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Induction of Nontrivial Supercharacter Theories for Finite Groups

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Induction of Nontrivial Supercharacter Theories for Finite Groups

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Contents

1	Introduction	2
2	Preliminaries	2
3	Fusion's Effects on Induced Class Functions	6
4	Crash Course in Pattern Groups	9
4.1	Definitions and Tools	10
4.2	Superclasses in Pattern Groups	13
5	Known Nontrivial Supercharacter Theories in Pattern Groups	19
6	New Nontrivial Supercharacter Theories	21

1 Introduction

The representation theories of finite groups provide a means to study such groups via linear algebra and combinatorics. Certain features in groups, such as conjugacy classes, have natural extensions in representation theory, where each conjugacy class corresponds to an irreducible representation and character. A character in particular, is a means of attaching a statistic to a group, which can make explicit the relationships between abstract group elements, as well as provide, via irreducible characters, an orthonormal basis for functions from the group to the complex numbers. The extension of character theory to supercharacter theory can provide even deeper information about groups by providing a change of basis for functions out of the group (along with other characterizing invariances), and has natural extensions to other areas of math, such as Fourier transformations. To derive nontrivial supercharacter theories in abstract groups, studying subgroups can provide the information needed to develop such a theory, by examining the inclusion of the subgroup into the group, and the additional structure this imposes in the ambient group. By studying well known supercharacter theories in matrix groups, this document is able to build a theory of some inclusion maps that induce nontrivial supercharacter theories in finite groups.

In particular, this study focuses on the partitions of a group that arise from: action by conjugation, a two sided multiplicative generalization of conjugation, and inclusion of a subgroup into the group. Since conjugacy classes correspond to irreducible characters, studying the partitions in a group compatible with conjugacy classes in the subgroup, and by analogy, studying the partition of a group compatible with superclasses in a subgroup, invariances in the group can be derived from the subgroup's simpler structure. The fusion of conjugacy classes, and superclasses, has some effects on the calculation of an induced and superinduced function. However, these effects do not change the partition of a group which arises from inducing a class function to it. Understanding why gives a clue to a crucial invariance in the induction process, and this means that induction behaves more like superinduction, which always gives rise to supercharacter theories. So this work aims at furthering Marberg and Thiem's work in [4], by presenting some conditions for induction and superinduction to be identical. In those cases, induced class functions give rise to nontrivial supercharacter theories, which as the title suggests, is the goal.

2 Preliminaries

A supercharacter theory is an intricate and abstract notion, which relies on information about groups and representations of groups, but is ultimately just an extension of standard character theory. This section is aimed at introducing those features of groups, and representations which are used in the study of supercharacter theory at a basic level. The inclusion of subgroups into their ambient group via different maps is quite important (though it is hard to see why until much later), and so certain properties of these inclusion maps are discussed in particular. We begin by introducing a simple means of partitioning a group into equivalence classes.

Given a group G it is possible to build an equivalence relation on G by considering the action of G on itself given by conjugation.

Definition 2.1. Let G be a group and let $x \in G$. We define the conjugacy class of x in G as the set

$$K_G(x) = \{gxg^{-1} \mid g \in G\}.$$

Distinct conjugacy classes are disjoint from one another, and since every element of a group is contained in one of its conjugacy class or other, and conjugacy classes are either equal or disjoint, the union of all conjugacy classes is equal to the group.

In a group G which has a proper subgroup H , it is possible that a conjugacy class in G splits into several conjugacy classes in H .

Definition 2.2. [2, Definition 9.9] Two conjugacy classes of H are defined to be fused if they are subsets of a single conjugacy class of G . Conversely, there is no fusion in H if for each $x, y \in H$ such that x is G -conjugate to y , x is already H -conjugate to y .

Example 1. For a small example in which no fusion occurs, consider the alternating groups A_3 and A_4 . The conjugacy classes of A_3 are represented in the following table:

Representative	1	(1 2 3)	(1 3 2)
Size	1	1	1
Centralizer	6	3	3

The conjugacy classes of A_4 are represented in the following table:

Representative	1	(1 2)(3 4)	(1 2 3)	(1 3 2)
Size	1	3	4	4
Centralizer	24	8	3	3

$K_{A_4}(1)$: The trivial conjugacy class $K_{A_4}(1)$ obviously only contains the conjugacy class $K_{A_3}(1)$, since both only have one element, the identity 1.

$K_{A_4}((1 2)(3 4))$: The conjugacy class of double transpositions does not contain any conjugacy classes of A_3 , and so vacuously satisfies that any two elements which are conjugate in this class are already conjugate in A_3 .

$K_{A_4}((1 2 3)(4))$: Since $K_{A_4}((1 2 3)(4)) \ni (1 2 3)(4)$, it is clear that this conjugacy class contains $K_{A_3}((1 2 3)) = (1 2 3)$. Then the only element of A_3 that is A_4 -conjugate to $(1 2 3)(4)$ is $(1 2 3)$ which is obviously A_3 -conjugate to itself. So $K_{A_4}((1 2 3)(4))$ only contains one conjugacy class of A_3 , and no fusing occurs in this class.

$K_{A_4}((1 3 2)(4))$: This conjugacy class behaves exactly the same way with A_3 as the previous did. The result is that $K_{A_4}((1 3 2)(4)) \supset K_{A_3}((1 3 2))$ and no other conjugacy class in A_3 . Hence no fusing occurs in this class either.

Example 2. Similarly, we can consider the alternating group A_3 and the symmetric group S_3 . Since for each partition of 3, there is exactly one conjugacy class of S_3 , the 3-cycles in S_3 form a single conjugacy class. But since A_3 is abelian, every conjugacy class in A_3 contains only one element, so that each 3-cycle in A_3 has a distinct conjugacy class. These must fuse in S_3 .

When groups become more abstract, it is possible to study them indirectly using the tools of combinatorics, by using the representation theory of groups.

Definition 2.3. A representation of a finite group G is a homomorphism

$$\phi : G \rightarrow GL_n(\mathbb{C}).$$

To be able to best use combinatorics to study abstract groups via representations, the notion of a character provides a statistic about representations.

Definition 2.4. Define the character χ of a representation $\phi : G \rightarrow GL_n(\mathbb{C})$ by

$$\chi(g) = \text{tr}(\phi(g)) \quad \text{where } \text{tr} \text{ denotes the trace of the matrix } \phi(g), \text{ for each } g \in G.$$

Example 3. The regular character γ_G on G is given by

$$\gamma_G(g) = \begin{cases} |G|, & g = e \\ 0, & o/w, \end{cases}$$

where e is the neutral element of G . Note that in particular, this character corresponds to the representation of G which arises from left multiplicative action of G on itself.

Characters are special sorts of class functions, which are defined next.

Definition 2.5. Given a group G , a class function is a map $\psi : G \rightarrow \mathbb{K}$ for some field \mathbb{K} , where ψ is constant on the conjugacy classes of G .

Definition 2.6. Denote the space of class functions on a group G by $cf(G)$.

Definition 2.7. Let \mathcal{P} be a partition of G in which for every $P \in \mathcal{P}$, P contains every conjugacy class of G that it intersects. We define the subspace of $cf(G)$ containing the class functions that are constant on the parts of \mathcal{P} by $cf(G; \mathcal{P})$.

Example 4. Finite groups have finite numbers of conjugacy classes, and so it is possible to explicitly construct them. A typical class function $\psi \in cf(A_3)$ is a function $\psi : A_3 \rightarrow \mathbb{C}$ given by

$$\psi(x) = \begin{cases} z_1, & x = 1 \\ z_2, & x \in K_{A_3}((1\ 2\ 3)) \\ z_3, & x \in K_{A_3}((1\ 3\ 2)) \end{cases}$$

where each $z_i \in \mathbb{C}$. Notice that these z_i need not be distinct from each other, but likewise may be.

To extend the notion of a character, and more generally, a class function, we can introduce an inner product between functions that are constant on equivalence classes in a set.

Definition 2.8. [5, Definition 1.9.1] Let $\chi, \psi : G \rightarrow \mathbb{C}$ be class functions on some finite group G . Define their inner product by

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)},$$

where given a complex number z , \bar{z} denotes its complex conjugate.

Next, we extend the notion of conjugacy classes and characters to a slightly more general setting.

Definition 2.9. [4, Definition 2.1] Let G be a finite group. A **supercharacter theory** for G is a partition \mathcal{S}^\vee of the set of elements of G and a set of characters \mathcal{S} , such that

- (a) $|\mathcal{S}| = |\mathcal{S}^\vee|$,
- (b) Each $S \in \mathcal{S}^\vee$ is a union of conjugacy classes (so each part of the partition of G contains every conjugacy class that it intersects),

(c) For each irreducible character γ of G , there exists a unique $\chi \in S$ such that

$$\langle \gamma, \chi \rangle > 0,$$

where \langle, \rangle is the usual inner product on class functions,

(d) Every $\chi \in S$ is constant on the elements of S^\vee .

The set S^\vee is called the set of **superclasses**, and S the set of **supercharacters**.

Every group has the two trivial supercharacter theories, the usual character theory, and the supercharacter theory with $S^\vee = \{\{1\}, G - \{1\}\}$ and $S = \{1, \gamma_G - 1\}$ where 1 is the trivial character on G , and γ_G is the regular character, [4, Preliminaries].

Remark. Throughout the document, denote the superclass of G that contains a particular $g \in G$ by $\text{SC}_G(g)$.

Remark. Throughout the document, denote the centralizer in G of a particular $g \in G$ by $Z_G(g)$.

Given a superclass function (which may be an ordinary class function) on some subgroup H of a finite group G there are two ways to extend the function into a class function on G .

Definition 2.10. Given a finite group G , a subgroup H , and a class function $\psi : H \rightarrow \mathbb{C}$ we define the induced function $\text{Ind}_H^G \psi : G \rightarrow \mathbb{C}$ by

$$\text{Ind}_H^G \psi(g) := \frac{1}{|H|} \sum_{a \in G} \dot{\psi}(aga^{-1}), \quad \text{where } \dot{\psi}(x) := \begin{cases} \psi(x), & x \in H \\ 0, & x \notin H. \end{cases}$$

Lemma 2.11. Given a finite group G , a subgroup H , and a class function $\psi : H \rightarrow \mathbb{C}$ the values of the induced function $\text{Ind}_H^G \psi : G \rightarrow \mathbb{C}$, for $g \in G$, are given by

$$\text{Ind}_H^G \psi(g) = \frac{|G|}{|H| |K_G(g)|} \sum_{a \in K_G(g)} \dot{\psi}(a).$$

Proof. An induced class function is intuitively, a scaled average over a conjugacy class in G as it intersects the subgroup H . We make this explicit by the following computations:

$$\begin{aligned} \text{Ind}_H^G \psi(g) &= \frac{1}{|H|} \sum_{a \in G} \dot{\psi}(aga^{-1}) && \text{by definition} \\ &= \frac{1}{|H|} \sum_{h \in H} \sum_{\substack{a \in G \\ aga^{-1}=h}} \psi(h) && \text{by definition of } \dot{\psi} \text{ and reindexing} \\ &= \frac{1}{|H|} \sum_{K_H \subseteq H} \sum_{h \in K_H} \sum_{\substack{a \in G \\ aga^{-1}=h}} \psi(a) && \text{where } K_H \text{ is a conjugacy class in } H \\ &= \frac{1}{|H|} \sum_{K_H \subseteq H \cap K_G(g)} \sum_{h \in K_H} |Z_G(g)| \psi(h) && \text{since } \psi \text{ is a class function} \\ &= \frac{1}{|H|} \sum_{C \subseteq H \cap K_G(g)} |C| |Z_G(g)| \psi(h) && \text{where } C \text{ is a class in } H \\ &= \frac{1}{|H|} \frac{1}{|K_G(g)|} \sum_{C \subseteq H \cap K_G(g)} |G| \psi(h) && \text{by class stabilizer} \\ &= \frac{|G|}{|H| |K_G(g)|} \sum_{a \in K_G(g)} \dot{\psi}(a). \end{aligned}$$

□

Definition 2.12. Given a finite group G , a subgroup H , and a class function $\chi \in cf(H)$, we define a superinduced class function by

$$\text{SInd}_H^G \chi(g) := \frac{|G|}{|H|} \frac{1}{|\text{SC}_G(g)|} \sum_{a \in \text{SC}_G(g)} \dot{\chi}(a), \quad \text{where } \dot{\chi}(x) := \begin{cases} \chi(x), & x \in H \\ 0, & x \notin H. \end{cases}$$

Definition 2.13. Given a finite dimensional, nilpotent, associative, nonunital algebra J over a finite field, the set G of formal objects $1+x$ for $x \in J$, under the operation $(1+x)(1+y) = 1+x+y+xy$ is defined to be an algebra group over the finite field.

Definition 2.14. [1] For an algebra group G and subalgebra group, H Isaacs and Diaconis define, for any $\chi : H \rightarrow \mathbb{C}$, a superinduced function by

$$\text{SInd}_H^G \chi(g) = \frac{1}{|G||H|} \sum_{a,b \in G} \dot{\chi}(a(g-Id)b+Id)$$

where Id is the identity in the vector space over which the algebras are defined, and $\dot{\chi}$ is defined as before.

Lemma 2.15. *The definitions 2.12 and 2.14 for the superinduced function are equivalent.*

Proof. Beginning with Diaconis and Isaacs' definition 2.14, we have

$$\begin{aligned} \text{SInd}_H^G \chi(g) &= \frac{1}{|G||H|} \sum_{a,b \in G} \dot{\chi}(a(g-Id)b+Id) \\ &= \frac{1}{|G||H|} \sum_{h \in H} \sum_{\substack{a,b \in G \\ a(g-Id)b+Id=h}} \dot{\chi}(h) \\ &= \frac{1}{|G||H|} \sum_{C \subseteq H} \sum_{h \in C} \sum_{\substack{a,b \in G \\ a(g-Id)b+Id=h}} \chi(h) \\ &= \frac{1}{|G||H|} \sum_{C \subseteq H \cap \text{SC}_G(g)} \sum_{h \in C} |\text{Stab}_{G \times G}(g-Id)| \chi(h) \\ &= \frac{1}{|G||H|} \sum_{C \subseteq H \cap \text{SC}_G(g)} |C| |\text{Stab}_{G \times G}(g-Id)| \chi(h) \\ &= \frac{|G|^2}{|G||H|} \frac{1}{|\text{SC}_G(g)|} \sum_{C \subseteq H \cap \text{SC}_G(g)} |C| \chi(h), \end{aligned}$$

which is exactly the definition 2.12. □

3 Fusion's Effects on Induced Class Functions

In this section, we examine the ways in which fusion of conjugacy classes influences an induced class function. The important aspect of this study will be the natural partition of the group G that the induced class function creates, since fused conjugacy classes change the coarseness of that partition.

Before digging into how or why this study matters, first we address what is meant by a natural partition arising from induction. Because an induced class function is itself a class function, there is a baseline coarseness to a partition of G which naturally arises from induction: the partition of a group into its conjugacy classes.

This motivates the idea that a nontrivial supercharacter theory is possible to derive with an induced class function, but serves mostly to illustrate the ways an induced function works, and which information about groups are required to explicitly induce class functions. First we will see an example of an induced class function when no fusion occurs, then one with exactly two conjugacy classes fused, and finally, an example with arbitrary amounts of fusion.

Example 5. To be very explicit, we begin with a tiny example, and compute $\text{Ind}_{A_3}^{A_4}\psi$ for a typical class function $\psi \in cf(A_3)$. That is

$$\psi(h) = \begin{cases} \psi_1, & h \in K_{A_3}(1) = \{1\} \\ \psi_2, & h \in K_{A_3}((1\ 2\ 3)) = \{(1\ 2\ 3)\}, \\ \psi_3, & h \in K_{A_3}((1\ 3\ 2)) = \{(1\ 3\ 2)\} \end{cases}$$

where $\psi_i \in \mathbb{C}$ is a constant. The values of the induced function are

$$\text{Ind}_{A_3}^{A_4}\psi(h) = \frac{1}{|A_3|} \sum_{a \in A_4} \psi(aha^{-1}).$$

So we need to consider 24 conjugates by $a \in A_4$ of a particular $h \in A_3$, evaluating the conjugate to be 0 if $aha^{-1} \in A_4 - A_3$. There are $\frac{|A_4|}{|K_{A_4}(h)|}$ repeated conjugates of h to consider. So

$$\text{Ind}_{A_3}^{A_4}\psi(h) = \frac{1}{|A_3|} \sum_{a \in A_4} \psi(aha^{-1}) = \frac{|A_4|}{|K_{A_4}(h)||A_3|} \sum_{k \in K_{A_3}(h)} \psi_i(k).$$

But the class function is must be constant on $K_{A_3}(h)$, and hence $\sum_{k \in K_{A_3}(h)} \psi_i(k) = |K_{A_3}(h)|\psi_i(h)$.

Then

$$\text{Ind}_{A_3}^{A_4}\psi(h) = \frac{|Z_{A_4}(h)|}{|Z_{A_3}(h)|} \psi_i(h),$$

where $Z_G(g)$ denotes the centralizer in G of g , by the orbit-stabilizer equation. Notice that $\text{Ind}_{A_3}^{A_4}\psi$ takes on four distinct values, partitioning A_4 into

$$\{K_{A_4}(1), K_{A_4}((1\ 2\ 3)), K_{A_4}((1\ 3\ 2)), K_{A_4}((1\ 2\ 3\ 4)) \cup K_{A_4}((1\ 2)(3\ 4))\}.$$

Lemma 3.1. *Let G be a finite group, and let H be a proper subgroup of G in which no fusion occurs. Let $h_1, \dots, h_n \in H$ and suppose $g_1, \dots, g_k \in G \setminus H$ satisfy $K_G(g_j) \cap H = \emptyset$. Then the coarsest partition of G that naturally arises from induction has form*

$$\{K_G(h_1), \dots, K_G(h_n), \bigcup_{k=1}^m K_G(g_k)\}.$$

Proof. Let G be a finite group, and let H be a proper subgroup of G in which no fusion occurs. If the conjugacy classes of H are denoted $K_H(h_1), \dots, K_H(h_n)$, for some $h_i \in H$, and $k \in G$ satisfies $gkg^{-1} \in H$ for some $g \in G$, then

$$(\text{Ind}_H^G\psi)(k) = \frac{|Z_G(k)|}{|Z_H(k)|} \psi(k).$$

Indeed, the conjugacy class $K_G(k) \subset G$ contains at most one conjugacy class in H by assumption, and ψ is a constant function on this class, with $|K_H(k)|$ distinct inputs, each evaluated

$$\frac{|G|}{|K_G(k)|} = |Z_G(k)|$$

times. So we have

$$\text{Ind}_H^G \psi(k) = \frac{|G||K_H(k)|}{|K_G(k)||H|} \psi(k) = \frac{|Z_G(k)|}{|Z_H(k)|} \psi(k).$$

Then $\text{Ind}_H^G \psi$ has one value for each distinct conjugacy class of H , which forces each $K_G(h_j)$ to have a distinct part in the partition of G . Finally, given any $g \in G$ such that $K_G(g) \cap H = \emptyset$

$$\text{Ind}_H^G \psi(g) = 0,$$

so that all such g are mapped to the same part in the partition of G . \square

Example 6. Now we examine a slightly larger example, in which exactly two conjugacy classes fuse, again brute force computing $\text{Ind}_{A_4}^{S_4} \psi$ for a prototypical $\psi \in cf(A_4)$. The class function ψ has four distinct values, but there is a major complication in the induced class function. This occurs only when $h \in K_{A_4}((1\ 2\ 3))$ or $K_{A_4}((1\ 3\ 2))$, which I will call the i th and j th conjugacy classes respectively, for convenience. Let $h \in K_{A_4}((1\ 2\ 3))$. Then

$$\text{Ind}_{A_4}^{S_4} \psi(h) = \frac{1}{|A_4|} \sum_{\pi \in S_4} \psi(\pi h \pi^{-1}),$$

as usual, but we do not have the luxury of summing over a single conjugacy class in A_4 , since S_4 -conjugates of $(3,1)$ -cycles include both the conjugacy classes $K_{A_4}((1\ 2\ 3)(4))$, $K_{A_4}((1\ 3\ 2)(4)) \subset A_4$. In fact, as we sum over S_4 , we hit two distinct classes worth of values, each repeated $|Z_{S_4}(h)|/2 = |Z_{A_4}(h)| = |Z_{A_4}((1\ 2)h(1\ 2))|$ times, so

$$\begin{aligned} \text{Ind}_{A_4}^{S_4} \psi(h) &= \frac{1}{|A_4|} \left[\sum_{a \in K_{A_4}((1\ 2\ 3))} |Z_{S_4}(h)| \frac{\psi_i(h)}{2} + \sum_{b \in K_{A_4}((1\ 3\ 2))} |Z_{S_4}(h)| \frac{\psi_j((1\ 2)h(1\ 2))}{2} \right] = \\ &= \frac{1}{|A_4|} [|Z_{A_4}(h)||K_{A_4}((1\ 2\ 3))| \psi_i(h) + |Z_{A_4}(h)||K_{A_4}((1\ 3\ 2))| \psi_j((1\ 2)h(1\ 2))]. \end{aligned}$$

But $|K_{A_4}((1\ 2\ 3))| = |K_{A_4}((1\ 3\ 2))|$, so this factors as

$$\text{Ind}_{A_4}^{S_4} \psi(h) = \frac{1}{|A_4|} \left[|Z_{A_4}(h)||K_{A_4}(h)| (\psi_i + \psi_j) \right] = \psi_i + \psi_j,$$

by the orbit-stabilizer equation.

For $k \in A_4$ such that k is not a $(3,1)$ -cycle, the induced class function is exactly as it was in the case of no fusion. That is for any such k

$$\text{Ind}_{A_4}^{S_4} \psi(k) = m_1(k) \psi_k, \quad \text{where } m_1(k) \text{ is the number of fixed points in } k,$$

so there is a part of the partition \mathcal{Q} of S_4 for permutations with the same number of fixed points as k has. The remaining parts of \mathcal{Q} include a part for the permutations $\mu \in S_4 - A_4$, or permutations

with cycle types with no fixed points, and the mysterious part(s) for the fused classes. When $h \in K_{A_4}((1\ 2\ 3))$, the resulting value of the induced class function is the same for $h \in K_{A_4}((1\ 3\ 2))$ by virtue of these classes being an equal split of the $(3, 1)$ -cycle class of S_4 . Then we see that the $m_1 = 1$ part of \mathcal{Q} contains all $(3, 1)$ -cycles in S_4 . This is reminiscent of when we induced class functions without fused classes. The final partition \mathcal{Q} of S_4 that arises naturally from the induced class function has a part $\mathcal{Q}_i = \{\gamma \in S_4 : m_1(\gamma) = i\}$ for each $i \in \{0, 1, 2, 4\}$, and clearly

$$A_4 = [m_1 = 4] \cup [m_1 = 1] \cup [m_1 = 2],$$

and $A_4 \trianglelefteq S_4$.

We can generalize this to a result which governs an arbitrary amount of fusion.

Theorem 3.2. *Let H be a subgroup of a finite group G in which an arbitrary amount of fusion of conjugacy classes occurs. Let $h_1, \dots, h_n \in H$ and let $g_1, \dots, g_k \in G \setminus H$ satisfy $K_G(g_j) \cap H = \emptyset$. Then for any class function $\psi \in \text{cf}(H)$, the coarsest partition of G with which $\text{Ind}_H^G \psi$ is compatible has form*

$$\{K_G(h_1), \dots, K_G(h_n), \bigcup_{k=1}^m K_G(g_k)\}.$$

Proof. There are two cases. If H is the trivial subgroup or G itself, no conjugacy classes are fused. In both cases, the value $\text{Ind}_H^G \psi(g) = \psi(g)$, and in fact, either evaluation of the two values the induced class function makes this obvious, depending on how you wish to consider that there is no fusion. If H is a proper subgroup with any number of conjugacy classes which fuse, recall that by Clifford's theorem, these classes all have equal sizes. Let $h \in G$ be such that $ghg^{-1} \in H$ for some $g \in G$. Then the induced class function is

$$\text{Ind}_H^G \psi(h) = \frac{1}{|H|} \left[|Z_G(g_1 h g_1^{-1})| |K_H(g_1 h g_1^{-1})| \psi(g_1 h g_1^{-1}) + \dots + |Z_G(g_k h g_k^{-1})| |K_H(g_k h g_k^{-1})| \psi(g_k h g_k^{-1}) \right],$$

so that over the k fused classes, the induced function takes one value. Otherwise, for $k \in H$ such that k is not H -conjugate to h nor $g_i h g_i^{-1}$ for any of the g_i specified, the value of the induced class function behaves as if there were no fusion. \square

It turns out that fusion has no effect on induced class functions. Because the coarsest partition of G which naturally arises from a prototypical class function on the subgroup H is simply the partition of G into its conjugacy classes whether or not there is fusion, as a functor, induction is independent of fusion.

4 Crash Course in Pattern Groups

Marberg and Thiem [4] use Diaconis' and Isaacs' [1] results about supercharacter theories in algebra groups to study a particular class of algebra groups: pattern groups. In [4], some nontrivial supercharacter theories in these pattern groups arise from induction of superclass functions. The key to their work was equating superinduction to induction in specific cases. The aim of this section is to cover basic facts about pattern groups which are needed in the coming induction of superclass functions, and an extension of Marberg's and Thiem's results.

4.1 Definitions and Tools

Combinatorics can be used to both define and analyze pattern groups, beginning with notions as simple as ordering. Certain sets admit for natural, total orderings, such as the natural numbers, but every set is also able to be at least partially ordered.

Definition 4.1. A poset, or partially ordered set, is a set S together with a relation \sim where \sim is reflexive, so for each $s \in S$, $s \sim s$

transitive, so $x \sim y$, and $y \sim z \Rightarrow x \sim z$, and

antisymmetric, so $x \sim y$, and $y \sim x \Rightarrow x = y$.

A pattern group marries the notions of posets with algebra groups. In particular, this document is concerned with groups of unipotent matrices, and closely related, associative nilpotent algebras, both of which consist of upper triangular matrices with coefficients in a finite field.

Definition 4.2. Define the group of $n \times n$ upper triangular matrices with entry 1 on the main diagonal as and other coefficients in a finite field \mathbb{F}_q for some prime q by

$$UT_n(\mathbb{F}_q) := \left\{ \left(\begin{array}{ccccc} 1 & * & \dots & * & * \\ 0 & 1 & * & \dots & * \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \mid * \in \mathbb{F}_q \right\},$$

where the operation in the group is the usual matrix multiplication. This is the group of upper triangular unipotent matrices.

Definition 4.3. Call an $n \times n$ matrix A with the property that $A^m = 0$, for some $m \in \mathbb{N}$, a nilpotent.

Example 7. Given some typical upper triangular unipotent $u \in UT_n(\mathbb{F}_q)$, the matrix $(u - Id)$ (where Id is the $n \times n$ identity matrix) is nilpotent, because $(u - Id)^n = 0$.

There is a nilpotent \mathbb{F}_q -algebra (products of n elements in the algebra are always 0) of matrices with form $(u - Id)$ for our unipotents $u \in UT_n(\mathbb{F}_q)$, defined by the ordinary matrix operations restricted to the set $\{u - Id \mid u \in UT_n(\mathbb{F}_q)\}$. Because the dimension n , and the field \mathbb{F}_q are both finite, the algebra is a finite dimensional vector space, and hence has a finite basis.

Definition 4.4. Let $e_{ij} \in M_n(\mathbb{F}_q)$ be defined as the matrix with 1 at position (i, j) and 0 elsewhere. Define a set of such e_{ij} matrices by

$$\mathcal{B} := \{e_{ij} \mid 1 \leq i < j \leq n\}.$$

The correspondence between posets on the set $\{1, \dots, n\}$ and $UT_n(\mathbb{F}_q)$ is a means of defining pattern groups combinatorially.

Definition 4.5. [4, Section 2.2] Given a poset \mathcal{P} on $\{1, \dots, n\}$, define the pattern group

$$UT_{\mathcal{P}}(\mathbb{F}_q) := \{U \in UT_n(\mathbb{F}_q) \mid U_{ij} \neq 0 \Leftarrow i \leq j \text{ in } \mathcal{P}\}.$$

Proof. We need to show that $Id \in UT_{\mathcal{P}}(\mathbb{F}_q)$, $UT_{\mathcal{P}}(\mathbb{F}_q)$ is closed under matrix multiplication restricted to this set, and $UT_{\mathcal{P}}(\mathbb{F}_q)$ is closed under inversion. Fix some $n \in \mathbb{N}$ and a poset \mathcal{P} on $\{1, \dots, n\}$. Because for each $i \in \{1, \dots, n\}$, $i \leq i$, it must be that $Id \in UT_{\mathcal{P}}(\mathbb{F}_q)$. Given typical $a, b \in UT_{\mathcal{P}}(\mathbb{F}_q)$, because for any $i, j \in \{1, \dots, n\}$

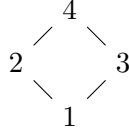
$$(ab)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

$x \not\leq y$ in \mathcal{P} means $ab_{xy} = \sum_{k=1}^n 0 = 0$, $ab \in UT_{\mathcal{P}}(\mathbb{F}_q)$. Finally, since $\det(a) \neq 0$ for any $a \in UT_{\mathcal{P}}(\mathbb{F}_q)$, any such a is invertible. Indeed, $a^{-1} \in UT_{\mathcal{P}}(\mathbb{F}_q)$ because given a pair $i \not\leq j$ in \mathcal{P} , we have $a_{ij} = 0$ and need to show $(a^{-1})_{ij} = 0$, but $(aa^{-1}) = Id$ so

$$(aa^{-1})_{ij} = 0 = \sum_{k=1}^n a_{ik}(a^{-1})_{kj}$$

where in particular, $0 = a_{ii}(a^{-1})_{ij} = (a^{-1})_{ij} = 0$, since $a_{ik} = 0$ for $i > k$ and $(a^{-1})_{kj} = 0$ for $k > j$. \square

Example 8. The poset \mathcal{P} on $\{1, 2, 3, 4\}$ with the following Hasse diagram



corresponds to the pattern group

$$UT_{\mathcal{P}} = \left\{ \left(\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid * \in \mathbb{F}_q \right) \right\}, \text{ since } 2 \not\leq 3 \text{ in } \mathcal{P}.$$

Now we have the tools to understand the decomposition of unipotent matrices into elementary ones, or equivalently, build nilpotent matrices from basis elements.

Lemma 4.6. *Given any $u \in UT_n(\mathbb{F}_q)$, there exist unique coefficients $\alpha_{i_k j_k} \in \mathbb{F}_q$ and an $m \in \{0, \dots, n-1\}$ such that*

$$u = \sum_{k=1}^m (\alpha_{i_k j_k} e_{i_k j_k} + Id).$$

Proof. Consider the poset which is given by the usual total ordering on $\{1, \dots, n\}$. Since every $i \in \{1, \dots, n\}$ satisfies $i \leq i \leq i+1 \leq i+2 \leq \dots \leq n$ and $i < j \Rightarrow j \not\leq i$ by definition of a poset, the total ordering on $\{1, \dots, n\}$ defines a set of upper triangular matrices as a pattern group. For typical e_{ij} , $e_{kl} \in \mathcal{B}$, the matrices correspond to the relations $i < j$, $k < l$ respectively, in the total order on $\{1, \dots, n\}$. If each relation $i \leq j$ is equipped with some $\alpha \in \mathbb{F}_q$ as a label, then linear combinations

$$\sum_{k=1}^m \alpha_{i_k j_k} e_{i_k j_k}$$

use the poset structure of the total order, equipped with labels, to span the set of nilpotent upper triangular matrices over the desired finite field. Then since simply adding the identity to each of the matrices spanned by the basis elements gives

$$\left(\sum_{k=1}^m \alpha_{i_k j_k} e_{i_k j_k}\right) + Id = u,$$

for each unipotent u . □

Next we want some pattern subgroups of $UT_n(\mathbb{F}_q)$ from which to induce superclass functions.

Definition 4.7. The pattern group $UT_n^{(k)}(\mathbb{F}_q)$ is the group of unipotent matrices corresponding to the poset on $\{1, \dots, n\}$ given by

$$1 \leq \dots \leq (k-1) \leq (k+1) \leq \dots \leq n$$

$$k$$

under the restricted operation of matrix multiplication to this set. Note that k is incomparable to any integer $i \in \{1, \dots, n\} - \{k\}$.

These matrices are just typical unipotent matrices where every nondiagonal entry in the k th row and k th column is 0. It turns out that these groups are less abstract than they appear, since the group of such matrices is in a canonical, structure preserving bijection with a group of smaller matrices.

Proposition 4.8. For any $k \in \{1, \dots, n\}$,

$$UT_{n-1}(\mathbb{F}_q) \cong UT_n^k(\mathbb{F}_q).$$

Proof. For slightly more concise notation, denote the n th row of an $n \times n$ matrix by M_{n*} and the n th column by M_{*n} . Consider the inclusion map

$$\phi_k : UT_{n-1} \rightarrow UT_n^{(k)}$$

given by

$$(\phi_k(M))_{ij} = \begin{cases} 1, & i = j \\ M_{ij}, & i, j < k \\ M_{i(j-1)}, & i < k \text{ and } j > k \\ M_{i-1j-1}, & i, j > k \\ 0, & i > j \text{ or } (i \text{ or } j = k, \text{ but not both}) \end{cases}.$$

Intuitively, the map adds a k th row and column with 1 on the diagonal and 0 elsewhere to a given $M \in UT_{n-1}$. Then ϕ_k is an isomorphism of groups, since

$\phi_k(M_1) = \phi_k(M_2) \Rightarrow (\phi_k(M_1))_{ij} = (\phi_k(M_2))_{ij}$ for all $(i, j) \Rightarrow (M_1)_{ij} = (M_2)_{ij}$ for all $(i, j) \Rightarrow M_1 = M_2$,
and

$$(\phi_k^{-1}(M))_{ij} := \begin{cases} 1, & i = j \\ M_{ij}, & i, j < k \\ M_{i(j+1)}, & i < k \text{ and } j > k \\ M_{i+1j+1}, & i, j > k \\ 0, & i > j \end{cases}$$

is injective, and an inverse mapping, and since

$$\phi_k(M_1 M_2)_{ij} = \langle \phi_k(M_{1_{i*}}), \phi_k(M_{2_{*j}}) \rangle = (\phi_k(M_1) \phi_k(M_2))_{ij}$$

for each index (i, j) , where \langle, \rangle denotes the usual inner product on vectors, by definition of matrix multiplication, the map is bijective and a homomorphism as desired. \square

Though pattern groups guarantee the existence of inverses for every matrix being considered, actually computing those inverses, and hence having the tools to build conjugacy classes, is not possible in general.

Example 9.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a_{12} & a_{12}a_{23} - a_{13} & a_{12}a_{24} - a_{14} + a_{13}a_{34} - a_{12}a_{23}a_{34} \\ 0 & 1 & -a_{23} & a_{23}a_{34} - a_{24} \\ 0 & 0 & 1 & -a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is not hard to imagine that as n grows, inverting these matrices requires much more computation, but to make that concrete, this 4×4 example is the largest inverse that can fit in a single line of this document, whereas a 5×5 inverse takes nearly half a page to typeset.

4.2 Superclasses in Pattern Groups

Because conjugation is so difficult, we define a two sided action of unipotent matrices on their corresponding nilpotents

$$UT_n(\mathbb{F}_q) \times (UT_n(\mathbb{F}_q) - Id) \times UT_n(\mathbb{F}_q) \rightarrow (UT_n(\mathbb{F}_q) - Id)$$

by the two sided products

$$A(u - Id)B$$

for $u, A, B \in UT_n(\mathbb{F}_q)$. That the range of the action is the nilpotent algebra $(UT_n(\mathbb{F}_q) - Id)$ is a product of the definition of matrix multiplication. Notice as well that all conjugates $A(u - Id)A^{-1}$ of the nilpotent $(u - Id)$ lie within its orbit under this two sided action. Hence the orbits of this action are a coarser partition than conjugacy classes in $UT_n(\mathbb{F}_q)$. It turns out these orbits actually form a well defined set of superclasses for this pattern group.

Example 10. The set

$$S := \left\{ \left(\begin{pmatrix} 1 & 0 & a_{12} & a_{12}b_{34} & a_{12}b_{35} \\ 0 & 1 & 1 & b_{34} & b_{35} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid a_{ij}, b_{xy} \in \mathbb{F}_q \right\} \subset UT_5(\mathbb{F}_q)$$

is the superclass of e_{23} in $UT_5(\mathbb{F}_q)$, because all matrices of form $A(e_{23})B$ for $A, B \in UT_5(\mathbb{F}_q)$ lie within S .

There is a combinatorial way to define the representatives of these superclasses.

Definition 4.9. [4] The set

$$\mathcal{R} := \{u - Id \mid u \in UT_n(\mathbb{F}_q) \text{ has at most one nonzero entry per row and column}\} \subset UT_n(\mathbb{F}_q)$$

is a complete set of representatives for superclasses in $UT_n(\mathbb{F}_q)$.

Before we can understand the combinatorics of these representatives, another definition is required.

Definition 4.10. [4] A labeled set partition is a set partition λ , where for each adjacent pair $i \frown j$, there is some nonzero label $\alpha \in \mathbb{F}_q^*$.

Example 11. The labeled partition of $\{1, 2, 3, 4\}$ into

$$\{1 \mid \overset{\alpha}{\frown} 2, 3, \overset{\beta}{\frown} 4 \}$$

corresponds to the representative

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

of a superclass in $UT_4(\mathbb{F}_q)$.

Remark. The important distinction between the use of labeled set partitions and posets is the transitivity of poset relations, which, in the previous example, would have placed a γ at $(2, 4)$, breaking the condition that every row and column of a representative has at most one nonzero entry.

Notice that since Bell numbers enumerate the number of set partitions of $\{1, \dots, n\}$, the n th Bell number is the number of superclasses of $UT_n(\mathbb{F}_2)$. For general \mathbb{F}_q , the n th Bell number is the number of combinations of pivot basis elements e_{ij} such that

$$\sum_{k=1}^m \alpha_{i_k j_k} e_{i_k j_k},$$

(where the $\alpha_{i_k j_k} \in \mathbb{F}_q$ and $m \in \{0, \dots, n-1\}$) is a representative for a superclass of $UT_n(\mathbb{F}_q)$.

As a final, and crucial note, different embeddings of $UT_{n-1}(\mathbb{F}_q) \leq UT_n(\mathbb{F}_q)$ via a map ϕ_k give rise to distinct supercharacter theories, since there will be different indices available for the representatives of superclasses.

Example 12. Although

$$UT_3(\mathbb{Z}_2) \cong UT_4^{(1)}(\mathbb{Z}_2) \cong UT_4^{(2)}(\mathbb{Z}_2) \cong UT_4^{(3)}(\mathbb{Z}_2)$$

the superclasses of the latter three groups have representatives

$$\{Id, e_{23}, e_{24}, e_{34}, e_{23} + e_{34}\},$$

$$\{Id, e_{13}, e_{14}, e_{34}, e_{13} + e_{34}\}$$

and

$$\{Id, e_{12}, e_{14}, e_{23}, e_{12} + e_{23}\}$$

respectively, and hence different superclasses.

It is possible to be very explicit about the structure of a superclass by examining the two sided orbit of a representative matrix. In general, if we express a given representative as a linear combination of distinct basis elements, such that

$$(u - Id) = \sum_{k=1}^m \alpha_{i_k j_k} e_{i_k j_k}, \quad \text{where } \alpha_{i_k j_k} \in \mathbb{F}_q, \text{ and } m \in \{0, \dots, n-1\}$$

then we have

$$(A(u - Id)B)_{xy} = \sum_{k=1}^m \alpha_{i_k j_k} A_{x i_k} B_{j_k y}. \quad (4.1)$$

Necessary and sufficient criteria for stabilizers in the two sided action follow from this result immediately and the two sided stabilizer of a representative is most easily presented in terms of those necessary and sufficient conditions.

Proposition 4.11. *Suppose a given representative $(u - Id) \in R$ may be written as a linear combination of m distinct basis elements for $m \in \{0, \dots, n-1\}$ and such that*

$$(u - Id) = \sum_{k=1}^m \alpha_{i_k j_k} e_{i_k j_k}.$$

Then

$$\text{Stab}_{UT \times UT}((u - Id)) = \{(A, B) \in UT \times UT \mid \sum_{k=1}^m \alpha_{i_k j_k} A_{x i_k} B_{j_k y} = 0 \text{ if } x < \min\{i_k\}, y \geq \max\{j_k\}\}.$$

Proof. Without loss of generality, we can assume that given

$$(u - Id) = \sum_{k=1}^m \alpha_{i_k j_k} e_{i_k j_k},$$

the i_k satisfy

$$i_1 < i_2 < \dots < i_m$$

since matrix addition commutes. For typical $A, B \in UT$, and $(u - Id) \in \mathcal{R}$,

$$(A(u - Id)B)_{xy} = \sum_{k=1}^m \alpha_{i_k j_k} A_{x i_k} B_{j_k y}.$$

Then it is clear that fixing the partial columns above each (i_k, j_k) th entry in A to be 0, and the partial column entries to the right of each (i_k, j_k) th entry in B to be 0, or fixing them to sum to 0 makes these $A, B \in UT$ into stabilizing elements. \square

Sometimes the superclass of a given representative is actually equal to its conjugacy class. Though not a complete set of necessary and sufficient conditions, Diaconis and Isaacs offer at least one sufficient condition for a conjugacy class in an algebra group to be equal to its superclass. First, a definition is required.

Definition 4.12. [1, Section 3] Given a nilpotent $x \in UT - Id$, the left and right annihilators of x are

$$L_x = \{y \in UT - Id \mid yx = 0\}$$

and

$$R_x = \{y \in UT - Id \mid xy = 0\}$$

respectively.

Example 13. The left and right annihilators of $e_{12} + e_{45}$ in $UT_5(\mathbb{F}_q)$ are the sets

$$L = \left\{ \left(\begin{pmatrix} 0 & * & * & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid * \in \mathbb{F}_q \right) \right\} \subset UT_5(\mathbb{F}_q), \quad R = \left\{ \left(\begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid * \in \mathbb{F}_q \right) \right\} \subset UT_5(\mathbb{F}_q)$$

respectively.

Theorem 4.13. [1, Theorem 3.1] Let $x \in UT - Id$ be a nilpotent. If $L_x + R_x = UT - Id$, then the superclass of $x + Id$ is equal to the conjugacy class of $x + Id$ in UT .

Remark. This statement is strictly not biconditional.

Example 14. Consider the conjugacy class

$$K_{UT_5(\mathbb{F}_q)}(e_{12} + e_{24} + e_{45} + Id)$$

whose elements are matrices of form

$$a(e_{12} + e_{24} + e_{45} + Id)a^{-1} = \begin{pmatrix} 1 & 1 & -a_{23} & a_{12} - a_{24} + a_{23}a_{34} & a_{14} - a_{25} - a_{12}a_{45} + a_{23}a_{35} + a_{24}a_{45} - a_{23}a_{34}a_{45} \\ 0 & 1 & 0 & 1 & a_{24} - a_{45} \\ 0 & 0 & 1 & 0 & a_{34} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

whereas given typical $a, b \in UT_5(\mathbb{F}_q)$ we have

$$a(e_{12} + e_{24} + e_{45})b = \begin{pmatrix} 0 & 1 & b_{23} & a_{12} + b_{24} & a_{14} + b_{25} + a_{12}b_{45} \\ 0 & 0 & 0 & 1 & a_{24}b_{45} \\ 0 & 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we build a matrix $a' \in UT_5(\mathbb{F}_q)$ such that

$$a'_{12} = a_{12}, \quad a'_{14} = a_{14}, \quad a'_{34} = a_{34},$$

$$a'^{-1}_{13} = b_{23}, \quad a'^{-1}_{24} = b_{24}, \quad a'^{-1}_{25} = b_{25}, \quad a'^{-1}_{45} = b_{45},$$

then since

$$(a'(e_{12} + e_{24} + e_{45})a'^{-1} + Id)_{xy} = \sum_{k=1}^3 a'_{xik} a'^{-1}_{jky}$$

it follows that

$$(a'(e_{12} + e_{24} + e_{45} + Id)a'^{-1}) = a(e_{12} + e_{24} + e_{45})b + Id$$

so that $(e_{12} + e_{24} + e_{45})$'s conjugacy class is in fact equal to its superclass in $UT_5(\mathbb{F}_q)$. However, the left and right stabilizers of $e_{12} + e_{24} + e_{45}$ do not span the nilpotent algebra $UT_5(\mathbb{F}_q) - Id$ because they have forms

$$L = \left\{ \left(\begin{array}{ccccc} 0 & 0 & * & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \mid * \in \mathbb{F}_q \right\} \subset UT_5(\mathbb{F}_q), \quad R = \left\{ \left(\begin{array}{ccccc} 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \mid * \in \mathbb{F}_q \right\} \subset UT_5(\mathbb{F}_q)$$

respectively, so that we see

$$L + R = \left\{ \left(\begin{array}{ccccc} 0 & * & * & * & * \\ 0 & 0 & * & \mathbf{0} & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \mid * \in \mathbb{F}_q \right\} \subset UT_5(\mathbb{F}_q) - Id.$$

Finally, since fusion describes conjugacy class behavior, but the two sided action on nilpotents subverts the need to conjugate in some cases, we need a generalized notion of fusion of superclasses for those cases when subversion fails.

Definition 4.14. Let $\mathcal{R}^{(k)}$ be the basis for the representatives of superclasses in $UT_n^{(k)}(\mathbb{F}_q)$. We define the superclasses of representatives $(u - Id)$, and $(v - Id) \in \mathcal{R}^{(k)}$, to be fused if there is some pair $A, B \in UT - UT^{(k)}$ such that

$$A(u - Id)B = (v - Id).$$

Theorem 4.15. *There is no fusion of superclasses for any pattern subgroup $UT^{(k)} \leq UT$.*

Proof. We begin with nilpotents $(u - Id) = e_{ij}$ for some $e_{ij} \in \mathcal{B}^{(k)}$, where $\mathcal{B}^{(k)}$ denotes the set of matrices e_{ij} as we are accustomed to, restricted to $UT^{(k)}$, so $i \neq k$ and $j \neq k$. There are three cases:

1. $i < k, j < k$: For $A, B \in UT$, and $u \in UT^{(k)}$ by equation 4.1 we have

$$A(u - Id)B = \begin{pmatrix} 0 & \dots & 0 & a_{1,k} & a_{1,k}b_{m,m+1} & \dots & a_{1,k}b_{m,n} \\ 0 & \ddots & 0 & a_{2,k} & a_{2,k}b_{m,m+1} & \dots & a_{2,k}b_{m,n} \\ \vdots & 0 & 0 & 1 & b_{m,m+1} & \dots & b_{m,n} \\ \vdots & \vdots & \vdots & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then suppose $A(u - Id)B \in UT^{(k)}$. All highlighted entries in $A(u - Id)B$ above need to be 0. Define

$$a := A|_{UT^{(k)}} \text{ and } b := B|_{UT^{(k)}}, \text{ or}$$

$$a = \begin{pmatrix} 1 & a_{1,2} & a_{1,3} & \dots & 0 & \dots & a_{1,n} \\ 0 & 1 & a_{2,3} & \dots & 0 & \dots & a_{2,n} \\ \vdots & 0 & 1 & \dots & 0 & \dots & \vdots \\ \vdots & \vdots & 0 & 1 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & b_{1,2} & b_{1,3} & \dots & 0 & \dots & b_{1,n} \\ 0 & 1 & b_{2,3} & \dots & 0 & \dots & b_{2,n} \\ \vdots & 0 & 1 & \dots & 0 & \dots & \vdots \\ \vdots & \vdots & 0 & 1 & 0 & \dots & \vdots \\ \vdots & \vdots & \vdots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Then we have

$$a(u - Id)b = \begin{pmatrix} 0 & \dots & 0 & a_{1,k} & 0 & \dots & a_{1,k}b_{m,n} \\ 0 & \ddots & 0 & a_{2,k} & 0 & \dots & a_{2,k}b_{m,n} \\ \vdots & 0 & 0 & 1 & 0 & \dots & b_{m,n} \\ \vdots & \vdots & \vdots & \ddots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = A(u - Id)B.$$

2. $i < k, j > k$: For $A, B \in UT$, and $u \in UT^{(k)}$ by equation 4.1 we have

$$A(u - Id)B = \begin{pmatrix} 0 & 0 & 0 & 0 & a_{1,k} & \dots & a_{1,k}b_{m,n} \\ \vdots & \ddots & \vdots & 0 & \vdots & \dots & \vdots \\ \vdots & \vdots & \ddots & 0 & 1 & \dots & b_{m,n} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and with the analogous restrictions

$$a := A|_{UT^{(k)}} \text{ and } b := B|_{UT^{(k)}},$$

so that we have

$$a(u - Id)b = A(u - Id)B.$$

3. $i > k, j > k$: For $A, B \in UT$, and $u \in UT^{(k)}$ by equation 4.1 we have

$$A(u - Id)B = \begin{pmatrix} 0 & 0 & 0 & 0 & a_{1,m} & \dots & a_{1,m}b_{m,n} \\ 0 & 0 & 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & a_{k-1,m-1} & \vdots & a_{k-1,m-1}b_{m,n} \\ 0 & 0 & 0 & \ddots & 1 & \dots & b_{m,n} \\ 0 & 0 & 0 & 0 & \ddots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and with analogous restrictions as in the previous two parts, we conclude

$$a(u - Id)b = A(u - Id)B.$$

Given $(u - Id) = \sum_{l=1}^m \alpha_l e_{i_l j_l}$ for m distinct elements, a typical element of the orbit of $(u - Id)$ has form

$$(A(u - Id)B)_{xy} = \sum_{k=1}^m \alpha_{i_k j_k} A_{x i_k} B_{j_k y},$$

for $A, B \in UT$. But since $(A(u - Id)B)_{k*} = 0 = (A(u - Id)B)_{*k}$, (since $(u - Id) \in UT^{(k)}$), if we restrict the columns $A_{*k} = 0 = B_{*k}$ and the rows $A_{k*} = 0 = B_{k*}$ to define

$$a := A|_{UT^{(k)}}, \text{ and } b := B|_{UT^{(k)}}$$

we can conclude

$$(a(u - Id)b) = (A(u - Id)B),$$

which completes the proof. \square

5 Known Nontrivial Supercharacter Theories in Pattern Groups

Marberg and Thiem present some sufficient conditions ([4]) for induced superclass functions to be equal to superinduced superclass functions in pattern groups. In theorems 3.1, 3.2 and 3.3 in [4], there are sufficient conditions for superinduced superclass functions to be induced superclass functions. The results are copied for reference.

Theorem 5.1. [4, Theorem 3.1] *Let H be a sub-pattern group (subgroup which is a pattern group in its own right) of an pattern group G and suppose*

- (1) *No two superclasses of H are in the same superclass of G ,*
- (2) *$x(h - 1) + 1 \in H$ for all $x \in G, h \in H$.*

Then for any superclass function χ of H we have

$$\text{SInd}_H^G(\chi) = \text{Ind}_H^G(\chi).$$

Theorem 5.2. [4, Theorem 3.2] *Suppose $G = H \times K$ where G, H and K are pattern groups. If $(k - 1)(h - 1) = 0$ for all $h \in H$ and $k \in K$, then*

$$\text{SInd}_H^G(\chi) = \text{Ind}_H^G(\chi) \quad \text{for all superclass functions } \chi \text{ of } H.$$

Theorem 5.3. [4, Theorem 3.3] *Let $U_{\mathcal{P}} \subseteq U_{\mathcal{R}}$ be pattern groups, and let*

$$I = \{u \in U_{\mathcal{R}} \mid u_{ij} \neq 0 \text{ implies } i < j \text{ in } \mathcal{R}/\mathcal{P}\}.$$

If $(l - 1)(u - 1) = 0$ for all $l \in I, u \in U_{\mathcal{R}}$, then

$$\text{SInd}_{U_{\mathcal{P}}}^{U_{\mathcal{R}}}(\chi) = \text{Ind}_{U_{\mathcal{P}}}^{U_{\mathcal{R}}}(\chi) \quad \text{for all superclass functions } \chi \text{ of } U_{\mathcal{P}}.$$

Note that $i \leq j$ in \mathcal{R}/\mathcal{P} if $i \leq j$ in \mathcal{R} and $i \not\leq j$ in \mathcal{P} . In [4], Marberg and Thiem show that

$$\text{SInd}_{UT_n^{(1)}}^{UT_n} \chi = \text{Ind}_{UT_n^{(1)}}^{UT_n} \chi$$

and

$$\text{SInd}_{UT_n^{(n)}}^{UT_n} \chi = \text{Ind}_{UT_n^{(n)}}^{UT_n} \chi$$

for any superclass function χ of the pattern subgroups $UT^{(1)}$ and $UT^{(n)}$ by theorem 5.1. This method extends to the following result.

Proposition 5.4. *For any superclass functions χ, ψ of the pattern subgroups $UT^{(2)}$ and $UT^{(n-1)}$ respectively, we have*

$$\text{SInd}_{UT_n^{(2)}}^{UT_n} \chi = \text{Ind}_{UT_n^{(2)}}^{UT_n} \chi$$

and

$$\text{SInd}_{UT_n^{(n-1)}}^{UT_n} \psi = \text{Ind}_{UT_n^{(n-1)}}^{UT_n} \psi.$$

Remark. This result has been proven before, but the following proof is original work.

Proof. There is no fusion of the superclasses of the pattern subgroups, as was proven in theorem 4.15. Then it remains to prove $(h-1)x+1 \in UT_n^{(2)}$ for all $x \in UT$, $h \in UT^{(2)}$ and $y(k-1)+1 \in UT^{(n-1)}$ for all $y \in UT$, $k \in UT^{(n-1)}$ to apply 5.1. For typical x, h, y, k

$$(h-1)x+1 = \begin{pmatrix} 0 & 0 & h_{13} & h_{14} & h_{15} & \dots & h_{1n} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & h_{34} & h_{35} & \dots & h_{3n} \\ 0 & 0 & 0 & 0 & h_{45} & \dots & h_{4n} \\ 0 & 0 & 0 & 0 & 0 & \dots & h_{5n} \\ 0 & 0 & 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} & x_{15} & \dots & x_{1n} \\ 0 & 1 & x_{23} & x_{24} & x_{25} & \dots & x_{2n} \\ 0 & 0 & 1 & x_{34} & x_{35} & \dots & x_{3n} \\ 0 & 0 & 0 & 1 & x_{45} & \dots & x_{4n} \\ 0 & 0 & 0 & 0 & 1 & \dots & x_{5n} \\ 0 & 0 & 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} + Id$$

$$\Rightarrow ((h-1)x+1)_{ij} = (h_{i1}, 0, h_{i3}, \dots, h_{in}) \begin{pmatrix} x_{1j} \\ x_{2j} \\ x_{3j} \\ \vdots \\ x_{nj} \end{pmatrix} = \begin{cases} 1, & i = j \\ \sum_{k=3}^j h_{ik} x_{kj}, & i < j \\ 0, & i = 2, j = 2, \text{ or } i > j \end{cases}$$

so $(h-1)x+1 \in UT^{(2)}$ as desired. Similarly,

$$y(k-1)+1 = \begin{pmatrix} 1 & \dots & y_{13} & y_{14} & y_{15} & \dots & y_{1n} \\ 0 & \ddots & y_{23} & y_{24} & y_{25} & \dots & y_{2n} \\ 0 & 0 & 1 & y_{34} & y_{35} & \dots & y_{3n} \\ 0 & \vdots & 0 & 1 & y_{45} & \dots & y_{4n} \\ 0 & \vdots & 0 & 0 & 1 & \dots & \vdots \\ 0 & \vdots & 0 & 0 & 0 & 1 & y_{(n-1)n} \\ 0 & \dots & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \dots & k_{13} & k_{14} & \dots & 0 & k_{1n} \\ 0 & \ddots & k_{23} & k_{24} & \dots & 0 & k_{2n} \\ 0 & 0 & 0 & k_{34} & \dots & 0 & k_{3n} \\ 0 & \vdots & 0 & 0 & \dots & 0 & k_{4n} \\ 0 & \vdots & 0 & 0 & 0 & 0 & k_{(n-2)n} \\ 0 & \vdots & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + Id$$

$$\begin{aligned} \Rightarrow (y(k-1) + 1)_{ij} &= \begin{cases} 1, & i = j \\ \sum_{l=1}^{j-1} y_{il} k_{lj}, & i < j \\ 0, & i = n-1, j = n-1 \text{ or } i > j \end{cases} \\ &\Rightarrow y(k-1) + 1 \in UT^{(n-1)}. \end{aligned}$$

By 5.1, the induced and superinduced superclass functions are equal. □

6 New Nontrivial Supercharacter Theories

So far, up to equivalence, two different embeddings

$$UT_n^{(k)}(\mathbb{F}_q) \leq UT_n(\mathbb{F}_q)$$

have been shown to allow

$$\text{Ind}_{UT_n^{(k)}}^{UT_n} \psi = \text{SInd}_{UT_n^{(k)}}^{UT_n} \psi \Rightarrow \text{Ind}_{UT_n^{(k)}}^{UT_n} \psi \in \text{scf}(UT_5)$$

for any superclass function $\psi \in \text{scf}(UT_n^{(k)}(\mathbb{F}_q))$. The procedure has thus far been to examine the inclusion maps which append a row and column to the top or bottom, left or right, of each given $M \in UT_n^{(k)}(\mathbb{F}_q)$ respectively, and then to change where this row and column are appended, by moving them towards the center of M . For some embeddings the induced superclass function is quite well behaved, since the superclasses of $UT_n^{(k)}(\mathbb{F}_q)$ are smaller in these inclusions. The next clear extension, and where the novel results in this area lie, is in moving the k appending (or inclusion if you like) index further towards the center of given M , and the smallest possible example of this is in the inclusion

$$UT_5^{(3)}(\mathbb{F}_q) \leq UT_5(\mathbb{F}_q).$$

Let us start this analysis by working out the superclasses of $UT_5^{(3)}(\mathbb{F}_q)$ in the table below.

Table 1: Superclasses in $UT_5^{(3)}(\mathbb{F}_q)$

Form	Representatives	Superclass $\stackrel{?}{=} \text{Conjugacy Class}$
0	Id	✓ by definition
e_{12}	αe_{12}	✓ by theorem 4.13
e_{14}	αe_{14}	✓ by theorem 4.13
e_{15}	αe_{15}	✓ by theorem 4.13
e_{24}	αe_{24}	✓ by theorem 4.13
e_{25}	αe_{25}	✓ by theorem 4.13
e_{45}	αe_{45}	✓ by theorem 4.13
$e_{12} + e_{24}$	$\alpha e_{12} + \beta e_{24},$	✓ by theorem 4.13
$e_{12} + e_{25}$	$\alpha e_{12} + \beta e_{25},$	✓ by theorem 4.13
$e_{24} + e_{15}$	$\alpha e_{15} + \beta e_{24}$	✓ by theorem 4.13
$e_{14} + e_{25}$	$\alpha e_{14} + \beta e_{25}$	✓ by theorem 4.13
$e_{24} + e_{45}$	$\alpha e_{24} + \beta e_{45}$	✓ by theorem 4.13
$e_{14} + e_{45}$	$\alpha e_{14} + \beta e_{45}$	✓ by theorem 4.13
$e_{12} + e_{24} + e_{45}$	$\alpha e_{12} + \beta e_{24} + \gamma e_{45}$	✓ by counterexample 14
$e_{12} + e_{45}$	$\alpha e_{12} + \beta e_{45}$	NO

To justify that the superclass of $e_{12} + e_{45}$ properly contains its conjugacy classes, consider the following:

Recall that $UT_5^{(3)}(\mathbb{F}_q) \cong UT_4(\mathbb{F}_q)$, so work in the latter as much as possible to simplify inverses somewhat.

Recall that the inverse of a typical $a \in UT_4(\mathbb{F}_q)$ has form

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a_{12} & a_{12}a_{23} - a_{13} & a_{12}a_{24} - a_{14} + a_{13}a_{34} - a_{12}a_{23}a_{34} \\ 0 & 1 & -a_{23} & a_{23}a_{34} - a_{24} \\ 0 & 0 & 1 & -a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The left and right annihilators of $e_{12} + e_{45}$ respectively, are

$$\left\{ \left(\begin{pmatrix} 0 & * & * & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid * \in \mathbb{F}_q \right) \right\} \text{ and } \left\{ \left(\begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mid * \in \mathbb{F}_q \right) \right\},$$

so the hypotheses of 4.13 are not satisfied.

Given $a, b \in UT_5^{(3)}(\mathbb{F}_q)$,

$$a(e_{12} + e_{45})b = \begin{pmatrix} 0 & 1 & 0 & b_{24} & a_{14} + b_{25} \\ 0 & 0 & 0 & 0 & a_{24} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

whereas

$$a(e_{12} + e_{45})a^{-1} = \begin{pmatrix} 0 & 1 & 0 & -a_{24} & a_{14} - a_{25} + a_{24}a_{45} \\ 0 & 0 & 0 & & a_{24} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Following the procedure of counterexample 14 there is no way to build an $a' \in UT_5(\mathbb{F}_q)$ such that

$$a'(e_{12} + e_{45} + Id)a'^{-1} = a(e_{12} + e_{45})b + Id$$

so this representative's superclass must properly contain its conjugacy classes. In particular, there will be q conjugacy classes of $e_{12} + e_{45}$ properly contained in the superclass.

Theorem 6.1. *For any superclass function $\psi \in scf(UT_5^{(3)}(\mathbb{F}_q))$ inducing and superinducing ψ to $UT_5(\mathbb{F}_q)$ we have*

$$\text{Ind}_{UT_5^{(3)}}^{UT_5} \psi = \text{SInd}_{UT_5^{(3)}}^{UT_5} \psi.$$

Proof. Since for most $u \in UT_5$ the superinduced function is equal to the induced function (since most conjugacy classes are equal to superclasses), we focus on the cases when this is not true, in particular, when

$$u \in K_{UT_5^{(3)}}((\alpha_{12}e_{12} + \alpha_{45}e_{45} + Id)),$$

for $\alpha_{12}, \alpha_{45} \in \mathbb{F}_q - \{0\}$. There are $q = |\mathbb{F}_q|$ conjugacy classes contained in $\text{SC}_{UT_5^{(3)}}(\alpha_{12}e_{12} + \alpha_{45}e_{45})$ since the sum

$$\alpha_{12}a_{14} - \alpha_{45}(a_{25} + a_{24}a_{45}) = a(\alpha_{12}e_{12} + \alpha_{45}e_{45})a^{-1},$$

for $a \in UT_5^{(3)}(\mathbb{F}_q)$, may take only $q = |\mathbb{F}_q|$ values. Recall that there is no fusion of superclasses in $UT_5^{(3)}(\mathbb{F}_q)$ into superclasses in $UT_5(\mathbb{F}_q)$, so the value of the superinduced superclass function on $\text{SC}_{UT_5^{(3)}}(\alpha_{12}e_{12} + \alpha_{45}e_{45})$ is constant. On any such u , since ψ is constant on $\text{SC}_{UT_5^{(3)}}(\alpha_{12}e_{12} + \alpha_{45}e_{45})$, the value of the superinduced superclass function is given by

$$\begin{aligned} \text{SInd}_{UT_5^{(3)}}^{UT_5} \psi(u) &:= \frac{|UT_5|}{|UT_5^{(3)}|} \frac{1}{|\text{SC}_{UT_5}(e_{12}+e_{45})|} \sum_{a \in \text{SC}_{UT_5}(g)} \psi(a) \\ &= \frac{|UT_5|}{|UT_5^{(3)}|} \frac{1}{|\text{SC}_{UT_5}(e_{12}+e_{45})|} \sum_{K_{UT_5^{(3)}} \subset \text{SC}_{UT_5}(e_{12}+e_{45})} \sum_{a \in K_{UT_5^{(3)}}} \psi(a) \\ &= \frac{|UT_5|}{|UT_5^{(3)}|} \frac{q|K_{UT_5^{(3)}}|}{|\text{SC}_{UT_5}(e_{12}+e_{45})|} \psi(u) \\ &= \frac{|UT_5|}{|UT_5^{(3)}|} \frac{|\text{SC}_{UT_5^{(3)}}(e_{12}+e_{45})|}{|\text{SC}_{UT_5}(e_{12}+e_{45})|} \psi(u) \\ &= \frac{|Z_{UT_5}(a_1(e_{12}+e_{45}+Id)a_1^{-1})|}{|Z_{UT_5^{(3)}}(a_1(e_{12}+e_{45}+Id)a_1^{-1})|} \psi(u) + \cdots + \frac{|Z_{UT_5}(a_q(e_{12}+e_{45}+Id)a_q^{-1})|}{|Z_{UT_5^{(3)}}(a_q(e_{12}+e_{45}+Id)a_q^{-1})|} \psi(u) \\ &= \text{Ind}_{UT_5^{(3)}}^{UT_5} \psi(u). \end{aligned}$$

□

Finally, any finite group which satisfies the characteristic invariances of the previous example can be shown to admit for a nontrivial supercharacter theory via induction.

Theorem 6.2. *Let G be a finite group, and $H \leq G$ be a subgroup. Then*

$$\text{SInd}_H^G \chi = \text{Ind}_H^G \chi$$

if and only if $\text{Ind}_H^G \chi$ is a superclass function of G , for all superclass functions χ on H .

Proof. Since $\text{SInd}_H^G \chi \in \text{scf}(G)$ for any $\chi \in \text{scf}(H)$ by definition, it is immediate that

$$\text{SInd}_H^G \chi = \text{Ind}_H^G \chi \Rightarrow \text{Ind}_H^G \chi \in \text{scf}(G).$$

Now suppose $\text{Ind}_H^G \chi \in \text{scf}(G)$ for each $\chi \in \text{scf}(H)$. Recall that

$$\text{Ind}_H^G \chi(g) := \frac{|G|}{|H|} \frac{1}{|K_G(g)|} \sum_{a \in K_G(g) \cap H} \chi(a).$$

Admitting any amount of fusion of conjugacy and/or superclasses in H into the superclass $\text{SC}_G(g)$, it follows that

$$\begin{aligned} \text{SInd}_H^G \chi(g) &:= \frac{|G|}{|H|} \frac{1}{|\text{SC}_G(g)|} \sum_{a \in \text{SC}_G(g) \cap H} \chi(a) \\ &= \frac{|G|}{|H|} \frac{1}{|\text{SC}_G(g)|} \sum_{C \subset \text{SC}_G(g)} \sum_{a \in C \cap H} \chi(a) \\ &= \frac{1}{|\text{SC}_G(g)|} \sum_{C \subset \text{SC}_G(g)} |C| \frac{|G|}{|H|} \frac{1}{|C|} \sum_{a \in C \cap H} \chi(a) \\ &= \frac{1}{|\text{SC}_G(g)|} \sum_{C \subset \text{SC}_G(g)} |C| \text{Ind}_H^G \chi(g) \\ &= \text{Ind}_H^G \chi(g). \end{aligned}$$

□

References

- [1] Persi Diaconis, I. M. Isaacs. Supercharacters and Superclass for Algebra Groups. American Mathematical Society Volume 360, Number 5, May 2008, Pages 2359-2392.
- [2] Isaacs, I. Martin. Algebra: a Graduate Course. American Mathematical Society, 2009.
- [3] James, G. D., and M. W. Liebeck. Representations and Characters of Groups. Cambridge University Press, 2012.
- [4] Eric Marberg, Nathaniel Thieme. Superinduction for Pattern Groups. arXiv:0712.1228 [math.RT]
- [5] Bruce Sagan. The Symmetric Group: Representations, Combinatorial Algorithms and Symmetric Functions. Graduate Texts in Mathematics, Springer Books. 2001. ISBN 978-1-4419-2869-6.