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Well-Posedness of the Damped Wave Equation with Nonlinear Source Terms

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1 Abstract

In general, existence of solutions to nonlinear wave equations heavily depends on the nonlinearity. The publication by Howard et al. proves global existence of solutions to wave equations with nonlinear damping and power source terms. However, the nonlinear damping can also act as a source term when the sign of the term is switched. In this paper, we prove local existence under linear damping combined with a nonlinear damping term. We also show that solutions which exist for only finite time must blow up. We prove this via fixed point iteration methods on Sobolev spaces.

2 Introduction

In 1865, James Clerk Maxwell published *A Dynamical Theory of the Electromagnetic Field*, [1], in which he formulated a unifying theory of electricity and magnetism. Today, we summarize his work with four partial differential equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{2.1}$$

$$\nabla \times \vec{E} = -\partial_t \vec{B} \tag{2.2}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.3}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \partial_t \vec{E} \tag{2.4}$$

Here, $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field, $\vec{J}$ is the current density in the medium, ρ is the charge density in the medium, and μ_0 and ϵ_0 are fundamental constants. From these equations, Maxwell derived the electromagnetic wave equation. In this paper, we concern ourselves with the properties of electromagnetic waves inside ohmic materials. That is, we will assume that Ohm's law holds everywhere in the medium and at all times:

$$\vec{J} = \sigma \vec{E} \tag{2.5}$$

where σ is the conductivity of the material, here assumed to be constant. For simplicity, we will set $\mu_0 = \epsilon_0 = \sigma = 1$. We will also assume that there is no charge density inside the medium ($\rho = 0$). Maxwell's equations then take the following form:

$$\nabla \cdot \vec{E} = 0 \tag{2.6}$$

$$\nabla \times \vec{E} = -\partial_t \vec{B} \tag{2.7}$$

$$\nabla \cdot \vec{B} = 0 \tag{2.8}$$

$$\nabla \times \vec{B} = \vec{E} + \partial_t \vec{E} \tag{2.9}$$

The electromagnetic wave equation can then be derived by differentiating equation (2.9) with respect to time:

$$\begin{gathered} \partial_t \left(\nabla \times \vec{B} \right) = \partial_t \left(\vec{E} + \partial_t \vec{E} \right) \\ \nabla \times \partial_t \vec{B} = \partial_t \vec{E} + \partial_t^2 \vec{E} \end{gathered} \tag{2.10}$$

Then, substituting in equation (2.7),

$$\begin{aligned} \nabla \times \left(-\nabla \times \vec{E}\right) &= \partial_t \vec{E} + \partial_t^2 \vec{E} \\ \Delta \vec{E} - \nabla(\nabla \cdot \vec{E}) &= \partial_t \vec{E} + \partial_t^2 \vec{E} \\ \partial_t^2 \vec{E} + \partial_t \vec{E} - \Delta \vec{E} &= 0 \end{aligned} \tag{2.11}$$

where we use a vector calculus identity to go from line 1 to line 2. The resulting equation neglects some physical effects due to the assumptions we made. For example, our assumption that the conductivity is constant and uniform is a good approximation for most metals, but some effects such as heating can cause the conductivity to change over time. However, perturbations to the system can be reintroduced by adding terms that may depend on position, time, and the solution itself, generalized as some function $F(\vec{E})$, which for our purposes will be a nonlinear source term. Also, since this vector equation can be computed component by component, it is easier to consider a single component and work with the equivalent scalar equation. This paper is thus concerned with the following equation:

Definition 2.1 (Damped Wave Equation)**.**

$$\begin{cases} \partial_t^2 u + \partial_t u - \Delta u = F(u), & (x,t) \in \mathbb{R}^n \times [0,\infty) \\ u(x,0) = g, & x \in \mathbb{R}^n \\ u_t(x,0) = h, & x \in \mathbb{R}^n \end{cases} \tag{2.12}$$

Because of the nonlinearity $F(u)$, we cannot find a closed-form formula for the solution. However, we can still show that the solution exists and that it must satisfy certain desirable properties. We generalize this with the notion of well-posedness:

Definition 2.2. *A partial differential equation is called* ***well-posed*** *if the following are satisfied:*

1. *The solution exists in some function space given initial data which is contained in the function space.*
2. *The solution is unique within this function space for given initial data.*
3. *The solution depends continuously on the initial data.*

Howard et al., [2], shows well-posedess for the damped wave equation in the case $F(u) = u|u|^p$ under nonlinear damping. Furthermore, they show that solutions cannot blowup in finite time under appropriate conditions. However, reversing the sign of the nonlinear damping can change it to a source term, inducing a blowup in finite time. In this paper, we consider a nonlinear "damping" source term in addition to a linear damping term. Specifically, we show that the Damped Wave Equation is well-posed in certain Sobolev spaces up to finite time for the case $F(u) = (\partial_t u)^2$, but finite time solutions must blowup. Similar work has been done on the Damped Wave Equation, obtaining $L^p - L^q$ estimates in [3], showing blowup of critical solutions in [4], and obtaining decay estimates of critical solutions in [5]. This paper establishes its own estimates which provide similar results to [3].

3 Notation

Here we present the notation which will be used throughout the paper.

1. u_t denotes $\partial_t u$.
   u_{tt} denotes $\partial_t^2 u$.

2. $\widetilde{\Box}$ denotes the damped wave operator $\partial_t^2 + \partial_t - \Delta$.

3. $\hat{u}$ denotes the spatial Fourier transform of u.

4. $a \lesssim b$ indicates $a \leq C \cdot b$, where $C \in \mathbb{R}$ is a nonnegative constant independent of the variables a and b.

4 Preliminaries

We show the existence of solutions within Sobolev spaces (H^s), defined below.

Definition 4.1 (Sobolev Space)**.** *The* ***H^s norm*** *of a function $f : \mathbb{R}^n \to \mathbb{R}$, denoted by $\|f\|_{H^s}$, is*

$$\|f\|_{H^s} = \left[\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\hat{f}|^2 d\xi\right]^{1/2} \tag{4.1}$$

where ξ is the Fourier transform variable of $\hat{f}$. If $\|f\|_{H^s} < \infty$, then $f \in H^s$.

For the sake of showing the existence of solutions, we would like to show that the H^s norm is finite for all time of existence. We measure this by considering a new function space, under which the norm is the supremum of the H^s norm in time. We call this X^s space, and define it below.

Definition 4.2 (X^s Space)**.** *The* ***X^s norm*** *of a function $u(x,t) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$, denoted by $\|u\|_{X^s}$, is*

$$\|u\|_{X^s} = \sup_{t \in [0,T]} \left(\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}\right) \tag{4.2}$$

where $T \leq \infty$ is a positive constant. If $\|u\|_{X^s} < \infty$, then $u \in X^s$.

Note that we leave the value of T undetermined for now. In the existence proof, we will bound T in terms of initial data. We will also need the standard L^p spaces at certain points in this paper:

Definition 4.3 (L^p Space)**.** *The* ***L^p norm*** *of a function $f : \mathbb{R}^n \to \mathbb{R}$, denoted by $\|f\|_{L^p}$, is*

$$\|f\|_{L^p} = \begin{cases} \left[\int_{\mathbb{R}^n} |f|^p dx\right]^{1/p}, & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f|, & p = \infty. \end{cases} \tag{4.3}$$

If $\|f\|_{L^p} < \infty$, then $f \in L^p$.

Here we present several theorems which will be necessary for proving our results:

Theorem 4.4 (Leibniz Integral Rule)**.** *For $-\infty < a(x) < b(x) < \infty$,*

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f\left(x, b(x)\right) \cdot \frac{d}{dx} b(x) - f\left(x, a(x)\right) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \partial_x f(x,t) dt. \tag{4.4}$$

Proof. See [7]. □

Theorem 4.5 (Bessel Potential)**.** *The Bessel potential, Λ^s, is a linear differential operator satisfying*

$$\begin{cases} \|u(t_0)\|_{H^s} = \|\Lambda^s u(t_0)\|_{L^2} \\ \|u(t_0)v(t_0)\|_{H^s} \lesssim \|u(t_0)\Lambda^s v(t_0)\|_{L^2} + \|v(t_0)\Lambda^s u(t_0)\|_{L^2} \end{cases} \tag{4.5}$$

for any functions $u(t_0), v(t_0) \in H^s$.

Proof. See [8], page 1. □

Theorem 4.6 (Sobolev Embedding)**.** *For any n-dimensional function $u(t_o) \in H^s$, if $s > n/2$, then*

$$\|u(t_0)\|_{L^\infty} \lesssim \|u(t_0)\|_{H^s}. \tag{4.6}$$

Proof. See [9], page 7. □

Theorem 4.7 (Banach Contraction-Mapping Principle)**.** *Let (X, d) be a complete metric space and $G : X \to X$ a map such that there exists $\theta \in [0, 1)$ satisfying $d(G(x), G(y)) \leq \theta d(x, y)$ for all $x, y \in X$. Then, there exists a unique $x_0 \in X$ such that $G(x_0) = x_0$.*

Proof. See [10], page 241. □

Theorem 4.8 (Gronwall's Inequality)**.** *Let $f(t)$ be a nonnegative, continuous function on $[0, T]$ which satisfies*

$$f(t) \lesssim \int_0^t f(s)ds \tag{4.7}$$

for all $t \in [0, T]$. Then, $f(t) = 0$ for all $t \in [0, T]$.

Proof. See [11], page 625. □

5 Solving the Damped Wave Equation

We begin by solving equation (2.12) on the Fourier transform side. This will provide the necessary formulas for bounding solutions in X^s space later in the paper.

Lemma 5.1. *If*

$$\hat{u_0} = \begin{cases} \hat{g}e^{-1/2t}\cosh(1/2t\sqrt{1-4|\xi|^2}) + \frac{\hat{g}+2\hat{h}}{\sqrt{1-4|\xi|^2}}e^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2}), & |\xi| < 1/2 \\ \hat{g}e^{-1/2t} + (1/2\hat{g} + \hat{h})te^{-1/2t}, & |\xi| = 1/2 \\ \hat{g}e^{-1/2t}\cos(1/2t\sqrt{4|\xi|^2-1}) + \frac{\hat{g}+2\hat{h}}{\sqrt{4|\xi|^2-1}}e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}), & |\xi| > 1/2 \end{cases} \tag{5.1}$$

then u_0 solves equation (2.12) for $F \equiv 0$ (the homogeneous case).

Proof. Taking the spatial Fourier transform of the homogeneous Damped Wave Equation yields the following:

$$\hat{u}_{tt} + \hat{u}_t + |\xi|^2\hat{u} = 0. \tag{5.2}$$

This is a second order ordinary differential equation with respect to time. This equation has the following characteristic equation:

$$r^2 + r + |\xi|^2 = 0 \tag{5.3}$$

to which the solution is

$$r = -1/2 \pm 1/2\sqrt{1-4|\xi|^2}. \tag{5.4}$$

The solution to equation (5.2) depends on whether the roots of the characteristic equation are real, imaginary, or double roots. Depending on the value of $|\xi|$ these are all possible, so the solution must be computed piecewise.

Case 1, $|\xi| < 1/2$: In this case, the roots are both real, and so the solution to equation (5.2) takes the form

$$\hat{u} = Ae^{-1/2t}\cosh(1/2t\sqrt{1-4|\xi|^2}) + Be^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2}). \tag{5.5}$$

Evaluating this at $t = 0$, we get

$$\hat{u}(\xi, 0) = A. \tag{5.6}$$

Since $u(x, 0) = g$, it follows that $\hat{u}(\xi, 0) = \hat{g}$, and so $A = \hat{g}$.
Taking the partial derivative with respect to time of equation (5.5) and evaluating at $t = 0$ gives

$$\hat{u}_t(\xi, 0) = -1/2A + 1/2B\sqrt{1-4|\xi|^2}. \tag{5.7}$$

From equation (2.12) we have $\hat{u}_t(\xi, 0) = \hat{h}$, so $B = \frac{\hat{g}+2\hat{h}}{\sqrt{1-4|\xi|^2}}$.

Case 2, $|\xi| = 1/2$: In this case, there is a double root, so the solution takes the form

$$\hat{u} = Ae^{-1/2t} + Bte^{-1/2t}. \tag{5.8}$$

Evaluating at $t = 0$ gives $A = \hat{g}$. Evaluating the partial derivative with respect to time at $t = 0$ gives $B = 1/2\hat{g} + \hat{h}$.

Case 3, $|\xi| > 1/2$: In this case, both roots are imaginary. The solution takes the form

$$\hat{u} = Ae^{-1/2t}\cos(1/2t\sqrt{4|\xi|^2-1}) + Be^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}). \tag{5.9}$$

Evaluating at $t = 0$ gives $A = \hat{g}$. Evaluating the partial derivative with respect to time at $t = 0$ gives

$$\hat{u}_t(\xi, 0) = -1/2A + 1/2B\sqrt{4|\xi|^2-1} \tag{5.10}$$

From equation (2.12) we have $\hat{u}_t(\xi, 0) = \hat{h}$, so $B = \frac{\hat{g}+2\hat{h}}{\sqrt{4|\xi|^2-1}}$.

□

Thus, we have found the homogeneous solution to equation (2.12) on the Fourier transform side. We now use Duhamel's principle to find a particular solution:

Lemma 5.2 (Duhamel's Principle)**.** *Suppose* $w = \int_0^t v(x, t-s; s)ds$*, where* v *solves*

$$\widetilde{\Box} v = 0, \;\; v(x, 0; s) = 0, \;\; v_t(x, 0; s) = F(x, s). \tag{5.11}$$

Then, w *solves equation* (2.12) *for* $g \equiv 0$ *and* $h \equiv 0$ *(a particular solution).*

Proof. Plugging w into equation (2.12) and using Theorem 4.4,

$$\begin{aligned}\widetilde{\Box}w &= \partial_t\left[v(x,0;s) + \int_0^t v_t(x,t-s;s)ds\right] + v(x,0;s) + \int_0^t v_t(x,t-s;s)ds - \int_0^t \Delta v(x,t-s;s)ds \\ &= v_t(x,t-s;s)|_{s=t} + \int_0^t \left[v_{tt}(x,t-s;s) + v_t(x,t-s;s) - \Delta v(x,t-s;s)\right]ds \\ &= v_t(x,0;t) + \int_0^t \widetilde{\Box}v(x,t-s;s)ds\end{aligned} \tag{5.12}$$

Then, by equation (5.11), $\widetilde{\Box}w = F(x,t)$. ☐

Corollary 5.3. *For v described as in Lemma 5.2,*

$$\hat{v}(\xi,t-s;s) = \begin{cases} \frac{2\hat{F}(\xi,s)}{\sqrt{1-4|\xi|^2}}e^{-1/2(t-s)}\sinh(1/2(t-s)\sqrt{1-4|\xi|^2}), & |\xi|<1/2 \\ \hat{F}(\xi,s)te^{-1/2(t-s)}, & |\xi|=1/2 \\ \frac{2\hat{F}(\xi,s)}{\sqrt{4|\xi|^2-1}}e^{-1/2(t-s)}\sin(1/2(t-s)\sqrt{4|\xi|^2-1}), & |\xi|>1/2. \end{cases} \tag{5.13}$$

Proof. This follows directly from Lemma 5.1 for $g \equiv 0$ and $h = F(x,s)$. ☐

We now have an explicit formula for the solution to equation (2.12) in terms of its Fourier transform:

Theorem 5.4. *If $u = u_0 + w$, with u_0 described as in Lemma 5.1 and w described as in Lemma 5.2, then u solves equation (2.12).*

Proof. u satisfies the initial conditions:

$$u(x,0) = u_0(x,0) + w(x,0) = g + 0 = g, \tag{5.14}$$

$$\partial_t u(x,0) = \partial_t u_0(x,0) + \partial_t w(x,0) = h + 0 = h. \tag{5.15}$$

Plugging u into equation (2.12) yields

$$\widetilde{\Box}u = \widetilde{\Box}(u_0+w) = \widetilde{\Box}u_0 + \widetilde{\Box}w = 0 + F = F \tag{5.16}$$

as shown in Lemmas 5.1 and 5.2. Since u is the sum of the homogeneous solution (u_0) and a particular solution (w), u solves equation (2.12). ☐

6 Well-Posedness of the Damped Wave Equation

First, we present several estimates which will be necessary for bounding the solution to equation (2.12):

Lemma 6.1.

$$e^{-1/2t}\cosh(1/2ty) \le 1 \tag{6.1}$$

for all $t \in [0,\infty)$ and all $y \in (0,1]$.

Proof. Since $\cosh(1/2ty)$ is strictly increasing on this domain, we can bound it by its value at $y = 1$:

$$\begin{aligned} e^{-1/2t}\cosh(1/2ty) &\leq e^{-1/2t}\cosh(1/2t) \\ &= 1/2e^{-1/2t}(e^{1/2t} + e^{-1/2t}) \\ &= 1/2(1 + e^{-t}) \end{aligned} \tag{6.2}$$

Since e^{-t} is strictly decreasing, we can bound it by its value at $t = 0$. Thus,

$$e^{-1/2t}\cosh(1/2ty) \leq 1. \tag{6.3}$$

□

Lemma 6.2.

$$\frac{e^{-1/2t}\sinh(1/2ty)}{y} < 1/2. \tag{6.4}$$

for all $t \in [0, \infty)$ and all $y \in (0, 1]$.

Proof. To start, we will bound $\frac{\sinh(1/2ty)}{y}$ above by finding its maximum in time on the domain:

$$\partial_y\left(\frac{\sinh(1/2ty)}{y}\right) = \frac{1/2yt \cdot \cosh(1/2ty) - \sinh(1/2ty)}{y^2} \tag{6.5}$$

Taking the derivative of the numerator in this equation,

$$\begin{aligned} \partial_y[1/2yt \cdot \cosh(1/2ty) - \sinh(1/2ty)] &= 1/2t \cdot \cosh(1/2ty) + 1/4yt^2 \cdot \sinh(1/2ty) - 1/2t \cdot \cosh(1/2ty) \\ &= 1/4yt^2 \cdot \sinh(1/2ty) \\ &\geq 0 \end{aligned} \tag{6.6}$$

on the domain. Thus, the minimum value of the numerator is assumed at $y = 0$:

$$\lim_{y\to 0} \frac{1/2yt \cdot \cosh(1/2ty) - \sinh(1/2ty)}{y^2} = \lim_{y\to 0} \frac{1/4yt^2 \cdot \sinh(1/2ty)}{2y} = 0. \tag{6.7}$$

Since the numerator of equation 6.5 is nonnegative, $\frac{\sinh(1/2ty)}{y}$ is non-decreasing on the domain. Therefore,

$$\frac{\sinh(1/2ty)}{y} \leq \left.\frac{\sinh(1/2ty)}{y}\right|_{y=1} = \sinh(1/2t). \tag{6.8}$$

Plugging this back into equation 6.4:

$$\begin{aligned} \frac{e^{-1/2t}\sinh(1/2ty)}{y} &\leq e^{-1/2t}\sinh(1/2t) \\ &= 1/2e^{-1/2t}(e^{1/2t} - e^{-1/2t}) \\ &= 1/2(1 - e^{-t}) \\ &< 1/2. \end{aligned} \tag{6.9}$$

□

Lemma 6.3.

$$\frac{\sqrt{1+|\xi|^2}}{\sqrt{4|\xi|^2-1}}e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}) \lesssim 1 \tag{6.10}$$

for all $t \in [0,\infty)$ and all $|\xi| > 1/2$.

Proof. For $1/2 < |\xi| \leq 1$,

$$\frac{\sqrt{1+|\xi|^2}}{\sqrt{4|\xi|^2-1}}e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}) \leq \frac{\sqrt{2}}{\sqrt{4|\xi|^2-1}}e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}). \tag{6.11}$$

Since $\sin(1/2tx)/x \leq 1/2t$ for any x and t,

$$\begin{aligned}\frac{\sqrt{2}}{\sqrt{4|\xi|^2-1}}e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}) &\leq \sqrt{2}/2te^{-1/2t} \\ &\lesssim 1.\end{aligned} \tag{6.12}$$

For $|\xi| > 1$, we can bound $e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1})$ by 1:

$$\frac{\sqrt{1+|\xi|^2}}{\sqrt{4|\xi|^2-1}}e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}) \leq \frac{\sqrt{1+|\xi|^2}}{\sqrt{4|\xi|^2-1}}. \tag{6.13}$$

The denominator grows faster than the numerator, so this is bounded by its value at $|\xi| = 1$, which is finite. Thus, the inequality holds for all $|\xi| > 1/2$. □

Lemma 6.4.

$$\frac{\sqrt{4|\xi|^2-1}}{\sqrt{1+|\xi|^2}}e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}) < 2 \tag{6.14}$$

for all $t \in [0,\infty)$ and all $|\xi| > 1/2$.

Proof. We can bound $e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1})$ by 1:

$$\frac{\sqrt{4|\xi|^2-1}}{\sqrt{1+|\xi|^2}}e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}) \leq \frac{\sqrt{4|\xi|^2-1}}{\sqrt{1+|\xi|^2}}. \tag{6.15}$$

The numerator grows faster than the denominator, so this is bounded by its value as $|\xi| \to \infty$, which is 2. □

Now, we will use these estimates to bound the H^s norms of our solution and its time derivative.

Theorem 6.5. *For the solution u (Theorem 5.4) to equation* (2.12),

$$\|u(t)\|_{H^s} \lesssim \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^t \|F(t')\|_{H^{s-1}}dt' \tag{6.16}$$

for all $t \in [0,\infty)$.

Proof. Beginning with the definition of the H^s norm,

$$\begin{aligned}
\|u(t)\|_{H^s} &= \left[\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{1/2} \\
&= \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi + \int_{|\xi|>1/2} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{1/2} \\
&\leq \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{1/2} + \left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{1/2}
\end{aligned} \tag{6.17}$$

by the Triangle Inequality. We will work with both ranges of $|\xi|$ separately. For $|\xi| < 1/2$ (by Theorem 5.4),

$$\begin{aligned}
\left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{1/2} &= \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s \left|\hat{g}e^{-1/2t}\cosh(1/2t\sqrt{1-4|\xi|^2})\right.\right. \\
&\quad + \frac{\hat{g}+2\hat{h}}{\sqrt{1-4|\xi|^2}} e^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2}) \\
&\quad \left.\left. + \int_0^t \frac{2\hat{F}(\xi,t')}{\sqrt{1-4|\xi|^2}} e^{-1/2(t-t')}\sinh(1/2(t-t')\sqrt{1-4|\xi|^2})dt'\right|^2 d\xi\right]^{1/2} \\
&\lesssim \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s \left|\hat{g}e^{-1/2t}\cosh(1/2t\sqrt{1-4|\xi|^2})\right.\right. \\
&\quad \left.\left. + \frac{\hat{g}}{\sqrt{1-4|\xi|^2}} e^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2})\right|^2 d\xi\right]^{1/2} \\
&\quad + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s \left|\frac{2\hat{h}}{\sqrt{1-4|\xi|^2}} e^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2})\right|^2 d\xi\right]^{1/2} \\
&\quad + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s \left|\int_0^t \frac{2\hat{F}(\xi,t')}{\sqrt{1-4|\xi|^2}} e^{-1/2(t-t')}\right.\right. \\
&\quad \left.\left. \cdot \sinh(1/2(t-t')\sqrt{1-4|\xi|^2})dt'\right|^2 d\xi\right]^{1/2}
\end{aligned} \tag{6.18}$$

Note that for $|\xi| \in [0,1/2)$, $\sqrt{1-4|\xi|^2} \in (0,1]$. Also, for $t' \in (0,t)$, $(t-t') \in (0,t)$ while

$t \in [0,\infty)$. Therefore, we can apply Lemmas 6.1 and 6.2 for $y = \sqrt{1-4|\xi|^2}$:

$$\begin{aligned}
\left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{1/2} &\lesssim \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s |\hat{g}|^2 d\xi\right]^{1/2} \\
&\quad + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s |\hat{h}|^2 d\xi\right]^{1/2} \\
&\quad + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s \left|\int_0^t \hat{F}(\xi,t')dt'\right|^2 d\xi\right]^{1/2} \\
&\leq \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s |\hat{g}|^2 d\xi\right]^{1/2} \\
&\quad + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} (1+|\xi|^2)|\hat{h}|^2 d\xi\right]^{1/2} \\
&\quad + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} (1+|\xi|^2) \int_0^t |\hat{F}(\xi,t')|^2 dt' \, d\xi\right]^{1/2} \\
&\lesssim \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^s |\hat{g}|^2 d\xi\right]^{1/2} \\
&\quad + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} |\hat{h}|^2 d\xi\right]^{1/2} \\
&\quad + \int_0^t \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} |\hat{F}(\xi,t')|^2 d\xi\right]^{1/2} dt' \\
&\lesssim \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^t \|F(t')\|_{H^{s-1}} dt'
\end{aligned} \tag{6.19}$$

This concludes the bounding of $\|u(t)\|_{H^s}$ for $|\xi| < 1/2$. For $|\xi| > 1/2$ (by Theorem 5.4),

$$
\begin{aligned}
\left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{1/2} &= \Bigg[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^s \Bigg|\hat{g}e^{-1/2t}\cos(1/2t\sqrt{4|\xi|^2-1}) \\
&\quad + \frac{\hat{g}+2\hat{h}}{\sqrt{4|\xi|^2-1}} e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1}) \\
&\quad + \int_0^t \frac{2\hat{F}(\xi,t')}{\sqrt{4|\xi|^2-1}} e^{-1/2(t-t')}\sin(1/2(t-t')\sqrt{4|\xi|^2-1})dt'\Bigg|^2 d\xi\Bigg]^{1/2} \\
&\lesssim \Bigg[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^s \Bigg|\hat{g}e^{-1/2t}\cos(1/2t\sqrt{4|\xi|^2-1}) \\
&\qquad + \frac{\hat{g}}{\sqrt{4|\xi|^2-1}} e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1})\Bigg|^2 d\xi\Bigg]^{1/2} \\
&\quad + \left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^s \left|\frac{2\hat{h}}{\sqrt{4|\xi|^2-1}} e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1})\right|^2 d\xi\right]^{1/2} \\
&\quad + \Bigg[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^s \Bigg|\int_0^t \frac{2\hat{F}(\xi,t')}{\sqrt{4|\xi|^2-1}} e^{-1/2(t-t')} \\
&\qquad \cdot \sin(1/2(t-t')\sqrt{4|\xi|^2-1})dt'\Bigg|^2 d\xi\Bigg]^{1/2} \\
&\lesssim \left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^s \left|\hat{g}te^{-1/2t}\right|^2 d\xi\right]^{1/2} \\
&\quad + \left[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^{s-1} \left|\frac{\hat{h}\sqrt{1+|\xi|^2}}{\sqrt{4|\xi|^2-1}} e^{-1/2t}\sin(1/2t\sqrt{4|\xi|^2-1})\right|^2 d\xi\right]^{1/2} \\
&\quad + \Bigg[\int_{|\xi|>1/2} \left(1+|\xi|^2\right)^{s-1} \Bigg|\int_0^t \frac{\hat{F}(\xi,t')\sqrt{1+|\xi|^2}}{\sqrt{4|\xi|^2-1}} e^{-1/2(t-t')} \\
&\qquad \cdot \sin(1/2(t-t')\sqrt{4|\xi|^2-1})dt'\Bigg|^2 d\xi\Bigg]^{1/2}
\end{aligned}
\tag{6.20}
$$

The domains for $|\xi|$, t, and t' allow us to use Lemma 6.3 and simplify as we did for the previous case:

$$\left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^s|\hat{u}(t)|^2d\xi\right]^{1/2} \lesssim \left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^s|\hat{g}|^2d\xi\right]^{1/2} + \left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}|\hat{h}|^2d\xi\right]^{1/2} + \int_0^t\left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}|\hat{F}(\xi,t')|^2d\xi\right]^{1/2}dt' \lesssim \|g\|_{H^s}+\|h\|_{H^{s-1}}+\int_0^t\|F(t')\|_{H^{s-1}}dt' \tag{6.21}$$

Plugging in the results for both ranges of $|\xi|$ into equation 6.17 completes the proof. □

Moving on to the bound of the time derivative, we have the following estimate:

Theorem 6.6. *For the solution u (Theorem 5.4) to equation* (2.12),

$$\|\partial_t u(t)\|_{H^{s-1}} \lesssim \|g\|_{H^s}+\|h\|_{H^{s-1}}+\int_0^t\|F(t')\|_{H^{s-1}}dt' \tag{6.22}$$

for all $t\in[0,\infty)$.

Proof. As was the case for Theorem 6.5,

$$\|\partial_t u(t)\|_{H^{s-1}} \leq \left[\int_{|\xi|<1/2}\left(1+|\xi|^2\right)^{s-1}|\partial_t\hat{u}(t)|^2d\xi\right]^{1/2}+\left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}|\partial_t\hat{u}(t)|^2d\xi\right]^{1/2} \tag{6.23}$$

and we will consider both ranges of $|\xi|$ separately. Again note that u is assumed to be a function of t. For $|\xi|<1/2$ (by Theorem 5.4),

$$\begin{aligned}
\partial_t\hat{u}(t) &= -1/2\hat{g}e^{-1/2t}\cosh(1/2t\sqrt{1-4|\xi|^2})+1/2\hat{g}\sqrt{1-4|\xi|^2}e^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2})\\
&\quad-\frac{\hat{g}+2\hat{h}}{2\sqrt{1-4|\xi|^2}}e^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2})+1/2(\hat{g}+2\hat{h})e^{-1/2t}\cosh(1/2t\sqrt{1-4|\xi|^2})\\
&\quad+\int_0^t\frac{2\hat{F}(t')}{\sqrt{1-4|\xi|^2}}\Big[-1/2e^{-1/2(t-t')}\sinh(1/2(t-t')\sqrt{1-4|\xi|^2})\\
&\qquad\qquad+1/2\sqrt{1-4|\xi|^2}e^{-1/2(t-t')}\cosh(1/2(t-t')\sqrt{1-4|\xi|^2})\Big]dt'\\
&= -1/2\hat{u}(t)+1/2\hat{g}\sqrt{1-4|\xi|^2}e^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2})\\
&\quad+1/2(\hat{g}+2\hat{h})e^{-1/2t}\cosh(1/2t\sqrt{1-4|\xi|^2})\\
&\quad+\int_0^t\hat{F}(t')e^{-1/2(t-t')}\cosh(1/2(t-t')\sqrt{1-4|\xi|^2})dt'
\end{aligned} \tag{6.24}$$

Note that $\sqrt{1-4|\xi|^2}\leq 1$ on this domain, so

$$\sqrt{1-4|\xi|^2}e^{-1/2t}\sinh(1/2t\sqrt{1-4|\xi|^2})\leq e^{-1/2t}\sinh(1/2t), \tag{6.25}$$

which is proportionally less than 1, as is shown in the proof of Lemma 6.2. We also apply Lemma 6.1 to the "cosh" terms to get the following:

$$\begin{aligned}
\left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} |\partial_t \hat{u}(t)|^2 d\xi\right]^{1/2} &\lesssim \|u(t)\|_{H^{s-1}} + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s-1} \frac{1+|\xi|^2}{1+|\xi|^2} |\hat{g}|^2 d\xi\right]^{1/2} \\
&\quad + \|h\|_{H^{s-1}} + \int_0^t \|F(t')\|_{H^{s-1}} dt' \\
&= \|u(t)\|_{H^{s-1}} + \left[\int_{|\xi|<1/2} \left(1+|\xi|^2\right)^{s} \left|\frac{\hat{g}}{\sqrt{1+|\xi|^2}}\right|^2 d\xi\right]^{1/2} \\
&\quad + \|h\|_{H^{s-1}} + \int_0^t \|F(t')\|_{H^{s-1}} dt' \\
&\lesssim \|u(t)\|_{H^s} + \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^t \|F(t')\|_{H^{s-1}} dt' \\
&\lesssim \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^t \|F(t')\|_{H^{s-1}} dt'
\end{aligned} \tag{6.26}$$

In the last line we use the result of Theorem 6.5 to remove $\|u(t)\|_{H^s}$. Now consider the case for $|\xi| > 1/2$. By Theorem 5.4,

$$\begin{aligned}
\partial_t \hat{u}(t) &= -1/2\hat{u}(t) - 1/2\hat{g}\sqrt{1-4|\xi|^2} e^{-1/2t} \sin(1/2t\sqrt{1-4|\xi|^2}) \\
&\quad + 1/2(\hat{g} + 2\hat{h}) e^{-1/2t} \cos(1/2t\sqrt{1-4|\xi|^2}) \\
&\quad + \int_0^t \hat{F}(t') e^{-1/2(t-t')} \cos(1/2(t-t')\sqrt{1-4|\xi|^2}) dt'
\end{aligned} \tag{6.27}$$

Bounding cosine above by 1 and applying Lemma 6.4 to the sine term yields the desired result:

$$\begin{aligned}
\left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}|\partial_t\hat{u}(t)|^2 d\xi\right]^{1/2} \lesssim & \left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s-1}\frac{1+|\xi|^2}{1+|\xi|^2}\right.\\
&\left.\cdot\left|\hat{g}(1+\sqrt{4|\xi|^2-1}\sin(1/2t\sqrt{4|\xi|^2-1}))\right|^2 d\xi\right]^{1/2}\\
&+\|u(t)\|_{H^{s-1}}+\|h\|_{H^{s-1}}+\int_0^t\|F(t')\|_{H^{s-1}}dt'\\
\lesssim & \left[\int_{|\xi|>1/2}\left(1+|\xi|^2\right)^{s}\right.\\
&\left.\cdot\left|\hat{g}\left(1+\frac{\sqrt{4|\xi|^2-1}\sin(1/2t\sqrt{4|\xi|^2-1})}{\sqrt{1+|\xi|^2}}\right)\right|^2 d\xi\right]^{1/2}\\
&+\|u(t)\|_{H^{s}}+\|h\|_{H^{s-1}}+\int_0^t\|F(t')\|_{H^{s-1}}dt'\\
\lesssim & \|u(t)\|_{H^s}+\|g\|_{H^s}+\|h\|_{H^{s-1}}+\int_0^t\|F(t')\|_{H^{s-1}}dt'\\
\lesssim & \|g\|_{H^s}+\|h\|_{H^{s-1}}+\int_0^t\|F(t')\|_{H^{s-1}}dt'
\end{aligned} \tag{6.28}$$

Plugging in the results for both ranges of $|\xi|$ into equation 6.23 completes the proof. □

Combining the results of the previous theorems, we obtain our primary estimate for proving well-posedness of equation (2.12):

Corollary 6.7. *For the solution u (Theorem 5.4) to equation* (2.12),

$$\|u\|_{X^s}\lesssim\|g\|_{H^s}+\|h\|_{H^{s-1}}+\int_0^T\|F(t')\|_{H^{s-1}}dt'. \tag{6.29}$$

Proof. Recall the definition of the X^s norm,

$$\|u\|_{X^s}=\sup_{t\in[0,T]}(\|u(t)\|_{H^s}+\|\partial_t u(t)\|_{H^{s-1}}). \tag{6.30}$$

Then, applying Theorems 6.5 and 6.6,

$$\begin{aligned}
\|u\|_{X^s}&\lesssim\sup_{t\in[0,T]}\left(\|g\|_{H^s}+\|h\|_{H^{s-1}}+\int_0^t\|F(t')\|_{H^{s-1}}dt'\right)\\
&=\|g\|_{H^s}+\|h\|_{H^{s-1}}+\int_0^T\|F(t')\|_{H^{s-1}}dt'.
\end{aligned} \tag{6.31}$$

□

Our results so far apply to equation (2.12) for any choice of $F(u)$. We are primarily interested in nonlinear damping acting as a source term, which is the case $F(u) = u_t^2$. We can now prove one of the main results of this paper:

Theorem 6.8 (Local Existence and Uniqueness of Solutions)**.** *Consider the Damped Wave Equation* (2.12) *in n-dimensions with $F = u_t^2$ and initial data $g \in H^s$, $h \in H^{s-1}$. If $s > \frac{n}{2} + 1$, then there exists a unique solution $u(x,t) \in X^s$.*

Proof. Let $B_R = \{u : \|u\|_{X^s} \leq R\}$. Define $G : B_R \to B_R$ to be the map $G(u) = u_0 + w$, where u_0 is the homogeneous solution with initial data g and h described in Lemma 5.1, and w is the particular solution with $F = u_t^2$ described in Lemma 5.2. Our goal is to show that there exists a unique u such that $G(u) = u$, which would show that u is the unique solution to equation (2.12) inside B_R. We will then show that this u is the unique solution in all of X^s. We begin by showing that G is well-defined. For any $u \in B_R$, we have a definite formula for $G(u)$ on the Fourier transform side (Lemmas 5.1 and 5.2). We can also show that $G(u) \in B_R$ for any $u \in B_R$. By Corollary 6.7,

$$\|G(u)\|_{X^s} \lesssim \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^T \|u_t^2(t)\|_{H^{s-1}} dt. \tag{6.32}$$

Then, by Theorem 4.5,

$$\begin{aligned} \|G(u)\|_{X^s} &\lesssim \|g\|_{H^s} + \|h\|_{H^{s-1}} + 2\int_0^T \|u_t(t)\Lambda^{s-1}u_t(t)\|_{L^2} dt \\ &\leq \|g\|_{H^s} + \|h\|_{H^{s-1}} + 2\int_0^T \|u_t(t)\|_{L^\infty}\|\Lambda^{s-1}u_t(t)\|_{L^2} dt \\ &= \|g\|_{H^s} + \|h\|_{H^{s-1}} + 2\int_0^T \|u_t(t)\|_{L^\infty}\|u_t(t)\|_{H^{s-1}} dt \end{aligned} \tag{6.33}$$

Since $u \in B_R$, we must have that $u_t(t)$ is contained in some closed ball of H^{s-1} by our definition of X^s. Since $s > \frac{n}{2} + 1$, we can use Theorem 4.6 to bound $\|u_t\|_{L^\infty}$ by $\|u_t\|_{H^{s-1}}$:

$$\begin{aligned} \|G(u)\|_{X^s} &\lesssim \|g\|_{H^s} + \|h\|_{H^{s-1}} + 2\int_0^T \left(\|u_t(t)\|_{H^{s-1}}\right)^2 dt \\ &\leq \|g\|_{H^s} + \|h\|_{H^{s-1}} + 2\left(\sup_{t\in[0,T]} \|u_t(t)\|_{H^{s-1}}\right)^2 \int_0^T dt \\ &\leq \|g\|_{H^s} + \|h\|_{H^{s-1}} + 2T\left(\|u\|_{X^s}\right)^2. \end{aligned} \tag{6.34}$$

Recalling our definition for "$\lesssim$", our bound so far is

$$\|G(u)\|_{X^s} \leq C_1\left(\|g\|_{H^s} + \|h\|_{H^{s-1}} + 2T\left(\|u\|_{X^s}\right)^2\right) \tag{6.35}$$

where $C_1 > 0$ is some real constant. We then define the size of R in terms of the initial data:

$$R = 2C_1\left(\|g\|_{H^s} + \|h\|_{H^{s-1}}\right) \tag{6.36}$$

Note that our definition of R is arbitrary; we can look for solutions inside any closed ball, and this merely imposes different restrictions on T. In this case, we restrict T to

$$T \leq \frac{1}{4C_1R}. \tag{6.37}$$

Then, substituting these restrictions into equation (6.35), we get the desired result:

$$\|G(u)\|_{X^s} \leq C_1 \left(\frac{R}{2C_1} + 2\left(\frac{1}{4C_1 R}\right) R^2 \right) = R. \tag{6.38}$$

Since $\|G(u)\|_{X^s} \leq R$, $G(u(0)) = g$, and $\partial_t G(u(0)) = h$, $G(u) \in B_R$, and so G is well-defined for the restrictions given.

We will now show that G is a contraction mapping on B_R. The natural metric on this space is the X^s norm, so given arbitrary $u, v \in B_R$, we have (by Corollary 6.7)

$$\begin{aligned} \|G(u) - G(v)\|_{X^s} &\lesssim \|g - g\|_{H^s} + \|h - h\|_{H^{s-1}} + \int_0^T \|u_t^2(t) - v_t^2(t)\|_{H^{s-1}} dt \\ &= \int_0^T \|\big(u_t(t) - v_t(t)\big)\big(u_t(t) + v_t(t)\big)\|_{H^{s-1}} dt \end{aligned} \tag{6.39}$$

Then, using Theorem 4.5 and the same methods as for showing G is well-defined,

$$\begin{aligned} \|G(u) - G(v)\|_{X^s} &\lesssim \int_0^T \Big[\|\big(u_t(t) - v_t(t)\big)\Lambda^{s-1}\big(u_t(t) + v_t(t)\big)\|_{L^2} \\ &\qquad + \|\big(u_t(t) + v_t(t)\big)\Lambda^{s-1}\big(u_t(t) - v_t(t)\big)\|_{L^2}\Big] dt \\ &\leq \int_0^T \Big[\|u_t(t) - v_t(t)\|_{L^\infty}\|u_t(t) + v_t(t)\|_{H^{s-1}} \\ &\qquad + \|u_t(t) + v_t(t)\|_{L^\infty}\|u_t(t) - v_t(t)\|_{H^{s-1}}\Big] dt \\ &\lesssim \int_0^T \Big[\|u_t(t) - v_t(t)\|_{H^{s-1}}\|u_t(t) + v_t(t)\|_{H^{s-1}} \\ &\qquad + \|u_t(t) + v_t(t)\|_{H^{s-1}}\|u_t(t) - v_t(t)\|_{H^{s-1}}\Big] dt \\ &= 2\int_0^T \|u_t(t) - v_t(t)\|_{H^{s-1}}\|u_t(t) + v_t(t)\|_{H^{s-1}} dt \\ &\leq 2\int_0^T \|u_t(t) - v_t(t)\|_{H^{s-1}}\big(\|u_t(t)\|_{H^{s-1}} + \|v_t(t)\|_{H^{s-1}}\big) dt \\ &\leq 2T \sup_{t\in[0,T]} \Big[\|u_t(t) - v_t(t)\|_{H^{s-1}}\big(\|u_t(t)\|_{H^{s-1}} + \|v_t(t)\|_{H^{s-1}}\big)\Big] \\ &\leq 2T\|u - v\|_{X^s}\big(\|u\|_{X^s} + \|v\|_{X^s}\big) \\ &\leq 4RT\|u - v\|_{X^s} \end{aligned} \tag{6.40}$$

Again, we reintroduce the proportionality constant that we have been ignoring to obtain

$$\|G(u) - G(v)\|_{X^s} \leq 4C_2 RT\|u - v\|_{X^s} \tag{6.41}$$

for some $C_2 > 0$. We will now restrict $T < \frac{1}{4CR}$, where $C = \max\{C_1, C_2\}$. Then, we see that $4C_2RT \leq 4CRT < 1$. This produces the desired result:

$$\|G(u) - G(v)\|_{X^s} \leq \theta\|u - v\|_{X^s} \tag{6.42}$$

for some $\theta \in [0,1)$. Thus, there exists a unique $u \in B_R$ such that $G(u) = u$ by Theorem 4.7. By our definition of G, this is the unique solution to equation (2.12) inside B_R. We can now show that this solution is unique in all of X^s. Let $w \in X^s$ be any function that solves equation (2.12). Then, by Theorems 6.5 and 6.6,

$$\begin{aligned} \|u(t)-w(t)\|_{H^s} + \|u_t(t)-w_t(t)\|_{H^{s-1}} &\lesssim \int_0^t \|u_t^2(t')-w_t^2(t')\|_{H^{s-1}}dt' \\ &\leq 2C\big(\|u\|_{X^s}+\|w\|_{X^s}\big)\int_0^t \|u_t(t')-w_t(t')\|_{H^{s-1}}dt' \\ &\leq 2C\big(\|u\|_{X^s}+\|w\|_{X^s}\big)\int_0^t \Big[\|u(t')-w(t')\|_{H^s} \\ &\qquad + \|u_t(t')-w_t(t')\|_{H^{s-1}}\Big]dt' \\ &\lesssim \int_0^t \Big[\|u(t')-w(t')\|_{H^s}+\|u_t(t')-w_t(t')\|_{H^{s-1}}\Big]dt'. \end{aligned} \tag{6.43}$$

Since $f(t) = \|u(t)-w(t)\|_{H^s} + \|u_t(t)-w_t(t)\|_{H^{s-1}}$ is nonnegative and continuous, we can apply Gronwall's Inequality:

$$\|u(t)-w(t)\|_{H^s} + \|u_t(t)-w_t(t)\|_{H^{s-1}} = 0 \tag{6.44}$$

for all $t \in [0,T]$. We conclude that $u(t) = w(t)$ for all $t \in [0,T]$, so the only solution to equation (2.12) in X^s is $u \in B_R$. □

Hence, we have confirmed that a unique solution exists inside X^s for nonlinear "damped" source terms. However, unlike the results of Howard et al., [1], we can show that these solutions must blowup in finite time if they do not exist for all time. We state this precisely in the following theorem.

Theorem 6.9 (Blowup Criterion). *Let $X^s[t_1,t_2]$ indicate X^s defined on the time interval $[t_1,t_2]$, i.e.*

$$\|u\|_{X^s[t_1,t_2]} = \sup_{t\in[t_1,t_2]} (\|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}). \tag{6.45}$$

Let $T_{max} = \sup\{t_0 > 0 :$ there exists a unique solution to equation (2.12) on $[0,t_0]\}$. Then, if $T_{max} < \infty$, $\|u\|_{X^s[0,T_{max}]} = \infty$.

Proof. We will prove this by contradiction. Suppose $T_{max} < \infty$ and $\|u\|_{X^s[0,T_{max}]} = M < \infty$. Let $t_1 = \max\{T_{max} - \frac{1}{16CR}, 0\}$, $t_2 = T_{max} + \frac{1}{16CR}$, and $R = 4CM$, where C is defined as in the proof of Theorem 6.8. Let $B_R = \{u : \|u\|_{X^s[t_1,t_2]} \leq R\}$ and $G : B_R \to B_R$ be the map $G(u) = u_0 + w$. We can show that G is well-defined using the same methods as the previous proof, and so we omit some of the details:

$$\begin{aligned} \|G(u)\|_{X^s[t_1,t_2]} &\lesssim \|u(t_1)\|_{H^s} + \|u_t(t_1)\|_{H^{s-1}} + \int_{t_1}^{t_2} \|u_t^2(t)\|_{H^{s-1}}dt \\ &\leq C\left(M + 2(t_2-t_1)\left(\|u_t\|_{X^s[t_1,t_2]}\right)^2\right) \\ &\leq C\left(\frac{R}{4C} + 2\left(T_{max}+\frac{1}{16CR} - \left(T_{max}-\frac{1}{16CR}\right)\right)R^2\right) \\ &= 1/2R \\ &< R. \end{aligned} \tag{6.46}$$

We can also show that G is a contraction mapping, again using the same methods as the previous proof. Let $u, v \in B_R$. Then,

$$\begin{aligned}
\|G(u) - G(v)\|_{X^s[t_1,t_2]} &\lesssim \int_{t_1}^{t_2} \|\big(u_t(t) - v_t(t)\big)\big(u_t(t) + v_t(t)\big)\|_{H^{s-1}} dt \\
&\leq 2(t_2 - t_1)\|u - v\|_{X^s[t_1,t_2]} \big(\|u\|_{X^s[t_1,t_2]} + \|v\|_{X^s[t_1,t_2]}\big) \\
&\leq 4CR(t_2 - t_1)\|u - v\|_{X^s[t_1,t_2]} \\
&\leq 4CR\left(T_{max} + \frac{1}{16CR} - \left(T_{max} - \frac{1}{16CR}\right)\right)\|u - v\|_{X^s[t_1,t_2]} \\
&= 1/2\|u - v\|_{X^s[t_1,t_2]}.
\end{aligned} \tag{6.47}$$

Thus, G is a contraction mapping, and so there exists a unique $u \in B_R$ which solves equation (2.12). Finally, by Theorems 6.5 and 6.6,

$$\begin{aligned}
\|u(t) - w(t)\|_{H^s} + \|u_t(t) - w_t(t)\|_{H^{s-1}} &\lesssim \int_{t_1}^{t_2} \|u_t^2(t') - w_t^2(t')\|_{H^{s-1}} dt' \\
&\leq 2C\big(\|u\|_{X^s[t_1,t_2]} + \|w\|_{X^s[t_1,t_2]}\big) \int_{t_1}^{t_2} \Big[\|u(t') - w(t')\|_{H^s} \\
&\qquad + \|u_t(t') - w_t(t')\|_{H^{s-1}}\Big] dt' \\
&\lesssim \int_{t_1}^{t_2} \Big[\|u(t') - w(t')\|_{H^s} + \|u_t(t') - w_t(t')\|_{H^{s-1}}\Big] dt'
\end{aligned} \tag{6.48}$$

for any $w \in X^s[t_1, t_2]$. By Gronwall's Inequality, $u \equiv w$, so u is the unique solution to equation (2.12) in all of $X^s[t_1, t_2]$. Since this solution exists up to $t_2 > T_{max}$, we have contradicted our definition of T_{max}, and so we must have $\|u\|_{X^s[0, T_{max}]} = \infty$. □

We now present our final result, which provides the last piece of well-posedness for equation 2.12.

Theorem 6.10 (Continuous Dependence on Initial Data)**.** *The solution to equation 2.12 in n-dimensions with $F = u_t^2$, initial data $g \in H^s$ and $h \in H^{s-1}$, and $s > \frac{n}{2} + 1$ is continuously dependent on the initial data.*

Proof. Let $S : H^s \times H^{s-1} \to X^s$ be the solution map for this equation. That is, given initial data g and h, $S(g, h) = u$, where u is the unique solution for the given initial data. By Theorem 6.8, S is well defined. Let $\epsilon > 0$ be a real constant. For any $g' \in H^s$ and $h' \in H^{s-1}$, if

$$\|g - g'\|_{H^s} + \|h - h'\|_{H^{s-1}} < \delta, \tag{6.49}$$

then by Corollary 6.7,

$$\begin{aligned}
\|S(g, h) - S(g', h')\|_{X^s} &\lesssim \|g - g'\|_{H^s} + \|h - h'\|_{H^{s-1}} + \int_0^T \|(u_t(t))^2 - (u_t'(t))^2\|_{H^{s-1}} dt \\
&< \delta + \int_0^T \Big\|\big(\partial_t S(g, h)\big)^2 - \big(\partial_t S(g', h')\big)^2\Big\|_{H^{s-1}} dt \\
&\leq C\delta + 4CRT\|S(g, h) - S(g', h')\|_{X^s}.
\end{aligned} \tag{6.50}$$

The steps between the second and third line are the same as in equation 6.39. We can simplify this statement to obtain

$$\|S(g,h) - S(g',h')\|_{X^s} \leq \frac{C\delta}{1-4CRT}. \tag{6.51}$$

From Theorem 6.8, we know that $0 < 4CRT < 1$ on the interval for which the solution exists, so the right-hand side is positive and finite. We can choose

$$\delta = \frac{\epsilon(1-4CRT)}{2C}. \tag{6.52}$$

Thus, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $\|g - g'\|_{H^s} + \|h - h'\|_{H^{s-1}} < \delta$, then $\|S(g,h) - S(g',h')\|_{X^s} \leq 1/2\epsilon < \epsilon$, so S is continuous at (g,h). □

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