M(\theta) \) uniformly on \([-\pi, \pi]\). Thus, for large enough \( n \), \( |\varphi_{M_n}\left(\frac{u}{\sqrt{n-k}}\right)| \leq 1 - \epsilon \) if \( |u| > \sqrt{n-k}\pi \). So

\[ |I_n^{(4)}(r)| \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{\sqrt{n-k\pi}<|u|<\sqrt{n-k\pi}} (1-\epsilon)(n-k) du \]

\[ \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} 2\pi \sqrt{n-k}(1-\epsilon)(n-k) \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Now look at \( I_n^{(1)}(A) \).

\[ |I_n^{(1)}(A)| = \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u|<A} \left( \gamma_n\left(\frac{u}{\sqrt{n-k}}\right)(n-k) - e^{-\frac{u^2+2r\pi}{2}} + \gamma_n\left(\frac{u}{\sqrt{n-k}}\right)(n-k)(e^{i\beta_n \frac{u}{\sqrt{n-k}}} - 1) \right) du \]

\[ \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u|<A} \left| \gamma_n\left(\frac{u}{\sqrt{n-k}}\right)(n-k) - e^{-\frac{u^2+2r\pi}{2}} \right| du \]

\[ + \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u|<A} \left| \gamma_n\left(\frac{u}{\sqrt{n-k}}\right)(n-k)(e^{i\beta_n \frac{u}{\sqrt{n-k}}} - 1) \right| du \]

\[ \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u|<A} \left| \gamma_n\left(\frac{u}{\sqrt{n-k}}\right)(n-k) - e^{-\frac{u^2+2r\pi}{2}} \right| du \]

\[ + \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u|<A} \left| e^{i\beta_n \frac{u}{\sqrt{n-k}}} - 1 \right| du. \]
First, note that \(|e^{ix} - 1| \leq |x|\) for all \(x \in \mathbb{R}\). By setting \(x = \beta_n \frac{u}{\sqrt{n-k}}\), we can see that

\[
(2.4) \leq \frac{\sigma_M}{\sqrt{2\pi}} \int_{|u|<A} |\beta_n| |u| \left| \frac{\sigma_M}{\sqrt{2\pi}} |\beta_n| A^2 \right| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Next, note that \(\gamma_n(0) = 1, \gamma_n^{(1)}(0) = 0, \) and \(\gamma_n^{(2)}(0) = -\sigma_M^2\).

Also, \(|\gamma_n^{(3)}(\theta)| = |(1 - i)^3 M_n^3 e^{-iM_n(\theta)} \varphi M_n(\theta) + (1 - i)^2 M_n^2 e^{-iM_n(\theta)} \varphi M_n^{(1)}(\theta) |

\[
+ 3(1 - i) M_n e^{-iM_n(\theta)} \varphi M_n^{(2)}(\theta) + e^{-iM_n(\theta)} \varphi M_n^{(3)}(\theta)\]

\[
= |i M_n^3 \varphi M_n(\theta) - 3 M_n^2 \varphi M_n^{(1)}(\theta) - 3 i M_n \varphi M_n^{(2)}(\theta) + \varphi M_n^{(3)}(\theta)|
\]

\[
\leq M_n^3 + 3 M_n^2 |e| X | + 3 M_n X^2 | + E[X^3] = 4(E[X])^3 + 3E[X]E[X^2] + E[X^3]
\]

\[
\leq 4E[X^3] + 3E[X^3] \frac{1}{2} (E[X^3]) \frac{1}{2} + E[X^3] = 8E[X^3] \leq \rho.
\]

So apply Taylor’s theorem to both real and imaginary parts of \(\gamma M_n(\theta)\) and get

\[
\gamma M_n(\theta) = 1 - \frac{1}{2} \sigma_M^2 \theta^2 + \frac{1}{6} \theta^3 \epsilon_n(\theta)
\]

(2.5)

where \(\text{Re} \epsilon_n(\theta) \leq \rho\) and \(\text{Im} \epsilon_n(\theta) \leq \rho\) uniformly in \(n\) and \(\theta\).

**Lemma 2.4.2.** If \(t \in [0, \infty)\) and \(z \in \mathbb{C}\) with \(|z| < t\), then for all \(n > t\)

\[
\left| (1 - \frac{t+z}{n})^n - e^{-t} \right| \leq |z| + \frac{t^2}{2n}.
\]

**Proof of lemma.** First, note \(|1 - \frac{t+z}{n}| \leq |1 - \frac{t}{n}| + \frac{|z|}{n} = 1 - \frac{t}{n} + \frac{|z|}{n} \leq 1\).

Now \(|(1 - \frac{t+z}{n})^n - e^{-t}| = |(1 - \frac{t+z}{n})^n - (e^{-\frac{t}{n}})^n| \]

\[
= |1 - \frac{t+z}{n} - e^{-\frac{t}{n} n} \sum_{k=0}^{n-1} \left(1 - \frac{t+z}{n}\right)^k e^{-\frac{t}{n}(n-k)}| \leq |1 - \frac{t+z}{n} - e^{-\frac{t}{n} n}|
\]

\[
= |1 - \frac{t+z}{n} - \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{t}{n}\right)^k| \leq |z| + \frac{1}{2n} \sum_{k=2}^{\infty} \frac{1}{k!} \left(\frac{t}{n}\right)^k|n|
\]

\[
\leq |z| + \frac{1}{2} \left(\frac{t}{n}\right)^2 n = |z| + \frac{t^2}{2n}
\]

Now by (2.5), we have (2.3) \(= \frac{\sigma_M}{\sqrt{2\pi}} \int_{|u|<A} |\gamma_n(\frac{u}{\sqrt{n-k}}(n-k) - e^{-\frac{u^2}{2}} |\ du
\]

\[
\leq \frac{\sigma_M}{\sqrt{2\pi}} \int_{|u|<A} \left(1 - \frac{1}{2} \sigma_M^2 \frac{u^2}{n-k} + \frac{1}{6} \frac{u^3}{(n-k)^{\frac{1}{2}} (n-k) \sqrt{n-k}} \right) \left(\frac{u}{\sqrt{n-k}}\right)^{(n-k)} - e^{-\frac{u^2}{2}} \right|du.
\]
If $n$ is large enough, namely $\sqrt{n - k} > \frac{32\rho A}{6\pi^2}$, so that $\frac{1}{2}\sigma_{M_n}^2 u^2 > \frac{1}{2}\sigma^2 u^2 > \frac{16\rho A u^2}{6\sqrt{n-k}} > \frac{1}{6} u^2 \epsilon_n(\frac{u}{\sqrt{n-k}})$ on $|u| < A$, then apply lemma 2.4.2 by setting $t = \frac{1}{2}\sigma_{M_n}^2 u^2$ and $z = -\frac{1}{6} u^2 \epsilon_n(\frac{u}{\sqrt{n-k}})$ we have

\begin{align*}
(2.3) & \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u|<A} \left| \frac{u^3}{6\sqrt{n-k}} \epsilon_n(\frac{u}{\sqrt{n-k}}) \right| + \frac{1}{2} \frac{\sigma_{M_n}^2 u^2}{n-k} \, du
\end{align*}

\begin{align*}
& \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{|u|<A} \left| \frac{16\rho A u^2}{6\sqrt{n-k}} \right| + \frac{1}{2} \frac{\sigma_{M_n}^2 u^2}{n-k} \, du \to 0 \text{ as } n \to \infty.
\end{align*}

Lastly, look at $I_n^{(3)}(A, r)$.

\begin{align*}
|I_n^{(3)}(A, r)| & \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{A < |u| < \sqrt{n-k-r}} \left| \gamma_n(\frac{u}{\sqrt{n-k}})^{(n-k)} e^{i\beta_n \frac{u}{\sqrt{n-k}}} \right| \, du
\end{align*}

Noting that for small $r$, namely $r < \frac{6\sigma^2}{64\pi}$,

\begin{align*}
|\gamma_n(\frac{u}{\sqrt{n-k}})| & = |1 - \frac{\sigma_{M_n}}{2} u^2 - \frac{1}{6} \frac{u^3}{(n-k)\sqrt{n-k}} \epsilon_n(\frac{u}{\sqrt{n-k}})|
\end{align*}

\begin{align*}
& \leq 1 - \frac{\sigma_{M_n}}{2} u^2 + \frac{1}{6} \frac{u^3}{(n-k)} \epsilon_n(\frac{u}{\sqrt{n-k}}) + e^{-\frac{2}{4n-k}} u^2
\end{align*}

on $A < |u| < \sqrt{n-k-r}$.

Thus, for small enough $r$, $|I_n^{(3)}(A, r)| \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{A < |u| < \sqrt{n-k-r}} e^{-\frac{2}{4(n-k)}} u^{(n-k)} \, du$

\begin{align*}
& = \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{A < |u| < \sqrt{n-k-r}} e^{-\frac{2}{4} u^2} \, du \leq \frac{\sigma_{M_n}}{\sqrt{2\pi}} \int_{A < |u|} e^{-\frac{3}{4} u^2} \, du
\end{align*}

\begin{align*}
& = \frac{1}{\sqrt{2\pi}} \int_{\sigma_{M_n} A < |x|} e^{-\frac{x^2}{4}} \, dx \leq \frac{1}{\sqrt{2\pi}} \int_{A < |x|} e^{-\frac{x^2}{4}} \, dx
\end{align*}

\begin{align*}
& = \frac{2}{\sqrt{2\pi}} \int_{x=\pi A} e^{-\frac{x^2}{4}} \, dx \to 0 \text{ as } A \to \infty.
\end{align*}

Thus, given $\epsilon > 0$, if we take $r$ small enough and $A < \infty$ large enough, then all four of $|I_n^{(1)}(A)|$, $|I_n^{(2)}(A)|$, $|I_n^{(3)}(A, r)|$, and $|I_n^{(4)}(r)|$ become less than $\epsilon$ as $n \to \infty$. \qed
Chapter 3

Infinitely many cities

In this chapter, we want to study the process with $|\Lambda| = \infty$. We will take $\Lambda = \mathbb{Z}$. However, we could take $\Lambda = \mathbb{Z}^d$ with no changes except for more complicated notation. Thus we want a process on $\{0, 1, \ldots, b\}^\mathbb{Z}$ with generator $\mathcal{U} f(\eta) = \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} c(\eta; x, y) (f(\eta^{x,y}) - f(\eta))$. If $b < \infty$, the domain of $\mathcal{U}$ should include all cylinder functions $f$, i.e., all functions $f$ such that there is a finite set $\Lambda$ and if $\eta(x) = \zeta(x)$ for $\forall x \in \Lambda$, then $f(\eta) = f(\zeta)$. If $b = \infty$, the domain should include all cylinder functions with the additional property that there is some $\beta < \infty$ such that if $\eta^\beta$ is defined by $\eta^\beta(x) = \eta(x) \wedge \beta$, $\forall x \in \mathbb{Z}$, then $f(\eta^\beta) = f(\eta)$ for $\forall \eta \in \{0, 1, \ldots\}^\mathbb{Z}$. The existence of such a process is guaranteed by [7], theorem 3.9, section 3 of Chapter 1, in the case $b < \infty$. If $b = \infty$, we know of no theorem in the literature that covers our case. However, the paper of Liggett [6] covers our process if we let $\lambda \to \infty$. In our case, the technique of Liggett in [6] can be used almost verbatim to obtain the desired existence theorem. As a part of proofs of the theorem mentioned above we have $T^\Lambda(t) f(\eta) \to T(t) f(\eta)$ as $\Lambda \nearrow \mathbb{Z}$ for all bounded continuous functions $f$, where $T^\Lambda(t)$ is the semigroup on the finite set $\Lambda$ and $T(t)$ is the semigroup on $\mathbb{Z}$. In the case $b = \infty$, the state space is only a subset of $\{0, 1, \ldots\}^\mathbb{Z}$. However, if we assume, in addition, that $\sum_{k=-\infty}^{\infty} p(0, k) e^{-|k|} < \infty$, then it contains all $\eta \in \{0, 1, \ldots\}^\mathbb{Z}$ for which $\sum_{k=-\infty}^{\infty} \eta(k) e^{-|k|} < \infty$.

3.1 Translation invariance

For the same Markov process $\eta$, now let $\Lambda = \mathbb{Z}$ so that the state space, denoted by $S^b = \{\eta: \mathbb{Z} \to \{0, 1, \ldots, b\}\}$ is infinite.
Theorem 3.1.1. If the initial probability measure $\mu_0$ is translation invariant, then $\mu_t$ is also translation invariant for $\forall t \in [0, \infty)$.

We prove the theorem using the following lemma.

Lemma 3.1.2. Let $\theta : S^b \to S^b$ to be such that $\theta(\eta)(k) = \eta(k + 1)$ for $\forall k \in \Lambda$ and $\Theta : C(S^b) \to C(S^b)$ to be $\Theta f(\eta) = f(\theta \eta)$ for $\forall f \in C(S^b)$. Then $\mathcal{U} \Theta = \Theta \mathcal{U}$ and $T(t) \Theta = \Theta T(t)$, where $T(t)$ is the semigroup for the infinitesimal generator $\mathcal{U}$, i.e., $\mathcal{U} \Theta f(\eta) = \Theta \mathcal{U} f(\eta)$ and $T(t) \Theta f(\eta) = \Theta T(t) f(\eta)$ for all cylinder functions $f$ in the domain of the generator $\mathcal{U}$.

Proof. First, show $(\theta \eta)^{x,y} = \theta(\eta^{x+1,y+1})$.

$$(\theta \eta)^{x,y}(k) = \begin{cases} 
\theta \eta(k) & \text{if } k \neq x, y \\
\theta \eta(y) + 1 & \text{if } k = y \\
\theta \eta(x) + 1 & \text{if } k = x 
\end{cases}$$

$$= \begin{cases} 
\eta(k + 1) & \text{if } k \neq x, y \\
\eta(y + 1) + 1 & \text{if } k = y \\
\eta(x + 1) & \text{if } k = x 
\end{cases}$$

$$= \eta^{x+1,y+1}(k + 1) = \theta(\eta^{x+1,y+1})(k).$$

Now to show $\Theta \mathcal{U} f(\eta) = \mathcal{U} \Theta f(\eta)$,

$$\Theta \mathcal{U} f(\eta) = \Theta(\mathcal{U} f)(\eta) = \mathcal{U} f(\theta \eta) = \sum_{x \in \Lambda} \sum_{y \in \Lambda} c(\theta \eta; x, y)(f(\eta^{x,y}) - f(\eta))$$

$$= \sum_{x \in \Lambda} \sum_{y \in \Lambda} P(x, y) \frac{\eta(x)(\eta(y)+\lambda)}{\eta(x)+\eta(y)+2\lambda} (f(\eta^{x,y}) - f(\eta))$$

$$= \sum_{x \in \Lambda} \sum_{y \in \Lambda} P(x, y) \frac{\eta(x)(\eta(y)+\lambda)}{\eta(x)+\eta(y)+2\lambda} (f(\eta^{x+1,y+1}) - f(\eta))$$

$$= \sum_{x \in \Lambda} \sum_{y \in \Lambda} P(x, y) \frac{\eta(x)(\eta(y)+\lambda)}{\eta(x)+\eta(y)+2\lambda} (f(\theta(\eta^{x,y})) - f(\eta))$$

$$= \sum_{x \in \Lambda} \sum_{y \in \Lambda} P(x, y) \frac{\eta(x)(\eta(y)+\lambda)}{\eta(x)+\eta(y)+2\lambda} (f(\eta^{x,y}) - \Theta f(\eta))$$

$$= \sum_{x \in \Lambda} \sum_{y \in \Lambda} P(x, y) c(\eta; x, y)(\Theta f(\eta^{x,y}) - \Theta f(\eta)) = \mathcal{U} \Theta f(\eta) = \mathcal{U} \Theta f(\eta).$$

Finally, let $\hat{T}(t) = \Theta^{-1} T(t) \Theta$. Note $\|\hat{T}(t)\| = \|T(t)\| \leq 1$.

Also, $\hat{T}(t)\hat{T}(s) f(\eta) = (\Theta^{-1} T(t) \Theta)(\Theta^{-1} T(s) \Theta) f(\eta) = \Theta^{-1} T(t) T(s) \Theta f(\eta)$.
First, note that

\[ \Theta^{-1}T(t+s)\Theta f(\eta) = \hat{T}(t+s)f(\eta). \]

Thus, \( \hat{T} \) is a positive contraction semigroup. Then the corresponding infinitesimal generator \( \mathcal{U} \) for \( \hat{T} \) is

\[
\mathcal{U} f(\eta) = \lim_{t \to 0^+} \frac{T(t)f(\eta) - f(\eta)}{t} = \lim_{t \to 0^+} \frac{\Theta^{-1}T(t)\Theta f(\eta) - \Theta^{-1}f(\eta)}{t} = \Theta^{-1} \mathcal{U} (\Theta f)(\eta) = \Theta^{-1} \mathcal{V} \mathcal{U} f(\eta) = \mathcal{U} f(\eta).
\]

So \( \mathcal{U} = \mathcal{G} \), which means that \( \hat{T}(t) \) and \( T(t) \) have the same infinitesimal generator. Because of the uniqueness of a semigroup and the corresponding infinitesimal generator, we must have \( \hat{T}(t) = T(t) \), or \( T(t)\Theta = \Theta T(t) \).

\[ \square \]

**Proof of theorem 3.1.1.** Since \( \mu_0 \) is translation invariant, \( \int f(\eta)\mu_0(\eta)\,d\eta = \int \Theta^n f(\eta)\mu_0(\eta)\,d\eta \) for \( \forall n \in \mathbb{N} \) and for \( \forall f \in \mathcal{D} \). Consider \( \int f(\eta)\mu_0(\eta)\,d\eta = \mathbb{E}_{\mu_0}[f(\eta)] = \mathbb{E}_{\mu_0}[T(t)f(\eta)] = \int T(t)f(\eta)\mu_0(\eta)\,d\eta = \int \Theta T(t)f(\eta)\mu_0(\eta)\,d\eta = \mathbb{E}_{\mu_0}[\Theta f(\eta)] = \int \Theta f(\eta)\mu_t(\eta)\,d\eta, \) which implies that \( \int f(\eta)\mu_t(\eta)\,d\eta = \int \Theta^n f(\eta)\mu_0(\eta)\,d\eta \) for \( \forall n \in \mathbb{N} \).

\[ \square \]

**Theorem 3.1.3.** If the initial probability measure \( \mu_0 \) is translation invariant and

\[ \mathbb{E}_{\mu_0}[\eta(0)] = M < \infty, \text{ then } \mathbb{E}_{\mu_0}[\eta(k)] = M \text{ for } \forall t \in [0, \infty) \text{ and } \forall k \in \Lambda = \mathbb{Z}. \]

**Proof.** First, note that \( \mu_t \) is translation invariant by theorem 3.1.1. So \( \mathbb{E}_{\mu_t}[\eta(k)] \) are the same for all \( k \in \Lambda = \mathbb{Z} \). So it suffices to show that \( \mathbb{E}_{\mu_t}[\eta(0)] = M. \)

For \( b < \infty \), let \( f(\eta) = \eta(0). \) Note that

\[
\mathcal{U} f(\eta) = \sum_{x \in \Lambda} \sum_{y \in \Lambda} c(\eta; x, y)(f(\eta^{x,y}) - f(\eta))
= \sum_{k \neq 0} c(\eta; k, 0)I_{\{\eta(0) < b\}} - \sum_{k \neq 0} c(\eta; 0, k)I_{\{\eta(k) < b\}}.
\]

Since we know that \( \mathbb{E}_{\mu_t}[f(\eta)] = \mathbb{E}_{\mu_0}[f(\eta)] + \int_0^t \mathbb{E}_{\mu_s}[\mathcal{U} f(\eta)]\,ds \), it is enough to show \( \mathbb{E}_{\mu_t}[\mathcal{U} f(\eta)] = 0. \)

\[
\mathbb{E}_{\mu_t}[\mathcal{U} f(\eta)] = \mathbb{E}_{\mu_0}\left[\sum_{k \neq 0} c(\eta; k, 0)I_{\{\eta(0) < b\}} - \sum_{k \neq 0} c(\eta; 0, k)I_{\{\eta(k) < b\}}\right]
= \sum_{k \neq 0} \mathbb{E}_{\mu_0}[c(\eta; k, 0)I_{\{\eta(0) < b\}}] - \sum_{k \neq 0} \mathbb{E}_{\mu_0}[c(\eta; 0, k)I_{\{\eta(k) < b\}}]
= \sum_{k \neq 0} \mathbb{E}_{\mu_0}[c(\eta; k, 0)I_{\{\eta(0) < b\}}] - \sum_{k \neq 0} \mathbb{E}_{\mu_0}[c(\eta; 0, -k)I_{\{\eta(-k) < b\}}] = 0 \text{ because } \mu_t \text{ is translation invariant, and } c(\theta^k \eta, x, y) = c(\eta, x + k, y + k) \text{ and } \theta^k \eta(-k) = \eta(0).
\]

For the case of \( b = \infty \), we prove the following two lemmas first.
Lemma 3.1.4. \( \mathbb{E}_{\mu_t}[\eta(k)] \leq e^t(M + \lambda) - \lambda < \infty \) for all \( t \in [0, \infty) \) and for all \( k \in \mathbb{Z} \).

Proof of lemma 3.1.4. Let \( f_n(\eta_s) = \eta_s(0) \land n \) so that \( f_n(\eta_s) \not\rightarrow \eta_s(0) \) as \( n \to \infty \). Note that \( \mathcal{W} f_n(\eta_s) = \sum_{k \neq 0} c(\eta_s; k, 0)I_{\eta_s(0) < n} - \sum_{k \neq 0} c(\eta_s; 0, k)I_{\eta_s(k) \leq n} \). Since the second term of the RHS is negative, \( \mathcal{W} f_n(\eta_s) \leq \sum_{k \neq 0} c(\eta_s; k, 0)I_{\eta_s(0) < n} \)

\[ = \sum_{k \neq 0} p(k, 0) \frac{\eta_s(k)(\eta_s(0) + \lambda)}{\eta_s(k) + 2\lambda} I_{\eta_s(0) < n} \leq \sum_{k \neq 0} p(k, 0)(\eta_s(0) + \lambda)I_{\eta_s(0) < n} \]

\[ = (\eta_s(0) + \lambda) \sum_{k \neq 0} p(k, 0) I_{\eta_s(0) < n} + \lambda \leq \eta_s(0)I_{\eta_s(0) < n} + \lambda \leq f_n(\eta_s) + \lambda \] for each \( n \).

Now consider \( \frac{d}{ds}\mathbb{E}_{\mu_s}[f_n(\eta_s)] = \frac{d}{ds}\mathbb{E}_{\mu_s}[f_n(\eta_s)] = \mathbb{E}_{\mu_s}[\mathcal{W} f_n(\eta_s)] \)

\[ \leq \mathbb{E}_{\mu_s}[f_n(\eta_s)] + \lambda, \text{ or } \frac{d}{ds}\mathbb{E}_{\mu_s}[f_n(\eta_s)] - \mathbb{E}_{\mu_s}[f_n(\eta_s)] \leq \lambda. \]

Equivalently, \( \frac{d}{ds}(e^{-s}\mathbb{E}_{\mu_s}[f_n(\eta_s)]) \leq \lambda e^{-s}. \)

Integrate from \( s = 0 \) to \( s = t \) to get \( e^{-t}\mathbb{E}_{\mu_s}[f_n(\eta_t)] - \mathbb{E}_{\mu_s}[f_n(\eta_0)] \leq \lambda(1 - e^{-t}) \) for each \( n \), i.e., \( \mathbb{E}_{\mu_s}[f_n(\eta_s)] \leq e^t\mathbb{E}_{\mu_s}[f_n(\eta_0)] + \lambda e^t - 1. \)

Let \( n \to \infty \). Then \( LHS \to \mathbb{E}_{\mu_0}[\eta(0)] = \mathbb{E}_{\mu_t}[\eta(0)] \) by Monotone Convergence Theorem and \( RHS \to e^t\mathbb{E}_{\mu_0}[\eta(0)] + \lambda(e^t - 1) = e^tM + \lambda(e^t - 1) < \infty \) for all \( t \in [0, \infty) \).

Thus, \( \mathbb{E}_{\mu_t}[\eta_s(0)] < e^t(M + \lambda) - \lambda \) for all \( t \in [0, \infty) \). Therefore, \( \mathbb{E}_{\mu_t}[\eta_s(k)] < e^t(M + \lambda) - \lambda \) for all \( k \in \mathbb{Z} \) and all \( t \in [0, \infty) \).

Lemma 3.1.5.

\[ \lim_{n \to \infty} \frac{1}{2n + 1} \sum_{i = -n}^{n} \sum_{|j| > n} p(j, i) = 0 \]

and

\[ \lim_{n \to \infty} \frac{1}{2n + 1} \sum_{i = -n}^{n} \sum_{|j| > n} p(i, j) = 0. \]

Proof of lemma 3.1.5. Since \( \sum_{|j| > n} p(j, i) \leq 1 \) for all \( n \in \mathbb{Z} \), for given \( \epsilon > 0 \), there exists \( l \in \mathbb{Z} \) such that \( \sum_{|j| > l} p(j, 0) < \epsilon. \) Take \( n > l \).

\[ \frac{1}{2n + 1} \sum_{i = -n}^{n} \sum_{|j| > n} p(j, i) \]

\[ = \frac{1}{2n + 1} \left[ \sum_{i = -n}^{n} \sum_{|j| > n} p(j, i) + \sum_{i = -n}^{n} \sum_{|j| > n} p(j, i) + \sum_{i = n}^{n} \sum_{|j| > n} p(j, i) \right] \]
Now let $t \to \infty$ \\
\[ t + \sum_{i=-n}^{n} \sum_{j=\max\{0,i\}}^{\infty} p(j-i,0) + \epsilon \] \leq \frac{1}{2n+1} \left[ 2t + (2(n-l) + 1)\epsilon \right]. \\
So \limsup_{n \to \infty} \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i) \leq \limsup_{n \to \infty} \left[ \frac{2t}{2n+1} + \frac{(2(n-l) + 1)\epsilon}{2n+1} \right] = \epsilon.

But $\frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i) \geq 0$ for $\forall n \in \mathbb{Z}$.

Therefore, $\lim_{n \to \infty} \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i) = 0$. Since $p(i,j)$ is symmetric, we also have $\lim_{n \to \infty} \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(i,j) = 0$.

Now back to the proof of the case $b = \infty$.

Consider $E_{\mu}[\frac{1}{2n+1} \sum_{i=-n}^{n} \eta(i)] = \frac{1}{2n+1} \sum_{i=-n}^{n} E_{\mu}[\eta(i)] = \frac{1}{2n+1} \sum_{i=-n}^{n} E_{\mu}[\eta(0)] = E_{\mu}[\eta(0)]$.

So it is enough to show $\lim_{n \to \infty} E_{\mu}[\frac{1}{2n+1} \sum_{i=-n}^{n} \eta(i)] = M$.

Now let $f_k^{(n)}(\eta) = (\frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i)) \wedge k$ so that $f_k^{(n)}(\eta) \nearrow \frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i)$ for each $n$ as $k \to \infty$ and $f_k^{(n)}(\eta) \in \mathcal{D}(\mathscr{U})$. First, note that for each $n$ $\mathcal{U} f_k^{(n)}(\eta)$ is bounded above by:

$\mathcal{U} f_k^{(n)}(\eta) = \frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i)I_{\{\frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i) \wedge k\}}$

$- \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} c(\eta_{i};j,i)I_{\{\frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i) \wedge k\}}$

$\leq \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} c(\eta_{i};j,i)I_{\{\frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i) \wedge k\}}$

$= \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i) \frac{\eta_{j}(\eta_{i}(i) + \lambda)}{\eta_{j}(\eta_{i}(i) + \lambda) + \eta_{i}(i) + 2\lambda}I_{\{\frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i) \wedge k\}}$

$\leq \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i) (\eta_{i}(i) + \lambda).$

Since $E_{\mu}[f_k^{(n)}(\eta)] = E_{\mu}[f_k^{(n)}(\eta)] = E_{\mu}[f_k^{(n)}(\eta)] + \int_{0}^{\epsilon} E_{\mu}[\mathcal{U} f_k^{(n)}(\eta)]ds,$

$E_{\mu}[f_k^{(n)}(\eta)] \leq E_{\mu}[f_k^{(n)}(\eta)] + \int_{0}^{\epsilon} E_{\mu}[\mathcal{U} f_k^{(n)}(\eta)]ds,$

$E_{\mu}[f_k^{(n)}(\eta)] = E_{\mu}[f_k^{(n)}(\eta)] + \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i)E_{\mu}[\eta_{i}(i) + \lambda]ds$

$= E_{\mu}[f_k^{(n)}(\eta)] + \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i)E_{\mu}[\eta_{i}(i) + \lambda]ds$

$\leq E_{\mu}[f_k^{(n)}(\eta)] + \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i)E_{\mu}[\eta_{i}(i) + \lambda]ds$ by lemma 3.1.4

$= E_{\mu}[f_k^{(n)}(\eta)] + \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i)(\epsilon^t - 1)(M + \lambda)$ for each $k, n$.

Let $k \to \infty$. By MCT,

$LHS \to E_{\mu}[\frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i)] = E_{\mu}[\eta(0)]$ and

$RHS \to E_{\mu}[\frac{1}{2n+1} \sum_{i=-n}^{n} \eta_{i}(i)] + (\epsilon^t - 1)(M + \lambda) \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i)(\epsilon^t - 1)$

$= E_{\mu}[\eta(0)] + (\epsilon^t - 1)(M + \lambda) \frac{1}{2n+1} \sum_{i=-n}^{n} \sum_{|j|>n} p(j,i)(\epsilon^t - 1)$ for each $n$ and $\forall t \in [0, \infty)$.

Now let $n \to \infty$, by lemma 3.1.5, we have $E_{\mu}[\eta(0)] \leq M.$
Lemma 3.1.6. In addition to the hypothesis of theorem 3.1.3, if $E_{\mu_0}[^0] < \infty$,
then $E_{\mu_t}[^k] < \infty$ for $\forall k \in \mathbb{Z}$ and $\forall t \in [0, \infty)$.

Proof. For $b = \infty$, let $g_n(\eta) = \eta(0)^2 \land n^2$ so that $g_n(\eta) < \infty$ and $g_n \rightarrow \eta(0)^2$. Then

\[
\mathbb{V} g_n(\eta) = \sum_{k \neq 0} c(\eta; k, 0)((\eta(0) + 1)^2 - \eta(0)^2)I_{\{\eta(0)^2 < n^2\}} + \sum_{k \neq 0} c(\eta; 0, k)((\eta(0) - 1)^2 - \eta(0)^2)I_{\{\eta(0)^2 < n^2\}} \leq \sum_{k \neq 0} c(\eta; k, 0)(\eta(0) + 1)^2 - \eta(0)^2)I_{\{\eta(0)^2 < n^2\}} \leq \sum_{k \neq 0} p(k, 0)(\eta(0) + 1)(2\eta(0) + 1)I_{\{\eta(0)^2 < n^2\}} \leq \sum_{k \neq 0} p(k, 0)(\eta(0) + 1)(2\eta(0) + 1)I_{\{\eta(0)^2 < n^2\}} \leq (2\lambda + 3)\eta(0)^2 I_{\{\eta(0)^2 < n^2\}} + \lambda$ since $\eta(0) \leq \eta(0)^2$ and $\sum_{k \neq 0} p(k, 0) \leq 1.

Similarly to lemma 3.1.4, $\frac{d}{dt} E_{\mu_0}[g_n(\eta)] - (2\lambda + 3)E_{\mu_0}[g_n(\eta)] \leq \lambda$ gives

\[
E_{\mu_0}[g_n(\eta)] \leq e^{(2\lambda + 3)t}E_{\mu_0}[g_n(\eta_0)] + \frac{\lambda}{2\lambda + 3}(e^{(2\lambda + 3)t} - 1) \text{ for each } n.
\]

Let $n \rightarrow \infty$. Then

\[
\text{LHS} \rightarrow E_{\mu_0}[\eta(0)^2] = E_{\mu_t}[\eta(0)^2] \text{ and } \\
\text{RHS} \rightarrow e^{(2\lambda + 3)t}E_{\mu_0}[\eta(0)^2] + \frac{\lambda}{2\lambda + 3}(e^{(2\lambda + 3)t} - 1).
\]

Since $E_{\mu_0}[\eta(0)^2] < \infty$ by the hypothesis and $\mu_t$ is translation invariant, $E_{\mu_0}[\eta(0)^2] < \infty$ for $\forall t \in [0, \infty)$. Thus, $E_{\mu_t}[\eta(k)^2] < \infty$ for $\forall k \in \mathbb{Z}$ and $\forall t \in [0, \infty)$. 

\[\square\]
3.2 Specific free energy

We have shown that free energy is nonincreasing function of time $t$ on a finite state space in section 2.2. In this section, we will use these results to show that in the case where $\Lambda = \mathbb{Z}$, all translation invariant stationary measures satisfying a technical condition (see Assumption 3.2.8) are averages of $\mathcal{P}_{m,b}$. We define specific free energy to be the free energy per site as $|\Lambda| \to \infty$. First, we show that the specific free energy is finite provided that the initial probability measure $\mu_0$ is translation invariant with $\mathbb{E}_{\mu_0}[\eta(0)] = M < \infty$ and $\mathbb{E}_{\mu_0}[\eta(0)^2] < \infty$. From now on, in this section, we will assume that the initial probability measure $\mu_0$ is translation invariant with $\mathbb{E}_{\mu_0}[\eta(0)] = M < \infty$ and $\mathbb{E}_{\mu_0}[\eta(0)^2] < \infty$.

Theorem 3.2.1. Let $H_\Lambda(\mu_t)$ be the free energy on $\mathcal{S}^b_{\Lambda,N}$ at time $t$. If the initial probability measure $\mu_0$ is translation invariant with $\mathbb{E}_{\mu_0}[\eta(0)] = M < \infty$ and $\mathbb{E}_{\mu_0}[\eta(0)^2] < \infty$, then $\lim_{\Lambda \to \mathbb{Z}} \frac{1}{|\Lambda|} H_\Lambda(\mu_t)$ is finite, denoted by $h(\mu_t)$, the specific free energy.

The proof of the theorem will be accomplished by the series of the following lemmas.

Lemma 3.2.2. For $x \in \Lambda$, the entropy at site $x$, $S_x$, is finite for $\forall t \in [0, \infty)$.

Proof. For $x \in \Lambda$, let $m_t(\eta(x))$ be the probability measure at time $t$. $\eta(x) = k$ takes the values $k = \{0, 1, \ldots, b\}$ and $m_t(\eta(x) = k) \in (0, 1)$ for $\forall t \in [0, \infty)$ and $\forall k \in \{0, 1, \ldots, b\}$.

Also, $-m_t(k) \ln(m_t(k))$ is positive and has the maximum value of $\frac{1}{e}$ at $m_t(k) = \frac{1}{e} \ln(m_t(k))$ if $b < \infty$, $S_x = -\sum_{k=0}^{b} m_t(k) \ln(m_t(k))$ is clearly finite. If $b = \infty$, let $B_1 = \{k : m_t(k) \geq e^{-1}\}$, $B_2 = \{k : e^{-k} < m_t(k) < e^{-1}\}$, and $B_3 = \{k : m_t(k) \leq e^{-k}\}$.

Now note that $-\ln(m_t(k)) \leq 1$ on $B_1$ and $-m_t(k) \ln(m_t(k)) \leq m_t(k)$ for $\forall k \in B_1$ and that $-\ln(m_t(k)) < k$ on $B_2$ and $-m_t(k) \ln(m_t(k)) \leq m_t(k)k$ for $\forall k \in B_2$. Since $-m_t(k) \ln(m_t(k))$ is increasing on $B_3$, $-m_t(k) \ln(m_t(k)) \leq -e^{-k} \ln(e^{-k}) = ke^{-k}$ for each $k \in B_3$.

Now the entropy $S_x = -\sum_{k=0}^{\infty} m_t(k) \ln(m_t(k)) = -\sum_{k \in B_1} m_t(k) \ln(m_t(k)) - \sum_{k \in B_2} m_t(k) \ln(m_t(k)) - \sum_{k \in B_3} m_t(k) \ln(m_t(k)) \leq \sum_{k \in B_1} m_t(k) + \sum_{k \in B_2} m_t(k)k + \sum_{k \in B_3} ke^{-k}$.
\[ \leq \sum_{k=0}^{\infty} m_t(k) + \sum_{k=0}^{\infty} m_t(k)k + \sum_{k=0}^{\infty} k e^{-k} \leq 1 + M + \frac{e}{(e-1)^2} < \infty. \]

\[ \square \]

**Lemma 3.2.3.** Let \( \Lambda_1 \) and \( \Lambda_2 \) such that \( \Lambda_1 \cap \Lambda_2 = \emptyset \).

For all \( \varepsilon \in [0, \infty) \), entropy \( S_{\Lambda_1 \cup \Lambda_2} \leq S_{\Lambda_1} + S_{\Lambda_2} \) and is finite.

**Proof.** The proof can be found in [4], pp.35–36.

\[ \square \]

**Lemma 3.2.4.** For all \( \varepsilon \in [0, \infty) \), \( \lim_{\Lambda \nearrow \mathcal{X}} \frac{S_{\Lambda}}{|\Lambda|} \) is finite.

**Proof.** The proof can be found in [4], pp. 47–49.

\[ \square \]

**Lemma 3.2.5.** If the hypothesis of theorem 3.2.1 holds, then

\[ \lim_{\Lambda \nearrow \mathcal{X}} \frac{1}{|\Lambda|} \sum_{\eta \in S_{\Lambda,N}} \mu_t(\eta) \ln(\mathcal{P}_{\Lambda,M}^b(\eta)) \]

is finite, where \( \mathcal{P}_{\Lambda,M}^b(\eta) = \prod_{k \in \Lambda} P_{m,b}(\eta(k)) \) with \( m = m(M,b) \).

**Proof.**

\[ \frac{1}{|\Lambda|} \sum_{\eta \in S_{\Lambda,N}} \mu_t(\eta) \ln(\mathcal{P}_{\Lambda,M}^b(\eta)) = \frac{1}{|\Lambda|} \sum_{\eta \in S_{\Lambda,N}} \mu_t(\eta) \ln(\prod_{k \in \Lambda} P_{m,b}(\eta(k))) = \frac{1}{|\Lambda|} \sum_{\eta \in S_{\Lambda,N}} \mu_t(\eta) \ln \left( \prod_{k \in \Lambda} \ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)} m^{\eta(k)}/Z) \right), \]

where \( Z = \sum_{i=0}^{\infty} \frac{\Gamma(i+\lambda)}{\Gamma(i+1)} m^i \), a constant.

\[ = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} \sum_{\eta \in S_{\Lambda,N}} \mu_t(\eta) \ln \left( \frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)} \right) + \eta(k) \ln(m) - \ln(Z) \]

\[ = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} \sum_{\eta \in S_{\Lambda,N}} H_\mu(\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})) + \eta(k) \ln(m) - \ln(Z) \]

\[ = \frac{1}{|\Lambda|} |\Lambda| H_\mu(\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})) + \eta(k) \ln(m) - \ln(Z) \]

\[ = H_\mu(\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})) + M \ln(m) - \ln(Z). \]

Since the last two terms are finite, it remains to show

\[ H_\mu(\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})) \]

finite for each \( \lambda > 0 \). To show \( H_\mu(\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})) \) is finite;

case 1) if \( \lambda = 1 \), \( \mu_t(\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})) = \mu_t(\ln(1)) = 0 \).

\[ (l+1) \leq \sum_{k=1}^{n+l-1} \ln(k) - \sum_{k=1}^{n+l-1} \ln(k) = \sum_{k=n+1}^{n+l-1} \ln(k) \leq \sum_{k=n+1}^{n+l-1} \ln(n + l - 1) = (l-1) \ln(n + l - 1) \leq (l-1)(n + l - 1). \]
Let \([\lambda]\) be the smallest integer that is greater than or equal to \(\lambda\), \(1 < \lambda \leq [\lambda] < \lambda + 1\).

\[
\mathbb{E}_\mu[\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})] = \mu_t(\eta(k) = 0) \ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)}) + \sum_{i=1}^b \mu_t(\eta(k) = i) \ln(\frac{\Gamma(i+\lambda)}{\Gamma(i+1)})
\]

\[
\leq \mu_t(\eta(k) = 0) \ln(\Gamma(\lambda)) + \sum_{i=1}^b \mu_t(\eta(k) = i) \ln(\frac{\Gamma(i+\lambda)}{\Gamma(i+1)})
\]

\[
\leq \ln(\Gamma(\lambda)) + \sum_{i=1}^b \mu_t(\eta(k) = i)(\lfloor |\lambda| - 1 \rfloor (i + |\lambda| - 1) < \ln(\Gamma(\lambda)) + \mathbb{E}_\mu[\lambda(\eta(k) + \lambda)]
\]

\[
= \ln(\Gamma(\lambda)) + \lambda(M + \lambda) < \infty.
\]

(case 3) if \(0 < \lambda < 1\), note that \(\ln(\Gamma(\lambda)) > 0\).

For \(\eta(k) \geq 2\), \(0 > \ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)}) > \ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1+\lambda)}) = \ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)}) = -\ln(\eta(k) + \lambda)
\]

\[
\geq -(\eta(k) + \lambda). \text{ So } \mathbb{E}_\mu[\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})] = \mu_t(\eta(k) = 0) \ln(\Gamma(\lambda)) + \mu_t(\eta(k) = 1) \ln(\Gamma(1 + \lambda))
\]

\[
+ \sum_{i=2}^b \mu_t(\eta(k) = i) \ln(\frac{\Gamma(i+\lambda)}{\Gamma(i+1)}) \geq -|\ln(\Gamma(1 + \lambda))| - \sum_{i=2}^b \mu_t(\eta(k) = i)(\eta(k) + \lambda)
\]

\[
\geq -|\ln(\Gamma(1 + \lambda))| - \mathbb{E}_\mu[\eta(k) + \lambda] = -|\ln(\Gamma(1 + \lambda))| - (M + \lambda) > -\infty. \text{ On the other hand, }
\]

\[
\mathbb{E}_\mu[\ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)})] \leq \ln(\Gamma(\lambda)) + |\ln(\Gamma(1 + \lambda))| \text{ since } \ln(\frac{\Gamma(\eta(k)+\lambda)}{\Gamma(\eta(k)+1)}) < 0 \text{ for } \forall \eta(k) \geq 2.
\]

To conclude the proof of theorem 3.2.1, recall that

\[
H_A(\mu_t) = \sum_{\eta \in S^b_{A,N}} \mu_t(\eta) \ln(\mathcal{B}^b_{A,M}(\eta)) = -S_A - \sum_{\eta \in S^b_{A,N}} \mu_t(\eta) \ln(\mathcal{B}^b_{A,M}(\eta))
\]

and \(\frac{H(\mu_t)}{|A|} = -S_A(\mu_t) - \frac{1}{|A|} \sum_{\eta \in S^b_{A,N}} \mu_t(\eta) \ln(\mathcal{B}^b_{A,M}(\eta))\). By lemma 3.2.4 and lemma 3.2.5, it is finite as \(A \not\subseteq \emptyset\).

Next is to show that under a certain assumption on \(\mu_o\) (see Assumption 3.2.8) \(\frac{d}{dt} H_A(\mu_t)|_{t=0}\) consists of nonpositive terms and a bounded term independent of the size of \(A\). For \(A \subseteq \emptyset\), define \(\tau : S^b \to S^b_A\) such that

\[
\tau(\eta(k)) = \begin{cases} 
\eta(k) & \text{if } k \in A, \\
0 & \text{otherwise}
\end{cases}
\]

Let \(B(\Lambda) = \{ f : S^b \to \mathbb{R} | f(\eta) = f(\tau \eta) \}\) and for \(\zeta \in S^b_A\), let \(A(\Lambda, \zeta) = \{ \eta \in S^b | \tau(\eta) = \zeta \}\). Note that

\[
H_A(\mu_t) = \sum_{\zeta \in S^b_A} \mu_o(A(\Lambda, \zeta)) \ln(\mathcal{B}^b_{A,M}(A(\Lambda, \zeta))) \text{ is finite and that } \ln(\mathcal{B}^b_{A,M}(A(\Lambda, \zeta))) \in B(\Lambda). \text{ Now look at }
\]

\[
\frac{d}{dt} H_A(\mu_t)|_{t=0} = \sum_{\zeta \in S^b_A} \int_{A(\Lambda, \zeta)} \frac{d}{dt}(\mu_t(d\eta)) \ln(\mathcal{B}^b_{A,M}(A(\Lambda, \zeta)))|_{t=0}
\]

\[
= \sum_{\zeta \in S^b_A} \int_{A(\Lambda, \zeta)} \mu_t(d\eta) \ln(\mathcal{B}^b_{A,M}(A(\Lambda, \zeta)))|_{t=0}
\]
Before we go on, let’s note that, for \( \xi \in S^b_{A \cup \{y\}} \) and \( x \in \Lambda \), if \( \xi(x) = 0 \) or \( \xi(y) = b \), then \( c(\xi; x, y) = 0 \).

If \( \xi(y) = 0 \), or \( \xi(x) = b \), then \( c(\xi; y, x) = 0 \).

Also, the map \( \{ \xi \in S^b_{A \cup \{y\}} : \xi(x) > 0, \xi(y) < b \} \to \{ \xi^x \xi : \xi(0) \in S^b_{A \cup \{y\}} \} \) is one-to-one and onto on \( \{ \xi \in S^b_{A \cup \{y\}} : \xi(x) < b, \xi(y) > 0 \} \). Thus, the last term above can be written as

\[
\sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{\xi \in S^b_{A \cup \{y\}}} \int_{A(\Lambda, \zeta^+)} \mu_0(d\xi)c(\xi; x, y)[\ln\left(\frac{\mu_0(A(\Lambda, \tau(x,y)))}{\mathcal{P}_{\Lambda, M}(A(\Lambda, \tau(x,y)))}\right) - \ln\left(\frac{\mu_0(A(\Lambda)))}{\mathcal{P}_{\Lambda, M}(A(\Lambda)))}\right)].
\]

Then \( \frac{d}{dt} H_\Lambda(\mu_0)_{|t=0} = I_\Lambda(\mu_0) + K_\Lambda(\mu_0) \), where

\[
I_\Lambda(\mu_0) = \sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{\xi \in S^b_{A \cup \{y\}}} \int_{A(\Lambda, \zeta^+)} \mu_0(d\xi)c(\xi; x, y)[\ln\left(\frac{\mu_0(A(\Lambda, \tau(x,y)))}{\mathcal{P}_{\Lambda, M}(A(\Lambda, \tau(x,y)))}\right) - \ln\left(\frac{\mu_0(A(\Lambda)))}{\mathcal{P}_{\Lambda, M}(A(\Lambda)))}\right)].
\]

and

\[
K_\Lambda(\mu_0) = \sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{\xi \in S^b_{A \cup \{y\}}} \left(\int_{A(\Lambda, \zeta^+)} \mu_0(d\xi)c(\xi; x, y) - \int_{A(\Lambda, \zeta^+)} \mu_0(d\xi)c(\xi; y, x)\right)
\]

\[
\times [\ln\left(\frac{\mu_0(A(\Lambda, \tau(x,y)))}{\mathcal{P}_{\Lambda, M}(A(\Lambda, \tau(x,y)))}\right) - \ln\left(\frac{\mu_0(A(\Lambda)))}{\mathcal{P}_{\Lambda, M}(A(\Lambda)))}\right)].
\]

We know that \( I_\Lambda(\mu_0) \) is nonpositive by lemma 2.2.2. Now we will show that \( K_\Lambda(\mu_0) \) is finite, independently of the size of \( \Lambda \) for \( \forall b \leq \infty \).

For \( \xi \in S^b_{A \cup \{y\}} \), let \( c(\xi; x, y) = p(x, y)r(\xi; x, y) \), i.e., \( r(\xi; x, y) = \frac{\xi(x)\xi(y) + \lambda}{\xi(x) + \xi(y) + 2\lambda} \).

Define \( \Gamma_+ (\Lambda, \zeta, x, y) = \int_{A(\Lambda, \zeta)} \mu_0(d\xi)r(\xi; x, y) \), where \( \xi \in S^b_{\Lambda} \). Then \( K_\Lambda(\mu_0) \) can be written as

\[
K_\Lambda(\mu_0) = \sum_{x \in \Lambda} \sum_{y \in \Lambda} p(x, y) \sum_{\xi \in S^b_{\Lambda}} \left[\Gamma_+ (\Lambda, \zeta, x, y) - \Gamma_+ (\Lambda, \zeta^+, y, x)\right]
\]

\[
\times [\ln\left(\frac{\mu_0(A(\Lambda, \tau(x,y)))}{\mathcal{P}_{\Lambda, M}(A(\Lambda, \tau(x,y)))}\right) - \ln\left(\frac{\mu_0(A(\Lambda)))}{\mathcal{P}_{\Lambda, M}(A(\Lambda)))}\right)].
\]
\[
= \sum_{x \in \Lambda} \sum_{y \in \mathcal{Z} \setminus \Lambda} p(x, y) \sum_{\zeta \in S^p_\Lambda} \left[ \Gamma_+ (\Lambda, \zeta, x, y) - \Gamma_+ (\Lambda, \zeta^{x-}, y, x) \right] \\
\times \left[ \left( \ln \left( \Gamma_+ (\Lambda, \zeta^{x-}, y, x) \right) \right) - \left( \ln \left( \Gamma_+ (\Lambda, \zeta, x, y) \right) \right) - \ln \left( \frac{\Gamma_+ (\Lambda, \zeta, x, y)}{\mu_0 (A(\Lambda, \zeta))} \right) \right] \\
+ \ln \left( \frac{\rho_p^{x,y} (A(\Lambda, \zeta))}{\rho_p^{x,y} (A(\Lambda, \zeta^{x-}))} \right) \right] \\
\leq \sum_{x \in \Lambda} \sum_{y \in \mathcal{Z} \setminus \Lambda} p(x, y) \sum_{\zeta \in S^p_\Lambda} \left[ \Gamma_+ (\Lambda, \zeta, x, y) - \Gamma_+ (\Lambda, \zeta^{x-}, y, x) \right] \\
\times \left[ - \ln \left( \Gamma_+ (\Lambda, \zeta^{x-}, y, x) \right) \right] + \ln \left( \Gamma_+ (\Lambda, \zeta, x, y) \right) + \ln \left( \frac{\Gamma_+ (\Lambda, \zeta, x, y)}{\mu_0 (A(\Lambda, \zeta))} \right) \right] \\
+ \sum_{x \in \Lambda} \sum_{y \in \mathcal{Z} \setminus \Lambda} p(x, y) \sum_{\zeta \in S^p_\Lambda} \left[ \Gamma_+ (\Lambda, \zeta, x, y) \right] \\
\times \left[ \left( \ln \left( \Gamma_+ (\Lambda, \zeta^{x-}, y, x) \right) \right) + \ln \left( \Gamma_+ (\Lambda, \zeta, x, y) \right) + \ln \left( \frac{\Gamma_+ (\Lambda, \zeta, x, y)}{\mu_0 (A(\Lambda, \zeta))} \right) \right] \\
+ \sum_{x \in \Lambda} \sum_{y \in \mathcal{Z} \setminus \Lambda} p(x, y) \sum_{\zeta \in S^p_\Lambda} \left[ \frac{\Gamma_+ (\Lambda, \zeta, x, y)}{\mu_0 (A(\Lambda, \zeta))} \right] \\
\times \left[ \left( \ln \left( \Gamma_+ (\Lambda, \zeta^{x-}, y, x) \right) \right) - \ln \left( \Gamma_+ (\Lambda, \zeta, x, y) \right) - \ln \left( \frac{\rho_p^{x,y} (A(\Lambda, \zeta))}{\rho_p^{x,y} (A(\Lambda, \zeta^{x-}))} \right) \right] \mu_0 (A(\Lambda, \zeta)).
\]

**Lemma 3.2.6.** \(\sum_{x \in \Lambda} \sum_{y \in \mathcal{Z} \setminus \Lambda} p(x, y) < \infty\), independently of the size of \(\Lambda\).

**Proof.** Let \(|\Lambda| = 2n + 1\) for some \(n \in \mathbb{N}\).

\[
\sum_{x \in \Lambda} \sum_{y \in \mathcal{Z} \setminus \Lambda} p(x, y) = \sum_{x=-n}^{n} \sum_{y=0}^{\infty} p(x, y) \\
= \sum_{x=-n}^{n} \sum_{y=-n}^{\infty} p(x, y) + \sum_{x=n+1}^{n} \sum_{y=-n}^{-1} p(x, y) \\
= \sum_{x=-n}^{n} \sum_{y=n+1}^{\infty} p(x, y) + \sum_{x=-n}^{-n} \sum_{y=n}^{-1} p(x, y) \\
= \sum_{x=-n}^{n} \sum_{y=n+1}^{\infty} p(x, y) + \sum_{x=-n}^{-n} \sum_{y=-n}^{n} p(x, y) \\
\leq 2 \sum_{x=-n}^{n} \sum_{y=n+1}^{\infty} p(x, y) = 2 \sum_{x=-n}^{n} \sum_{y=1}^{\infty} p(x, y) \\
= 2 \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} p(0, y-x) = 2 \sum_{x=0}^{\infty} \sum_{y=1}^{\infty} p(0, y+x) \\
= 2 \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} p(0, y+x) = 2 \sum_{x=0}^{\infty} \sum_{y=1}^{\infty} p(0, y) \\
= \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} p(0, x) + \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} p(x, y) \\
= \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} p(0, y) + \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} p(0, y) \\
= \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} p(0, x) + \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} p(0, x) \\
= \sum_{y=1}^{\infty} \mathbb{P}(|X| \geq y) = \mathbb{E}[|X|] < \infty, \text{ where } X \text{ has the distribution of } p(0, x).
Lemma 3.2.7. If $\zeta \in S^b_{\Lambda}$ and $\zeta(x) > 0$ for $x \in \Lambda$, $\ln(\frac{\mathcal{P}_A^b(\Lambda(\zeta))}{\mathcal{P}_A^b(\Lambda(\zeta^x))})$ is bounded below and above for each $\lambda \in (0, \infty)$.

Proof. Since $\zeta(k) = \zeta^x(k)$ if $k \neq x$, $\frac{\mathcal{P}_A^b(\Lambda(\zeta))}{\mathcal{P}_A^b(\Lambda(\zeta^x))} = m^\zeta(x) \frac{\Gamma((\zeta(x)+\lambda))}{\Gamma((\zeta(x)+1)+\lambda)} = m^\zeta(x) \lambda \frac{\Gamma((\zeta(x)+\lambda))}{\Gamma((\zeta(x)+1)+\lambda)}$.

For a fixed $m < \infty$, if $0 < \lambda < 1$, $0 < \lambda \leq (1 + \frac{\lambda}{\zeta(x)}) < 1$. If $\lambda = 1$, $(1 + \frac{\lambda}{\zeta(x)}) = 1$. If $\lambda > 1$, $1 < (1 + \frac{\lambda}{\zeta(x)}) \leq \lambda$. So for all $\lambda > 0$, $m(1 + \frac{\lambda}{\zeta(x)})$ is positive and bounded below and above. Thus, $\ln(\frac{\mathcal{P}_A^b(\Lambda(\zeta))}{\mathcal{P}_A^b(\Lambda(\zeta^x))}) = \ln(m(1 + \frac{\lambda}{\zeta(x)}))$ is bounded below and above for any given $\lambda > 0$ and $M < \infty$, i.e., there exist $B_1(\lambda, M)$ and $B_2(\lambda, M)$ such that $B_1(\lambda, M) < \ln(\frac{\mathcal{P}_A^b(\Lambda(\zeta))}{\mathcal{P}_A^b(\Lambda(\zeta^x))}) < B_2(\lambda, M)$.

Assumption 3.2.8. Suppose there exists $\delta > 0$ such that, for all $\Lambda$ and $y \notin \Lambda$,

$$\mathbb{P}_{\mu_0}(\eta(y) = 0 | \mathcal{F}_\Lambda) \leq 1 - \delta$$

and that $\mathbb{P}_{\mu_o}(\eta(y) = b | \mathcal{F}_\Lambda) \leq 1 - \delta$ if $b < \infty$, where $\mathcal{F}_\Lambda$ is a sigma algebra generated by $\{\eta(x) : x \in \Lambda\}$.

Note this assumption guarantees that $\mu_0(\eta \equiv 0) = 0$.

Lemma 3.2.9. If $\zeta \in S^b_{\Lambda}$ and $\zeta(x) > 0$ for $x \in \Lambda$, then under assumption 3.2.8 $\frac{\Gamma^+(\Lambda, \zeta, x, y)}{\mu_0(\Lambda(\zeta^x))}$ are bounded below by $\lambda \frac{1}{1+2\delta}$.

Proof. Let $\xi = (\zeta, k)$, where $\zeta \in S^b_{\Lambda}$ and $\zeta(y) = k$, $k \in \{0, 1, \ldots, b\}$.

Note that $\sum_{k=0}^{b-1} \int_{A(\zeta^x)} \mu_0(d\xi) = \mu_0(\Lambda(\zeta^x))$.

Write $\Gamma^+(\Lambda, \zeta, x, y) = \sum_{k=0}^{b-1} \int_{A(\zeta^x)} \mu_0(d\xi) r(\zeta; x, y)$

$$\geq \min_{0 \leq k \leq b-1} \sum_{k=0}^{b-1} \int_{A(\zeta^x)} \mu_0(d\xi)$$

$$= \min_{0 \leq k \leq b-1} \int_{A(\zeta^x)} \mu_0(d\xi)$$

Simple calculus will show that $\min_{0 \leq k \leq b-1} \int_{A(\zeta^x)} \mu_0(d\xi) = \lambda \frac{1}{1+2\delta}$.

Thus we have $\Gamma^+(\Lambda, \zeta, x, y) \geq \lambda \frac{1}{1+2\delta} \sum_{k=0}^{b-1} \int_{A(\zeta^x)} \mu_0(d\xi)$ and

$$\Gamma^+(\Lambda, \zeta, x, y) \geq \frac{\lambda}{1+2\delta} \sum_{k=0}^{b-1} \int_{A(\zeta^x)} \mu_0(d\xi)$$

$$= \frac{\lambda}{1+2\delta} (1 - \mathbb{P}_{\mu_o}(\eta(y) = b | \mathcal{F}_\Lambda)(\zeta))$$

$$\geq \frac{\lambda}{1+2\delta} (1 - (1 - \delta)) = \frac{\lambda}{1+2\delta}.$$
On the other hand, \( \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} = \mathbb{E}_{\mu_o}[r(\xi; x, y)|\mathcal{F}_A](\zeta) \). So we have
\[
\frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} = \mathbb{E}_{\mu_o}[r(\xi; x, y)|\mathcal{F}_A](\zeta) \geq \frac{\lambda}{1 + 2\delta}. \]
Similarly, \( \frac{\Gamma_+(\Lambda, \zeta^-, y, x)}{\mu_o(A(\Lambda, \zeta^-))} = \mathbb{E}_{\mu_o}[r(\xi, y, x)|\mathcal{F}_A](\zeta^-) \geq \frac{\lambda}{1 + 2\delta}. \)

\[\square\]

**Lemma 3.2.10.** For \( x \in \Lambda \) and \( y \notin \Lambda \),
\[
\sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} \ln \left( \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} \right) \mu_o(A(\Lambda, \zeta)) \leq \mathbb{E}_{\mu_o}[\zeta(0)^2] \text{ and }
\sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \frac{\Gamma_+(\Lambda, \zeta^-, y, x)}{\mu_o(A(\Lambda, \zeta^-))} \ln \left( \frac{\Gamma_+(\Lambda, \zeta^-, y, x)}{\mu_o(A(\Lambda, \zeta^-))} \right) \mu_o(A(\Lambda, \zeta^-)) \leq \mathbb{E}_{\mu_o}[\zeta(0)^2], \text{ independently of the size of } \Lambda.
\]

**Proof.** For any \( \Lambda \subset Z \),
\[
\sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} \ln \left( \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} \right) \mu_o(A(\Lambda, \zeta)) = \sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \mathbb{E}_{\mu_o}[r(\xi; x, y)|\mathcal{F}_A](\zeta) \ln \left( \mathbb{E}_{\mu_o}[r(\xi; x, y)|\mathcal{F}_A](\zeta) \right) \mu_o(A(\Lambda, \zeta))
\leq \sum_{\{\xi \in S^\zeta(H)(x) > 0\}} (\mathbb{E}_{\mu_o}[r(\xi; x, y)|\mathcal{F}_A](\zeta))^2 \mu_o(A(\Lambda, \zeta))
\leq \sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \mathbb{E}_{\mu_o}[r(\xi; x, y)^2|\mathcal{F}_A](\zeta) \mu_o(A(\Lambda, \zeta))
\leq \sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \mathbb{E}_{\mu_o}[\zeta(0)^2] |\mathcal{F}_A](\zeta) \mu_o(A(\Lambda, \zeta)) \leq \mathbb{E}_{\mu_o}[\zeta(0)^2], \text{ independently of the size of } \Lambda.
\]

\[\square\]

**Lemma 3.2.11.** For \( x \in \Lambda \) and \( y \notin \Lambda \), under assumption 3.2.8
\[
\sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} \left( - \ln \left( \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} \right) \right) \mu_o(A(\Lambda, \zeta)) \leq \ln \left( \frac{1 + 2\lambda}{\lambda \delta} \right) M \text{ and }
\sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \frac{\Gamma_+(\Lambda, \zeta^-, y, x)}{\mu_o(A(\Lambda, \zeta^-))} \left( - \ln \left( \frac{\Gamma_+(\Lambda, \zeta^-, y, x)}{\mu_o(A(\Lambda, \zeta^-))} \right) \right) \mu_o(A(\Lambda, \zeta^-)) \leq \ln \left( \frac{1 + 2\lambda}{\lambda \delta} \right) M, \text{ independently of the size of } \Lambda.
\]

**Proof.** For any \( \Lambda \subset Z \),
\[
\sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} \left( - \ln \left( \frac{\Gamma_+(\Lambda, \zeta, x, y)}{\mu_o(A(\Lambda, \zeta))} \right) \right) \mu_o(A(\Lambda, \zeta)) \leq \ln \left( \frac{1 + 2\lambda}{\lambda \delta} \right) \sum_{\{\xi \in S^\zeta(H)(x) > 0\}} \mathbb{E}_{\mu_o}[r(\xi; x, y)|\mathcal{F}_A](\zeta) \mu_o(A(\Lambda, \zeta))
\]
Similarly, $\sum_{\langle \zeta \in S_\Lambda : \zeta(x) > 0 \rangle} \frac{\Gamma_+(\Lambda, \zeta^x - y, x)}{\mu_0(\Lambda(\zeta^x - y))} \left( - \ln \left( \frac{\Gamma_+(\Lambda, \zeta^x, y)}{\mu_o(A(\Lambda, \zeta^x))} \right) \mu_o(A(\Lambda, \zeta^x)) \right) \leq \ln \left( \frac{1 + 2\Lambda}{M} \right) M$. $\square$

Lemma 3.2.12. For $x \in \Lambda$ and $y \not\in \Lambda$,

$$\sum_{\langle \zeta \in S_\Lambda : \zeta(x) > 0 \rangle} \frac{\Gamma_+(\Lambda, \zeta^x, y, x)}{\mu_o(A(\Lambda, \zeta^x))} \ln \left( \frac{\mathcal{P}_b^b(\Lambda, \zeta^x)}{\mathcal{P}_b^b(\Lambda, A(\Lambda, \zeta^x))} \right) \mu_o(A(\Lambda, \zeta^x)) \leq B_2(\lambda, M) M \text{ and}$$

$$\sum_{\langle \zeta \in S_\Lambda : \zeta(x) > 0 \rangle} \frac{\Gamma_+(\Lambda, \zeta^x - y, x)}{\mu_o(A(\Lambda, \zeta^x - y))} \ln \left( \frac{\mathcal{P}_b^b(\Lambda, \zeta^x - y)}{\mathcal{P}_b^b(\Lambda, A(\Lambda, \zeta^x - y))} \right) \mu_o(A(\Lambda, \zeta^x - y)) \leq B_2(\lambda, M) M, \text{ independently of the size of } \Lambda. \text{ Here, } B_2(\lambda, M) \text{ is as in lemma 3.2.7.}$$

Proof. For any $\Lambda \subset Z$, $\sum_{\langle \zeta \in S_\Lambda : \zeta(x) > 0 \rangle} \frac{\Gamma_+(\Lambda, \zeta^x, y, x)}{\mu_o(A(\Lambda, \zeta^x))} \ln \left( \frac{\mathcal{P}_b^b(\Lambda, \zeta^x)}{\mathcal{P}_b^b(\Lambda, A(\Lambda, \zeta^x))} \right) \mu_o(A(\Lambda, \zeta^x)) \leq B_2(\lambda, M) \sum_{\langle \zeta \in S_\Lambda : \zeta(x) > 0 \rangle} \mathbb{E}_{\mu_t}[r(\xi; x, y)] \mathcal{P}_\Lambda \mu_o(A(\Lambda, \zeta^x))$ by lemma 3.2.7

$$= B_2(\lambda, M) \mathbb{E}_{\mu_t}[r(\xi; x, y)] \leq B_2(\lambda, M) \mathbb{E}_{\mu_t}[\zeta(x)] = B_2(\lambda, M) M < \infty. \quad \square$$

By lemma 3.2.10, lemma 3.2.11, and lemma 3.2.12, we have $K_\lambda(\mu_o)$ is finite, independently of the size of $\Lambda$. So $\frac{d}{dt} H_\Lambda(\mu_t)$ is bounded above, independently of the size of $\Lambda$, i.e., $\exists K < \infty, \exists \frac{d}{dt} H_\Lambda(\mu_t) \leq I_\Lambda(\mu_o) + K$.

Lemma 3.2.13. Suppose $\Lambda = \Lambda_1 \cup \Lambda_2$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$. Then $I_\Lambda(\mu_o) \leq I_{\Lambda_1}(\mu_o) + I_{\Lambda_2}(\mu_o)$, i.e., $-I_\Lambda(\mu_o)$ is superadditive in $\Lambda$.

Proof. $I_\Lambda(\mu_o) = \sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{\zeta \in S_\Lambda} f_{A(\Lambda, \zeta)}(d(\zeta)c(\zeta; x, y)[\ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b^b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) - \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right)]$ \times c(\zeta; x, y)$

$$= \sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{\zeta \in S_\Lambda} \left[ \frac{\mu_o(A(\Lambda, \zeta))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta))} - \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right] \left[ \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) - \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) \right]$$

$$\times c(\zeta; x, y) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta))$$

$$+ \sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{\zeta \in S_\Lambda} \left[ \frac{\mu_o(A(\Lambda, \zeta))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta))} - \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right] \left[ \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) - \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) \right]$$

$$\times c(\zeta; x, y) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta))$$

$$\leq \sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{\zeta \in S_\Lambda} \left[ \frac{\mu_o(A(\Lambda, \zeta))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta))} - \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right] \left[ \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) - \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) \right]$$

$$\times c(\zeta; x, y) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta))$$

$$+ \sum_{x \in \Lambda} \sum_{y \in \Lambda} \sum_{\zeta \in S_\Lambda} \left[ \frac{\mu_o(A(\Lambda, \zeta))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta))} - \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right] \left[ \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) - \ln \left( \frac{\mu_o(A(\Lambda, \zeta^y))}{\mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y))} \right) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta^y)) \right]$$

$$\times c(\zeta; x, y) \mathcal{P}_b(\Lambda, A(\Lambda, \zeta))$$
Now let $\xi \in S^b_N$, write $\xi = (\xi_1, \xi_2)$ where $\xi_1 \in S^b_{\Lambda_1}$ and $\xi_2 \in S^b_{\Lambda_2}$ and look at the first sum of the last inequality:

$$\sum_{x \in \Lambda_1} \sum_{y \in \Lambda_1} \sum_{\zeta \in S^b_N} \left[ \frac{\mu_1(A(\zeta, \xi))}{\mathcal{P}_{\Lambda,M}(A(\zeta, \xi))} - \frac{\mu_1(A(\zeta^\infty, \xi))}{\mathcal{P}_{\Lambda,M}(A(\zeta^\infty, \xi))} \right] \ln\left( \frac{\mu_1(A(\zeta, \xi))}{\mathcal{P}_{\Lambda,M}(A(\zeta, \xi))} \right) - \ln\left( \frac{\mu_1(A(\zeta^\infty, \xi))}{\mathcal{P}_{\Lambda,M}(A(\zeta^\infty, \xi))} \right)$$

$$\times c(\zeta; x, y) \mathcal{P}_{\Lambda,M}(A(\Lambda, \zeta))$$

$$= \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_1} \sum_{\xi_1 \in S^b_{\Lambda_1}} c(\xi_1; x, y) \sum_{\xi_2 \in S^b_{\Lambda_2}} \left[ \frac{\mu_1(A(\Lambda_1, \xi_2))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1, \xi_2))} - \frac{\mu_1(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2)))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2)))} \right]$$

$$\times \ln\left( \frac{\mu_1(A(\Lambda_1, \xi_2))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1, \xi_2))} \right) - \ln\left( \frac{\mu_1(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2)))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2)))} \right) \mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2)))$$

$$\leq \sum_{x \in \Lambda_1} \sum_{y \in \Lambda_1} \sum_{\xi_1 \in S^b_{\Lambda_1}} c(\xi_1; x, y) \left[ \frac{\mu_1(A(\Lambda_1, \xi_1))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1, \xi_1))} - \frac{\mu_1(A(\Lambda_1, \xi^\infty, \xi_1))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1, \xi^\infty, \xi_1))} \right]$$

$$\times \ln\left( \frac{\mu_1(A(\Lambda_1, \xi_1))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1, \xi_1))} \right) - \ln\left( \frac{\mu_1(A(\Lambda_1, \xi^\infty, \xi_1))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1, \xi^\infty, \xi_1))} \right) \mathcal{P}_{\Lambda,M}(A(\Lambda_1)) = I_{\Lambda_1}(\mu_0).$$

The last inequality is achieved by Jensen’s inequality because $f(u, v) = (u - v)(\ln v - \ln u)$ with $u, v > 0$ is concave.

Here, set $u = \frac{\mu_1(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2)))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2)))}$ and $v = \frac{\mu_1(A(\Lambda_1 \cup \Lambda_2, (\xi_1^\infty, \xi_2)))}{\mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1^\infty, \xi_2)))}$, and note that $\mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2))) \leq \mathcal{P}_{\Lambda,M}(A(\Lambda_1, \xi_1)) \mathcal{P}_{\Lambda,M}(A(\Lambda_2, \xi_2))$ and

$$\sum_{\xi_2 \in S^b_{\Lambda_2}} \mu_1(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2))) \mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2))) = \mu_1(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2))) = \mu_0(A(\Lambda_1, \xi_1))$$

$$\sum_{\xi_2 \in S^b_{\Lambda_2}} \mu_1(A(\Lambda_1 \cup \Lambda_2, (\xi_1^\infty, \xi_2))) \mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1^\infty, \xi_2))) = \mu_1(A(\Lambda_1 \cup \Lambda_2, (\xi_1, \xi_2))) \mathcal{P}_{\Lambda,M}(A(\Lambda_1 \cup \Lambda_2, (\xi_1^\infty, \xi_2)))$$

Similar manipulation will give the second term is less than or equal to $I_{\Lambda_2}(\mu_0)$. Hence, $I_\Lambda(\mu_0) \leq I_{\Lambda_1}(\mu_0) + I_{\Lambda_2}(\mu_0)$.

\[ \square \]

**Theorem 3.2.14.** Suppose that $\mu_0$ is stationary for $\eta_t$ and that $\mu_0$ is translation invariant. In addition, $\mu_0$ satisfies Assumption 3.2.8 and $\mathbb{E}_{\mu_0}[\eta(0)] = M \in (0, b)$ and $\mathbb{E}_{\mu_0}[\eta(t)^2] < \infty$. Then for all $\Lambda \subset \mathbb{Z}$, $I_\Lambda(\mu_0) = 0$.

**Proof.** Suppose not. Then there exists an interval $\lambda \subset \mathbb{Z}$ such that $I_{\lambda_n}(\mu_0) < 0$. Moreover, there exists $K < \infty$ such that $H_{\Lambda}(\mu_\ell)|_{\sigma = 0} \leq I_\Lambda(\mu_0) + K$ for all finite $\Lambda$.

Now define, for $\Lambda \subset \mathbb{Z}$, $\Lambda + k = \{x + k : \forall x \in \Lambda\}$. Let $|\Lambda_0| = n_0$ and $\Lambda_n = \Lambda_0 \cup (\Lambda_0 + n_0) \cup (\Lambda_0 + 2n_0) \cup \cdots \cup (\Lambda_0 + (n - 1)n_0)$. Then for $\forall n$, 

0 = \frac{d}{dt}H_{\Lambda_n}(\mu_t)|_{t=0} \leq I_{\Lambda_n}(\mu_0) + K \leq nI_{\Lambda_0}(\mu_0) + K$ by lemma 3.2.13. But since $I_{\Lambda_0}(\mu_0) < 0$, for large enough $n$, $I_{\Lambda_n}(\mu_0) + K < 0$, which is contradiction.

Note that by the ergodic theorem (see [2], p.465) if $\mu_o$ is translation invariant, then $\mu_o(\lim_{\Lambda \rightarrow Z} N_{\Lambda} \upharpoonright \Lambda \Lambda) \exists$. exists $= 1$.

**Lemma 3.2.15.** If the hypotheses of theorem 3.2.14 hold, then $\sum_{x \in \Lambda} \eta(x)$ converges to something in $(0,b)$ a.s. as $\Lambda \rightarrow Z$, i.e., $\mu_o(\lim_{\Lambda \rightarrow Z} \frac{N_{\Lambda} \upharpoonright \Lambda \Lambda}{|\Lambda|} = 0) = \mu_o(\lim_{\Lambda \rightarrow Z} \frac{N_{\Lambda} \upharpoonright \Lambda \Lambda}{|\Lambda|} = b) = 0$, where $N_{\Lambda} = \sum_{x \in \Lambda} \eta(x)$.

**Proof.** First, note the following claim 3.2.16.

**claim 3.2.16.** For $0 < \delta \leq 1$, $\frac{1 - \delta + \frac{\delta^4}{4}}{(\frac{\delta^4}{4})^{\frac{1}{3}}} < 1$.

**proof of claim 3.2.16.** Note that $\frac{1 - \delta + \frac{\delta^4}{4}}{(\frac{\delta^4}{4})^{\frac{1}{3}}} > 0$ for $\forall \delta \in (0,1]$. Thus, $\frac{1 - \delta + \frac{\delta^4}{4}}{(\frac{\delta^4}{4})^{\frac{1}{3}}} < 1$ $\iff$ $\ln(1 - \delta + \frac{\delta^4}{4}) - \frac{\delta^2}{4}(-\ln 4 + 3\ln(\delta)) < 0$. Now let $f(\delta) = -\delta + \frac{\delta^4}{4}$.

Since $f'(\delta) = -1 + 3\delta^3 \leq 0$ on $(0,1]$ and $f(0) = 0$ and $f(1) = \frac{3}{4}$. $0 \geq f(\delta) = -\delta + \frac{\delta^4}{4} \geq -\frac{3}{4}$ on $(0,1]$. Thus, $\ln(1 - \delta + \frac{\delta^4}{4}) \leq (-\delta + \frac{\delta^4}{4})$ on $(0,1]$. So if $(-\delta + \frac{\delta^4}{4}) - \frac{\delta^2}{4}(-\ln 4 + 3\ln(\delta)) < 0$ on $(0,1]$, then $\ln(1 - \delta + \frac{\delta^4}{4}) - \frac{\delta^2}{4}(-\ln 4 + 3\ln(\delta)) < 0$ on $(0,1]$. Further, note that $(-\delta + \frac{\delta^4}{4}) - \frac{\delta^2}{4}(-\ln 4 + 3\ln(\delta)) = \delta((-1 + \frac{\delta^3}{4}) - \frac{\delta}{4}(-\ln 4 + 3\ln(\delta)))$. So to show $\frac{1 - \delta + \frac{\delta^4}{4}}{(\frac{\delta^4}{4})^{\frac{1}{3}}} < 1$ on $(0,1]$, show $g(\delta) = (-1 + \frac{\delta^3}{4}) - \frac{\delta}{4}(-\ln 4 + 3\ln(\delta)) < 0$ on $(0,1]$.

$\lim_{\delta \rightarrow 0^+} g(\delta) = -1$ and $g'(\delta) = \frac{3}{4}(\delta^2 - \ln \delta) + \frac{1}{2}\ln 2 - \frac{3}{4}$. Since $g'(\delta)$ has minimum of $\frac{7\ln 2}{8} - \frac{3}{8} > 0$ at $\delta = \frac{1}{\sqrt{2}}$ on $(0,1]$, $g'(\delta) > 0$ on $(0,1]$. So $g(\delta)$ is increasing on $(0,1]$ and $g(1) = -\frac{3}{4} + \frac{1}{2}\ln 2 < 0$. Hence, $g(\delta) < 0$ on $(0,1]$.

Now to show $\mu_o(\lim_{\Lambda \rightarrow Z} \frac{N_{\Lambda} \upharpoonright \Lambda \Lambda}{|\Lambda|} = 0) = \mu_o(\lim_{\Lambda \rightarrow Z} \frac{N_{\Lambda} \upharpoonright \Lambda \Lambda}{|\Lambda|} = b) = 0$, Notice that

$\mathbb{E}_{\mu_o}[e^{-s\sum_{k=1}^{n} \eta(k)}] \geq e^{-sn\frac{1}{4}\delta^2} \mu_o(\sum_{k=1}^{n} \eta(k) \leq n\frac{1}{4}\delta^2)$. Also, under Assumption 3.2.8,

$\mathbb{E}_{\mu_o}[e^{-sn\sum_{k=1}^{n} \eta(k)}] = \mathbb{E}_{\mu_o}[e^{-s\sum_{k=1}^{n} \eta(k)}] \mathbb{E}_{\mu_o}[e^{-sn\sum_{i=1}^{n} \eta(k)} | \mathcal{F}_{\{1,2,...,n-1\}}]$
Thus, for a fixed $\sum i=1 \eta(k)$ the value of $\ln(\frac{\sum i=1 \eta(k)}{n}) = \ln(\frac{1}{4})$. Take $s = -\ln(\frac{1}{4} + e^{-s})$. Then by claim 3.2.16 we have $\mu_o(\sum k=1 \eta(k)) \leq \frac{n}{4}$. This implies $\mu_o(\sum k=1 \eta(k)) \leq \frac{1}{4} \leq \frac{n}{4}$. Finally, since $\lim_{n \to \infty} \sum x \in \Lambda \eta(x) = \lim_{n \to \infty} \sum x \in \Lambda \eta(x) = \lim_{n \to \infty} \mu_o(\sum x \in \Lambda \eta(x)) = 0$. When $b < \infty$, we can similarly show that $\mu_o(\lim_{n \to \infty} \sum x \in \Lambda \eta(x)) = 0$.

**Theorem 3.2.17.** Suppose that $\mu_o$ is stationary for $\eta_l$ and that $\mu_o$ is translation invariant and ergodic for translation. In addition, $\mu_o$ satisfies Assumption 3.2.8 and $E_{\mu_o}[\eta(0)] = M \in (0, b)$ and $E_{\mu_o}[\eta(0)^2] < \infty$. Then $\mu_o = \mathcal{P}_m(M, b)$.

**Proof.** Consider $\mu_o(\cdot | N_\Lambda = n)$, where $N_\Lambda = \sum x \in \Lambda \eta(x)$.

notice that if $\sum k=1 \eta(k) = n$ is fixed, $\ln(\mathcal{P}_\Lambda^b(\eta)) = \ln(\prod_{k=1}^{\sum i=1 \eta(k)} \frac{m(M, b)^{\eta(k)}}{\Gamma(\eta(k) + 1)}) = \sum_{k=1}^{\sum i=1 \eta(k)} \ln(\frac{\Gamma(\eta(k) + 1)}{\Gamma(\eta(k) + 1)}) + n \ln(m(M, b)) - |\Lambda| \ln(Z)$, where $Z = \sum \frac{\Gamma(\eta(k) + 1)}{\Gamma(\eta(k) + 1)} (m(M, b))^i$; regardless of the value of $m(M, b)$, the last two terms are the same for all $\eta \in S^b_\Lambda$ with $N_\Lambda = n$.

Thus, for a fixed $\sum k=1 \eta(k) = n$ and $x, y \in \Lambda$, $\ln(\frac{\mathcal{P}_\Lambda^b(\eta)}{\mathcal{P}_\Lambda^b(\eta)}) = \ln(\frac{\prod_{k=1}^{\sum i=1 \eta(k)} \frac{m(M, b)^{\eta(k)}}{\Gamma(\eta(k) + 1)}}{\prod_{k=1}^{\sum i=1 \eta(k)} \frac{m(M, b)^{\eta(k)}}{\Gamma(\eta(k) + 1)}}) = \ln(\frac{\Pi^b_{\Lambda, n}(\eta)}{\Pi^b_{\Lambda, n}(\eta)}) = \ln(\Pi^b_{\Lambda, n}(\eta) - \ln(\Pi^b_{\Lambda, n}(\eta)))$.

Now write $I_\Lambda(\mu_o) = \sum x \in \Lambda \sum y \in \Lambda \sum \mu_o(\cdot | N_\Lambda = n) \ln(\frac{\mu_o(\cdot | N_\Lambda = n)}{\mathcal{P}_\Lambda^b(\eta)}) = \sum x \eta \in \Lambda \sum y \in \Lambda \sum \mu_o(\cdot | N_\Lambda = n) \ln(\frac{\mu_o(\cdot | N_\Lambda = n)}{\mathcal{P}_\Lambda^b(\eta)}) = \sum_{n=0}^{\infty} I_\Lambda(\mu_o(\cdot | N_\Lambda = n)) \mu_o(N_\Lambda = n) = 0$ by theorem 3.2.14 for all finite $\Lambda$.

So either $\mu_o(N_\Lambda = n) = 0$, or $I_\Lambda(\mu_o(\cdot | N_\Lambda = n)) = 0$ for all $\forall n$. Thus, if $\mu_o(N_\Lambda = n) > 0$, then by lemma 2.2.2, $\mu_o(\cdot | N_\Lambda = n) = \Pi^b_{\Lambda, n}$. Let $\Lambda_o \subset \Lambda$ and $\zeta \in S^b_\Lambda$ be fixed. Then
\( \mu_o(\eta(x_i) = \zeta(x_i), \forall x_i \in \Lambda_o) = \sum_{n=0}^{\lambda_b} \Pi_{\Lambda,N_\lambda}(\eta(x_i) = \zeta(x_i), \forall x_i \in \Lambda_o) \mu_o(N_\lambda = n) \)

\( = \mathbb{E}_{\mu_o}[F(\Lambda,N_\lambda)], \) where \( F(\Lambda,N_\lambda) = \Pi_{\Lambda,N_\lambda}(\eta(x_i) = \zeta(x_i), \forall x_i \in \Lambda_o). \) Recall that we have shown in section 2.4, if \( \frac{N_\lambda}{|\Lambda|} \to M \in (0, b) \) as \( N_\lambda \to \infty \) and \( \Lambda \to \mathbb{Z}, \) then \( F(\Lambda,N_\lambda) \to \mathcal{P}(M, b, \cdot), \) a.s. as \( \Lambda \to \mathbb{Z}, \) where \( \mathcal{F} \) is the Borel field of invariant sets. Since \( \frac{N_\lambda}{|\Lambda|} \) converges to \( \mathbb{E}_{\mu_o}[\eta(0)] \) with probability 1, we have \( \mu_o(\eta(x_i) = \zeta(x_i), \forall x_1 \in \Lambda_o) = \mathbb{E}_{\mu_o}[\mathcal{P}(E_{\mu_o}[\eta(0)], b, \cdot), \cdot(\eta(x_i) = \zeta(x_i), \forall x_i \in \Lambda_o)]. \) In particular, since \( \mu_o \) is ergodic for translation, \( \frac{N_\lambda}{|\Lambda|} \to \mathbb{E}_{\mu_o}[\eta(0)] = \mu_o \) a.s. Hence, \( \mu_o = \mathcal{P}(M, b, \cdot). \)

\[ \square \]

**Remark 1:** If \( \mu_o \) satisfies all assumptions in theorem 3.2.17 except for ergodicity, then the proof of theorem 3.2.17 shows that \( \mu_o(\cdot) = \mathbb{E}_{\mu_o}[\mathcal{P}(E_{\mu_o}[\eta(0)], b, \cdot)], \) i.e., if \( \mu_o \) is not ergodic, \( \mu_o \) is a linear combination of some \( \mathcal{P}(m, b) \) for \( m \in (0, \infty) \) if \( b < \infty, \) or \( m \in (0, 1) \) if \( b = \infty. \)

**Remark 2:** Let \( W_1 = \{ \mu : \Theta \mu = \mu \}, \ W_2 = \{ \mu : \mathbb{E}_\mu[\eta(0)] < \infty \}, \ W_3 = \{ \mu : \mathbb{E}_\mu[\eta(0)^2] < \infty \}, \ W_4 = \{ \mu : \text{assumption 3.2.8 holds for some } \delta \}. \) We have shown that \( W_1 \cap W_2 \cap W_3 \) is invariant under \( T(t), \) the semigroup of \( \eta. \) If we knew that \( W_1 \cap W_4 \) were invariant under \( T(t), \) then we could strengthen the theorem 3.2.17 by adding \( h(\mu_t) \) is decreasing to the conclusion.
4.1 Return to finitely many cities

In the connection with the city sizes with finitely many cities in chapter 2, we will discuss the expected values of the ratio $\frac{X_{(k)}}{X_{(1)}}$, the rank-size distribution. Social scientists and linguists are interested in the rank-size distribution because the rank-size distribution describes, among other things, the distribution of city sizes and frequencies of word usage. These phenomena are known to follow Zipf’s law, or the Pareto distribution [5],[10]. In the case of the population sizes of cities with finite number of cities, $E[\frac{X_{(k)}}{X_{(1)}}]$ would be inversely proportional to its rank, or to some power of its rank. For example, the rank 5 city would have a fifth of the population of the largest city if it is inversely proportional to its rank. The following theorem may describe such distribution.

**Theorem 4.1.1.** Let $X_1, X_2, \ldots, X_n$ be i.i.d. with continuous distribution $F$ and $X_i \geq 0$ for all $i$. Suppose that the order statistics of $X_1, X_2, \ldots, X_n$ are $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where $X_{(1)} \geq X_{(2)} \geq \ldots \geq X_{(n)}$ and $E[\frac{X_{(k)}}{X_{(1)}}] = \alpha \in (0, 1)$ for all $n$. Then $X_i$s have the density $f(x) = c_0^{\alpha/(1-\alpha)} \frac{x^{\alpha}}{1-\alpha x^{\alpha/(1-\alpha)}}$ for $0 < c_0 < x < \infty$.

**Proof.** Let $U$ be a $\mathcal{U}(0, 1)$. Let $F^{-1}(u) = \inf\{z : F(z) \geq u\}$. Then $X = F^{-1}(U)$ has the distribution $F$ and $F^{-1}$ is increasing and left continuous. Now let $U_1, U_2, \ldots, U_n$ be i.i.d. with $\mathcal{U}(0, 1)$ and $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ be the order statistics of $U_i$s.

Then $F^{-1}(U_{(1)}), F^{-1}(U_{(2)}), \ldots, F^{-1}(U_{(n)})$ have the same distribution as $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$.

Let $E[\frac{X_{(2)}}{X_{(1)}}] = \alpha$ for $0 < \alpha < 1$ for every $n \in \mathbb{Z}$. Since the distribution of $(U_{(1)}, U_{(2)})$ has the
density \( g(x, y) = \left(\frac{n}{n-2}\right) 2y^{n-2} = n(n-1)y^{n-2} \) for \( 0 < y < x < 1 \),

\[
E(X_{(2)}^{X} | X_{(1)}) = \int_0^1 \int_y^1 F^{-1}(x) F^{-1}(y) n(n-1) y^{n-2} dxdy = n(n-1) \int_0^1 y^{n-2} F^{-1}(y) \int_y^1 \frac{1}{F^{-1}(x)} dxdy = \alpha.
\]

Let \( G(y) = F^{-1}(y) \int_y^1 \frac{1}{F^{-1}(x)} dx \) and we have \( \int_0^1 y^{n-2} G(y) dy = \frac{\alpha}{n(n-1)} \).

Note that \( \int_0^1 y^{n-2} \alpha(1-y) dy = \frac{\alpha}{n(n-1)} \). Since \( \{1, y, y^2, \ldots\} \) spans \( L^2[0, 1] \) and \( \alpha(1-y) \) is continuous, and \( G(y) \) is left continuous, \( G(y) = \alpha(1-y) \). Thus, \( G(y) = F^{-1}(y) \int_y^1 \frac{1}{F^{-1}(x)} dx = \alpha(1-y) \).

Now let \( H(y) = \int_y^1 \frac{1}{F^{-1}(x)} dx \). Then \( H'(y) = -\frac{1}{F^{-1}(y)} \) and we have \( \frac{H'(y)}{H(y)} = \frac{1}{\alpha(1-y)} \),

or \( H(y) = \frac{1}{\alpha(1-y)} \) for some constant \( c > 0 \) and \( H'(y) = -\frac{1}{\alpha \cdot (1-y)} \).

So we have \( F^{-1}(y) = c (1-y)^{1/(\alpha-1)} \) for \( 0 < y < 1 \), or \( F(x) = 1 - \left(\frac{ac}{x}\right)^{\alpha/(\alpha-1)} \) for \( ac < x < \infty \).

Thus the density \( f(x) = (\alpha c)^{\alpha/(\alpha-1)} \cdot \frac{\alpha}{1 - \alpha x^{1/(\alpha-1)}} \) for \( ac < x < \infty \). Note that \( \int_0^\infty f(x) dx = 1 \) for any \( c > 0 \), i.e., \( c \) does not depend on \( \alpha \). So let \( ac = c_0 \) and write \( f(x) = c_0^{\alpha/(\alpha-1)} \cdot \frac{\alpha}{1 - \alpha x^{1/(\alpha-1)}} \) for \( c_0 < x < \infty \). Now suppose that \( X_1, X_2, \ldots, X_n \) be i.i.d. with the density \( f(x) = c_0^{\alpha/(\alpha-1)} \cdot \frac{\alpha}{1 - \alpha x^{1/(\alpha-1)}} \) for \( c_0 < x < \infty \).

Consider
\[
E(X_{(k)}^{X} | X_{(1)}) = \left(\frac{n}{n-k,k-1,1}\right) \int_0^1 \int_y^1 y^{n-k}(x-y)^{k-2} F^{-1}(y) \int_y^1 \frac{1}{F^{-1}(x)} dxdy
\]

= \[
\left(\frac{n!}{(n-k)!(k-2)!}\right) \int_0^1 \int_y^1 y^{n-k}(x-y)^{k-2} \frac{c_0(1-y)^{1-(1/\alpha)}}{c_0(1-x)^{1-(1/\alpha)}} dxdy
\]

= \[
\left(\frac{n!}{(n-k)!(k-2)!}\right) \int_0^1 \int_y^1 y^{n-k} y^{k-2} \left(\frac{1-c}{1-y}\right)^{1/(\alpha-1)} dxdy
\]

= \[
\left(\frac{n!}{(n-k)!(k-2)!}\right) \int_0^1 y^{n-k} y^{k-2} \left(\frac{1-c}{1-y}\right)^{1/(\alpha-1)} dxdy
\]

where \( \beta = (1/\alpha) - 1 \) and let \( w = \frac{x-y}{1-y} \)

\[
= \frac{\Gamma(n+1)}{\Gamma(n-k+1)} \int_0^1 y^{n-k} \frac{1}{1-y} \left(1-y^2 \right)^{-\beta} (1-y) \left(1-w^2 \right)^{\beta} dw.
\]

= \[
\frac{\Gamma(n+1)}{\Gamma(n-k+1)} \frac{\Gamma(k+1)}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k+1)} = \frac{\Gamma(\beta+1)\Gamma(k)}{\Gamma(k+\beta)} = \frac{\Gamma(1/\alpha)\Gamma(k)}{\Gamma(k+1/\alpha-1)}.
\]

Note that the last integrals were evaluated using the fact \( \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \) if \( a, b > 0 \). In particular, if \( \alpha = 1/2 \), then
\[
E(X_{(k)}^{X} | X_{(1)}) = \frac{\Gamma(1/2)\Gamma(k)}{\Gamma(k+1/2)} = \frac{\Gamma(2)\Gamma(k)}{\Gamma(k+2)} = \frac{k!}{k} = \frac{1}{k}.
\]

Note that the probability distribution found in theorem 4.1.1 is, in fact, the Pareto distribution with the probability density given by \( g(x) = \gamma x^{\gamma-2} \) for \( x > x_m \) with parameters \( x_m > 0 \) and \( \gamma > 0 \). If we set \( \gamma = \frac{\alpha-1}{\alpha} \), the density \( f(x) \) in theorem 4.1.1 and the density \( g(x) \) for the Pareto
distribution are identical. Also, we can easily see from the density function \( g(x) = \gamma \frac{x^\gamma}{x^\gamma + 1} \) that it requires \( \gamma > 1 \) for the first moment to exist and \( \gamma > 2 \) for the second moment to exist. Thus, if \( \alpha = \frac{1}{2} \), or \( \gamma = 1 \), then \( \mathbb{E}[X_i] = \infty \). Further, since the city populations described in chapter 2 requires that the first moment and the second moment to be finite, we need \( \gamma > 2 \), or \( \alpha > \frac{2}{3} \). In other words, \( \mathbb{E}[\frac{X_{i(2)}}{X_{i(1)}}] > 2/3 \).

According to the generalized Zipf’s law [10], for some \( B > 0 \), we have \( r^B f_r = c \), where \( r \) is the rank, \( f_r \) is the frequency, or the population size, of the rank \( r \), and \( c \) is a positive constant. This means that \( \frac{f_k}{f_1} = \left( \frac{k}{1} \right)^B \), or \( \left( \frac{f_k}{f_1} \right)^{1/B} = \frac{1}{k} \) for all \( k \). Since the actual observed frequencies are random, we do not expect this to hold exactly. However, we may have \( \mathbb{E}[\frac{X_{i(k)}}{X_{i(1)}}] = \left( \frac{1}{k} \right)^B \), or possibly \( \mathbb{E}[\left( \frac{X_{i(k)}}{X_{i(1)}} \right)^{1/B}] = \frac{1}{k} \). The following lemma will show that \( B = \frac{1}{\alpha} - 1 = \frac{1-\alpha}{\alpha} \) if the distribution is as in theorem 4.1.1.

**Lemma 4.1.2.** Let \( X_1, X_2, \ldots, X_n \) be i.i.d. with the density \( f(x) = c_o^{\alpha/(1-\alpha)} \frac{x^{\alpha}}{x^{\alpha} + 1} \left( \frac{1}{1-x} \right)^{1-\alpha} \) for \( 0 < c_o < x < \infty \). Suppose that the order statistics of \( X_1, X_2, \ldots, X_n \) are \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \), where \( X_{(1)} \geq X_{(2)} \geq \ldots \geq X_{(n)} \). Then \( \mathbb{E}[\left( \frac{X_{(k)}}{X_{(1)}} \right)^{\alpha/(1-\alpha)}] = \frac{1}{k} \).

**Proof.** Similarly to the proof of theorem 4.1.1,

\[
\mathbb{E}[\left( \frac{X_{(k)}}{X_{(1)}} \right)^{\alpha/(1-\alpha)}] = \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k-1)} \int_0^1 y^{n-k}(x-y)^{k-2}(\frac{F^{-1}(y)}{F^{-1}(x)})^{\alpha/(1-\alpha)} \, dx dy
\]

\[
= \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k-1)} \int_0^1 y^{n-k} \int_0^1 (x-y)^{k-2}(\frac{1}{1-y})^{(1/\alpha-1)} \, dx \, dy
\]

\[
= \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k-1)} \int_0^1 y^{n-k} (1-y)^{k-1} \, dy \int_0^1 w^{k-2}(1-w)^1 \, dw
\]

\[
= \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k-1)} \Gamma(n+1) \Gamma(k-1) \Gamma(2) = \frac{\Gamma(k)}{\Gamma(k+1)} = \frac{1}{k}.
\]

\( \square \)

### 4.2 Generalization

Recall that \( c(\eta; x, y) = \eta(x)p(x, y)\frac{\eta(y)+\lambda}{\eta(x)+\eta(y)+2\lambda} \). We identified \( p(x, y) \) as the probability of each individual in city \( x \) considering to move to city \( y \) and \( \frac{\eta(y)+\lambda}{\eta(x)+\eta(y)+2\lambda} \) as the probability of each individual actually moving to city \( y \). In particular, we took \( \frac{\eta(x)}{\eta(x)} = 1 \) to be the rate that an individual in city \( x \) decides upon moving. Here, the rate of each individual in city \( x \) thinking about moving
does not depend on the size of the city that an individual is moving from. Suppose that the rate of each individual moving from city \( x \) does depend on the size of the city an individual is moving from so that \( c(\eta; x, y) = \eta(x)p(x, y)\frac{\eta(y)+\lambda}{\eta(x)+\eta(y)+2\lambda}g(\eta(x)) \), where \( g(\eta(x)) \) is a function of \( \eta(x) \) that determines the rate of an individual thinking about moving. We claim that the density for a stationary distribution we found in section 2.3 changes to

\[
P_{m,b}(\eta(x) = k) = \begin{cases} \frac{\Gamma(k+\lambda)}{\Gamma(k+1)\prod_{j=1}^b g(j)} m^k \prod_{j=1}^b b_j^{\eta(j)} & \text{for } k = 0, 1, \ldots, b, \\ 0 & \text{otherwise.} \end{cases}
\]

If \( b = \infty \), we assume \( m < \lim \inf(\prod_{j=1}^i g(j))^{1/i} \) so that the denominator of \( P_{m,b} \) above is forced to converge. Note that the denominator of \( P_{m,b} \) may also converge in some cases when \( m = \lim \inf(\prod_{j=1}^i g(j))^{1/i} \).

If \( \mathcal{A}_{\Lambda,M} = \prod_{x \in \Lambda} P_{m,b} \), then lemma 2.1.1 should hold with this new \( P_{m,b} \). Indeed, it holds:

For \( \eta(x) > 0 \) and \( \eta(y) < b \), Consider \( \mathcal{A}_{\Lambda,M}(\eta) \)

\[
\mathcal{A}_{\Lambda,M}(\eta) = \frac{\prod_{x \in \Lambda} x^{\eta(x)} g(x)^{\eta(x)}}{\prod_{x \in \Lambda} x^{\eta(x)+1} g(x)} = \frac{\prod_{x \in \Lambda} x^{\eta(x)+1} g(x)^{\eta(x)+1}}{\prod_{x \in \Lambda} x^{\eta(x)+1} g(x)^{\eta(x)+1}}
\]

Thus, the ratio \( \mathcal{A}_{\Lambda,M}(\eta) \) does not depend on the size of the city an individual is moving from. We will need, in addition to condition at the beginning of chapter 3, \( \max_k k g(k) - (k+1)g(k+1) < \infty \).

These conditions guarantee the existence of the process when \( \Lambda = \mathbb{Z} \), see Liggett [6].

Now suppose that we set \( g(j) = \frac{(j+2)(j-1+\lambda)}{(j+\lambda)} \) and \( m = \lim_{j \to \infty} g(j)^{1/j} = 1 \), and let \( b = \infty \).

Then \( P_{1,b}(k) = \frac{\Gamma(k+\lambda)}{\Gamma(k+1)\prod_{j=1}^b g(j)} m^k \prod_{j=1}^b b_j^{\eta(j)} \) for \( k = 0, 1, \ldots, b \). Note that this distribution does
not have the first moment, so our result do not apply when \( \Lambda = \mathbb{Z} \). However, in the case where 
\(|\Lambda| < \infty \) we get the analogous results from section 2.3. Noting that \( \lim_{j \to \infty} g(j) = 1 \), we see that 
this model would behave very much like those in section 2.3 if the size of city is large.
Bibliography


