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A Classical Technique to Prove the h-Cobordism Theorem

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A Classical Technique to Prove the h-Cobordism Theorem

by

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B.A. Mathematics, San José State University, 2007

A thesis submitted to the
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Department of Mathematics

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This thesis entitled:
A Classical Technique to Prove the h-Cobordism Theorem
written by Masaya Sato
has been approved for the Department of Mathematics

Carla Farsi

Prof. Martin Walter

Prof. Markus Pflaum

Date _____

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Sato, Masaya (M.A. Mathematics)

A Classical Technique to Prove the h-Cobordism Theorem

Thesis directed by Prof. Carla Farsi

Let W be an m dimensional compact smooth manifold with boundary $\partial W = \partial_- W \sqcup \partial_+ W$, where submanifolds $\partial_- W$ and $\partial_+ W$ are closed and disjoint. Then suppose that W , $\partial_- W$, and $\partial_+ W$ are all simply connected, $\dim W \geq 6$, and $H_*(W, \partial_- W) = 0$. The *h-Cobordism Theorem* states that W is diffeomorphic to a product cobordism.

In this paper we will follow a classical technique developed by John Milnor in his “Lectures on the h-Cobordism Theorem” half a century ago.

Dedication

To all people who believe Japan will recover from the heart-rending tragedy in no time.

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Chapter 1

Introduction

1.1 Differential Manifolds

We will start with preliminary definitions which are used throughout this paper.

Definition 1 (Manifolds). *A topological manifold M of dimension $m \geq 0$ is a second countable Hausdorff topological space so that the following properties are satisfied:*

- (1) *A family of open sets $\{U_\alpha\}_{\alpha \in \Lambda}$ covers M , where Λ is an index set,*
- (2) *For each $\alpha \in \Lambda$, (U_α, ϕ_α) is called a coordinate chart, where $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^m$ is a homeomorphism, and*
- (3) *For all α and β in Λ , there is a homeomorphism called a transition map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\beta \cap U_\alpha) \rightarrow \phi_\alpha(U_\beta \cap U_\alpha)$ whenever $U_\alpha \cap U_\beta \neq \emptyset$.*

Note that if each transition map is in $C^p(\mathbb{R}^m, \mathbb{R}^m)$ for a nonnegative integer $p \geq 0$ or in $C^\infty(\mathbb{R}^m, \mathbb{R}^m)$, then M is called a C^p -manifold or *smooth* manifold, respectively. Throughout this paper we assume that manifolds are equipped with a smooth structure. See Milnor[6] for the detail.

Note also that an *atlas* \mathcal{A} , the set of coordinate charts $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ is called *maximal* if one adds an extra chart to \mathcal{A} , then the property (3) fails. Now let us observe that we can always determine a maximal atlas \mathcal{M} . For if we define a partial ordering on \mathcal{A} by the set inclusion, then the *Zorn's Lemma* guarantees that a maximal atlas \mathcal{M} is determined. Therefore, without loss of generality, we always suppose that a manifold M is equipped with a maximal atlas \mathcal{M} .

For two smooth manifolds M and N of dimension m and n , respectively, we say that a map $f : M \rightarrow N$ is smooth if $\psi_\alpha \circ f \circ \phi_\beta^{-1} : \phi_\beta(U_\beta) \rightarrow \psi_\alpha(U_\alpha)$ is a smooth map, where $\{(U_\alpha, \psi_\alpha)\}$ and $\{(U_\beta, \phi_\beta)\}$ are charts for N and M , respectively.

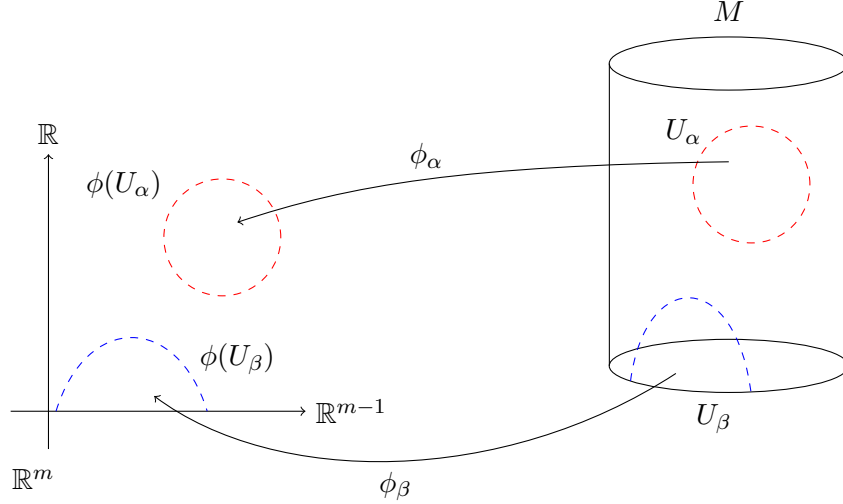


Figure 1.1: A smooth manifold M with boundary ∂M .

For an upper half m -dimensional Euclidean space $\mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m | x_m \geq 0\}$, one defines a (smooth) manifold with *boundary* by allowing charts homeomorphic to \mathbb{H}^m . And the set of points $x \in M$, where each m -th coordinate of $\phi_\beta(x)$ is 0, is called the *boundary* of M and denoted by ∂M . And the boundary ∂M is of dimension $m - 1$. On the other hand, a manifold without a boundary is called *closed*.

Definition 2 (Tangent Spaces and Tangent Bundles). *The tangent space $T_p M$ to M at p is the set of all germs at p $X_p : C^\infty(M, \mathbb{R}) \rightarrow \mathbb{R}$ such that for all $\alpha, \beta \in \mathbb{R}$ and all $f, g \in C^\infty(M, \mathbb{R})$*

$$(1) X_p(\alpha f + \beta g) = \alpha X_p f + \beta X_p g \text{ (linearity),}$$

$$(2) X_p(fg) = (X_p f)g(p) + f(p)(X_p g) \text{ (Leibnitz rule),}$$

with the operations in T_pM defined by

$$(X_p + Y_p)f = X_pf + Y_pf \quad (1.1)$$

$$(\alpha X_p)f = \alpha(X_pf). \quad (1.2)$$

Then a tangent bundle is defined to be a collection of all tangent spaces, i.e.

$$TM = \{X_p \in T_pM | p \in M\} = \bigcup_{p \in M} T_pM. \quad (1.3)$$

Moreover a smooth map $X : M \rightarrow TM$ such that $\pi(X) = \text{id}$, where π and id denote the projection and identity map respectively, is called a vector field on M . And the push-forward of a vector field X on M by some diffeomorphism $f : M \rightarrow N$ is the vector field $f_*X_{f(p)}$ on N defined by $T_{f(p)}f(X_p)$.

For a smooth manifold M of dimension m , recall that M is *orientable* if the determinants of the Jacobian of all transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are positive whenever $U_\alpha \cap U_\beta \neq \emptyset$ for any charts (U_α, ϕ_α) and (U_β, ϕ_β) . Or equivalently, the orientation $\langle M \rangle$ of M is given by an m -frame of vector fields $\langle \zeta_1, \dots, \zeta_m \rangle$.

1.2 Cobordisms

Definition 3 (Cobordisms). For a smooth compact m -dimensional manifold W with a boundary $\partial W = \partial_-W \sqcup \partial_+W$, where both ∂_-W and ∂_+W are closed submanifolds, W is called a cobordism from ∂_-W to ∂_+W and denoted by a triple $(W; \partial_-W, \partial_+W)$.

A compact manifold M with boundary ∂M has a neighborhood V of ∂M with a diffeomorphism $g : \partial M \times [0, 1) \rightarrow V$. Such a neighborhood V is called a *collar neighborhood* of the boundary ∂M .

For two cobordisms $(W; \partial_-W, \partial_+W)$ and $(W'; \partial_-W', \partial_+W')$, if there is a diffeomorphism $h : \partial_+W \rightarrow \partial_-W'$ defined, we can glue or attach those two cobordisms via the map h . In other words, we have a new cobordism $W \cup_h W'$ from ∂_-W to ∂_+W' , where

$$W \cup_h W' = W \sqcup W' / \sim_h, \quad (1.4)$$

with an equivalence relation \sim_h generated by $x \sim_h h(x)$ for every $x \in M$. Two boundaries $\partial_- W$ and $\partial_+ W$ are glued together via the diffeomorphism h along their collar neighborhoods.

Note that there exists a unique smooth structure \mathcal{B} , which is compatible with the given structures \mathcal{A} and \mathcal{A}' on W and W' , respectively. See Milnor [6] for the detail.

Chapter 2

Morse Functions

2.1 Morse Functions

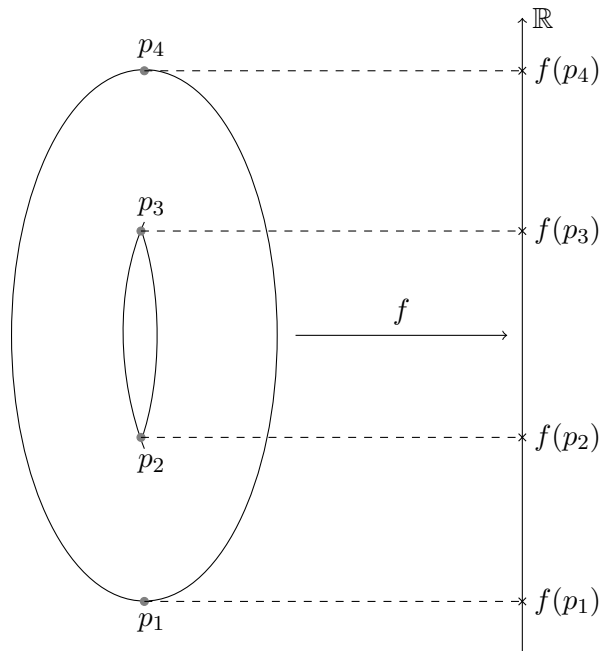


Figure 2.1: A torus of genus 1.

Let us begin with an example as Figure 2.1 shows. For a torus T^2 we can think of the height function f as a Morse function.

Definition 4 (Critical Points of A Smooth Function). *Let M be an m -dimensional manifold without boundary and $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ an atlas for M . Then for a smooth scalar function $f : M \rightarrow \mathbb{R}$, $p \in M$*

is called a critical point of f if

$$\frac{\partial(f \circ \phi_\alpha^{-1})}{\partial x_i}(\phi_\alpha(p)) = 0 \quad \text{for } i = 1, \dots, m, \quad (2.1)$$

where (U_α, ϕ_α) is a coordinate chart that contains p .

Let us observe that a critical point $p \in M$ does not depend on the choice of a coordinate system. For two distinct charts (U_α, ϕ_α) and (U_β, ϕ_β) , assume that $p \in U_\alpha \cap U_\beta$. Then $\phi_\alpha(p) = (x_1, \dots, x_m)$ and $\phi_\beta(p) = (y_1, \dots, y_m)$ and thus

$$\frac{\partial(f \circ \phi_\beta^{-1})}{\partial y_i}(\phi_\beta(p)) = \sum_{j=1}^m \frac{\partial x_j}{\partial y_i} \cdot \frac{\partial(f \circ \phi_\alpha^{-1})}{\partial x_j}(\phi_\alpha(p)) = 0. \quad (2.2)$$

So for simplicity, from now on, we write $\frac{\partial f}{\partial x_i}$ to denote each partial derivative $\frac{\partial(f \circ \phi_\alpha^{-1})}{\partial x_i}$.

Then we define a symmetric bilinear form of a smooth function f as follows.

Definition 5 (The Hessian Matrix). *Let $f : M \rightarrow \mathbb{R}$ be a smooth function defined on a smooth manifold M equipped with an atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in \Lambda}$ and $p \in M$ a critical point of f . Then the Hessian $H_f(p)$ of f at p is an $m \times m$ matrix whose entries are the second order partial derivatives, i.e.*

$$H_f(p) = \left[\frac{\partial^2(f \circ \phi_\alpha^{-1})}{\partial x_i \partial x_j}(\phi(p)) \right]. \quad (2.3)$$

Moreover, a critical point p is called non-degenerate if $\det(H_f(p)) \neq 0$. Otherwise, it is called degenerate.

Lemma 6. *A non-degenerate critical point $p \in M$ does not depend on the choice of a coordinate system at p .*

Proof. Let (x_1, \dots, x_m) and (y_1, \dots, y_m) be two local coordinate systems at p . Then

$$\frac{\partial^2 f}{\partial y_{i'} \partial y_{j'}} = \sum_{i,j=1}^m \frac{\partial x_j}{\partial y_{i'}} \cdot \frac{\partial x_i}{\partial y_{j'}} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \quad (2.4)$$

$$= \sum_{i,j=1}^m \frac{\partial x_j}{\partial y_{i'}} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot \frac{\partial x_i}{\partial y_{j'}}. \quad (2.5)$$

This implies that

$$H'_f(p) = J(p)^T H_f(p) J(p), \quad (2.6)$$

where $J(p) = \left[\frac{\partial x_i}{\partial y_j} \right]$ is the matrix of coordinate change, and $H_f(p)$ and $H'_f(p)$ are the Hessian matrices with respect to coordinate systems (x_1, \dots, x_m) and (y_1, \dots, y_m) , respectively. Since $J(p)$ is invertible, $\det(J(p)) \neq 0$ and

$$\det(H'_f(p)) = \det(J(p)^T H_f(p) J(p)) = \det(J(p)^T) \cdot \det(H_f(p)) \cdot \det(J(p)) \quad (2.7)$$

$$= \det(J(p))^2 \cdot \det(H_f(p)). \quad (2.8)$$

So $\det(H'_f(p)) \neq 0$ if and only if $\det(H_f(p)) \neq 0$. Therefore any non-degenerate critical point p is independent of the choice of coordinate systems. \square

So from now on we write $H_f(p) = \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$ for simplicity.

Definition 7 (Morse Functions). *A function $f : M \rightarrow \mathbb{R}$ is a Morse function if every critical point of f is non-degenerate.*

Next we observe that a Morse function is expressed as a quadratic form around small neighborhood of a non-degenerate critical point.

Lemma 8 (Morse's Lemma). *If $f : M \rightarrow \mathbb{R}$ is a Morse function with a non-degenerate critical point $p \in M$, then there exists a coordinate system $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ of p such that the function f can be expressed as the form*

$$f(x) = -x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_m^2 + c \quad (2.9)$$

around a neighborhood U of p , where $c \in \mathbb{R}$ is a constant. Furthermore the uniquely determined non-negative integer λ is called the index of p .

Proof. Since $p \in M$, $\phi(p) = x = (x_1, \dots, x_m)$ for some coordinate chart $\phi : U \rightarrow \mathbb{R}^m$. And without loss of generality, let us suppose that p corresponds to $(0, \dots, 0)$ via ϕ and $f(p) = (0, \dots, 0)$. Then

by the *Fundamental Theorem of Calculus* (FTC),

$$f(x) = f(x) - f(p) = \int_0^1 \frac{df}{dt}(tx) dt \quad (2.10)$$

$$= \int_0^1 \frac{df}{dt}(tx_1, \dots, tx_m) dt \quad (2.11)$$

$$= \int_0^1 \sum_{i=1}^m x_i \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_m) dt \quad (2.12)$$

$$= \sum_{i=1}^m x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_m) dt. \quad (2.13)$$

Now let $g_i : M \rightarrow \mathbb{R}$ be smooth functions defined by $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx) dt$. Then

$$f(x) = f(x_1, \dots, x_m) = \sum_{i=1}^m x_i g_i(x_1, \dots, x_m), \quad (2.14)$$

where $g_i(0, \dots, 0) = \frac{\partial f}{\partial x_i}(0, \dots, 0) = 0$. Moreover, by applying the FTC again, there are smooth functions $h_{ij} : M \rightarrow \mathbb{R}$ so that

$$g_i(x) = \sum_{j=1}^m x_j h_{ij}(x) \quad (2.15)$$

with $h_{ij}(0, \dots, 0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0)$. Then

$$f(x) = f(x_1, \dots, x_m) = \sum_{i,j=1}^m x_i x_j h_{ij}(x_1, \dots, x_m). \quad (2.16)$$

Now let $H_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$, and we have

$$f(x) = f(x_1, \dots, x_m) = \sum_{i,j=1}^m x_i x_j H_{ij}(x_1, \dots, x_m) \quad (2.17)$$

and $H_{ij}(x_1, \dots, x_m) = H_{ji}(x_1, \dots, x_m)$. Moreover,

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(0, \dots, 0) = 2H_{ij}(0, \dots, 0). \quad (2.18)$$

Now observe that the Hessian $H_f(p)$ is a bilinear, symmetric, and non-degenerate form. So $H_f(p)$ is diagonalizable by the *Gram-Schmidt orthonormalization process*, and moreover the diagonalized $H_f(p)$ contains a negative definite maximal submatrix. The size of such a submatrix corresponds to the index λ of the critical point. \square

We can immediately show the following corollary since each partial derivative $\frac{\partial f}{\partial x_i}$ of the standard form of a Morse function f is either $-2x_i$ or $2x_i$ on some open neighborhood U of a non-degenerate critical point p .

Corollary 9. *A non-degenerate critical point $p \in M$ of a Morse function $f : M \rightarrow \mathbb{R}$ is isolated.*

Observe also that every compact manifold M can be covered by finitely many charts $\{U_i, \phi_i\}_{i=1}^k$ and the following holds true.

Corollary 10. *If M is a compact manifold, then a Morse function $f : M \rightarrow \mathbb{R}$ has only finitely many non-degenerate critical points.*

For an open set $U \subset \mathbb{R}^m$, consider a smooth function $f : U \rightarrow \mathbb{R}$. Recall that the image $f(C) \subset \mathbb{R}$ has Lebesgue measure zero, where C is the set of critical points of f , by the *Sard's Lemma*. We will use the Sard's Lemma to prove the following lemma that constructs a Morse function from an arbitrary smooth function f by perturbing f using an appropriate linear function on \mathbb{R}^m .

Lemma 11. *Let $U \subset \mathbb{R}^m$ be an open set and $f : U \rightarrow \mathbb{R}$ a smooth function. Then there exist some $a_1, \dots, a_m \in \mathbb{R}$ so that*

$$f(x_1, \dots, x_m) - (a_1x_1 + \dots + a_mx_m) \tag{2.19}$$

is a Morse function.

Proof. Define a map $h : U \rightarrow \mathbb{R}$ by

$$h(x_1, \dots, x_m) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m} \right)^T. \tag{2.20}$$

Then observe that the Jacobian $J_h(p)$ of h at $p \in \mathbb{R}^m$ is the Hessian $H_f(p)$ of f at p , *i.e.*

$$J_h(p) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \right] = H_f(p). \tag{2.21}$$

So $\det(J_h(p)) = \det(H_f(p))$ and p is a critical point of h if and only if $\det(H_f(p)) = 0$. Now by the *Sard's Lemma* let us choose $a = (a_1, \dots, a_m)^T \in \mathbb{R}^m$ such that a is not a critical value of h .

Then define $\tilde{f} : U \rightarrow \mathbb{R}^m$ by

$$\tilde{f}(x_1, \dots, x_m) = f(x_1, \dots, x_m) - (a_1x_1 + \dots + a_mx_m). \quad (2.22)$$

We claim that \tilde{f} is a Morse function. So if p is a critical point of \tilde{f} , then

$$0 = \frac{\partial \tilde{f}}{\partial x_i}(p) = \frac{\partial f}{\partial x_i}(p) - a_i \quad (2.23)$$

for $i = 1, \dots, m$. This implies that

$$h(p) = (a_1, \dots, a_m)^T \quad (2.24)$$

and p is not the critical point of h . Therefore $\det(J_h(p)) \neq 0$ and thus $\det(H_{\tilde{f}}(p)) = \det(H_f(p)) \neq 0$. Hence p is a non-degenerate critical point of \tilde{f} and \tilde{f} is a Morse function. \square

Definition 12. Let $(W; \partial_-W, \partial_+W)$ be a cobordism. A Morse function $f : W \rightarrow [0, 1]$ on the cobordism is a smooth function such that

- (1) $\partial_-W = f^{-1}(0)$ and $\partial_+W = f^{-1}(1)$, and
- (2) Every non-degenerate critical point p of f exists in $\text{Int } W = W - \partial W$.

The following lemma lets us construct a smooth function without any critical points near the boundary of W . Note that such a smooth function is constructed with the help of a partition of unity. A *partition of unity* is a family of non-negative smooth functions $\{\psi_i : W \rightarrow \mathbb{R}_{\geq 0}\}$ such that the closure \bar{V}_i of $V_i = \{x \in W \mid \psi_i(x) > 0\}$ form a locally finite cover of W , and such that $\sum_i \psi_i(x) = 1$ for every $x \in W$.

Lemma 13. There exists a smooth function $f : W \rightarrow [0, 1]$ on a cobordism $(W; \partial_-W, \partial_+W)$ such that f has no critical points in any neighborhood of $\partial W = \partial_-W \sqcup \partial_+W$.

Proof. Let $\{U_i, \phi_i\}_{i=1}^k$ be an atlas for W so that no U_i intersects with both ∂_-W and ∂_+W . Then

define $f_i : U_i \rightarrow [0, 1]$ by

$$f_i(p) = \begin{cases} \pi_m \circ \phi_i(p) = x_m & \text{if } U_i \cap \partial_- W \neq \emptyset \\ 1 - \pi_m \circ \phi_i(p) = 1 - x_m & \text{if } U_i \cap \partial_+ W \neq \emptyset \\ \frac{1}{2} & \text{if } U_i \cap \partial W = \emptyset, \end{cases} \quad (2.25)$$

where $\pi_m : \mathbb{R}^m \rightarrow \mathbb{R}$ is the projection onto the m -coordinate. Now choose a partition of unity $\{\psi_i : M \rightarrow \mathbb{R}\}_{i=1}^k$ dominated by the open cover $\{U_i\}_{i=1}^k$, i.e. the $\bar{V}_i \subset U_i$, where $V_i = \psi_i^{-1}((0, 1])$. Then define a map $f : W \rightarrow [0, 1]$ by

$$f(p) = \psi_1(p)\tilde{f}_1(p) + \cdots + \psi_k(p)\tilde{f}_k(p), \quad (2.26)$$

where each $\tilde{f}_i : W \rightarrow [0, 1]$ is the smooth extension of $f_i : U_i \rightarrow [0, 1]$ such that $\tilde{f}_i|_{U_i} = f_i$ and $\tilde{f}_i|_{W - U_i} = 0$. Observe that f is well-defined and smooth on W . Moreover $f^{-1}(0) = \partial_- W$ and $f^{-1}(1) = \partial_+ W$. Finally we claim that the derivative of f is non-zero on $\partial_- W \sqcup \partial_+ W$. So let $q \in \partial_- W$. Then $q \in U_j$ and $\phi_j(q) > 0$ for some j . Also let $\phi_j(p) = (x_1, \cdots, x_m)$ and consider the coordinate system. Then

$$\frac{\partial f}{\partial x_m} = \frac{\partial}{\partial x_m} \sum_{i=1}^k \phi_i \tilde{f}_i = \sum_{i=1}^k \frac{\partial}{\partial x_m} \phi_i \tilde{f}_i \quad (2.27)$$

$$= \sum_{i=1}^k (\tilde{f}_i \frac{\partial \phi_i}{\partial x_m} + \phi_i \frac{\partial \tilde{f}_i}{\partial x_m}). \quad (2.28)$$

Since $f_j(q) = 0$ and $f_j(q) = 0$ for $j \neq i$, the first summand is zero. So $\frac{\partial f}{\partial x_m}(q) = \sum_{i=1}^k \phi_i(q) \frac{\partial \tilde{f}_i}{\partial x_m}(q)$. Because of construction of \tilde{f}_j , $\frac{\partial \tilde{f}_j}{\partial x_m}(q) = 1$ or 0 for $q \in \partial_- W$. Therefore $\frac{\partial f}{\partial x_m}(q) \neq 0$. Similarly, if $q \in \partial_+ W$, then $\frac{\partial f}{\partial x_m}(q) \neq 0$. Hence the derivative of f is non-zero on $\partial_- W \sqcup \partial_+ W$. \square

Note that a polynomial of first and second order partial derivatives can be approximated by another polynomial, and the following lemmas hold true.

Lemma 14. *For an open set $U \subset \mathbb{R}^m$, a compact set $K \subset U$, and a smooth function $f : U \rightarrow \mathbb{R}$, if f has only non-degenerate critical points in K , then there is some constant $\delta > 0$ such that if*

$g : U \rightarrow \mathbb{R}$ is smooth on K with

$$\left| \frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i} \right| < \delta \quad \text{and} \quad \left| \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 g}{\partial x_i \partial x_j} \right| < \delta \quad (2.29)$$

for all $i, j = 1, \dots, m$ then g has only non-degenerate critical points.

Lemma 15. Let U and U' be open sets in \mathbb{R}^m and $h : U \rightarrow U'$ be a diffeomorphism such that $K' = h(K) \subset U'$ for some compact set $K \subset U$. Then for every $\epsilon > 0$ there is some $\delta > 0$ such that if $f : U' \rightarrow \mathbb{R}$ is smooth with

$$|f| < \delta, \quad \left| \frac{\partial f}{\partial x_i} \right| < \delta, \quad \text{and} \quad \left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right| < \delta \quad (2.30)$$

for all $i, j = 1, \dots, m$ on $K' \subset U'$, then

$$|f \circ h| < \epsilon, \quad \left| \frac{\partial}{\partial x_i} f \circ h \right| < \epsilon, \quad \text{and} \quad \left| \frac{\partial^2}{\partial x_i \partial x_j} f \circ h \right| < \epsilon \quad (2.31)$$

on $K \subset U$.

Now let us define C^∞ -topology on the set of smooth functions $C^\infty(M, \mathbb{R})$. So for a compact and closed manifold M let us consider a finitely many charts $\{(U_i, \phi_i)\}_{i=1}^k$ and a compact refinement $\{C_i\}_{i=1}^k$ of the open cover $\{U_i\}_{i=1}^k$, i.e. each compact C_i is contained in U_i and the refinement still covers W . Then for every $\delta > 0$ define a neighborhood $N(\delta)$ of $C^\infty(M, \mathbb{R})$ by

$$N(\delta) = \{g \in C^\infty(M, \mathbb{R}) \mid |g_l^{(n)}| < \delta\} \quad (2.32)$$

on $\phi_l(C_l)$, where $g_l = g \circ \phi_l^{-1}$, and $g_l^{(n)}$ denotes the partial derivative of order n for all $n \geq 0$. In other words, $N(\delta)$ is an open neighborhood of the zero function.

Definition 16. The topology on $C^\infty(M, \mathbb{R})$ generated by $N(f, \delta) = f + N(\delta)$, where $f \in C^\infty(M, \mathbb{R})$ is called the C^∞ -topology.

Theorem 17. On any cobordism $(W; \partial_- W, \partial_+ W)$, there exists a Morse function $f : W \rightarrow \mathbb{R}$

Proof. Let U be an open neighborhood of $\partial W = \partial_- W \sqcup \partial_+ W$. Since W is compact and Hausdorff, there is an open set V such that $\partial_- W \subset \bar{V} \subset U$. Then let us take a finite open cover $\{U_i\}_{i=1}^k$

for W such that either $U_i \subset U$ or $U \subset W - \partial W$, and consider its compact refinement $\{C_i\}_{i=1}^k$ of $\{U_i\}_{i=1}^k$. Now let

$$C = \bigcup_{i' \in \{1, \dots, k\}} C_{i'} \quad \text{such that} \quad C_{i'} \subset U_{i'} \subset U. \quad (2.33)$$

By Lemma 14 and Lemma 15, for a small neighborhood $N \subset C^\infty(W, \mathbb{R})$ of f , no function in N has critical points in C . Observe also that $0 < f < 1$ on $W - V$. Let $N' \subset C^\infty(M, \mathbb{R})$ be a neighborhood of f such that every $g \in N'$ is $0 < g < 1$ on $W - V$. Then let $N_0 = N \cap N'$. By Lemma 14 and 15, there exists $f_1 \in N_0$ such that f_1 has only non-degenerate critical points on C_1 , and a neighborhood $N_1 \subset N_0$ of f_1 such that every $g_1 \in N_1$ has only non-degenerate critical points on C_1 . By repeating this recursive procedure, at the k -stage, there is $f_k \in N_k \subset N_{k-1} \subset \dots \subset N_0$ so that f_k has only non-degenerate critical points in

$$C \cup C_1 \cup \dots \cup C_k = M. \quad (2.34)$$

Since $f_k \in N_0 \subset N'$, $f_k|_V = f|_V$, $f_k^{-1}(0) = \partial_- W$, $f_k^{-1}(1) = \partial_+ W$, and f_k has no critical points in any neighborhood of ∂W . Therefore f_k is a Morse function on $(W; \partial_- W, \partial_+ W)$. \square

We can slightly perturb a Morse function f to get another Morse function with distinct critical values as the following lemma states.

Corollary 18 (Non-resonant Morse Functions). *Let $(W; \partial_- W, \partial_+ W)$ be a cobordism and $f : W \rightarrow [0, 1]$ a Morse function with finitely many critical points p_1, \dots, p_k . Then there exists another Morse function $g : W \rightarrow [0, 1]$ that approximates f and*

$$g(p_i) \neq g(p_j) \quad (2.35)$$

for $i \neq j$. And such a Morse function g is called non-resonant.

Corollary 19. *For a Morse function $f : W \rightarrow [0, 1]$ on a cobordism $(W; \partial_- W, \partial_+ W)$, suppose that $c \in (0, 1)$ is not a critical value of f . Then $f^{-1}([0, c])$ and $f^{-1}([c, 1])$ are both smooth compact manifolds with boundary.*

Definition 20. *The Morse number μ of a cobordism $(W; \partial_-W, \partial_+W)$ is the minimum number of critical points of all Morse functions $f : W \rightarrow [0, 1]$.*

Moreover we have the following corollary since Corollary 18 lets us choose a Morse function with distinct critical values.

Corollary 21. *Any cobordism $(W; \partial_-W, \partial_+W)$ can be decomposed into a composition of cobordisms with Morse number $\mu = 1$.*

2.2 Gradient-like Vector Fields

In this section we define a vector field $X : W \rightarrow TW$ with special properties as follows.

Definition 22 (Gradient-like Vector Fields). *For a cobordism $(W; \partial_-W, \partial_+W)$, let $f : W \rightarrow \mathbb{R}$ be a Morse function with k critical points p_1, \dots, p_k . Then a vector field $X : W \rightarrow TW$ is called a gradient-like vector field associated to f if*

$$Xf > 0 \text{ on } W - \{p_1, \dots, p_k\}$$

and for each critical point $p \in W$ of index λ there exists an open neighborhood U of p such that the coordinate of the vector field X around U is given by

$$(-x_1, \dots, -x_\lambda, x_{\lambda+1}, \dots, x_n),$$

and such that X at $p \in W$ has the form

$$X_p = -2x_1 \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} \cdots + 2x_m \frac{\partial}{\partial x_m}.$$

Recall that an *integral curve* $\varphi_p : I \rightarrow W$ of X on a compact manifold W is a C^∞ -diffeomorphism such that for every $t \in I$ and $p \in M$ $\varphi_p(0) = p$ and

$$\frac{d}{dt}(f \circ \varphi_p) = Xf, \tag{2.36}$$

where $I = [0, 1]$ denotes the unit interval. Then the lemma discussed below lets us construct a gradient-like vector field X from an arbitrary Morse function f .

Lemma 23. For every Morse function $f : W \rightarrow \mathbb{R}$ there exists a gradient-like vector field $X : W \rightarrow TW$ for f .

Proof. For simplicity we will show the case f has only one critical point $p \in W$ of index λ . Using the standard form of a Morse function f , there exists an open neighborhood U_0 of p and the coordinate system $(x_1, \dots, x_\lambda, x_{\lambda+1}, \dots, x_m)$ such that

$$f(x) = f(p) - x_1^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_m^2 \quad (2.37)$$

on U_0 . Now let $U \ni p$ be an open set so that $p \in \bar{U} \subset U_0$. Since each partial derivative is nonzero on $W - U_0$, the *Implicit Function Theorem* guarantees that for every non-critical point $p' \in W - U_0$ there exists a coordinate system (x'_1, \dots, x'_m) in an open neighborhood $U' \subset W - U_0$ such that

$$f(x) = x'_1 + k,$$

where k_i is constant. Since $W - U_0$ is compact, it is covered by a finite open cover $\{U_1, \dots, U_k\}$ such that

(1) $U_0 \cap U_i = \emptyset$ for $i = 1, \dots, k$, and

(2) U_i has coordinates $x^i = (x_1^i, \dots, x_m^i)$ and $f(x^i) = x_1^i + k_i$, where each k_i is a constant, on U_i .

On the neighborhood U_0 , there exists the gradient-like vector field $X^0 : U_0 \rightarrow TW$ associated to the Morse function f so that

$$X^0 = -2x_1 \frac{\partial}{\partial x_1} \dots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} \dots + 2x_m \frac{\partial}{\partial x_m}.$$

Moreover on each U_i there is a vector field $X^i : U_i \rightarrow TW$ such that

$$X^i = \frac{\partial}{\partial x_1^i} \text{ for } i = 1, \dots, k.$$

Now consider a partition of unity $\{\psi_i : W \rightarrow \mathbb{R}\}_{i=0}^k$ subordinate to the open cover $\{U_i\}_{i=0}^k$. Therefore $X : W \rightarrow TW$ defined by

$$X = \psi_0 X^0 + \psi_1 X^1 + \dots + \psi_k X^k$$

is a desired gradient-like vector field for f .

□

Chapter 3

Product and Elementary Cobordisms

3.1 Product Cobordisms

We first investigate the simplest cobordism called a *product cobordism*.

Definition 24 (Product Cobordisms). *A cobordism $(W; \partial_-W, \partial_+W)$ is called a product cobordism if it is diffeomorphic to $(\partial_-W \times I; \partial_-W, \partial_-W)$, where $I = [0, 1]$.*

Theorem 25. *If the Morse number μ of $(W; \partial_-W, \partial_+W)$ is zero, then $(W; \partial_-W, \partial_+W)$ is a product cobordism.*

Proof. Note that every Morse function $f : W \rightarrow \mathbb{R}$ has no critical points since $\mu = 0$. Also observe that there is a gradient-like vector field $Y : W \rightarrow TW$ for f . So $Yf > 0$ on W . Then define another gradient-like vector field $X : W \rightarrow TW$ by rescaling

$$X = \frac{1}{Yf}Y.$$

Xf is constant on W , *i.e.*

$$Xf = \frac{1}{Yf}Yf = 1.$$

For $p \in \partial W = \partial_-W \sqcup \partial_+W$ and an open neighborhood U of p , $f|_U : U \rightarrow \mathbb{R}$ extends to a smooth map $\tilde{f}|\tilde{U} : \tilde{U} \rightarrow \mathbb{R}$, where \tilde{U} is homeomorphic to an open set in \mathbb{R}^m . Then the gradient-like vector field $X|_U : U \rightarrow TW$ also extends to $\tilde{X} : \tilde{U} \rightarrow TW$. Because $Xf = 1$ on a compact manifold W , there is an integral curve $\varphi_p : I \rightarrow W$ such that $f \circ \varphi_p(t) = t + c$, where c is a constant. Now define

a C^∞ -diffeomorphism $\psi_p : I \rightarrow W$ by $\psi_p = \varphi_p(t - c)$. Then

$$f \circ \psi_p(t) = t \quad \forall t \in I$$

and $\psi_p(0) = p$. Therefore $h : \partial_- W \times I \rightarrow W$ defined by

$$h(p, t) = \psi_p(t)$$

is a diffeomorphism with the inverse $h^{-1}(p) = (\psi_p(0), f(p))$. \square

3.2 Elementary Cobordisms and Surgery Theory

Definition 26 (Elementary Cobordisms). *An elementary cobordism is a cobordism $(W; \partial_- W, \partial_+ W)$ that admits a Morse function $f : W \rightarrow \mathbb{R}$ with exactly one critical point $p \in W$.*

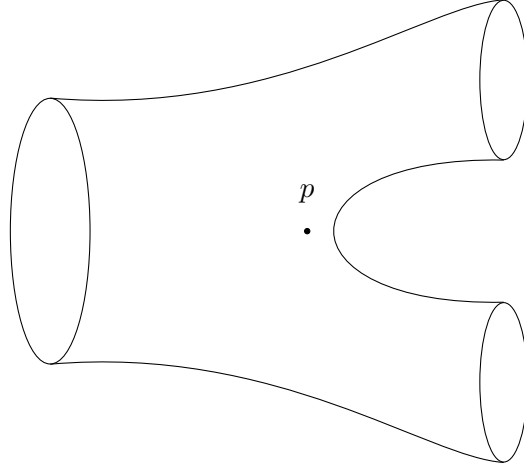


Figure 3.1: An elementary cobordism with a critical point p .

For an $(m - 1)$ -dimensional manifold M consider an embedding $\iota : S^{\lambda-1} \times \text{Int } D^{m-\lambda} \rightarrow M$.

Let $\chi(M, \iota)$ denote the manifold obtained by

$$M - \iota(\partial D^\lambda \times \mathbf{0}) \sqcup \text{Int } D^\lambda \times S^{m-\lambda-1} = M - \iota(\partial D^\lambda \times \mathbf{0}) \cup \text{Int } D^\lambda \times S^{m-\lambda-1} / \sim_\iota,$$

where \sim_ι is the equivalence relation generated by $\iota(u, \theta v) \sim_\iota (\theta u, v)$ for $u \in S^{\lambda-1}$ and $v \in S^{m-\lambda-1}$, where $\theta \in (0, 1)$. Then $\chi(M, \iota)$ is obtained from M by *surgery of type $(\lambda, m - \lambda)$* .

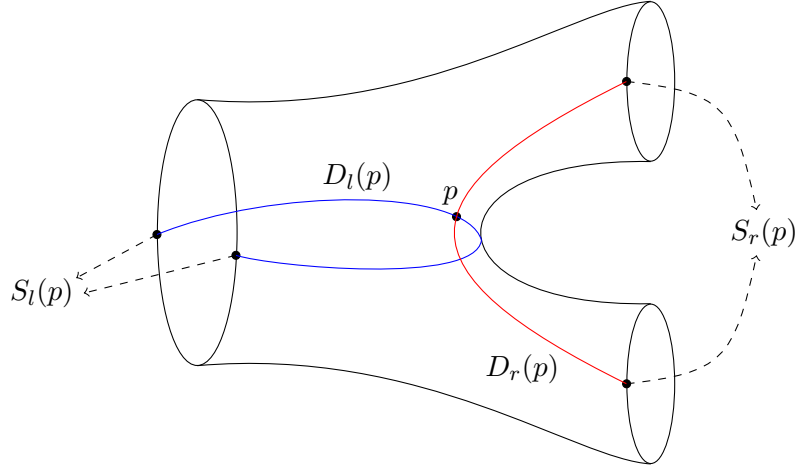


Figure 3.2: An elementary cobordism with embedded disks and spheres.

Theorem 27. *A cobordism $(W; \partial_- W, \partial_+ W)$ with $\partial_+ W = \chi(\partial_- W, \iota)$ is an elementary cobordism and there exists a Morse function $f : W \rightarrow \mathbb{R}$ with exactly one critical point $p \in W$ of index λ .*

Observe that $\phi : S^{\lambda-1} \times (D^{m-\lambda} - \mathbf{0}) \rightarrow (\text{int } D^\lambda - \mathbf{0}) \times S^{m-\lambda-1}$ that maps $(u, \theta v)$ to $(\theta u, v)$ is a diffeomorphism. So $\chi(M, \iota)$ is a smooth manifold since the smooth structure is inherited by the smooth structure of both $M - \iota(S^{\lambda-1} \times \mathbf{0})$ and $D^\lambda \times S^{m-\lambda-1}$.

Now let us construct a cobordism whose boundary is $M \sqcup \chi(M, \iota)$. So consider

$$L = \{(x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \mid -1 \leq -\|x\|^2 + \|y\|^2 \leq 1\} \cap \\ \{(x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \mid \|x\| \cdot \|y\| \leq \cosh 1 \cdot \sinh 1\}.$$

Because of being a subspace of \mathbb{R}^m , L is a smooth manifold whose boundary is given by $\partial_- L = \{(x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \mid -\|x\|^2 + \|y\|^2 = -1\}$ and $\partial_+ L = \{(x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \mid -\|x\|^2 + \|y\|^2 = 1\}$. Now observe that $\partial_- L$ is diffeomorphic to $S^{\lambda-1} \times \text{int } D^{m-\lambda}$ via correspondence between $(u \cosh \theta, v \sinh \theta)$ and $(u, \theta v)$ for $\theta \in [0, 1)$. Similarly, $\partial_+ L$ is diffeomorphic to $\text{int } D^\lambda \times S^{m-\lambda-1}$ via correspondence between $(u \sinh \theta, v \cosh \theta)$ and $(u, \theta v)$. Moreover, for the hypersurfaces $-\|x\|^2 + \|y\|^2 = c$ and the map $t \mapsto (e^{-t}x, e^t y)$ gives orthogonal trajectories through these hypersurfaces. If $x = 0 \in \mathbb{R}^\lambda$ or $y = 0 \in \mathbb{R}^{m-\lambda}$, then they are straight line segments. Otherwise, the maps parametrize a hyperbola from some $(u \cosh \theta, v \sinh \theta) \in \partial_- L$ to $(u \sinh \theta, v \cosh \theta) \in \partial_+ L$.

Next construct the manifold $\omega(M, \iota)$ as follows. So, for $u \in S^{\lambda-1}$, $v \in S^{m-\lambda-1}$, $\theta \in (0, 1)$, and $c \in [-1, 1]$, consider the disjoint union

$$(M - \iota(S^{\lambda-1} \times \mathbf{0})) \times [-1, 1] \sqcup L / \sim_\iota, \quad (3.1)$$

where the equivalence relation \sim_ι generated by identification of $(\iota(u, \theta v), c)$ with the unique point $(x, y) \in L$ so that

$$-\|x\|^2 + \|y\|^2 = c$$

and

$$(x, y) \text{ is on the orthogonal trajectory through } (u \cosh \theta, v \sinh \theta).$$

This correspondence defines a diffeomorphism

$$\iota(S^{\lambda-1} \times (\text{int } D^{m-\lambda} - \mathbf{0})) \times [-1, 1] \rightarrow L \cap ((\mathbb{R}^\lambda - \mathbf{0}) \times (\mathbb{R}^{m-\lambda} - \mathbf{0})),$$

and we get a differential manifold $\omega(M, \iota)$ with boundary. So observe the following two cases, $c = -1$ and $c = 1$.

Case 1: ($c = -1$). If $c = -1$, then we identify points of $(\iota(S^{\lambda-1} \times (\text{int } D^{m-\lambda} - \mathbf{0}))) \times \{-1\}$ with the points of $\partial_- L$ that corresponds to $S^{\lambda-1} \times (\text{int } D^{m-\lambda} - \mathbf{0})$. So one boundary of the manifold $\omega(M, \iota)$ is given by M .

Case 2: ($c = 1$). If $c = 1$, then we identify the points of $(\iota(S^{\lambda-1} \times (\text{int } D^{m-\lambda} - \mathbf{0}))) \times \{1\}$ with the points of $\partial_+ L$ that corresponds to $(\text{int } D^\lambda - \mathbf{0}) \times S^{m-\lambda-1}$. So another boundary of $\omega(M, \iota)$ is $\chi(M, \iota)$.

Therefore $\omega(M, \iota)$ is a cobordism whose boundary is $M \sqcup \chi(M, \iota)$.

Proposition 28. *There exists a Morse function $f : \omega(M, \iota) \rightarrow [-1, 1]$ with $f^{-1}(-1) = M$, $f^{-1}(1) = \chi(M, \iota)$, and there is exactly one critical point of index λ in the interior of $\omega(M, \iota)$.*

Proof. Define a map $f : \omega(M, \iota) \rightarrow [-1, 1]$ by

$$f(u) = \begin{cases} c & \text{if } u = (z, c) \in (M - \iota(S^{\lambda-1} \times \mathbf{0})) \times [-1, 1] \\ -\|x\|^2 + \|y\|^2 & \text{if } u \in L \end{cases}$$

By construction of the manifold $\omega(M, \iota)$, f agrees on the overlap and thus the map is *well-defined* on $\omega(M, \iota)$. Moreover observe that $x \in \mathbb{R}^\lambda$ and $y \in \mathbb{R}^{m-\lambda}$. So f is a Morse function with only one critical point of index λ as desired. \square

Theorem 29. *Let $(W; \partial_-W, \partial_+W)$ be an elementary cobordism with an embedding $\iota : S^{\lambda-1} \times \text{Int } D^{m-\lambda} \rightarrow \partial_-W$. Then $(W; \partial_-W, \partial_+W)$ is diffeomorphic to $(\omega(\partial_-W, \iota); \partial_-W, \chi(\partial_-W, \iota))$.*

Proof. For simplicity we assume $f(p) = 0$, where p is the critical point of the Morse function f . By Morse Lemma, there is a small neighborhood U around p so that f can be written as

$$f(x, y) = -\|x\|^2 + \|y\|^2,$$

where $x \in \mathbb{R}^\lambda$ and $y \in \mathbb{R}^{m-\lambda}$. Now let $\epsilon > 0$ such that

$$\begin{aligned} L_\epsilon &= \{(x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \mid -\epsilon \leq -\|x\|^2 + \|y\|^2 \leq \epsilon\} \cap \\ &\quad \{(x, y) \in \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda} \mid \|x\| \cdot \|y\| \leq \epsilon \cosh 1 \cdot \sinh 1\}. \end{aligned}$$

satisfying $L_\epsilon \subset U'$, where $\phi : U \rightarrow U' \subset \mathbb{R}^m$ is the local coordinate chart near p . Then $i := \phi^{-1}|_{\partial_-L_\epsilon} : \partial_-L_\epsilon \rightarrow f^{-1}(-\epsilon)$ is an embedding, and $W_\epsilon = f^{-1}([-\epsilon, \epsilon])$ is a cobordism that is identified as $\omega(f^{-1}(-\epsilon), \phi^{-1}|_{\partial_-L_\epsilon})$. Moreover $f^{-1}(-\epsilon)$ and $f^{-1}(\epsilon)$ are diffeomorphic to ∂_-W and ∂_+W , respectively. \square

For a topological space X and its subspace A with the inclusion $i : A \hookrightarrow X$, recall that A is called a *deformation retract* of X if there exists some continuous map $r : X \rightarrow A$ such that

$$r \circ i = \text{id}_A \text{ and } i \circ r \simeq \text{id}_X. \tag{3.2}$$

Observe by definition above that A and X have the same homotopy type if A is a deformation retract of X . Moreover if A and X have the same homotopy type, then the homology group $H_n(A)$ is isomorphic to $H_n(X)$.

Theorem 30. *For a cobordism $(W; \partial_-W, \partial_+W)$ and a Morse function $f : W \rightarrow \mathbb{R}$ with one critical point $p \in W$ of index λ , let $D_l(p)$ denote the left-hand λ -disk associated to a gradient-like vector field $X : W \rightarrow TW$ for f . Then $\partial_-W \cup D_l(p)$ is a deformation retract of W .*

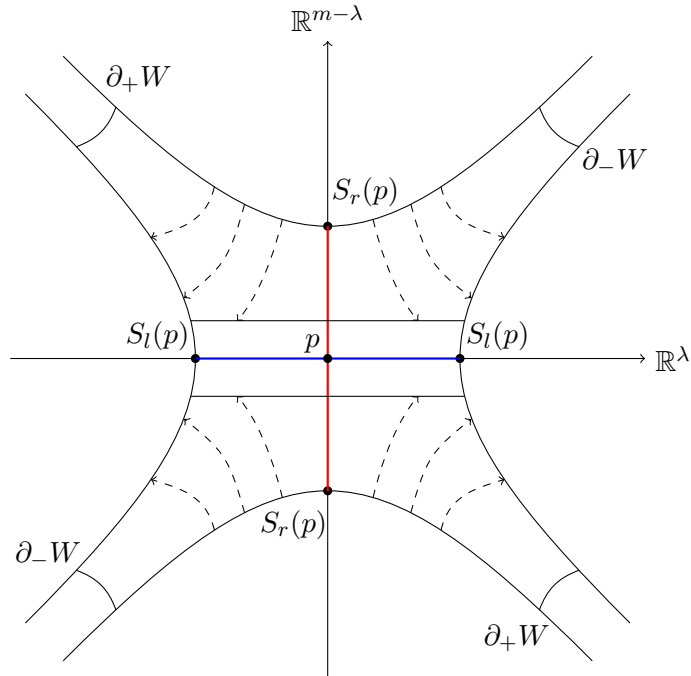


Figure 3.3: The first retraction r from W to $\partial_-W \cup C$.

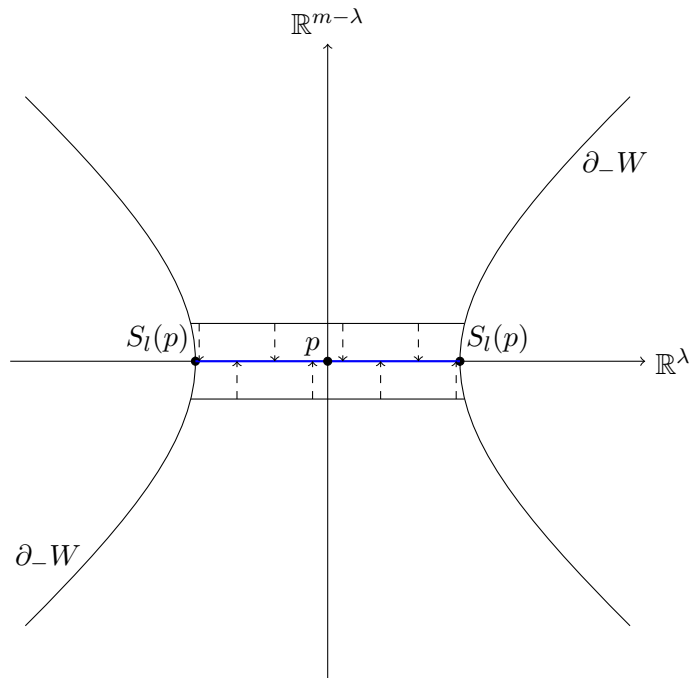


Figure 3.4: The second retraction r' from $\partial_-W \cup C$ to $\partial_-W \cup D_l(p)$.

Proof. By Theorem 29, observe that $(W; \partial_- W, \partial_+ W)$ is diffeomorphic to

$$(\omega(\partial_- W, \iota); \partial_- W, \chi(\partial_- W, \iota)) \quad (3.3)$$

for some embedding $\iota : S^{\lambda-1} \times \text{Int } D^{m-\lambda} \rightarrow \partial_- W$. Also there is a Morse function $f : \omega(\partial_- W, \iota) \rightarrow [-1, 1]$ on $\omega(\partial_- W, \iota)$. For L in ??, let $D_l(p) = \{(x, y) \in L \mid \|y\| = 0\}$ denote an embedded disk. Also let $C = \{(x, y) \in L \mid \|y\| \leq \frac{1}{10}\}$ denote an open neighborhood of $D_l(p)$. Now define a retraction $r_t : \omega(\partial_- W, \iota) \rightarrow \partial_- W \cup C$ as follows. Recall from the equation 29 above that

$$(\partial_- W - \iota(S^{\lambda-1} \times \mathbf{0})) \times [-1, 1] \cup_\iota L.$$

So for $(u, c) \in \partial_- W - \iota(S^{\lambda-1} \times \mathbf{0}) \times [-1, 1]$ define

$$r_t(u, c) = (u, c - t(c + 1)) \quad \forall t \in [0, 1] \quad (3.4)$$

and for $(x, y) \in L$ define

$$r_t(x, y) = \begin{cases} (x, y) & \text{if } \|y\| \leq \frac{1}{10} \\ (\frac{x}{\rho}, \rho y) & \text{if } \|y\| \geq \frac{1}{10} \end{cases}, \quad (3.5)$$

where $\rho = \rho(x, y, t) = \max\{\frac{1}{10\|y\|}, \zeta\}$ and ζ is the unique positive real solution of the equation

$$-\frac{\|x\|^2}{\zeta^2} + \zeta^2 \|y\|^2 = (-\|x\|^2 + \|y\|^2)(1 - t) - t. \quad (3.6)$$

So if $\|y\| \geq \frac{1}{10}$ then ρ determines a trajectory from (x, y) to some point in $\partial_- W$.

Next construct another retraction $r'_t : \partial_- W \cup C \hookrightarrow \partial_- W \cup D_l(p)$ as follows. For $(x, y) \in C$ define

$$r'_t(x, y) = \begin{cases} (x, (1 - t)y) & \text{if } \|x\|^2 \leq 1 \\ (x, \rho'y) & \text{if } 1 \leq \|x\|^2 \leq 1 + \frac{1}{100} \end{cases}, \quad (3.7)$$

where $\rho' = \rho'(x, y, t) = (1 - t) + t \frac{\sqrt{\|x\|^2 - 1}}{\|y\|^2}$. Since $(x, (1 - t)y)$ and $(x, \rho'y)$ coincide when $\|x\|^2 = 1$, r'_t is well-defined and thus a retraction of $\partial_- W \cup C$ to $\partial_- W \cup D_l(p)$. Therefore $r'_t \circ r_t$ is a deformation retract of W to $\partial_- W \cup D_l(p)$. \square

Corollary 31. $H_n(W, \partial_- W) \cong \mathbb{Z}$ if $n = \lambda$. Otherwise $H_n(W, \partial_- W) \cong 0$.

Proof. By Theorem 30, there is an embedding

$$j : (D^\lambda, S^{\lambda-1}) \rightarrow (W, \partial_- W) \quad (3.8)$$

such that $j^{-1}(\partial_- W) = S^{\lambda-1}$ and $\partial_- W \cup j(D^\lambda)$ is a deformation retract of W . So

$$H_*(W, \partial_- W) \cong H_*(\partial_- W \cup j(D^\lambda), \partial_- W). \quad (3.9)$$

Moreover $j : S^{\lambda-1} \rightarrow \partial_- W$ extends to an embedding $\tilde{j} : S^{\lambda-1} \times \text{Int } D^{m-\lambda} \rightarrow \partial_- W$ and let $U = \text{im } \tilde{j}$.

Therefore

$$H_*(W, \partial_- W) \cong H_*(\partial_- W \cup j(D^\lambda), \partial_- W) \quad (3.10)$$

$$\cong H_*(\partial_- W \cup j(D^\lambda) - (\partial_- W - U), \partial_- W - (\partial_- W - U)) \quad (3.11)$$

$$\cong H_*(U \cup j(D^\lambda), U) \quad (3.12)$$

$$\cong H_*(j^{-1}(U \cup j(D^\lambda)), j^{-1}(U)) \quad (3.13)$$

$$\cong H_*(D^\lambda, S^{\lambda-1}) \quad (3.14)$$

because of excision and homotopy invariance. Hence

$$H_n(W, \partial_- W) \cong H_n(D^\lambda, S^{\lambda-1}) \cong \begin{cases} \mathbb{Z} & \text{if } n = \lambda \\ 0 & \text{if } n \neq \lambda \end{cases}. \quad (3.15)$$

□

Now we can generalize Theorem 30 and Corollary 31 as follows. Suppose that a cobordism $(W; \partial_- W, \partial_+ W)$ admits a Morse function $f : W \rightarrow \mathbb{R}$ with k critical points p_1, \dots, p_k of indices $\lambda_1, \dots, \lambda_k$, respectively. Moreover suppose that those critical points have the same critical value, *i.e.* $f(p_1) = \dots = f(p_k)$. By perturbing the Morse function f and the associated gradient-like vector field X , obtain k embeddings $\iota_i : S^{\lambda_i-1} \times \text{Int } D^{m-\lambda_i} \rightarrow V$ and construct the smooth manifold $\omega(V; \iota_1, \dots, \iota_k)$ as given above for each $i = 1, \dots, k$. This manifold $\omega(\partial_- W; \iota_1, \dots, \iota_k)$ is diffeomorphic to W and moreover $\partial_- W \cup D_l(p_1) \cup \dots \cup D_l(p_k)$ is a deformation retract of W .

Therefore if $\lambda_1 = \dots = \lambda_k = \lambda$, then $H_n(W, \partial_- W) \cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ (k summands) for $n = \lambda$. Otherwise $H_n(W, \partial_- W) \cong 0$ for $n \neq \lambda$.

Chapter 4

Rearrangement of Cobordisms

4.1 Rearrangement of Critical Values

In this section we will consider a cobordism $(W; \partial_- W, \partial_+ W)$. with exactly two critical points p and p' of indices λ and λ' , respectively. However their indices are not necessarily same or consecutive.

Let $\varphi : W \times \mathbb{R} \rightarrow W$ denote a *flow* of the vector field X . For a critical point $p \in W$ of a Morse function $f : W \rightarrow \mathbb{R}$ define

$$W^s(p; X) = \{x \in W \mid \lim_{t \rightarrow +\infty} \varphi(x, t) = p\}$$

and

$$W^u(p; X) = \{x \in W \mid \lim_{t \rightarrow -\infty} \varphi(x, t) = p\}$$

the *stable* and *unstable manifold* of p with respect to X . Observe that the stable manifold $W^s(p; X)$ is equivalent to the left-hand disk $D_l(p)$ of the critical point p . Similarly, $W^u(p; X)$ is equivalent to the right-hand disk $D_r(p)$.

The following theorem states that we can perturb a Morse function f by an isotopy so that two critical points of f change their critical values in some small neighborhoods of those two critical points.

Theorem 32 (Preliminary Rearrangement Theorem). *For a cobordism $(W; \partial_- W, \partial_+ W)$ let $f : W \rightarrow [0, 1]$ be a non-resonant Morse function with exactly two critical points $p, p' \in W$. Suppose*

for simplicity that $f(W) = [0, 1]$ and $f(p) < f(p')$. Moreover, for some gradient-like vector field X for f , suppose that

$$K_p = W^s(p; X) \cup W^u(p; X) \quad (4.1)$$

and

$$K_{p'} = W^s(p'; X) \cup W^u(p'; X). \quad (4.2)$$

are disjoint. Then there exists another Morse function $g : W \rightarrow [0, 1]$ satisfying the following properties:

- (1) X is a gradient-like vector field for g ,
- (2) $g(p) = a$ and $g(p') = a'$ for some $a, a' \in [0, 1]$,
- (3) $f - g$ is a constant function in an open neighborhood of $\{p, p'\}$, and
- (4) g coincides with f near $\partial_- W \cup \partial_+ W$.

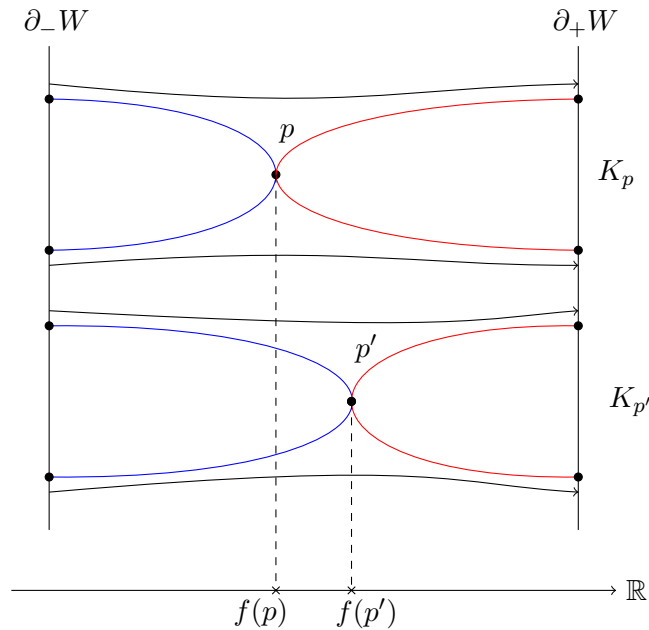


Figure 4.1: Rearrangement of critical values.

Proof. Note first that $\partial_- W = f^{-1}(0)$ and $\partial_+ W = f^{-1}(1)$, and let

$$S_l(p) = \partial_- W \cap W^s(p; X) \text{ and } S_l(p') = \partial_- W \cap W^s(p'; X) \quad (4.3)$$

denote the left-hand spheres of p and p' , respectively.

Observe that all integral curves outside $K = K_p \cup K_{p'}$ proceed from $\partial_- W$ to $\partial_+ W$. For each $x \in W - K$ consider the uniquely determined integral curve $\varphi : W \times [0, 1] \rightarrow W$ such that

$$\varphi(x, 0) = x. \quad (4.4)$$

Then let $\tau : M \rightarrow \mathbb{R}$ so that $\varphi(x, \tau(x)) \in \partial_- W$. So, by the *Immersion Theorem* the map $\pi : W - K \rightarrow \partial_- W$ that assigns $x \in W - K$ the unique point $\varphi(x, \tau(x))$ is well-defined and smooth. Moreover, if x is near K , then $\pi(x)$ is near K as well.

Now let $\mu : \partial_- W \rightarrow [0, 1]$ be a smooth function defined by

$$\mu(x) = \begin{cases} 0 & \text{if } x \in U_0 \\ 1 & \text{if } x \in U_1, \end{cases} \quad (4.5)$$

where U_0 and U_1 are small open neighborhoods of x such that $U_0 \cap S_l(p') \neq \emptyset$ and $U_1 \cap S_l(p) \neq \emptyset$.

Using $\pi : W - K \rightarrow \partial_- W$, μ extends to a smooth function $\tilde{\mu} : W \rightarrow [0, 1]$ defined by

$$\tilde{\mu}(x) = \begin{cases} 0 & \text{if } x \in \tilde{U}_0 \\ 1 & \text{if } x \in \tilde{U}_1 \\ k & \text{elsewhere,} \end{cases} \quad (4.6)$$

where $k \in \mathbb{R}$ is a constant, and \tilde{U}_0 and \tilde{U}_1 are small open neighborhoods of x such that $\tilde{U}_0 \cap K'_p \neq \emptyset$ and $\tilde{U}_1 \cap K_p \neq \emptyset$.

Now define a smooth function $G : [0, 1] \times [0, 1] \rightarrow [0, 1]$ with the following properties:

(1) For all x and y , $\frac{\partial G}{\partial x} > 0$ and G increases from 0 to 1 as x increases from 0 to 1.

(2) $G(f(p), 0) = a$ and $G(f(p'), 1) = a'$,

- (3) $G(x, y) = x$ for x near 0 and 1 and for all y ,
- (4) $\frac{\partial G}{\partial x}(x, 0) = 1$ for x near $f(p)$, and
- (5) $\frac{\partial G}{\partial x}(x, 1) = 1$ for x near $f(p')$.

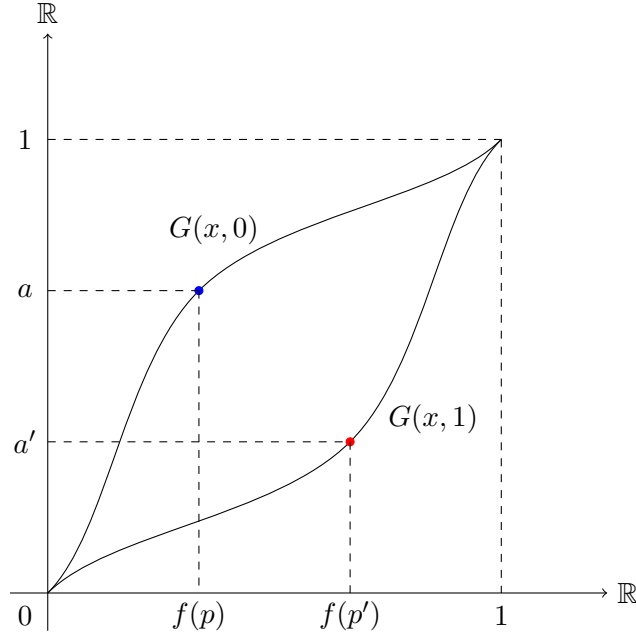


Figure 4.2: The graph of G .

Define a smooth function g by $g(x) = G(f(x), \tilde{\mu}(x))$. Observe that $f - g$ is constant near p and p' by properties (4) and (5). Thus X is also a gradient-like vector field for g as well near the critical points p and p' . By property (1) X is a gradient-like vector field for g away from critical points. Property (2) shows that $g(p) = a$ and $g(p') = a'$. Moreover by property (3) $g = f$ near the boundaries $\partial_- W$ and $\partial_+ W$. Therefore g is a Morse function with the desired property. \square

Corollary 33. *Suppose that the Morse function f has k critical points p_1, \dots, p_k such that*

$$\{p_1, \dots, p_l\} \subset f^{-1}(b)$$

and

$$\{p_{l+1}, \dots, p_k\} \subset f^{-1}(b'),$$

where $\{p_1, \dots, p_l\}$ and $\{p_{l+1}, \dots, p_k\}$ have indices λ and λ' , respectively. Let $p = \{p_1, \dots, p_l\}$ and $p' = \{p_{l+1}, \dots, p_k\}$ for simplicity. Then the Theorem 32 is still valid.

4.2 Rearrangement of Spheres

Definition 34 (Product Neighborhoods). *Let M be an m dimensional manifold and N an r dimensional submanifold of M . Then an open neighborhood U of N , which is diffeomorphic to $N \times \mathbb{R}^{m-n}$, is called a product neighborhood of N in M .*

Lemma 35. *Let N and N' be n and n' dimensional submanifolds of a manifold M of dimension m , respectively. Suppose that N has a product neighborhood in M and that $n + n' < m$. Then there exists some diffeomorphism $h : M \rightarrow M$ such that h is smoothly isotopic to the identity $\text{id}_M : M \rightarrow M$ and $h(N) \cap N' = \emptyset$.*

Proof. Let $k : N \times \mathbb{R}^{m-n} \rightarrow U \subset M$ be a diffeomorphism so that $k(N \times 0) = N$ for $0 \in \mathbb{R}^{m-n}$. Moreover let $N_0 = U \cap N$, $\pi : N \times \mathbb{R}^{m-n} \rightarrow \mathbb{R}^{m-n}$ a canonical projection, and $g = \pi \circ k^{-1}|_{N_0} : N_0 \rightarrow \mathbb{R}^{m-n}$. Observe that $k(N \times x) \cap N' \neq \emptyset$ if and only if $x \in g(N_0)$. And if N_0 is nonempty, then $\dim N_0 = n < m - n'$. By Sard's Lemma $g(N_0)$ has the Lebesgue measure 0. So choose $u \in \mathbb{R}^{m-n} - g(N_0)$.

Now we construct a diffeomorphism h of M onto itself such that h is isotopic to the identity $\text{id}_M : M \rightarrow M$ and $h(N)$ to $k(N \times u)$. So define a smooth vector field $X : \mathbb{R}^{m-n} \rightarrow \mathbb{R}^{m-n}$ such that

$$X = \begin{cases} u & \text{if } |x| \leq |u| \\ 0 & \text{if } |x| \geq 2|u| \end{cases} \quad (4.7)$$

for every $x \in \mathbb{R}^{m-n}$. By construction of the vector field, X has a compact support. Furthermore $\partial\mathbb{R}^{m-n} \neq \emptyset$. This implies that the integral curves $\psi(x, t)$ are defined for all $t \in [0, 1]$. Then $\psi(x, 0) = x = \text{id}_M(x)$, $\psi(x, 1)$ is a diffeomorphism that carries 0 to u , and thus $\phi : \mathbb{R}^{m-n} \times [0, 1] \rightarrow \mathbb{R}^{m-n}$ is an isotopy from id_M to the diffeomorphism $\psi(x, 1)$. This isotopy leaves all points fixed

outside a bounded set U in \mathbb{R}^{m-n} , so define an isotopy $h_t : M \rightarrow M$ by

$$h_t(w) = \begin{cases} k(q, \psi(x, t)) & \text{if } w = k(q, x) \in U \\ w & \text{if } w = M - U \end{cases}. \quad (4.8)$$

Therefore $h = h_1$ is the desired diffeomorphism such that $h(N) \cap N' = \emptyset$. \square

Theorem 36. *Suppose that $\lambda \geq \lambda'$ and let $h : W \rightarrow W$ be a diffeomorphism with the property given in Lemma 35. Then there exists a gradient-line vector field \tilde{X} for f so that $h(S_r(p))$ is disjoint from $S_l(p')$ in some open neighborhood U of $V = f^{-1}(\frac{1}{2})$ and \tilde{X} coincides with X outside U .*

Proof. Observe first that $S_r(p)$ has a product neighborhood in V . By Lemma 35 there exists a diffeomorphism $h : V \rightarrow V$ such that h is smoothly isotopic to the identity $\text{id}_V : V \rightarrow V$ and $h(S_r(p)) \cap S_l(p') = \emptyset$. We use this diffeomorphism to construct new gradient-like vector field \tilde{X} . So choose $a < \frac{1}{2}$ so that $f^{-1}([a, \frac{1}{2}])$ does not contain p . Then the integral curves of $Y = \frac{1}{Xf}X$ determine a diffeomorphism $\varphi : [a, \frac{1}{2}] \rightarrow f^{-1}([a, \frac{1}{2}])$ such that $f(\varphi(q, t)) = t$ and $\varphi(\frac{1}{2}, q) = q \in V$. Now define a diffeomorphism $H : [a, \frac{1}{2}] \times V \rightarrow [a, \frac{1}{2}] \times V$ by $H(t, q) = (t, h_t(q))$, where h_t is a smooth map from $[a, \frac{1}{2}] \times V$ to V and isotopy from id_M to h such that

$$h_t = \begin{cases} \text{id}_M & \text{for } t \text{ near } a \\ h & \text{for } t \text{ near } \frac{1}{2} \end{cases}. \quad (4.9)$$

Then $\tilde{Y} = (\varphi \circ H \circ \varphi^{-1})_* Y$ is a smooth vector field defined on $f^{-1}([a, \frac{1}{2}])$, which coincides with Y near $f^{-1}(a)$ and V . Moreover $\tilde{Y}f = 1$ identically. Finally the vector field \tilde{X} on W defined by

$$\tilde{X} = \begin{cases} (Xf)\tilde{Y} & \text{on } f^{-1}([a, \frac{1}{2}]) \\ X & \text{elsewhere} \end{cases} \quad (4.10)$$

is a new gradient-like vector field for f .

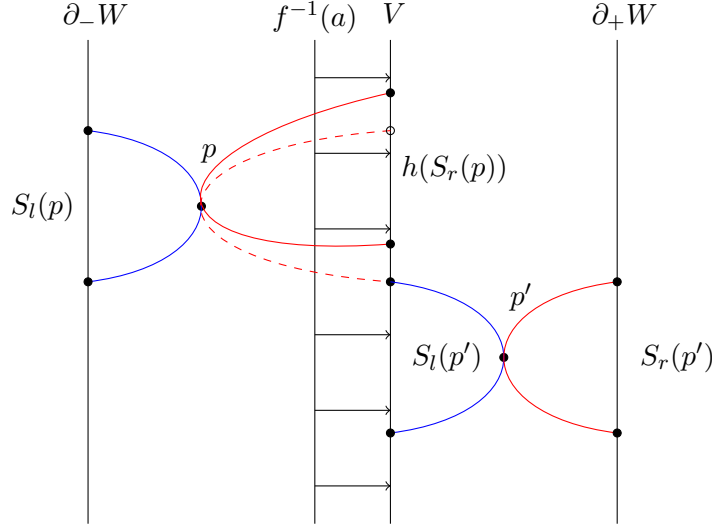


Figure 4.3: Construction of an isotopy to make $S_r(p)$ disjoint from $S_l(p')$.

Now for every fixed $q \in V$ $\varphi(t, h_t(q))$ is an integral curve of \tilde{X} from $\varphi(a, q) \in f^{-1}(a)$ to $\varphi(\frac{1}{2}, h(q)) = h(q) \in V$. It follows that the right-hand sphere $\varphi(a \times S_r(p))$ in $f^{-1}(a)$ is carried to $h(S_r(p))$ in V . Thus $h(S_r(p)) \cap S_l(p') = \emptyset$. \square

Observe that Theorem 36 is generalized as follows. If c denotes a cobordism that admits a Morse function f with k critical points p_1, \dots, p_k of index λ , and c' a cobordism with l critical points p'_1, \dots, p'_l of index λ' of f , then the new gradient-like vector field \tilde{X} is constructed from X on a small open neighborhood of V such that right-hand spheres $S_r(p_i)$ and $S_l(p'_j)$ are disjoint by some isotopy of V .

Corollary 37. *If $\{p_1, \dots, p_l\}$ and $\{p_{l+1}, \dots, p_k\}$ are the sets of critical points of indices λ and λ' , respectively, then the gradient-like vector field X can be changed so that $h(S_r(p_i)) \cap S_l(p'_j) = \emptyset$ for $i = 1, \dots, l$ and $j = l + 1, \dots, k$.*

By Theorem 32 and 36, we should be able to construct a various types of isotopies to make two critical points have the same critical values, and to make spheres disjoint. So we can state the following theorem as a result.

Theorem 38 (Final Rearrangement Theorem). *Any cobordism $c = (W; \partial_- W, \partial_+ W)$ factors into a composition*

$$c = c_0 c_1 \cdots c_m, \tag{4.11}$$

where $m = \dim W$, and each c_λ admits a Morse function f possessing critical points of index λ with the same critical value.

Chapter 5

First Cancellation Theorem

In this section we assume that a cobordism $(W; \partial_- W, \partial_+ W)$ has exactly two critical points p^λ and $p^{\lambda+1}$ in W of indices λ and $\lambda + 1$.

Next consider embedded submanifolds N and N' of M whose dimensions are n and n' , respectively. Recall that N and N' intersect *transversely* if

$$T_p M = T_p N + T_p N'$$

for every $p \in N \cap N'$. In other words, two submanifolds N and N' intersect transversely if the tangent spaces at each point of the intersection span the tangent space of M .

Let N and N' be orientable submanifolds of M with dimensions r and s , respectively, so that $m = r + s$. Furthermore let f be a non-resonant Morse function defined on a cobordism $(W; \partial_- W, \partial_+ W)$ with two critical points p^λ and $p^{\lambda+1}$ of indices λ and $\lambda + 1$, respectively, such that $f(p^\lambda) < \frac{1}{2} < f(p^{\lambda+1})$. In the f -fiber $V = f^{-1}(\frac{1}{2})$, a gradient-like vector field X associated with f determines a right-hand sphere $S_r(p^\lambda)$ of p^λ and a left-hand sphere $S_l(p^{\lambda+1})$ of $p^{\lambda+1}$.

At the end of Chapter 4, Lemma 35 and 36 let us construct an isotopy of V to make two spheres $S_r(p)$ and $S_l(p')$ disjoint. In the similar way, we can define a isotopy of V so that those spheres have transverse intersection in V . Then we have the following theorem.

Theorem 39. *The gradient-like vector field X can be changed to another gradient-like vector field \tilde{X} so that $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$ intersect transversely in V .*

Now observe that V is compact and

$$\dim S_r(p^\lambda) + \dim S_l(p^{\lambda+1}) = (m - \lambda - 1) + \lambda = m - 1 = \dim V. \quad (5.1)$$

So for each $q \in S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ there exists some local coordinate system (x_1, \dots, x_m) on an open neighborhood U of q in V such that q corresponds to $(0, \dots, 0) \in \mathbb{R}^{m-1}$ and that

$$\begin{cases} x_1 = \dots = x_\lambda = 0 \text{ on } U \cap S_r(p^\lambda) \\ x_{\lambda+1} = \dots = x_{m-1} = 0 \text{ on } U \cap S_l(p^{\lambda+1}). \end{cases} \quad (5.2)$$

By this construction q a unique point contained in $S_r(p^\lambda) \cap S_l(p^{\lambda+1}) \cap U$. Therefore we should be able to assume that the intersection $S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ consists of finitely many points.

Theorem 40 (First Cancellation Theorem). *Assume that $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$ intersect transversely in V and $S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ is a single point, $\{q\} = S_r(p^\lambda) \cap S_l(p^{\lambda+1})$. On an arbitrary small neighborhood U of the single integral curve $\varphi_q : \mathbb{R} \rightarrow W$ from p^λ to $p^{\lambda+1}$, a gradient-like vector field X can be altered to a nowhere zero vector field X' so that all integral curves proceed from $\partial_- W$ to $\partial_+ W$. Furthermore X' is a gradient-like vector field for another Morse function f' such that f' coincides with f near $\partial_- W \cup \partial_+ W$.*

Idea of Proof. Observe first by existence and uniqueness of the ODEs, there is a unique integral curve $\varphi_q : \mathbb{R} \rightarrow W$ such that $\varphi_q(0) = q$. Then φ_q has the following property: $\lim_{t \rightarrow -\infty} \varphi_q(t) = p^\lambda$ and $\lim_{t \rightarrow +\infty} \varphi_q(t) = p^{\lambda+1}$. Since the index of p^λ is λ by assumption, the gradient-like vector field X at p^λ is of the form

$$X_{p^\lambda} = -2x_1 \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} \cdots + 2x_m \frac{\partial}{\partial x_m} \quad (5.3)$$

on some open neighborhood U_p^λ of p^λ with a coordinate system (x_1, \dots, x_m) . Now consider the appropriate coordinate change

$$(x_1, \dots, x_\lambda, x_{\lambda+1}, \dots, x_n) \mapsto (x_{\lambda+1}, \dots, x_\lambda, x_1, \dots, x_n). \quad (5.4)$$

Then X can be rewritten as

$$X = 2x_1 \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} - 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} \cdots + 2x_m \frac{\partial}{\partial x_m}. \quad (5.5)$$

Similarly, on some neighborhood $U_{p^{\lambda+1}}$ of $p^{\lambda+1}$, there is a coordinate system (x'_1, \dots, x'_m) and X at $p^{\lambda+1}$ has the form

$$X = -2x'_1 \frac{\partial}{\partial x'_1} \cdots - 2x'_\lambda \frac{\partial}{\partial x'_\lambda} - 2x'_{\lambda+1} \frac{\partial}{\partial x'_{\lambda+1}} \cdots + 2x'_m \frac{\partial}{\partial x'_m}. \quad (5.6)$$

Then find a local coordinate system (x_1, \dots, x_m) in a neighborhood U with the following two properties:

- (1) The coordinates of p^λ and $p^{\lambda+1}$ correspond to $(0, 0, \dots, 0)$ and $(1, 0, \dots, 0)$, respectively, and
- (2) X has the form

$$X = 2v_1(x_1) \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} - 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} + 2x_{\lambda+2} \frac{\partial}{\partial x_{\lambda+2}} + \cdots + 2x_m \frac{\partial}{\partial x_m} \quad (5.7)$$

on U , where $v_1 : [-2\delta, 1 + 2\delta] \rightarrow \mathbb{R}$ is a smooth scalar function defined as follows. For some small $\delta > 0$, v_1 is defined by

$$v_1(x_1) = \begin{cases} x_1 & \text{if } x_1 \in U_0, \\ 1 - x_1 & \text{if } x_1 \in U_1, \end{cases} \quad (5.8)$$

where U_0 and U_1 are small open neighborhoods of 0 and 1, respectively. Also $v_1(x_1) > 0$ for $0 < x_1 < 1$.

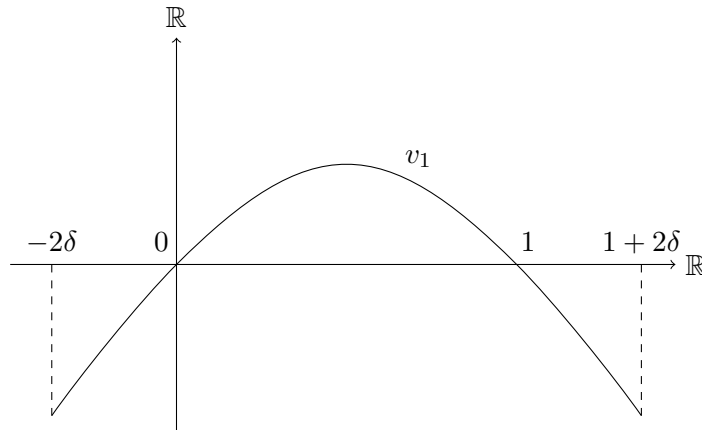


Figure 5.1: The graph of v_1 .

By construction of the vector field, X in 5.7 coincides with 5.5 and 5.6 in open neighborhoods U_0 and U_1 , respectively. Finally perturb the vector field X to construct a nowhere zero vector field \tilde{X} in U . To do this consider the family of smooth scalar functions $\{v_{1-t} : [-2\delta, 1 + 2\delta] \rightarrow \mathbb{R}\}_{t \in [0,1]}$ with the following properties:

- (1) $\{v_{1-t}\}_{t \in [0,1]}$ depends smoothly on t ,
- (2) v_1 is the function defined in Definition 5.8,
- (3) $v_0 < 0$ on $[-2\delta, 1 + 2\delta]$, and
- (4) $v_0(x_1) = v_1(x_1)$ if $x_1 < \frac{\delta}{2}$ or $x_1 > 1 + \frac{\delta}{2}$.

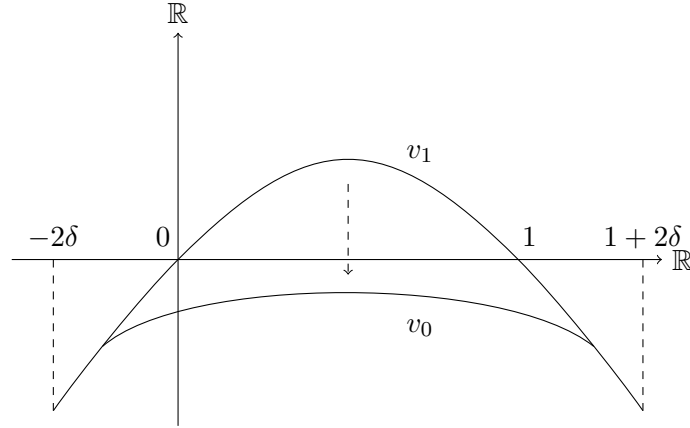


Figure 5.2: Deformation of v_1 to v_0 .

This deforms v_1 to v_0 smoothly and the values of v_0 are all negative on $[-2\delta, 1 + 2\delta]$. Then define a vector field $\tilde{X} : W \rightarrow TW$ by

$$\tilde{X} = 2v_\rho(x_1) \frac{\partial}{\partial x_1} \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} - 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} + 2x_{\lambda+2} \frac{\partial}{\partial x_{\lambda+2}} + \cdots + 2x_m \frac{\partial}{\partial x_m}, \quad (5.9)$$

where $\rho = x_2^2 + \cdots + x_n^2$. By its construction \tilde{X} is a nowhere zero vector field on U , which coincides with X in 5.7 outside U . Moreover every integral curve φ proceeds from $\partial_- W$ to $\partial_+ W$. Hence \tilde{X} and φ determine a smooth function $\tilde{f} : W \rightarrow \mathbb{R}$ such that \tilde{f} coincides with f outside U and $\frac{d\tilde{f}}{dt}(\varphi) = \tilde{X}\tilde{f}$. Thus \tilde{f} is a Morse function with no critical points in W . \square

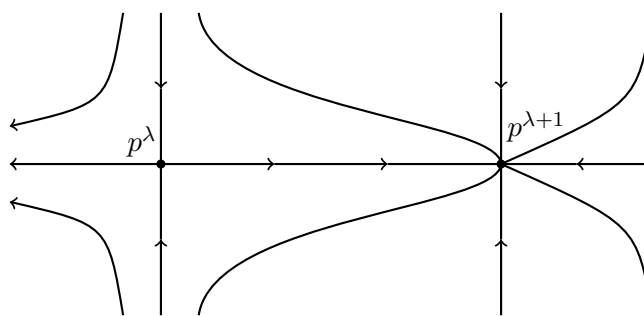


Figure 5.3: The two critical points p^λ and $p^{\lambda+1}$ in the vector field \tilde{X} .

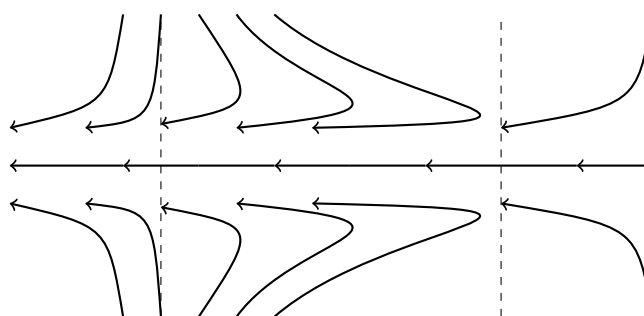


Figure 5.4: Cancellation of critical points p^λ and p^λ .

Chapter 6

Second Cancellation Theorem

Recall first from the *Whitney embedding theorem* that every compact smooth manifold M can be embedded in some Euclidean space \mathbb{R}^n . See Milnor [?, Milnor1997] or the proof.

Definition 41 (Normal Bundles). *Suppose that an m -dimensional manifold M is embedded in \mathbb{R}^n for some n . Then $v \in \mathbb{R}^n$ is called perpendicular to M at $p \in M$ if $X_p \cdot v = 0$ for all $X_p \in T_p M$, where \cdot is the inner product defined on \mathbb{R}^n . Moreover the normal bundle $\nu(M)$ of M in \mathbb{R}^n is defined by*

$$\nu(M) = \{(p, v) \in M \times \mathbb{R}^{n-m} \mid v \text{ is perpendicular to } M \text{ at } p\}. \quad (6.1)$$

For an m -dimensional smooth manifold M , let N and N' be submanifolds of dimension r and s , respectively, such that $r + s = m$. Suppose that N and N' intersect in finitely many points $p_1, \dots, p_k \in M$, transversely. Suppose also that M and the normal bundle $\nu(N')$ of N' in M are both oriented. Because

$$T_{p_i} M = T_{p_i} N \oplus T_{p_i} \nu(N')$$

at each p_i , where $T_{p_i} N$ has a positively oriented r -frame $\langle \zeta_1, \dots, \zeta_r \rangle$ of linearly independent vectors generating $T_{p_i} N$, $\langle \zeta_1, \dots, \zeta_r \rangle$ is a basis for the fiber at p_i of $\nu(N')$.

Definition 42 (Intersection Numbers). *The sign of intersection $\epsilon(p_i)$ at each p_i is defined to be either $+1$ or -1 according to a positively or negatively oriented basis for the fiber at p_i of $\nu(N')$.*

And the intersection number $\langle N \rangle \cdot \langle N' \rangle$ is defined by

$$\langle N \rangle \cdot \langle N' \rangle = \sum_{i=1}^k \epsilon(p_i).$$

Note that for an orientable manifold M every submanifold N of M is orientable if and only if $\nu(N)$ of N is orientable. Moreover, given an orientation for N , we have a canonical way to orient $\nu(N)$ and *vice versa*.

Lemma 43. *Let a manifold M and its submanifold N' be both compact and connected without boundary. Then there exists a isomorphism*

$$\psi : H_0(N') \rightarrow H_r(M, M - N').$$

We will use the lemma above without proof since it is easy to verify. Then the following theorem is based on the *Thom Isomorphism Theorem* and *Tubular Neighborhood Theorem*. Readers are encouraged to consult Kosinski [9] for the detail.

Lemma 44. *For the sequence*

$$H_r(N) \xrightarrow{i_*} H_r(M) \xrightarrow{j_*} H_r(M, M - N'), \quad (6.2)$$

where j_* and i_* are both induced by the inclusion map, $i_* \circ j_*(\langle M \rangle) = \langle N' \rangle \cdot \langle N \rangle \psi(\alpha)$.

Theorem 45. *Let N and N' be smooth closed submanifolds of dimensions r and s , respectively, such that N and N' intersect transversely in the smooth closed $(r + s)$ -dimensional manifold M . Suppose that N and the normal bundle $\nu(N')$ in M are both oriented. Moreover suppose that $r + s \geq 5$, $r \geq 3$, and suppose that the inclusion $i : M - N \hookrightarrow M$ induces the injective map $i_* : \pi_1(M - N) \rightarrow \pi_1(M)$ if $s = 1$ or $s = 2$.*

Let $p, q \in N \cap N'$ be a pair of intersection points with opposite intersection numbers such that there exists some loop γ connecting p and q so that γ is contractible in M . Suppose that γ does not contain any other intersection points in $N \cap N' - \{p, q\}$.

Under the assumption given above, there exists some isotopy $h_t : M \rightarrow M$, where $t \in [0, 1]$, such that

- (1) $h_0 = \text{id}_M : M \rightarrow M$,
- (2) h_t fixes id_M near $N \cap N' - \{p, q\}$, and
- (3) $h_1(N) \cap N' = N \cap N' - \{p, q\}$.

Theorem 45 lets us construct an isotopy of M such that a pair of intersection points with opposite intersection numbers is cancelled. See Milnor[6] for the detail construction of such an isotopy.

Theorem 46 (Second Cancellation Theorem). *For a cobordism $(W; \partial_-W, \partial_+W)$, suppose W , ∂_-W , and ∂_+W are simply connected, $\lambda \geq 2$, and $\lambda + 1 \leq n - 3$. If $\langle S_r(p^\lambda) \rangle \cdot \langle S_l(p^{\lambda+1}) \rangle = \pm 1$, then X can be altered near the fiber V so that $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$ in V intersect at a single point, transversely. Then the First Cancellation Theorem applies and W is therefore diffeomorphic to $\partial_-W \times [0, 1]$.*

Proof. Observe first that $\dim S_r(p^\lambda) = n - \lambda - 1$, $\dim S_l(p^{\lambda+1}) = \lambda$, and $\dim V = n - 1$. So $\dim S_r(p^\lambda) \geq 3$ and $\dim V = \lambda - 1$. Moreover $\pi_1(V) = 0$ by the *Seifert-van Kampen Theorem*. Then consider the following two cases.

Case 1: $\lambda \geq 3$. Then $\dim S_l(p^{\lambda+1}) \geq 3$ and $\dim V \geq 6$. Then the assumptions of Theorem 45 are satisfied.

Case 2: $\lambda = 2$. Then $\dim S_l(p^{\lambda+1}) = 2$ and $\dim V \geq 5$. Since $\{x \in W | a < f(x) < b\}$, where $a = f(\partial_-W - S_l(p^{\lambda+1}))$ and $b = f(V - S_r(p^\lambda))$, does not contain any critical points, the gradient-like vector field X for f determines a diffeomorphism between $\partial_-W - S_l(p^{\lambda+1})$ and $V - S_r(p^\lambda)$. Thus $\pi_1(V - S_r(p^\lambda)) \cong \pi_1(\partial_-W - S_l(p^{\lambda+1}))$. Now let U be a product neighborhood of S_l in ∂_-W . Since $\dim S_l(p^{\lambda+1}) = n - \lambda - 1 \geq 3$, $\pi_1(U - S_l(p^{\lambda+1})) \cong \mathbb{Z}$. Moreover there is a diagram for the fundamental groups given below. Thus $i_* : \pi_1(V - S_r(p^\lambda)) \rightarrow \pi_1(V)$ induced by the inclusion $i : V - S_r(p^\lambda) \hookrightarrow V$ is injective.

So for both cases the assumptions of Theorem 45 are satisfied. Then the *First Cancellation Theorem* directly applies if $S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ consists of a single point, for $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$

intersect transversely at a singleton in V . So suppose that $S_r(p^\lambda) \cap S_l(p^{\lambda+1})$ is not a single point. Since $\langle S_r(p^\lambda) \rangle \cdot \langle S_l(p^{\lambda+1}) \rangle = \pm 1$, the number of intersection points is odd, say $2k + 1$. Then there are pairs of intersection points $\{p_1, q_1\}, \dots$, and $\{p_k, q_k\}$ so that $\epsilon(p_i) = +1$ and $\epsilon(q_i) = -1$ for $i = 1, \dots, k$. By Theorem 45, each pair $\{p_i, q_i\}$ can be eliminated by isotopies of W . Thus W can be deformed so that $S_r(p^\lambda)$ and $S_l(p^{\lambda+1})$ have a transverse intersection at a single point in V . This completes the proof.

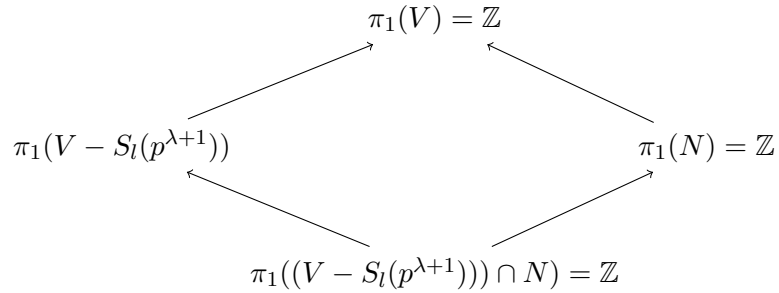


Figure 6.1: A diagram of fundamental groups induced by inclusions.

□

Chapter 7

Cancellation of Critical Points of Indices λ with $2 \leq \lambda \leq m - 2$

7.1 A Chain Complex and Homology of Manifolds

We focus our attention on homology groups with integer coefficients. So suppose first that M is an m dimensional compact smooth and orientable manifold with boundary ∂M . Observe that the orientation of M given by an orientation of its tangent bundle TM corresponds to an orientation of M specified its orientation generator $\langle M \rangle$ of $H_n(M)$.

Lemma 47. *Let M be an oriented closed smooth manifold of dimension λ embedded in $\partial_- W$ and $\langle M \rangle \in H_\lambda(M)$ the orientation generator. Let $i_* : H_\lambda(M) \rightarrow H_\lambda(W, \partial_+ W)$ be the map induced by the inclusion $h : \cdot$. Then*

$$i_*(\langle M \rangle) = (\langle S_r(p_1) \rangle \cdot \langle M \rangle) \langle D_l(p_1) \rangle + \cdots + (\langle S_r(p_l) \rangle \cdot \langle M \rangle) \langle D_l(p_l) \rangle. \quad (7.1)$$

Corollary 48. *With respect to the basis $\{\langle D_l(p_1) \rangle, \dots, \langle D_l(p_l) \rangle\}$, the boundary map $\partial : H_{\lambda+1}(W \cup W', W) \rightarrow H_\lambda(W, \partial_- W)$ for the triple $\partial_- W \subset W \subset W \cup W'$ is a linear map, i.e. ∂ is represented by the $k \times l$ matrix A whose entries are $a_{ij} = \langle S_r(p_i) \rangle \cdot \langle S_l(p'_j) \rangle$.*

$$\begin{array}{ccccc}
& & & & H_\lambda(S_i(p'_j)) \\
& & & & \downarrow i_* \\
H_{\lambda+1}(W \cup W', W) & \longrightarrow & H_{\lambda+1}(W', \partial_+ W) & \longrightarrow & H_\lambda(\partial_+ W) \\
& \searrow & & & \downarrow \\
& & & & H_\lambda(W) \\
& \searrow \partial & & & \downarrow \\
& & & & H_\lambda(W, \partial_- W)
\end{array}$$

By Theorem 38 any cobordism $c = (W; \partial_- W, \partial_+ W)$ decomposes into a composition of c_λ , *i.e.*

$$c = c_0 c_1 \cdots c_m$$

such that each c_λ has a Morse function f with critical points of index λ on the fiber $f^{-1}(a)$ for some $a \in \mathbb{R}$. Or equivalently, we can think of a Morse function $f : W \rightarrow \mathbb{R}$ with the property called *self-indexing*, *i.e.* $f(p^\lambda) = \lambda \in \mathbb{Z}_{\geq 0}$ for every critical point p^λ of index λ . Let $W_k = f^{-1}([-\frac{1}{2}, k + \frac{1}{2}])$ and $V_{k+} = f^{-1}(k + \frac{1}{2})$ for a non-negative integer k . So setting $W_{-1} = \partial_- W$, we have the sequence

$$\partial_- W = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_m = W.$$

Now let $C_\lambda = H_\lambda(W_\lambda, W_{\lambda-1})$ and $\partial : C_\lambda \rightarrow C_{\lambda-1}$ the boundary homomorphism defined in Corollary 48, and consider the long exact sequence of the triple $(W_{\lambda+1}, W_\lambda, W_{\lambda-2})$

$$\cdots \rightarrow H_{\lambda+1}(W_{\lambda+1}, W_\lambda) \rightarrow H_\lambda(W_\lambda, W_{\lambda-2}) \rightarrow H_\lambda(W_{\lambda+1}, W_{\lambda-2}) \rightarrow 0$$

and another exact sequence of the triple $(W_\lambda, W_{\lambda-1}, W_{\lambda-2})$

$$0 \rightarrow H_\lambda(W_\lambda, W_{\lambda-2}) \rightarrow H_\lambda(W_\lambda, W_{\lambda-1}) \rightarrow H_{\lambda-1}(W_{\lambda-1}, W_{\lambda-2}) \rightarrow \cdots.$$

Observe that there is a commutative diagram of pairs of spaces given by two triples given above.

Then define a λ -th homology group by

$$H_\lambda(W_\lambda, W_{\lambda-1}) = \ker \partial / \text{im } \partial, \tag{7.2}$$

and it is well known that $C_* = (C_\lambda, \partial)$ is a chain complex and $H_\lambda(C_*) = H_\lambda(W, \partial_- W)$ for every λ with $0 \leq \lambda \leq m$.

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
C_{\lambda+1} = H_{\lambda+1}(W_{\lambda+1}, W_\lambda) & \longrightarrow & H_\lambda(W_\lambda, W_{\lambda-2}) & \longrightarrow & H_\lambda(W_{\lambda+1}, W_{\lambda-2}) & \longrightarrow & 0 \\
& \searrow \partial & \downarrow & & & & \\
& & C_\lambda = H_\lambda(W_\lambda, W_{\lambda-1}) & & & & \\
& & \downarrow \partial & & & & \\
& & C_{\lambda-1} = H_{\lambda-1}(W_{\lambda-1}, W_{\lambda-2}) & & & &
\end{array}$$

7.2 Cancellation of Critical Points in the Middle Dimensions

Construct a Morse function \tilde{f} as follows. Using Theorem 32 perturb the Morse function f so that \tilde{f} coincides with f outside a small open neighborhood U_1 of p_1 , $\tilde{f}(p_1) > f(p_1)$, and \tilde{f} has the same critical points p_1, \dots, p_k and the gradient-like vector field as f . Now let $t \in \mathbb{R}$ such that $\tilde{f}(p_1) > t > f(p_1)$, and let $V = \tilde{f}^{-1}(t)$.

The left-hand $(\lambda-1)$ -sphere $S_l(p_1)$ and the right-hand $(m-\lambda-1)$ -spheres $S_r(p_i)$ for $2 \leq i \leq k$ in V are disjoint. Let $a \in S_l(p_1)$ and $b \in S_r(p_2)$. Then since W is connected, V is connected and hence there exists some embedding $\iota : (0, 3) \rightarrow V$ such that the image $\iota(0, 3)$ intersects with $S_l(p_1)$ and $S_r(p_2)$ exactly once, transversely, in $a = \iota(1)$ and $b = \iota(2)$ respectively, and such that $\iota(0, 3) \cap S_r(p_i) = \emptyset$ for $i = 3, \dots, k$. Then extend the embedding ι as follows. See Milnor [6] for details.

Lemma 49. *There is an embedding $\tilde{\iota} : (0, 3) \times \mathbb{R}^{\lambda-1} \times \mathbb{R}^{m-\lambda-1} \rightarrow V$ such that*

$$(1) \quad \tilde{\iota}(t, 0, 0) = \iota(t) \quad \forall t \in (0, 3),$$

$$(2) \quad \tilde{\iota}^{-1}(S_l(p_1)) = 1 \times \mathbb{R}^{\lambda-1} \times 0, \quad \tilde{\iota}^{-1}(S_r(p_2)) = 2 \times 0 \times \mathbb{R}^{m-\lambda-1}, \quad \text{and}$$

(3) $\text{im } \tilde{\iota} \cap S_r(p_i) = \emptyset$ for $i = 3, \dots, k$. Moreover ι maps $1 \times \mathbb{R}^{\lambda-1} \times 0$ into $S_l(p_1)$ with positive orientation and $\iota((0, 3) \times \mathbb{R}^{\lambda-1} \times 0)$ intersects $S_r(p_2)$ at $\iota(2, 0, 0) = b$ with intersection number $+1$.

Theorem 50 (Basis Theorem). *On a cobordism $(W; \partial_-W, \partial_+W)$ of dimension m , let $f : W \rightarrow \mathbb{R}$ be a Morse function with k critical points p_1, \dots, p_k such that*

$$f(p_1) = \dots = f(p_k), \quad (7.3)$$

and let X be a gradient-like vector field for f . Suppose that all critical points are of the same index λ with $2 \leq \lambda \leq m - 2$. Moreover suppose that W is connected. Then for every basis for $H_\lambda(W, \partial_-W)$ there exists another Morse function f' and another gradient-like vector field X' for f' with the following properties:

- (1) f' and X' both coincide with f and X respectively in some open neighborhood U of $\partial_-W \cup \partial_+W$,
- (2) f' has the same critical points p_1, \dots, p_k with $f'(p_1) = \dots = f'(p_k)$, and
- (3) the suitably oriented left-hand disks $D_l(p_1), \dots, D_l(p_k)$ forms a basis for $H_\lambda(W, \partial_-W)$.

Proof. Let $\{b_1, \dots, b_k\}$ be a basis for $H_\lambda(W, \partial_-W) \cong \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$ (k summands), where each b_j is homologous to the left-hand disk $D_l(p_j)$ with some fixed orientation. Then let $\nu(D_r(p_1)), \dots, \nu(D_r(p_k))$ denote the normal bundles so that

$$\langle D_r(p_i) \rangle \cdot \langle D_l(p_j) \rangle = \delta_{ij}, \quad (7.4)$$

where δ_{ij} denotes the Kronecker delta sign. So for every λ -disk D embedded in W such that $D \subset \partial_-W$ its representation is given by

$$\alpha_1 b_1 + \dots + \alpha_k b_k \in H_\lambda(W, \partial_-W) \quad (7.5)$$

for some integers $\alpha_1, \dots, \alpha_k$. So D is homologous to $\alpha_1 D_l(p_1) + \dots + \alpha_k D_l(p_k)$ and

$$\langle D_r(p_i) \rangle \cdot \langle D \rangle = \langle D_r(p_i) \rangle \cdot (\alpha_1 \langle D_l(p_1) \rangle + \dots + \alpha_k \langle D_l(p_k) \rangle) \quad (7.6)$$

$$= \alpha_1 \langle D_r(p_i) \rangle \cdot \langle D_l(p_1) \rangle + \dots + \alpha_k \langle D_r(p_i) \rangle \cdot \langle D_l(p_k) \rangle \quad (7.7)$$

$$= \sum_{j=1}^k \alpha_j \langle D_r(p_i) \rangle \cdot \langle D_l(p_j) \rangle \quad (7.8)$$

$$= \sum_{j=1}^k \alpha_j \delta_{ij} \quad (7.9)$$

$$= \alpha_i. \quad (7.10)$$

So $D = (\langle D_r(p_i) \rangle \cdot \langle D \rangle) b_1 + \dots + (\langle D_r(p_k) \rangle \cdot \langle D \rangle) b_k$.

Now construct a Morse function f' and a gradient-like vector field X' such that $D'_l(p_1)$, $D_l(p_2)$, \dots , and $D_l(p_k)$ are the new left-hand disks with

$$\langle D_r(p_1) \rangle \cdot \langle D'_l(p_1) \rangle = 1, \quad \langle D_r(p_2) \rangle \cdot \langle D'_l(p_1) \rangle = 1, \quad (7.11)$$

and

$$\langle D_r(p_i) \rangle \cdot \langle D'_l(p_1) \rangle = 0 \quad (7.12)$$

for $i = 3, \dots, k$. This implies that $\{b_1 + b_2, b_2, \dots, b_k\}$ is the new basis for $H_\lambda(W, \partial_- W)$. In order to verify this elementary row operation of the basis, we will need to construct an isotopy as follows.

For some fixed $\delta > 0$ let $\alpha : \mathbb{R} \rightarrow [1, \frac{3}{2}]$ be a smooth function such that

$$\alpha(x) = \begin{cases} \frac{9}{4} & \text{if } x \leq \delta \\ 1 & \text{if } x \geq 2\delta. \end{cases} \quad (7.13)$$

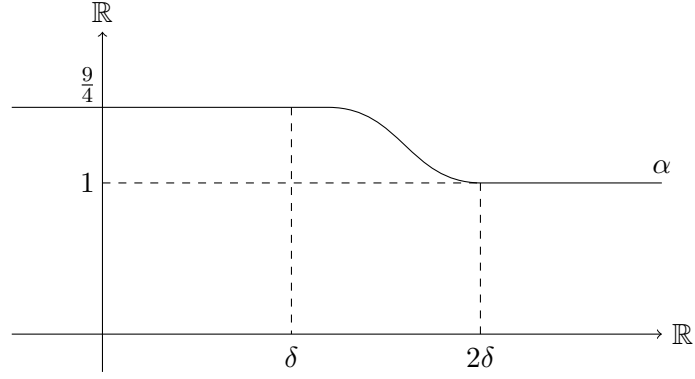


Figure 7.1: The graph of α .

Then construct a smooth isotopy $H_t : (0, 3) \times \mathbb{R}^{\lambda-1} \times \mathbb{R}^{m-\lambda-1} \rightarrow (0, 3) \times \mathbb{R}^{\lambda-1} \times \mathbb{R}^{m-\lambda-1}$ such that

- (1) $H_t = \text{id}$ outside some compact set, where $0 \leq t \leq 1$.
- (2) $H_t(1, x, 0) = (t\alpha(\|x\|^2) + (1-t), x, 0)$ for $x \in \mathbb{R}^{\lambda-1}$.

Now define a new isotopy F_t of V by

$$F_t(v) = \begin{cases} \tilde{t} \circ H_t \circ \tilde{t}^{-1}(v) & \text{for } v \in \text{im } \tilde{t} \\ v & \text{otherwise} \end{cases} \quad (7.14)$$

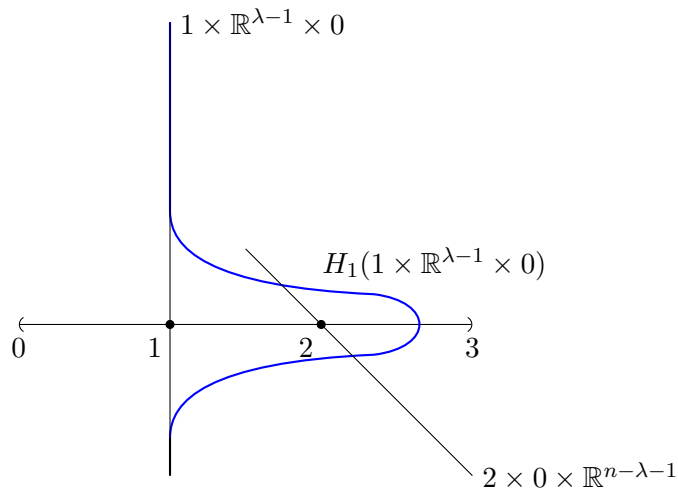
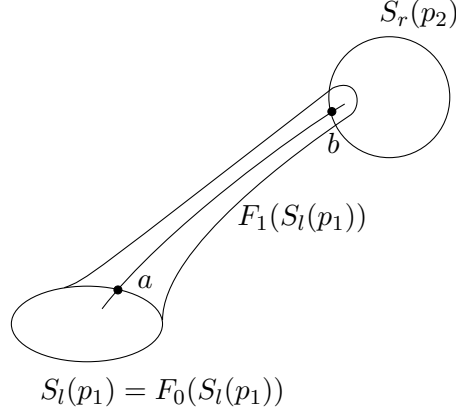


Figure 7.2: The graph of H_t .

Figure 7.3: Deformation by the isotopy F_t .

□

Theorem 51. *For a cobordism $(W; \partial_-W, \partial_+W)$ of dimension $m \geq 6$, let f be a Morse function without any critical points of indices $0, 1, m-1$, or m . Suppose that W , ∂_-W , and ∂_+W are all simply connected. Moreover suppose that $H_*(W, \partial_-W)$ is trivial. Then $(W; \partial_-W, \partial_+W)$ is a product cobordism.*

Proof. Let c denote the cobordism $(W; \partial_-W, \partial_+W)$. By the Final Rearrangement Theorem it follows that c decomposes into the factors $c = c_2 c_3 \cdots c_{m-2}$, and there exists a Morse function f on c such that each restriction $f|_{c_\lambda}$ contains all critical points of index λ with the same critical value. Then consider the sequence of the following chain complex

$$C_{m-2} \xrightarrow{\partial} C_{m-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_{\lambda+1} \xrightarrow{\partial} C_\lambda \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_2. \quad (7.15)$$

For each λ choose a basis $\{z_1^{\lambda+1}, \dots, z_{k_{\lambda+1}}^{\lambda+1}\}$ for the cycle $\ker \partial$ of $\partial : C_{\lambda+1} \rightarrow C_\lambda$. Because $H_*(W, \partial_-W)$ is trivial by assumption, the sequence given above is exact and there exist k_λ elements $b_1^{\lambda+1}, \dots$, and $b_{k_\lambda}^{\lambda+1}$ in C_λ such that $\partial(b_i^{\lambda+1}) = z_i^\lambda$ in $C_{\lambda+1}$ for $i = 1, \dots, k_\lambda$. Thus $\{z_1^{\lambda+1}, \dots, z_{k_{\lambda+1}}^{\lambda+1}, b_1^{\lambda+1}, \dots, b_{k_\lambda}^{\lambda+1}\}$ is a basis for $C_{\lambda+1}$.

Since $2 \leq \lambda \leq \lambda+1 \leq m-2$ by assumption, by Basis Theorem there exist a Morse function f' and gradient-like vector field X' defined on c such that the left-hand disks contained in c_λ and $c_{\lambda+1}$ are the bases for C_λ and $C_{\lambda+1}$, respectively.

Now let $p \in c_\lambda$ and $q \in c_{\lambda+1}$ be the critical points that correspond to the basis elements z_1^λ and $b_1^{\lambda+1}$. Perturb f' in open neighborhoods U_p of p and U_q of q so that

$$f(p_{i_\lambda}^\lambda) < f(p) < f(q) < f(p_{i_{\lambda+1}}^{\lambda+1}), \quad (7.16)$$

where $p_{i_\lambda}^\lambda$ and $p_{i_{\lambda+1}}^{\lambda+1}$ are the critical points of indices λ and $\lambda + 1$, respectively. Then $c_\lambda c_{\lambda+1}$ factors into $c'_\lambda c_p c_q c'_{\lambda+1}$, where both c_p and c_q are elementary cobordisms containing critical points p and q , respectively. Let $t \in \mathbb{R}$ such that $f(p) < t < f(q)$ and let $W' = c_p c_q$. Then consider the fiber $V = f^{-1}(t)$. Observe that all W' , $\partial_- W'$, and $\partial_+ W'$ are simply connected. Since $z_1^\lambda = \partial(b_1^\lambda)$ by construction of ∂ , $S_r(p)$ and $S_l(q)$ have the intersection number ± 1 , *i.e.* $\langle S_r(p) \rangle \cdot \langle S_l(q) \rangle = \pm 1$. By the Second Cancellation Theorem, the critical points p and q in $c_p c_q$ are eliminated and $c_p c_q$ is a product cobordism. Repeating this process and hence all critical points in the cobordism c are eliminated. Thus $(W; \partial_- W, \partial_+ W)$ is a product cobordism. \square

Chapter 8

Cancellation of Critical Points of index 0 and 1

8.1 Index 0 Cancellation

We first eliminate all critical points of 0 from a cobordism. The idea is to find out a transverse intersection between right-hand and left-hand spheres of critical points of indices 0 and 1, respectively, so that the First Cancellation Theorem applies.

Theorem 52. *Suppose that $H_0(W, \partial_- W) = 0$. Then the critical points of index 0 are cancelled against the same number critical points of index 1.*

Proof. Consider the homology groups with coefficients from $\mathbb{Z}_2 = \{0, 1\}$. Since $H_0(W, \partial_- W) = 0$, the boundary homomorphism $\partial : H_1(W_1, W_0) \rightarrow H_0(W_0, \partial_- W)$ is surjective. Observe that ∂ is given by the matrix whose (i, j) -th entries are

$$\langle S_r(p_i^0) \rangle \cdot \langle S_l(p_j^1) \rangle \pmod{2}, \quad (8.1)$$

where p_i^0 and p_j^1 denote critical points of indices 0 and 1, respectively. Observe that every right-hand $(m-1)$ -sphere $S_r(p_i^0)$ has at least one left-hand 0-sphere $S_l(p_j^1)$ such that $\langle S_r(p_i^0) \rangle \cdot \langle S_l(p_j^1) \rangle \not\equiv 0$ under modulo 2 because ∂ is surjective. $S_r(p_i^0) \cap S_l(p_j^1)$ cannot contain more than two points and thus it consists of an odd number of points. Therefore $S_r(p_i^0)$ and $S_l(p_j^1)$ intersects transversely in a single point and the First Cancellation Theorem applies. \square

8.2 Index 1 Cancellation

We will need the two lemmas introduced below to prove Theorem 56, which eliminates critical points of index 1.

Lemma 53. *For $0 \leq \lambda < m$, there exists a smooth function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ such that*

- (1) $f(x_1, \dots, x_m) = x_1$ outside of a compact set, and
- (2) f has exactly two non-degenerate critical points p^λ and $p^{\lambda+1}$ of indices λ and $\lambda + 1$ respectively with $f(p^\lambda) < f(p^{\lambda+1})$.

Proof. Identify \mathbb{R}^n with $\mathbb{R} \times \mathbb{R}^\lambda \times \mathbb{R}^{m-\lambda-1}$ and let (x, y, z) be a point in the product. For simplicity let y^2 and z^2 denote $\|y\|^2$ and $\|z\|^2$, respectively. Moreover let $s : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with compact support such that $s(x) + x$ has exactly two non-degenerate critical points a_1 and a_2 .

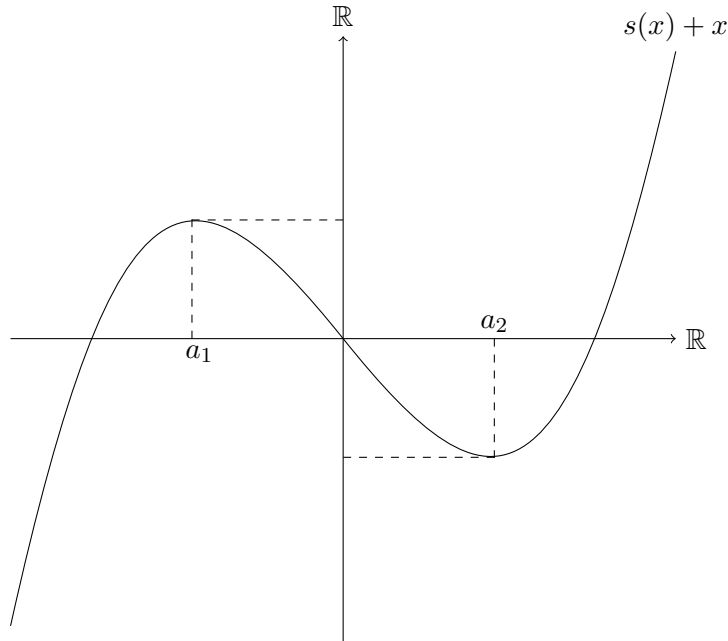


Figure 8.1: The graph of $s(x) + x$.

Next construct smooth functions $\alpha, \beta, \gamma : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ with compact supports so that

- (1) $\alpha(x) = 1$ if $|x| \leq 1$,
- (2) $|\alpha'(x)| < \frac{1}{\sup|s(x)|}$ for all x ,
- (3) $\beta(x) = 1$ if $\alpha(x) \neq 0$,
- (4) $\gamma(x) = 1$ if $s'(x) \neq 0$, and
- (5) $|\gamma'(x)| < \frac{1}{\sup(x\beta(x))}$.

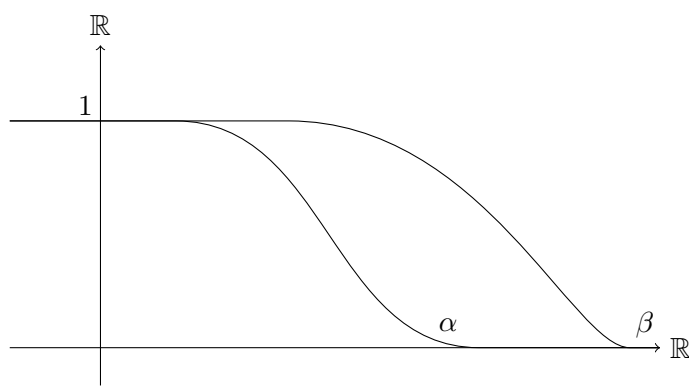


Figure 8.2: The graphs of α and β .

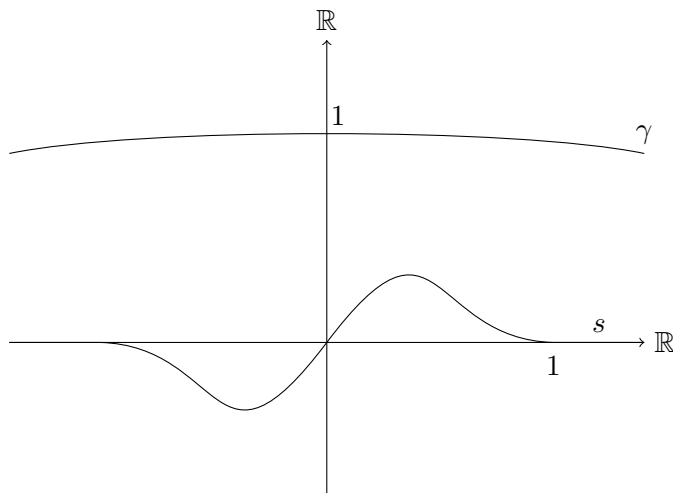


Figure 8.3: The graphs of s and γ .

Define $f : \mathbb{R}^m \rightarrow \mathbb{R}$ by $f(x, y, z) = x + s(x)\alpha(y^2 + z^2) + (-y^2 + z^2)\gamma(x)\beta(-y^2 + z^2)$. Observe by construction of f that

- (1) $f(x) - x$ has compact support,
- (2) On the open interval such that $\alpha = 1$ and $\gamma = 1$, f corresponds to $s(x) + x$.

Observe also that

$$\frac{\partial f}{\partial x} = 1 + s'(x)\alpha(y^2 + z^2) + (-y^2 + z^2)\gamma'(x)\beta(-y^2 + z^2) \neq 0 \quad (8.2)$$

if $s'(x) \neq 0$ or $\gamma'(x) \neq 0$. Observe also that the gradient of f vanishes only if $y = 0$ and $z = 0$. In this case $\alpha = 1$, and therefore $f(x)$ reduces to $s(x) + x$. \square

Lemma 54. *If S_r^{m-2} is a right-hand sphere in V_{1+} , then there exists a 1-sphere embedded in V_{1+} such that it intersects with S_r^{m-2} transversely but does not meet with any other right-hand spheres in V_{+1} .*

Proof. Choose a sufficiently imbedded 1-disk $D^1 \subset V_{1+}$ so that D^1 intersects with S_r^{m-2} at the midpoint q of D^1 and that D^1 does not intersect with any other right-spheres in V_{1+} . Then translate the end points a and b of D^1 left along the integral curves φ and φ' of the gradient-like vector field X to $\varphi(a)$ and $\varphi'(b)$ in V_{0+} . $V_{0+} = \partial_- W$ is connected and of dimension $m - 1 \leq 2$, there exists a smooth path γ so that γ joins $\varphi(a)$ and $\varphi'(b)$, and that γ does not intersect with any left-hand 0-spheres in V_{0+} . Then $\varphi^{-1} \circ \gamma$ is a smooth path joining a and b in V_{0+} and avoids all right-hand spheres. Note that $\dim V_{0+} = m - 1 \geq 3$, and by Theorem define a smooth function $g : S^1 \rightarrow V_{1+}$ by the following properties:

- (1) $g^{-1}(q) = a \in S^1$ and g embeds smoothly a closed set A containing a onto some neighborhood of q in D .
- (2) $g(S^1 - a)$ does not intersect with any right-hand $(m - 2)$ -spheres.

This completes the proof of theorem. \square

Theorem 55. *Let M and N be smooth manifolds of dimension m and n such that $n \leq 2m + 3$. If two smooth embeddings i and j of M into N are homotopic, then i and j are smoothly isotopic.*

See Whitney [14] for the proof of the theorem.

Theorem 56. *Suppose that W and ∂_-W are simply connected, and $m \geq 5$. Moreover suppose that $(W; \partial_-W, \partial_+W)$ has no critical points of index 0. Then for each critical point p_1 of index 1 there exist a pair of critical points q_2 of auxiliary index 2 and p_3 of index 3 such that p_1 is cancelled against q_2 .*

Proof. Note first that V_{2+} is simply connected, i.e. $\pi_1(V_{2+}) = \pi_1(W) = 0$. Note also from generalization of Theorem 30 that

$$\mathcal{D}_r^{m-1} \cup \mathcal{D}_r^{m-2} \cup V_{2+} \cup \mathcal{D}_l^3 \cup \mathcal{D}_l^4 \cdots \cup \mathcal{D}_l^m, \quad (8.3)$$

where \mathcal{D}_r^λ and \mathcal{D}_l^λ denote the collection of λ -disks attached to V_{2+} , is a deformation retract of W .

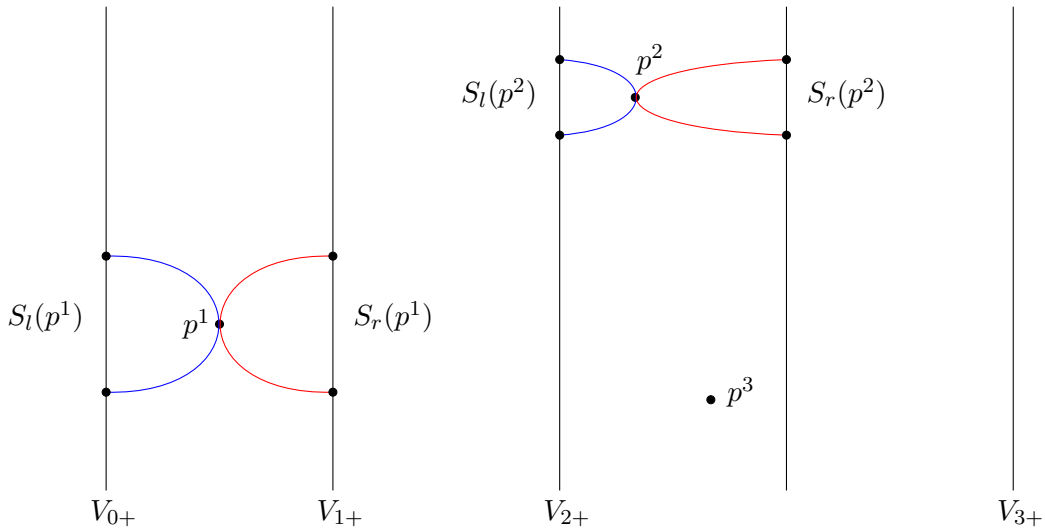


Figure 8.4: Auxiliary critical points p^2 and p^3 .

Now for every critical point p^1 construct an 1-sphere $S(p^1)$ embedded in V_{1+} by Lemma 54. Moreover by Theorems 36 the gradient-like vector field X can be adjusted so that $S(p^1)$ does not

intersect with any left-hand spheres in V_{1+} . So we can translate $S(p^1)$ right to V_{2+} . Extend a collar neighborhood to the right of V_2 . Apply the *Implicit Function Theorem* to choose an embedded open set $U \subset \mathbb{R}^m$ and some coordinate system (x_1, \dots, x_m) such that $f(x_1, \dots, x_m)|_U = x_m$. Then by Lemma 53 construct another Morse function \tilde{f} such that \tilde{f} coincides with f outside U with extra non-degenerate critical points p^2 and p^3 of indices 2 and 3, respectively.

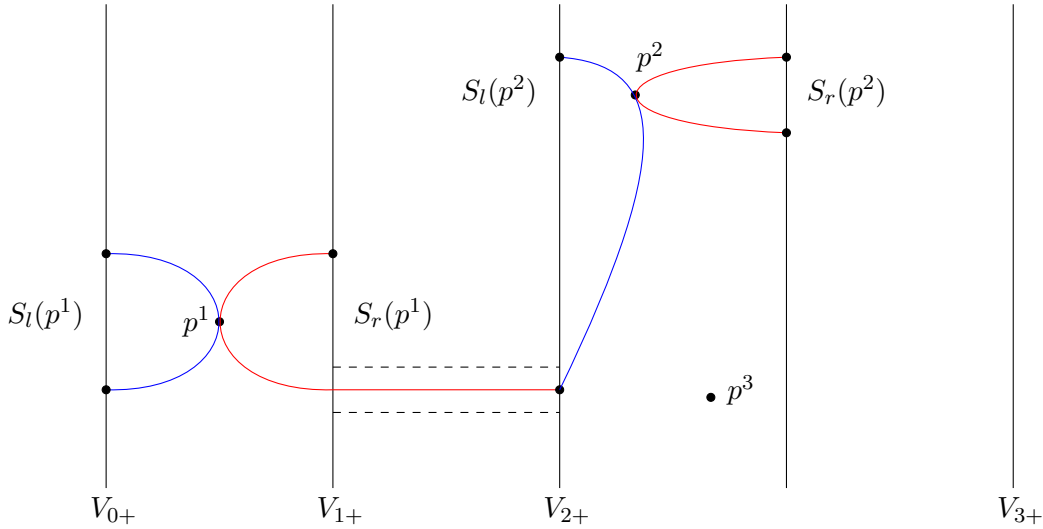


Figure 8.5: Auxiliary critical points p^2 and p^3 .

Denote by $S_l(p^2)$ the left-hand sphere of p^2 in V_{2+} , and construct a smooth isotopy $h : V_{2+} \rightarrow V_{2+}$ such that $h(S_l(p^2)) = S(p^1)$. Then adjust X so that $S_l(p^2)$ and $S(p^1)$ coincide. Observe by construction that $S_l(p^2)$ intersects with $S(p)$ in a single point transversely. Therefore the *First Cancellation Theorem* applies and two critical points p^1 and p^2 are eliminated. Then perturb f so that p^3 is on the fiber $f^{-1}(3) \subset V_{3+}$, and this completes the proof of the theorem. \square

Chapter 9

The h-Cobordism Theorem

In Chapter 5, 6, and 7, we studied how to eliminate pairs of critical points of consecutive indices. Moreover Chapter 8 made us cancel critical points of indices 0 and 1. Finally we should claim that the following main theorem so called the *h-Cobordism Theorem* holds true.

Theorem 57 (The h-Cobordism Theorem). *Let $(W; \partial_-W, \partial_+W)$ be a cobordism such that*

- (1) *All W , ∂_-W , and ∂_+W are simply connected.*
- (2) *$H_*(W, V) = 0$.*
- (3) *$\dim W \geq 6$.*

Then W is diffeomorphic to $\partial_-W \times [0, 1]$ and thus a product cobordism.

Proof. Let a Morse function $f : W \rightarrow \mathbb{R}$ be self-indexing, i.e. $f(p^\lambda) = \lambda \in \mathbb{Z}$ for each non-degenerate critical point $p^\lambda \in W$. By Theorem 52 and 56 all critical points of indices 0 and 1 are eliminated. Then consider the function $-f$. It is easy to observe that $-f$ is also a Morse function that contains the exactly same critical points with opposite indices. In other words a critical point p^λ of index λ of f is of index $m - \lambda$ of $-f$. So all critical points of indices m and $m - 1$ are also cancelled. Finally apply Theorem 51 and other critical points are all eliminated. \square

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