A Compendium of Riemann Surfaces and Algebraic Curves

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A Compendium of Riemann Surfaces and Algebraic Curves

for Advanced Undergraduates

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1 Introduction

When I took an undergraduate course in complex analysis of a single variable I struggled immensely with the topic. However, the structure of the complex numbers and their functions captivated me. I asked my professor, one day after he mentioned analogues to the complex plane in higher dimensions, about where I could learn more about complex analysis. He pointed me to Riemann surfaces, the next step up from the complex plane—the manifolds that behaved like the complex plane. This thesis was born from my study of the topic with Dr. Sebastian Casalaina-Martin from the book *Algebraic Curves and Riemann Surfaces* by Rick Miranda. Then main goal of this thesis is to introduce—at the level of an advanced undergraduate in mathematics—the deep and surprising theory of Riemann surfaces, culminating in the statements of the Riemann-Roch theorem and Abel’s theorem. The reader should leave with an understanding of what is meant by the theorems, and a basic intuition for the basics of Algebraic Geometry.

Much like the area of differential geometry develops the theory of manifolds that look like $\mathbb{R}^n$ locally, we can develop an analogous (but ultimately different in important ways) theory about manifolds that locally look like $\mathbb{C}^n$. In other words, Riemann surfaces are manifolds of complex dimension one. Hence, we call them curves. Many of the results that we talk about, namely the equivalence of algebraic curves and compact Riemann surfaces, ultimately fail when we look at manifolds of higher complex dimension. Therefore this thesis, and really the field of Riemann surfaces is a special case of complex geometry where things are reasonably nice. Many of the definitions given, such as divisors in Section 5 have generalized definitions for higher dimensions, where here they are defined by points, rather than, say, some codimension-1 subvarieties. However, I hope that this thesis gives some intuition for Complex Algebraic Geometry; just be aware that this intuition might not always hold true in higher dimension. The sections of this thesis are laid out in a similar order to Miranda’s book, and each section ends with an easy exercise taken from Miranda’s book that should help the reader apply the basic definitions given in each section. These are not meant to be hard, but simply require the reader to flex their new definition knowledge and perhaps a theorem or two from the chapter.

The tangible result of this thesis will be the classification of compact Riemann surfaces of genus 0, 1, 2, and 3 using both Riemann-Roch and Abel’s theorem. This thesis will build the tools and understanding of the tools needed to complete this classification.

The second section (the introduction here is the first section) will be introductory, beginning with the definition/construction of a Riemann surface, including some basic examples. These examples include Elliptic Curves (a Torus), the Riemann sphere, complex projective space, smooth projective curves, and intersection curves. Basic Topology and Complex analysis is assumed. The reader should refer to the appendix if in need of reminders of the basics of Complex analysis and Topology for this section. I note a few theorems there...
Section three will cover maps between Riemann surfaces. It will be an extension of familiar ideas from Complex Analysis to Riemann surfaces and their theorems. These include holomorphic and meromorphic functions on Riemann surfaces, Laurent series, and examples/classifications of these on different Riemann surfaces. We will state Hurwitz’s Formula and define a holomorphic embedding of a Riemann surface. This section will also examine morphisms between Riemann surfaces, defining isomorphisms and some important theorems we use to prove genus 0 compact curves are spheres. We end the section with one more important example of a Riemann surface—hyperelliptic curves—that become important in the classifications that come at the end of the thesis.

The fourth section will introduce forms and integration on Riemann surfaces. We will define forms, the wedge product, and differentiation and integration of these forms. It will briefly cover Chains and Homology, and is geared toward understanding the statement of Abel’s theorem. Here the reader should be reminded of Differential Geometry, if they have seen it before, otherwise, the reader should gain enough of an intuition to understand homology and how we integrate. (Historically, this question of integration of ‘forms’ with no closed answer over \( C \) gave impetus to the study of Riemann surfaces. For me the impetus was different.)

The fifth section will introduce arguably the most important idea to proving the important theorems I wish to show: divisors and linear systems of divisors. These sort of objects are really at the heart (along with meromorphic functions) of Algebraic Geometry, that information about divisors on our space encapsulates important information about our manifold. Over complex manifolds in this case. We will define Canonical and Principal divisors, and explore their relationships with ordering and equivalence properties. This section will help the reader understand the statement of the Riemann-Roch theorem.

The sixth section will define algebraic curves, maps to projective space from surfaces, linear systems associated to these maps and base points. Using Chow’s theorem we state the equivalence of algebraic curves to compact Riemann surfaces. And then we state the Riemann-Roch theorem. We do not prove it, however. This is the first major theorem in the thesis that the reader should by now be in a reasonably good place to understand.

The seventh section will deal with the Jacobian of our Riemann surfaces. We will state a general classification of jacobians of genus \( g \) curves. From there, we will quickly develop an understanding of the Abel-Jacobi map taking the set of divisors into the jacobian. This culminates, naturally, in Abel’s theorem, which we state without proof.

The eighth and final section of the thesis will be the classifications of genus 0, 1, 2, and 3 curves, using the major theorems and some littler ones from the building sections. With that, the thesis ends with appendices that offer orienting theorems and facts from topology and complex analysis which I have assumed for the first section (the only section where the reader should need them).
2 Defining a Riemann Surface and Finding Examples

2.1 The Basic Definition and a First Example

We wish to consider general spaces that look locally like some open set in $\mathbb{C}$. In this thesis, a space is a collection of points with a topology. For our space $M$ and some set in it $U$, we can think of local condition as an assignment, for each $p \in U \subseteq M$, to some point $z_p$ in some open subset of $\mathbb{C}$. More than that, we want these sets to look the same topologically, meaning that they are homeomorphic. An example of such a space is $\mathbb{C}$. To do this properly in more generality, we need to define two things: (1) complex charts for some space, and (2) complex atlas for this space.

**Definition 2.1.** A complex chart on a topological space $X$ is a homeomorphism $\phi : U \to V$, where $U \subset X$ and $V \subset \mathbb{C}$ are open sets in their respective spaces.

We often want 0 to be in $V$ for ease of computations, and if so, we say that $\phi$ is centered at $p \in U$ if $\phi(p) = 0$. We also often omit the $\phi$ notation, and simply write $z$, since writing $z = \phi(p)$ means that we can suppress the notation with a minimum of ambiguity. I will be very clear to start about whenever I am omitting this notation, but will quickly fully suppress the notation in favor of simply $z$.

Here is an easy example of a chart:

**Example 2.2.** Let $X = \mathbb{R}^2$ and define $\phi : X \to \mathbb{C}$ by $\phi(x, y) = x + iy$.

This is obviously a homeomorphism, and so is a chart. However, this is slightly misleading, because this chart works everywhere in $X$, when generally we can only define charts with respect to smaller open sets in our space—not on the whole space at once!

If you'll recall your basic calculus, one of the tools used is a change of coordinates. One of the key ideas there is that the structure we impose with the new coordinate is not any different than the original structure. We wish the same to be true if there are two complex charts $\phi, \psi$ which both contain some $p \in X$. These two charts (which we think of giving a change of coordinates on $p$) should be structurally compatible. It wouldn’t make sense to have two different notions of the ‘structure’ of an open set of $p$ in our space $X$. Thus we come to another basic definition, that of chart compatibility.

**Definition 2.3.** Let $\phi : U \to V$ and $\psi : U' \to V'$ be two charts on $X$. We say that $\phi$ and $\psi$ are compatible if $\psi \circ \phi^{-1} : \phi(U \cap U') \to \psi(U \cap U')$ is holomorphic. This definition is vacuously true if the intersection $U \cap U'$ is empty.

This is essentially saying that on their chart domains, if they intersect, then the coordinates restricted to their intersection can be deformed smoothly to each other. One bit of notation: the map $T := \psi \circ \phi^{-1}$ is called the chart transition map. So another way to define compatibility is to say that the chart transition map is holomorphic.

We’d like to be able to give such a coordinate to every point of $X$. This means that everywhere $X$ looks like $\mathbb{C}$, like we originally wanted. Moreover, if some point is in two charts, meaning there are possibly two coordinates for it, they should be compatible with each other. We define an object that allows us to find charts at any point of $X$.

**Definition 2.4.** A complex atlas $A$ on a space $X$ is a set of charts $A = \{ \phi_\alpha : U_\alpha \to V_\alpha \}$ such that for any two indexes $\alpha, \beta$ the charts $\phi_\alpha$ and $\phi_\beta$ are compatible, and such that $X = \bigcup_\alpha U_\alpha$.

Notice how this means that for any point in $X$ we can pick some coordinate $\phi = z$ in $\mathbb{C}$ and if there are two such coordinates, the transition between them is holomorphic.

We can also develop a notion of a smooth structure on $X$. This is essentially an equivalence class of atlases (we’ll refer to complex atlases by atlases unless specified), where two atlases are equivalent if all their charts are compatible. However, we may always select some representative from these classes, so we do not lose any clarity for this paper if we think of a smooth structure as an atlas. For a further discussion of smooth structures look at [Mir97] p.3-4.

**Definition 2.5.** A smooth structure on $X$, for the purposes of this paper, is an atlas.

Now we are ready to define a Riemann Surface, the object we wish to work with for the rest of the paper. Recall the definitions of Hausdorff, second countable, and connected given in Appendix [B].
Definition 2.6. A Riemann surface is a connected, second countable, Hausdorff space $X$ that has a smooth structure.

2.1.1 Example: The Riemann Sphere

Example 2.7. Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} = S^2$ be the unit two sphere, then an atlas for $X$ is the northern projection $\pi_N$ discussed in Appendix A together with the opposite projection from $S = (0, 0, -1)$ with a chart given by

$$\pi_S(x, y, z) = \frac{x}{1 + z} - i \frac{y}{1 + z}.$$

Notice how the open sets of the charts, $S^2 - \{N\}$ and $S^2 - \{S\}$, cover $S^2$ and that their charts are homeomorphisms. We also note that, topologically, $S^2$ is connected, and second countable because its topology is the subspace topology. It is also closed and a subspace of a Hausdorff space, so is Hausdorff. The proof that they are compatible follows from that if $\pi_N = z$ then $\pi_S = 1/z$ in coordinates. Then $T(z) = 1/z$ is holomorphic in $S^2 - \{N\} \cap S^2 - \{S\}$. Therefore, by our definition it is a Riemann surface, and an important one for reasons you will see later on. Also note that $S^2$ is compact, so it is an example of a compact Riemann surface, which is a Riemann surfaces that is also compact. Riemann surfaces can also have holes, and the number of holes is called the genus.

One can extend the definition of a Riemann surface to surfaces that look like higher dimensional complex space called complex manifolds; in fact, the definition is word for word the same, with the caveat that $C$ becomes $\mathbb{C}^n$. The aforementioned definitions of compatibility and smooth structures also easily generalize, however we will only focus on Riemann surfaces, where $n = 1$.

2.2 More Examples of Riemann Surfaces

Before we jump in to the exploration of exactly what spaces are Riemann surfaces, we’d like an easier way to find them. One powerful tool is that sometimes the information in an Atlas suffices to characterize a Riemann surface. To wit, you do not always have to start with a set $X$ that satisfies all of the topological requirements of a Riemann surface. It is enough to start with a countable collection $\{U_i\}$ of subsets of $X$ and if you can find a bijection $\phi_i : U_i \rightarrow V_i \subset \mathbb{C}$ for each $i$, then the set $\{(U_i, \phi_i)\}$ is an Atlas for $X$.

Such a construction leaves some checks, however. For one you have to be able to impose a topology on $X$ from the $\{U_i\}$ that is Hausdorff, second countable, and connected. Then you must check that the charts you have found are pairwise compatible. For now, we will only check compatibility of our charts $\text{Mir97}$ p.8], however, just know that there are typically more things to check using this method.

The upshot is that we now have an easier way of finding Riemann surfaces! The first one we will construct is the complex Projective Line $\mathbb{P}^1$. We want to first think about what the set is here. Projective space generally is the collection of all 1-dimensional sub-spaces of some space.

2.2.1 Complex Projective Space

If $(z, w) \in \mathbb{C}^2$ is non-zero, then its span (the one-dimensional subspace it is in) is denoted $[z : w]$, following the conventions of projective space in the real case. One thing to note is that this notation is basically selecting a representative for the line, in fact, any non-zero point on the line is an appropriate choice for a representative since

$$[z : w] = [\lambda z : \lambda w]$$

for $\lambda \neq 0 \in \mathbb{C}$. This is because the equation $(\lambda z, \lambda w) = \lambda(z, w)$ is exactly the criterion for $(\lambda z, \lambda w)$ to be on the line $[z, w]$.

Definition 2.8. Complex projective space $\mathbb{P}^1$ is the set of all one dimensional subspaces of $\mathbb{C}^2$.

So let $X = \{[z, w] \mid z, w \in \mathbb{C}\}$. Now we want to find subsets covering $X$ where we can define a bijection to a subset of $\mathbb{C}$. So consider the sets $U_1 = \{[z : w] \in \mathbb{P}^1 \mid z \neq 0\}$ and $U_2 = \{[z, w] \in \mathbb{P}^1 \mid w \neq 0\}$; their union is obviously all of $X$. So what we need now are charts for each of these sets. Define $\phi_1 : U_1 \rightarrow \mathbb{C}$ to
be \( \phi_1([z : w]) = \frac{w}{z} \) and similarly let \( \phi_2 : U_2 \to \mathbb{C} \) be such that \( \phi_2([z : w]) = \frac{z}{w} \). There are two things we need to check. First, each chart is well defined: by our previous observation, the representatives differ by a common non-zero multiple \( \lambda \). So if \([z : w] = [\lambda z : \lambda w]\) we see for our first chart that
\[
\phi_1([\lambda z : \lambda w]) = \frac{\lambda w}{\lambda z} = \frac{z}{w} = \phi_1([z : w]).
\]

We are also never dividing by 0 because of our assumption on \( z \), so the charts are well defined. For another check, we must ask if we have a bijection onto \( \mathbb{C} \). To do this we compute an inverse. For any \( z \in \mathbb{C} \), (using the first chart as an example) \( \phi_1^{-1}(z) = [1 : z] \). We compute easily that this is an inverse because \( \phi_1(\phi_1^{-1}(z)) = \phi_1([1 : z]) = z \) and for \([z : w] \in U_1\) we have \( \phi_1^{-1}(\phi_1([z : w])) = \phi_1^{-1}(\frac{w}{z}) = [1 : \frac{w}{z}] = [z : w] \) since \( z \neq 0 \). A similar computation shows that \( \phi_2 \) is also a bijection. We have chart compatibility because \( \phi_2(\phi_1^{-1}(z)) = \phi_2([1 : z]) = \frac{1}{z} \), which is holomorphic since \( z \neq 0 \). Once we know that \( U_1 \) and \( U_2 \) are open sets in \( \mathbb{P}^1 \) then we know the bijections we have found are homeomorphisms, compatible, and hence charts. Together they form a smooth structure on \( X = \mathbb{P}^1 \). (Recall that we are omitting the proofs that our Riemann surfaces are second countable, Hausdorff and connected in most cases). Thus we have the following:

**Proposition 2.9.** Complex Projective Space \( \mathbb{P}^1 \) satisfies the definition of a Riemann Surface with a smooth structure given by charts \( (U_1, \phi_1) \) and \( (U_2, \phi_2) \) constructed above.

### 2.2.2 Elliptic Curves

We can find another example in the complex torus \( T \), otherwise known as an elliptic curve. To do this, a definition is required.

**Definition 2.10.** Let \( w_1, w_2 \) be two complex numbers linearly independent over \( \mathbb{R} \). A **lattice** \( L \) is the additive subgroup \( L = \mathbb{Z}w_1 + \mathbb{Z}w_2 = \{k_1w_1 + k_2w_2 \mid k_1, k_2 \in \mathbb{Z}\} \) of \( \mathbb{C} \).

![Figure 1: Example of a lattice. Note how the lattice is actually only the points, not the dashed lines.](image)

We can now define the complex torus as the quotient space \( \mathcal{T} = \mathbb{C}/L \) with a topology given by the quotient topology afforded by the quotient map \( \pi : \mathbb{C} \to \mathbb{C}/L \), again, see Appendix B. The idea for showing that \( \mathcal{T} \) has an atlas is to invert \( \pi \) at \( p \) using some small open sets. Therefore we wish to find for every \( z \in \mathbb{C} \) an open set \( D_z \subseteq \mathbb{C} \) such that \( \pi|_{D_z} \) is a homeomorphism. Then \( (\pi|_{D_z})^{-1} \) will be the desired chart. I will gloss over the topological arguments for why this map will map open sets to open sets in both directions, but I will show that it is a bijection.

First note that for any open set \( \pi|_{U} \) is onto its image, which we will take for granted as an open set. Therefore we only need to check injectivity. Because of the way we defined the quotient, the only way that...
such a restriction could fail on an open set $U$ if there are $z_1, z_2 \in U$ such that $z_2 - z_1 \in L$. Meaning that two points differ by some lattice point, and since $\pi$ collapses points differing by a lattice point, our map would not be injective. However, we can pick our neighborhood small enough to ensure that this does not happen.

**Definition 2.11.** A subset $U \subseteq X$ is discrete if for all $p \in U$ there exists an open set $V \in X$ such that $V \cap U = \{p\}$.

It is clear from the picture that $L$ is discrete, and hence so will the set $z + L$ of points that differ from $z$ by a lattice point. This means that we can find an open set $D_z$ such that $\pi|_{D_z}$ is injective and surjective. If you want more justification, consider the linearly independent complex numbers defining the lattice, $w_1, w_2$ and let $D_z = B_\epsilon(z)$ where $\epsilon = \frac{|w_1-w_2|}{2}$, the ball of radius $\epsilon$ about $z$. Notice by the way we picked $B_\epsilon(z)$, it contains only one point of the lattice $z + L$. Indeed, if we consider the epsilon ball about some $z_0$, we have for any $z \in B_\epsilon(z_0)$ that

$$|z - z_0| \leq \frac{|w_1-w_2|}{2} < |w_1 - w_2|.$$

So certainly the difference is not in the lattice for any points (If it were the difference $|z - z_0|$ would be an integer multiple of $|w_1-w_2|$). Hence, if we let out charts be $\phi_p = (\pi|_{D_z})^{-1}$ where $\pi(z) = p$, we get that $T$ has a smooth structure. We must, of course, check compatibility: Suppose we have two charts $\phi_1, \phi_2$ and their corresponding open sets $D_{z_1}, D_{z_2}$. Let $z \in U = \pi(D_{z_1}) \cap \pi(D_{z_2})$ and $T(z) = \phi_1 \circ \phi_2^{-1}(z)$, we wish to know if $T(z)$ is holomorphic. We can show that $\pi(T(z)) = \pi(z)$, which only happens, as we have noted before, when $T(z) - z \in L$. As a function, $T(z) - z$ is continuous and maps to a discrete set, so is locally constant. Hence $T(z) - z = w$ for some $w \in L$, and so $T(z) = z + w$ is obviously holomorphic. So the charts are compatible. Hence we have the following:

**Proposition 2.12.** The elliptic curve—complex torus—$T = \mathbb{C}/L$, constructed from the quotient of $\mathbb{C}$ with a lattice $L$, is a Riemann surface. The torus is an example of a compact Riemann surface of genus one (It looks like a doughnut).

### 2.2.3 Graphs of Holomorphic Functions on $\mathbb{C}$

One example particularly helpful in visualizing the process of assigning a complex chart to a neighborhood of a point in a Riemann surface $X$, graphs of holomorphic functions are unsurprisingly Riemann surfaces. Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be the graph of a holomorphic function $G = \{(z, f(z)) \mid z \in U\}$ in $\mathbb{C}^2$. As long as $U$ is open, we have that $\pi : G \to \mathbb{C}$ given by $\pi(z, f(z)) = z$, is a chart for $G$. Indeed, its inverse sends $z \to (z, f(z))$, which is a homeomorphism. Therefore $G$ satisfies the definition of a Riemann surface. Indeed, this is just projection of our surface $G$ onto $\mathbb{C}$. This can also easily generalize to graphs of surfaces in higher dimensions, and the chart is the same projection chart. Therefore we have that:

**Proposition 2.13.** If $g_1,...,g_n$ are holomorphic functions $g_i : U \to \mathbb{C}$ for some open set $U$ and $1 \leq n$ then the graph $G = \{(z, g_1(z),...,g_n(z)) \mid z \in U\}$ is a Riemann surface with the chart $\pi$ which projects onto the first coordinate $z$.

We might call these types of Riemann surfaces curves on account of their being graphs of functions.

The example above simplifies the image of Riemann surfaces; the assignment of coordinates becomes more concrete. Still, the charts for a graph are globally defined, but as we have discussed, usually we can only find charts locally. We got close to this with $\mathbb{CP}^1$, as we needed two charts, but the next example shows how one may be even more limited in defining charts on a neighborhood of a point $p \in X$.

We recall that some things look only locally like graphs, so we turn to an analogue of the real (i.e. over $\mathbb{R}$) implicit function theorem. This is enough to get us charts for some locus (a collection of points) $X \subseteq \mathbb{C}^2$ that is locally the graph of a function.

**Theorem 2.14.** (The Implicit Function Theorem). Let $f(z, w)$ be a complex polynomial and let $X = \{(z, w) \mid f(z, w) = 0\}$ be its zero locus. If $p_0 = (z_0, w_0)$ is a zero of $f$ (i.e in $X$) such that

$$\frac{\partial f}{\partial w}(p_0) \neq 0$$
then there exists a holomorphic function \( g(z) \) defined in a neighborhood of \( z_0 \) such that near \( z_0 \), \( X \) is the graph \( w = g(z) \), and \( g' = -\frac{\partial f}{\partial z} \frac{\partial w}{\partial f} \). If \( \frac{\partial f}{\partial z}(p_0) = 0 \), one might still be able to apply the theorem to the other variable. We now have an impetus for another definition.

**Definition 2.15.** A smooth affine plane curve is a locus of zeroes in \( \mathbb{C}^2 \) of a complex polynomial \( f(z, w) \) such that for every locus point \( p \) one of \( \frac{\partial f}{\partial z} \) or \( \frac{\partial f}{\partial w} \) is not zero at \( p \).

**Proposition 2.16.** A smooth affine plane curve \( X \) is a Riemann surface.

**Proof.** Let \( X = \{(z, w) \mid f(z, w) = 0\} \) be a smooth affine plane curve, and without loss of generality, assume \( \frac{\partial f}{\partial w}(p_0) \neq 0 \). By the Implicit function theorem, at every \( p \in X \) there exists a holomorphic \( g_p(z) \) such that in an open neighborhood \( U \) of \( p \), \( X \) is the graph \( w = g_p(z) \). By the previous example, we know the projection which sends \( p = (z, w) \rightarrow z \) is a chart. Another chart is the one that sends \( p = (z, w) \rightarrow w = g_p(z) \), since \( g \) is holomorphic (i.e. the charts on \( X \) are the projections). These charts are compatible and if \( f \) is irreducible then \( X \) is connected and is obviously Hausdorff as a subspace of \( \mathbb{C}^2 \), so is a Riemann surface. Irreducibility of our polynomial implied that its zero locus is connected. \( \square \)

### 2.3 Smooth Projective Plane Curves and Local Complete Intersections

We will find many important examples of Riemann surfaces embedded in projective space. One example we have already seen of this is the projective line \( \mathbb{P}^1 \), which is itself a projective space. However, some embed into higher dimensional projective space, not all embed into \( \mathbb{P}^1 \). But we can construct higher dimensional projective space in an analogous way to the way we constructed \( \mathbb{P}^1 \).

**Definition 2.17.** Let \( \mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^* \) be the set of one dimensional subspaces of \( \mathbb{C}^{n+1} \). We call this projective \( n \)-space. In the case \( n = 1 \) we call this the projective line, and for \( n = 2 \) we call this space the projective plane.

**Smooth Projective Plane Curves.** We can even talk about how polynomials behave in projective space. In the second simplest case, this is for polynomials on \( \mathbb{P}^2 \). In general, however, the linear properties of projective space prevent us from having a well-defined notion about the behavior of general polynomials on projective space.

**Definition 2.18.** A polynomial \( F \) is homogeneous if every term has the same degree. A homogeneous polynomial with every term of degree \( d \) is said to be homogeneous of degree \( d \).

For example, the polynomial \( F(x, y, z) = x^2 + y^2 - 4yz + xz \) is homogeneous of degree 2. We can also check that, for \([x : y : z] \in \mathbb{P}^2\), then \([\lambda x : \lambda y : \lambda z] = [x : y : z] \), but

\[
F(\lambda x, \lambda y, \lambda z) = \lambda^2 F(x, y, z).
\]

In general this means that we cannot evaluate our polynomial on \( \mathbb{P}^n \), but we have a well-defined notion of when \( F \) is zero on \( \mathbb{P}^n \). This motivates our next example of Riemann Surfaces.

**Definition 2.19.** Let \( F \) be homogeneous of degree \( d \). Let \( x = x^1, y = x^2, z = x^3 \) and set \( X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\} \). We call \( X \) a Projective Plane Curve.

Further, for \( 1 \leq i \leq 3 \) define \( X_i \) to be the intersection of \( X \) with \( U_i = \{[x^i] \mid x^i \neq 0\} \). We can show that, for example, \( X_2 \cong \{(a, b) \in \mathbb{C}^2 \mid F(a, 1, b) = 0\} \). These are open sets, and their union covers \( X \); therefore they offer us prime real estate to find chart domains. However, to show that such an \( X \) is a Riemann surface, we need additionally that \( F \) to be non-singular.

**Definition 2.20.** A homogeneous polynomial \( F \) is nonsingular if there are no nonzero solutions to

\[
F = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0
\]

in \( \mathbb{P}^2 \).
It is a lemma in Miranda’s book that the $X_i$ are smooth affine plane curves if and only if the polynomial $F$ is non-singular (using that the polynomial is non-singular, one can show that the partials of $f(a, b) = F(1, a, b)$ are non-singular). This affords us a new definition.

**Definition 2.21.** A **Smooth Projective Plane Curve** is a Projective plane curve such that each of the $X_i$’s are smooth affine plane curves.

**Example 2.22.** Let $F(x, y, z) = x^3 + y^3 + z^3$, if $X$ is the smooth projective curve defined by a degree 3 homogeneous polynomial $F$ then we call $X$ a **plane cubic**. If $F$ has degree 4 then we call $X$ a **smooth plane quartic**.

**Proposition 2.23.** **Smooth Plane Curves are Riemann Surfaces.**

**Proof.** We need to find charts and check their compatibility. We will take for granted that the zero set of a polynomial evaluated on Projective space is connected, second-countable, and Hausdorff.

Let $X = \{[x : y : z] \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$ be a Smooth Projective Plane Curve. This means that each of $X_1, X_2, X_3$ are smooth affine plane curves. We find charts on the $X_i$. Therefore, suppose without loss of generality that at $[x : y : z] = p \in X$ is such that $x_0 \neq 0$ then $F(x : y : z) = 0$ and $p \in X_1$. Then, since we are working in projective space, we have the following equality:

$$[x_0 : y_0 : z_0] = [1 : y/x : z/x]$$

so $F(1/y/x, z/x) = f(y/x, z/x) = 0$. Our assumption on $X_1$ ensures that at $(y/x, z/x) \in \mathbb{C}^2$ there is a neighborhood on $X_1 \subset X$ and a holomorphic function $g$ such that either $y/x = g(z/x)$, or $z/x = g(y/x)$. From the proof of lemma 1 we know that the projections $p \to y/x$ and $p \to z/x$ give charts on $X_1$. The other cases for $X_2$ and $X_3$ give charts (written by suppressing the usual notation as we noted we often do after Definition 1) $x/y$ and $z/y, x/z$ and $y/z$ respectively.

To check compatibility, we don’t have to check compatibility if both charts come from the same $X_i$ as we already did that for affine plane curves, but we do need to check them for different charts. We can do this by example, as all cases here are similar. Suppose that $p = [x : y : z]$ is in both $X_1$ and $X_2$: $x, y \neq 0$. Charts around $p$ might look like $\phi_1 = z/x$ and $\phi_2 = x/y$. Because these are on smooth affine plane curves, we can assume that locally $X_2$ is the graph $\{(w, g(w)) \mid g \text{ is holomorphic}\}$ so $\phi_2^{-1}(w) = [1 : w : g(w)]$. Then we can compute

$$\phi_1 \circ \phi_2(w) = \phi_1([1 : w : g(w)]) = g(w)/w.$$  

This case typifies the checks for compatibility: all end up being ratios of holomorphic things. And since we can do this for every point in $X$, the chart domains obviously cover $X$, so we have a smooth structure. Therefore every smooth projective plane curve is a Riemann Surface.

**Local Complete Intersections.** This next example is perhaps the most important because of a powerful theorem later on which states that any holomorphically embedded Riemann surface (don’t worry we haven’t defined this yet, but we will) is actually one from this example.

**Definition 2.24.** A **local complete intersection curve** in projective $n$-space is a locus $X \subset \mathbb{P}^n$ given by the vanishing of a set $\{F_{a_i}\}$ of homogeneous polynomials where in a small neighborhood of every $p \in X$, $X$ is described by $n - 1$ of the polynomials

$$F_{a_1} = F_{a_2} = \cdots = F_{a_{n-1}} = 0$$

satisfying the condition that the $(n - 1) \times (n - 1)$ matrix of partial derivatives $(\partial F_{a_i}/\partial x_j)$ has maximal rank at $p$.

This definition relies on a complex analogue of higher dimension Implicit Function Theorem, which is treated in [Rud87]. However, it is enough to ensure that each of the $X_i$ which cover $X$ has charts given by the projection from the local neighborhood defined by the graph of some holomorphic function, with coordinates given by some ratio of the homogeneous coordinates $x_i/x_j$, ensuring we have a Riemann surface.

**Exercise 1.** Prove that the zero locus of a homogeneous polynomial $F$ in $\mathbb{P}^n$ is well-defined.

$$Z = \{[z] \in \mathbb{P}^n \mid F(z) = 0\}$$
3 Maps Between Riemann Surfaces

Thus far we have lots of examples of Riemann surfaces, but now we would like some of those theorems from Complex Analysis to extend to Riemann surfaces. Many of these theorems had to do with functions, so let’s develop some ideas about functions between Riemann surfaces.

We already know about holomorphic and meromorphic maps when considering maps \( \mathbb{C} \to \mathbb{C} \). Appendix 1 already know about holomorphic and meromorphic maps with \( \mathbb{C} \to \mathbb{C} \). Recall that we are already equipped with a tool—charts—which give us local complex sets for sets on \( X \). We use these sets to define what holomorphic and meromorphic mean on general Riemann surfaces.

3.1 Holomorphic Maps

We begin by defining what holomorphic maps between two Riemann surfaces, then we give important terminology used to describe these maps and their properties.

**Definition 3.1.** Let \( f: W \subseteq X \to Y \) be a map from an open set of one Riemann Surface to another Riemann surface. We say \( f \) is a **holomorphic map** if for all \( p \in X \) and all charts \( \phi: U \to V \) on \( X \) with \( p \in U \) and every chart \( \psi: U_1 \to V_1 \) on \( Y \) with \( f(p) \in U_1 \) the map \( \tilde{f} = \psi \circ f \circ \phi^{-1} \) is holomorphic as a map from \( \mathbb{C} \to \mathbb{C} \). If \( Y = \mathbb{C} \), we say \( f \) is a **holomorphic function on** \( W \). The set of holomorphic functions on \( X \) is denoted \( \mathcal{O}(X) \).

If both \( Y = \mathbb{C} \) and \( X = \mathbb{C} \), then this definition matches the definition of holomorphic in basic complex analysis.

3.1.1 Two Important Theorems about Holomorphic Functions on Riemann Surfaces

**Theorem 3.2.** Assume \( F: X \to Y \) is non-constant and holomorphic between compact surfaces. We have that:

1. All such maps are open maps, i.e. they map open sets to open sets.
2. Preimages are discrete, i.e. for all \( y \in Y \) the set \( F^{-1}(y) \) is discrete in \( X \).
3. The map \( F \) is a surjection.

**Theorem 3.3** (Local Normal Form). Let \( F: X \to Y \) be a nonconstant holomorphic map. There is a unique integer \( m \geq 1 \) such that for every chart \( \phi_2: U_2 \to V_2 \) on \( Y \) centered at \( F(p) \) for \( p \in X \) there is a chart \( w = \phi_1: U_1 \to V_1 \) on \( X \) such that \( \phi_2(F(\phi_1^{-1}(w))) = z^m \).

**Definition 3.4.** We call the integer \( m \) the **multiplicity** of \( F \) at \( p \), denoted \( \text{mult}_p(F) \). We say \( p \in X \) is a **ramification point** if \( \text{mult}_p(F) \geq 2 \), and we say \( y \in Y \) is a **branch point** if there is some ramification point \( x \in X \) such that \( F(x) = y \).

**Definition 3.5.** The **degree** of a function \( F: X \to Y \) at \( y \) is the integer

\[
\deg(F) = d_yF = \sum_{p \in F^{-1}(y)} \text{mult}_p(F).
\]

**Lemma 3.6.** The degree of a map \( F: X \to Y \) is independent of the point \( y \) chosen when both \( X,Y \) are compact Riemann surfaces. (One should reference [Mir97, p.41] for the proof.)

3.2 Example of a Holomorphic Function

Some familiar examples of holomorphic functions are charts: they are holomorphic functions from our Riemann surface to \( \mathbb{C} \). We easily find others by simply composing charts with know holomorphic functions from \( \mathbb{C} \to \mathbb{C} \). But what about more examples? How about Holomorphic Functions on a projective plane curve?

Let \( X \) be a smooth projective plane curve defined by the vanishing of a non-singular, homogeneous polynomial \( F \), i.e. \( F(x,y,z) = 0 \). Let \([x : y : z] \) be in \( X_1 = \{ [x : y : z] \in \mathbb{P}^2 \mid x \neq 0 \} \). From the proof
of Proposition 2.23 we know that $y/x$ and $z/x$ are charts on $X$. Hence they are holomorphic maps. This means that any polynomial $g(y/x, z/x)$ is holomorphic because it consists of sums and powers of holomorphic maps. By clearing denominators, we find that

$$g(y/x, z/x) = \frac{G(x, y, z)}{x^d}$$

for some homogeneous polynomial $G$ of degree $d$. Noting that $x^d$ is a homogeneous polynomial of degree $d$ which does not vanish on $X$, you might suspect that any ratio $G/H$ of homogeneous polynomials of the same degree is holomorphic on $X$ as long as the denominator does not vanish, and you would be correct. The same generalizes immediately to local complete intersection curves inside $\mathbb{P}^n$.

**Proposition 3.7.** Let $X$ be a smooth projective plane curve or a local complete intersection curve, then a ratio of homogeneous polynomials where the denominator does not vanish at $p$ is a holomorphic function.

### 3.3 Meromorphic Functions

Recall from complex analysis that holomorphic functions were only the first step. We could also talk about functions that have poles, and that they were equally valid and important to the study of complex analysis.

**Definition 3.8.** We say a function $f : U \setminus \{p\} \to \mathbb{C}$ has a removable singularity (pole/essential singularity) at $p$ if at $p$ there exists a chart $\phi$ containing $p$ such that $\tilde{f} = f \circ \phi^{-1}$ has a removable singularity (pole/essential singularity) at $\phi(p)$ as a map from $\mathbb{C}$ to $\mathbb{C}$.

**Definition 3.9.** A function $f : X \to \mathbb{C}$ is meromorphic at $p$ if it is either holomorphic, has a removable singularity, or has a pole at $p$. The set of meromorphic functions on $X$ is denoted $\mathcal{M}(X)$.

Meromorphic functions on Riemann surfaces inherit a couple theorems from complex analysis including the Identity Theorem, the Maximum Modulus Principle. One theorem that I will state about preimages of zeros and poles of meromorphic functions proves to be useful.

**Theorem 3.10.** Let $f : X \to \mathbb{C}$ be a meromorphic function on an open set $W \subseteq X$ of a Riemann surface. If $f$ is not identically zero, then the zeros and poles of $f$ form a discrete subset of $W$.

Meromorphic functions are no more difficult to find than holomorphic functions, and we can entirely classify meromorphic functions for certain Riemann surfaces. One example of this is the Riemann sphere, and the other is on $\mathbb{P}^1$:

**Theorem 3.11.** This theorem collects classifications of two of the Riemann surfaces we work with in this paper.

1. Any rational function $f(z)/g(z)$ is meromorphic on $\mathbb{C}_\infty$ the Riemann sphere, in fact, these are the only meromorphic functions on $\mathbb{C}_\infty$.
2. Every meromorphic function on $\mathbb{P}^1$ is a ratio of homogeneous polynomials in $z$ and $w$ of the same degree $d$.

The proof of this requires some machinery we do not have yet, so we will take it as fact.

Recall that another useful tool we used to talk about functions $\mathbb{C} \to \mathbb{C}$ were Laurent series. Unsurprisingly, we can define Laurent series for functions on Riemann surfaces. Let $f$ be a meromorphic function on a subset $W \subseteq X$ of a Riemann surface, let $p \in W$, and let $\phi : U \to V$ be a chart containing $p$. Writing $z_0 = \phi(p)$, and so $z = \phi$, we can suppress the chart notation thinking of $z$ as a local coordinate in a neighborhood of $p$. Then we expand $f = f(\phi^{-1}(z))$ as

$$\sum_{k=m}^{\infty} c_k (z - z_0)^k.$$

For some $m \in \mathbb{Z}$ and we require $c_m \neq 0$. We do this in the same way we typically would expand a meromorphic function in complex analysis.
Definition 3.12 (Laurent Series). We call the polynomial
\[ \tilde{f} = \sum_{k=m}^{\infty} c_k (z - z_0)^k \]
constructed above the Laurent series for \( f \) around \( p \) with respect to the local coordinate \( z \).

This definition depends heavily on the local coordinate selected, but it does afford us a well-defined and independent of choice of local coordinate idea about what the order of a meromorphic function should mean.

Definition 3.13. Let \( f \) be a meromorphic function on a Riemann surface \( X \), with the Laurent series with respect to the local coordinate \( z \)
\[ f(z) = \sum_{n} c_n (z - z_0)^n. \]

The order of \( f \) at \( p \), \( \text{ord}_p(f) = \min\{n \mid c_n \neq 0\} \).

Well-Defined. We wish to show that \( \text{ord}_p(f) \) does not depend on our choice of local coordinate. Therefore, consider two local coordinates \( z, w \) near \( p \). We know that they must be compatible, to that the transition function \( T(w) = z \) is holomorphic. Write \( z_0, w_0 \) as the coordinate for \( p \) in these local coordinates. First, a lemma.

Lemma 3.14. Transition functions have non-zero derivative.

Proof of Lemma 1. Transition charts are invertable with smooth inverse, so there is a smooth function \( S(z) = w \) such that \( S(T(w)) = w \). Therefore, by the chain rule we have that \( S'(T(w))T'(w) = 1 \). Particularly, \( T'(w) \neq 0 \).

We now can write \( T \) in power series form with respect to these local coordinates.
\[ z = T(w) = z_0 + \sum_{n \geq 1} a_n (w - w_0)^n \]
where using Lemma 3, we have that \( a_1 \) is nonzero. (Note that since \( T \) is holomorphic, all of the terms are of non-negative power.) Now suppose we have a meromorphic function \( f \) with a Laurent series \( c_{n_0} (z - z_0)^{n_0} + \text{higher order terms} \) (and where \( c_{n_0} \neq 0 \)). Then \( \text{ord}_p(f) = n_{n_0} \) with respect to \( z \). However, rearranging the transition function gives that \( z - z_0 = \sum_{n \geq 1} a_n (w - w_0)^n \), and so we can substitute into our Laurent series to discover that
\[ c_{n_0} (z - z_0)^{n_0} + \text{higher order terms} = c_{n_0} \left( \sum_{n \geq 1} a_n (w - w_0)^n \right)^{n_0} + \text{higher order terms} = c_{n_0} a_1 (w - w_0)^{n_0} + \text{higher order terms} \]

Since \( n_0 \) is the smallest power possible, we need only check that its coefficient is non-zero, which it is since we assumed that \( c_{n_0} \neq 0 \) and since we showed that \( a_1 \neq 0 \). So with respect to \( w \), \( \text{ord}_p(f) = n_0 \) as well.

The order of a function at a point actually gives us the structure (form) theorem: Theorem 3.3. We simply take \( z^n \) to be the highest power of \( z \) in the laurent series we can factor out of it.

We should remark here that the only possible non-zero orders of a meromorphic function happen at poles and zeros. We can therefore talk about the sums of orders on compact Riemann surfaces since the discreteness of zeros and poles means on compact spaces that they are finite, so we need not worry about convergence. We find that if we add the orders of zeros and poles together that they cancel.
**Example 3.15 (Sums of orders of points).** Consider the Riemann sphere. We know that all meromorphic functions on it are the rational functions. So let our local coordinate be \( z \), and our meromorphic function be \( f(z) = g(z)/h(z) \) be a rational function in \( z \). We can factor \( f \) into roots \( \lambda_i \) as \( f(z) = c \prod_i (z - \lambda_i)^{e_i} \), where \( c \) is a non-zero complex number and the \( \lambda_i \) are the roots of the polynomials \( g, h \). The \( e_i \) are the orders of the roots in each polynomial, and \( e_i \) is negative if \( \lambda_i \) is a zero of \( h \).

By definition and properties of order,

\[
\text{ord}_{\lambda_i}(f) = \sum_j \text{ord}_{\lambda_i}(z - \lambda_j)^{e_j} = \text{ord}_{\lambda_i}(z - \lambda_i)^{e_i}
\]

since wherever \( i \neq j \) we have \( \lambda_i \neq \lambda_j \) (they are distinct) and so \( \text{ord}_{\lambda_i}(z - \lambda_j)^{e_j} = 0 \) since \( \lambda_i \) is neither a zero nor a pole of that polynomial.

At \( \infty \), we have the local coordinate \( 1/z \) so we get

\[
f(1/z) = c \prod_i (1/z - \lambda_i)^{e_i} = \frac{c}{z^{(\sum e_i)}} + \text{higher order stuff}.
\]

The point \( \infty \) is obviously a zero of \( 1/z \), so the Laurent expansion above (about \( \infty \)) gives that

\[
\text{ord}_{\infty}(f) = -\sum_i e_i.
\]

Note that for every other point in \( \mathbb{C}_\infty \) we can evaluate them using the coordinate \( z \), meaning that \( \text{ord}_p(f) = 0 \) for \( p \in X \) not a \( \lambda_i \) or \( \infty \). Then we have just shown that for \( \mathbb{C}_\infty \),

\[
\sum_{p \in \mathbb{C}_\infty} \text{ord}_p(f) = \sum_i e_i - \sum_i e_i = 0
\]

for an arbitrary meromorphic function \( f \) on \( \mathbb{C}_\infty \).

We can also compute that \( \sum_{p \in \mathbb{P}^1} \text{ord}_p(f) = 0 \) for any meromorphic function \( f \) on \( \mathbb{P}^1 \). This statement about the sums of orders of meromorphic functions generalizes to any compact Riemann surface, and have almost all the requisite machinery to prove it.

Since meromorphic functions have only zero’s and poles, we can easily associate a map from our Riemann surface to \( \mathbb{C}_\infty \). This essentially separates the zeros and poles giving us a useful way to talk about these structural properties of our meromorphic function without explicit reference to it.

**Definition 3.16.** Given a meromorphic function \( f \) on \( X \), the **associated holomorphic map** (\( F \)) to \( \mathbb{C}_\infty \) is defined to be

\[
F(x) = \begin{cases} 
\infty & \text{if } x \text{ is a pole} \\
 f(x) & \text{else.}
\end{cases}
\]

As implied in the name, this map is holomorphic. This follows from the fact that the only place this function might fail to be holomorphic are at the poles. However, the chart at infinity is \( 1/z \) so that these poles become zeros instead. The following lemma and definition encapsulates the usefulness of this definition.

**Lemma 3.17.** Let \( f \) be a meromorphic function, and let \( F \) be the associated map to \( \mathbb{C}_\infty \). Then

1. if \( p \in X \) is a zero of \( f \), then \( \text{mult}_p(F) = \text{ord}_p(f) \)
2. if \( p \in X \) is a pole of \( f \), then \( \text{mult}_p(F) = -\text{ord}_p(f) \)
3. if \( p \in X \) is neither a zero nor a pole of \( f \), then \( \text{mult}_p(F) = \text{ord}_p(f - f(p)) \)

Using this lemma, and the invariance of \( \text{deg}(F) \) for the associated map to \( \mathbb{C}_\infty \), we can prove a theorem.

**Theorem 3.18.** Let \( f \) be a non-constant meromorphic function on a compact Riemann surface \( X \). Then

\[
\sum_{p \in X} \text{ord}_p(f) = 0.
\]
Proof. Let \( F : X \to \mathbb{C}_\infty \) be the associated holomorphic map to \( \mathbb{C}_\infty \). Let \( d = \deg(F) \), and let the set \( \{x_i\} \) denote the zeros of \( F \), and the set \( \{y_j\} \) denote its poles. Be the definition of degree and its invariance on compact Riemann surfaces (Lemma 3.6), we have that \( d = \sum_i \text{mult}_{x_i}(F) \) and \( d = \sum_j \text{mult}_{y_j}(F) \). By the above lemma, we have that \( \text{mult}_{x_i}(F) = \text{ord}_{x_i}(f) \), and \( \text{mult}_{y_j}(F) = -\text{ord}_{y_j}(f) \) and at all other points \( \text{ord}_p(f) = 0 \). Then,
\[
\sum_{p \in X} \text{ord}_p(f) = \sum_i \text{ord}_{x_i}(f) + \sum_j \text{ord}_{y_j}(f) = \sum_i \text{mult}_{x_i}(F) - \sum_j \text{mult}_{y_j}(F) = d - d = 0.
\]

This is a fantastic theorem because in some sense it means that the orders of poles and zeroes pair up in equal but opposite magnitudes. This means that much of the information about functions are encoded in the zeroes and poles. In fact, this quickly becomes the trend as we continue to discuss the subject, zeroes and poles come to nearly characterize our Riemann surfaces.

The first glimpse of this more algebraic side of Riemann surfaces comes from Hurwitz’s formula, which I do not prove, but state. For this let \( g(X) \) denote the genus of the surface \( X \).

**Theorem 3.19** (Hurwitz’s Formula). Let \( F : X \to Y \) be a non-constant holomorphic map between compact surfaces. Then,
\[
2g(X) - 2 = \deg(F)(2g(Y) - 2) + \sum_{p \in X} [\text{mult}_p(F) - 1].
\]

With this formula, and given sufficient information about the genus and multiplicities, one can deduce the degree, and vice versa. The discussion which truly justifies this is edifying and I recommend reading up on it yourself in [Mir97, Ch. 2.4]. For the purposes of this paper, however, it is not necessary to see the proof.

### 3.4 Isomorphisms

Lastly, how should we talk about Riemann surfaces being the same? This gives rise to the notion of Riemann surface **isomorphism**. Mostly, isomorphism allows us to move freely between different viewpoints of our Riemann surfaces. Often, one isomorphic representation gives computationally useful charts while another gives some easier used for theoretical discussions.

**Definition 3.20.** An **isomorphism** between two Riemann surfaces \( X, Y \) is a holomorphic map \( f : X \to Y \) that has a holomorphic inverse \( f^{-1} : Y \to X \). If such a map exists, \( X \) and \( Y \) are **isomorphic**, and we write \( X \cong Y \).

We will not see much use for isomorphisms in my thesis until the very end, however, one must be aware that many surfaces are isomorphic and so we do not think of them as being essentially different. For example, \( \mathbb{C}_\infty \cong \mathbb{P}^1 \).

**Proposition 3.21.** The map \( F : \mathbb{P}^1 \to \mathbb{C}_\infty \) defined by
\[
F([z : w]) = (2\text{Re}(z\overline{w}), 2\text{Im}(z\overline{w}), |z|^2 - |w|^2)/(|z|^2 + |w|^2)
\]
is an isomorphism.

We do have some useful theorems that will help us classify Riemann surfaces in the end. For one, a 1-1 holomorphic map is automatically a bijection of \( X \) with its image.

**Proposition 3.22.** Let \( F : X \to Y \) be a 1-1 holomorphic map between Riemann surfaces, then \( X \cong F(X) \) if \( F(X) \) is a Riemann surface.
3.5 Example: Hyperelliptic Riemann Surface

If we know that we already have two Riemann surfaces $X, Y$ and an isomorphism $\phi : U \to V$ between open subsets of them then the identification space $Z = (X \cup Y)/\phi$ talked about in any introductory topology book, is also a Riemann surface, as long as $Z$ is Hausdorff.

It can be shown that if $h(x)$ is a polynomial with distinct roots that $f(x, y) = y^2 - h(x)$ is a smooth affine plane curve. Suppose that $h$ has degree $2g + 1 + \epsilon$ where $\epsilon \in \{0, 1\}$ with distinct roots. Form the smooth affine plane curve (Definition 2.15) $X$ defined by the equation $y^2 = h(x)$. Then define an open set $U := \{(x, y) \in X \mid x \neq 0\} \subseteq X$. Then define a new affine plane curve $Y$ by first making a new polynomial $k(z) = z^{2g+1}h(1/z)$ (which has distinct roots because $h$ did) and define $Y$ by the equation $w^2 = k(z)$. Let $V := \{(z, w) \in Y \mid z \neq 0\}$. We have an isomorphism $\phi : U \to V$ given by $\phi(x, y) = (1/x, y/x^g+1)$. This can be verified to be an isomorphism by noting that as a map $U \to V$, $\phi$ is basically self-inverse. Then let $Z$ be the identification space given by these two affine plane curves and the map $\phi$.

**Definition 3.23.** Let $Z$ be given above. Then we call this Riemann surface a **hyperelliptic Riemann surface** defined by the equation $y^2 = h(x)$.

This definition comes in handy when we start classifying compact Riemann surfaces at the end of this thesis.

**Lemma 3.24.** Hyperelliptic Riemann surfaces are compact and genus $g$ surfaces with a map $\pi : Z \to \mathbb{P}^1$ of degree 2.

**Proof.** The fact that they are compact follows from $Z$ being the union of two compact sets

$$\{(x, y) \in X \mid ||x|| \leq 1\}, \{(z, w) \in Y \mid ||z|| \leq 1\}.$$  \hfill (3.1)

To show this, simply note that these two sets are clearly contained in $Z$, and that any point $p \in Z$ can be shown to be in one of these sets. Consider the projection map $\pi : X \to \mathbb{C}$ Given by $(x, y) \to x$. This is holomorphic on $X$. If we extend it to a map $\pi_z : Z \to \mathbb{C} \to \mathbb{P}^1$ we’ll get a holomorphic map of degree 2. For this, consider that degree of a holomorphic map is independent of the point $q$ in the image by Lemma 3.6. So let’s pick $q = 0$ clearly by (3.1) we have that $\pi_z^{-1}(0) = \{(x_0, x_1) \in Z \mid x_0 = 0\}$ since no matter what coordinate we use we have $x_1^2 = j(x_0)$ where $j$ is either $h$ or $k$ by what coordinate we are using, we see that the points $(0, \pm \sqrt{j(0)})$ are exactly the preimage. (We can assume that 0 is not a root of $j$ since we could have easily looked at the preimage of a point that was not a root.) Hence $\pi_z$ has degree 2 (how many preimages there are of a general point). Now, by Hurwitz’s formula (Theorem 3.19) we have that

$$2g(Z) - 2 = 2(2g(\mathbb{P}^1) - 2) + \sum_{p \in Z} [\text{mult}_p(\pi_z) - 1].$$

Now, since we only have ramification (where $\text{mult}_p(\pi_z) > 1$) over the $2g + 2$ roots of $h(x)$ if $\text{deg}(h) = 2g + 2$ (or over the $2g + 1$ roots and one at $\infty$ if $\text{deg}(h) = 2g + 1$) and by degree of the map these are at worst multiplicity 2 we see that we get

$$2g(Z) - 2 = 2(2g(\mathbb{P}^1) - 2) + 2g + 2 = 2(-2) + 2g + 2 = 2g - 2$$ \hfill (3.2)

So that $g(Z) = g$ since $\mathbb{P}^1 \cong \mathbb{C}_\infty$ has genus 0. \hfill $\square$

It turns out, using a thing called Monodromy representation, that there is an equivalent way to define a hyperelliptic Riemann surface. Recall above how $\pi$ induced a holomorphic map of degree 2 to $\mathbb{P}^1$, as it turns out, a kind of converse is true, if $F : X \to \mathbb{P}^1$ is a holomorphic map of degree 2, then $X$ is a hyperelliptic curve. The proof of this is the content of [Mir97] Proposition 4.11 pg. 92].

**Proposition 3.25** (Equivalent Definition of Hyperelliptic). A hyperelliptic Riemann surface is a Riemann surface with a degree 2 map to $\mathbb{P}^1$.

**Exercise 2.** Let $f(z) = z^3/(1 - z^2)$ be a meromorphic function on $\mathbb{C}_\infty$. Find the orders of points, verify that their sum is zero, and consider the associated map $F$ for this function and verify Hurwitz’s formula for $F$. Taken from [Mir97] pg. 53].
4 Forms and Integration

In complex analysis, we integrated functions. However, in the modern terminology, one integrates forms. We can generalize these to Riemann surfaces, after all, they are manifolds.

4.1 Forms

For anyone familiar with forms already, you will be aware that there are many types of forms for manifolds. We therefore also have forms on Riemann surfaces since they are manifolds. We definitely would like to be able to 'integrate' holomorphic and meromorphic functions. But what do these look like as forms? First we treat the \( \mathbb{C} \) case, and then use charts to talk about them on manifolds.

**Definition 4.1.** A holomorphic (meromorphic) 1-form on an open subset \( V \subseteq \mathbb{C} \) is a formal expression \( w = f(z)dz \) where \( f(z) \) is a holomorphic (meromorphic) function on \( V \). We call \( \Omega^1(X) \) the set of holomorphic 1-forms on \( X \). The set of meromorphic 1-forms is denoted \( \mathcal{M}^{(1)}(X) \).

We define forms formally, because it is not important to understand what exactly the \( dz \) mean, though there are more mathematical justifications for this definition. Recall that, using charts, our coordinate \( z \) might change slightly, but we do not want to think about our form being fundamentally different (on our Riemann surface) under such a change in coordinate. The following definition is highly suggestive of generalization to Riemann surfaces.

**Definition 4.2.** Suppose \( w = T(z) \) is a change of coordinate. A 1-form \( w_1 = f(z)dz \) transforms under \( T \) to the 1-form \( w_2 = g(w)dw \) if \( g(w) = f(T(w))T'(w) \). The reason for this is we would like for \( dw = d(T(z)) = T'(z)dz \) by a chain rule.

With this we can generalize a 1-form to our surface \( X \). Recall that we have lots of charts to work with, and we have a way to talk about their overlaps being compatible. If our one form somehow sat inside a single chart \( z \), then it’d be enough to give a single 1-form \( f(z)dz \), but the domain might also be in the chart \( w \) and be given by \( g(w)dw \). They should transform into each other; the definition we concocted for compatibility of forms in \( \mathbb{C} \) is exactly right for this (it works with definition of chart compatibility).

**Definition 4.3.** A holomorphic (meromorphic) 1-form on a Riemann surface \( X \) is a collection \( \{w_\phi\} \) of holomorphic (meromorphic) 1-forms on \( \mathbb{C} \), for every chart in the atlas \( \{\phi\} \), such that if \( \phi, \psi \) overlap and have transition function \( T = \phi \circ \psi^{-1} \) then \( w_\phi \) transforms to \( w_\psi \) under \( T \).

We can actually define our 1-forms with respect to a single chart, rather than in Miranda’s book [Mir97] the justification for this relies on the identity theorem for meromorphic functions and forms: if they are equal on overlapping open sets, then they are the same. Still, one must be diligent in checking that this single formulation of your form actually exists on all of \( X \). The example Miranda gives is the form \( \exp(z)dz \) on \( \mathbb{C}_\infty \). This form makes perfect sense everywhere except the point \( \infty \), where it cannot be expressed as a meromorphic form locally. Despite this, most examples of forms I will give are perfectly fine, and so we will not check them. Instead, we will think of forms simply in a single chart. Giving the basis for this note:

**Note 4.4.** Such a one form \( \{w_\phi\} \) is simply written \( w = f(z)dz \) for a single chart. I adopt this notation presently and from here on out. And the order of a 1-form \( f(z)dz \) is the order of the function \( f \).

4.2 \( C^\infty \) 1-Forms

One might be more familiar with the notation \( f(x)dx \) or \( f(y)dy \) when talking about forms because we already use these in basic calculus. The forms we have been talking about are not the same because we have asked that \( f \) be wither holomorphic or meromorphic. Actually, relaxing our requirements that the function \( f \) be holomorphic or meromorphic, and instead simply requiring smoothness, we can obtain forms familiar to the ones used in calculus all the time. Smooth forms (i.e. \( C^\infty \)-forms) look like: \( f(x,y)dx + ig(x,y)dy \). That is, if we are given a chart \( z = x + iy \). However, it is much more advantageous to work in terms of \( z \) and \( \overline{z} \) rather than the underlying real coordinates. We do this not only because we are working with the complex coordinates anyway, but also because it simplifies notation. If \( z = x + iy \), then \( dz \) ought to be \( dx + idy \), and
likewise $d\overline{z} = dx - idy$. It is easy to compute that $dx = (dz + d\overline{z})/2$ and $dy = (dz - d\overline{z})/2$. Therefore we can re-parametrize our smooth form to $F(z,\overline{z})dz + G(z,\overline{z})d\overline{z}$. It is not immediately clear why using $dz$ and $d\overline{z}$ simplifies things at all. To wit, consider how one would compute partial derivatives of a function $f(x,y)$. Take the derivative with respect to $\overline{z}$. By the chain rule, we have

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \overline{z}} = \frac{1}{2} \frac{\partial f}{\partial x} - \frac{1}{2i} \frac{\partial f}{\partial y}$$

Where we use the relations $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$ to find the coefficients. From the above computation, we find that a simple equation $\partial f/\partial \overline{z} = 0$ is equivalent to the Cauchy-Riemann criterion. Using $dz$ also prevents “mixing” of cross terms when changing coordinates when we get to generalizing these forms.

Definition 4.5. A $C^\infty$ 1-form on an open set $V \subset \mathbb{C}$ is an expression $w = f(z,\overline{z})dz + g(z,\overline{z})d\overline{z}$, where $f, g \in C^\infty(V)$. For shorthand, I refer to these simply as 1-forms on $\mathbb{C}$.

We generalize this definition in an analogous way to which we generalized the other types of 1-forms. Given charts, we have a collection of one forms that transform to each other under another transformation rule. This rule is given below, and the logic behind it requires another basic multi-variable chain rule.

Definition 4.6. Let $z = T(w)$. Two $C^\infty$ 1-forms $w_1 = f_1(z,\overline{z})dz + g_1(z,\overline{z})d\overline{z}$ and $w_2 = f_2(z,\overline{z})dz + g_2(z,\overline{z})d\overline{z}$ transform to each other if $f_1(T(w),\overline{T(w)})T'(w) = f_2(z,\overline{z})$ and $g_1(T(w),\overline{T(w)})T'(w) = g_2(z,\overline{z})$.

Definition 4.7. Given a Riemann surface $X$, and given an atlas $\{\phi\}$, a $C^\infty$ (smooth) 1-form is a collection $\{w_\phi\}$ of 1-forms on $\mathbb{C}$ such that if $\phi_1, \phi_2$ overlap with transition function $T$ then $w_{\phi_1}$ transforms to $w_{\phi_2}$ under $T$. We denote the set of all of these on a surface $X$ as $C^\infty(X)$.

The same rule where we could define our forms with a single chart applies here. So we typically just say we have a form rather than worry about all the technicalities.

A 1-form $w$ is of type $(1,0)$ if it is locally of the form $f(z,\overline{z})dz$, and of type $(0,1)$ if it is locally of the form $g(z,\overline{z})d\overline{z}$. In this notation, smooth 1-forms are of type $(1,1)$ generally.

Example 4.8. Holomorphic 1-forms on $X$ are of type $(1,0)$.

We use 1-forms as the thing we integrate over curves in $X$. However, we also want to integrate over areas, for which need a 2-form. We first introduce the wedge product between two one forms, a way to construct a 2-form from two 1-forms.

Definition 4.9. The wedge product of two 1-forms $w_1 = fdz + gd\overline{z}$ and $w_2 = hdz + ld\overline{z}$ is the formal expression $w_1 \wedge w_2 = (fh - gl)dz \wedge d\overline{z}$.

Formally, the wedge operator $\wedge$ misses the functions and only operates on $dz, d\overline{z}$ under the rules that (1) $dz \wedge dz = d\overline{z} \wedge d\overline{z} = 0$ and (2) $dz \wedge d\overline{z} = -d\overline{z} \wedge dz$. The wedge product, however, does distribute over sums. Again, these have meaning beyond the formal ideas I have given, but truthfully their full interpretation belongs in a differential geometry course, so I exclude them here. A great resource is [Lee13] for those interested. For those not, do not fear, there are very easy interpretations.

Basic intuition behind (1) comes from needing two different directions to be able to integrate an area, while (2) has to do with what happens if the orientation of your surface is inverted. This leaves us primed to define a 2-form on a Riemann surface. By now, the typical construction pathway, from subsets of $\mathbb{C}$ then to transformations between forms, then to charts and collections of forms, should be fairly standard and you should believe that these are the proper things to do. With this justification, I leave the details of the construction unstated and jump directly to a definition. One thing that should be noted before this is that these forms, just like 1-forms, two forms typically can be deduced from a single one form in a single chart, so that is the form I define them in (you must practice equal caution here to ensure that it is well-defined everywhere on your Riemann surface, but all of the forms given here are nice enough).
Definition 4.10. A smooth 2-form on $X$ is a formal expression $\eta = f(z, \overline{z})dz \wedge d\overline{z}$, where $f \in C^\infty(X)$ is a smooth function.

Let’s practice with these wedges with this definition of a 2-form. How should this form transform if we used a coordinate $w$ and not $z$.

Example 4.11. Let $z = T(w)$ be the transition function. Then,

$$\eta = f(z, \overline{z})dz \wedge d\overline{z} \quad (4.1)$$

$$= f(T(w), \overline{T(w)})d(T(w)) \wedge d(\overline{T(w)}) \quad (4.2)$$

$$= f(T(w), \overline{T(w)})T'(w) \cdot \overline{T'(w)}dw \wedge d\overline{w} \quad (4.3)$$

$$= f(T(w), \overline{T(w)})||T'(w)||^2dw \wedge d\overline{w} \quad (4.4)$$

Incidentally, (4.4) is the criterion in [Mir97] for two 2-forms to transform into each other; nothing more is going on than ensuring that this definition plays nicely with different coordinates on our Riemann surfaces. These 2-forms we use for integrals over areas.

Remark 1. By how we defined the wedge of two 1-forms, we could have just as well defined a 2-form as a wedge of two 1-forms.

All of this development in this section has been leading to integration, but we are not there yet. A few form operations are needed.

4.3 Operations on Forms

New forms can be defined based on old forms, or even based on functions. This principle most importantly allows for transporting forms into easier spaces in which to work. In practice, we can essentially manipulate forms in three ways. This section collects them.

4.3.1 Multiplying by Smooth Functions

Given a smooth function $h$ on $X$ and a smooth 1-form $w = fdz + gd\overline{z}$, the 1-form $hw = hfdz + hgd\overline{z}$ is also a smooth 1-form. The same holds for a 2-form $\eta = fdz \wedge d\overline{z}$; the form $h\eta$ is also a 2-form, and moreover, these are well defined on $X$. Properties of our forms preserve under multiplication by smooth functions. For example, if $w$ is of type $(1, 0)$, then so is $hw$; if further $h$ is holomorphic, then so is $hw$. These are summarized in the following proposition, which needs almost no proof.

Proposition 4.12. Let $w$ be a 1-form, and let $h$ be a smooth function. Then,

- If $w$ is of type $(1, 0)$ or $(0, 1)$, then so is $hw$.
- If $w$ and $h$ are holomorphic, then so is $hw$.
- If $w$ and $h$ are meromorphic, then so is $hw$.
- If both $w$ and $h$ are meromorphic at $p \in X$, then $\text{ord}_p(hw) = \text{ord}_p h + \text{ord}_p w$.

4.3.2 Differentiating Functions and Forms

Functions: “Differentiating” functions also yields 1-forms. I put this in quotes because differentiation here is not the same as one would expect in a calculus course, but one would expect to say this in a differential geometry course—so I use the word regardless. Given a smooth function $f(z, \overline{z})$, we can define $\partial f := \frac{\partial f}{\partial z} dz$ and $\overline{\partial} f := \frac{\partial f}{\partial \overline{z}} d\overline{z}$. Together, we get $df$, the differential of $f$, defined by

$$df := \partial f + \overline{\partial} f.$$ 

This 1-form satisfies a Leibniz rule $d(fg) = (df)g + f(dg)$, because partial derivatives satisfy the Leibniz rule. This justification also answers: Why must these be smooth 1-forms? They are because of the chain
rule for partial derivatives (for well-definedness between coordinates), and because of the Leibniz rule for partial derivatives. If \( df = \omega \), we call \( \omega \) exact. This formulation becomes important when we start talking about Homology on our Riemann surface (which I introduce at the end of this chapter), in this way bridging more general topology with our analytic approach to Riemann surfaces.

**Definition 4.13.** A 1-form \( \omega \) is said to be **exact** if there exists a smooth function \( f \in C^\infty(X) \) such that \( df = \omega \).

**Smooth 1-forms:** We have already seen how to wedge two 1-forms together to get a 2-form. and we have seen how differentiation of functions, or 0-forms, takes us into the space of 1-forms. Thus one should expect to differentiate 1-forms and get to 2-forms. We define this analogously to how we differentiated functions.

Consider a smooth 1-form \( \omega = f(z, \bar{z})dz + g(z, \bar{z})d\bar{z} \), lets just try to take the partial derivative of this with respect to \( z \). (Recall the notation \( \partial \) for with respect to \( z \), and \( \bar{\partial} \) for \( \bar{z} \)).

\[
\partial w = \frac{\partial f}{\partial z}dz \wedge dz + \frac{\partial g}{\partial z}dz \wedge d\bar{z}
\]
\[
= \frac{\partial g}{\partial z}dz \wedge d\bar{z}
\]

\[
\bar{\partial} w = \frac{\partial f}{\partial \bar{z}}d\bar{z} \wedge dz + \frac{\partial g}{\partial \bar{z}}d\bar{z} \wedge d\bar{z}
\]
\[
= \frac{\partial f}{\partial \bar{z}}d\bar{z} \wedge dz
\]
\[
= -\frac{\partial f}{\partial \bar{z}}dz \wedge d\bar{z}
\]

Hence if \( d = \partial + \bar{\partial} \) we get that

\[
dw = \left( \frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right)dz \wedge d\bar{z}.
\]

**Example 4.14.** For example, a holomorphic 1-form \( fdz \) has the property that \( d(fdz) = 0 \), this is called being \( d \)-closed.

### 4.3.3 Pullbacks of Forms

Pullbacks allow for mapping forms between manifolds, often one easier to explicitly compute integrals on than another. Consider two Riemann surfaces and a holomorphic map between them \( F : X \to Y \). Given a smooth 1-form \( \omega = f(z, \bar{z})dz + g(z, \bar{z})d\bar{z} \) on \( Y \), we define the pullback, written \( F^*(\omega) \).

**Definition 4.15.** Given a coordinate representation of \( F \), say \( z = h(w) \), the pullback of \( \omega \) to \( X \) is

\[
F^*(\omega) := f(h(w)\bar{h}(\bar{w}))h'(w)dw + g(h(w)\bar{h}(\bar{w}))\bar{h}'(\bar{w})d\bar{w}.
\]

Pullbacks are the nicest operation on forms you can deal with. They preserve all of the information about your form–form types, and whether they are holomorphic or meromorphic, all remain the same under the pullback. More than that, they commute with \( d \), meaning that \( d(F^*(\omega)) = F^*(d\omega) \) and \( F^*(df) = d(F^*(f)) \) (where \( F^* \) of a function is simply composition \( F^*(f) = f \circ F \)).

Pullbacks of 2-forms work the same way. Let \( \eta = f(z, \bar{z})dz \wedge d\bar{z} \), then

\[
F^*(\eta) = f(h(w)\bar{h}(\bar{w})) |h'(w)||\bar{h}'(\bar{w})|dw \wedge d\bar{w}.
\]

In this context we begin integration.

### 4.4 Integration on Riemann Surfaces

Integration of 1-forms in \( \mathbb{C} \) take place over paths. The same thing happens on a Riemann surface, we integrate smooth 1-forms over paths, contours.
Definition 4.16. A path $\gamma$ on $X$ is a continuous function 
\[ \gamma : [a, b] \to X. \]

Points $\gamma(a), \gamma(b)$ are called the initial and terminal points, respectively. And if $\gamma(a) = \gamma(b)$, then the path $\gamma$ is closed.

Given any path, we can re-parameterize, meaning if $\alpha : [c, d] \to [a, b]$ is a continuous (and positive) then $\gamma \circ \alpha : [c, d] \to X$ is also a path on $X$. Essentially, we can always define a path from [0, 1] to our Riemann surface by re-parametrizing, hence the paths I give are from this domain. Also, given a path on $X$ and a holomorphic map between Riemann surfaces, $F : X \to Y$, we get a path $F_{\gamma} := F \circ \gamma$, called the push-forward of the path. It is possible to reverse and also concatenate paths; reversal means starting at the end point–going backward– and concatenation is stringing together paths if one path’s endpoint is the beginning of another (with possible reparametrization). The reverse of a path is denoted $-\gamma$.

What if there is some point in $X$ that has some undesirable property (say being a pole of a meromorphic function), but we would still like to integrate near it? We get the definition of a small path about $p$.

Definition 4.17. Let $S$ be some subset of $X$ whose closure does not contain $p$. A small path enclosing $p$ and not enclosing any point of $S$ is a closed path $\gamma_s$ such that

1. $\gamma_s([a, b])$ is 1-to-1 with $[a, b]$ and lies within a single chart $\phi : U \to V$.
2. The closed path $\phi \circ \gamma_s$ has winding number 1 about $\phi(p)$.
3. For all $s \in S \cap U$, the winding number of $\phi \circ \gamma_s$ about $\phi(s)$ is 0.

The second and third requirements allow the picking-off of these ‘bad’ points from potentially other bad points with a path that only goes around the point once. In practice, these bad points are zeros and poles of some function, which are discrete (hence we can find such a small path). Further, if the charts are chosen carefully enough, we can center the chart at $p$ and ensure that the path $\phi \circ \gamma_s = re^{2nit}$ (given that $\gamma_s$ is reparametrized to the interval $[0, 1]$). From here on out, assume that this is the case already, since changes to chart and reparametrization are easily enough dealt with. These small paths will give us ways to calculate integrals on paths enclosing poles, and also residues of functions.

As you may imagine, some paths may be in the domains of many charts and you might guess that this is something to overcome. This is not really a problem, we simply consider the path in a segmented way, called a partition of $\gamma$ with respect to our atlas. That these exist is pretty easy to show.

Lemma 4.18. Given a path $\gamma : [a, b] \to X$, and an atlas $\{\phi_i\}$, there exists a partition $P = \{a < x_1 \cdots < x_n < b\}$ of $[a, b]$ such that $\gamma_i := \gamma|_{[x_i, x_{i+1}]}$ is contained in a single chart, giving a finite collection $\{\gamma_i\}$ of curves (whose concatenation is $\gamma$) called a partition of gamma. Any two partitions have a common subpartition (a partition which contains all the points of both partitions, consider the union).

Justification for Lemma 3.18. The biggest thing is that this partition is finite. We could clearly pick points in each chart on the curve and find the $\phi_i$ that corresponds to them, but finiteness is not necessarily guaranteed. For compact Riemann surfaces, which we work with in this paper, the result is perhaps more believable but still not obvious. However, on $\mathbb{C}$ there is only one chart, and this is the place we often integrate.

Finally, we have the tools to integrate. The easiest part, defining what exactly we mean by an integral only needs a partition of a path. The gist is that we define our integral to be exactly what we want it to be under the partition, a sum of integrals we know how to do.

Definition 4.19. Let $w$ be a smooth 1-form, $\gamma$ a partitioned path in $X$ with partition $\{\gamma_i\}$ with domains in the set of charts $\{\phi_i\}$. Write our local coordinates as $z = z(t)$ be defined by the paths $\gamma_i$ and $w_i = f_i(z, \bar{z})dz + g_i(z, \bar{z})d\bar{z}$ be the local part of $w$ (w in each chart). Then the integral of $w$ along $\gamma$ is

$$\int_{\gamma} w = \sum_i \int_{\gamma_i} \left[ f_i(z(t), \bar{z}(t))z'(t) + g_i(z(t), \bar{z}(t))\bar{z}'(t) \right] dt.$$  

In a single chart, we have that, for chart $\phi$ where $w = f(z, \bar{z})dz + g(z, \bar{z})d\bar{z}$, then $\int_{\gamma} w = \int_{\phi_{\gamma}} f(z, \bar{z})dz + g(z, \bar{z})d\bar{z}$. The right hand side is the normal contour integral from complex analysis.
Everything required of forms thus far ensures well-definedness of this value–independent of our choices of charts. Also notice that the sum is finite, so there are no convergence considerations. This integral is \( \mathbb{C} \)-linear and path reversal gives the negative of our integral,

\[
\int_{-\gamma} w = -\int_{\gamma} w.
\]

We also have the following lemma:

**Lemma 4.20.** Let \( f \) be a smooth function, and \( w \) a smooth 1-form on \( Y \) and \( F : X \to Y \) a holomorphic map. Then

1. \( \int_{\gamma} df = f(\gamma(b)) - f(\gamma(a)) \)
2. \( \int_{F \ast \gamma} w = \int_{\gamma} F \ast w. \)

4.4.1 Chains

An equivalent way to think about integration over paths has to do with chains.

**Definition 4.21.** A **chain** on a Riemann surface \( X \) is a finite formal sum of paths on \( X \) with integer coefficients. A chain typically looks like \( \sum n_i \gamma_i \) (not to be confused with a partition of a single path, these \( \gamma_i \) may be any paths). The set of all chains on \( X \) is a free abelian group, denoted \( CH(X) \).

The integral of some 1-form \( w \) over a chain \( \gamma = \sum n_i \gamma_i \) is defined to be

\[
\int_{\gamma} w = \sum n_i \int_{\gamma_i} w.
\]

This definition is equivalent to our formulation earlier, in that case, a path \( \gamma \) is represented as a formal sum of its partition (with coefficients 1). Also notice that the integral over the chain \((-1) \cdot \gamma\) is

\[
\int_{(-1) \cdot \gamma} w = -1 \int_{\gamma} w = \int_{-\gamma} w.
\]

So integration does not see the difference between the reverse \(-\gamma\) and the chain \((-1) \cdot \gamma\). Integration also does not, as we have claimed above, see the distinction between integration over a partition or over a chain made with the partition. This will prove to be an advantageous way to think about integration, so we will use it interchangeably.

4.4.2 Integrating 2-forms

Recall that the motivation for 2-forms was to be able to integrate over areas on \( X \). We can do this, at the easiest, over some triangle \( T \) inside \( X \).

**Definition 4.22.** A **triangle** in a Riemann surface is a set homeomorphic to a triangle in \( \mathbb{C} \). Usually inside some chart. A closed set \( S \subset X \) is **triangulable** is if can be written as a (possibly infinite) union of triangles.

Most closed sets that we deal with are triangulable, so we do not fuss too much about this. Some examples of triangulable sets are any disk in the plane, and things homeomorphic to disks (which are almost everything we might want to deal with). We define the following inside a single chart, because we can generally triangulate small enough to ensure that each triangle is inside only one chart.

**Definition 4.23.** Let \( \eta = f(z, \overline{z})dz \wedge d\overline{z} \) in a chart \( \phi \) and let \( T \) be a triangle inside \( \phi \). Recall that \( dz = dx + idy \) and \( d\overline{z} = dx - idy \), meaning that \( dz \wedge d\overline{z} = (-2i)dx \wedge dy \). We define the integral of a 2-form over a triangle as

\[
\int T f(z, \overline{z})dz \wedge
= \int_{\phi(T)} (\cdot -2i) \cdot f(x + iy, x - iy)dx \wedge dy.
\]

Where the last integral is the normal double integral in \( \mathbb{C} = \mathbb{R}^2 \).
In fact, the general notation of double integral digns has fallen out of fashion. The most important thing is that we are integrating over some surface, and the surface will generally determine how many \( \int \)'s we will need to compute. Therefore we drop the double integral symbol from here on out, and instead focus on which type of surface we integrate over. The evaluation of these is understood to be the normal way we integrate. Hence

\[
\int \int_T \omega = \int_T \omega
\]  

(4.5)

which is justified by the fact that \( T \) is some triangulable area.

More generally, we can integrate a 2-form over a triangulable subset \( D \); in which case

\[
\int_D \eta = \sum_i \int_{T_i} \eta
\]

is the sum of the form on all of the triangles which triangulate \( D \), i.e. \( \cup_i T_i = D \).

There are potential issues about well-definedness of these definitions, such as \textit{Is this sum finite?}, but the way we were careful in defining how our forms reacted to change in coordinates makes these problems vanish. So we do not worry about it here.

Chains come up here too. We can define a chain based on a triangulable set. For a single triangle, let \( \partial T \) denote the boundary (a path) of \( T \). This is a chain.

**Definition 4.24.** Let \( D \) be a trianguable set, and let its triangulation be \( \{T_i\} \). The \textit{boundary chain for} \( D \) is the chain

\[
\partial D = \sum_i \partial T_i.
\]

This definition is highly dependent on the triangulation, but when we incorporate this definition into integration, we miss the dependency on the triangulation.

### 4.4.3 Theorems Involving Integration

Analogous theorems to integration in \( \mathbb{C} \) exist for integration on Riemann surfaces. Recall Greens theorem in \( \mathbb{C} \).

**Theorem 4.25.** Let \( \partial D \) be the boundary of some triangle in \( \mathbb{C} \). Let \( w = f dx + gdy \) be a smooth 1-form. Then,

\[
\int_{\partial D} w = \int_D dw = \int_D \left( \frac{\partial f}{\partial x} - \frac{\partial g}{\partial y} \right) dx \wedge dy.
\]

We use this theorem to prove the following.

**Theorem 4.26** (Stokes' Theorem). Let \( D \) be a triangulable set on a Riemann surface \( X \) and \( w = f dz + gd\overline{z} \) be a 1-form on \( X \). Then,

\[
\int_{\partial D} w = \int_D dw.
\]

**Proof.** This is a sketch of a proof, we do it in a single coordinate. Recall that \( dw = (\frac{\partial f}{\partial z} - \frac{\partial g}{\partial \overline{z}}) dz \wedge d\overline{z} \). By Green’s theorem and a little re-organization of variables to get \( \overline{z} \) out,

\[
\int_{\partial D} f dz + gd\overline{z} = \int_D \left( \frac{\partial f}{\partial z} - \frac{\partial g}{\partial \overline{z}} \right) dz \wedge d\overline{z} = \int_D dw.
\]

The residue theorem is a powerful tool in complex analysis, and its generalization to compact Riemann surfaces is even nicer. If a meromorphic function \( f \) has a laurent series \( \sum_{n=-\infty}^{\infty} c_n z^n \) then the meromorphic 1-form \( w = f dz \) has a laurent series \( w = (\sum_{n=-\infty}^{\infty} c_n z^n) dz \).
Definition 4.27. The residue of $w$ is the complex number $\text{Res}_p(w)c_{-1}$. This definition is independent of choice of laurent series.

Lemma 4.28. Let $w$ be a meromorphic 1-form in a neighborhood of a pole $p$, and let $\gamma$ be a small path enclosing $p$ and no other pole of $w$. Then,

$$\text{Res}_p(w) = \frac{1}{2\pi i} \int_\gamma w.$$  

We outline the proof below:

Proof. Outline. Let $z$ be the coordinate centered at $p$ whose domain encloses $\gamma$. Then $w = f(z)dz$ in the chart. Then by the usual residue theorem this is $2\pi ic_{-1}$.

Theorem 4.29 (Residue Theorem). On a compact $X$, given a $w \in \mathcal{M}^{(1)}(X)$ we have

$$\sum_{p \in X} \text{Res}_p(w) = 0.$$  

Proof. Since $X$ is compact, the number of poles is finite. Let $\{p_i\}$ (this set is closed) be the poles and pick for each a small path enclosing the pole but no other pole let this set be $\{\gamma_i\}$. By the lemma above,

$$\int_{\gamma_i} w = 2\pi i \text{Res}_{p_i}(w).$$  

Let $D = X - \bigcup_i \text{int}(\gamma_i)$ (int being the interior) so that $\partial D = \sum_i \gamma_i$.

$$\sum_{p \in X} \text{Res}_p(w) = \frac{1}{2\pi i} \sum_i \int_{\gamma_i} w$$

$$= \frac{1}{2\pi i} \int_{\sum_i \gamma_i} w$$

$$= \frac{1}{2\pi i} \int_{\partial D} w$$

$$= \frac{1}{2\pi i} \int_D dw = 0.$$  

as in $D$ the form $w$ is holomorphic, so $dw = 0$.

As you can see, all of these follow from the results in complex analysis by the way we have defined how things work on our Riemann surface.

4.5 Homology

Riemann surfaces are, as we have seen, nothing more than special cases of normal manifolds. On manifolds you can compute homology, which one can think of as giving linear spaces that count the number of holes in our manifolds. Therefore we can do this also on Riemann surfaces.

The set of chains on $X$ is denoted $CH(X)$ and is a free abelian group. There is a group homomorphism to the free abelian group of points on $X$

$$\phi : CH(X) \to F(X)$$

defined by sending $\gamma \to \gamma(b) - \gamma(a)$, the difference of the terminal and end points. This homomorphism has a kernel $CLCH(X)$ the set of all closed chains (meaning that each $\gamma_i$’s terminal point is canceled by another’s initial point), which is a subgroup.
Example 4.30. For a triangulable set $D$, $\partial D$ is a closed chain. Let $BCH(X)$ denote the set of all boundary chains $\partial D$ for triangulable sets $D$. Then $BCH(X) \leq CLCH(X)$ is a subgroup. Therefore the quotient $CLCH(X)/BCH(X)$ makes sense.

Definition 4.31. The quotient group $CLCH(X)/BCH(X)$ is called the first homology group of $X$, denoted $H_1(X)$.

4.5.1 The Period Map

Suppose that $w$ is a closed (i.e. $dw = 0$) smooth 1-form. By Stokes’ Theorem $\int_{\partial D} w = 0$ for any boundary chain. Using this, we can induce a homomorphism from closed chains to $\mathbb{C}$. This is

$$\int_* : CLCH(X) \to \mathbb{C}$$

defined by sending $\sum_i n_i \gamma_i \to \sum_i n_i \gamma_i w$. Again, the observation we made using Stokes’ Theorem, we see that $BCH(X) \subseteq \ker(\int_*)$. So we another induced homomorphism, this time from $H_1(X)$:

$$\int_- : H_1(X) \to \mathbb{C},$$

defined by sending

$$\sum_i n_i \gamma_i + BCH(X) \to \sum_i n_i \gamma_i w.$$

This works the same as the first homomorphism, but instead ignores the boundary chains, and called the period mapping for $w$. It is well defined because of Stokes’ Theorem.

Exercise 3. Let $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined by $w = z^n$ for some integer $n \geq 2$ where $w$ is a coordinate in the range and $z$ is one in the domain. Compute the pullback $F^*((1/w)dw)$ of the form $(1/w)dw$. Taken from \cite{Mir97 pg. 117}
5 Divisors, Linear Equivalence, and Linear Systems of Divisors

What is not obvious to us now, though it is of the utmost importance, is that we can tell a lot about Riemann surfaces based on the poles and zeroes of the meromorphic functions on them. We will take for granted that all Riemann surfaces have Meromorphic functions, and that compact ones have non-constant meromorphic functions, but they do!

We therefore organize the zeros and poles of some meromorphic function into a single object called a divisor. The first notion of a divisor is as a function. Recall that the support of a function \( D: X \rightarrow \mathbb{Z} \) is the set \( \{ p \in X \mid D(p) \neq 0 \} \). A divisor is one such function \( D: X \rightarrow \mathbb{Z} \) from a surface to the integers such that the support is a discrete set. **The set of divisors on** \( X \) **is the group** \( \text{Div}(X) \), under pointwise addition. However, a secondary and easier notion is that a divisor can be thought of as a formal sum of weighted points on \( X \). So a divisor \( D \) can be written as

\[
D = \sum_{p \in X} D(p) \cdot p.
\]

Excusing the double notation of \( D \) we note that the weight for each point is defined to be the integer value of that point in the divisor’s function definition. On a compact \( X \), this sum becomes a finite sum since \( D(p) = 0 \) for all points in \( X \) except a discrete (and hence finite) subset. This summation notation we will use in this paper when talking about divisors. **The degree** of a divisor \( D \) on a compact \( X \) is the sum \( \deg(D) = \sum_{p \in X} D(p) \) of the coefficients. This value is well-defined since on compact \( X \) this sum is finite.

We have many different types of divisors based on what idea we use to get the integer coefficients. For example, the order at a point of a meromorphic function gives rise to a special type of divisor. This type of divisor organizes the poles and zeros of a meromorphic function in the desired way.

5.1 Principal Divisors

Let \( f \in \mathcal{M}(X) \) be a meromorphic function not identically zero on a Riemann Surface \( X \) (and for the sake of ease let \( X \) be compact, though this definition also makes sense for any Riemann surface). We define a divisor based on the familiar order function as follows.

**Definition 5.1.** A principal divisor is the divisor

\[
\text{div}(f) = \sum_{p \in X} \text{ord}_p(f) \cdot p.
\]

And the set of all such divisors is called \( \text{PDiv}(X) \).

**Example 5.2.** On the Riemann Sphere \( c_\infty \) let \( f(z) \) be any rational function, which we already know encompass all examples of meromorphic functions by Theorem 3.11. And let

\[
f(z) = c \prod_{i=1}^{n} (z - \lambda_i)^{e_i}
\]

be its factorization in a chart. Then the principal divisor of \( f \) is

\[
\text{div}(f) = \sum_{i} e_i \cdot \lambda_i - (\sum_{i} e_i) \cdot \infty.
\]

5.2 Canonical Divisors

We can use the order function again to define another type of divisor, this time for meromorphic 1-forms. Letting us organize poles and zeros of forms too. Let \( \omega \) be a meromorphic 1-form.

**Definition 5.3.** A canonical divisor is a divisor

\[
\text{div}(\omega) = \sum_{p \in X} \text{ord}_p(\omega) \cdot p.
\]

The set of all of these divisors on a Riemann surface \( X \) is denoted \( \text{KDiv}(X) \).
Example 5.4. Consider the meromorphic 1-form $dz$ on $\mathbb{C}_\infty$. What is $\text{div}(dz)$? Well, clearly this form has no zeroes, and no poles on $\mathbb{C}_\infty - \{\infty\}$ where the chart $z$ makes sense. So let’s change coordinates using the chart $w = 1/z$. Then

$$dz = d(1/w) = -1/w^2 dw$$

which has a pole of order $-2$ at $\infty$. Therefore, $\text{div}(dz) = -2 \cdot \infty$ since all other points have order zero.

Lemma 5.5. We can add and subtract divisors to get new divisors.

Proof. Unions of discrete sets are discrete. 

Recall that we can combine a meromorphic function $f$ and a meromorphic 1-form $\omega$ to get a meromorphic one form $f \omega$. The divisor given by this 1-form is actually just the sum of the principal and canonical divisors given by $f$ and $\omega$.

Proposition 5.6. Given $f \in \mathcal{M}(X)$ and $\omega \in \Omega^{(1)}(X)$, $\text{div}(f \omega) = \text{div}(f) + \text{div}(\omega)$.

Proof. We know that $\text{div}(f) = \sum_{p \in X} \text{ord}_p(f) \cdot p$ and that $\text{div}(\omega) = \sum_{p \in X} \text{ord}_p(\omega) \cdot p$. Then, by properties of the order function we have that

$$\text{div}(f \omega) = \sum_{p \in X} \text{ord}_p(f \omega) \cdot p$$

$$= \sum_{p \in X} \text{ord}_p(f) \cdot p + \text{ord}_p(\omega) \cdot p$$

$$= \text{div}(f) + \text{div}(\omega).$$

In other words, this proposition shows that adding a principal divisor to a canonical divisor yields another canonical divisor, namely the divisor of $f \omega$. What we’d like to say then, is that we can always add some principal divisor to a canonical divisor to get whatever canonical divisor we want. This ends up being possible, with the following (equivalent) proposition, which we do not prove here, but [Mir97] does it very quickly in a single chart.

Proposition 5.7. On a Riemann surface and given two meromorphic 1-forms $\omega_1, \omega_2$ there exists an $f \in \mathcal{M}(X)$ such that $\omega_1 = f \omega_2$.

The work already given then proves that you can always add a principal divisor to a canonical divisor to get any other canonical divisor. A corollary to this is that $\text{KDiv}(X)$ is a coset of the subgroup $\text{PDiv}$.

Corollary 5.8. $\text{KDiv}(X) = \text{div}(\omega) + \text{PDiv}(X)$ for any meromorphic 1-form $\omega$.

5.2.1 Degrees of Canonical Divisors on Compact Surfaces

Believe it or not, but we are already in a place to state and understand something in the vein of Compact riemann surfaces. The proof of this uses Hurwitz’s formula, pullback divisors, and theorems about degrees of pullback divisors, and is omitted here but can be found in [Mir97, p.132].

Proposition 5.9. Let $X$ be a compact genus $g$ Riemann surface. Given a meromorphic 1-form $\omega \in \mathcal{M}^{(1)}(X)$ the divisor $\text{div}(\omega)$ has degree $2g - 2$.

Example 5.10. Let $K$ be canonical and let $D$ be any old divisor. Then $\text{deg}(K - D) = 2g - 2 - \text{deg}(D)$.
5.3 Other Types of Divisors

5.3.1 Boundary Divisors

Looking back, we realize that we have already seen divisors. Recall the boundary chain; every boundary chain is a divisor. Let $\gamma = \sum_i n_i \gamma_i$. Its boundary is $\partial \gamma = \sum_i n_i [\gamma_i(1) - \gamma_i(0)]$, which is easily seen to be a divisor. This we call the boundary divisor of the chain $\gamma$. From this we can make two extra observations: for one, $\deg(\partial \gamma) = 0$ and for another, since $X$ is path connected (recall the definition of Riemann Surfaces) we can see that any divisor of degree 0 is the boundary divisor of some chain.

5.3.2 Pullback Divisors

In the general spirit thus far in the paper, we’d like to be able to pull back divisors from one Riemann surface to another Riemann surface. Let $F : X \to Y$ be a holomorphic map between compact surfaces. We’d simply like to pull the support of the divisor $D$ on $Y$ back to the preimage of the support in $X$ with the same coefficients. However, ramification might cause issues.

We begin by remembering that the preimage of any single point is a finite discrete set. This set we can make into a divisor if we scale by the multiplicity, so that we get that only ramified points show up. Recall that a single point is itself a divisor.

Definition 5.11. An inverse image divisor of the point $q$ in $Y$ is the divisor

$$F^*(q) = \sum_{p \in F^{-1}(q)} \text{mult}_p(F) \cdot p.$$ 

If the point $q$ is not a branch point then this divisor is zero. We need this definition to finally pull back divisors themselves. We need it chiefly to deal with that potential ramification.

Definition 5.12. Let $D = \sum_i n_i \cdot q_i$ be a divisor on $Y$. The pullback divisor if the divisor $F^*(D) = \sum_{q \in Y} n_q \cdot F^*(q)$

where $n_q$ is the coefficient of the point $q$.

Pullbacks, as ever is their benefit, play nicely with operations.

Lemma 5.13. Let $X, Y$ be Riemann surfaces with a holomorphic map $F : X \to Y$ between them.

1. $F^* : \text{Div}(X) \to \text{Div}(Y)$ is a group homomorphism.
2. Pullbacks of principal divisors are principal:

$$F^*(\text{div}(f)) = \text{div}(F^*(f)) = \text{div}(f \circ F).$$

3. If $X, Y$ are compact so that degree is defined, we have that

$$\deg(F^*(D)) = \deg(F) \cdot \deg(D).$$

A last example of a divisor, which we will be useful to understand some theorems later, is an intersection divisor. This is a way to collect the zeros of some homogeneous polynomial in projective space.

Definition 5.14. Let $X$ be a smooth projective curve (i.e. a Riemann surface embedded in $\mathbb{P}^n$) and let the homogeneous coordinates be $[x_0 : \cdots : x_n]$. Given a homogeneous polynomial $G(x_0, \ldots, x_n)$, we define the intersection divisor of $G$, denoted $\text{div}(G)$ to be the sum of points $p$ where $G(p) = (x_0, \ldots, x_n) = 0$ with coefficients counting the order of vanishing.

The way to define this well is to pick for some $p \in X$ a non-vanishing homogeneous polynomial $H$ of the same degree of $G$ and define $\text{div}(G)(p) = \text{ord}_p(G/H)$, where we set $\text{div}(G)(q) = 0$ if $G(q) \neq 0$ (i.e. wherever we are not on the zero locus of $G$). A good choice for $H$ is typically $H = x_t^d$ where $x_t(p) \neq 0$ and $d = \deg(G)$. This gives a well-defined divisor.
Proof. Pick another such \( H' \). Then since \( \text{ord}_p(H/H') = 0 \) we get that
\[
\text{ord}_p(G/H) = \text{ord}_p(G/H \cdot H/H') = \text{ord}_p(G/H').
\]

For intersection divisors we have that\[
\text{div}(G_1 G_2) = \text{div}(G_1) + \text{div}(G_2)
\]
by properties of order, and a nice relation to principal divisors on a projective curve. On curves in \( \mathbb{P}^n \) meromorphic functions are ratios of homogeneous polynomials. This gives the following, which I state as a theorem as I do not prove the statement that makes it so.

**Theorem 5.15.** Given a meromorphic function \( f \) on some projective curve, \( f = G_1/G_2 \) is a ratio of homogeneous polynomials of the same degree and so the principal divisor \( \text{div}(f) \) is the difference of the two intersection divisors \( \text{div}(f) = \text{div}(G_1) - \text{div}(G_2) \).

### 5.4 Partial Ordering on Divisors

A **partial ordering** on a set \( S \) is a relation \( \leq \) that is transitive, antisymmetric, and reflexive, but not all elements need have a relationship to each other. For divisors, there exists such a partial ordering, namely if \( D = \sum n_p \cdot p \) and \( E = \sum m_p \cdot p \) such that \( n_p \geq m_p \) for every point \( p \) then we say \( D \geq E \). In other words, the coefficients for one are larger than those of the other, pointwise. We formalize this below:

**Definition 5.16.** For a divisor \( D \) on \( X \) we say \( D \geq 0 \) if \( D(p) \geq 0 \) for all \( p \). Similarly we say \( D > 0 \) if \( D \geq 0 \) and \( D \neq 0 \). Then we say that \( D_1 \geq D_2 \) if the divisor \( D_1 - D_2 \geq 0 \) (and for \( > \)).

These relations put a partial ordering on the set \( \text{Div}(X) \). This idea comes to play a major part in understanding some of the big theorems of this paper.

**Example 5.17.** A meromorphic function \( f \) is holomorphic if and only if \( \text{div}(f) \geq 0 \).

**Definition 5.18.** We call a divisor \( D \) **effective** if \( D > 0 \).

### 5.5 Linear Equivalence of Divisors

Recall that the difference of any two canonical divisors is a principal divisor. One might be tempted to make the claim that all divisors differ by a principal divisor, but this simply is not true. However, it does happen enough to motivate the following definition, that of divisors being linearly equivalent.

**Definition 5.19.** Two divisors \( D_1, D_2 \) are **linearly equivalent** if their difference \( D_1 - D_2 \) is a principal divisor. This is written \( D_1 \sim D_2 \).

**Lemma 5.20.** A couple easy lemmas.

1. The symbol \( \sim \) is an equivalence relation on divisors.
2. \( D \sim 0 \) if and only if \( D \) is principal.
3. On a compact \( X \), \( D_1 \sim D_2 \) implies that \( \deg(d_1) = \deg(d_2) \).

The first comes as no real surprise, the second is by definition, and the last is just the statement that on compact surfaces the sum of the degrees of meromorphic functions is zero. Therefore we can see that a linear equivalence class is exactly a coset for \( \text{PDiv}(X) \).

**Example 5.21.** Some examples will illustrate this. Any two canonical divisors are linearly equivalent, and any divisor such that \( D \sim K \) is linearly equivalent to a canonical divisor is also canonical. For example if \( F : X \to \mathbb{C}_\infty \) is the associated homolorphic map to the Riemann sphere then the inverse image divisors \( F^*(\lambda) \) are \( \sim \) and points on \( \mathbb{C}_\infty \) are \( \sim \).

Using the above ideas and we can prove that

**Proposition 5.22.** For a divisor \( D \) on the Riemann sphere, \( D \) is principal if and only if \( \deg(D) = 0 \).

We have already established the necessity of this in the sum of orders on compact surfaces being zero. The proof is not too enlightening, so we do not prove it here.
5.6 Spaces of Functions and Forms Associated to Divisors

We began this section with the goal of organizing the zeros and poles of some meromorphic function or form. The good in this is that we can find spaces of functions on $X$ whose poles and zeroes are no worse than those defined by the divisor.

**Definition 5.23.** The space of meromorphic functions with poles bounded by $D$ is the complex vector space of meromorphic functions $L(D) = \{ f \in \mathcal{M}(X) \mid \operatorname{div}(f) \geq -D \}$.

The point of this space is that if $D(p) = n > 0$ then $\operatorname{ord}_p(f) \geq -n$ forcing the orders of zeroes and poles to be no worse than the divisor $D$ allows. On a compact surface, the space $L(D)$ is finite dimensional. Since we have established in example 5.17 that a meromorphic function is holomorphic if and only if $\operatorname{div}(f) \geq 0$ we can see that

$$L(0) = \mathcal{O}(X) = \{ \text{holomorphic functions on } X \}.$$  

If $X$ is compact then $L(0) \cong \mathbb{C}$ since the only non-zero holomorphic functions on a compact surface are the constants.

**Lemma 5.24.** If $\deg(D) < 0$ on a compact surface, then $\dim(L(D)) = 0$.

**Proof.** Let $\deg(D) < 0$. Then $L(D) = \{ f \in \mathcal{M}(X) \mid \operatorname{div}(f) \geq -D \}$. In particular, this implies that $\operatorname{ord}_p(f) \geq -D(p)$, but since $D < 0$, we have that $-D(q) > 0$ for some $q$, so $\operatorname{ord}_q(f) > 0$. This means that $f$ is non-constant and holomorphic (since $\operatorname{div}(f) \geq -D > 0$). Since we are on a compact surface, we see that there are no such $f$’s, so that $L(D)$ is empty.

We can also organize divisors into a space based on a single divisor $D$. We use linear equivalence to do this. The complete linear system of $D$ is the complex vector space

$$|D| = \{ E \in \text{Div}(X) \mid E \sim D \text{ and } E \geq 0 \}.$$  

In other words $|D|$ is the set of positive divisors linearly equivalent to $D$.

**Example 5.25.** We compute $|K|$ for a canonical divisor. This system is called the canonical system. Consider a canonical divisor $K$. Then

$$|K| = \{ E \in \text{Div}(X) \mid E \sim K \text{and } E \geq 0 \}.$$  

So let $E \in |K|$. That $E \sim K$ implies that $E - K = \operatorname{div}(f)$ for some meromorphic function $f \in \mathcal{M}(X)$. Then $E = \operatorname{div}(f) + K$. Since by Corollary 5.8 we have that $K \text{Div}(X) = \operatorname{div}(\omega) + \text{PDiv}(X)$ adding a principal divisor to $K$ gives us another canonical divisor. Hence $E$ is a canonical divisor. Because Corollary 5.8 is for any meromorphic 1-form, we can get any canonical divisor this way. Therefore

$$|K| = \{ E \in K \text{Div}(X) \mid E \sim K \text{and } E \geq 0 \}.$$  

We can relate this space to the projectivization of $L(D)$ (which explains why we need $E \geq 0$). Recall that the projectivization of a complex vector space $V$ is the set of all one dimensional subspaces of that space. We denote this with $\mathbb{P}(V)$. Define a function $S : \mathbb{P}(L(D)) \rightarrow |D|$ defined by sending $\text{span}(f) \rightarrow \operatorname{div}(f) + D$. This map is well-defined since $\operatorname{div}(\lambda f) = \operatorname{div}(f)$.

**Proposition 5.26.** On a compact $X$, the map $S$ is a bijection.

**Proof.** Surjective: Let $E \in |D|$. Then $E \sim D$ means that there exists a meromorphic function $f$ on $X$ such that $E = \operatorname{div}(f) + D$, and since $E \geq 0$ we must have $\operatorname{div}(f) \geq -D$, so $f \in L(D)$ and $S(f) = E$.

Injective: Suppose $S(f) = S(g)$ for some meromorphic functions $f, g$. This means that $\operatorname{div}(f) = \operatorname{div}(g)$ so $\operatorname{div}(f/g) = 0$ meaning that $f/g$ is holomorphic so that $f/g = \lambda$ since $X$ is compact. Hence $f = \lambda g$ so they are in the same span.

**Corollary 5.27.** The dimension of the linear space is one more than the complete system of divisors: $\dim(L(D)) - 1 = \dim(|D|)$.
Proof. The dimension of $\mathbb{P}(L(D)) = \dim(L(D)) - 1$, and by the proposition above, this is exactly the dimension of $|D|$.

Note that for any $X$ the proof for Proposition 5.26 can show surjectivity. For linearly equivalent divisors, we have the following lemma.

**Lemma 5.28.** If $D_1 \sim D_2$ then for some $h \in M(X)$ the function $\mu_h : L(D_1) \to L(D_2)$ sending $f \to hf$ is an isomorphism.

The proof of this is a quick application of definitions and typical isomorphism arguments (Hint: Use symmetry).

Now that we have looked at spaces based one functions, we can construct analogous spaces based on 1-forms. The space of meromorphic 1-forms with poles bounded by $D$ is the complex vector space

$$ L^{(1)}(D) = \{ \omega \in M^{(1)}(X) \mid \text{div}(\omega) \geq -D \}. $$

We make the same observation that

$$ L^{(1)}(0) = \Omega(X) = \{ \text{global holomorphic 1-forms on } X \}. $$

Many similar theorems follow for this space, such as:

**Lemma 5.29.** The map $\mu_h : L^{(1)}(D_1) \to L^{(1)}(D_2)$ is an isomorphism if $D_1 \sim D_2$ where $h$ is a function such that $\omega_1 = h\omega_2$ for $\omega_i \in L^{(1)}(D_i)$.

We can even relate a space of meromorphic 1-forms to a space of meromorphic functions. Let $K = \text{div}(\omega)$ for some 1-form $\omega$. Then suppose $f \in L(D + K)$ for some divisor $D$. This means that $\text{div}(f) + D + K \geq 0$. Recalling that $f\omega$ is a 1-form with $\text{div}(f\omega) = \text{div}(f) + \text{div}(\omega) = \text{div}(f) + K$, we see that $\text{div}(f\omega) + D \geq 0$, meaning that $f\omega \in L^{(1)}(D)$. Therefore we have a map

$$ \mu_\omega : L(D + K) \to L^{(1)}(D) $$

sending $f \to f\omega$.

**Proposition 5.30.** The above map is an isomorphism.

**Proof.** That this map is linear and bijective is not proven here, but is an exercise at the end of this section.

Now we have enough of a setup for understanding the language in the major theorems, we end this section with a computation of $L(D)$ for a divisor on $C_\infty$.

**Example 5.31.** Let $D$ be a divisor on the Riemann sphere with $\text{deg}(D) \geq 0$. In other words, let

$$ D = \sum_{i=1}^{n} e_i \lambda_i + e_\infty \infty $$

such that $\sum_{i=1}^{n} e_i + e_\infty \geq 0$. Consider the meromorphic polynomial function $f_D(z) = \prod_{i=1}^{n} (z - \lambda_i)^{e_i}$.

**Proposition 5.32.** With the above notation,

$$ L(D) = \{ g(z) f_D(z) \mid g(z) \text{ is a polynomial of degree at most } \text{deg}(D) \}. $$

**Proof.** Let $g$ be a polynomial of degree $\text{deg}(D) = d$ and note that we have that $\text{div}(g) \geq -d \cdot \infty$ since at worst at infinity $g$ has a pole of degree $d$. Then we note that $\text{div}(f_D(z)) = \sum_{i=1}^{n} e_i \lambda_i + (\sum_{i=1}^{n} e_i) \infty$. Taken together, these two facts give that

$$ \text{div}(g \cdot f_D) + D = \text{div}(g) + \text{div}(f_D) + D $$

$$ \geq -d \cdot \infty - \sum_{i=1}^{n} e_i \lambda_i + (\sum_{i=1}^{n} e_i) \infty + \sum_{i=1}^{n} e_i \lambda_i + e_\infty \infty $$

$$ = (\sum_{i=1}^{n} e_i + e_\infty - d) \cdot \infty $$

$$ = (\text{deg}(D) - d) \cdot \infty \geq 0. $$

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Which proves that
\[ \{ g(z)f_D(z) \mid g(z) \text{ is a polynomial of degree at most } \deg(D) \} \subseteq L(D). \]

For the other containment, let \( h \in L(D) \) and consider \( g = h/f_D \). Then \( \text{div}(g) = \text{div}(h) - \text{div}(f_D) \). Since \( \text{div}(h) \geq -D \) we have that
\[
\text{div}(g) \geq -D - \left( \sum_{i=1}^{n} -e_i \lambda_i + \sum_{i=1}^{n} e_i \infty \right)
= \left( - \sum_{i=1}^{n} e_i - e_\infty \right)
= -\deg(D) \cdot \infty
\]
showing that \( g \) can have no poles in \( \mathbb{C} \) and a pole of degree at most \( \deg(D) \) at \( \infty \), meaning that \( g \) is a polynomial of degree at most \( \deg(D) \), and that means that \( h = gf_D \) showing that
\[ L(D) \subseteq \{ g(z)f_D(z) \mid g(z) \text{ is a polynomial of degree at most } \deg(D) \}. \]
And therefore we have equality of the two sets. \( \square \)

**Corollary 5.33.** On \( \mathbb{C}_\infty \) let \( D \) be a divisor. Then
\[
\dim(L(D)) = \begin{cases} 
0 & \text{deg}(D) < 0 \\
1 + \deg(D) & \text{deg}(D) \geq 0.
\end{cases}
\]

**Exercise 4.** Prove Proposition 5.30. \( \#31 \)
6 The Riemann-Roch Theorem and Algebraic Curves

6.1 The Riemann-Roch Theorem

The first major theorem of the paper, which we now have enough to understand it, but not enough to prove it fully, is Riemann-Roch. This theorem is essentially a theorem about the $L$ spaces of a Riemann surface.

**Theorem 6.1** (Riemann-Roch). Let $X$ be an compact Riemann surface of genus $g$. Then for any divisor $D$ and any canonical divisor $K$ we have

$$\dim(L(D)) - \dim(L(K - D)) = \deg(D) + 1 - g.$$  

Equivalently,

$$\dim|D| - \dim|K - D| = \deg(D) + 1 - g.$$  

If we have a divisor of high enough degree $(2g - 1)$ then we can force $\dim L(K - D) = 0$ so that

$$\dim(L(D)) = \deg(D) + 1 - g.$$  

**Proof.** We only prove the second statement. Consider a divisor of degree $2g - 1$, for example $(2g - 1) \cdot p$ for some $p \in X$. Then consider a canonical divisor $K$, by definition we have that $L(K - D) = \{ f \in M(X) | \text{div}(f) + K - D \geq 0 \}$. Since by Proposition we have that the degree of $K$ is $2g - 2$, and the degree of any principal divisor on a compact curve is zero divisor is zero, then the degree of

$$\text{div}(f) + K - D = 0 + (2g - 2) - (2g - 1) = -1$$  

which contradicts the definition, so forces this set to be empty and degree zero. Hence we reduce Riemann-Roch to the statement we wanted to show. 

6.2 Maps $\phi$ to $\mathbb{P}^n$

As alluded to early on, all examples of Riemann surfaces can be found inside $\mathbb{P}^n$ for some $n$. These are called smooth projective curves. The idea is that we can holomorphically embed any Riemann surface into some projective space. In this first part of the section, we make this precise. For the rest of the section, we talk about what these embeddings look like. Once we know we can embed them, we can say that they are actually a different type of construction: algebraic curves.

We typically want to map our surfaces to $\mathbb{P}^n$ because projective space has nice properties that we can exploit when working with our surfaces.

**Definition 6.2.** A map $\phi : X \to \mathbb{P}^n$ is holomorphic at $p \in X$ if there exist holomorphic functions $g_0, \ldots, g_n$ defined on $X$ near $p$ and not all zero at $p$ such that $\phi(x) = [g_0(x) : \cdots : g_n(x)]$ for $x$ close to $p$. It is holomorphic on $X$ if it is holomorphic at all $p \in X$.

Note that if $g_i$ is nonzero at $p$ it will be nonzero in a neighborhood of $p$ so this is well-defined. Now, we might be tempted to have these sorts of maps being the basis (the word basis being highly indicative, as we will come to see) of our definition of an embedding. However, recall that the only non-constant meromorphic functions on a compact surface are the constants. Therefore we cannot expect that, if we want an embedding, a single collection $\{g_i\}$ of holomorphic functions will work for all points of $X$.

We accordingly expand our notion here to meromorphic maps to $\mathbb{P}^n$, as this allows us to fix some functions $\{g_i\}$ that should work for all of $X$.

Consider $n + 1$ meromorphic functions on $X$, $\{f_0, \ldots, f_n\}$. Define

$$\phi_f := [f_0 : \cdots : f_n].$$

Note that this is a fine map to $\mathbb{P}^n$ from $X$ if we are not at a pole of any of the $f_i$ or not at a common zero of the $f_i$. It is a lemma that in fact, we can use the properties of projective space to deal with both of these potential issues and transform this meromorphic map to a holomorphic one.
Lemma 6.3. The map $\phi$ extends to a holomorphic map.

Proof. Let $m = \min_i(\text{ord}_p(f_i))$ in some coordinate $z$ centered at $p$. We may assume that, since zeros of meromorphic functions are discrete, that $p$ is not a common zero of the $\{f_i\}$. Then $z^{-m}$ is nonzero near $p$ and we have that $\phi_f = [z^{-m}f_0 : \cdots : z^{-m}f_n]$ by the properties of projective space. Notice that all of the $g_i := z^{-m}f_i$ are holomorphic in $z$, and that at least one of them is non-zero at $p$. If we repeat this process for all the other coordinates on $X$, we can therefore extend $\phi_f$ to a holomorphic map on $X$. \qed

Taking ratios of homogeneous coordinates, one can discern ([Mir97, p.135]) that every holomorphic map $\phi : X \to \mathbb{P}^n$ can be defined in the way above.

Proposition 6.4. Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map. Then there exists $n + 1$ meromorphic functions $f_0, \ldots, f_n$ such that $\phi = \phi_f$, the extended holomorphic map discussed above.

These holomorphic maps are how we wish to embed our surfaces, so a natural question is when are they 1-1 (and then by Proposition 3.22 $X \cong \phi(X)$ means we have successfully embedded $X$)?

6.2.1 The Associated Linear System to $\phi$

Since for every holomorphic map to projective space we can get a set of meromorphic functions, we can use divisors to associate a linear system to $\phi$. This will be a useful idea for deciding when $\phi$ defines an embedding.

Suppose that $\phi = [f_0 : \cdots : f_n]$ is our holomorphic map (written, with the caveat that we must extend them, with meromorphic functions $f_i$). Define $D = -\min_i(\text{div}(f_i))$ so that $-D(p) \leq \text{ord}_p(f_i)$ for all $i$. This, in turn implies that $\{f_i\} \subset L(D)$. So let $V_f$ be the subset of $L(D)$ spanned by the $f_i$.

Definition 6.5. We call the set $|\phi| = \text{div}(g) + D \mid g \in V_f$ the linear system of $\phi$, and this is a $\mathbb{C}$-linear subspace of $|D|$.

This definition, as it stands, might depend on the $f_i$, but it turns out that in fact it only depends on $\phi$. Therefore we take this to be the definition.

6.3 Base Points of Linear Systems

Given some linear system of divisors $Q$ on $X$, we say that $p$ is a base point of $Q$ if for every divisor $E \in Q$ we have that $E \geq p$ where we think of $p$ as a divisor. We say that a system is base-point free if it has no base points, or equivalently there exists some divisor $E < p$.

Another way to describe base-point free are as follows (taken from Proposition [Mir97, Prop 4.9 p 158]):

Proposition 6.6. Let $D$ be a divisor on a compact Riemann surface. Then $p \in X$ is a base point of $|D|$ if and only if $\dim L(D - p)) = \dim L(D))$. The system $|D|$ is base point free if and only if for every $p \in X$ we have that $\dim L(D - p)) = \dim L(D)) - 1$.

Interestingly, the only criterion on a compact $X$ for $Q = |\phi|$ to be a linear system of some holomorphic map $\phi : X \to \mathbb{P}^n$, is that $Q$ be base-point free. This we will take as a fact, without proof because it is involved and the result is the only important aspect of it.

Proposition 6.7. On a compact $X$, base point free systems of dimension $n$ on $X$ are all $|\phi|$ for some holomorphic map, and all $|\phi|$ are base point free.

Proof. We accepted as true the necessity. But why the sufficiency? Consider a $\phi : X \to \mathbb{P}^n$ and $D = -\min_i(\text{ord}_p(f_i))$. Fix a $p \in X$. Then let $D(p) = -k = -\text{ord}_p(f_j)$ for some $j$. We define a new divisor $E = \text{div}(f_j) + D$. We can see that $E \in |\phi|$ but that $E(p) = \text{ord}_p(f_j) + D(p) = 0$ so $E < p$. This, as we have noted means that $|\phi|$ must be base-point free. \qed

Hence, we can go the other way with holomorphic maps: Given a divisor $D$ such that $|D|$ is base point free, we can always associate a holomorphic map $\phi_D : X \to \mathbb{P}^n$, $n = \dim L(D(D))$.

The big question now is, When is $\phi_D$ an embedding? Since we boiled down discussions about these maps to discussions about divisors, we can use facts about them to prove things about our holomorphic maps to $\mathbb{P}^n$.

The criterion is as follows:
Theorem 6.8. Let $X$ be a compact Riemann surface and $D$ a divisor whose linear system is base-point free. Then $\phi_D$ is a holomorphic map and an isomorphism onto $\phi_D(X)$ (i.e., an embedding) if and only if for every $p, q \in X$ we have that
\[ \dim(L(D - p - q)) = \dim(L(D)) - 2. \]

We call a divisor $D$ with $\phi_D$ an embedding very ample. This definition is essentially saying that not only is $|D|$ base point free, but that $|D - p|$ is base point free for any $p$.

Proof. The idea of the proof is as follows. Since $|D|$ is base-point free, we know that $\phi_D = \phi_f$ for some functions $\{f_0, \ldots, f_n\}$. At two separate points $p, q$ we want to check if $\phi_D(p) = \phi_D(q)$ implies that $p = q$, but the first equation implies that there is some $f_i$ such that $\text{ord}_p(f_0) \leq \text{ord}_q(f_i)$ (after potential rearrangement) this is a condition on divisors based around $p$, and since $|D|$ is base point free we can deduce that $L(D - p) = L(D - q) = L(d - p - q)$. If we assume that the map is injective, and after some counting, we can restate in the way it is in the proof. A full proof is found, obviously, in [Mir97] p.162-163.

Definition 6.9. We say a surface $X$ is embedded in $\mathbb{P}^n$ if there is an embedding $\phi_D$.

6.4 Hyperplane Divisors

Now that we can embed our surfaces into $\mathbb{P}^n$, we would like an algebraic tool for $\mathbb{P}^n$ to help describe our embedded guys. Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map. Recall that we defined $|\phi|$ to be the linear system of divisors $\text{div}(f_i)$ given from the set $\{f_0, \ldots, f_n\}$ of meromorphic functions that define $\phi$. We can actually generate $|\phi|$ in a more geometric way.

Let $H \subset \mathbb{P}^n$ be a hyperplane, i.e. defined by $F = 0$ for some homogeneous polynomial of degree 1. Suppose further that $X$ is not entirely contained in $H$ (where we identify $X$ with $\phi(X)$). Then we define a divisor, meant to count the number if intersection points of $X \cap H$, called the hyperplane divisor.

Fix $p \in X$ and let $L$ be the homogeneous linear equation for $H$. By our containment assumption, $L$ does not vanish identically on $X$. Fix a linear equation $M$ that does not vanish at $\phi(p)$, then consider the function $h = (L/M) \circ \phi$ defined near $p$. This is holomorphic near $p$ by our assumptions on $L, M$. Indeed, near $p$ we have
\[ h(z) = (L/M) \circ [g_0(z) : \cdots : g_n(z)] = L(g_0(z), \ldots, g_n(z))/M(g_0(z), \ldots, g_n(z)) \]
is a ratio of linear combinations of holomorphic functions, and the denominator is non-zero in a neighborhood of $z$.

Definition 6.10. We define the hyperplane divisor of $H$, denoted $\phi^*(H)$ by
\[ \phi^*(H)(p) = \text{ord}_p(h) \]
for the $h$ determined above. Basically, it counts the number of times that $H$ intersects $X \cong \phi(X) \subset \mathbb{P}^n$.

Lemma 6.11. This divisor is effective and $\text{ord}_p(h) > 0$ if and only if $\phi(p) \in H$

That this is well defined is similar to the justification for why intersection divisors are well defined (indeed, the two are very similar in definition). It is also independent of $L$, and only depends on $H$. Indeed, if $K$ is also a linear equation for $H$ then we can show $K = cL$ for some $c \in \mathbb{C}$, which will not change the order of $p$ in $h$.

It is a fact that the set of hyperplane divisors generates $|\phi|$. Reference Corollary [Mir97] Cor 4.14 p 159.

Proposition 6.12. Let $\phi : X \to \mathbb{P}^n$ be a holomorphic map. The set $\{\phi^*(H)\}$ of hyperplane divisors forms the linear system $|\phi|$.

Corollary 6.13. The system $|\phi|$ is base point free.

Proof. We can always find a hyperplane $H$ in $\mathbb{P}^n$ that avoids potential base points $p \in X$ thought of as $\phi(p)$. Hence by Proposition 6.12 $|\phi|$ has no base points. □
6.5 Algebraic Curves

Now we get to the main point of all of these sections. We will find that all the examples of compact Riemann Surfaces can be embedded somehow into $\mathbb{P}^n$, projective space, and they all are examples of a single kind of surface.

**Definition 6.14.** Given $n - 1$ homogenous polynomials $\{F_1, \ldots, F_n\}$ in $n + 1$ complex variables $z_0, \ldots, z_n$. Let $Z$ be the common zero set of these polynomials in $\mathbb{P}^n$, $Z$ not a finite collection of points. In other words $Z$ is the set of non-zero points $p = [z_0 : \cdots : z_n]$ where $F_1(p) = \cdots = F_n(p) = 0$. Recall that $p$ being non-zero means that one of the $z_i \neq 0$, so we may assume that $z_0 \neq 0$. A (Complex) **Algebraic Curve** is a $Z$ such that the jacobian matrix

$$
\begin{pmatrix}
\frac{\partial F_1}{\partial z_1} & \cdots & \frac{\partial F_1}{\partial z_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial z_1} & \cdots & \frac{\partial F_n}{\partial z_n}
\end{pmatrix}
$$

has rank $n - 1$ everywhere on $Z$.

This last part of the definition ensures that the inverse function theorem applies so that $Z$ has charts to $\mathbb{C}$. From this definition, it is clear to see that we have already defined it in the first chapter as a local complete intersection curve, and hence is a Riemann surface. We alluded to the fact that actually, every compact Riemann Surface takes this form, we therefore see that every Riemann surface can be embedded into projective space. Note that Algebraic curves are not clearly embedded by the definition given. We develop brief reasoning why all Riemann surfaces are algebraic below, but it has everything to do with being embeddable into $\mathbb{P}^n$.

6.5.1 Chows Theorem

We identify every Riemann surface with an algebraic curve. Here we talk about why, although some of the following statements we state without proof. All of the following discussion is adapted from [GH78].

**Definition 6.15.** An open subset $U \subseteq \mathbb{P}^n$ is called an **analytic variety** if for any $p \in U$ there is an open set $V$ containing $p$ such that $V \cap U$ is the zero locus of $k$ holomorphic functions $\{f_0, \ldots, f_k\}$.

Analytic varieties are generally smooth, connected, Hausdorff, and so are manifolds. Discussions about these varieties are found near [GH78, pg. 12].

**Definition 6.16.** Let $x = [x_0, \ldots, x_n]$ be a homogeneous coordinate. An **algebraic variety** is a subset $U \subseteq \mathbb{P}^n$ defined by the zero locus of a collection of homogeneous polynomials $\{F_\alpha(x) = 0\}_{\alpha \in I}$.

For us, we can see that embedded Riemann surfaces are analytic varieties, and algebraic curves are algebraic varieties. Importantly, these two are equivalent in projective space:

**Theorem 6.17.** (Chows Theorem) Any analytic variety (of $\mathbb{P}^n$) is algebraic.

We see this as immediately showing Theorem 6.18 because we will see that all compact surfaces are embeddable into $\mathbb{P}^n$. The content of Chow’s Theorem uses sections of line bundles (don’t worry, you don’t need to know them for this paper), and can be found in [GH78 Ch. 1.3]. Now we come to the big statement, which we do not prove, and Miranda [Mir97] also does not prove right away. But we can at least understand how to use Chow’s to prove this.

**Theorem 6.18.** Every compact Riemann surface is an algebraic curve.

From now on we refer to Riemann surfaces as algebraic curves.
6.6 Algebraic Curves Embed into Projective Space

We can immediately find [Mir97, Prop 1.2 p.195] (one should note that Miranda’s proof of this uses something called Laurent Tail divisors, which I have chosen not to cover, but their treatment is found [Mir97, Ch. 6.2 p. 78]), that, using the criterion for $\phi_D$ to be an embedding, Riemann-Roch, and Tail Divisors we can get a good idea of when a divisor will definitely be an embedding.

**Lemma 6.19.** On an Algebraic Curve $X$ of genus $g$, any divisor of degree $\deg(D) \geq 2g + 1$ is very ample.

This, in turn, leads us to the observation that our one example of Algebraic Curves–smooth projective curves–are actually the only ones.

**Proposition 6.20.** Algebraic Curves can be embedded in projective space.

*Proof.* This only requires, by Lemma 6.19, a divisor of degree $2g + 1$. For any point $p \in X$ the divisor $D = (2g + 1) \cdot p$. Then $\phi_D$ is an embedding. \hfill \atab

**Exercise 5.** Show that if Riemann-Roch is true for any divisor $D$ then it is also true for $K - D$. (Hint: What is the degree of $K - D$? Check Example 5.10.) Taken from [Mir97, pg. 193].
7 The Jacobian and Abel’s Theorem

The other big theorem is one that answers some of our questions about integration on surfaces and how it coincides with the algebraic descriptions of our surfaces. Recall the earlier section on integration, chains, and homology.

7.1 The Jacobian Jac(X)

Let’s recall the definition of the dual space, from linear algebra, of some vector space $V$ over a field $F$ (F is not a polynomial this time).

**Definition 7.1.** We call the set of all linear functions $\lambda : V \to F$ the **dual space**, denoted $V^*$. Sometimes we call $\lambda$ a linear functional.

Remember that for a closed holomorphic 1-form $\omega \in \Omega'(X)$ and $T \subseteq X$ a triangulable set we have that

$$\int_{\partial T} \omega = \int_T d\omega = 0.$$ 

So any integral of $w$ on a boundary chain is zero. Meaning that, for $\gamma$ a closed chain, $\int_\gamma \omega$ depends only on the the parts of $\gamma$ that are not in $BCH(X)$. Said another way,

$$\int_\gamma \omega = \int_{[\gamma]} \omega$$

where $[\gamma]$ is the homology class of $\gamma$. Since holomorphic 1-forms are closed, integrals of holomorphic 1-forms are well defined, and we therefore get a functional for every class $[c]$

$$\int_{[c]} : \Omega'(X) \to \mathbb{C}$$

defined by $\omega \to \int_{[c]} \omega$.

**Definition 7.2.** We call a linear functional $\lambda : \Omega'(X) \to \mathbb{C}$ a **period** if $\lambda = \int_{[c]}$ for some class. The set of all periods is denoted $\Lambda$.

It is clear that $\Lambda \subseteq \Omega'(X)^*$ is a subspace of the dual space of forms. Therefore we quotient the dual space by the periods to get something new.

**Definition 7.3.** The **Jacobian** of a compact Riemann surface (algebraic curve) $X$ is the quotient

$$Jac(X) = \frac{\Omega(X)^*}{\Lambda}$$

i.e. it is the space of duals that are not periods.

If we pick a basis (we can do this don’t worry) $w_1, \ldots, w_g$ for $\Omega'(X)$ the dual is the space of column vectors

$$\mathbb{C} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_g \end{bmatrix} = \mathbb{C}^g$$

and we associate $\lambda$ to this by the correspondence

$$\lambda \to \begin{bmatrix} \lambda(w_1) \\ \vdots \\ \lambda(w_g) \end{bmatrix}$$
Every period is given by
\[
\begin{bmatrix}
\int_{c_1}(w_1) \\
\vdots \\
\int_{c_l}(w_g)
\end{bmatrix}
\]
so that we can define \(\text{Jac}(X)\) also by
\[
\text{Jac}(X) = \mathbb{C}^g / \Lambda.
\]

**Example 7.4.** On a torus \(\mathbb{C}/L\), \(\text{Jac}(X) = X\).

**Note 7.5.** It is a general fact [Mir97, Def 4.9, Pg 263] that for a genus \(g\) algebraic curve \(X\) that \(\text{Jac}(X)\) is a \(g\)-dimensional complex torus. This means, in particular, that for a genus 1 curve that \(\text{Jac}(X) \cong \mathbb{C}/L\) for some lattice \(L\) (which we can think of as being the linearly independent basis periods of \(\Lambda\)).

### 7.2 The Abel-Jacobi Map

We can obtain a map between some surface to its Jacobian with something called the **Abel-Jacobi Map**. This is a map \(A : X \to \text{Jac}(X)\) which essentially sends a point \(p\) to some non-period dual. First we find a map (which we also call \(A\)) \(A : X \to \Omega'(X)^*\). Pick a base point \(p_0\) for \(X\) and a path \(\gamma_p\) with a base point at \(p_0\) and an endpoint at \(p\). Then define \(A(p)\) to act on \(\omega \in \Omega'(X)\) by \(A(p)(\omega) = \int_{\gamma_p} \omega\). Note that, a priori, this map is not well defined, since another curve \(\gamma'_p\) might integrate differently. However note that these values differ by integrating along the closed chain \(\gamma_p - \gamma'_p\). Hence we get a well-defined map instead to \(\text{Jac}(X)\).

This is the **Abel-Jacobi Map**:
\[
A : X \to \text{Jac}(X)
\]
defined by \(A(p) = [\int_{\gamma_p} \omega]\) where the brackets denote the equivalence via closed curves on integration. This does depend on the base point. In a basis for \(\Omega'(X)^* = \mathbb{C}^g, w_1, \ldots, w_g\), we can write the Abel-Jacobi map as
\[
A(p) = (\int_{p_0}^p w_1, \ldots, \int_{p_0}^p w_g)^\top \mod \Lambda.
\]

As we noted, this definition on duals depends on our base point. We want one that is independent. First we note that we can extend this map into a base-point independent map on \(\text{Div}(X)\) to \(\text{Jac}(X)\). The extension is easy. Define \(A : \text{Div}(X) \to \text{Jac}(X)\) in the following way. Given a \(D \in \text{Div}(X)\) we can write \(D = \sum n_p \cdot p\). Then define
\[
A(D) := \sum n_p \cdot A(p).
\]
This induces a group homomorphism \(\text{Div}(X) \to \text{Jac}(X)\) because it is linear and pointwise. We also call this map the **Abel-Jacobi Map**
\[
A : \text{Div}(X) \to \text{Jac}(X).
\]
As I alluded to, we want something base point independent. This map above is not. However, if we restrict it to degree zero divisors it is.

**Proposition 7.6.** Let \(A_0 : \text{Div}_0(X) \to \text{Jac}(X)\) be the restriction of the Abel-Jacobi map to the divisors of degree zero. Then \(A_0\) is independent of base point.

**Proof.** Suppose \(p'_0\) is a different base point. Then let \(\gamma\) be a path from \(p_0 \to p'_0\).
\[
A_0(p) = (\int_{p_0}^p w_1, \ldots, \int_{p_0}^p w_g)^\top \mod \Lambda
\]
\[
= (\int_{p_0}^{p'_0} w_1 + \int_{p_0}^p w_1, \ldots, \int_{p_0}^{p'_0} w_g + \int_{p_0}^p w_g)^\top \mod \Lambda
\]
From this we see that the answer differs by a factor of
\[
k = [\int_{p_0}^{p'_0} w_1, \ldots, \int_{p_0}^{p'_0} w_g]^\top
\]
and therefore differs by something independent of $p$. Hence we can also write

$$A(D) = \sum n_p \cdot A(p) + \sum n_p \cdot k$$

$$= \sum n_p \cdot A(p) + k \sum n_p$$

But since $D \in \text{Div}_0(X)$ we have that $\sum n_p = 0$. So clearly $A(D)$ is independent of base point.

\[\square\]

### 7.3 Abel’s Theorem

This brings us to our final major theorem. Abel’s Theorem is the strengthening of the statement that all principal divisors are degree zero. Not all degree zero divisors are principal, but this theorem gives us a criterion for them to be.

**Theorem 7.7** (Abel’s Theorem). Let $X$ be a compact genus $g$ surface. Let $D \in \text{Div}_0(X)$ Then $D = \text{div}(f)$ for some meromorphic function on $X$ if and only if $A_0(D) = 0$ in $\text{Jac}(X)$.

We don’t prove this, as it is the subject of [Mir97, Ch. 8.3-8.5]. Yet, the reader should be able to follow Miranda’s proof with the knowledge the reader has thus far in the paper.

**Exercise** 6. Show that the chosen base point $p_0$ (as a divisor) is sent by the Abel-Jacobi Map $A$ to the origin of $\text{Jac}(X)$. Taken from [Mir97, pg. 250].
8 Classifications of Algebraic Curves of genus $g=0,1,2,3$

This section uses the big results 6.1 and 7.7 to classify curves of up to genus 3.

8.1 Classification of genus 0 curves

First a lemma.

Lemma 8.1. Let $X$ be a compact Riemann surface. Then if $\dim(L(p)) > 1$ we get $X \cong \mathbb{C}_\infty$.

Proof. If $\dim(L(p)) > 1$ then there is a non-constant meromorphic function $h$ on $X$ allowed a simple pole at $p$. This means that the associated holomorphic map $F : X \to \mathbb{C}_\infty$ has degree one. By Proposition 3.22 this means that $F$ is an isomorphism.

Corollary 8.2. If $D$ is a divisor of degree 2 such that $\dim(L(D)) = 2$ on a genus $g \geq 1$ curve, then $L(D)$ is base point free.

Proof. Suppose $D$ is degree 2 and $\dim(L(D)) = 2$. If $[D]$ has a base point, then $\dim(L(D-p)) = \dim(L(D))$ so that $\dim(L(D-p)) = 2$. Since we can generate $[D]$ with the divisor $D = p + q$, if $q$ is the base point this implies that $\dim(L(p)) = 2 > 1$ so $X \cong \mathbb{P}^1$ by the above lemma, but this contradicts $g \geq 1$.

All that’s left to do is prove that for a point $p$ in our Riemann Surface $\dim(L(p)) > 1$.

Proposition 8.3. Let $X$ be an algebraic curve of genus 0, then $X \cong \mathbb{C}_\infty$.

Proof. Fix a $p \in X$. Recall that the form $\omega = dz$ has a pole of order $-2$ at $p$ by Example 5.4. Then if $K = \text{div}(\omega)$ we have that $\deg K = -2$. This means that $\deg(K-p) = -3$ at $p$. Since this is less than $2g-2 = -2$, we have $\dim(L(K-p)) = 0$ by Lemma 5.24. Using again Riemann-Roch 6.1 we have then that

$$\dim(L(p)) = \deg(p) + 1 - g + \dim(L(K-p)) = 2.$$

By the above Lemma 8.1 this means that $X \cong \mathbb{C}_\infty$.

8.1.1 The Canonical System is Base Point Free

Recall the canonical system $|K|$. The lemma we used to prove that genus 1 curves are tori gives us somewhat more.

Proposition 8.4. The canonical system $|K|$ is base point free.

Proof. If $g = 0$, then $|K|$ is empty so is vacuously base point free, so we assume $g \geq 1$. Fix $p \in X$, we want to show $\dim(L(K-p)) \neq \dim(L(K))$. By Riemann-Roch,

$$\dim(K-p) - \dim(L(p)) = \deg(K-p) + 1 - g = 2g - 3 + 1 - g = g - 2.$$

Since by Lemma 8.1 if $\dim(L(p)) > 1$ then $X \cong \mathbb{P}^1$, which would contradict our assumption on the genus. So $\dim(L(p)) = 1$ since we always have the constants. Therefore $\dim(L(K-p)) = g - 1 = \dim(L(K)) - 1$ this is the definition of base point free.

This leads us to arguably the most important map for classification of low genus curves (in fact Miranda also claims that this map is one of the most important in algebraic geometry).

8.1.2 The Canonical Map

Definition 8.5. Consider the canonical system $|K|$. Recall from Example 5.22 that $|K|$ is the set of divisors of meromorphic 1-forms on $X$. Given a basis of $\{f_0, \ldots, f_n\} \subset \mathcal{M}(X)$ that define the forms in $|K|$, we can define a map called the canonical map $\phi_K = |K| : X \to \mathbb{P}^n$ by setting $|K|(z) = [f_0(z) : \cdots : f_n(z)]$. This we will show to be well-defined later on when we show that $|K|$ is base point free.
Let’s explore this map’s properties. This map has not shot of being an embedding of an algebraic curve if the genus is less than three (by some of the classifications to follow) so assume that \( g(X) \geq 3 \). Recall that any holomorphic map \( \phi_D \) associated to a divisor \( D \), such that \(|D|\) is base point free, fails to be an embedding if and only if for some \( p,q \in X \) we have that \( \dim(L(D - p - q)) \neq \dim(L(D)) - 2 \). Since \(|K|\) is base point free, this only happens if \( L(K - p) \) has a base point at \( q \), so \( \dim(L(K - p - q)) = \dim(L(K - p)) \). Hence \( \dim(L(K - p - q)) = \dim(L(K - p)) = \dim(L(K)) - 1 = g - 1 \) again since \(|K|\) is base point free. This is the only way it fails to embed \( X \). Then by Riemann-Roch, we have that \[
\dim(L(K - p - q)) = \deg(K - p - q) + 1 - g + \dim(L(p + q)) = 2g - 4 + 1 - g + \dim(L(p + q)) = g - 3 + \dim(L(p + q)).
\]
Therefore by the above statements, \(|K|\) fails to embed if and only if \( \dim(L(p + q)) = 2 \). But then we’d get a degree two map to \( \mathbb{P}^1 \), so \( X \) would be hyperelliptic by Proposition \( 3.25 \). Otherwise, \(|K|\) embeds \( X \) into \( \mathbb{P}^{g-1} \). Therefore we have:

**Proposition 8.6.** If \( X \) is a curve of genus \( g \geq 3 \) is not hyperelliptic, then the canonical system \(|K|\) defines an embedding \( \phi_K \) of \( X \) into \( \mathbb{P}^{g-1} \).

### 8.2 Classification of genus 1 curves

**Proposition 8.7.** Every compact genus 1 curve is a complex torus.

**Proof.** We show that for a genus 1 curve, the Abel-Jacobi map is an isomorphism. Let \( p_0 \) be the base point, and suppose that \( A(p) = A(q) \), then \( A(p - q) = 0 \) by definition of the Abel-Jacobi map. By Abel’s Theorem, we see have that \( p - q \sim 0 \) as divisors. This means \( p - q = \text{div}(f) \) for some meromorphic function \( f \) on \( X \); i.e., \( \text{div}(f) + q = p \geq 0 \), so that either \( q = p \) (and \( f \) is constant) or \( \dim(L(p)) > 1 \). If \( \dim(L(p)) > 1 \) then by Lemma 8.1 this would mean that \( X \cong \mathbb{P}^1 \), but \( g = 1 \) so this is not possible. Hence \( p = q \). Since by Theorem 3.2 any non-constant map between compact surfaces is surjective, we get that \( A \) is an isomorphism, so \( X \cong \text{Jac}(X) \), which we have stated to be a complex 1-dimensional torus in Note 7.5.

Before we classify them in a different way, we first prove a little proposition about the zero set of \( F(x, y) = 0 \) in \( \mathbb{P}^1 \). This will also be useful for classifying genus 3 curves.

**Lemma 8.8.** An irreducible homogeneous polynomial \( F(w, z) \) of degree \( d \) has exactly \( d \) roots in \( \mathbb{P}^1 \).

**Proof.** Consider the line \([w : 1]\). Then \( F(w, 1) = w^m \sum_{i=0}^{r} a_i w^i \) such that \( r + w \leq d \). Then we homogenize with \( z \) as follows: \( F(w, z) = z^{d-(m+r)} w^m \sum_{i=0}^{r} a_i w^i z^{-i} \). We note that on \([w : 1]\) we have \( m + r \) roots, and we potentially only missed something on the line where \( z = 0 \). But on this line we only look at \( z^{d-(m+r)} \) for zeros since the polynomial \( w^m \sum_{i=0}^{r} a_i w^i z^{r-i} \) is non-zero for \( z = 0 \) (look at \( w^{m+r} \) term). Hence we have exactly \( d - (m + r) \) zeros on \( z = 0 \) and \( m + r \) zeros else (irreducibility gives us exact counts). Therefore there are \( d \) roots of \( F \) in \( \mathbb{P}^1 \).

**Proposition 8.9.** Every compact genus 1 curve is a plane cubic.

**Proof.** Consider an effective divisor of degree 3 \( D = p_1 + p_2 + p_3 \) where it is possible that these points are the same. Then by Lemma 6.19 \( \phi_D : X \to \mathbb{P}^n \) is an embedding for some \( n \). By Riemann-Roch we get that \[
\dim(L(D)) - \dim(L(K - D)) = \deg(D) + 1 - g = 3 + 1 - 1 = 3.
\]
And since \( \deg(K - D) = 2g - 2 - 3 = 2 - 2 - 3 = -3 \) we have by Lemma 5.24 \( \dim(K_D) = 0 \). Hence \( \dim(L(D)) = 3 \). Therefore \( n = 3 - 1 = 2 \). Hence by Definition 6.14 and Theorems 6.17 and 6.18, \( X \) is an algebraic curve defined by 2 - 1 = 1 homogeneous polynomials in \( n + 1 = 3 \) variables. Hence it is described by \( \{(x : y : z) \in \mathbb{P}^2 \mid F(x, y, z) = 0\} \) for a single degree \( d \) polynomial. Consider the hyperplane \( H \) defined by \( L = 0 \) not passing through the points \( p_1, p_2, p_3 \). Then we can find that \( F|_L \) is a degree \( d \) polynomial in \( \mathbb{P}^1 \) (intersecting with a plane intuitively drops dimension by 1) has \( d \) roots by Lemma 8.8 so we have exactly \( d \) intersection points. Since the hyperplane divisor \( \phi^*(H) \) is in \(|D|\) by Proposition 6.12 we see that \( \deg(\phi^*(H)) = 3 \) since \( \deg(D) = 3 \). Therefore \( d = 3 \) and this means that \( X \) is a smooth plane cubic. Recall Example 2.22 about smooth projective curves. \( \square \)
Proposition 8.10. Every compact genus 1 curve is a hyperelliptic curve that is the projective completion of a plane curve of the form

\[ y^2 = x^3 + a_2x^2 + a_1x + a_0. \]

Proof. We try to see if there is a degree 2 map to \( \mathbb{P}^1 \). Consider \( L(p + q) \) by Riemann-Roch we have that

\[ \dim(L(p + q)) - \dim(K) = \deg(p + q) + 1 - g = 2 + 1 - 2 = 1. \]

since \( \deg(K) = 2g - 2 = -2 \), \( \deg(p + q) = 0 \). Hence \( \dim(p + q) = 2 \) and is base point free by Corollary 8.2. Therefore \( L(p + q) \neq L(p) \), so there must be a meromorphic function \( f \) with a pole at both \( p, q \). Hence \( f \) induces a degree two map 2 by the associated holomorphic map \( F \). By Proposition 3.25 this means that \( X \) is isomorphic to some the projective completion (homogenization) of a hyperelliptic curve defined by \( y^2 = h(x) \). Where \( \deg(h) \in \{3, 4\} \). Since we have already that \( X \) ought to be a cubic, we can easily see that \( \deg(h) = 3 \), or else homogenization would give a quartic. Hence we have three roots where \( h(x) = 0 \), so that we can write \( h(x) = (x-a)(x-b)(x-c) \). So we are done.

8.3 Classification of genus 2 curves

Proposition 8.11. Compact genus two curves are hyperelliptic.

Proof. Let \( X \) be a compact algebraic curve of genus two. Consider a canonical divisor \( K \) with degree \( 2g - 2 = 2 \), which we have by 5.9. Since

\[ \dim(L(K)) = \deg(K) + 1 = 2 + 1 = 3, \]

and \( \dim(L(0)) = 1 \) (the only holomorphic functions on compact curves are constants), we see that \( \dim(L(K)) = 2 \) (so that the map \( \phi_K \) sends \( X \) to \( \mathbb{P}^1 \)). Therefore \( \dim(K) = 2 - 1 \) by Corollary 3.27 so there is some effective divisor \( E \in |K| \) by definition; without loss of generality, we replace \( K \) with \( E \). We have already seen that \( |K| \) is base point free by Proposition 8.4 so then the map \( \phi_K \) defines a degree 2 map to \( \mathbb{P}^1 \) by Proposition 6.12 when considering the hyperplane \( L = \{[z : w] \in \mathbb{P}^1 | z = 0 \} \), hence by Proposition 3.25 this means that \( X \) is hyperelliptic.

8.4 Classification of genus 3 curves

We know that we can find examples of hyperelliptic curves of genus 3, we know that we have some genus 3 curves that are hyperelliptic and defined by \( y^2 = h(x) \) where \( \deg(H(x)) \in \{7, 8\} \). Otherwise, we know that the canonical map is an embedding.

Proposition 8.12. If \( g(X) = 3 \) and \( X \) is not hyperelliptic, then \( X \) is a smooth plane quartic.

Proof. Consider a canonical divisor \( K \) of degree \( 2g - 2 = 6 - 2 = 4 \). Since \( X \) is not hyperelliptic Proposition 8.6 tells us that \( |K| \) embeds. Then by Riemann-Roch,

\[ \dim(L(K)) = \deg(K) + 1 - g + \dim(L(K - K)) = 4 + 1 - 3 + 1 = 3 \]

hence \( \phi_K : X \hookrightarrow \mathbb{P}^2 \) is an embedding. By Chow’s Theorem 6.17 this is defined by a single homogeneous polynomial \( F(x, y, z) = 0 \) of degree \( d \). By a similar argument we consider a hyperplane divisor to count \( d \) and find that it is in \( |K| \), so must have degree 4 because \( K \) does and \( |K| \) is base point free by Proposition 8.4. Therefore \( X \) is a smooth plane quartic. Recall Example 2.22 about smooth projective curves.

Therefore we have show the following.

Proposition 8.13. Any genus 3 curve is either hyperelliptic and described by \( y^2 = h(x) \) for some \( h(x) \) of degree 7 or is a smooth plane quartic.

The canonical map can be used to classify curves of up to genus 5 very well; beyond that there are simply too many possibilities for the same arguments shown here to fully classify the curves.
A Review of Complex Analysis

Consider the field $\mathbb{C}$. This section collects basic review results from analysis on this field. The results can all be found in [SS03].

First we note that we can project $S^2 \subset \mathbb{R}^3$ onto $\mathbb{C}$. We do this with the northern projection map $\pi_N$. Let $N = (0,0,1)$. Recall that $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Note that $N \in S^2$ is the ‘north-pole’.

**Definition A.1.** Define $\pi_N : S^2 \rightarrow \mathbb{C}$ by

$$\pi_N(x, y, z) := \frac{x}{1-z} + i \frac{y}{1-z},$$

this map maps every point of $S^2$ to $\mathbb{C}$ except for $N$. However, as $|\frac{x}{1-z} + i \frac{y}{1-z}|$ increases, we get closer and closer to $N$. Hence we often write $S^2 = \mathbb{C} \cup \{\infty\}$ or simply $\mathbb{C}_\infty$. This is called the Riemann sphere.

Functions $f(z) : \mathbb{C} \rightarrow \mathbb{C}$ can be written as (given the usual interpretation that $z = x + iy$) $f(x, y) = u(x, y) + iv(x, y)$.

**Definition A.2.** A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic at $z_0$ if they obey the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

at $z_0 = x_0 + iy_0$. It is holomorphic in an open set if it is holomorphic at every point in that open set.

The above definition is the same as complex differentiable. It is a major theorem that holomorphic functions are analytic, in [SS03] they define this to be complex differentiable, but we take it to mean there is some disk near $z_0$ such that $f$ can be described by some Taylor series. The point is that in $\mathbb{C}$ all of these are equivalent, so it doesn’t matter how we define things.

Meromorphic functions are defined in a similar way, where now we allow them to have poles on some finite subset of our open set. Instead of Taylor series, we have to use Laurent series to write a power series representation of our function.

**Definition A.3.** A Laurent series for a meromorphic function at $z_0$ is a series $\sum_{i=-k}^{\infty} a_n (z - z_0)^n$ such that (1) $a_{-k} \neq 0$ and (2) $f(z) = \sum_{i=-k}^{\infty} a_n (z - z_0)^n$ near $z_0$.

Hence we can talk about the residue of a meromorphic function.

**Definition A.4.** Given a Laurent series for a meromorphic function $f$ near $p$, we define the residue $\text{res}_p(f) := a_{-1}$ the $-1$st coefficient given by the Laurent series.

Now we can talk about some integration.

**Definition A.5.** A path is a continuous function $\gamma : [a, b] \rightarrow \mathbb{C}$. If $\gamma(a) = \gamma(b)$ we call $\gamma$ a loop.

**Definition A.6.** The integral of a holomorphic function $f$ along a path $\gamma$ is defined to be

$$\int_\gamma f \, dz := \int_a^b f(\gamma(t)) \gamma'(t) \, dt.$$

If the path is not smooth, we partition it into differentiable parts, integrate, and then sum the integrals.

In a simply connected set $D$ (means that it is contractible to a point, technical definition has to do with the fundamental group, reference [Mun13] for information about fundamental groups) and a loop $\Gamma$ inside $D$, if a function $f$ is analytic in $D$ we have:

**Theorem A.7** (Cauchy’s Integral Theorem).

$$\int_{\Gamma} f \, dz = 0.$$

We also have the residue theorem.

**Theorem A.8** (Residue Theorem). Let $D$ be simply connected and let $\Gamma$ be a loop inside $D$ (positively oriented, reference [SS03]—for us it is not that important). If $f$ is analytic inside $\Gamma$ except possibly at $z_0, \ldots, z_k$, then

$$\int_{\Gamma} f \, dz = 2\pi i \sum_{i=0}^{k} \text{res}_{z_i}(f).$$
B Review of Topology

Here are some definitions from Topology the reader should know. All is adapted from [Mun13], which is the standard topology book.

Definition B.1. A **topology** on a set $X$ is a collection $\mathcal{T}$ of subsets of $X$ satisfying the following properties:

1. $\emptyset$ and $X$ are in $\mathcal{T}$
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$

Example B.2. Let $X = \{a, b, c\}$. Then the collection $\mathcal{T} = \{\{a\}, \{a, b\}, \{a, b, c\}\}$ is a topology. With a topology, we call $X$ a **topological space**.

Actually, the powerset $\mathcal{P}(X)$ is a topology called the **discrete topology**. We call a set $U \in \mathcal{T}$ an **open set**. For example, the open sets of $\mathbb{R}$’s standard topology are all sets of the form $(a, b)$ up to unioning.

The example of $\mathbb{R}$’s topology (one of its topologies I should say), is that smaller open sets can generate the whole topology.

Definition B.3. A **basis** for a topology on $X$ is a collection $\mathcal{B}$ of subsets of $X$ called basis elements such that

1. For each $x \in X$, there is at least one basis element $B$ containing $x$.
2. If $x$ belongs to the intersection of two basis elements $B_1, B_2$ then there is a basis element $B_3$ containing $x$ such that $B_3 \subset B_1 \cap B_2$.

Basis elements need not be open sets, but they can be. We can often find a basis that generates a topology we want, there is an exact condition, reference [Mun13, Lemma 13.2 p. 78].

Example B.4. A basis for the familiar topology on $\mathbb{R}$ is the collection $\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$.

There are also closed sets.

Definition B.5. A **closed set** in $X$ is the compliment $X - U$ of some open set $U$.

We have a countability axiom that Riemann surfaces need to satisfy.

Definition B.6. A topological space $X$ is **second countable** if it has a countable basis for its topology.

We also have a separation axiom that Riemann surfaces need to satisfy.

Definition B.7. A topological space $X$ is said to be **Hausdorff** if for every pair of points $p \neq q \in X$ there exists two open sets $U, V \in \mathcal{T}$ with $p \in U$ and $q \in V$ but $U \cap V = \emptyset$.

We also want Riemann surfaces to be connected.

Definition B.8. A **separation** of a topological space $X$ is a pair $U, V \in \mathcal{T}$ such that $X = U \cup V$ but $U \cap V = \emptyset$. A space $X$ is said to be **connected** is there is no separation of $X$.

Example B.9. The space $S^2 \cong C_{\infty}$ is connected. So is $\mathbb{R}^n$, and so are smooth curves. These are also hausdorff.

We can map topological spaces to each other also. Usually we want them to be continuous to preserve topological properties.

Definition B.10. A map $f : X \to Y$ is said to be **continuous** if for every open set $V$ in the topology on $Y$, denoted $\mathcal{T}_Y$, the preimage $f^{-1}(V)$ is open in $X$ (i.e. $f^{-1}(V) \in \mathcal{T}_X$). If the inverse $f^{-1}$ exists and is continuous, then we call $f$ a **homeomorphism**, meaning that the two topological spaces are isomorphic in the category of topological spaces.

Suppose, on the other hand, that we want to paste together two topological spaces to make another. (We do this when defining hyperelliptic curves [3.23].)
Definition B.11. Let $X, Y$ be two topological spaces; let $p : X \to Y$ be a surjective map. We say $p$ is a quotient map if a subset $U \in Y$ is open in $Y$ if and only if $p^{-1}(U)$ is open in $X$. Conversely, if $f$ is a continuous map, we may define a topology on $Y$ given by the criterion that $U$ is open in $Y$ if the preimage $f^{-1}(U)$ is open in $X$. This is called the quotient topology given by $f$.

Example B.12. Consider two topological spaces $X, Y$ with open sets $U, V$ and a homeomorphism $f : U \to V$. Then we may form a new set by declaring $Z = X \sqcup Y/f$ to be the set $X \sqcup Y$ with the equivalence relation given by $x \sim y$ if $x \in U$ and $y \in V$ and $f(x) = y$. This set $Z$ is given the quotient topology with this identification.

The quotients do not always inherit topological characteristics like second countable or Hausdorff, but in the one case we use it, they do.
References


