Numerical Investigations of Photoevaporative Disks: Processes Relevant to Planet and Regular Satellite Formation

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Numerical Investigations of Photoevaporative Disks: processes relevant to planet and regular satellite formation

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A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
Department of Astrophysical & Planetary Sciences
2014
This thesis entitled:
Numerical Investigations of Photoevaporative Disks: processes relevant to planet and regular satellite formation
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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
Mitchell, Tyler Robert (Ph.D., Astrophysical & Planetary Sciences)

Numerical Investigations of Photoevaporative Disks: processes relevant to planet and regular satellite formation

Thesis directed by Research Associate Dr. Glen R. Stewart

Traditional models of disks around young planets and stars make a number of simplifying assumptions. These include the use of ad hoc radial temperature profiles, or isothermal disks. Another common assumption is in regard to the treatment of the outer boundary, which is allowed to expand to infinity, or neglected completely. There has also been a lack of time-dependent viscous models that include the affects of photoevaporation and/or ongoing accretion. We alleviate many of these issues by adapting numerical methods for solving propagating phase change problems to astrophysical disks in a completely novel way. These models are all viscous, time-dependent models that include a self-consistent treatment of mass loss via photoevaporation at the disk outer edge. In the case of circumplanetary disks, they also include continued accretion from the solar nebula.

I present investigations of disks around young planets and stars, made using a variety of numerical models. The investigations are primarily focused on how disk structure and evolution affect the growth and migration of growing satellite and planetary embryos. Another focus is to assess what, if any, processes are responsible for angular momentum transport in circumplanetary disks. I present detailed descriptions of these models as well as the results of applying these models to both the solar nebula and to disks around giant planets, in which regular satellites formed.

Photoevaporation can substantially truncate disks and has a similar level of affect on disk evolution and morphology as variations in the viscosity parameter, $\alpha$. All of the solar nebula models were truncated, yet none of them match the steep radial surface density profile inferred from the compact configuration of the giant planets in the Nice model. Furthermore, photoevaporation has the ability to remove gas and dissipate disks on very short timescales. Despite their evolving nature, we find that giant planets and satellites can form in the evolving disks produced by these models.
We conclude that steady-state circumstellar disk models are lacking and the traditional way of treating the outer boundary needs to be reexamined.

With regard to circumplanetary disks, magnetorotational instability is not a viable mechanism for angular momentum transport in the detailed 1+1D model presented here. However, temperature and density dependent opacities produce non-power law radial profiles. The deviations from power-law cause there to be increases in the radial entropy gradient. This allows for the generation of baroclinic instabilities that can be sustained and amplify. These results help alleviate the long-standing problem of angular momentum transport in circumplanetary disks and differentiate between competing models of circumplanetary disk structure.
Dedication

This thesis is dedicated to my parents, without whom nothing would be possible. Their unwavering support has always been a blessing. One for which I will forever be grateful.
Acknowledgements

First and foremost, I would like to acknowledge my advisor, Dr. Glen R Stewart, who is partially supported by NASA’s Planetary Geology and Geophysics Program. His assistance with the development and analysis of this thesis was invaluable. I would also like to acknowledge NASA’s Earth and Space Science Fellowship program who have provided financial support for this thesis with fellowship #10-Planet10F-0057. Along these lines, I would like to thank Dr. Larry Esposito and the Cassini Project for their generous financial support as well. I would like to thank Dr. Steve Desch for his insights and illuminating discussions. I must also acknowledge Dr. Neal Turner for providing me with tables of diffusion rates that made my disk analysis possible. Finally, a big thanks to Monte Lunacek and everyone at CU Boulder’s research computing group. This work utilized the Janus supercomputer, which is supported by the National Science Foundation (award number CNS-0821794) and the University of Colorado Boulder. The Janus supercomputer is a joint effort of the University of Colorado Boulder, the University of Colorado Denver and the National Center for Atmospheric Research. Lastly, I owe a debt of gratitude to Michelle Hamernick and all of the wonderful women in the APS department office.
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1.1 The History of Nebular Theory

The nebular hypothesis as we know it today was first formulated in the late 18th century. Immanuel Kant and Pierre Simon, Marquis de Laplace independently applied the newly developed, Newtonian physics to explain the origin of our solar system. They both sought to explain four observed properties of the solar system. (1) The planets and their known satellites orbit in the equatorial plane of the Sun. (2) Generally, planets rotate in the same direction that the Sun does, and that this is in the same direction as the aforementioned orbits. (3) The nearly circular orbit of the planets. (4) The large eccentricities of cometary orbits. They succinctly explained these features by invoking a spinning remnant disk of gas, left over after the Sun formed, from which all of the other solar system bodies formed.

The field took a large step forward in the 1970’s and 1980’s, when renewed interest in solar system formation sparked many important innovations in the field. The two most notable of which, in terms of theory, were the development of the $\alpha$-viscosity [101] model and the development of the minimum mass solar nebula [119, 39]. With regard to numerical modeling, many innovations were made by Cameron & Pine [16], who modeled the collapse of a spherical cloud of dust into a flat disk in which the vertical and radial forces balanced in each respective direction.

It was understood early on that the kinematic viscosity was not due to the molecular viscosity of the gas in the disk. Estimates of the amount of angular transport that results from molecular viscosity was orders-of-magnitude too small to account for the accretion rates observed in young,
low-mass T Tauri stars.

The poorly understood, complex nature of viscosity led Shakura & Sunyaev to parameterize the kinematic viscosity using their now-famous $\alpha$-viscosity parameter $[101]$. They formulated a simple model wherein the viscosity is proportional to the product of the scale height and the sound speed, $c_s$, in the disk. They then bounded the relation by multiplying these two quantities by $\alpha$, a non-dimensional constant between $0 - 1$. It is from this constant that the method received its name. The $\alpha$-viscosity scales the kinematic viscosity, $\nu$, such that,

$$\nu = \alpha c_s H.$$  \hspace{1cm} (1.1)

The scale height, $H$, can be found using the relation, $H = c_s \Omega^{-1}$. In protoplanetary disks, the value for $\alpha$ has been constrained to be from $10^{-4} - 10^{-2}$ by modeling the accretion onto young, T-Tauri stars and matching the observed luminosities of these objects $[112, 13]$.

The minimum mass solar nebula is an empirical result that came about by trying to understand the distribution of mass in the primordial disk of gas and dust that surrounded the young Sun, the solar nebula. The minimum mass solar nebula was developed by Weidenschilling $[119]$, and similarly by Hayashi $[39]$, by augmenting the planets’ estimated heavy element component to solar composition. The necessary mass was then distributed in annuli centered on the planets’ current semi-major axes and a single power law was fit to the derived surface density constraints.

$$\Sigma(r) \approx 1700 \left( \frac{r}{1\text{AU}} \right)^{-3/2} \text{ g cm}^2,$$  \hspace{1cm} (1.2)

where $\Sigma(r)$ is the surface mass density at a radius $r$. Because the minimum mass solar nebula is strictly an empirical result, further work was required to connect this result to the physical properties in the protoplanetary disk. An example of this connection is that the mass contained in each region should be inversely proportional to the vertically averaged viscosity at that location in the disk.

A giant step forward occurred when Balbus & Hawley showed that, given a sufficient level of ionization and hence coupling with the magnetic field, a magnetorotational instability (MRI)
can occur and persist in a differentially rotating disk [10]. They also showed that under ideal conditions the MRI could sustain a level of turbulence sufficient to produce the observed accretion rates. However, the conditions in circumstellar, and especially in circumplanetary disks, has been shown to be far from ideal. Large dead zones, where the ionization is too low for the gas to couple effectively with the magnetic field, have been shown to exist in large portions of both types of disk [31, 66].

In recent years, circumplanetary disks have begun to receive more attention than ever before. Despite the surge in circumplanetary disk research, detailed modeling of their structure and evolution is still in its infancy. Although it is compelling to make direct comparisons with circumstellar disks, circumplanetary disks are fundamentally different for a number of reasons: 1) The environments in which they form and evolve are markedly different; including the ongoing accretion of material from the solar nebula, external irradiation from the central star and their location in gaps formed by tidal interactions of the host planet with the solar nebula. 2) The structure of these disks is also different from their circumstellar analogs because the combined effect of infall from the solar nebula and the removal of mass at the outer boundary is thought to produce steady state disks that persist for $\sim 10^5$ years. 3) Despite the low surface densities generally assumed for circumplanetary disks, the smaller scales result in cooler and denser disk conditions. The small scales also result in dynamical timescales that are much shorter than those in circumstellar disks.

The co-planar, prograde orbits of regular satellites of giant planets, as well as the low-eccentricity and low-inclination of these orbits, indicate that they formed in situ in circumplanetary disks. These disks are thought to be fed by the solar nebula in which they are embedded. The small radial extent of their regular satellite systems indicates that they are truncated by some mechanism, presumably photoevaporation or tidal torques. Although photoevaporation has been widely investigated in the context of circumstellar disks, investigations into its role in shaping circumplanetary disks is new.

The similarity of the satellite systems of giant planets, with Jupiter in particular, to the solar system has contributed to the development of formation theories that employ a “minimum
mass subnebula” (MMSN) [67], analogous to the “minimum mass solar nebula” [119]. Summing the mass of heavy elements contained within the satellites and augmenting it to solar composition determines the mass of this subnebula. However, this leads to a variety of problems when applied to the regular satellite systems of Jupiter and Saturn. In an effort to rectify these problems, two competing theories of circumplanetary disk structure.

For one, the resulting nebula is too warm to condense ice, yet, with the exception of Io, condensed volatiles are known to be a major component of the Galilean satellites. In such a dense nebula, the effect of type I migration would be strong enough to migrate Callisto into Jupiter before it had sufficient time to grow to its current size [115]. Even if type I migration did not operate, Callisto would accrete on such a short timescale it would melt and become differentiated, which is not supported by Galileo observations [7]. Recent gravity measurements by the Cassini spacecraft indicate that Titan is also partially differentiated which would imply a long accretion timescale for the regular satellites of Saturn as well [43, 12].

The giant planets in our solar system have similar ratios of the total mass contained within the satellite systems to the mass of the host planets ($M_{\text{satellites}} \sim 10^{-4} M_{\text{planet}}$) [19]. In general, the mass of a satellite increases with increasing distance from the planet, reaches a maximum, and then decreases with distance. A striking difference is seen in the mass distributions of the satellite systems of Jupiter and Saturn. Jupiter has four large moons of relatively the same size, whereas Saturn has one large moon that contains almost all of the mass in the entire satellite system. The similarity of these systems indicate that they formed from similar processes, yet these processes have acted in such a way as to produce very different outcomes.

Numerical simulations show that Jupiter is large enough to open a complete gap in the solar nebula cutting off infall onto the Jovian subnebula on a $10^2 - 10^4$ yr timescale [99]. In contrast, Saturn is too small to have opened a complete gap in the solar nebula and infall onto the Saturnian subnebula would have halted on the $\sim 10^6$ yr timescale for the dissipation of the solar nebula as a whole. Although the assumption that Jupiter opened a gap and Saturn did not depends on the assumed viscosity and scale height of the protoplanetary disk, the critical mass for opening a gap is
larger in the outer regions of the disk. It is a reasonable assumption using standard disk parameters and further supported by the current masses of Jupiter and Saturn. For these reasons, Sasaki et al. [99] assume in their models that Jupiter opened a gap in the solar nebula and Saturn did not.

Sasaki et al. [99] have recently published results from a suite of simulations in which the growth and dynamical evolution of protosatellite embryos was modeled. They seek to explain why the Jovian regular satellite system consists of four nearly equal-mass satellites whereas the Saturnian system contains one large satellite. They propose that the Jovian satellite system may have been “frozen” in place when Jupiter grew sufficiently large that a gap was opened in the solar nebula. In the Saturnian system, where only an incomplete gap may have formed, the longer shutoff timescale for material infalling from the solar nebula would have allowed its satellites to continue to dynamically evolve. The typical end result is significant depletion of solids in the inner disk and the retention of a single large satellite in the outer disk that is similar to Titan.

1.2 Photoevaporation

Star formation in giant molecular clouds implies that the majority of stars in our galaxy were born in clusters rather than in isolation. The same is most likely true for the Sun. It is estimated that 90% of stars born in clusters are born into rich clusters, with 100 or more members of mass in excess of 50 M⊙[60]. The Sun likely formed in a cluster of 1000 – 10000 stars, which in turn implies an average external FUV flux that is a few thousand times the non-cluster background, but with a standard deviation that is comparable to the average value [28, 1]. The compelling reasons for the Sun being formed in a cluster of this size are: (1) The abundance of the short-lived radioactive nuclide ⁶⁰Fe derived from meteoric samples cannot be produced by spallation in the solar system and can only be explained by an extrasolar nucleosynthetic origin. The capture of this extrasolar ⁶⁰Fe into the early solar system is more likely if the Sun’s birth cluster contained a sufficient number of massive supernovae [113, 1]. (2) Sedna’s orbit requires a stellar encounter at a distance of less than or equal to 400 AU [49, 80, 15]. A smaller birth cluster would not yield such a close stellar encounter during its lifetime and cannot provide the necessary amount of ⁶⁰Fe
whereas a larger birth cluster would give such a large FUV flux that it would photoevaporate the disk before the giant planets can form [1].

Observations of low mass young stellar objects near the Trapezium cluster in Orion show disks silhouetted against the background nebula, so called “proplyds” (PRoto PLanetary DiskS) [11, 74]. Initial modeling of these disks invoked photoevaporation as a result of ionizing radiation from an embedded central source. Observations of the same cluster shows that mass loss must be due to neutral flows generated at the disks’ surfaces. The outflows then become ionized at some distance from the base of the flow [47]. A natural explanation for the observed neutral outflows is that the disks are heated by far ultraviolet (FUV) radiation with energies in the range of $6 - 13.6$ eV. The neutral outflows then expand and are ionized by extreme ultraviolet (EUV) radiation with $E > 13.6$ eV. Models of external irradiation by nearby, massive stars were successful in explaining the observations of proplyds in Orion [47].

The presence of early type stars in the vicinity of the solar nebula would have exposed it to FUV radiation. FUV radiation would have heated the periphery of the disk. Gas heated to sufficient temperatures would then have become unbound from the disk. The gravitational radius, $r_g$, is defined as the radius at which the sound speed of the heated gas, $a_s$, equals the escape speed from the system. Gas beyond the gravitational radius will escape from the system.

$$r_g = \frac{GM_s\langle\mu\rangle}{kT} \tag{1.3}$$

Here $G$ is the gravitational constant, $M_s$ is the mass of the central star, $k$ is the Boltzmann constant and $T$ is the temperature of the super-heated atmosphere, or what we will refer to as the envelope temperature, $T_{env}$. The gravitational radius is the canonical radius beyond which gas heated to a temperature $T_{env}$ will escape from the disk. In actuality, gas can escape from the disk at radii substantially smaller than $r_g$.

In reality, the heated atmosphere has a depth dependent temperature and the heating and resultant outflow are complicated processes. Because of these complexities, it is useful to employ a simplified model with an isothermal atmosphere. Consider a disk irradiated and heated by external
FUV radiation. Depending on the strength of the FUV flux, the heated gas will reach temperatures in the range $100 \, \text{K} < T < 3000 \, \text{K}$ [2]. As the gas heats, it expands generating a neutral outflow. The expanding outflow begins subsonically but becomes supersonic by the time it reaches the gravitational radius. This outflow is generally isotropic, but the majority of mass loss is dominated by mass loss from the outer edge of the disk. The isotropic, neutral outflow serves to shield the disk from EUV radiation that would ionize the disk and heat it to $\approx 10,000 \, \text{K}$. A diagram of a disk of radius $r_d$ around a star of mass $M_*$ illuminated by an external source and the subsequent resultant outflow was presented by Adams et al. [2] and is reproduced in Figure 1.1. This outflow is, in effect, a super heated atmosphere that can be characterized by a single envelope temperature, $T_{\text{env}}$.

Until recently, only mass loss beyond the gravitational radius has been considered. Analytic arguments and numerical experiments have shown that gas can be removed down to a radius of $(0.2 \cdot r_g)$ [62, 2]. Although the heated gas at these radii is prevented from directly escaping from the disk, there exists an atmosphere which can extend beyond $r_g$. This atmosphere can be photoevaporated away and a resultant outflow will develop. The outflow will behave very much like a Parker wind. As mass is lost from the out-flowing atmosphere, it is replenished from the disk and mass is effectively lost at $r < r_g$. Assuming a temperature of $T_{\text{env}} = 1000 \, \text{K}$ for the heated atmosphere of the solar nebula, $r_g \sim 100 \, \text{AU}$. At $(0.2 \cdot r_g) = 20 \, \text{AU}$, the formation of planets would be effected by the photoevaporative outflow. For a more thorough discussion of subcritical mass loss see Adams et al. [2] and Hollenbach et al. [41].

In an effort to calculate the probability that a solar type star would experience sufficient photoevaporation from external irradiation that it would effect giant planet formation, Adams et al. [2] investigated the mass loss rates from circumstellar disks due to external FUV radiation. They studied the previously unexplored subcritical regime, where the outer radius of the disk, $r_d$, is smaller than the gravitational radius. Adams et al. [2] used a photodissociation region (PDR) code to determine the depth-dependent temperature of the gas based on the optical depth, density and FUV flux. Their PDR code also included 46 chemical species and 222 chemical reactions.
Figure 1.1: (Figure taken from Adams et al. [2].) Schematic of a disk with radius $r_d$ around a star of mass $M_*$, illuminated by the FUV (and perhaps EUV) radiation from nearby stars of greater mass. The disk is inclined so that the top and edge are exposed. The disk scale height is $H_d$ at the outer radius $r_d$. In the subcritical regime, where $r_d < r_g$, the bulk of the photoevaporation flow (the radial flow) originates from the disk edge, which marks the inner boundary. The flow begins subsonically at $r_d$, with speed $v_d$ and density $n_d$. The flow accelerates to the sound speed at $r_s$ (the sonic point), which lies inside the critical escape radius $r_g$. Beyond the sonic point, the flow attains a terminal speed and the density falls roughly as $n \propto r^{-2}$. Although some material is lost off the top and bottom faces of the disk (the vertical flow), its contribution to the mass-loss rate is secondary to that from the edges. Nonetheless, the polar regions are not evacuated, the star is fully enveloped by the circumstellar material, and the incoming FUV radiation will be attenuated in all directions.
The chemistry is critical for determining the cooling rate of the gas. For a given a radiation field strength $G_0$, disk size $r_d$, disk temperature $T_d$, and stellar mass $M_*$, an iterative procedure was used to determine the density at the base of the flow $n_d$ as well as the flow speed at the inner boundary. These two quantities then determine the mass-loss rate.

In order to understand the results of their detailed numerical model, Adams et al. [2] developed simple analytical models for the photoevaporative mass loss rates for cases in which $r_d$, the location of the outer edge of the disk, is both inside and outside the gravitational radius. These models are characterized by a single temperature $T_{\text{env}}$. Although the strength of the FUV radiation field $G_0$ does not directly enter into these equations it specifies the envelope temperature which determines a unique sound speed, $a_s$ for the isothermal atmosphere.

\[
\dot{M} = C_0 N_C \langle \mu \rangle a_s r_g \left( \frac{r_g}{r_d} \right) \exp \left( -\frac{r_g}{2 r_d} \right) \quad r_d < r_g \quad (1.4)
\]

\[
\dot{M} = 4 \pi \mathcal{F}(\mu) \sigma_{\text{FUV}}^{-1} a_s r_d \quad r_d > r_g \quad (1.5)
\]

The first equation is for subcritical disks, and the second equation is for supercritical disks. $N_C$ is the critical surface density of the flow and $\sigma_{\text{FUV}}$ is the cross section for dust grains interacting with FUV radiation. The dust optical depth is given by $\tau_{\text{FUV}} = \sigma_{\text{FUV}} \cdot N_H$. For an optical depth of order unity, $\sigma_{\text{FUV}}^{-1} \approx N_H$, where $N_H$ is evaluated at the critical density $N_C \sim 10^{-21} \, \text{cm}^{-2}$.

The factor $\mathcal{F}$ is the fraction of the solid angle subtended by the flow and is $\sim 1$ because the flow from the disk surface and edge merge at roughly $r_d - 2r_d$ creating a nearly spherically symmetric outflow.

The factor $C_0$, in the second equation, is a constant of order unity that is used by Adams et al. [2] to match their numerical and analytical solutions. Using a value of $C_0 = 5.8$, I have matched the analytic solutions for sub- and supercritical disks. I have matched them at a radius of $r_d/r_g = 0.25$. Johnstone et al. [47] consider it likely that the mechanism for supercritical disks can operate inwards of $r/r_g \sim 0.5$ and estimate that it may even operate when $r/r_g \sim 0.2$. Therefore a matching of these two solutions at $r/r_g \sim 0.25$ seems justified. The value of $C_0 = 5.8$ also allows
for a smooth transition between sub- and supercritical regimes.

The FUV flux $G_0$ heats a column density, $N_c$, defined as $N_c = \int_{r_{\text{in}}}^{\infty} n(r) dr$. Then, set this equal to $\sigma_{\text{FUV}}^{-1}$, the inverse of the cross-section for dust grains interacting with FUV photons. $G_0$ is a dimensionless quantity expressed in terms of the Habing field (1 Habing field = $1.2 \times 10^{-4}$ erg cm$^{-2}$ s$^{-1}$ sr$^{-1}$) and where $G_0 = 1.7$ Habing fields for the local interstellar FUV field [107].

Using the equations of mass loss from Adams et al. [2], Guillot & Hueso [35] were able to explain the over abundance of noble gases in Jupiter’s atmosphere as a result of the loss of hydrogen and helium through the photoevaporative process. They used a simplified 1D evolutionary disk model, in which the vertical structure was averaged, to determine the enrichment of the solar nebula in heavy elements. The noble gases are assumed to be trapped within solids in the cold outer disk while hydrogen and helium are removed by photoevaporation. As solids migrated inward due to gas drag, they are then released as gases in the inner disk and incorporated into the giant planets. This scenario differs from previous models that required the noble gases to be accreted while trapped in solid planetesimals. Removing the requirement that noble gases be delivered in solids further loosens constraints on nebular temperatures in the outer solar system. Guillot & Hueso [35] modeled photoevaporation from both EUV generated by an embedded early Sun and by ambient FUV generated by neighboring cluster members. They found that the scenario involving EUV was problematic due to the long timescale involved in the removal of nebular gas and the high constant value of EUV flux from an “unidentified mechanism”. In the FUV scenario, they found that the observed enrichment in Jupiter’s atmosphere could be matched with a wide range of values in their free parameters. They also found that for disk atmospheres heated to $T_{\text{env}} > 100$ K, the disks dissipated on Myr timescales, consistent with observations.

The mass loss rates from irradiation by the central star are generally lower than those due to external irradiation, but depending on the assumed FUV flux they can be of comparable magnitude. The FUV environment of any given star in a young cluster is highly uncertain. Due to the incomplete sampling of the stellar initial mass function, the variety of cluster sizes and the $1/r^2$ dependence of flux on the radial distance from the center of the cluster, where the most massive
stars reside, the standard deviation of predicted FUV fields can be comparable to their mean values [93, 1]. When the highly variable nature of young stellar emission is also taken into account, the ratios of internal to external FUV flux becomes even more uncertain. Given these uncertainties, it is generally not until late times that the internal source begins to affect the inner disk; opening an inner gap and rapidly dispersing the system [22, 34]. For these reasons we have neglected UV irradiation from the Sun in our study of the solar nebula.

Many models of circumstellar disks assume static disk models. This may be true for early stages of the solar nebula where continued accretion form the Sun’s birth cloud can replenish the circumstellar disk, but at later times this assumption is a poor one. One of our goals was to investigate the morphology and temporal evolution of the solar nebula as it is eroded by photoevaporation in a system that is closed off from the replenishment from the solar nebula. We wanted to know not only how steep of a profile could be produced in such a model, but how an adjustment of our key parameters, $\alpha$ and $T_{\text{env}}$ affect this evolution.

We have chosen to model irradiation from an external source because of the success of Johnstone et al. [47] in modeling proplyds in Orion as well as the need for a truncated disk discovered by Desch [26]. Adams et al. [2] have shown that any incident EUV is attenuated very rapidly in the disk atmosphere at several disk radii. It photoionizes a portion of the disk atmosphere but is unable to penetrate deeply and effect the disk itself. EUV radiation can perhaps effect disk evolution at late stages when it could help to clear the gas on very short timescales. Therefore, our research has been focused on FUV radiation and as of yet has neglected the effects of EUV radiation.

It is fairly certain that photoevaporative mass loss would have affected the solar nebula. We assume that it played a similar role in shaping the circumplanetary subnebulae that surrounded the giant planets. High-mass stars in the birth cluster would have irradiated them in a similar fashion as the solar nebula. Despite this contribution, it has been shown that significant radiation from the star can be scattered into the gap formed by Jupiter to heat it and affect its evolution [109].
1.3 The Stefan Problem

Stefan problems include, but are not limited to, melting/freezing and heat ablation. These problems are often solved on a finite domain, where one of the boundaries is the location at which the phase change is occurring. The phase-change boundary is allowed to move according to the rate at which the phase change occurs. These problems require the simultaneous solution of both a parabolic PDE, known as the heat equation, and a first order ODE that governs the rate at which the phase-change boundary moves. In order to self-consistently model the diffusive transport of mass in the disk, as well as the photoevaporation at the outer edge, we adapted the Stefan problem from the material sciences. In our application of the formalism of the Stefan problem, we substitute mass for heat as the transported quantity. The phase-change is the transition between bound and unbound gas, caused by photoevaporation, at the outer edge of the disk.

A general form for the Stefan problem, in one dimension, is the following system of two coupled differential equations, where \( u \) is the quantity being evolved and \( x \) and \( t \) are the spatial and temporal coordinates. In the classical heat equation this quantity corresponds to temperature. In my system, the mass surface density will fulfill this role. \( K(x, u) \) is the diffusivity constant and \( S(x, t, u, \frac{\partial u}{\partial x}) \) is a source term. The two coupled equations in their most general form are,

\[
\frac{\partial u}{\partial t} = K(x, u) \frac{\partial^2 u}{\partial x^2} + S(x, t, u, \frac{\partial u}{\partial x}) \tag{1.6}
\]

and

\[
\frac{ds}{dt} = f(u, x, t, \frac{\partial u}{\partial x}, etc., \ldots). \tag{1.7}
\]

As with the general heat equation, the Stefan problem becomes increasingly difficult when the diffusion coefficient isn’t constant, but depends on the local conditions. When this occurs, the equation becomes nonlinear and sophisticated methods are required to solve it.

I will now briefly show that the viscous disk equation matches the form of the Stefan problem. Under the thin disk approximation, the continuity equation for a small annulus of \( \Delta r \) located at
radius \( r \), as \( \Delta r \to 0 \) for a disk with surface density, \( \Sigma(r,t) \), and viscosity, \( \nu \), is:

\[
r \frac{\partial \Sigma}{\partial t} = \frac{\partial}{\partial r}(r \Sigma v_r) \tag{1.8}
\]

Where \( v_r \) is the net radial velocity of material transported via viscous processes in the disk. In this same disk the conservation of angular momentum is described by:

\[
\frac{\partial}{\partial t}(\Sigma r^2 \Omega) + \frac{1}{r} \frac{\partial}{\partial r}(\Sigma r^3 \Omega v_r) = \frac{1}{r} \frac{\partial}{\partial r} \left( \nu \Sigma r^3 \frac{d\Omega}{dr} \right), \tag{1.9}
\]

where \( \Omega \) is the angular Keplerian angular velocity at that radius, \( \Omega = \left( \frac{GM_P}{r^3} \right)^{1/2} \). Now, define the variables \( h \) and \( g \); the specific angular momentum and viscous couple (torque) [37].

\[
h = r^2 \Omega = (GM_P r)^{1/2} \tag{1.10}
\]

and

\[
g = -2\pi r \Sigma \nu r^2 \frac{d\Omega}{dr} = 3\pi \nu h \Sigma. \tag{1.11}
\]

By expanding the angular momentum equation in terms of \( h \) and \( g \), it can be shown that they obey the following relation:

\[
\frac{\partial g}{\partial h} = -\dot{M}_{\text{disk}}, \tag{1.12}
\]

where \( \dot{M} \) is the outward mass flux in the disk.

Using the above relation the continuity equation becomes,

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{2\pi r} \frac{\partial \dot{M}_{\text{disk}}}{\partial h} \tag{1.13}
\]

which reduces to

\[
\frac{\partial g}{\partial t} = \frac{3\nu}{4} \left( \frac{GM_P}{h} \right)^2 \frac{\partial^2 g}{\partial h^2}. \tag{1.14}
\]

This is analogous to the equation governing the temporal evolution of the mass surface density of a thin accretion disk in its standard form.

\[
\frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[ r^{1/2} \frac{\partial}{\partial r} \left( \nu \Sigma r^{1/2} \right) \right] \tag{1.15}
\]
A source term must be added onto the right-hand side of Equation 1.14, because the circumplanetary disk models include the addition of material from the disk. This will account for accretion from the solar nebula onto the circumplanetary disk.

\[
\frac{\partial g}{\partial t} - \frac{3\nu}{4} \left( \frac{GM_P}{h} \right)^2 \frac{\partial^2 g}{\partial h^2} = 3\pi \nu h \Gamma
\]

(1.16)

where \(\Gamma\) is the infall rate of surface density per unit time and has units of \(\text{[g cm}^2 \text{s}^{-1}]\). One can now see that this equation is of the same form as Equation 1.6. Given another equation for the temporal evolution of the outer boundary, Equation 1.7, methods developed to solve Stefan problems will apply. I present these equations in Sections 3.1 and 7.1.1.

1.4 Thesis Organization

A brief history of nebular theory, photoevaporation and the formalism of the Stefan problem were all presented in this chapter. This sets the stage for the rest of this thesis. The goal of which is to model the circumstellar and circumplanetary disks from which planets and regular satellites form. The purpose of this modeling is to evaluate the affects of photoevaporation on viscously accreting disks, gain a better understanding of the viability of various viscous processes, and assess the growth and migration of planetesimals and growing satellites in the resultant models.

The remainder of this thesis is organized in the following manner. Chapter 2 gives a detailed description of recent developments regarding the solar nebula and the growth of giant planets that have influenced the work presented here. A general description of the 1D model and the numerical technique used is given in Chapter 3. The results of the 1D model, applied to the solar nebula and the growth of giant planets, is presented in Chapter 4. A description of the current state of circumplanetary disk structure, and regular satellite formation, research is presented in Chapter 5. The results of our modeling of circumplanetary disks, using the 1D model, is presented in Chapter 6.

As a follow up to our initial investigation, a 1+1D disk model was developed. A detailed description of this model, as well as the Keller box method used to solve it, is presented in Chapter
7. This improved model was then used to model the circum-Jovian subnebula at the tail end of accretion, when the last generation of regular satellites was though to form. The results of these calculations, as well as an analysis to assess the viability of magnetorotational and baroclinic instabilities, is presented in Chapter 8.

Lastly, a review of the key finding presented here are summarized in Chapter 9. Chapters 2, 3 and 4 were published in Mitchell & Stewart [78]. Chapters 5 and 6 were published in Mitchell & Stewart [79]. Chapters 7 and 8 are new, and are yet to be presented elsewhere.
Chapter 2

The Solar Nebula and Growth of the Giant Planets

The recent discovery of hundreds of extrasolar planets has sparked renewed interest in solar system formation. The diversity of these systems has driven researchers to re-investigate the importance of planetary migration as driving factor in the architecture of planetary, and satellite, systems. In light of these recent developments, the Nice model, in which the giant planets formed in a much more compact configuration than we see today, was developed. It was inferred from this that the minimum mass solar nebula was steeper and, and more massive, that previously thought. We describe each of these developments in detail below.

2.1 Nice Model

Since its inception, many shortcomings have been recognized in the minimum mass solar nebula. Observations of intermediate age (2.5 – 30 Myr) clusters indicate a mean disk lifetime of \(\sim 6 \) Myr; consistent with gas dissipation timescales for circumstellar disks of \(\sim 1 – 10\) Myr \([36, 40]\). It must be emphasized that the minimum mass solar nebula, by definition, contains the minimum amount of mass necessary to build the planets at their current semi-major axes. Therefore, any proposed solar nebula more massive than the minimum mass solar nebula is allowable. Despite this, the canonical minimum mass solar nebula has been used for decades as the initial conditions for both disk evolution and planet formation simulations. It is difficult to grow the cores of the giant planets within the time constraint of the gas dissipation timescale given the low mass surface densities predicted for the minimum mass solar nebula.
The recent development of the Nice model has shed new light on the process of giant planet formation in the solar system [33, 81, 108]. The model assumes the giant planets formed in a much more compact configuration than they now reside. Simulations suggest that the giant planets migrated, subsequent to gas dissipation, through scattering interactions with small planetesimals exterior to Neptune. These interactions caused the inward migration of Jupiter and the outward migration of the other giant planets. In their simulations chaotic behavior in the outer solar system is initiated by the crossing of Jupiter and Saturn through their 2:1 mean motion resonance (MMR). The chaotic behavior causes the rapid outward migration of Uranus and Neptune. Uranus and Neptune switch places in about half of their simulations which nicely explains why Neptune is more massive than Uranus [108]. The mass and location of this planetesimal disk play critical roles in the outcome of their simulations.

Observations of diverse extrasolar planetary systems in recent years make it evident that planetary migration plays a significant role in the planet formation process. Two types of migration have been widely investigated, commonly known as type I and type II migration [114]. Type I migration occurs when spiral density waves are launched in a disk from an orbiting body. The density waves exert unbalanced inward and outward torques on the orbiting body. The sum of these torques is generally negative, causing a body undergoing type I migration to spiral inward on a relatively short timescale compared to the viscous timescale of the disk. Type II migration occurs when a body grows sufficiently large to open a gap in the disk. The body is then drawn along and its orbit decays on the viscous timescale of the disk. Type I migration effects smaller bodies and type II migration effects larger bodies.

2.2 The Desch Model

The known inadequacies of the MMSN model and the development of the Nice model led Desch [26] to re-investigate the primordial solar nebula. Within the context of the Nice model, Desch [26] has developed an new MMSN with Jupiter located at 5.45 AU, Saturn at 8.18 AU, Neptune at 11.5 AU and Uranus at 14.2 AU. He assumes that Uranus and Neptune switched places
during the chaotic period following the crossing of Jupiter and Saturn through their 2:1 MMR. These four mass constraints were combined with mass constraints from chondrules, the asteroid belt and the disk of primordial planetesimals laying outside the orbit of Uranus to develop a single power law profile for the primordial solar nebula.

\[ \Sigma(r) = 343 \left( \frac{f_p}{0.5} \right)^{-1} \left( \frac{r}{10 \text{AU}} \right)^{-2.168} \text{g cm}^{-2}, \]  

(2.1)

where \( f_p \) is the fraction of the mass of condensable solids in planetesimals.

The surface density profile of Equation 2.1 is not only more massive but much steeper than the canonical MMSN (\( \Sigma \propto r^{-3/2} \)). In a steady state thin \( \alpha \)-disk, the surface density follows the relation \( \Sigma \propto 1/\nu \). Using these assumptions, along with the \( \alpha \)-viscosity prescription, would imply that \( \Sigma \propto T(r)^{-1}r^{-3/2} \). A surface density profile consistent with the MMSN would therefore imply a constant temperature profile throughout the disk. It is generally thought that disks are flared due to irradiation from the central star. It was shown that a flared disk with an internal radial temperature distribution of \( T(r) = 150 r_{\text{AU}}^{-3/7} \) can produce a spectral energy distribution that is consistent with observations of T Tauri stars [20]. This has lead many researchers to use a temperature profile of the form \( T(r) \propto r^{-1/2} \). Again, using the same assumptions about the disk, this implies that the surface density profile should be \( \Sigma \propto r^{-1} \).

Desch [26] first investigated whether a viscously spreading accretion disk of the type studied by Lynden-Bell & Pringle [68] would adequately match his surface density constraints as well as timing constraints on planet formation. Although the surface density constraints could be matched with a viscously spreading disk, such a disk evolves too rapidly to satisfy the constraints for planet formation. At 10 AU, planetesimals should have formed by 0.03 Myr, but at this time he finds densities that are an order of magnitude lower than those implied by the augmented mass of Saturn. The viscously spreading disk also has trouble matching the density profile of the new MMSN (Equation (2.1)) at small radii for early times and at large radii for late times. Desch [26] concludes that the various constraints on the surface density and timing of the solar nebula are
best matched with a steady state profile.

Following up on his conclusion, Desch [26] re-examined the equations for viscously evolving steady state disks. For a viscous disk with surface density $\Sigma$ and viscosity $\nu$. Conservation of mass yields

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r} \frac{\partial \dot{M}}{\partial r}$$

(2.2)

and the conservation of angular momentum yields

$$\frac{\partial}{\partial t} (\Sigma r^2 \Omega) = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\dot{M}}{2\pi} r^2 \Omega + r^3 \Sigma \nu \frac{d\Omega}{dr} \right),$$

(2.3)

where $\Omega$ is the angular frequency. $\dot{M}$ is the net flow of mass through an annulus of the disk with a negative mass flux corresponding to inward accretion. Integrating Equation (2.3) results in

$$-\frac{\dot{M}}{2\pi} r^2 \Omega - r^3 \Sigma \nu \frac{d\Omega}{dr} = \text{const.}$$

(2.4)

The constant is evaluated by choosing an appropriate boundary condition. This is where Desch [26] diverges from previous derivations. In the past, the equation has been solved by assuming the dominant boundary is the inner boundary and the evolution of the disk is governed by the mass flux across the inner boundary while the outer boundary is allowed to expand indefinitely such that angular momentum is conserved.

Assuming a temperature profile of the form $T(r) \propto r^{-q}$ and a viscosity of the form $\nu = \alpha c_s/\Omega$, the standard $\alpha$-viscosity prescription, he finds a general solution for the surface density profile,

$$\Sigma(r) = \frac{\dot{M}}{3\pi \nu (r_0)} \left( \frac{r}{r_0} \right)^{-3/2-\alpha} \frac{\dot{m}}{3\pi \nu (r_0)} \left( \frac{r}{r_0} \right)^{-2}.$$

(2.5)

Instead of using boundary conditions to determine the constants of integration $\dot{m}$ and $r_0$, Desch [26] matches this solution to his derived surface density profile (Equation (2.1)). For a steady state disk to be consistent with a surface density profile $\Sigma(r) \propto r^{-2.168}$ the mass flux $\dot{M} < 0$. This implies that the solar nebula must be a decretion disk, with significant mass loss from the outer disk edge, rather than an accretion disk. Unlike a standard accretion disk which only requires sufficient outflow for the removal of angular momentum, a decretion disk is characterized
by an significant outward net flow of mass. The outward mass flow is driven by photoevaporation which requires a constant supply of mass from the inner solar system to replenish mass lost at the outer edges. Depending on local conditions, mass loss from the outer edge can dominate over accretion onto the central star necessitating an outward mass transport throughout the majority of the disk. It is also implied that the disk must be truncated at an outer edge, $r_d$. Truncation can be naturally explained by invoking photoevaporation by an external source.

Although Desch’s MMSN is an improvement over the original MMSN, it still suffers its own limitations. For one, Desch [26] is unable to constrain his disk’s outer radius to better than within $30 - 100$ AU. A recent paper by Crida et al. [23] also points out that the large surface densities present in Desch’s model would cause substantial migration of the giant planets. Using the hydro code FARGO in its 2D1D version, their models show Jupiter quickly falling into the regime of type III runaway migration and rapidly falling into the Sun. Even though Desch [26] was able to construct a steady state solution, a steady state profile is an oversimplification and is inconsistent with a disk eroded by photoevaporation if the planetary accretion timescale is a significant fraction of the disk lifetime.

Desch [26] derived a steady state decretion disk that matched surface density constraints derived by assuming the four giant planets formed in the compact configuration of the Nice model. We extend this line of investigation by modeling a time-dependent disk that experiences mass loss by photoevaporation from an external FUV source. Radiation from the central star certainly also played some role in disk evolution. Recent numerical simulations which combine photoevaporation from the central star with viscous evolution show that the majority of mass loss is dominated by loss from the outer regions of the disk, where less tightly bound material can easily escape from the system [34].
Chapter 3

1D Viscous Disk Model with Photoevaporation

The one-dimensional disk model presented here uses the simplifying assumption that the viscosity is proportional to the radius. This reduces the nonlinear PDE that governs the temporal evolution of the surface mass density to a diffusion equation with a constant coefficient that is much easier to solve.

Using the $\alpha$-viscosity prescription, and the assumption that the viscosity is proportional to the radius of the disk,

$$\nu = \nu_0 (r/R_0).$$

(3.1)

The linear dependence of viscosity on radius in our model implies that the temperature profile in my disk of the form $T_{\text{disk}}(r) = T_0 (r/R_0)^{-1/2}$. In all of our simulations, $T_0 = 150$ K and $R_0 = 1$AU. This is consistent with earlier works, and in particular with that of Desch [26]. He uses a temperature profile from Chiang et al. [20] that is of the form $T_{\text{disk}}(r) = 150 \cdot (r/R_0)^{-0.429}$. K.

Evaluating the temperature profile above at $r = 1$ AU, $T_{\text{disk}}(1 \text{ AU}) = 150$ K and using the alpha prescription for viscosity enables the viscosity scaling constant to be determined.

$$\nu_0 = \alpha \sqrt{\frac{kT_{\text{disk}}}{\langle \mu \rangle}}$$

(3.2)

where $k$ is the Boltzmann constant and $\alpha$ is the viscosity parameter (1.1).

By substituting in the functional form of viscosity, and adding a source term, the viscous
disk equation can be written as

\[
\frac{\partial g}{\partial t} - \frac{3}{4} \frac{\nu_0 GM \partial^2 g}{R_0} \frac{\partial h^2}{R_0} = \frac{3\pi \nu_0}{GM R_0} h^3 \Gamma(h, t),
\]

(3.3)

where \( \Gamma(h, t) \) is the infall rate in \([\text{g cm}^{-2}\text{s}^{-1}]\) as a function of specific angular momentum.

In order to non-dimensionalize these equations, the following three transformations for \( t, h \) and \( g \) are used.

\[
\tilde{t} = \frac{3\nu_0 t}{4R_0^2}
\]

(3.4)

\[
\tilde{h} = \frac{h}{(GM R_0)^{1/2}}
\]

(3.5)

\[
\tilde{g} = \frac{g}{3\sqrt{2\pi N_\mu \nu_0 \sqrt{GM R_0}}}
\]

(3.6)

By substituting these non-dimensionalized quantities, a non-dimensional form of viscous disk equation, as well as the equations necessary to self-consistently solve for the locations of the disk’s outer boundary, was derived.

\[
\frac{\partial \tilde{g}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{g}}{\partial \tilde{h}^2} + \frac{4\pi R_0^2}{3\sqrt{2\pi N_\mu \nu_0}} \tilde{h}^3 \Gamma(\tilde{h}, \tilde{t}),
\]

(3.7)

and

\[
\frac{d\tilde{h}_d}{dt} = -\frac{1}{\tilde{h}_d^3} \frac{\partial \tilde{g}}{\partial \tilde{h}_d} - \frac{C_0 a_s r_g}{3\sqrt{2\pi \nu_0}} \frac{r_g}{h_d^5} \frac{r_g}{R_0} \exp \left[ -\frac{r_g}{2R_0 h_d^2} \right]
\]

(3.8)

or

\[
\frac{d\tilde{h}_d}{d\tilde{t}} = -\frac{1}{\tilde{h}_d^3} \frac{\partial \tilde{g}}{\partial \tilde{h}_d} - \frac{\sqrt{8\pi}}{3 \nu_0} \frac{a_s R_0}{\tilde{h}_d}.
\]

(3.9)

### 3.1 Variable Space Grid Method

To solve the Stefan problem, Kutluay et al. [59] adopt a numerical method with a variable space grid (VSG) first proposed by Murray & Landis [85]. The VSG method employs a fixed number of grid points with a variable grid size at each time step. This method involves solving two coupled differential equations at each time step, one for the location of the outer boundary and
one for the diffusive evolution of the disk. Once the location of the outer boundary is found the abscissa is re-scaled and the diffusive evolution calculated.

I will now briefly describe the VSG method. For a non-dimensional diffusion equation,

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

(3.10)

with the boundary condition

$$\frac{ds(t)}{dt} = f(x, U)$$

(3.11)

where \(s(t)\) is the location of the boundary at time \(t\). Equation (3.10) can be differentiated with respect to time and, for the \(i\)th grid point,

$$\frac{\partial U}{\partial t} \bigg|_i = \left[ \frac{\partial U}{\partial x} \right]_i \frac{dx}{dt} + \left[ \frac{\partial U}{\partial t} \right]_t$$

(3.12)

assuming the grid point \(x_i\) is moved by

$$\frac{dx_i}{dt} = \frac{x_i}{s(t)} \cdot \frac{ds}{dt}$$

(3.13)

In order to self-consistently solve for the location of the outer edge of the disk, the viscous evolution equation must simultaneously be solved with another differential equation governing the motion of the outer boundary. The equation that governs the location of the outer boundary comes in two flavors. One is for a supercritical disk, where the outer boundary, \(r_d\), is greater than the gravitational radius, \(r_g\), and another for subcritical disks, where \(r_d < r_g\).

The derivation is begun using mass conservation at the outer boundary.

$$\dot{M}_{\text{boundary}} = \dot{M}_{\text{viscous spreading}} - \dot{M}_{\text{photo-evaporation}}$$

(3.14)

For both flavors of disk, mass loss rates from Adams et al., Equations 1.4 and 1.5 are used to determine the motion of the outer boundary in the following manner.

$$\dot{M}_{\text{photo-evaporation}} = \begin{cases} 
4\pi F(\mu)\sigma_{\text{FUV}}^{-1}a_\nu r_d & r_d < r_g \\
C_0N_c(\mu)a_\nu r_g \left( \frac{r_g}{r_d} \right) \exp\left( -\frac{r_g}{2r_d} \right) & r_d > r_g 
\end{cases}$$

(3.15)
The first equation is for supercritical disks, and the second equation is for subcritical disks. The mass flux due to viscous spreading is found by taking the derivative of the viscous couple with respect to the specific angular momentum, or

$$\dot{M}_{\text{viscous spreading}} = -\frac{\partial g}{\partial h}.$$  (3.16)

The mass flux due to the motion of the boundary is the mass contained within an annulus at the disk’s outer edge multiplied by the time rate of change of that edge. It is,

$$\dot{M}_{\text{boundary motion}} = 2\pi r_d \Sigma_d (dr_d/dt).$$  (3.17)

The equations governing the location of the outer boundary was derived by substituting these three expressions into Equation 3.14. Depending on the location of the outer boundary, I will either be in the subcritical regime,

$$\frac{dh_d}{dt} = -\frac{\sqrt{8\pi}}{3h_d^3} \frac{\partial g}{\partial h_d} - \frac{C_0 a_s \mu_g^2}{\sqrt{2\pi} 3\nu_0 R_0 h_d^5} \exp\left[\frac{-r_g}{2R_0 h_d^2}\right] \quad r < r_g,$$  (3.18)

or the supercritical regime,

$$\frac{dh_d}{dt} = -\frac{\sqrt{8\pi}}{3h_d^3} \frac{\partial g}{\partial h_d} + \frac{\sqrt{2\pi} a_s R_0^3}{3\nu_0 h_d^3} \quad r > r_g.$$  (3.19)

By substituting Equation (3.13) into Equation (3.12), the heat equation (Equation (3.10)) can be reformulated as

$$\frac{\partial U}{\partial t} = \frac{x_i}{s(t)} \frac{ds(t)}{dt} \frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}.$$  (3.20)

In our particular case, the term $\frac{\partial^2 U}{\partial x^2}$ can be replaced by the RHS of Equation 10. Since they have both been non-dimensionalized, this is a valid substitution. This results in,

$$\frac{\partial \tilde{g}}{\partial t} = \frac{\tilde{h}_i}{s} \frac{ds}{dt} \frac{\partial \tilde{g}}{\partial \tilde{h}} + \frac{\partial^2 \tilde{g}}{\partial \tilde{h}^2} + \frac{4\pi R_0^2}{3\sqrt{2\pi} N_c(\mu)\nu_0} \frac{\tilde{h}_i^3}{\tilde{h}_d^3} \Gamma(\tilde{h}, t).$$  (3.21)

Following the example of Kutluay et al. [59], Equation 3.21 has been discretized in the following manner,

$$\tilde{g}_i^{m+1} = \tilde{g}_i^m + \frac{\tilde{h}_i^m}{2} \frac{\Delta \tilde{h}}{s_m} \frac{\Delta \tilde{t}}{\tilde{g}_i^m - \tilde{g}_i^{m-1}} + r(\tilde{g}_i^m - 2\tilde{g}_i^{m-1} + \tilde{g}_i^{m-2}) + \frac{4\pi R_0^2}{3\sqrt{2\pi} N_c(\mu)\nu_0} \frac{\tilde{h}_i^m}{\tilde{h}_d^3} \Gamma(\tilde{h}, t),$$  (3.22)
where \( r \) is defined as \( \Delta \tilde{t}/(\Delta \tilde{h})^2 \).

Von Neumann’s method of stability analysis results in the following criterion on the maximum size of the time step, \( \Delta \tilde{t} \), that can be taken [59].

\[
\Delta \tilde{t} \leq \frac{2\Delta \tilde{h}^2}{4 + (\Delta \tilde{h} \dot{s})^2} \quad (3.23)
\]

### 3.2 Boundary Conditions

As with most previous disk models, a zero torque inner boundary condition is used. This condition allows for accretion from the disk onto the central object. The out-flowing material carries some finite amount of angular momentum away from the system and must therefore exert a torque on the outer boundary of the disk. My model uses the torque exerted on the disk by the photoevaporating outflow as my outer boundary condition.

The viscous couple (torque) is defined as

\[
g = 3\pi \nu \Sigma h = -2\pi r \Sigma \nu r^2 \frac{d\Omega}{dr} \quad (3.24)
\]

which must be evaluated at the outer boundary.

\[
g_d = 3\pi N_c \langle \mu \rangle \nu_0 \left( \frac{r_d}{R_0} \right) (GM_p r_d)^{1/2} \quad (3.25)
\]

In terms of the non-dimensionalized variables the outer boundary condition becomes

\[
\tilde{g} = \left( \frac{r_d}{R_0} \right)^{3/2} = \tilde{h}_d^{3/2} \quad (3.26)
\]

It must be noted that these two boundary conditions are used to solve the equation governing the temporal evolution of the mass surface density. As mentioned before in Section 1.3, I have, in effect, a second outer boundary condition that governs the temporal evolution of the location of the outer boundary.

Following the example of Kutluay et al. (1997) I let \( \tilde{h}_d \to s \), which results in

\[
\frac{s_{m+1} - s_m}{\Delta \tilde{t}} = -\frac{3\tilde{g}_N^m - 4\tilde{g}_{N-1}^m + \tilde{g}_{N-2}^m}{2 \Delta \tilde{h} s_m^3} - \frac{C_0 a_0 r_s^2}{3\sqrt{2\pi} \nu_0 R_0 s_m^5} \left[ -\frac{r_s}{2R_0 s^2} \right] \quad (3.27)
\]
and

\[
\frac{s_{m+1} - s_m}{\Delta t} = -\frac{3\tilde{g}_N^m - 4\tilde{g}_{N-1}^m + \tilde{g}_{N-2}^m}{2 \Delta \tilde{h}} \frac{\tilde{s}_m^3}{s_m} - \frac{\sqrt{8\pi}}{3\nu_0} \frac{a_x R_0}{s_m}.
\] (3.28)

3.3 Initial Conditions

The initial conditions, for models using the VSG method, are the same as those used by Clarke [21], which were taken from Lynden-Bell and Pringle [68]. Namely,

\[
\Sigma_{\text{init}}(r) = \frac{M_0}{2\pi R_1 r} \exp \left[ -\frac{r}{R_1} \right],
\] (3.29)

where \(R_1\) as the initial disk scaling radius and \(M_0\) as the initial disk mass. This is generally a good initial condition to use for a disk where \(\nu \propto r\) because it is based on the similarity solution of Lynden-Bell and Pringle [68]. It can be scaled by any choice of initial disk mass and, in my case, will be extended outward until the surface mass density reaches \(N_c\).

In terms of my non-dimensionalized quantities,

\[
\tilde{\Sigma}_{\text{init}} = \frac{M_0}{\sqrt{8\pi N_c(\mu) R_1 R_0}} \exp \left[ -\tilde{h}^2 \left( \frac{R_0}{R_1} \right) \right]
\] (3.30)

Although I present the initial conditions to the reader, it must be stated that they are relatively unimportant. The disks rapidly evolve away from the initial conditions and any information about them is lost.
Chapter 4

1D Model Applied to the Solar Nebula and Planet Formation

The one-dimensional model, using the VSG method, was first used to investigate the evolution of the solar nebula. One of the primary questions of this investigation was whether our photoevaporative disk model would be able to reproduce the disk model derived by Desch [26]. Another focus of this study was to determine how strong of an effect changing the rate of photoevaporation had and if it was of the same magnitude as the effect of varying the viscosity. The viscosity is important because it directly affects the temperature and surface density profiles of the disk. The temperature and surface density profiles are, in turn, very important for the growth and migration of objects growing in the disk.

We also investigated how rapidly the solar nebula disperses, and what consequences that may have on the growth and migration of planetary embryos. From these simulations we determined the time evolution of the surface density at the locations of the four giant planets in the compact configuration of the Nice model. Then, using the time-dependent surface densities we estimated the growth rates of the giant planets’ cores. The growth rates and decaying surface density profiles were then used to calculate the migration rates of the giant planets.

Five simulations were conducted to investigate the role played by $T_{\text{env}}$ and $\alpha$ in affecting disk morphology and evolution timescales and the relative importance of the two. The various input parameters and resultant timescales are tabulated in Table 4.1. The fiducial model best matches the parameters; temperature, viscosity, etc. those typically used in canonical disk models. The fiducial model has an envelope temperature of $T_{\text{env}} = 600$ K and a viscosity parameter of $\alpha = 0.001$. 
Table 4.1: Input parameters and power-slopes for various models.

<table>
<thead>
<tr>
<th>run</th>
<th>$T_{env}$</th>
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<th>normalized timescale</th>
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<tr>
<td>LV</td>
<td>600</td>
<td>0.0001</td>
<td>3.6</td>
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<tr>
<td>LT</td>
<td>100</td>
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<td>fiducial</td>
<td>600</td>
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<td>HT</td>
<td>3000</td>
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<tr>
<td>HV</td>
<td>600</td>
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In addition to my fiducial model, four more simulations were completed in order to investigate the effect that the adjustment of key parameters has on the evolution of the disk and on planet formation timescales within that disk. For these simulations, the envelope temperature and viscosity parameter $\alpha$ were varied to their extreme values as predicted by canonical disk and complex photodissociation region models.

The viscosity parameter, $\alpha$, in protoplanetary disks is considered to lie within an order of magnitude of 0.001. Therefore, only values of $\alpha$ between $10^{-4} - 10^{-2}$ were considered. According to complex photodissociation region models that have been used to determine temperature and density profiles of photoevaporating outflows, the envelope temperature lies in the range $100 - 3000$ K [2]. The given range of temperatures corresponds to FUV fields with $300 < G_0 < 3000$ [2]. In an effort to constrain the behavior only extremes in these values were used. The models have been labeled according to which parameter(s) have been varied, with L or H referring to either low or high values of either the viscosity (V) or the temperature (T).

In general, the evolution of disks produced in these numerical simulations begin with a short contraction of the outer boundary, as it is rapidly eroded by photoevaporation. The evolution continues with a prolonged phase where the outer boundary expands as viscosity transports mass outward from the massive inner disk. The final phase of disk evolution is characterized by the slow erosion of the outer boundary and a nearly self similar shape of the disk’s radial mass surface density profile. One feature all disks have in common is a mass front located at the truncated outer
4.1 Comparison with the Desch Model

The Nice model predicts that the giant planets formed in a compact configuration and underwent a third type of migration, a planetesimal-driven migration, to their current locations. Desch [26] developed a new steady state disk model using the assumption of a compact configuration. The predicted disk has a much steeper profile and much larger surface densities than that predicted by the MMSN. Desch [26] found that a truncated *decretion* disk, characterized by outward mass transport, is required in order to maintain the steep profile in a quasi steady state. Photoevaporation was invoked as a natural mechanism for truncating the disk and removing mass at the outer boundary. In this section I compare our solar nebula models to the model proposed by Desch [26].

With the exception of the fiducial model, the models were run until the mass, $M_{\text{comp}}$, of the gas disk within the region of giant planet formation, $2 \text{ AU} < r < 30 \text{ AU}$, had reached a given value. I have chosen this region because it allows us to compare my models to that of Desch [26]. His steady state disk model contains $\sim 0.07\, \text{M}_\oplus$ in the comparison region. These models were run until $M_{\text{comp}} = 0.07\, \text{M}_\oplus$, $0.035\, \text{M}_\oplus$ and $0.0175\, \text{M}_\oplus$. The fiducial model was additionally run until $M_{\text{comp}} = 0.00875\, \text{M}_\oplus$. It took the fiducial model 0.70 Myr to evolve from a mass of 0.07 M$_\oplus$ to a mass of 0.035 M$_\oplus$ and 0.38 Myr to evolve from there to a disk mass of 0.0175 M$_\oplus$. All times listed for a given specified model are relative to the time when that particular model contains $M_{\text{comp}} = 0.07\, \text{M}_\oplus$.

Snapshots of the evolving mass surface density of the fiducial model are shown in Figure 4.1 at four different times that span a 1.3 Myr interval. The surface density constraints from Desch [26], as well as his derived surface density profile, have been over-plotted for comparison. The uncertainty in the solid component of the two inner giant planets, with Jupiter in particular, are large because of the inability of hydrostatic models, which rely on high-pressure equations of state, to constrain current core masses. The solid components of Uranus and Neptune are better
constrained because of their smaller core to atmosphere mass ratios.

By inspection, one can see that as the disk evolves it matches with the surface density constraints of Desch [26] at various radii at various times. The inner surface density constraints are matched early on and the outer constraints are matched at later times. The ad hoc profile of Equation 2.1 is plotted with the dotted line. It is interesting to note that although our first output profile contains the same amount of mass as his profile between 2 and 30 AU, his profile is much steeper than our model profiles. Our surface density profiles have $\Sigma(r) \propto r^{-1.25_{-0.33}^{+0.88}}$, where the range of exponent here is not the uncertainty but represents rather a range of profile slopes. All profiles were fit with a power-law slope through the giant planet forming region, 5 AU $< r < 15$ AU.

The steepest profile, and my most contracted ($r_d \approx 19$ AU), is from the simulation LV and has a radial surface density profile of $r^{-2.1}$, which is as steep as that derived by Desch [26]. Although the slope of the radial surface density profile of simulation LV matches that derived by Desch [26], it matches at only very late times and does not maintain a steep slope such as implied by Desch’s quasi steady-state model. The differences between most of the derived surface density profiles of this work and of Desch [26] probably arise from the different assumptions that were made in these models. This will be discussed further in Section 4.6.

4.2 Disk Morphology and Evolution

Generally speaking, there are two families of disks, those with contracted radial surface density profiles and those with extended profiles. The contracted disks are dominated by photoevaporation and have slopes with an average power-law slope of $-1.6$, whereas the extended profiles are dominated by viscous spreading and have an average slope of $-0.94$. The models with contracted profiles and those with extended profiles both have slopes that slightly increase with time as the outer disk radius moves inward.

The outer edges of these disks typically extend many 100’s of AU from the Sun. In this respect, our models differ from some earlier photoevaporation models which have truncated disks at roughly 30 – 40 AU [2]. Previous photoevaporation models have arbitrarily imposed an outer
Figure 4.1: The radial mass surface density at four times for the fiducial model. Outputs for all simulations look very similar these profiles due to our constraint on the mass contained within the planet forming region. These surface densities evolved over a 1.3 Myr time interval. The times plotted, relative to our first output, are as follows; 0.70 Myr, 1.1 Myr, and 1.3 Myr. The surface density constraints inferred from the Nice model are over plotted [26]. The dotted line is the surface density derived by Desch [26].
boundary location through the use of timescale arguments. By equating the evaporation timescales and disk accretion timescales they are able to determine an outer boundary radius where these processes are in balance. They do not self-consistently solve for the outer boundary location in the rigorous, time-dependent fashion that we have. However, it must be cautioned that these large outer disk radii may be an artifact of our assumption that the viscosity is linearly proportional to the radius, which artificially increases the rate of viscous spreading at large radii.

The initial mass is different for each of my models. This was necessary such that each model exhibited the same behavior at the times of interest. This has however required us to use some disks with unrealistically large masses, some as large as a 0.5 M⊙. Such large disks would likely be susceptible to gravitational instabilities. An additional simulation with a small initial disk mass (M₀ = 0.1 M⊙) was performed, so that the behavior exhibited by the high-mass models can be verified under more physically realistic conditions. A plot of the radial surface density profiles at various times (times shown in inset) is shown in Figure 4.2. The elapsed time between the first and last outputs is 2.0 Myr. As with all other models, this model uses the similarity solutions of Lynden-Bell & Pringle [68] as the initial conditions. In this case, the initial disk mass is 0.1 M⊙ and the scaling radius, R₁, has been set to 10 AU.

As before, the disk begins with a rapid expansion of the outer boundary. As the disk spreads the slope of its radial surface density profile quickly approaches a power law with an exponent ≈ −1.05. This is similar to the average slope from all other models presented here. At t ∼ 6.6 × 10⁵ yr the outer boundary of the disk reaches its maximum value then begins the shrink. The disk then shrinks, both in radius and in overall magnitude, as the outer edge is eaten away by photoevaporation. The disk maintains a nearly self-similar shape until the outer boundary shrinks considerably. At which point, the slope of the radial surface density profile steepens slightly. At the end of this simulation the disk contains 0.0035 M⊙. We infer from this that the behavior is independent of disk mass.

Simulations LT and HT were designed to test the effect of FUV radiation on the evolution of the disk and in turn on planetary formation timescales. There are few, if any, strong constraints on
Figure 4.2: The radial mass surface density at five times for the a model with an initial disk mass of 0.1 $M_\odot$. These surface densities evolved over a 2.0 Myr time interval. The times plotted, relative to my first output, are shown in the inset box. This model also uses the similarity solutions of [68] as the initial conditions.
the FUV environment of the early solar system. Therefore, a range of disk envelope temperatures was explored to investigate its effect on the timescale for disk evolution. Compared to the fiducial model, the evolutionary timescale was in fact larger in LT and smaller in HT. It took the disk in LT 3.6 times as long to evolve from a mass of 0.07 M\(_\odot\) to a mass of 0.0175 M\(_\odot\) as the fiducial model disk, whereas the disk evolution in HT was faster than that in the fiducial model by a factor of 0.45.

Given the wide range of \(\alpha\) generally used in solar nebula models, the value of \(\alpha\) was varied to see how it affects planetary growth timescales. The viscosity parameter \(\alpha\) was varied to a value of 0.0001 for LV and to a value of 0.01 for HV. As expected, the simulations evolved more rapidly for increasing values of \(\alpha\). The evolutionary timescale of LV was 3.6 times longer than in my fiducial model and the timescale of HV was 0.81 times shorter than that of the fiducial model. The viscosity parameter has been changed by an order of magnitude and shows that, at the fiducial temperature, the disk’s temporal evolution is just as dependent on the viscosity as the temperature.

It is interesting to note that while LV and LT both produce longer evolutionary timescales than the fiducial model, the surface density profiles generated by them are strikingly different. The low viscosity of model LV prevents the mass in the inner regions of the disk from spreading outward and maintains the outer edge of the disk within a few times 10 AU. In contrast, the relatively high fiducial viscosity (\(\alpha = 0.001\)) of LT allows for the massive inner disk to rapidly spread outward to > 100 AU where it is slowly eroded by the photoevaporative outflow. The radial profiles of LV and LT can be seen in Figure 4.3, along with the fiducial model for comparison. Each disk radial surface density profile corresponds to a time when the mass contained within the planet forming region, 2 AU < \(r< 30\) AU, is 0.035 M\(_\odot\). These correspond to times of 2.0 Myr, 2.1 Myr, and 0.70 Myr for LV, LT, and the fiducial model respectively. Here one can clearly see the difference in the radial distribution between these two models. A similar dichotomy is seen with simulations HV and HT. They both produce short evolutionary timescales relative to my fiducial model, but are opposite with regards to the radial extent of the surface density profiles they produce.

The evolutionary timescales of the model disks can be sped up or slowed down by altering the
Figure 4.3: Surface density profiles of the extended disk for LT, LV and the fiducial model. The fiducial model is shown in solid, LT with a dotted line and LV with a dashed line. Each disk radial surface density profile corresponds to a time when the mass contained within the planet forming region, $2 \text{ AU} < r < 30 \text{ AU}$, is $0.035 \, M_\odot$. These correspond to times of 2.0 Myr, 2.1 Myr, and 0.70 Myr for LV, LT, and the fiducial model respectively.
amount of far ultraviolet flux or the viscosity parameter $\alpha$. Slowing the evolutionary timescale by decreasing the incident far ultraviolet flux, or similarly by decreasing $\alpha$, can help to grow planets more rapidly, but at the cost of decreased migration timescales. In the low viscosity and low envelope temperature models we were able to grow all of our giant planet embryo cores to sufficient mass.

Despite their differences, these disks are all characterized by outward mass transport, mass loss at the outer edge and a truncated outer boundary. The transport of mass from small to large radii can potentially prevent the rapid inward migration of Jupiter and Saturn, while at the same time supply enough mass to the outer regions of the disk for the formation of Uranus and Neptune.

### 4.3 Embryo Growth

If one assumes that planets form via the accumulation of smaller bodies and not through direct gravitational collapse, the early stage of planetesimal accretion is characterized by a period of runaway growth [122]. During runaway growth the velocity distribution of planetesimals is dominated by interactions with other planetesimals. During this time the velocity dispersion of planetesimals is low and gravitational focusing is effective. While gravitational focusing is effective, the largest bodies grow much more rapidly than smaller bodies and a bimodal size distribution is achieved.

This is followed by a phase of oligarchic growth, where the velocity distributions are dominated by interactions of the larger planetary embryos. During oligarchic growth the presence of large bodies enhances the velocity dispersion of smaller bodies and decreases the velocity dispersion of the largest bodies. The increased dispersion in velocities of the smaller planetesimals decreases the effect of gravitational focusing and the largest bodies begin to decrease their growth rate. The system becomes dominated by a few large bodies, an oligarchy, separated by a few mutual Hill radii. Since our simulations take place while large amounts of gas are present, we only consider runaway growth.

Early analytical models, by Safronov [98] and others, overestimated the growth timescale of
planets by upwards of five orders of magnitude and were inconsistent with observational constraints of proto-planetary systems that showed the removal of gas in $\sim 5 - 10$ Myr. Lissauer [63] developed an analytic model for the runaway growth of planetary embryos (Equation 3 from Lissauer [63]).

To evaluate the growth rates of the giant planet embryos we use Equation (14) of Lissauer [64]. The growth rate $\frac{dM_e}{dt}$ is defined for an embryo of mass $M_e$ and radius $R_e$ and escape velocity $v_{esc}$ embedded in a swarm of planetesimals with a local surface density $\Sigma_p$ and velocity dispersion $\sigma$.

$$\frac{dM_e}{dt} = \sqrt{\frac{3}{2}} \Sigma_p(t) \Omega_{kep} \pi R_e^2 \left(1 + \frac{v_{esc}^2}{\sigma^2}\right)$$

(4.1)

The numerical prefactor depends on the velocity distribution of planetesimals and many values have been quoted in the literature, the value used here of $\sqrt{3}/2$ is due to an isotropic velocity distribution. We make the conservative assumption that the surface density of solids, $\Sigma_p$, is 0.014 times the surface mass density of gas and that the solids-to-gas ratio does not change with time or radius.

One can see from Equation 4.1 that the growth rate is dependent on the geometric radius of the embryo, $\pi R_e^2$, enhanced by a gravitational focusing factor, $(1 + \frac{v_{esc}^2}{\sigma^2})$. The exact value of the gravitational focusing factor has been the subject of much study over the years and is still much debated. It has been studied with both analytical and numerical studies in a variety of different regimes including gas-free and gas-damped accretion.

Numerical experiments show that the eccentricities and inclinations of the planetesimals in a swarm are damped due to the interactions with gas in a disk [54, 55, 56, 57]. The damping of inclination and eccentricity due to gas drag causes, at least at the small end of the size distribution, the planetesimals to be in the shear-dominated regime where gravitational focusing is important [96]. We can define a characteristic velocity

$$v_H = \sqrt{\frac{GM_e}{r_H}},$$

(4.2)

based on the definition of the Hill radius,

$$r_H = a \left(\frac{M_e}{3M_\odot}\right)^{1/3},$$

(4.3)
where $a$ is the semi-major axis of the embryo. Our characteristic velocity marks a transition between the shear-dominated and dispersion-dominated regimes. When the velocity dispersion of planetesimals is smaller than our characteristic velocity, $\sigma < v_H$, accretion will proceed in the shear-dominated regime [8]. We adopt the characteristic velocity, $v_H$, for the value of the velocity dispersion of planetesimals, $\sigma$, in all of our calculations of embryo growth. When $\sigma < v_H$, the system is in the shear dominated regime and 3-body dynamics become important. Therefore, Equation 4.1 is not strictly valid as it is derived considering only 2-body effects.

Also, as the embryos grow the system will transition to a dispersion-dominated regime where the embryos will grow in an oligarchic fashion. Due to the uncertainties in when this transition occurs we have focused only on runaway growth. Owing to the large variation in estimates planetary formation timescales and the wide array of unknown parameters; disk mass, viscosity, gas/solid ratio, etc., our calculations are not meant to definitively describe embryo growth but to illustrate how various surface density profiles determined by viscous evolution and photoevaporation effect planet growth. In this regard, the following results on embryo growth should be treated with some caution.

In order to determine the cores’ growth rates it was necessary to determine the time evolution of the mass surface density. The temporal evolution of surface densities at the location of each core were fit with a decaying exponential. This seemed to give a good fit to the data. These fits were then used in Equation 4.1 to determine the masses of the giant planet embryos as functions of time. It should be noted that in our models it is the time-dependent surface density that determines the growth rates and hence the embryo masses. In our models, the embryos initially grow rapidly because the surface densities are large, but the growth rate then begins to wane because the disk evolves and the surface densities become small. This differs from most models in which the growth rates are large throughout the duration of the simulations because they assume unrealistically large, steady state surface densities. Our research indicates that, because of the similarity of relevant timescales, planet formation models must take into account the time-dependent behavior of the solar nebula.
Embryo masses for the four giant planets as a functions of time, determined for our reference model, are shown in Figure 4.4. We were able to quite easily build the core of Jupiter to $> 10 \, M_⊕$ well within the $\sim 5 - 10$ Myr window implied by gas dissipation. One can see from Figure 4.4 that, in our reference model, we are unable to grow the cores of Saturn, Neptune and Uranus to $10 \, M_⊕$ within the $\sim 5 - 10$ Myr time constraint. The gas simply dissipated too quickly for them to form in the allotted time. Desch [26] was able to grow cores of sufficient mass in his models because he relied on steady state models with surface mass densities that remained large throughout the planetary growth process. We feel that our decaying, time-dependent model is a more realistic representation of the solar nebula.

Figure 4.5 shows the growth of planetary embryos in simulation LT; a disk with a heated envelope temperature of 100 K. Because of a smaller mass loss rate at the outer boundary, the evolutionary timescale of LT is a factor of 3.6 over the evolutionary timescale for the disk in the reference model. In this model the cores of Jupiter and Saturn were both able to grow cores of 10 $M_⊕$ or more during the first 10 Myr of evolution. This is not surprising considering the prolonged temporal evolution of the disk in the low-temperature model. Despite the success at the growth of the cores of two innermost giant planets, the cores of Neptune and Uranus are unable to grow large enough during the duration of the simulation.

Figure 4.6 shows the growth of planetary embryos for LV ($\alpha = 0.0001$). The embryos in this model grow faster than the embryos in the reference model, and similarly to LT. All embryos, with the exception of that of the outermost giant planet, are able to grow to masses of 10 $M_⊕$ within the allocated $\sim 5 - 10$ Myr. As seen before in the low-temperature model, the prolonged temporal evolution of the low-viscosity model provided a sufficiently high surface mass density for a long enough time for the three innermost cores to grow to sufficient masses. The effect of varying the viscosity has nearly the same effect on the evolutionary timescale as in the above case where $T_{\text{env}}$ was varied, but the truncated disk in LV provides more mass in the giant planet forming region. The extra mass provides a more conducive environment for embryo formation as seen with the success in growing the core of Neptune to $> 10 \, M_⊕$. 
Figure 4.4: Growth of giant planet embryos in our reference model. The masses of planet embryos, plotted from top to bottom, are Jupiter, Saturn, Neptune and Uranus.
Figure 4.5: Growth of giant planet embryos in LT, where the envelope temperature, $T_{\text{env}}$, has been reduced to 100 K. The masses of planet embryos, plotted from top to bottom, are Jupiter, Saturn, Neptune and Uranus.
Figure 4.6: Growth of giant planet embryos in LV, where the viscosity parameter, $\alpha$, has been reduced 0.0001. The masses of planet embryos, plotted from top to bottom, are Jupiter, Saturn, Neptune and Uranus.
As stated earlier, the surface density of solids, $\Sigma_p$, was assumed to be 0.014 times the surface mass density of gas. This estimate is based on the canonical gas/solid ratio of 70 derived from composition of Comet Halley [46]. It should be noted that this estimate is based solely on the content of $\text{H}_2\text{O}$ ice. Observations of the ejecta of Comet 9P/Tempel 1 during Deep Impact showed significant amounts of CO, CO$_2$ and CH$_3$OH [3]. These ices would certainly be present at the locations of Neptune and Uranus and would result in a higher solid/gas ratio. Combined models of viscous disk evolution and kinetic ice formation show an increase in the solid/gas ratio with radius. By following a chemical reaction network tracing the formation and freeze out of ices in a viscously evolving disk, it was found that the solid surface density at Saturn (9.5 AU) is roughly 3 times that used in previous models of planet formation and that the solid surface density at Uranus (20 AU) was higher by a factor of nearly 4.5 [27].

An increase in solid surface density can also facilitate the formation of planets in a more dramatic fashion. The increase in solids would certainly decrease the formation timescale of the outermost giant planets. Settling of dust to the disk midplane and preferential photoevaporation of gas can lead to a significant increase in the dust-to-gas surface density ratio. This increase in solid surface density can potentially become unstable to gravitational collapse and trigger rapid planet formation [106].

### 4.4 Migration Timescales

Viscous torques from density waves launched in a disk from an orbiting planet are thought to cause migration [114, 115]. Using numerical results, Tanaka et al. [103] were able to constrain analytical models for the torque exerted by corotation and Lindblad resonances on a body orbiting in an isothermal disk. The net torque in 3-D is

$$\Gamma = (1.364 + 0.541 \frac{d\Sigma_e}{da_e}) \left( \frac{M_e a_e \Omega_e}{M_\odot c_s} \right)^2 \Sigma_e a_e^4 \Omega_e^2$$

where the subscript e indicates the values of these variable at location of the embryo. Here $a_e$ refers to the embryo’s semi-major axis and $c_s$ is the local sound speed of the disk. The local orbital
velocity $\Omega_e$ is approximated by the Keplerian orbital velocity $\Omega_{\text{kep}}$. Equation 4.4 can then be used to determine the type I migration timescale using

$$T_{\text{mig}} = \frac{L_e}{2\Gamma} = \frac{M_e (G M_\odot R_e)^{1/2}}{2\Gamma}. \quad (4.5)$$

where $L_e$ is the angular momentum of the planet, $L_e = M_e (G M_\odot R_e)^{1/2}$.

Figure 4.7 shows Jupiter’s migration timescales for LT and HT, models where the envelope temperature, $T_{\text{env}}$, has been varied along with our reference model for comparison. They have been calculated keeping the semi-major axis of Jupiter fixed at 5.45 AU. In general, the simulations that grew planets the fastest also suffered the shortest migration timescales. At early times the migration timescale decreases due to the growth of the planet, but at late times the decaying surface density of the disk causes the migration timescale to increase. In our reference model it is unlikely that Jupiter would survive orbital decay due to type I migration. However, if it can survive through the period in which type I migration timescales reach a minimum it has a chance for long term survival.

Figure 4.7 shows that the migration timescales of Jupiter for the reference model are intermediate to those for in LT and HT. The migration rates in these simulations are mainly effected by the large timescale variations in the evolution of the surface density of the disk. The long (short) timescales produced by lowering (raising) the disk envelope temperature allow planets to grow faster (slower) and maintain surface densities at higher (lower) levels. The combined effect of larger (smaller) planets and higher (lower) surface densities combine to cause shorter (longer) migration timescales than in our reference model. The migration rates of LV and HV are similar to those in LT and HT. Figure 4.8 again shows that models with short evolutionary timescales produce smaller embryos in a less massive disk and therefore these planetary cores have longer migration timescales.

The migration rates derived using Equation (4.5) are for type I migration and are only valid when the Hill radius of an embryo is smaller than the scale height of the disk. When the planet mass exceeds some critical value, the migration switches to type II migration and the evolution of
Figure 4.7: Jupiter’s migration timescales for the reference model and models where the envelope temperature has been varied. From top to bottom, the migration rates shown are from HT, the reference model and LT.
Figure 4.8: Jupiter’s migration timescales for our reference model and models where the viscosity parameter $\alpha$ has been varied. From top to bottom, the migration rates shown are from HV, the reference model and LV.
the embryo’s semi-major axis becomes locked into the viscous evolution of the disk. This happens rather late in most of our calculations, generally later than 10 Myr but in some cases can occur much earlier. Our disk, with its large amount of outward transport of material and truncated outer radius, could cause an embryo to migrate either inward or outward depending on the semi-major axis of a given embryo [116].

4.5 Outward Mass Transport: A Decretion Disk

In a truncated disk with outward mass transport there is a critical radius inward of which the mass moves inward and is accreted onto the central object and outward of which the material is transported outward and eventually out of the system. The survival of a growing embryo against the effects of type II migration depends on which side of the critical radius it is. To investigate the location of the critical radius in our model we have calculated the radial velocity of the material in our disk.

\[
v_r = -\frac{3}{\Sigma r^{1/2}} \frac{\partial}{\partial r}[\nu \Sigma r^{1/2}] \tag{4.6}
\]

Which in terms of our non-dimensional variables is

\[
v_r = -\frac{3 \nu_0}{R_0} \left[ \tilde{h} \frac{\partial \tilde{g}}{\partial \tilde{h}} + \frac{3}{2} \right] \tag{4.7}
\]

The results of these calculations for the reference model can be seen in Figure 4.9. We have plotted the radial velocity of the flow for the four times shown in Figure 4.1. The critical radius lies at 38 AU, 25 AU, 18 AU, and 14 AU at the four times shown. The times shown occur at 0 Myr, 0.70 Myr, 1.1 Myr and 1.3 Myr respectively. The critical radius, where the curves intersect \( v_r = 0 \), moves inward with time. The critical radius at the times shown is exterior to the orbit of Jupiter, but it is rapidly moving inward and will at some point transition to a radius smaller than Jupiter’s semi-major axis. If this happens early enough it could save Jupiter from migrating into the Sun. This could also affect the radial diffusion of dust particles due to turbulent fluctuations in the disk.
Figure 4.9: The radial velocity of material in the disk in the reference model. The four times plotted in Figure 4.1 have also been plotted. They are sequential in time from right to left. The critical radius lies at 38 AU, 25 AU, 18 AU, and 14 AU at the four times shown. The times shown occur at 0 Myr, 0.70 Myr, 1.1 Myr and 1.3 Myr respectively.
4.6 Summary and Discussion

We adapted a method of solving propagating phase change problems and developed a 1-D viscous disk model that self-consistently tracks the location of the outer boundary under the influence of photoevaporation from an external source. Our application of the formalism of the Stefan problem to astrophysical disks is a novel approach. Our model is, as far as we know, the first model to track the location of the outer boundary in a fully self-consistent manner.

We present five simulations designed to test the effects that varying the strength of the FUV flux and altering the strength of the viscosity have on the temporal evolution of the disk. It took the disk in LT 3.6 times as long to evolve from a mass of 0.07 M⊙ to a mass of 0.0175 M⊙ as the disk in our reference model, whereas the disk evolution in HT was faster than of our reference model by a factor of 0.45. The evolutionary timescale of LV was 3.6 times longer than the timescale of our reference model and the timescale of HV was 0.81 times shorter than that of our reference model. The embryos in LT and LV grow faster than the embryos in our reference model, and in LV the cores of the three innermost giant planets were able to grow to > 10 M⊕ within the allocated ~ 5 – 10 Myr.

Photoevaporation has, for some time now, been invoked as a mechanism for the rapid dispersal of protoplanetary disks [11, 47, 22, 2]. It has recently been invoked as a possible mechanism for the truncation of the solar nebula in such a fashion as to produce the steep surface density profile required to produce the giant planets in the compact configuration of the Nice model [26]. We have performed a number of simulations to test the relative importance of external photoevaporation vs. viscous evolution. We also investigate whether or not photoevaporative mass loss from the outer edge of an evolving protoplanetary disk can produce the steep surface density profile posited by Desch [26]. We find, for reasonable disk parameters, that the viscous evolution and photoevaporation play equally important roles in determining disk evolution timescales and morphology.

The five simulations presented here were designed to have comparable masses throughout the planet forming region (2 AU < r < 30 AU). Although their radial surface density profiles
are nearly identical within the giant planet forming region, their profiles are very different in the outer regions of the disk. The various disk models have very different radial profiles at large radii, but one must keep in mind that there is in actuality very little mass in these regions. Despite the very low mass surface densities at large radii, these disks can extend, in some cases, many hundred of AU from the Sun and may have strong implications for the development and evolution of the Kuiper belt and Trans-Neptunian objects [2].

By evolving the surface density and the location of the outer boundary in a self-consistent fashion we are able to track the exact location of the outer boundary with time. One shortfall of Desch’s model is the inability to constrain the location of the outer boundary to better than $30 - 100$ AU. In our models, the radii of the outer boundaries are typically larger than $100$ AU. Although our boundaries are generally quite large, the surface densities beyond $100$ AU are mostly less than $25$ g cm$^{-2}$ and beyond $150$ AU they are all less than $10$ g cm$^{-2}$. Observations of the T-Tauri stars DM Tau and AM Aur have constrained their disks to have radii as large as $> 500$ AU, and even upwards of $850$ AU [51, 25]. These measurements are made with observations of CO and more tenuous, less dense parts of these disks may lie at greater radii, beyond the sensitivity of the instruments. Although our disks tend to be quite large, they seem to be consistent with observations of extrasolar circumstellar disks.

Despite many of our simulations having large outer disk radii, two simulations, HT and LV, have outer disk radii of roughly $20 - 30$ AU. This is within the current semi-major axis of Neptune. The simulation HT has a disk evolution timescale that is much shorter than our reference model and fails to grow the giant planets within the timescale of gas dissipation. In contrast, LV had a longer disk evolution timescale than our reference model and was more effective at growing the giant planet embryos. The combination of low viscosity and relatively high FUV flux prevented gas from spreading outward and created a steeper more compact disk. These results show that photoevaporation can effect the solar nebula in the giant planet formation region in a significant manner.

Our model is a further exploration of the steady state decretion disk proposed by Desch
[26]. With the exception of LV, all of our surface density profiles are shallower than the surface density profile of Equation 2.1, with a typical dependence on radius of $\Sigma(r) \propto r^{-1.25^{+0.88}_{-0.33}}$. Despite the mismatch, our profiles seem conducive to planet formation and do match the surface density constraints derived by Desch [26] at certain radii at certain times.

We ran our models assuming a constant external FUV flux. There are compelling reasons to believe that the incident flux is not constant. Individual stars in the solar birth cluster the would certainly have experienced motion relative to one another. Relative motion between the Sun and any of its high mass brethren would certainly have caused the FUV flux to vary [93, 1]. The observed spread in stellar ages within young clusters is usually $\sim 1$ Myr, but subgroups within the same cluster have been observed to have a roughly 10 Myr difference in ages [45]. This would imply a highly varying UV environment as new stars are born into clusters with the early type stars rapidly moving onto the main sequence on timescales of $10^4 - 10^5$ yr.

The lifetimes of early type stars are comparable to the lifetimes of circumstellar disks and therefore any changes in luminosity experiences during their short lives will effect the local UV environment. The FUV flux from early type stars can vary substantially on very short timescales ($10^4$ yr) as they transition through the luminous blue variable (LBV) stage. LBV stars undergo a phase marked by high mass loss and instability. During the LBV phase, a constant bolometric luminosity ($L \approx 10^5 - 10^6 L_\odot$) is maintained, but the dense stellar wind absorbs EUV such that the majority of the flux escapes in the FUV. Observations of proplyds in the Carina Nebula (NGC 3372) suggest these outbursts can have dramatic consequences for nearby protoplanetary disks [102]. This would imply direct consequences on disk evolution and survival. In the future, it would be interesting to model a variety of FUV irradiation scenarios and investigate the effect of variable flux rates.

In our models, $\alpha$ was held constant throughout the entire disk and $\nu \propto r$ which follows from $T \propto r^{-1/2}$. In reality, $\nu$ should not be a simple monotonic function if the ionization fraction of the disk varies with radius leading to dead zones where the magnetorotational instability is suppressed. By allowing the viscosity to vary throughout the disk it could substantially alter the rates of planet
formation in various regions [58, 69]. Without further investigation, it is hard to say exactly what the effect of allowing the viscosity to depend on the local qualities of the disk would have on the growth rates and chances for survival against migration of growing embryos.

Furthermore, the use of a constant $\alpha$ could explain the discrepancy found between the slope of the radial surface density profile of our models and that derived by Desch [26]. If $\alpha$ was allowed to be lower in the inner portion of the disk than in the outer regions, then the shallow profile that our models have produced would probably not develop. In this case, it is possible that the steep profile derived by Desch [26] would develop. The 1D numerical model presented here is unable to handle a change of $\alpha$ in the inner portion of the disk.

At any heliocentric distance, the timescale for the growth of dust grains into planetesimals is a few thousand times the local orbital period [120]. Therefore, planetesimals formed in inner regions of a disk form faster than planetesimals in outer regions. Coincidentally, in the inner regions of the planet forming zone our disk in the reference model matches with the surface density constraints from [26] at early times and in the outer planet forming regions at late times. Any planetesimals formed early on in the inner region, large enough to decouple from the disk yet not so large that they undergo significant migration, will be present and available for planet formation at later times. These planetesimals would effectively maintain the surface density of solids at the high levels needed to match Desch’s [26] constraints after the gas has been transported elsewhere.

Farther out in the disk, where planetesimals form more slowly and at later times, our surface density in this region also is consistent with surface density constraints at later times. Although we have begun the growth of all of the giant planets in our simulations at the same time in each model, there is no reason for this to be the case. It could very well be that planet formation was delayed in one region or another.

In the simulations LV and LT we are able to successfully grow the embryos of the outer most giant planets within the given $\sim 5 - 10$ Myr time constraint. It is important to again stress the uncertainties involved in planetary growth from the accumulation of planetesimals. At some point, a transition to oligarchic growth and a clearing of the embryos’ feeding zones would slow embryo
growth. On the other hand, other processes exist which would increase embryo growth rates. We model growth without atmospheres, from a single size distribution. We also neglect any tidal effects that could dissipate energy from the impacting planetesimals. These are but a few of the major uncertainties in core accretion models. Any of these and others could easily alter the growth rates by a factor of two [26].

We have also made the simplifying assumption that the solids-to-gas ratio is constant with radius. The solids-to-gas ratio should increase with radius as exotic ices condense in the cold outer regions of the solar nebula. The increase of solids beyond the snow line would provide more material for planet growth and decrease the growth timescale of the giant planets, especially of Neptune and Uranus. Photoevaporation of hydrogen and helium from the disk would also tend to increase the solids-to-gas ratio of the outer disk. Despite these uncertainties, we have shown that given reasonable model parameters the cores of the outermost giant planets can successfully be built in a decretion disk with a truncated outer boundary. It is the outward flow of mass and removal at some outer radius that provides sufficient mass to the outer regions of the disk for planetary core growth.

We placed our growing embryos in the ad hoc compact configuration of the Nice model, but one must keep in mind that the Nice model begins after gas has dissipated. Planet-disk interactions would likely have lead to some amount of radial migration while gas was still present in sufficient quantities. This would imply that the giant planets could have begun to grow elsewhere in the disk and migrated to a more compact configuration before the gas dissipated. Our placement of the giant planets at the locations of the Nice model is most likely incorrect, but it is a good proxy for testing planet formation in a compact configuration. A more realistic treatment would require coupling an N-body code to our viscous evolution code and evolving it with migration such that the giant planets end up in the compact configuration of the Nice model. We plan to investigate this avenue of study in the near future.

It was our aim to simply illustrate how the migration timescales are effected by the decreasing surface density. One must also keep in mind that type I migration is a poorly understood
phenomenon. Most of the studies to date have investigated type I migration in isothermal disks. Accurately coupling an N-body code to our viscous disk model with migration will be difficult enough without the inherent uncertainties in migration processes themselves. It has been shown that in non-isothermal disks with high opacities the induced net torque may have opposite sign and act to push planets outward [89]. That is to say, it is uncertain in which direction type I migration would force a planet. Furthermore, recent simulations suggest that the inward scattering of planetesimals could drive the outward migration of growing embryos [61].

Type II migration is better understood. It is likely that the largest giant planet Jupiter would quickly fall into the regime of type II migration and be swept inward while on the inside of the critical radius, where the radial gas flow transitions from inward to outward. If however, the planetary embryo does not completely open a gap the type II migration rate can be reduced or even reversed [24]. In our model, the critical radius is continually moving inward. In our reference model it moves from 38 AU to 14 AU over the 1.3 Myr of disk evolution. At some point, the critical radius should overtake Jupiter and reverse the course of its migration outward. It may be that the decreasing surface density and outward mass transport could save Jupiter from being lost into the Sun.

The disks produced with our numerical model are all characterized by outward mass transport, mass loss at the outer edge and a truncated outer boundary. The outer boundary is characterized by substantial mass loss due to photoevaporative heating. This mass loss drives outward mass flow from the critical radius to the outer edge of the disk. The transport of mass from small radii to large can potentially prevent the rapid inward migration of Jupiter and Saturn, while at the same time supply enough to the outer regions of the disk for the formation of Uranus and Neptune.
Chapter 5

Circumplanetary Disks and Regular Satellite Formation

Recent work indicates that the MMSN model proposed by Lunine & Stevenson results in disk models that are too dense for sufficient ionizing radiation to penetrate them [65, 66, 110]. The uncertainty in the amount of angular momentum transport derives from the uncertainty in the ionization state of circumplanetary disks. A number of authors have begun research into the ionization states of these disks [110, 65, 66].

To date, circumplanetary dead zone studies have focused solely on MRI as a mechanism for generating turbulence. Depending on disk structure, other non-linear instabilities such as baroclinic instabilities may occur. Recent three-dimensional, compressible simulations have been able to create long-lived stable vortices generated by baroclinic instabilities [70]. Lubow and Martin [65] found that dead zones do exist and that mass builds up in these zones as it is transported inward from MRI active regions at large radii. Dead zones, if they exist, would be potential places for condensable solids to accumulate and grow, and therefore ideal places for satellite formation. Dead zones have been extensively investigated in the context of planet formation, but have yet to be fully explored in the context of satellite formation.

Ward & Canup [117] developed a comprehensive model that follows the formation and evolution of a giant planet, and subsequent circumplanetary nebula, from cloud collapse through the contraction phase. Their model is composed of three elements: (1) an inflow model describing the properties of the in-flowing material from the circumstellar nebula, (2) a quasi-steady state disk model, and (3) a planet growth and contraction model. It is their quasi-steady state viscous disk
model that most concerns us.

The quasi-steady state viscous disk model requires an in-plane flux as well as a mass loss mechanism at the disk’s outer boundary, $r_d$ [117]. Despite the comprehensive nature of their model, Ward & Canup [117] have yet to identify an appropriate mass loss mechanism. In their own words, “the outer edge of the disk, $r_d$, is not well defined other than it be much further out than the centrifugal radius.” Given a moderate external far-ultraviolet (FUV) flux from either the central star or nearby high-mass stars, photoevaporation provides a natural mechanism for both mass loss and truncation at the outer edge of circumplanetary disks.

In recent years, the impact of photoevaporation on the evolution of protoplanetary disks has received much attention [47, 11, 2, 34]. Recent observations confirm that there is sufficient FUV and X-ray flux from young, solar-type stars to drive photoevaporative mass loss in the surrounding nebulae [44]. Although photoevaporation has been investigated in detail in the context of planet formation, it has yet to be applied to circumplanetary disks and the formation of regular satellites.

5.1 Competing Circumplanetary Disk Models

The first of the two competing models is a static, two-component model that invokes an optically thick, massive inner disk and an optically thin, low-mass outer disk. This model has been advocated primarily by Mosquiera, I. and Estrada, P. and will hereafter be referenced as the ME model [82, 83]. This model is similar to the MMSN model in that it is a closed model that does not include continued inflow of material from the solar nebula. However, it differs from the MMSN model in that it has an increased solid-to-gas ratio and therefore less mass in gas than the MMSN model. The ME model assumes a $1/r$ temperature profile that has been normalized such that $T_{\text{disk}} = 250$ K at Ganymede’s orbit.

The ME model depends on the validity of a two-component subnebula. The inner component of the subnebula is formed when accretion from the solar nebula begins to wane, due to the opening of a gap in the solar nebula, and the distended giant planet begins to contract and a spin-out disk is formed. The inner component of the disk has gas mass surface densities in the range,
$10^4 - 10^5 \text{ g cm}^{-2}$. The outer, low mass component of the ME disk model is formed as an undefined amount of material trickles in through the gap in the solar nebula and has gas mass surface densities in the range $10^2 - 10^3 \text{ g cm}^{-2}$. The total combined mass of these two components is derived using the same approach as in the MMSN model, that of augmenting the masses of the satellites and distributing this mass about their orbital locations. Based on angular momentum conservation arguments, the outer disk component extends out to $r_H/5$, where

$$r_H = a \left( \frac{M_p}{3M_\odot} \right)^{1/3}.$$  

(5.1)

The Hill radius, $r_H$, is the distance from the planet at which the gravitational influence from the Sun balances that of the planet. It is defined in terms of the planet’s mass, $M_p$, the mass of the Sun, $M_\odot$, and the semi-major axis of the planet, $a$. The ability of the subnebula to maintain these two components is contingent on the assumed low viscosities, which prevent the diffusion of mass and spreading of the system. The massive inner disk is assumed to have a viscosity parameter of $\alpha = 10^{-6} - 10^{-4}$, where $\alpha$ is the $\alpha$-viscosity prescription of Shakura & Sunyaev [101].

Despite the relatively large surface densities in the ME model, the low viscosities allow for the formation of ices and the slow formation of satellites. This is especially true in the outer component of the disk, which is expected to be quiescent, with very low viscosity. In this outer disk component, Callisto is calculated to form in $10^5 - 10^6$ years; allowing for it’s partially differentiated state. The low viscosities in the ME model also prevent the rapid inward migration of satellites that is believed to occur in the MMSN model.

The second model also assumes the regular satellites formed around the giant planets during the very end of the gas accretion phase, but invokes a viscously accreting disk. This model has been proposed by Canup, R. and Ward, W. and will hereafter be referenced as the CW model [18, 19, 117]. The CW model assumes the circumplanetary subnebula is a very low-mass disk that is actively accreting and continually resupplied, either through a partially or completely opened gap in the solar nebula, such that a steady state is reached where mass loss is balanced by accretion from the solar nebula. The CW model employs larger values of the viscosity parameter than the
ME model that are in the range $\alpha \approx 10^{-4} \sim 10^{-3}$. The temperature profile and resultant surface densities in the CW model depend on the assumption that the opacity is constant with temperature and radius and that the assumed value of $\alpha$ can be reached in the disk.

The CW model is particularly attractive because it features an explanation for the nearly constant mass ratio of the satellite systems of the giant planets to their host planets. Canup & Ward [19] show how the mass ratio of the regular satellites in these two systems to their host planets, $10^{-4}$, might arise naturally as subsequent generations of satellites are formed and lost through migration into the host planet. The question then arises as to why there are four, near equal mass satellites around Jupiter, yet about Saturn all of the mass is concentrated in the single large satellite, Titan. This problem is referred to as the Jupiter/Saturn dichotomy. Canup & Ward [18] assume the same circumplanetary disk evolution for both Jupiter and Saturn and that the dichotomy present in their satellite systems is explained by the stochastic timing of formation/migration and the depletion of the solar nebula [19].

The two competing theories of circumplanetary disks make very different assumptions about the level of angular momentum transport. Much of this uncertainty is due to uncertainty in the ionization state of the disk gas, which in turn affects the potential for MRI to occur. It is uncertain whether this mechanism can operate in the dense, dusty environment of circumplanetary disks.

The ME and CW models presented above require a number of assumptions to be made about the temperature profile, opacity and the level of angular momentum transport in the disk. These assumptions affect the resultant surface density profile as well as the growth and migration rate of satellite embryos in these models. Menou & Goodman [76] showed that the opacity changes in protoplanetary disks can significantly effect the migration rate of planetary embryos.

### 5.2 Infall from the Solar Nebula

A major difference between circumplanetary disks and their circumstellar counterparts is the continual resupply of mass from the solar nebula onto the circumplanetary subnebula. The partial differentiation of Callisto implies a long formation timescale was one factor that led Canup & Ward
[18] to develop their “gas-starved disk model”. They assert that the 0.002 Jupiter masses required for formation the Galilean satellites need not be present all at once. This material may have slowly flowed through the system over an extended period of time. They introduce a timescale for the addition of mass to the system.

\[
\tau_G = \frac{M_P}{\dot{M}_P}
\]  

(5.2)

In this thesis I present two circumplanetary disk models, a 1D model and a 1+1D model. The 1D model uses the results from Machida [71] to parameterize the infall from the solar nebula. The newer 1+1D model uses more recent results from Tanigawa et al. [105]. I give a brief description of each of these results below. A common feature of these models, that was not discovered until 3D simulations were utilized to study the inflow of material onto circumplanetary disks, is that the in-flowing material is unable to accrete through the midplane, but rather falls vertically onto the disk surface. It seems the majority of the in-flowing material contains too much angular momentum to accrete onto the circumplanetary disk and flows outward rather than inward near the disk. This outflow, combined with the shock formed by the infalling material sets up a thermal pressure gradient that directs gas upward out of the midplane, near the Hill radius, where it can then rain down on the disk surface.

Machida [71] ran high resolution three-dimensional simulations of the accretion of angular momentum onto a protosatellite system. Using their results, we estimated the centrifugal radius, \( r_c \), the location at which the bulk of the in-falling gas lands on the circumplanetary nebula. The angular momentum contained within the in-falling material is used to constrain the radii where infall occurs, so that the added disk mass has the same angular momentum as the local circular orbit where it is added. For a Jupiter mass protoplanet, the in-falling material has an average specific angular momentum consistent with a Keplerian orbit at \( r_c \approx 25 R_P \). Were this not the case, and the in-falling material contained less angular momentum than the local circular orbit where it was added, it would not be rotationally supported and would rapidly fall inward and be redistributed
at smaller radii corresponding to its angular momentum content. A very recent, high-resolution simulation of mass accreted from the solar nebula onto circumplanetary disks indicates that infalling mass does, if fact, intercept the disk at radii (∼ 50R_P) greater than that which corresponds to its angular momentum content and is redistributed inward [104]. While these results are too new to have been included in the 1D model, they were taken into account in the newer 1+1D model presented in Chapter 7.

We adopted the following functional form, which peaks at 25 R_P, for the infalling mass onto circumplanetary nebulae.

\[
\Gamma(r, t) = -\left(\frac{r}{R_P}\right)^{28} \exp\left[\frac{(r/R_P) - 28}{3}\right] \text{g cm}^{-2}\text{s}^{-1},
\]

(5.3)

where \(\Gamma\) is the infall of surface density per unit time and where \(r\) is measured in units of \(R_P\), the planet’s current equatorial radius. This equation places most of the infalling mass at ∼ 25R_P. Normalizing this equation according to the infall timescale \(\tau_G = \frac{M_P}{\dot{M}_P}\) gives a reasonable infall at a rate appropriate for the “gas-starved disk model”. The results from Machida et al. [71] were used in Mitchell & Stewart [79].

The prescription for the infalling material has roughly 90% of the infalling mass accreting onto the circumplanetary disk at radii between 16 \(R_P < r < 28 \) \(R_P\) and peaks at 25 \(R_P\). This prescription for infall is also consistent with the previous work of Canup & Ward [18, 19] in which the infall is limited to the inner region of the disk. There is no accretion of material onto the circumplanetary disks from the solar nebula external to \(r = 28 \) \(R_P\).

Tanigawa et al. [105] published high resolution results, from nested-grid simulations, which model the inflow of gas from the solar nebula onto the circumplanetary disk about a giant planet. They not only calculated the mass inflow rate onto the circumplanetary disk as a function of radius, but the angular momentum content of the material as well. This is important because the material is assumed to rapidly redistribute itself in the disk based on its angular momentum content.

Their results show that the mass flux as a function of radius is \(\dot{M} \propto r^2\). However, when they account for the redistribution of mass due to the angular momentum content and place the
material on Keplerian orbits they find that the mass flux onto the disk is \( \dot{M}_{\text{kep}} \propto r \). Applying these results, I use a mass surface density flux onto the disk of the form,

\[
\Gamma(r) = \Gamma_0 r^{-1} \quad \text{for} \quad r < 100R_p. \tag{5.4}
\]

The parameterization of my infall, for a rate of \( \tau_G = 5 \times 10^6 \), is shown in Figure 5.1. This corresponds to a mass infall rate of \( 1.9 \times 10^{-10}M_\odot \text{yr}^{-1} \).

Although Tanigawa et al. [105] have presented high-resolution simulations that are arguably the state of the art, they should still be taken with some caution. The calculations were performed under the assumption that the disk is isothermal and inviscid. Furthermore, it was done without including magnetic fields or self-gravity. The authors state that while these assumptions may quantitatively change their results, the overall qualitative aspect of their results should remain unchanged. In both cases, the angular momentum contained in the infalling material is used to constrain the radii where infall occurs. Initial investigations indicate that as long as the viscosity is large enough, the exact location of infall doesn’t matter as the material spreads on relative short timescales.

Once the gas giant planets grow large enough they begin to open gaps in the solar nebula as a result of resonant interactions with the disk. As mentioned previously, variations in the gap opening timescale between Jupiter and Saturn may have caused the dichotomy seen in their respective satellite systems. In order to investigate this effect and see whether photoevaporation can clear circumplanetary subnebulae rapidly enough to be consistent with satellite formation simulations. The gap-clearing timescale can be estimated by assuming that it would occur on the viscous timescale to spread across the scale height in the protoplanetary disk [99].

\[
\tau_{\text{gap}} \sim \frac{H_{\text{PD}}^2}{\nu} \sim (10^{-3} - 10^{-4}) \times (1 - 10) \text{ Myr} \tag{5.5}
\]

The actual timescale for a growing planet to open a gap is likely longer than this estimate, but it must be substantially shorter than the planet’s accretion timescale in order to limit the final mass of the planet. A reasonable estimate for the gap opening timescale would be to assume a median value of \( \tau_{\text{gap}} \approx 2.5 \times 10^3 \text{ yr} \).
Figure 5.1: Parameterization of infall from the solar nebula shown as surface mass density as a function of radius. The functional form is taken from the results of Tanigawa et al. [105]. The functional form is $\Sigma \propto r^{-1}$ and is contained within $100R_p$. 

![Graph showing infall rate as a function of radius](image-url)
The infall rate is assumed to decay exponentially over a timescale, $\tau_{\text{off}}$.

$$\Gamma(r, t) = \Gamma(r) \exp \left( \frac{t}{\tau_{\text{off}}} \right)$$  (5.6)

This assumed form for the infall decay rate is also made by other authors in similar investigations [17, 111]. The infall decay time is further assumed to be of the same order as the gap opening timescale, $\tau_{\text{off}} = \tau_{\text{gap}} = 2.5 \times 10^3$ yr.

5.3 Models of Satellite Growth and Migration

In the context of our model, the solid material necessary for the growth of satellites is slowly delivered over the duration of the infall in small particles. Large particles in the circumstellar disk, which feeds the circumplanetary disk, settle rapidly to the midplane. However, small dust and ice particles will be entrained in the gas and delivered to the circumplanetary disk. This supply of small particles will not only enhance the disk in small particles, but provide the necessary material to build the multiple generations of satellites though to have formed around the giant planets.

As mentioned in Section 5.1, one advantage of the Canup & Ward model is the natural production of satellites that fit the common mass ratio seen in the regular satellite systems of the giant planets in our solar system. However, these authors were not able to produce the differences seen in the Jovian and Saturnian systems, or the resonances exhibited by the innermost three Galilean satellites.

Sasaki et al. [99] are able to produce four or five similarly sized satellites in the Jovian system in 80% of their runs whereas, in the Saturnian system only one large satellite remains in 70% of their runs. Their models, however, rely on a rapid dispersal mechanism for circumplanetary gas once the gap in the solar nebula has been opened. Photoevaporation could provide just such a mechanism as well as help to determine a natural outer disk boundary. As with Ward & Canup [117], the models of Sasaki et al. [99] rely on an ad hoc outer disk boundary.

This outcome may also be a natural result of the inside-out clearing of the solar nebula that occurs in many simulations which include irradiation from a central source. In particular, we
would like to draw the reader’s attention to (Figure 4) in Gorti et al. [34]. The snap-shots of the mass surface density in Figure 4 show the photoevaporation front sweeping outward past Jupiter’s location rather rapidly, but taking much longer to sweep past Saturn’s location. Although it is difficult to determine from the snap-shots shown in (Figure 4) in Gorti et al. [34], the outward-traveling photoevaporation front is moving at approximately $8.8 \text{ AU Myr}^{-1}$ through the region where Jupiter is located. This is roughly twice as fast as it is moving through the region in which Saturn is located, which is approximately $4.5 \text{ AU Myr}^{-1}$. The outward-traveling front not only sweeps past Saturn at a slower rate than Jupiter, but nearly 1 Myr later as well.

The inside-out clearing of the solar nebula may have exposed Jupiter’s circumplanetary disk to much more solar flux at an earlier time than Saturn. Also, at Saturn’s location the surface density seems to be decreasing globally on a similar timescale over which the front moves outward. If so, it may mean that the shutoff of the infall from the solar nebula onto the Jovian subnebula, because of the local depletion of the solar nebula due to inside-out clearing, would have happened sooner and on a shorter timescale than in the Saturnian system. In our scenario, the slow infall rates required by [18, 117] would occur as a result of gap opening, or partial gap opening in the solar nebula. Slow infall from the solar nebula, onto the circumplanetary nebulae, would continue until the outward-traveling photoevaporation front passes their respective locations, globally clearing the solar nebula and ceasing any further infall. In this scenario, the final structure of the regular satellite systems of the giant planets that we see today would have been determined by the rate and timing of the inside-out clearing of the solar nebula.

Despite the success of these models the resonant configuration of the Galilean satellites could not be maintained until an inner cavity was introduced [87]. As the satellites migrate inward, their migration is halted as the innermost satellite reaches the cavity. This causes all of the satellites, migrating in lock-step, to halt and remain in the resonant configuration that we see today.
5.4 Truncation of the Jovian-Subnebula

The regular satellites of Jupiter and Saturn extend to 0.06 \( r_H \). This value is much smaller than the extent of the circumstellar disk predicted by considering the angular momentum content of the accreting gas. By estimating the angular momentum of accreting gas as it travels from the solar nebula, through the Lagrange points, and onto Jupiter, Quillen & Trilling [94] estimated that the truncation radius is \( \sim r_H/3 \). A similar outer disk radius was found in numerical simulations which consider only the angular momentum content of accreting gas [9]. The recent, high-resolution simulation of Tanigawa et al. [104] indicates that accreting gas intercepts the Jovian circumstellar disk at \( \sim r_H/15 \) and that much of the gas has an angular momentum content that corresponds to even smaller radii than this.

Tidal truncation simulations produce circumplanetary disks that are truncated to radii that occur at \( \sim r_H/4 \) [73]. However, these outer disk radii are too large to explain the compact configurations of the regular satellite systems of Jupiter and Saturn, which extend to less than \( \sim r_H/20 \). The radial extent of the gas disk may have been truncated by photoevaporation; a process known to contribute to circumstellar disk evolution [21, 4].

Adams et al. [2] demonstrate how photoevaporation creates a subsonic outflow of gas in the disk that is well inside the gravitational radius, \( r_g \), where the thermal velocity of the hot disk atmosphere equals the planet’s escape velocity. We apply the [2] photoevaporation model to circumplanetary disks and find that the disks are truncated well inside the gravitational radius, \( r_g \). It is important to note that photoevaporation only needs to remove gas from the planet’s Hill sphere in order to truncate the circumplanetary disk; it does not need to remove the gas from the solar system.

Given a nominal FUV flux, photoevaporation can easily truncate the Jovian subnebula to 50 Jupiter radii and the Saturnian subnebula to 18 Saturnian radii. In the Saturnian system this is well within the orbit of Titan and, although in the Jovian system this is nearly twice the orbital radius of Callisto, the disk structure at the locations of the Galilean satellites will certainly be affected.
by the truncated disk. As stated above, amount the FUV flux at Jupiter’s location would be more than twice as large as the incident flux at Saturn’s location which would place the truncation radius at the location of Callisto. It is important to note that photoevaporation only needs to remove gas from the planet’s Hill sphere in order to truncate the protosatellite disk; it does not need to remove the gas from solar orbit.

The radial extent of the dust may be much less than the gas disk depending on how it is transported and where it grows. These processes may be controlled by the existence and location of dead zones. For the outer disk radii to set the scale of the present regular satellite systems, the solids from which the satellites formed must have been small enough to be coupled to the gas disk. This could occur if small solids are continually being supplied from the solar nebula or through collisional processes within the proto-satellite disk. However, if the solids grow on a timescale that is much shorter than the timescale for orbital decay due to gas drag, then the solids will decouple from the gas and the radial extent of the resulting satellite system would be set by the centrifugal radius, $r_c$. 
Chapter 6

Application of the 1D Model to Circumplanetary Disks: Satellite Formation

The study of the solar nebula presented in Chapter 4 showed that, for a reasonable range of parameters, photoevaporation has an effect similar in magnitude to that of changing the viscosity parameter. After confirming that photoevaporation plays a significant role in protoplanetary disks, we applied the 1D disk model, with the inclusion of infall from the solar nebula, to circumplanetary disks around Jupiter and Saturn.

Essentially, the same model that was used to investigate the evolution of the solar nebula is now applied to circumplanetary disks. The only significant change is that the model now includes infall from the solar nebula. As stated in Section 5.2, we use a parameterized infall that was developed using the results of Machida et al. [71]. Infall from the solar nebula allows these disks to attain a steady state, in which mass added from the solar nebula is balanced by mass loss onto the planet and to photoevaporation at the disk’s outer edge.

One of our primary goals with this investigation was to see if we could reproduce the gas-starved disks of Canup & Ward with a viscously accreting, photoevaporating model [18]. We were also interested to see whether photoevaporation could truncate circumplanetary disks and help provide an explanation for the compact regular satellite systems that we see around the giant planets today. Lastly, we wanted to investigate if the circumplanetary subnebulae produced in my models would be rapidly dispersed when infall from the solar nebula was terminated. I present the results from this study below.

We ran a number of simulations of circumplanetary disks about Jupiter and Saturn. As in
our study of the solar nebula study, both $\alpha$ and $T_{\text{env}}$ were varied to their extremes to see how these parameters affects the morphology and evolution of circumplanetary subnebulae. Here I will focus only on the Jovian models as they are pertinent to this thesis. For a detailed description of the Saturnian models and comparisons with Jupiter, please refer to Mitchell & Stewart [79].

6.1 Steady State Disks

All of the simulations presented in this chapter agree well with the low surface densities suggested by Canup & Ward [18]. As in the simulations of Machida [71], surface density enhancements are seen at roughly $25 r_p$, as a result of infall from the solar nebula. Such enhancements are not seen in recent, high-resolution simulations in which the infall occurs over a wide range of radii [104]. These enhancements likely arise from my choice of infall shape (Equation 5.3) and the fact that my viscosity depends only on radius and not on the local conditions in the circumplanetary disk. If such density enhancements are real, they may have implications for satellite growth. I will further explore the significance of these density enhancements in Section 6.3.

As stated in Section 1.2, recent observations confirm that the FUV emission of young, solar-type stars is sufficient for photoevaporation [44]. The UV flux can produce a wide range of envelope temperatures ($100 \text{ K} \text{–} 3000 \text{ K}$) depending on the magnitude of the flux [2, 78]. Figure 6.1 shows the radial mass surface density from three simulations in which the envelope temperature has been varied. The solid curve is of the fiducial model with $T_{\text{env}} = 600 \text{ K}$, whereas the dotted and dashed curves are for simulations with $T_{\text{env}} = 100 \text{ K}$ and $T_{\text{env}} = 3000 \text{ K}$ respectively. These simulation were all run with $\alpha = 10^{-3}$. One significant feature of these simulations is the enhancement in mass at $r \approx 25 R_J$ mentioned above. The truncation radii of these simulations ranges from $73 R_J$ in my high temperature simulation to $324 R_J$ in my low temperature simulation. In terms of Hill’s radius, these disk radii range from $0.098 - 0.44 r_H$ with the fiducial model’s outer radius at $0.17 r_H$.

The higher envelope temperature causes more erosion at the outer boundary and therefore causes the steady state disk to have a much smaller truncation radius. Despite having a much smaller truncation radius, the three runs presented in Figure 6.1 all have disk masses that lie in a
Figure 6.1: Steady state radial mass surface density of Jupiter’s circumplanetary disk. The three curves are for three different values of the isothermal, heated envelope. The solid curve is of the fiducial model with $T_{env} = 600$ K, whereas the dotted and dashed curves are for simulations with $T_{env} = 100$ K and $T_{env} = 3000$ K respectively. The current location of Callisto, $26.3 \, R_J$, has been marked with a dot-dashed line.
narrow range from $3.3 \times 10^{-5} - 1.2 \times 10^{-4} \, M_J$.

A suite of simulations was also run in which the viscosity parameter, $\alpha$, was varied from $10^{-4} - 10^{-2}$. Simulations of steady state circum-Jovian disks are presented in Figure 6.2. Again, the solid curve is of the fiducial model with $\alpha = 10^{-3}$, whereas the dotted and dashed curves are for simulations with $\alpha = 10^{-4}$ and $\alpha = 10^{-2}$ respectively. These simulation were all run with $T_{\text{env}} = 600$ K. A striking difference between the varied viscosity runs and those for a varied envelope temperature, presented in Figure 6.1, is that the outer disk radius is independent of the strength of the viscosity. It depends only on the mass loss rate at the outer boundary which, in my models, is controlled solely by the envelope temperature.

Another difference between the temperature and viscosity runs is in the masses of the steady state disks. The disk masses in the runs presented in Figure 6.2 each differ by an order of magnitude and lie in the range of $5.5 \times 10^{-6} - 5.4 \times 10^{-4} \, M_J$. This suggests that the steady state disk mass is dominated by the strength of the viscosity and not by the amount of photoevaporative mass loss.

The time-dependent, viscous evolution of my models allow us to accurately track the transfer of mass throughout my disks. The transfer of mass is important for the formation of satellites. The rate of mass transfer through the disk, $\dot{M}$ can be calculated at any radius by simply taking the derivative if the torque with respect to the specific angular momentum.

$$\dot{M} = -\frac{\partial g}{\partial h}$$  \hfill (6.1)

Disk models are often characterized by the slope of the radial mass surface density using a power law of the form

$$\Sigma(r) \propto r^{-q}$$  \hfill (6.2)

The slope of the radial surface mass density profiles of the models presented in Figures 6.2 are roughly power laws with $q = 1.3$. This slope is steeper than the slopes used in the disk models of [18, 99], who both assume $q = 3/4$. However, it is in accordance with my assumed radial temperature dependence of $r^{-1/2}$, which predicts $q = 1$. The fact that $q$ is slightly steeper than $q = 1$ can be attributed to the truncation of the outer boundary by photoevaporation. It must
Figure 6.2: Steady state radial mass surface density of Jupiter’s circumplanetary disk. The three curves are for three different values of the viscosity parameter, $\alpha$. The solid curve is of the fiducial model with $\alpha = 10^{-3}$, whereas the dotted and dashed curves are for simulations with $\alpha = 10^{-4}$ and $\alpha = 10^{-2}$ respectively. The current location of Callisto, $26.3 \ R_J$, has been marked with a dot-dashed line.
be kept in mind that all of the slopes mentioned here are \textit{ad hoc} and have been assumed by the various authors. It is unclear how steep these disks would actually be given a more comprehensive model with realistic viscosity that depends on the local conditions in the disk.

Figure 6.3 shows the mass transfer rate as a function of radius in a steady state circum-Jovian disk. This analysis was conducted on my fiducial model, with an envelope temperature of 600 K and a viscosity parameter $\alpha = 10^{-3}$.

Nearly equal mass is transferred inward and outward in the disk, with slightly more mass accreted onto the planet than is lost through the outer edge due to photoevaporation. Of the $2 \times 10^{-7} \, M_J \, \text{yr}^{-1}$ of material being accreted from the solar nebula, $1.25 \times 10^{-7} \, M_J \, \text{yr}^{-1}$ is accreted onto Jupiter and $0.75 \times 10^{-7} \, M_J \, \text{yr}^{-1}$ is lost through photoevaporation at the disk’s outer edge. Assuming a solar dust-to-gas mass ratio of 0.014, this is sufficient to provide enough mass in solids through the outer regions of the disk to build Callisto over the $10^5 \, \text{yr}$ required for it to remain undifferentiated \cite{18}.

\begin{equation}
M_{\text{tot}} = f \dot{M} \tau_{\text{acc}} = 1.9 \, M_{\text{Callisto}}
\end{equation}

where $f$ is the assumed dust-to-gas mass ratio of 0.014 and $\tau_{\text{acc}}$ is the accretion timescale of $10^5 \, \text{yr}$.

6.2 Decaying Infall

Another goal of this work was to validate photoevaporation as a potential mechanism for the rapid dispersal of circumplanetary nebulae as infall from the solar nebula wanes because of gap opening. I have performed one such simulation that investigates the decrease in infall rate onto the Jovian subnebula over a $2.5 \times 10^3 \, \text{yr}$ timescale (see Section 5.2). Figure 6.4 shows the temporal evolution of Jupiter’s circumplanetary disk as the infall is abated.

Similarly to Sasaki et al. \cite{99}, this simulation was carried out in the context of the “gas-starved” disk model of Canup & Ward \cite{18}. This is done because gap opening would likely occur in the final stage of giant planet accretion. Therefore, the simulation presented in Figure 6.4 was begun with the steady state solution from my Jovian fiducial model with $\tau_G = 5 \times 10^6 \, \text{yr}$. This value differs
Figure 6.3: Mass transfer rate in a steady state circum-Jovian disk as a function of radius. This analysis was conducted on my fiducial model, with an envelope temperature of 600 K and a viscosity parameter $\alpha = 10^{-3}$. Nearly equal mass is transferred inward and outward in the disk, with slightly more mass accreted onto the Jupiter than is lost through the outer edge due to photoevaporation. The mass accretion rate in the outer region of the disk is characterized by a constant mass flux rate. The region of constant flux begins exterior to $28 \, R_J$. Interior to $28 \, R_J$, the mass flux rate is decreases and at $\sim 24 \, R_J$ it becomes negative. The region inward of $28 \, R_J$ corresponds to the region over which infall from the solar nebula occurs. Inward of $\sim 24 \, R_J$ the mass flux rate is negative and is being accreted onto the planet.
Figure 6.4: Temporal evolution of Jupiter’s circumplanetary disk as the infall from the solar nebula exponentially decays. The radial mass surface density is shown at 500yr increments, with the solid bold lines indicating $t = \tau_{\text{off}}$, $2 \cdot \tau_{\text{off}}$ and $3 \cdot \tau_{\text{off}}$. 
slightly from that used by Sasaki et al. [99] of $2 \times 10^6$ yr, but as they have shown the distribution and composition of final satellites is insensitive to the exact value of $\tau_G$. The radial surface density is shown at 500 yr increments, with the solid bold lines indicating $t = \tau_{\text{off}}$, $2 \cdot \tau_{\text{off}}$ and $3 \cdot \tau_{\text{off}}$. The total mass of the circumplanetary disk decreased nearly two orders of magnitude from $5.5 \times 10^{-5}$ $M_J$ to $7.6 \times 10^{-7}$ $M_J$ over the course of this simulation.

This simulation was done using a nominal FUV flux, which corresponds to an envelope temperature of 600 K. Envelope temperatures can range from 100 K to 3000 K [2]. This shows that, even for a moderate FUV flux, photoevaporation is able to clear the Jovian subnebula on the very short timescale over which the infall wanes due to gap opening.

**6.3 Summary and Discussion**

We modeled and analyzed circumplanetary, protosatellite disks with a 1D numerical model that includes the combined influence of viscous forces and photoevaporation. The model also includes mass infall from the solar nebula, allowing for steady state solutions. These models were developed in the context of the “gas-starved” disk models put forth by Canup & Ward [18], during the late stages of giant planet growth, when the accretion rate from the solar nebula is limited. The limited accretion rate may have been a result of gap opening, the global depletion of the solar nebula, or a combination of both. With these models, I present a new mechanism for the truncation of circumplanetary disks.

Our models show that photoevaporation can truncate circumplanetary disks to radii that are in consistent with the locations of the regular satellites of the giant planets. The reference model produces outer disk radii truncated at $0.057 \ r_H$ and $0.17 \ r_H$ for Saturn and Jupiter respectively. These small outer disk radii provide a natural explanation for the locations of the regular satellites of Jupiter and Saturn.

Recently, the radial extent of the regular satellites of Jupiter and Saturn has been explained by the rapid accumulation of solids at the location of the infalling material, $r_c$ or by the truncation of the protosatellite disk by solar tides [18, 73]. Photoevaporation is invoked to naturally explain
the location of regular satellites. Models produce a large range of outer disk radii depending on the choice of envelope temperature (FUV flux). These outer disk edges range in radii from \(0.035 \, r_H\) to \(0.44 \, r_H\), with a mean value of \(0.16 \, r_H\). Even though my models show a wide range of outer disk edge radii based on envelope temperature, the masses of these disk all lie in a narrow range from \(3.3 \times 10^{-5} \, - \, 1.2 \times 10^{-4} \, M_J\) in the Jovian models. This narrow range of disk masses is to be expected because these simulations were all run with the same alpha-viscosity parameter of \(10^{-3}\).

These small outer disk radii may provide an obvious explanation for the locations of the regular satellites of Jupiter and Saturn. For the outer disk radii to set the scale of the present regular satellite systems, the solids from which the satellites formed must have been small enough to be coupled to the gas disk. This would occur if small solids are continually being supplied from the solar nebula or through collisional processes within the proto-satellite disk. However, if the solids grow on a timescale that is much shorter than the timescale for orbital decay due to gas drag, then the solids will decouple from the gas and the radial extent of the resulting satellite system would be set by the centrifugal radius, \(r_c\). The location and transport of solids in an actively supplied, photoevaporating circumplanetary disk is an important issue that merits further investigation.

The constant outward mass flux in the outer regions of the circum-Jovian disk, presented in Figure 6.3, indicate that while very low surface densities exist in the outer regions of these disks there is still significant amounts of gas being transported to this region. Assuming that solids are carried along with the gas as it is transported outward, there is sufficient mass present for satellite formation. Given a solar abundance of solids, I calculated that there is nearly twice as much mass in solids transported outward over a \(10^5\) yr time period than is needed to form Callisto. There are reasons to believe that the solids-to-gas mass ratio would be higher than solar and therefore I take this value as an underestimate.

Another aspect of these simulations that may have consequences on satellite formation is the density enhancement seen in the simulations at \(\sim 25 \, r_p\), the location of peak mass infall. A similar density enhancement was seen in Machida et al. [71], but was not present in more recent
simulations which include radiative transfer [9]. Similar features have not been seen in the newer 1+1D model, when applied to the Jovian subnebula (chapters 7 and 8). If real, these enhancements would have a significant impact on satellite formation. Density enhancements such as these are accompanied by pressure maxima. It has been shown that migrating solids can be trapped in such pressure maxima and rapidly grow into satellitesimals [58].

While the strength of the viscosity plays no role in the location of a disk's outer boundary, it does play a significant role in the total mass contained in a given disk. In a steady state, the mass accretion rate, $\dot{M} \propto \nu \Sigma$, is constant. A constant mass accretion rate implies that the mass surface density must be proportional to the inverse of the viscosity. This means the surface density is inversely proportional to the viscosity parameter, $\alpha$. One might naively assume that a larger surface density would result in larger satellites, but in actuality the opposite is true due to the increased rate of migration. A more massive, lower viscosity disk results in a less massive satellite system. Canup & Ward [18] found that satellites will only survive against type I migration for values of $\alpha \geq 10^{-3}$, assuming a solar gas-to-solid ratio for the infalling material.

By varying the viscosity parameter, $\alpha$, these models produce a wide variety of total integrated disk masses. These masses range from $2.3 \times 10^{-6} - 1.9 \times 10^{-4} M_S$ in the Saturnian system to $5.5 \times 10^{-6} - 5.4 \times 10^{-4} M_J$ in the Jovian system. These radii are all many of orders of magnitude smaller than the $\sim 0.02 M_J$ inferred for the “MMSN” approach. Such low mass surface densities are a result of the $5 \times 10^6$ yr timescale over which the infall occurs. Again, these conditions are necessary to produce the ice-rich compositions of the regular satellites of Jupiter and Saturn as well as the incomplete differentiation of Callisto and Titan.

The current 3-D hydrodynamical simulations used to model inflow from the solar nebula onto circumplanetary disks have insufficient resolution to identify the location at which the inflow intercepts the circumplanetary disk. In an effort to test what effect the location at which infalling material intersects the disk has on steady state disk morphology, I performed one test simulation in which the peak of the infalling material occurred at $35 r_p$ rather than at $25 r_p$. In the test simulation, the location of the disk outer edge was shifted farther out by $\sim 3\%$ and the total disk
mass increased by $\sim 5\%$. The changes are a result of a greater fraction of the infalling mass being transported outward rather than inward. However, in a disk in which the viscosity is calculated locally, this material may be rapidly redistributed and the location of deposition may be irrelevant.

As discussed earlier, Jupiter is expected to open a gap in the solar nebula as a result of resonant interactions. Regardless, of whether the gap opened is complete or partial, it would seriously inhibit the flow of gas onto circumplanetary, satellite forming disks. The simulation presented in Figure 6.4 shows that if such a restriction in infall is accompanied by a rapid dispersal mechanism, such as photoevaporation, it will cause the rapid removal of the circumplanetary disk as well. The rapid dispersal may also be aided by the increased ionization fraction, and increased viscosity, that may occur as a result of lower surface densities. These differences may help to account for the Jupiter/Saturn dichotomy.

The difference in solar flux, due to the difference in semimajor axes, may further account for the different evolutionary histories of Jupiter and Saturn. Saturn’s greater distance from the Sun would cause the incident solar flux to be a factor of $(a_J/a_S)^2$ less at Saturn than at Jupiter, where $a_J$ and $a_S$ are the semimajor axes of Jupiter and Saturn, respectively. Even in the compact configuration proposed in the Nice model, the difference in incident flux would be $(5.45\text{ AU}/8.18\text{ AU})^2 = 0.44$, implying that the incident flux would have been more than twice as strong at Jupiter’s location than Saturn’s [108, 81]. The greater amount of incident flux at Jupiter would have caused a greater amount of photoevaporative mass loss in the Jovian system. Furthermore, the increased rate of photoevaporation would have caused the Jovian subnebula to be more drastically truncated than the Saturnian subnebula given the same solar luminosity.
Chapter 7

An Improved 1+1D Disk Model

After our initial investigation with the one-dimensional model we developed a 1+1D model, that includes the vertical dimension. The new model includes vertical structure calculations at each radial grid point that include realistic opacities. This inclusion should produce surface density and temperature profiles that are very different than the smooth, monotonic ones previously assumed. A detailed model such as this allows the viscosity, using the $\alpha$-viscosity prescription, to be calculated based on the local conditions in the disk. Furthermore, this method avoids any \textit{a priori} assumptions about the radial temperature (or viscosity) profile. I describe this model, including the Keller box scheme and vertical structure calculation methods, below.

Begin with the viscous disk equation presented in Chapter 1, Equation 1.16. In order to non-dimensionalize these equations, I use the following three transformations for $t$, $h$ and $g$. Note that these transformations are different than those used for the non-dimensionalization in the VSG method (Equations 3.4, 3.5 and 3.6).

$$\tilde{t} = \frac{3\nu_0 t}{4R_0^2} \quad (7.1)$$

$$\tilde{h} = \frac{h}{(GMR_0)^{1/2}} \quad (7.2)$$

$$\tilde{g} = \frac{g}{4\pi N_c \langle \mu \rangle \nu_0 \sqrt{GMR_0}} \quad (7.3)$$

By substituting these non-dimensionalized quantities, one can derive a non-dimensional form for the viscous disk equation.
However, unlike before, no assumptions have been made about the radial dependence of the viscosity in this model. This causes the diffusion equation to be non-linear and a more sophisticated numerical technique is needed to solve the equations.

### 7.1 Keller Box Method

Meek & Norbury [75] developed a method of solving nonlinear Stefan problems based on the original discretization of Keller [48]. They solve problems of the form shown in Equation 1.6. The method is ideal for this problem for a variety of reasons. First, it is able to handle the inclusion of a source term, allowing for infall from the solar nebula to be included. Second, it solves for the outer boundary self-consistently. This is important because the photoevaporative mass loss is dependent on the location of the boundary and can be equal to the amount of mass accreted onto the planet [79]. Finally, and most important for my application, the numerical method is suitable for nonlinear problems, allowing us to calculate the viscosity based in the local conditions in the disk.

The derivation begins with the nondimensional diffusion equation, including the source term, Equation 7.4, for a thin disk in terms of the viscous couple and specific angular momentum. Then, let

\[
\frac{\partial \tilde{g}}{\partial \tilde{t}} = \left( \frac{\nu}{\nu_0} \right) \frac{1}{\tilde{h}^2} \frac{\partial^2 \tilde{g}}{\partial \tilde{h}^2} + \frac{R_0^2 \nu \tilde{h} \Gamma(\tilde{h}, \tilde{t})}{\nu_0^2 N_c \langle \mu \rangle} \tag{7.4}
\]

\[
\frac{\partial \tilde{g}}{\partial \tilde{t}} = \left( \frac{\nu}{\nu_0} \right) \frac{1}{\tilde{h}^2} \frac{\partial^2 \tilde{g}}{\partial \tilde{h}^2} + \frac{R_0^2 \nu \tilde{h} \Gamma(\tilde{h}, \tilde{t})}{\nu_0^2 N_c \langle \mu \rangle} \tag{7.4}
\]

However, unlike before, no assumptions have been made about the radial dependence of the viscosity in this model. This causes the diffusion equation to be non-linear and a more sophisticated numerical technique is needed to solve the equations.

\[
\frac{\partial \tilde{g}}{\partial \tilde{t}} = \left( \frac{\nu}{\nu_0} \right) \frac{1}{\tilde{h}^2} \frac{\partial^2 \tilde{g}}{\partial \tilde{h}^2} + \frac{R_0^2 \nu \tilde{h} \Gamma(\tilde{h}, \tilde{t})}{\nu_0^2 N_c \langle \mu \rangle} \tag{7.4}
\]

The derivation begins with the nondimensional diffusion equation, including the source term, Equation 7.4, for a thin disk in terms of the viscous couple and specific angular momentum. Then, let

\[
K(\tilde{h}, \tilde{g}) = \left( \frac{\nu}{\nu_0} \right) \frac{1}{\tilde{h}^2} \tag{7.5}
\]

and

\[
S(\tilde{h}, \tilde{g}, \tilde{t}) = \left( \frac{\nu}{\nu_0} \right) \frac{\tilde{h} R_0^2 \Gamma}{\nu_0 N_c \langle \mu \rangle} \tag{7.6}
\]

where

\[
\nu = \nu(\tilde{h}, \tilde{g}) \tag{7.7}
\]

and

\[
\Gamma = \Gamma(\tilde{h}, \tilde{t}) \tag{7.8}
\]
This results in
\[ \frac{\partial \tilde{g}}{\partial t} = K(\tilde{h}, \tilde{g}) \frac{\partial^2 \tilde{g}}{\partial \tilde{h}^2} + S(\tilde{h}, \tilde{g}, \tilde{t}). \] (7.9)

From here forth, the tildes will be omitted. It is now clear that this equation matches the form of Equation 1.6, as required by Meek & Norbury [75].

\[ \frac{\partial g}{\partial t} = K(h, g) \frac{\partial^2 g}{\partial h^2} + S(h, g, t) \] (7.10)

The Keller box scheme is an implicit method and is unconditionally stable. Thus, in theory there is no limit to the size of the time-step that can be taken. In practice, the method is limited by the Courant-Friedrichs-Lewy (CFL) condition. In essence, the CFL condition limits the size of the time-step by limiting it to an interval in which material with velocity \( v \) can move across a grid cell of width \( dr \) in a time \( dt \). It is expressed as,

\[ dt \leq \frac{dr}{v}, \] (7.11)

where \( dr \) and \( dt \) are the radial mesh size and time step size respectively. In terms of my nondimensional variables, the CFL criterion is,

\[ dt \leq \left( \frac{3\nu_0}{2R_0} \right) \frac{dh}{v}. \] (7.12)

Please refer to Section 6.1 to see how the radial velocity is calculated.

The Keller box scheme is a two-level scheme that solves two systems of linear algebraic equations at each time step to produce second-order approximations to the solution. Before the equations are discretized the following coordinate transformation is applied.

\[ \sigma = \frac{h}{s(t)} \] (7.13)

and

\[ \tau = t, \] (7.14)

where \( s \) is defined as the location of the moving boundary. The coordinate transformation results in

\[ s^2 \frac{\partial g}{\partial \tau} - \sigma ss \frac{\partial g}{\partial \sigma} = K(\sigma, g) \frac{\partial^2 g}{\partial \sigma^2} + s^2 S(\sigma, g, \tau). \] (7.15)
Now, split the second-order equation into two first-order PDE’s using

\[ v = \frac{\partial g}{\partial \sigma} \]  
\[ (7.16) \]

The two coupled first-order equations that result are

\[ s^2 \frac{\partial g}{\partial \tau} - \sigma s \dot{v} = K(\sigma, g) \frac{\partial v}{\partial \sigma} + s^2 S(\sigma, g, \tau) \]
\[ (7.17) \]
and

\[ v = \frac{\partial g}{\partial \sigma} \]  
\[ (7.18) \]

Depending on how these two equations are discretized, the two equations needed at both the first and second steps can be derived. Please refer to Meek & Norbury [75] for a thorough description of the discretization. After discretization, the equations can be thought of as a system of equations, such that

\[ F(y_{j+1}) = 0, \]  
\[ (7.19) \]

where \( y_{j+1} \) is a vector composed of the unknown quantities at the subsequent time-step \( j + 1 \) (or \( j + 1/2 \) for the first half-step of the method),

\[ y_{j+1} = (u_{0j+1}, v_{0j+1}, u_{1j+1}, ..., u_{Nj+1}, v_{Nj+1}, s_{j+1}). \]  
\[ (7.20) \]

Then, apply Newton’s method to the discretized system of equations resulting in,

\[ y_{j+1}^{(k+1)} = y_{j+1}^{(k)} - J^{-1} (y_{j+1}^{(k)}) F(y_{j+1}^{(k)}), \quad k = 0, 1, 2, ... \]  
\[ (7.21) \]

where \( J \) is the Jacobian matrix for the discretized equations. Next, perturb each unknown in the Jacobian and expand each element about the unknown in a Taylor series to generate the matrix equations.

The first step involves solving a matrix based on the two equations presented below. They are,

\[ g_{i-1j+1/2} + \left( \frac{\Delta \sigma}{2} \right) v_{i-1j+1/2} = g_{ij+1/2} + \left( \frac{\Delta \sigma}{2} \right) v_{ij+1/2} \]  
\[ (7.22) \]
\[
\begin{align*}
g_{i-1j+1/2} + \left( \frac{\Delta \tau}{s_j^2} \right) \left[ \frac{K}{2 \Delta \sigma} - \frac{\sigma_{i-1/2}s_j S_j}{4} \right] v_{i-1j+1/2} \\
+ g_{ij+1/2} - \left( \frac{\Delta \tau}{s_j^2} \right) \left[ \frac{K}{2 \Delta \sigma} + \frac{\sigma_{i-1/2}s_j S_j}{4} \right] v_{ij+1/2} \\
= g_{i-1j} - \left( \frac{\Delta \tau}{s_j^2} \right) \left[ \frac{K}{2 \Delta \sigma} - \frac{\sigma_{i-1/2}s_j S_j}{4} \right] v_{i-1j} \\
+ g_{ij} + \left( \frac{\Delta \tau}{s_j^2} \right) \left[ \frac{K}{2 \Delta \sigma} + \frac{\sigma_{i-1/2}s_j S_j}{4} \right] v_{ij} + \frac{S}{\Delta \tau} \tag{7.23}
\end{align*}
\]

The second step is also solved using a matrix made up of a system of equations. The two equations are developed using the same steps listed above; discretization, creating a matrix equation by applying Newton’s method and perturbing the Jacobian and expanding about the unknown quantities in a Taylor series. The result is the following two equations.

\[
\frac{1}{\Delta \sigma} g_{i-1j+1} + \frac{1}{2} v_{i-1j+1} - \frac{1}{\Delta \sigma} g_{ij+1} + \frac{1}{2} v_{ij+1} \tag{7.24}
\]

\[
\left( -\frac{s_{j+1/2}}{2 \Delta \tau} + \left[ \frac{1}{4 \Delta \sigma} \frac{\partial K}{\partial g} (v_{ij+1/2} - v_{i-1j+1/2}) + \frac{s_{j+1/2}}{4} \frac{\partial S}{\partial g} \right] \right) g_{i-1j+1} \\
+ \left[ -\frac{\tilde{K}}{2 \Delta \sigma} + \frac{\sigma_{i-1/2}s_{j+1/2}}{2 \Delta \tau} (s_{j+1/2} - s_j) \right] v_{i-1j+1} \\
+ \left( -\frac{s_{j+1/2}}{2 \Delta \tau} + \left[ \frac{1}{4 \Delta \sigma} \frac{\partial K}{\partial g} (v_{ij+1/2} - v_{i-1j+1/2}) + \frac{s_{j+1/2}}{4} \frac{\partial S}{\partial g} \right] \right) g_{ij+1} \\
+ \left[ \frac{K}{2 \Delta \sigma} + \frac{\sigma_{i-1/2}s_{j+1/2}}{2 \Delta \tau} (s_{j+1/2} - s_j) \right] v_{ij+1} \\
= - \left[ \frac{s_{j+1/2}}{\Delta \tau} (g_{ij+1/2} - g_{ij} + g_{i-1j+1/2} + g_{i-1j}) - \frac{\sigma_{i-1/2}}{2 \Delta \tau} (v_{ij+1/2} + v_{i-1j+1/2}) (2s_{j+1/2} - s_j) - s_{j+1/2} S \right] s_{j+1} \\
= - \left[ \frac{\tilde{K}}{2 \Delta \sigma} + \frac{\sigma_{i-1/2}s_{j+1/2}}{2 \Delta \tau} (s_{j+1/2} - s_j) \right] v_{ij} - \left[ -\frac{\tilde{K}}{2 \Delta \sigma} + \frac{\sigma_{i-1/2}s_{j+1/2}}{2 \Delta \tau} (s_{j+1/2} - s_j) \right] v_{i-1j} \\
- s_{j+1/2} \tilde{S} - \frac{s_{j+1/2}}{2 \Delta \tau} (g_{ij} + g_{i-1j}) \\
- \left[ \frac{1}{4 \Delta \sigma} \frac{\partial \tilde{K}}{\partial g} (v_{ij+1/2} - v_{i-1j+1/2}) + \frac{s_{j+1/2}}{4} \frac{\partial \tilde{S}}{\partial g} \right] (-2g_{ij+1/2} + g_{ij} - 2g_{i-1j+1/2} + g_{i-1j}) \\
+ \left[ \frac{s_{j+1/2}}{\Delta \tau} (g_{ij+1/2} - g_{ij} + g_{i-1j+1/2} + g_{i-1j}) \right] \\
- \frac{\sigma_{i-1/2}}{2 \Delta \tau} (v_{ij+1/2} + v_{i-1j+1/2}) (2s_{j+1/2} - s_j) - s_{j+1/2} \tilde{S} \right] (-2s_{j+1/2} + s_j) \tag{7.25}
\]
The matrix for the second step includes the equations above for the boundary conditions on $g$, but because $s_{j+1}$ is assumed as another unknown, another equation to close the system is needed. Depending on whether the disk outer boundary is in the sub- or supercritical regime, either Equation 3.18 or Equation 3.19 is used.

These boundary conditions are included in the last line of the matrix presented in Meek & Norbury [75] as Equation (2.20). In this matrix they use the variables $\Omega_1$, $\Omega_2$, $p$ and $q$. They are defined as follows:

$$\Omega_1 = -\frac{1}{2} \left[ -\frac{\sqrt{8\pi}}{3} \left( \frac{\nu_0}{\nu} \right) \frac{1}{s_j^4} \right]$$

$$\Omega_2 = 1$$

$$p = s_{j+1}$$

subcritical:

$$q = \frac{1}{2} \left[ -\frac{\sqrt{8\pi}}{3} \left( \frac{\nu_0}{\nu} \right) \frac{1}{s_j^4} \right] v_{Nj} - \frac{C_0 a_s r_k^2}{\sqrt{3\pi} \nu_0 R_0 s_j^{5/2}} \exp \left[ -\frac{r_k}{2 R_0 s_j^2} \right] + s_j$$

supercritical:

$$q = \frac{1}{2} \left[ -\frac{\sqrt{8\pi}}{3} \left( \frac{\nu_0}{\nu} \right) \frac{1}{s_j^4} \right] v_{Nj} - \frac{\sqrt{2\pi} a_s R_0}{3\nu_0 R_0 s_j} + s_j$$

7.1.1 Boundary Conditions

This model also uses a zero torque inner boundary condition and the torque exerted on the disk by the photoevaporating outflow as my outer boundary condition. In terms of the non-dimensionalized variables the outer boundary condition becomes

$$\tilde{g} = \left( \frac{r_d}{R_0} \right)^{3/2} = \tilde{h}_d^3$$

inner boundary:

$$g = 0$$

outer boundary:

$$g = \frac{3h_d^3}{4} = \frac{3s^3}{4}$$
As before, a second set of boundary conditions are needed. This second set controls the rate at which the outer boundary moves. In terms of my non-dimensionalized quantities for the Keller box scheme, the mass loss equations have the form,

\[
\dot{M} = \begin{cases} 
4\pi \langle \mu \rangle N_c a_s R_0 \tilde{h}_0^2 \\
C_0 N_c \langle \mu \rangle a_s \frac{r_g^2}{\rho_b^4} \exp\left(-\frac{r_g}{2R_0 \tilde{h}_0^2}\right)
\end{cases} 
\tag{7.34}
\]

### 7.1.2 Initial Conditions

The models using the Keller box scheme are initialized using the initial conditions from Dodson-Robinson et al. [27]. They are of the form,

\[
\Sigma = \Sigma_0 r^{-3/2}.
\tag{7.35}
\]

This initial condition can be used to create a disk that is both stable against gravitational instabilities (Toomre’s \(Q > 1\)) and massive enough such that Jupiter and Saturn form within mean disk lifetime of \(2 - 3\) Myr.

First I need to scale the disk to the correct mass by finding \(\Sigma_0\) above for a given total initial disk mass, \(M_0\).

\[
M_0 = \int_0^{r_d} 2\pi r \Sigma_0 r^{-3/2} \, dr
\tag{7.36}
\]

let \(A = 2\pi \Sigma_0\). Then,

\[
A = 2\pi \Sigma_0 = \frac{M_0}{\int_0^{r_d} r^{-1/2} \, dr}
\tag{7.37}
\]

which of course must be accomplished numerically at each time step.

In terms of the non-dimensionalized variables, the initial conditions can be expressed as the following.

\[
\tilde{g}_{init}(\tilde{h}) = \frac{A}{2\sqrt{2\pi} N_c \langle \mu \rangle R_0^{-3/2}}
\tag{7.38}
\]

This means that the initial conditions are such that the torque is constant with radius.

Although my models have initial conditions, and a time-dependent model, the models presented here are steady-state models. Furthermore, the steady-state models are independent of the
initial conditions and quickly evolve away from them. The models are begun using a viscosity that is proportional to the radius in the disk. Then, the vertical structure calculations are turned on, using a thermodynamic gradient that is purely adiabatic. Finally, the full vertical structure is turned on slowly over a 100 yr timescale.

### 7.2 Vertical Structure

In order to use the Keller Box scheme the viscosity in the disk needs to be calculated based on the local conditions. This is accomplished by calculating the vertical structure following the example in Dodson-Robinson et al. [27]. First, hydrostatic equilibrium is assumed in the vertical direction. This assumption is valid because the timescale to adjust to equilibrium is the shortest timescale in the disk. The vertical structure is calculated by integrating three coupled, first-order ODE’s that govern the temperature, pressure and energy flux from the disk surface to the midplane. The solution is then checked and a final solution is converged upon using an iterative, multi-dimensional Newton-Raphson method. The three coupled differential equations are,

\[
\frac{\partial P}{\partial z} = -\rho \Omega^2 z, \tag{7.39}
\]

\[
\frac{\partial F}{\partial z} = \frac{9}{4} \nu \Omega^2 \rho \tag{7.40}
\]

and

\[
\frac{\partial T}{\partial z} = \frac{\partial P}{\partial z} \frac{T}{P}. \tag{7.41}
\]

The temperature, pressure and energy flux, \( T, P \) and \( F \) respectively, are integrated downward in the negative \( z \)-direction. The system of equations is closed using the ideal gas law,

\[
P = \frac{\rho k_B T}{\langle \mu \rangle}. \tag{7.42}
\]

The Schwarzschild criterion is used to calculate the thermodynamic gradient, \( \nabla = d\ln T/d\ln P \).

\[
\nabla = \begin{cases} 
\nabla_{\text{rad}} & \nabla_{\text{rad}} \leq \nabla_{\text{ad}} \\
\nabla_{\text{rad}} & \nabla_{\text{rad}} > \nabla_{\text{ad}}
\end{cases} \tag{7.43}
\]
The adiabatic gradient is taken to be that of a diatomic gas, \( \nabla_{\text{ad}} = 2/7 \). The radiative gradient is calculated using

\[
\nabla_{\text{rad}} = \frac{3}{4} \frac{\kappa PF}{a c \Omega^2 z T^4},
\]

where \( a \) is the radiation density constant, \( c \) is the speed of light and \( \kappa \) is the Rosseland mean opacity. Opacities are taken from tables provided by Semenov et al. [100]. These opacities include both dust grain and gas phase opacities. They allow for layered compositional models and nonspheroidal shapes. A 5-layer spherical, composite dust particles employing the normal silicate dust model with \( \frac{Fe}{Fe+Mg} = 0.3 \) is used in the model presented here.

Mixing length theory is used to calculate \( \nabla_{\text{conv}} \). Follow the example of Milsom et al. [77, 50],

\[
W \equiv \nabla_{\text{rad}} - \nabla_{\text{ad}}
\]

and

\[
U \equiv \frac{3acT^3}{c_P \rho^2 \kappa \ell_m^2} \sqrt{8\varepsilon},
\]

where \( \ell_m \) is the mixing length, which is defined as the minimum of the convection zone top, or the pressure scale height \( H \). The heat capacity at constant pressure, \( c_P \), is calculated using,

\[
c_P = \frac{7R}{2\mu}
\]

Then define another variable such that,

\[
\xi = \nabla_{\text{conv}} - \nabla_{\text{ad}} + U^2 = \nabla_{\text{conv}} - \nabla_{\varepsilon} + U,
\]

where \( \nabla_{\varepsilon} \) is the thermodynamic gradient of a single convective element. Then, the following equation, which is cubic in \( \xi \), must be solved.

\[
(\xi - U)^2 + \frac{8U}{9} (\xi^2 - U^2 - W) = 0
\]

Boundary conditions for the three, coupled first-order equations that govern the vertical structure are needed. We use mixed boundary conditions that are initialized at the disk surface and are integrated down to the midplane using a shooting method. The integration is begun with
guesses for the height of the $\tau = 0.03$ surface, $z_{\text{surf}}$, the surface density, $\rho_{\text{surf}}$ and temperature at the disk’s surface, $T_{\text{surf}}$. Using the assumed surface temperature, $T_{\text{surf}}$, calculate the accretion temperature, $T_{\text{acc}}$, which is one of the variables to be integrated downward.

$$T_{\text{surf}}^4 = T_{\text{amb}}^4 + \frac{3}{4} \left[ \tau + f(\tau) \right] T_{\text{acc}}^4,$$

where $f(\tau)$ is the Hopf function, with an assumed value of 0.601242 at $\tau = 0.03$. $T_{\text{amb}}$ is the temperature of the atmosphere in which the circumplanetary disk is embedded. Next, initialize the other two variables of integration, the pressure and the energy flux. The pressure is calculated using the ideal gas law, Equation 7.42 and the energy flux is calculated using,

$$F = \sigma T_{\text{acc}}^4.$$

Now that the integration variables have been initialized, they can be integrated to the midplane. Once the midplane is reached, the procedure is iterated using a multidimensional Newton-Raphson method. This method requires conditions to be evaluated for a goodness of fit. The first condition is that the flux goes to zero at the midplane, $F_{\text{midplane}} = 0$. The second condition is such that the integrated density matches the mass surface density from my disk model,

$$2 \int_{z=0}^{z_{\text{surf}}} \rho dz = \Sigma.$$

The final condition allows the height of the $\tau = 0.03$ surface to be self-consistently determined. This is accomplished using the following equation from Dodson-Robinson et al. [27],

$$\frac{1}{\Omega^2 z_{\text{surf}}} = \frac{\tau}{\kappa(\rho_{\text{surf}}, T_{\text{surf}}) P}.$$

Once a solution is converged upon, the midplane temperature is used to calculate the kinematic viscosity at the radial location in question. The viscosity is calculated using Equation 1.1. The midplane viscosity is then used to further evolve the mass surface density. The entire process listed above must be repeated at each subsequent time-step.
Chapter 8

Application of the 1+1D to the Jovian Subnebula

We continue our investigation into the physical conditions that affect regular satellite formation by applying the 1+1D model to the Jovian system and model a late-stage circum-Jovian disk. This model is a steady state disk model. The steady state is possible because infall form the solar nebula is balanced by mass loss at the inner and outer edges of the disk. The mass surface density as a function of radius is shown in Figure 8.1. It is apparent from the figure that the radial surface density profile is consistent with the models proposed by Canup & Ward.

The most notable features of the model are the abrupt changes that occurs in mass surface density at $r \sim 18R_J$ and $r \sim 55R_J$. Although these changes are slight, they are deviations from the smooth profiles assumed by many authors in the past and may have significant impact on satellite growth and migration [18, 82, 83, 66]. These deviations from a smooth power-law profile occur because of our assumed smooth radial temperature profile. They were not seen previously because of a lack of a temperature and density dependent viscosity.

The abrupt changes in the mass surface density that are seen in the 1+1D model of the Jovian subnebula are accompanied by corresponding abrupt changes in the radial midplane temperature profile. The midplane temperature profile, along with the aforementioned abrupt changes is presented in Figure 8.2. The abrupt changes have been marked with dotted lines. Please refer to Section 8.2 for a detailed description of how these changes in temperature result from our choice of opacity model.

A key aspect of our 1+1D model is the ability to evaluate many of the assumptions made in
Figure 8.1: Mass surface density vs. radius from the steady state circum-Jovian disk model produced by the 1+1D model. The radius is shown in terms of Jovian radii.
Figure 8.2: Midplane temperature vs. radius from the steady state circum-Jovian disk model produced by the 1+1D model. The radius is shown in terms of Jovian radii. Sharp changes in the temperature profile can be seen at $r \sim 18 \, R_J$ and $\sim 55 \, R_J$ and are marked with dotted lines.
this, and earlier models. Our main purpose was to evaluate our assumptions about disk viscosity and angular momentum transport, but we can also assess one of our assumptions about basic disk structure. A quick analysis shows that the thin disk approximation is only weakly valid. The thin disk approximation assumes the scale height at any radius in the disk is much smaller than the radius, such that $H/r \ll 1$. Figure 8.3 shows the scale height in our 1+1D Jovian subnebula model. In the vast majority of the disk, the ratio of $H/r$ is bounded between $0.1 - 0.2$. Although the value for $H/r$ may be outside the range of the thin disk approximation, it is not so large that we have to concern ourselves with it. It should be noted that these values are consistent with the disk models presented in Canup & Ward [18], but that their scale height increases with radius, while our decreases.

### 8.1 Comparison with Existing Model

The vertical structure calculation in the 1+1D model requires significant computing power and cause the code to be very computationally expensive. Thus, in order to validate the model we decided to compare it to a similar model in the literature. Under slightly different assumptions, Alibert et al. [6] produced a circum-Jovian disk model that included both viscous diffusion and vertical very much like my own. Theirs is also an $\alpha$ disk model, but with $\alpha = 2 \times 10^{-4}$. Their goal was to investigate the thermodynamic conditions present in their models and see if they are consistent with what is known about the compositions of the Galilean satellites. The thermodynamic conditions in their model allowed the authors to determine which volatile species will condense and which will remain in the gas phase. They then compared these results to the known composition of the Galilean satellites [6, 84].

The vertical structure calculations performed in their model assumed hydrostatic equilibrium, and use nearly the same set of equations that we do. The primary difference being that they neglect convection and include only the radiative transfer of energy. The opacities in their models are both temperature and density dependent in a similar fashion to our own. For a detailed description of the differences between the opacities used in their model and our own, please see Semenov et al. [100].
Figure 8.3: Scale height vs. radius from the steady state circum-Jovian disk model produced by the 1+1D model. The radius is shown in terms of Jovian radii.
In short, the differences arise due to the difference in the compositional models of the dust and the evaporation temperatures used. Primarily, the opacities used in their work are systematically lower than those used in our own.

Other than the minor differences in the vertical structure calculations, the primary difference in our models are about how and when mass is added to the subnebula from the Solar nebula. Their model assumes that the circumplanetary subnebula undergoes two phases. In phase one the disk is actively accreting mass from the solar nebula directly onto the disk’s outer edge. The accretion rate isn’t constant. It is decreasing, based on the results of Jupiter formation models [5]. The outer boundary of the disk is held at $150R_J$ during phase one. In phase two, accretion from the solar nebula terminates because of its dispersal and the disk’s outer edge is allowed to expand. The transition occurs when the mass accretion rate onto the subnebula is $\dot{M} = 9 \times 10^{-7} M_J/yr$. After the transition the disk’s outer edge is held at $700R_J$. The subnebula then dissipates as it is slowly drained onto the planet.

There are a few other minor differences that prevent a direct comparison, but probably don’t strongly affect the disks produced in either model. We address one of those here. The difference lies in how the viscosity is calculated. Alibert et al. [6] use a vertically averaged viscosity and we use a midplane viscosity. The lower temperatures at large $z$ would cause the viscosities calculated in Alibert et al. [6] to be systematically higher than those in my model. This would cause their model to evolve faster than our own. However, the lower opacities in their models may be producing systematically lower temperatures and counterbalance this effect. It is hard to say, because of the coupled nature of temperature and viscosity in the disk. Regardless, the radial dependence of the viscosity should be relatively unchanged. The differences in outer assumed alpha also effect this in the same manner and the effects of each are hard to disentangle from one another.

Regardless of the differences in the model presented here and those in Alibert et al. [6], there are striking similarities between the two. When both models are plotted on a log-log plot, one can immediately see that the two models appear very similar overall. Figure 8.6 shows the circum-Jovian model presented above re-plotted on a log-log plot. Despite the differences at the
Figure 8.4: Midplane temperature profiles presented in Alibert et al. [6]. This is Figure 7 in the aforementioned work.
Figure 8.5: Mass surface density profiles presented in Alibert et al. [6]. This is Figure 8 in the aforementioned work.
inner and outer boundary, which arise from the different assumptions and modeling techniques, the overall shapes correspond nicely. Because of the decaying nature of their model, comparison is a bit tricky. However, the second curve from the bottom in Figure 8.5 has roughly the same mass surface density as our steady-state model and will be used for comparison. Both models have the same overall shape, including the concave down shape and the large drop-off at the outer edge.

The biggest similarity, other than the overall shape, are the abrupt changes in the slope of the radial mass surface density profile. The breaks in their model occur at various locations at various times, but a large break can be seen, at $r \sim 18R_J$, in the second lowest curve in Figure 8.5. The surface density of their model at that time roughly corresponds to the steady-state model presented here. It appears that the temperature dependent opacity in their model has also produced sharp changes in the radial temperature gradient as can be seen in the second to lowest curve in Figure 8.4. Since this discontinuity occurs at just over 100 K, it is safe to assume that it is due a drop in opacity caused by the sublimation of ice.

Although each component was tested and verified individually, it wasn’t possible to test the code in its entirety against any analytic model because one simply does not exist that includes all of the necessary physics. Despite the many differences in these two models they have produced strikingly similar outputs. The similarities in these two models gives us confidence in the numerical techniques presented in this thesis.

8.2 Opacity Effects

Unlike most previous circumplanetary disk models, our model uses opacities that are both temperature and density dependent [100]. The dependence on temperature plays a key role in the model presented here. Small changes in temperature cause phase changes in the gas, where various species condense out of the gas phase. These condensation fronts cause there to be sharp transitions in the opacity.

The abrupt changes in opacity seen in Figure 8.7 are caused by the condensation of various species. The increase in opacity is due to the increased amount of solids that occurs as a result
Figure 8.6: Mass surface density vs. radius from the steady state circum-Jovian disk model produced by the 1+1D model plotted on a log-log plot.
of the condensation. Sharp changes in the opacity cause there to be a sharp increase in the radial temperature profile at certain radii. These increases happen at \( r \sim 6, 20, \) and \( 60R_J \) and can be seen in the midplane temperature profile shown in Figure 8.2. As the temperature changes, different species are included as dust in the opacity calculations provided by Semenov [100]. It has been shown that surface density features caused by opacity changes in circumstellar disks can greatly affect the growth and migration of planetary embryos [76].

Figure 8.8 shows the temperature in the disk as functions of both radius and height. The contour lines shown are the temperatures at which various species are included in as dust in the opacity calculations. Please refer to Table 8.1 for a list of species and their respective sublimation temperatures. It is apparent from Figures 8.1 and 8.8, that these sublimation fronts correspond well with the locations at which the mass surface density changes abruptly.

Table 8.1: Dust component sublimation temperatures in opacity model [100].

<table>
<thead>
<tr>
<th>Species</th>
<th>( T_{\text{sub}} ) [K]</th>
<th>( \log(T_{\text{sub}}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ice</td>
<td>100</td>
<td>2.00</td>
</tr>
<tr>
<td>Volatile organics</td>
<td>275</td>
<td>2.44</td>
</tr>
<tr>
<td>Triolite</td>
<td>425</td>
<td>2.63</td>
</tr>
</tbody>
</table>

The abrupt changes in the temperature and surface mass density that occur as a result of the temperature dependent opacities used in this model have a large impact on the stability of the disk and the dynamics of growing solids. These features were not present in earlier models that employed constant opacities and/or power-law radial temperature profiles. I discuss some of the impacts of our choice of opacity below.

8.3 Stability Analysis

The transfer of material through astrophysical disks requires some mechanism for the transfer of angular momentum. A variety of mechanisms have been proposed to transfer angular momentum
Figure 8.7: Opacity as functions of radius and height in the 1+1D circum-Jovian disk model. All opacities are taken from Semenov [100].
Figure 8.8: Disk temperature as a function of radius and height in the 1+1D Jovian subnebula model. The sublimation temperatures of key species are shown with contours. Reference Table 8.1 for a description of temperatures and species.
in protoplanetary disks, with the magnetorotational instability being the most successful. The uncertainty in which mechanism is responsible is even more pronounced in circumplanetary disks.

We evaluate the circum-Jovian disk produced by our numerical 1+1D model for the viability that two possible instabilities can arise and grow. These instabilities lead to turbulence, which in turn leads to the transport of mass and angular momentum. The two instabilities that we focus on are magnetorotational instabilities, baroclinic instabilities. We neglect gravitational instabilities here because the Toomre parameter is never near its critical value. The low mass of the disk at each radius, combined with the relatively fast orbital speeds prevent any gravitational instabilities from occurring.

### 8.3.1 Magnetorotational Instability

The magnetorotational instability was first invoked as a means of generating turbulence in disks by Balbus & Hawley [10]. Until then, there was no viable mechanism, derived form first principles, that could explain the necessary level of turbulence required in astrophysical disks. Their proposed mechanism requires a differentially rotating disk and a weak poloidal magnetic field. The instability can be understood by considering an outwardly displaced parcel of gas in a differentially rotating disk with a vertical magnetic field threading it. The magnetic field tries to keep the disk in solid body rotation and will speed up the outwardly displaced parcel. This adds angular momentum to the parcel of gas, causing it to be displaced outward even further. Herein lies the heart of the instability. The force of the magnetic field on displaced parcels allows for high and low angular momentum components to interpenetrate and act as a viscous couple.

The instability, as investigated by Balbus & Hawley [10], is capable of sustaining turbulence and transporting angular momentum in a wide variety of conditions. However, they assumed the disk gas to be fully ionized and well coupled to the magnetic field. Gammie [31] put forth the idea that T Tauri disks may be insufficiently ionized outside of some critical radius ($R \sim 0.1$ AU) and that accretion may only occur through surface layers that are ionized by cosmic
rays. This may be true for large regions of circumplanetary disks as well, because they are cooler and denser than circumstellar disks. Various authors have made assumptions about the ability of circumplanetary disks to transport angular momentum, but very few have conducted detailed calculations to investigate the various instabilities that may occur.

The 1+1D model allows us to assess if our disks are sufficiently thermally ionized to couple with the magnetic field and if it is consistent with the assumptions made about the level of angular momentum transport and accretion through the disk. An estimation of the level of MRI responsible for driving turbulence is done by combining density and temperature in our model, along with assumptions about the dust-to-gas ratio, with the results from magnetohydrodynamic shearing-box calculations (Turner, Private Communication). These calculations include non-thermal ionization from FUV and X-ray radiation as well as the decay of radioactive nuclei.

A simple criterion for MRI driven turbulence has been derived, and can be determined by calculating the Elsasser number. The instability will occur if,

$$\Lambda \equiv \frac{v^2_{A z}}{\eta \Omega} > 1 \quad (8.1)$$

where $v^2_{A z}$ is the square of the Alfvén speed in the $z$-direction and $\eta$ is the resistivity [111]. If the instability occurs, the flow will be turbulent and $\alpha$ will be in the high regime and be in the range of $\alpha = 10^{-4} - 10^{-1}$. Otherwise, $\alpha$ will be in the low regime, where $\alpha < 10^{-6}$.

I evaluate this criterion using look-up tables provided by Neal Turner (Private Communication). His tables give the Hall, Ohmic and ambipolar diffusion rates, $v^2_A/\eta$. The diffusion rates can be interpolated from the table based on given values for the density, temperature, ionization rate and plasma beta. The density and temperature will be taken directly from our models, but we must make assumptions about the ionization rate and plasma beta. We assume the ionization rate to be, $\zeta = 10^{-18}$, based on the work presented by Fujii et al. [29]. The value for the plasma beta is dependent on our assumed value for the magnetic field strength in the disk. The field strength in protoplanetary disks can be anywhere from 10 mG to 10 G [94, 118]. Following Turner et al. [111], we assume the magnetic pressure is 0.1% of the midplane gas pressure. This assumption results in
plasma betas in our model that range between \( \approx 29 - 2.5 \times 10^6 \).

An analysis of the 1+1D circum-Jovian disk model shows that the diffusion rates for Hall and ambipolar diffusion result in Elsasser numbers which are orders of magnitude too small to produce, let alone sustain, MRI. The only Elsasser numbers which approach unity, the value required to trigger an instability, results from Ohmic diffusion. This can be seen in Figure 8.9, which shows the Elsasser numbers approach unity in the surface layers in the disk. This result is not surprising and is one that has been seen in other circumplanetary disk models [66, 111].

### 8.3.2 Baroclinic Instability

Baroclinic instabilities have long been studied in the context of planetary weather. The mechanism is responsible for much of the energy transport from the equator northward and is responsible for generating cyclones and anticyclones at mid-latitudes. The most striking example in our solar system is the Great Red Spot on Jupiter. The main difference between the instability in disks and the classical planetary analog is the presence of Keplerian shear. Baroclinic instabilities may provide a source of turbulence where MRI cannot function [53, 52, 91, 92, 95].

The baroclinic instability occurs when there is a misalignment between the pressure and density gradients in the flow,

\[
\nabla P \times \nabla \rho \neq 0. \tag{8.2}
\]

This is different from a barotropic flow where the pressure and density gradients are aligned. Unlike, barotropic flows, the vorticity is not conserved in baroclinic flows and vortexes can form. These vortexes then act as a form of turbulence and transport angular momentum.

Unlike in planetary atmospheres, where the vertical structure is considered important, it has been shown that the baroclinic instability can arise in disks simply due to a radial entropy gradient. The radial entropy gradient is defined as \( \beta \) where,

\[
\beta = -\frac{d \ln s}{d \ln r}. \tag{8.3}
\]

Klahr & Bodenheimer [53] were the first to show that non-isothermal disks should be baro-
Figure 8.9: Dimensionless Elsasser number for Ohmic dissipation in the 1+1D circum-Jovian disk model. These values were calculated under the assumption that $P_{\text{mag}} = 0.001P_{\text{gas, mid}}$. 
clinic. These vortexes transport angular momentum and allow for mass transport. Using a simple two-dimensional simulation, they showed that a baroclinic disk is unstable and forms strong geostrophic turbulence [53]. They found that the instability develops when the radial entropy gradient is larger than, $\beta = 0.57$. Klahr [52] performed a local linear stability analysis and found a transient linear instability that can amplify the initial instability only up to a certain point. He came to the conclusion that only nonlinear effects can lead to significant amplification.

Following up on these earlier studies, Raettig et al. [95] carried out 3D shearing sheet simulations and measured the Reynolds stresses. They found that angular momentum transport occurred for values of $\beta$ as low as 0.5 and that the amplification rate scales as $\beta^2$. They were able to translate their measured values of the Reynolds stresses into values of $\alpha$ in the context of the $\alpha$-viscosity model. Their models produce values of $\alpha \approx 10^{-3}$ for entropy gradients as low as $\beta = 0.5$ and values as large as $\alpha \approx 10^{-2}$ for radial entropy gradients as steep as $\beta = 2.0$. Circumplanetary disks are thought to have radial entropy gradients with $\beta = 0.5-2.0$. This has not been investigated in the context of circumplanetary disks.

The formation and growth of baroclinic instabilities is highly dependent on the temperature profile of the disk and its ability to cool [91, 92, 95]. Simulations of vorticity production show that vortex strength increases with increased background temperatures, larger background temperature gradients and larger initial temperature perturbations. However, the cut-off criteria for these values is uncertain as they decrease with increasing resolution [91]. The conditions necessary for baroclinic feedback, where temperature perturbations create vorticity which then reinforces the the temperature perturbation, are also sensitive to the radial temperature gradient as well as the heating and cooling of the gas. If the gas cools too quickly, no entropy is transported and the gas is essentially isothermal. If the gas cools too slowly, the gas is essentially adiabatic with constant entropy across the vortex [92, 95].

We analyze our models to see if the stability criterion is met. This will once again allow us to see if our assumptions about the level of turbulence are justified. An estimation of the strength
of this instability can be accomplished by calculating the global radial entropy gradient in the disk:

\[
\frac{ds}{dr} = -\frac{d\ln T}{d\ln r} + (\gamma_{2D} - 1)\frac{d\ln \Sigma}{d\ln r}
\]  

(8.4)

where \( s \) is the entropy, \( \Sigma \) is the mass surface density, and \( \gamma_{2D} \) is the 2-dimensional adiabatic index [95].

The minimum criterion for a baroclinic to occur is that the radial entropy gradient be greater than zero, \( \beta > 0 \). However, if the instability is to be amplified, a necessary condition for sustained turbulence, the entropy gradient must be higher than that. A parameter study, conducted using 2D shearing-box calculations, was used to determine the amount of angular momentum transport which occurred for various strengths of baroclinic instability. The authors found that values of beta \( \beta = 2 \) correspond to \( \alpha = 10^{-2} \) and that for \( \beta = 1.0 \) and even as low as \( \beta = 0.5 \) correspond to \( \alpha = 10^{-3} \) [95].

Using Equation 8.4, we calculated the entropy gradient in the 1+1D circum-Jovian disk model. The results of these calculations are presented in Figure 8.10. The entropy gradient in innermost portion of the disk \( (< 50 R_J) \) is at, or below, the threshold from instability, \( \beta = 0.5 \). However, there are two spikes in the entropy gradient that occur at the two locations where the abrupt changes in temperature and surface density mentioned above occur. Beyond 50 \( R_J \), the disk has a relatively large entropy gradient and may have substantial levels of turbulence, \( 10^{-3} < \alpha < 10^{-2} \). However, it must be remembered that the overall turbulence in our model is governed not only by the value of \( \alpha \), but on the sound speed and scale height in the disk; both of which would be very low because of the low temperatures, \( T_{\text{mid}} < 100 \) K, present in the outer disk.

Like the temperature and mass surface density, the spikes in the radial entropy gradient in the inner disk correspond to changes in the opacity that are caused by the condensation of various species. The spike at \( r \sim 20 R_J \) is due to the condensation of triolite, an iron sulfide mineral. Inward of this radius, only silicates and iron are able to condense out of the gas. The second spike in the inner region, at \( r \sim 30 R_J \), is due to the condensation of refractory organics such as tholins. The ice condensation front at the midplane lies at roughly \( r \sim 75 R_J \), well beyond the current
Figure 8.10: Radial entropy gradient in the 1+1D circum-Jovian disk model, calculated using Equation 8.4. The minimum criterion for a baroclinic instability to occur is that entropy gradient, $\beta > 0$. 

![Graph showing radial entropy gradient](image-url)
orbit of Callisto. However, in the surface layers the ice condensation front extends inward to nearly $r \sim 35 \ R_J$.

The location of the ice condensation front in our Jovian circumplanetary disk model indicates that our assumed choice of $\alpha$ is too large, especially in the satellite-forming region. The preceding assessment of baroclinic instabilities also indicates that the viscosity is too high in this region. The value for the radial entropy gradient below 50 $R_J$ corresponds to a value of $\alpha = 10^{-4}$, an order of magnitude smaller than the value that we used [95]. Were our model fully consistent with our baroclinic stability analysis, I would predict the temperatures in this region to be much lower than currently seen. The lower level of turbulence would produce less heat, providing for a lower midplane temperature more consistent with the ice/rock fraction seen in the Galilean satellites.

8.4 Summary & Discussion

The Jovian subnebula model presented here demonstrates that a temperature and density dependent opacity produces disk models with radial profiles that do not follow simple power laws. Even though this was an expected result, it has only been investigated in the context of circumplanetary disks in limited context. Although the deviations from a power law are small, they have large implications for the growth and migration of solids as well as the generation of turbulence and angular momentum transport.

Because of the large computational expense of this model we were only able to produce a single run at this point. Therefore, we have compared our model to the closest thing we could find to our own in the literature. Despite the differences in the assumptions made, namely in where and how mass is added to the disk, our model correspond well with an earlier model presented by Alibert et al. [6]. The agreement in these two models gives confidence that the 1+1D model is working correctly.

Overall, the circum-Jovian disk model presented here is consistent with those presented in Canup & Ward [18]. When compared to their slow-inflow, low-opacity model, our model has surface densities that decrease faster with radius and temperatures that decrease slower with radius. This
means that growing satellites in our model would experience higher temperatures and lower surface
densities than those in their model. This is true in general, but it is unclear how the non-power-law
profiles in our models would affect the rate, and direction, of growing satellite embryos. Despite the
differences, and uncertainties, the similarity of their model and our own leads us to the conclusion
that the low surface densities and relatively low temperatures would be able to produce the Galilean
satellites and the slow growth of Callisto.

In our model, ice can only form beyond $r \sim 75 \, R_J$, in the outer regions of this disk. Even
if we consider ice condensation in surface layers, ice can only form beyond the current location of
Callisto. However, ice is known to be a large constituent of the three outermost Galilean satellites.
If our model is to believed, this tells us that either ice was transported inward efficiently and was
accreted then onto the satellites, or that they formed farther out in the disk than where they are
seen today and that they migrated inward subsequent to formation. Migration has already been
used to successfully explain the common mass ratio of the giant planets and their satellites, as well
as the Laplace resonance of the inner three Galilean satellites. It is therefore very likely that the
Galilean satellites formed farther out in the circum-Jovian disk, where water ice could condense,
and then migrated to where they are seen today.

It was mentioned in the previous section that the large midplane temperature in our models is
not consistent with the composition of the Galilean satellites and that we can potentially lower these
temperatures in future models by using an $\alpha$ that is an order of magnitude smaller than the current
value. One ramification of this choice would be an increase in the steady-state surface density in this
region. This would arise because of the inverse relationship between viscosity and surface density in
steady-state disks. It is uncertain how large the surface density would increase, but it would likely
increase by an order of magnitude. Surface densities in the range of $10^4 \, \text{g cm}^{-2}$ are incompatible
with the constraint of an undifferentiated Callisto. A likely remedy to this problem would be to
decrease our infall rate. Canup & Ward [18] showed that the surface density of a steady-state,
circumplanetary disk is proportional to the infall rate and inversely proportional to the viscosity.
An order of magnitude decrease in the infall rate would increase satellite growth timescales beyond
the lifetime of the solar nebula. This may indicate that a decaying infall is needed to keep the midplane cool enough to form ices and still grow the satellites in a reasonable amount of time.

One of the primary simplifying assumptions that was made in the 1+1D model is that $\alpha$ is constant throughout the disk. Our stability analysis showed that there are indeed regions in which the viscosity should be higher than others. These regions correspond to the peaks in the radial entropy gradient shown in Figure 8.10. It would be fairly straightforward, yet computationally expensive, to evaluate the entropy gradient at each time-step and adjust $\alpha$ correspondingly on the fly. It would be very interesting to investigate the affect that this may have on the disk structure.

Turner et al. [111] investigated magnetic coupling in circumplanetary disks for both minimum mass and gas-starved models. They found that the minimum mass models are all MRI dead throughout and that the gas-starved starved models, in general, have MRI active surface layers. They concede that if the surface density in minimum mass models drops off considerable beyond the orbit of Callisto, the outer regions could become MRI active and resemble the two-component ME model. They conclude that circumplanetary disks have conductivities sufficient for MRI to be active [111]. However, the MRI may not be sufficient to sustain well-developed turbulence [30]. Fujii et al. [30] point out, that because of the compact nature of circumplanetary disks, the orbital timescales are a minimum of two orders of magnitude shorter than in protoplanetary disks. These high orbital velocities prevent the magnetic field from coupling to the disk gas efficiently, and any surface layers that develop will not couple well with the midplane of the disk. They conclude that the disk mass would increase until it becomes gravitationally unstable unless a mechanism other than MRI can be found for generating turbulence.

We found no evidence that MRI can occur in the circum-Jovian disk model presented here, except perhaps in the very surface layer in the disk. They temperatures are too low for a sufficient level of ionization, limiting the coupling with the magnetic field. We did however find evidence that the baroclinic instabilities can play a role in generating turbulence and shaping circumplanetary disks. There are peaks in the radial entropy gradient in the inner disk high enough for sustained turbulence. In the outer disk, the radial entropy gradient is also large enough for baroclinic instabilities to be
sustained and grow. The low levels of turbulence in the inner disk, combined with the high levels of turbulence in the outer disk, inferred from our analysis of baroclinic instabilities may lend credence to the ME model. It would be very interesting to see if a fully self-consistent model, as described in the previous paragraph, would result in a disk structure consistent with the ME model.

Another example of the baroclinic instability, not investigated here, is the vertical shear instability investigated by Nelson et al. [86]. The vertical shear instability is a linear instability and may act to trigger the non-linear baroclinic instability presented above. The vertical shear instability can be triggered for much lower values of the radial entropy gradient. It has even been shown to develop in simulations with zero radial entropy gradient, provided that the radial density gradient is steep enough and that the thermal relaxation time is short enough that the original temperature gradient is maintained. It is unclear whether this holds true in the inner regions of circumplanetary disk where the orbital and dynamical timescales can be quite short.

One source of turbulence that we have neglected is that due to turbulence generated as the infalling, pre-shocked material intercepts the disk. This causes a shock and, in some circumstances, can generate substantial turbulence. We have neglected this contribution, because simulations indicate that the shock occurs at some distance above the disk surface and facilitates the transfer of infalling material from the circumstellar disk onto the circumplanetary disk, rather than transport within the circumplanetary disk.

In this analysis we have neglected heating from accretion onto Jupiter. The contributions from various sources to the heating of the disk has been investigated using semi-analytic models and the results indicate that radiation from a young Jupiter would affect the inner-most region of the disk ($< 25 R_J$), but that the contribution drops drastically. However, at most, its contribution to disk heating is an order of magnitude less than the contribution from viscous dissipation within the disk [72].
Chapter 9

Summary

I have presented here a variety of numerical simulations of viscous disks around young planets and stars. One feature that they all have in common is a detailed treatment of the disks outer boundary that includes both viscous spreading and photoevaporative mass loss. By treating the outer boundary in this fashion, these models differ significantly from most earlier viscous disk models in that the mass transport is not dominated my inward accretion, but also have a significant portion of the mass being transported outward as well.

I developed two separate, but similar computer models for this thesis research. The first is a time-dependent, one-dimensional, viscous disk model that uses the formalism of the Stefan problem to self-consistently treat the balance between viscous spreading and photoevaporative mass loss at the disks outer edge. The first model made the simplifying assumption that the viscosity at any point in is proportional to the radial location of said point in the disk. In this model we were able to include the continual infall of gas and dust from the solar nebula onto circumplanetary disks by including a source term in out viscous disk equation. These models were developed to investigate the affect that the removal of mass, and subsequent outward mass transport, has on the structure and evolution of circumstellar and circumplanetary disks.

The first, and simpler, of the two models was first used to model the solar nebula to investigate disk morphology and evolution, as well as the growth and survival of giant planets in such disks. It was found that a 1D time-dependent disk model, with a truncated outer boundary, produces shallower profiles than those predicted for a steady state disk. The evolutionary timescales of
the model disks can be sped up or slowed down by altering the amount of far ultraviolet flux or the viscosity parameter $\alpha$. Although they similarly affect relevant timescales, changes in the far ultraviolet flux or $\alpha$ produce disks with drastically different outer radii.

The strength of the viscosity and the amount of FUV flux (envelope temperature) were both able to affect the evolutionary timescales of the disks produced in various simulations. A small FUV flux or low viscosity were both found to produce longer evolutionary timescales than our reference model. Both models were more successful than our reference model in growing the giant planet cores, but they produced very different radial surface density profiles. The low FUV flux model produced a radially extended disk whereas the low viscosity model produced a radially contracted disk with a higher surface density in the giant planet forming region. The differences in these two models may provide a natural explanation for the location of the outer edge of the solar system.

Even though the outer-disk radii which result from these simulations vary widely, many of them are consistent with the current outer edge of our solar system, the Kuiper Belt. This may provide a natural explanation for the size of our solar system. A feature that can not be explained by models that make the standard assumption about the outer edge, a zero-torque boundary that is allowed to expand to infinity.

It was expected that the radial surface density profiles would be $\propto r^{-1}$, because of the $r^{1/2}$ temperature profile that was assumed in these models. Given a high enough FUV flux, and corresponding large envelope temperature, FUV driven PE can truncate the outer boundaries of circumstellar disk, which in turn steepens the radial surface density profile, such that $\Sigma(r) \propto r^{-1.25 \pm 0.88}_{+0.33}$. Although this is steeper than expected, it isn’t as steep the profile predicted by Desch [26].

Regardless of the choice of envelope temperature, or viscosity parameter $\alpha$, all of our disk models evolved and decayed on relatively short timescales. There were large changes in both overall disk mass and radial mass distribution as these disk evolved. This is not surprising. Such disk have been observationally constrained to have lifetimes of 5 – 10 Myr, which is comparable to the timescale over which giant planets are though to form. While not surprising, these models
confirm that it is insufficient to model giant planet formation in steady-state disks, as has been by so many authors in the past.

In order to further investigate the affect that an evolving disk, with a truncated outer boundary and outward mass transport has on growing planets, we did analytic calculations of the growth and migration rate of giant planet embryos using the models which resulted from our numerical code. Like Desch [26], we assume the giant planets formed in the compact configuration of the Nice model [108]. Despite the decaying surface densities, and taking migration into account, we find that the it is possible to grow the giant planets in some of our models. It is a bit counter-intuitive. Although a high PE rate, or high viscosity, limits the the amount of material available for accretion, the decaying surface density prolongs migration rates and gives the giant planets sufficient time to form.

After some slight modification, the 1D model was applied to circumplanetary disks and regular satellite formation. Other than scaling down the central object, the largest modification that was made was including a source term that allowed for the continual accretion of material from the solar nebula onto a circumplanetary disk. The most important effect that this has is to allow for steady state solutions, where the additional mass gained through the inflow is balanced by mass lost at the inner and outer edges. These models were done in the context of the CW model and were conducted to verify their findings with a detailed, fully numerical model. Not surprisingly, all of our resultant disk models were consistent with the low surface densities and truncate outer boundary posited in the CW model.

A major constraint on the formation of the Galilean satellites is the partially differentiated state of Callisto. It must form slow enough (> $10^5$ yrs) that the heat from accretion is insufficient to melt it completely. Our surface densities are low enough for such to occur. However, in such a low surface density disk it must be verified that there is sufficient solid material delivered to Callisto’s orbital location for it to form. The large outward mass transport in the disk models presented here delivers such solids the the outer regions of the disk from where it is delivered by the infall from the solar nebula. Assuming a solar solid-to-dust ratio, outward mass transport delivers enough solid
material to Callisto’s orbital location for it to form in $10^5$ yr, the minimum time required for its formation.

The Jupiter/Saturn dichotomy can be explained by varying the timescales over which the planetary subnebulae dissipate. Titan may have been produced as the circumplanetary disk of Saturn slowly dissipated, causing the existent satellites to dynamically evolve and merge into a single large satellite. In contrast, Jupiter’s subnebula is thought to have dissipated rapidly, thereby causing the existent satellites to remain dynamically “frozen”, with the innermost three remaining in their 4:2:1 Laplace resonance. The source term in these models was set up such that it could be turned off over a given timescale, $\tau_{\text{off}}$. These models show that even given a nominal value for the FUV flux, the circumplanetary nebula can be dissipated very rapidly. Essentially, the PE mass loss, combined with accretion onto the host planet, is large enough that the subnebula will decay on whatever timescale over which infall from the solar nebula wanes.

Until now, the models by Shigeru Ida and his research group have used static ad hoc disk models. The possibility has been discussed of a collaboration in which he uses our detailed circumplanetary disk models to conduct his satellite formation simulations. Ideally, our two models would be coupled and any interactions between the gas disk and growing embryos would be included. It would be very interesting to see how migration would work in the decretion disks that we have produced here. As far as I know, migration has not been explored in disks where a large portion of the gas is being transported outward, rather than inward.

The second model was very similar to the first, except that no assumption about the viscosity. Although the viscosity was modeled using the very common Shakura & Sunyaev $\alpha$—viscosity prescription, the scale height and sound speed were determined using the local conditions in the disk based on detailed vertical structure calculations. This model was used to investigate viable methods of generating turbulence in circumplanetary disks. Magnetorotational and baroclinic instabilities were the two that I focused on in this work. In this study, this model was only used to model the circum-Jovian subnebula.

Unlike most circumplanetary disk models used today, our model includes an opacity that is
dependent on the local density and temperature conditions in the disk. This prevents the need to make assumptions about the temperature, and hence the viscosity, profile in the disk. Most models to date use assume isothermal disks, or disks with power-law radial temperature profiles. These models produce only smooth, power-law profiles for the surface mass density. The opacity choice in our models allows for surface density enhancements in regions where the opacity changes abruptly. These density enhancements could be ideal places for the solids to accumulate and satellite embryos to grow. More interestingly, they are places where baroclinic instabilities arise and turbulence is generated. This helps rectify the problem with the apparent lack of MRI.

Up to this point we have always assumed the viscosity parameter, \( \alpha \), to be constant. In the 1+1D model it would be possible to adjust \( \alpha \) based on the amount of turbulence we believe is being generated at each radial location. In the solar nebula, as well as in circumplanetary disks, this may cause the inner and outer regions of the disk to have very different viscosity profiles. This is exactly what is seen in our analysis of baroclinic instabilities in our circum-Jovian disk model. Desch (private communication) believes this may help produce the steep profiles that he has predicted for the solar nebula. With regard to circumplanetary disks, this is the type of disk predicted in the ME model, which assumes a low viscosity inner disk, surrounded by a larger viscosity outer disk.

In regard to this last point, it should be noted that the Jovian circumplanetary disk model presented here resembles a hybrid of the two competing disk models of Canup & Ward and Mosquiera & Estrada. Our model is essentially a gas-starved model, but our temperature profile, as well as our baroclinic instability analysis indicate that the viscosity in the inner region of the disk should be much lower than in the outer regions. The similarity to CW is not surprising as we essentially built a viscously accreting model, with slow infall from the solar nebula. However, it is unclear how closely our model would resemble that of CW if \( \alpha \) was allowed to vary in a manner self-consistent with the radial entropy gradient. I suspect it would resemble even more that of ME. Our results seem to indicate that the most likely disk model is a hybrid between the two, what I call a gas-starved, two-component subnebula.

Now that I have these two models in good working order, especially the newest 1+1D model,
there a number of aspects of planetary accretion disks that I would like to investigate. With regard to the solar nebula, it would be very interesting to allow for variable FUV sources. This could perhaps mimic a cluster member going through the asymptotic blue giant phase, where it would bathe the Sun’s birth cluster in a large amount of FUV radiation. Variable FUV could also model the radiation field experienced by the Sun as it orbited throughout its birth cluster, with increasing FUV flux as it neared the center and less as it moved to the periphery.

Another interesting thing to investigate would be the transport and growth of solids. This would apply to both circumstellar and circumplanetary disks. The large amount of outward mass transport, along with the truncated PE boundary at the outer edge may cause a large amount of solids to be transported to the disk outer edge, where they could grow into larger particles. Such an accumulation of solids would almost cause a back reaction of the gas. This would be another interesting aspect to investigate. The detailed vertical structure calculations in the 1+1D model would facilitate this without too much hassle.

Lastly, I would also like to apply my 1+1D model to the Saturnian system and look at how the Jovian and Saturnian systems compare and contrast in the context of our model. It would be interesting to identify where any sharp changes in opacity may be in the Saturnian subnebula. I could Compare this to where they occur in the Jovian subnebula and see if any natural explanations for the Jupiter/Saturn dichotomy arise naturally in our models.
Bibliography


Appendix A

Analytic Derivation of Photoevaporative Mass Loss

The supercritical mass loss rate due to photoevaporation by an external FUV source can be readily derived using a simple, spherically symmetric model of outflow from an infinitely dense cloud of radius $r_c$. Assuming that gas, of mean molecular weight $\langle \mu \rangle$, is driven outward at a constant speed, $v_w$, the mass loss rate is

$$\dot{M} = 4\pi r^2 \langle \mu \rangle v_w n(r)$$  \hspace{1cm} (A.1)

where $n(r)$ is the number density of the flow at radius $r$. The column density of the outflow, $N_H$, is given by

$$N_H = \int_{r_c}^{\infty} n(r) dr.$$  \hspace{1cm} (A.2)

Equation (A.1) can then be solved for $n(r)$, substituted into Equation (A.2) and integrated.

$$N_H = \int_{r_c}^{\infty} \frac{\dot{M}}{4\pi r^2 \langle \mu \rangle v_w} dr = \frac{\dot{M}}{4\pi \langle \mu \rangle v_w r_c}$$  \hspace{1cm} (A.3)

The result is then solved for the mass loss rate, $\dot{M}$. The mass loss rate is dependent on the column density of attenuation, $N_H$, and proportional to the radius, $r$.

$$\dot{M} = 4\pi r_c \langle \mu \rangle v_w N_H$$  \hspace{1cm} (A.4)

The outflow velocity can then be set equal to the sound speed in the FUV heated disk “atmosphere” which is analogous to the isothermal atmosphere in the simplified model that was derived by Adams et al. [2]. This results in the supercritical mass loss rate differing only by the geometric factor $F$, which incidentally is of order unity.
A visual magnitude of extinction $A_v$, of order unity, typically requires the column density $N(H) \approx 5 \times 10^{21}\text{cm}^{-2}$. A column density of roughly $10^{21}\text{cm}^{-2}$ is a generally accepted value for complete extinction [47, 2]. If the disk atmosphere is heated to a temperature of 1000 K [2], then the gravitational radius, beyond which heated gas can escape from the host planet, is about 100 AU (see Equation 1.3). Our estimate of the mass loss rate is therefore $\dot{M} \approx 1.5 \times 10^{-7}\text{M}_\odot\text{yr}^{-1}$, which is certainly enough to clear the circumplanetary subnebula on the $10^6\text{yr}$ timescale needed to match observations.
Appendix B

Analytic Test of Variable Space Grid Method

Various methods, such as similarity solutions, have been used to solve Stefan problems analytically. I used these solutions to test the numerical code. One such test was done on the following system, a Stefan problem of transient heat conduction in a melting slab [14]. The one-dimensional, finite slab is insulated at $x = 0$ and has a propagating phase change at $x = X(t)$, where heat flows into the melting face at a rate $H(t)$.

$$
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < X(t)
$$  \hspace{1cm} (B.1)

with the boundary condition

$$
H(t) = \frac{\partial u}{\partial x} - \frac{dX}{dt}
$$  \hspace{1cm} (B.2)

Given the following boundary and initial conditions

$$
u(0, t) = 0, \quad x = 0 \hspace{1cm} (B.3)$$

$$
u(X(t), t) = 0, \quad x = X(t) \hspace{1cm} (B.4)$$

and

$$
u(x, 0) = g(x) \hspace{1cm} (B.5)$$

and assuming $X(t) = 1 - t$, the exact solution is of a self-similar form

$$
u(x, t) = \exp(\pi^2) \frac{\sin(\pi \xi)}{(1 - t)^{1/2}} \exp\left(\frac{-\pi^2}{1 - t} - \frac{\xi^2(t - 1)}{4}\right) \hspace{1cm} (B.6)$$
where $\xi = x/X(t)$.

The exact analytic, self-similar solution was used to check the VSG method. I tested the code for convergence against the exact analytical solutions by increasing the spatial grid resolution. These tests were all done with the same sized time steps. The results of these tests have been tabulated in Table B. The numerical results are in good agreement with the exact solution and exhibits the expected convergence as the number of grid point, $N$, increases.

Table B.1: Analytic test for the VSG method. Results of the tests for convergence against exact, analytic self-similar solution at a final time of $t = 0.25$ are shown.

<table>
<thead>
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<th>$x/X(t)$</th>
<th>$u$ (actual)</th>
<th>$u$ ($N = 50$)</th>
<th>$u$ ($N = 100$)</th>
<th>$u$ ($N = 200$)</th>
</tr>
</thead>
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<td>0.000000000</td>
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<td>0.02546055</td>
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<td>0.04215575</td>
</tr>
<tr>
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</tr>
<tr>
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<td>0.000000000</td>
<td>0.000000000</td>
<td>0.000000000</td>
<td>0.000000000</td>
</tr>
</tbody>
</table>

As a further test of convergence on the problem at hand I have completed a number of simulations with a variety of grid sizes. I have checked for both spatial and temporal convergence. Figure B shows the spatial convergence as the number of grid spaces increases. The convergence has been calculated by differencing the surface density of each of the lower resolution simulations from the surface density of the highest resolution simulation ($N = 800$) and then normalizing by the innermost available grid space. The computed convergence is shown with plus symbols connected by solid lines. For comparison, the expected $1/N^2$ convergence is shown over-plotted with x’s connected by dotted lines. It can clearly be seen that the actual convergence is very close to the expected convergence. I have chosen to use $N = 200$ for my number of grid spaces. This allows for simulations that complete in a reasonable amount of time and is acceptably accurate, to within less than 0.5% of the highest resolution simulation. Due to the large uncertainties in many model parameters, I feel this level of convergence is acceptable.

I have also investigated the temporal convergence of my model. For this test I have divided
Figure B.1: The spatial convergence of a number of simulations with various numbers of grid spacings. The calculated convergence is shown with plus symbols connected by solid lines. The expected $1/N^2$ convergence is shown with x’s connected by dotted lines. In all of my simulations I use $N = 200$ grid spaces.
the end times of each simulation by the end time of the highest resolution simulation. Again, the highest resolution simulation has $N = 800$. The temporal convergence was calculated by dividing the final time of each lower resolution simulation by the final time of the highest resolution simulation. It can be seen in Figure B that the final time of the simulation with $N = 200$ is within 2% of the final time of the highest resolution simulation. As with the spatial convergence, I feel that this is sufficient considering the large uncertainties in many of my model parameters.
Figure B.2: The temporal convergence of a number of simulations with various numbers of grid spacings. In all of my simulations I use $N = 200$ grid spaces.
Appendix C

Corrections to Meek & Norbury

It must be noted that Meek & Norbury [75] has a couple of significant typos that render their method useless. I was able to re-derive their method, find the errors, and correct them. The first typo occurs in Equation (2.15b). There is a missing dot over one of the $s_j$’s. It should read,

$$\frac{K}{2\Delta\sigma}(v_{ij+1/2} + v_{ij} - v_{i-1,j+1/2} - v_{i-1,j})$$

$$= s_j^2 \frac{K}{\Delta\tau}(u_{ij+1/2} - u_{ij} + u_{i-1,j+1/2} - u_{i-1,j})$$

$$- \sigma_{i-1/2}s_j \frac{1}{4}(v_{ij+1/2} + v_{i-1,j+1/2} + v_{ij} + v_{i-1,j}). \quad (C.1)$$

The second typo occurs, in Equation (??), on page 888, just before their description of the numerical method. There are three equations, one each for $\eta$, $\xi$ and $\gamma$. The one for $\xi$ should be for $\varepsilon$ and should read,

$$\varepsilon_{j+1} = v_{j+1} - 2v_{j+1/2} + v_{j}. \quad (C.2)$$

The third and final typo is in Equation (2.22) and involves a couple of missing tildes. It should read,

$$\tilde{A}_{N+1}x_{N+1} + p\tilde{s}_{N+1} = \tilde{f}_{N+1}. \quad (C.3)$$
Appendix D

Analytic Test of Keller Box Method

The Keller box scheme of Meek & Norbury [75] is valid for equations of the form, Equations 1.6 & 1.7. However, their derived numerical method did not include a source term. I had to derive the Keller box scheme with a source term in order to include infall from the solar nebula onto the circumplanetary disk.

In order to test that the version of the Keller box scheme algorithm I derived was functioning properly, I tested it against an analytic solution. I follow the example in Ruden [97], who use a Green’s function approach to develop an analytic solution for a photoevaporating disk which is losing mass through the disk surface. They use a diffusion equation of the form,

$$\frac{\partial S}{\partial \tau} - \frac{3}{4} \frac{\partial^2 S}{\partial x^2} = -f_m,$$

where $x = (r/r_g)^{1/2}$, $\tau = t/t_{\text{vis}}$, $S = \Sigma x^3$ and $f_m = t_{\text{vis}}x^3F_m$. The viscous timescale is $t_{\text{vis}} = r_g^2/\nu_g$, where $r_g$ is the gravitational radius defined in the photoevaporation section and $\nu_g$ is the viscosity evaluated at $r_g$. The source term, $f_m$, is defined in terms of the wind mass flux, $F_m$, based on the work of Hollenbach et al. [42] such that

$$F_m \approx \begin{cases} 
0 & r < r_g \\
F_0 \left( \frac{r}{r_g} \right)^{-5/2} & r \geq r_g
\end{cases}$$

where the constant $F$ is $4.9 \times 10^{-12}(\phi/10^{40} \text{ s}^{-1})^{1/2} (r_g/10^{13} \text{ cm})^{-3/2} \text{ g cm}^{-2} \text{ s}^{-1}$, where $\phi$ is the ionizing photon luminosity illuminating the disk. The wind mass flux is also related to the mass
flux such that,

\[ \dot{M} = \int_0^\infty 2\pi F_m dr = 4\pi F_0 r_g^2. \]  

(D.3)

A Green’s function approach was used to solve to equation. I begin by solving the homogeneous equation, without the source term. Then, the homogeneous solution is used to solve for the particular solution, including the source term. The Green’s function \( G(x, \tau; x_0, \tau_0) \) represents the spatial and temporal diffusion of a delta function introduced at \( x_0 \) at time \( \tau_0 \). I will test the Keller box scheme under the assumption that the viscosity is proportional to the radius, \( \nu \propto r \). The particular solution to zero-torque boundary condition is,

\[ G(x, \tau; x_0, \tau_0) = \frac{1}{\sqrt{3\pi(t-\tau)}} \left( \exp\left[\frac{-(x-x_0)^2}{3(t-\tau)}\right] - \exp\left[\frac{-(x+x_0)^2}{3(t-\tau)}\right] \right). \]  

(D.4)

I can now use the particular solution to the homogeneous equation to generate the general solution.

\[ S(x, \tau) = \int_0^\infty dx_0 G(x, \tau; x_0, 0) S(x_0, 0) - \int_0^\tau d\tau_0 \int_0^\infty G(x, \tau; x_0, \tau_0) f_m(x_0, \tau_0) \]  

(D.5)

The solution in this form allows us to separate and identify the effects of viscous diffusion and evaporative mass loss. The first term corresponds to the viscous diffusion of the initial mass surface density distribution, \( S(x, 0) = x^3 \Sigma(x, 0) \). The second term corresponds to the loss of mass through an photoevaporative disk wind. The second term can also be written as,

\[ \Delta S_w(x, \tau) = -\int_0^\infty dx_0 \Gamma(x, \tau; x_0) f_m(x_0), \]  

(D.6)

where \( \Gamma(x, \tau; x_0) \) is the “wind Green’s function” and can be determined using,

\[ \Gamma(x, \tau; x_0) = \int_0^\tau G(x, \tau; x_0, \tau_0) d\tau_0. \]  

(D.7)

Using Equation D.6, I can now determine the analytic solution for a viscously evolving disk
with photoevaporative mass loss from the disk surface.

\[ S(x, \tau) = \left[ -\int_{0}^{\infty} 2x_0 f_m(x_0) dx_0 ight. \\
+ \int_{0}^{\infty} \sqrt{\frac{3\tau}{\pi}} \left( \exp \left[ \frac{-(x - x_0)^2}{3\tau} \right] - \exp \left[ \frac{(x + x_0)^2}{3\tau} \right] \right) f_m(x_0) dx_0 \\
+ \int_{0}^{\infty} (x - x_0) \text{erf} \left[ \frac{x - x_0}{\sqrt{3\pi}} \right] f_m(x_0) dx_0 \\
\left. - \int_{0}^{\infty} (x + x_0) \text{erf} \left[ \frac{x + x_0}{\sqrt{3\tau}} \right] f_m(x_0) dx_0 \right] \] (D.8)
Appendix E

Heated Envelope Temperature

It is assumed in this model that the disk is embedded in a FUV heated atmosphere. It is this heated atmosphere that drives the photoevaporation process. The concept of a heated atmosphere that can be modeled by a single temperature was taken from Adams et al. [2]. They say that the gas temperature of the disk is $100\text{K} \leq T_{\text{env,dust}} \leq 3000\text{K}$. However, the authors also state that the dust is not thermally well coupled to the gas and should have a temperature of, $T_{\text{env,dust}} \approx 10-50\text{K}$. The gas is thermally less well coupled to the disk and much hotter than the dust because it radiates very inefficiently in the IR. It is only able to radiate in a few fine-structure lines, primarily O[I] emission at 63µm and C[II] emission at 158µm. The dust, which can cool through gray-body emission, is a much more efficient radiator. The high IR radiation efficiency of the dust is why it is much cooler than the gas and why it should be the primary contributor to disk heating.

In more evolved disks, such as the ones in my models, the dust-to-gas ratio may be significantly different than that considered by Adams et al. [2]. Due to settling and coagulation the gas-to-dust ratio may be increased by orders of magnitude in the more evolved disks considered here, especially in the atmosphere that is supplied by inflow from the solar nebula. A sufficient increase in the gas-to-dust ration could cause the gas to become the primary IR radiation despite it’s low efficiency. Because of these uncertainties I conducted some calculations to determine the relative emergent intensity, in the IR, of the gas and dust components separately and found that even if the dust-to-gas ratio is decreased by a factor of 1000, the amount of energy radiated in the IR by the dust is still at least an order of magnitude greater than that radiated by the gas.
First, assume that the emissions from the gas is a simple, two-level system. Under this approximation calculate the populations of the upper and lower states.

\[
\frac{n_u}{n_l} = \frac{(g_u/g_l)\exp\left[\frac{E_{ul}}{kT}\right]}{1 + (n_{cr}/n_H)}
\]

(E.1)

Here, \(n_u\) and \(n_l\) are the populations of the upper and lower states respectively. The statistical weights of each state are \(g_u\) and \(g_l\). The energy difference between the two states is \(E_{ul}\), \(n_H\) is the number density of hydrogen and \(n_{cr}\) is the critical density of the transition in question. Then, close this equation and solve for the populations of the two states using the relation,

\[
n_u + n_l = A_j n,
\]

(E.2)

where \(A_j\) is the abundance of element \(j\).

Once the level populations have been calculated, one can calculate the optical depth of gas at the wavelengths of interest. This is to verify that the gas is not optically thick which would prevent the radiation from escaping and further complicate my calculations. The optical depth at the wavelength of interest, \(\tau_{ul}\), is calculated using

\[
\tau_{ul} = \frac{A_{ul}c^3}{8\pi\nu_{ul}b/\Delta z} \left[ \frac{n_l g_u}{n_u g_l} - 1 \right],
\]

(E.3)

where \(A_{ul}\) is the the Einstein “\(A\)” coefficient, \(c\) is the speed of light and \(\nu_{ul}\) is the frequency of the transition. The broadening parameter, \(b\), is \(\sqrt{2}\sigma_v\), where \(\sigma_v\) is the velocity dispersion. The velocity dispersion is assumed to be the same as the sound speed of the gas and calculate it thus. The term \(\Delta z\) is the integrated path length and, because of the large scale height assumed for circumplanetary disks, is assumed to be roughly the radial position of interest.

My calculations show that, for appropriate parameters, the gas is optically thin at both of the transitions of interest. Given that, it is straightforward to calculate the emergent intensity of radiation of the gas at the specific wavelengths of interest. For optically thin gas,

\[
I = \frac{n^2 A_j \gamma_{ul} h\nu_{ul} \Delta z}{2\pi},
\]

(E.4)
where $\gamma_{ul}$ is the collision-rate coefficient and $h$ is Planck’s constant.

Now that the emergent intensity for the gas is calculated, it is possible to calculate the emergent intensity for the dust component. As before, the optical depth needs to be calculated before calculating the emergent intensity. The dust optical depth, at wavelength $\lambda$, over a given path length, $L$, is defined as

$$
\tau_d(\lambda) = L \int_{a_-}^{a_+} n_d(a) C_{\text{ext}}(a, \lambda) da, \quad (E.5)
$$

where $n_d$ is the number density of dust with grain size $a$. The extinction cross-section, $C_{\text{ext}}$, is defined as

$$
C_{\text{ext}}(a, \lambda) = Q_{\text{ext}}(a, \lambda) \sigma_d. \quad (E.6)
$$

Here, $Q_{\text{ext}}(a, \lambda)$ is the absorption efficiency and $\sigma_d$ is the geometric cross-section of the dust. The absorption efficiency is approximated with the following power law,

$$
Q(\lambda) = Q_0 \left( \frac{\lambda_0}{\lambda} \right)^\beta, \quad (E.7)
$$

where $Q_0 = 1$, $\lambda_0 = 2\pi a$ and $\beta = 2$.

Before the emergent intensity of IR radiation from the dust component can be calculated, an assumption about the grain size distribution needs to be made. Let

$$
n_d(a) da = A a^{-q} da, \quad (E.8)
$$

where $q$ is the power law slope of the grain size distribution, where $q = 3.5$. $a$ is determined by solving

$$
M_d = \frac{3}{4} \pi \rho_d a^3 n_d(a) da, \quad (E.9)
$$

where $M_d$ and $\rho_d$ are the total mass and mass density of dust.

Then determine the optical depth of the dust, at wavelength $\lambda$,

$$
\tau_d = A L \left( \frac{2\pi}{\lambda} \right)^2 \pi \left( \frac{2}{3} \right) \left[ a^{3/2} \right]_{a_-}^{a_+}, \quad (E.10)
$$
with
\[
A = \frac{3n_H \langle \mu \rangle}{8\pi \rho_d f_{g/d} \left( a_+^{1/2} - a_-^{1/2} \right)}.
\] (E.11)

Here, \( f_{g/d} \) is the gas-to-dust mass ratio.

Now that the optical depth has been fully specified, the emergent intensity, at wavelength \( \lambda \), can be calculated. Begin by looking at the definition of \( I(\lambda) \),
\[
I(\lambda) = B(T_d, \lambda),
\] (E.12)

where the Planck, or blackbody, function is,
\[
B(T_d, \lambda) = \frac{2hc^2}{\lambda^5} \left( \frac{1}{e^{hc/\lambda k_B T_d} - 1} \right)
\] (E.13)

Therefore,
\[
I(\lambda) = \frac{2hc^2}{\lambda_{ul}} \left( \frac{1}{e^{hc/\lambda_{ul} k_B T_d} - 1} \right) \tau_d(\lambda_{ul})
\] (E.14)

The calculations of the relative emergent intensities of the dust and gas components of the heated atmosphere require a few assumptions about the relevant parameters. The temperature of the dust is assumed to be 50 K, the highest that it should be according to Adams et al. [2]; This assumes the dust will radiate the more at higher temperatures and then reach the lower equilibrium temperature assumed here. As previously stated, we assume that the path length is equal to the radial distance in the disk and let \( r = 10 \, R_J \) as a fiducial distance. However, the relative emergent intensities of the gas and dust do not depend on this choice, as long as the path length is the same for both components.

The final, and most important assumption, is with regard the size distribution of the dust. Here, assume that the size distribution is a power law of the form \( a^{-3.5} \), where the size distribution is limited such that \( 0.001 \, \mu m \leq a \leq 0.1 \, \mu m \). The power law slope of \(-3.5\) is standard for a collisionally dominated size distribution. The range of sizes was chosen because it is only the smallest particles that will be entrained in the gas from the solar nebula and constantly replenished in the heated atmosphere. Any particles that grow much larger will rapidly settle into the disk.
midplane. Under these assumptions, the emergent intensity from the dust dominates that from the
gas by at least an order of magnitude until the gas-to-dust ratio becomes $\sim 2 \times 10^5$, roughly 200
times that generally assumed for the solar nebula.