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Hawking Radiation with the WKB and Graviational WKB approximations

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Calculated Near the Horizon and Extended to $r < 2M$

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Abstract

The purpose of this thesis is to calculate the Hawking temperature for a Schwarzschild black hole, using the gravitational WKB approximation, in the local coordinate frame of the near horizon approximation. Although tunneling models of Hawking radiation have been able to reproduce Hawkings original temperature calculation for a number of different coordinate choices, the coordinates of the near horizon approximation still have not been used to calculate a temperature that is consistent with Hawkings original calculation. In this thesis, it is shown that the Hawking temperature can be calculated from the near horizon approximation to yield a result that is consistent with Hawking. This is done by using a sequence of coordinate transformations that make the metric compatible with the gravitational WKB approximation.

Contents

Introduction

In 1975 Stephen Hawking showed that quantum field theory predicts the emission of thermal radiation by black holes, which over a long period of time causes them to completely evaporate out of existence [1]. Hawking's original derivation of black hole radiance assumed a fixed and curved background spacetime geometry. This assumption of the geometry being fixed led to the prediction of a perfectly thermal spectrum, with a characteristic temperature that is now known as the Hawking temperature. However, the assumption of a fixed background metric does not take conservation of energy into account, because it assumes that the black hole's mass remains constant as it radiates.

This problem was successfully addressed in the year 2000, when Parikh and Wilczek demonstrated that the quantum mechanical WKB approximation can be used to calculate Hawking radiation tunneling rates through an approach that assumes conservation of energy [2]. This model has since been extended to numerous other contexts for Hawking radiation as a quantum tunneling mechanism [3–6].

A more recent alternative approach, that utilizes quantum field theory's field description, is known as the gravitational WKB approximation. Unlike the quantum mechanical WKB method, the gravitational version includes a contribution from the action's time component. An attempt to calculate the Hawking temperature with the gravitational WKB method in the Rindler coordinates, was made by Kim in 2007 [7]. The calculation predicted a temperature equal to twice the Hawking temperature. This was followed by a gravitational WKB approximation that successfully calculated the Unruh temperature by Gilldouglas in 2010 [8].

In section 8, I'll show that by applying two successive coordinate transformations to the near horizon approximation, the approximation's coordinates can be transformed into a set of local Rindler coordinates that are compatible with the gravitational WKB approximation. Making use of the fact that this local metric is conformal to the Minkowski metric, the coordinate relations, when written in terms of surface gravity κ and Schwarzschild time t, can be shown to be symmetric under the interchange of $t, \kappa \leftrightarrow \tau, a$, to the coordinates that were used by Gilldouglas in [8] in calculating the Unruh temperature. This allows the same method that was used for calculating the Unruh temperature to be used to obtain a correct calculation for the Hawking temperature.

Furthermore, I point out that the one-to-one correspondence between a Rindler observer's proper acceleration and the proper distance to the Rindler horizon, can be used to extend the formula for Hawking temperature to $r < 2M$, by noting a symmetric correspondence between κ and ρ. This extension of the tunneling model to $r < 2M$ led to a very interesting finding: In order for an object to hit the event horizon of a black hole, it must also come into contact with its own Rindler horizon. And in order for an object to hit the geometric singularity of a Schwarzschild black hole, it must come into contact with the Rindler horizon analog that is determined by replacing proper acceleration a with surface gravity κ.

Chapter 1

About Hawking radiation

1.1 Some numbers to put things in perspective

Measured quantities in black hole physics are generally taken to be normalized by a "gravitational redshift" factor. These quantities are defined to be redshifted in the same way that a photon gets redshifted as it climbs out of a graviational potential, on its way to $r = \infty$. Many quantities in black hole physics would end up being infinite if this standard of normalization were not used. Quantities that are not assumed to be normalized in this way are typically referred to as being "proper." Two quantities that are important in the study of Hawking radiation are the Hawking temperature and the lifetime of a black hole.

First, a comment on notation. Greek indices are used to denote 4-vectors a^μ, and superscript i is used to denote spatial 3-vector components a^i. This thesis assumes $\hbar, G, c, k_B = 1$ unless otherwise stated.

1.1.1 Temperature of a black hole

I've replaced the fundamental constants for this section since it is computational. The Hawking temperature of a black hole is

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} \tag{1.1}$$

The Hawking temperature is defined to be the thermal temperature corresponding to the characteristic wavelength of a photon of Hawking radiation, that is detected by an observer located infinitely far away from the black hole.

Mass	Temperature (K)
m_P	5.6×10^{30}
$M_\odot$	6.1×10^{-8}
$4 \times 10^6 M_\odot$	1.5×10^{-14}

Table 1.1: Black hole temperatures. Smaller black holes have larger temperatures. The smallest possible black hole has a mass equal to the Planck mass $m_P = 0.0218$ mg, and has the largest possible Hawking temperature. The black hole that is thought to be at the center of our galaxy, Sagittarius A^*, has a mass of about $4 \times 10^6\ M_\odot$, and correspondingly has a very low Hawking temperature.

The temperatures shown in table 1.1 indicate that small black holes have large temperatures and large black holes have small temperatures. This is caused by the fact that smaller black holes radiate much higher energy photons than large black holes.

1.1.2 Life time of a black hole

Realistically, any black hole that forms from the collapse of a star will have to be about 6 $M_\odot$ or greater, where one solar mass $M_\odot = 1.989 \times 10^{30}$ kg. An excellent discussion on why this is the case and how it occurs can be found in [9].

The lifetime of a black hole, as predicted by thermodynamics, can be calculated using the Stefan-Boltzmann law for the power radiated by a black body,

$$P = \frac{dE}{dt} = \sigma A T^4 \tag{1.2}$$

where A is the surface area of the black body, T is the black hole's Hawking temperature, and $\sigma = 5.67037 \times 10^{-8}\ \frac{W}{m^2 \cdot K^4}$ is the Stefan-Boltzmann constant. Taking the radiated energy E to be equal to the radiated rest mass Mc^2, the black hole loses mass at a rate

$$\begin{aligned} \frac{d(Mc^2)}{dt} &= -\sigma A T_H^4 \\ &= -\sigma 4\pi \left(\frac{2GM}{c^2}\right)^2 \left(\frac{\hbar c^3}{8\pi G M k_B}\right)^4 \end{aligned} \tag{1.3}$$

where I've replaced A with a standard measure of the black hole's surface area $A = 4\pi {R_s}^2 = 4\pi \left(\frac{2GM}{c^2}\right)^2$. The minus sign in equation (1.3) comes from the fact that the change in the black hole's mass is negative. Rearranging constants, multiplying both sides of equation (1.3) by dt, and integrating,

$$\begin{aligned}\tau &= \int_0^\tau dt \\ &= -\frac{256\pi^3 {k_B}^4 G^2}{\sigma \hbar^4 c^6} \int_M^0 M'^2 dM' \\ &= \frac{256\pi^3 {k_B}^4 G^2}{3\sigma \hbar^4 c^6} M^3 \end{aligned} \tag{1.4}$$

where τ is the lifetime of the black hole. As indicated by table 1.2, the M^3 factor in equation (1.4) plays an enormous role in determining the lifetime of a black hole. This M^3 dependence, causes more massive black holes to be more stable against evaporation.

Black hole mass	Lifetime (yrs)
m_P	3.45×10^{-48}
$M_\odot$	2.21×10^{67}
$4 \times 10^6 M_\odot$	1.41×10^{87}

Table 1.2: Black hole lifetimes. A one Planck mass ($m_P = 0.0218$ mg) black hole evaporates in about 8.8×10^{-40} seconds. The black hole that is thought to be at the center of our galaxy, Sagittarius A^*, has a mass of about 4×10^6 $M_\odot$, which is an example of a realistic large black hole mass. These numbers assume a background temperature of near absolute zero.

Under the assumption that space and time are quantized by Planck units, the smallest possible black hole has a mass of one Planck mass $m_P = 0.0218$ mg, and evaporates within about 8.8×10^{-40} seconds of reaching the Planck mass. According to the numbers in table 1.2, a black hole on the order of one solar mass or larger will last at least 10^{57} times longer than the age of the universe. However, these numbers assume a cosmic microwave background (the thermal reservoir) temperature very close to 0 K. So realistically, currently existing black holes are expected to take longer to evaporate than the predicions shown in table 1.2. The lifetimes of currently existing black holes are prolonged by other factors as well, such as accretion of matter onto the black hole.

1.1.3 Non-thermality of the Hawking radiation spectrum

One of the nice features of Parikh and Wilczek's tunneling calculation is that it incorporated conservation of energy. Hawking's original calculation [1] was made under the assumption of a static background geometry. This means that he did not take into account the fact that black hole's lose mass as they radiate. Enforcing total energy conservation requires the black hole's mass to decrease as it evaporates. As I will show here, nonthermality of the Hawking radiation spectrum results in a radiative intensity that is larger than would be predicted if the radiation were perfectly thermal. The Planck spectrum is

$$I(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1} \tag{1.5}$$

Using Wein's displacement law, which states that $\lambda T = constant$, we can replace $T \to b\lambda^{-1}$ for some constant b, in equation (1.5),

$$I = \frac{2hc^2T^5}{b^5}\frac{1}{e^{\frac{hc}{bk_B}}} \propto T^5 \tag{1.6}$$

From equation (1.1),

$$T_H = \frac{\hbar c^3}{8\pi GMk_B} \propto \frac{1}{M} \tag{1.7}$$

Then equations (1.6) and (1.7) imply $I \propto \frac{1}{M^5}$. But conservation of energy requires that the black hole's mass decreases by an amount ω as it radiates a Hawking photon. This can be accounted for with the replacement $M \to M - \omega$ into equation (1.7). Then we have

$$I_{thermal} \propto \frac{1}{M^5} < \frac{1}{(M-\omega)^5} \propto I_{non-thermal} \tag{1.8}$$

Equation (1.8) indicates that the true intensity of the Hawking radiation spectrum is larger than it would be if it had a perfectly thermal spectrum.

Chapter 2

Quantum mechanical preliminaries

2.0.1 The WKB approximation

The WKB approximation is used to find approximate general solutions to linear differential equations. The WKB approximation allows the step of solving the differential equation to be skipped. The WKB approximation can be applied to differential equations that have solutions with either constant, or slowly varying coefficients. The assumption of slowly varying coefficients in the present derivation is a result of truncating the action after a first order approximation. However, a much more instructive explanation can be found in the derivation by Griffiths in [10], where the approximation is made by dropping a second order derivative of the wavefunction's coefficient. This very straightforwardly explains why the coefficient is assumed to be slowly varying.

We can drop the time-dependence of the wavefunction because the spacelike contribution to the tunneling event occurs instantaneously. Therefore we can write the wavefunction as

$$\psi(x) = Ae^{iS(x)/\hbar} \tag{2.1}$$

where S is the classical action. The general plane wave solutions in equation (2.1) can be inserted into the time-independent Schrödinger equation (TISE),

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) \tag{2.2}$$

which describes the probability distribution of ψ in the presence of a potential energy distribution $V(x)$.

Solutions to equation (2.2) are of the form $\psi \sim e^{\frac{i}{\hbar}S(x)}$. Differentiating ψ with respect to position,

$$\psi' = \frac{i}{\hbar}S'\psi \tag{2.3a}$$

$$\psi'' = \left(\frac{i}{\hbar}S'' - \frac{1}{\hbar^2}S'^2\right)\psi \tag{2.3b}$$

and substituting its derivatives into the TISE (2.2) gives

$$-\frac{\hbar^2}{2m}\left(\frac{i}{\hbar}S'' - \frac{1}{\hbar^2}S'^2\right) - [E - V(x)] = 0 \tag{2.4}$$

where I've divided ψ out of the equation. The system's total energy $E = \frac{p^2}{2m} + V(x)$ can be rearranged to obtain $p^2 = 2m\,(E - V(x))$, which can then be substituted into equation (2.4) to give

$$i\hbar S'' - S'^2 - p^2 = 0 \tag{2.5}$$

From this point, the WKB approximation can be derived by Taylor expanding the classical action $S(x)$ in powers of $\hbar$, and then truncating the power series after linear order. This method of approximation is called "semiclassical" because the quantum mechanical effects are retained only to linear order in $\hbar$. It's worth pointing out that this is a very reasonable approximation: $\hbar \sim 10^{-34}$, $\hbar^2 \sim 10^{-64}$, $\hbar^3 \sim 10^{-102}, \ldots$ Taylor expanding the classical action in powers of $\hbar$ gives

$$S(x) = S_0(x) + S_1(x)\hbar + S_2(x)\hbar^2 + \ldots \tag{2.6}$$

Differentiating equation (2.6) and putting it into equation (2.5) gives

$$i\hbar\left(S_0'' + S_1''\hbar + S_2''\hbar^2 + \ldots\right) - \left(S_0' + S_1'\hbar + S_2'\hbar^2 + \ldots\right)^2 - p^2 = 0 \tag{2.7}$$

Collecting like-powers in $\hbar$ gives,

$$-\left(p^2 + {S_0'}^2\right) + \left(iS_0'' - 2S_0'S_1'\right)\hbar + \left(iS_1'' - {S_1'}^2 - 2S_0'S_2'\right)\hbar^2 + \ldots = 0 \tag{2.8}$$

Since the right-hand side of equation (2.8) is equal to zero, the coefficient of each power in $\hbar$ on the left-hand side must also be equal to zero. Taking the zeroth order term in equation (2.8) gives

$$p^2 = -{S_0'}^2 \implies S_0(x) = \pm\int_{x_0}^{x} p(x)dx \tag{2.9}$$

The first order term in equation (2.8) gives

$$\frac{i}{2}S_0'' = S_0'S_1' \tag{2.10a}$$

$$\frac{i}{2}p' = pS_1' \quad \text{(using the zeroth order equation } S_0 = \pm p\text{)} \tag{2.10b}$$

$$\frac{i}{2}\int\frac{dp}{p} = \int dS_1 \tag{2.10c}$$

$$S_1(x) = \frac{i}{2}\ln|p| \tag{2.10d}$$

Equations (2.9) and (2.10a) can be used to find a first order semiclassical approximation for ψ,

$$\begin{aligned}\psi(x) &= \exp\left[\tfrac{i}{\hbar}S(x)\right]\\ &= \exp\left[\tfrac{i}{\hbar}\left(S_0(x)+S_1(x)\hbar+S_2(x)\hbar^2+...\right)\right]\\ &\approx \exp\left[\tfrac{i}{\hbar}\left(S_0(x)+S_1(x)\hbar\right)\right]\\ &= \exp\left[\pm\frac{i}{\hbar}\int_{x_0}^{x}p(x)dx-\frac{1}{2}\ln|p|\right] \end{aligned} \tag{2.11}$$

Which, after taking the second term as a coefficient, gives approximate general solutions to the time-independent Schrödinger equation

$$\psi(x)\approx\frac{1}{\sqrt{|p(x)|}}\left[C_+e^{+\frac{i}{\hbar}\int_{x_0}^{x}p(x)dx}+C_-e^{-\frac{i}{\hbar}\int_{x_0}^{x}p(x)dx}\right],\quad p(x)\equiv\sqrt{2m(E-V(x))} \tag{2.12}$$

where I've restored the constant coefficients $C_\pm$ in the probability amplitudes. Equation (2.12) is the WKB approximation for $\psi(x)$.

An important note is that some derivations of the WKB approximation, such as [10, 11], take the integrand in the exponent of equation (2.12) to be an absolute value (i.e. $\int|p(x)|dx$ instead of $\int p(x)dx$). This can be done to simplify the equation for the probability of transmission of the tunneling particle from $T\sim\exp\left(-2\text{Im }S\right)$ to $T\sim\exp\left(-2S\right)$.

WKB approximation at the event horizon

General plane wave solutions to the TISE equation, with slowly varying coefficients, can be approximated as

$$\psi(x)\approx\frac{C_\pm}{\sqrt{|p(x)|}}e^{\pm\frac{i}{\hbar}\int_{x_0}^{x}p(x)dx},\quad p(x)\equiv\sqrt{2m(E-V(x))} \tag{2.13}$$

where m, E, and $p(x)$ are the mass, total energy, and classical momentum of the tunneling particle. $V(x)$ is the potential barrier that the particle must tunnel through. For the case of a particle tunneling across the event horizon, $V(x)$ can be thought of as being associated with the gravitational potential energy barrier that the particle must overcome in order to escape to future lightlike infinity. As was assumed in the original tunneling model of Hawking radiation [2, 12], the particle must tunnel against an energy barrier that is determined by the particle's own total energy. Note that the momentum $p(x)$ in equation (2.13) implies that both energy and momentum are conserved if $V(x)$ is made to be equal to the particle's own self-energy.

The integral in the exponent of $\psi(x)$, from equation (2.13), is equal to the classical action

$$S(x)=\int_{x_i}^{x}p(x')dx' \tag{2.14}$$

$S(x)$ is real-valued when the particle is in a classically allowed region, because $E > V(x)$ implies $p(x)$ is real. $S(x)$ is imaginary when the particle is in a region with $V(x) > E$, since $V(x) > E$ implies $p(x)$ is imaginary.

If the particle is massless then $p(x)$ must be expressed in a way that does not explicitly assume the particle to have a mass m. This can be done by recalling that quantum mechanics has a Hamiltonian formalism. This means that $p(x)$ can be viewed under the more general context as the particle's canonical momentum

$$p(x) = \frac{\partial \mathscr{L}}{\partial \dot{x}} \tag{2.15}$$

where $\mathscr{L}$ is the Lagrangian. Equation (2.15) does not make any assumptions about the particle's mass. A particle tunneling across the event horizon is assumed to travel along the radial coordinate axis. Indicating this with the replacement $x' \to r$ into equation (2.14) and relabeling the momentum to be interpreted as a component of a momentum 4-vector,

$$S = \int_{r_i}^{r_f} p_r dr \tag{2.16}$$

Equation (2.16) can be used as a starting point for calculating the tunneling rates of Hawking radiation.

Probability current

The total probability of any set of possible outcomes is always equal to one. This is a very simple conservation law can be modified to give a conservation law that is useful when applied to the wave function, whose value has a statistical interpretation. The probability density $\rho =$ for a wavefunction Ψ is given by $\rho = \Psi^*\Psi$. The time rate of change of probability density is called the probability current.

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\Psi^*\Psi) = \dot{\Psi}^*\Psi + \Psi^*\dot{\Psi} \tag{2.17}$$

where the dot denotes a time derivative. The Schrödinger equation for Ψ and its complex conjugate are respectively given by

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial \Psi}{\partial t} \tag{2.18a}$$

$$-\frac{\hbar^2}{2m}\nabla^2\Psi^* + V\Psi^* = -i\hbar\frac{\partial \Psi^*}{\partial t} \tag{2.18b}$$

Division of the Schrödinger equation and its complex conjugate by $\pm i\hbar$ gives the corresponding time derivatives for Ψ and Ψ^*

$$\frac{\partial \Psi}{\partial t} = i\frac{\hbar}{2m}\nabla^2\Psi - \frac{i}{\hbar}V\Psi \tag{2.19a}$$

$$\frac{\partial \Psi^*}{\partial t} = -i\frac{\hbar}{2m}\nabla^2\Psi^* + \frac{i}{\hbar}V\Psi^* \tag{2.19b}$$

Replacing the time derivatives of Ψ and Ψ^* in equation (2.17) with equations (2.19a) and (2.19b) gives

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= -i\frac{\hbar}{2m}(\nabla^2\Psi^*)\Psi + \frac{i}{\hbar}V\Psi^*\Psi + i\frac{\hbar}{2m}\Psi^*\nabla^2\Psi - \frac{i}{\hbar}V\Psi^*\Psi \\ &= \frac{i\hbar}{2m}\left(\Psi^*\nabla^2\Psi - \nabla^2\Psi^*\Psi\right) \\ &= \frac{i\hbar}{2m}\left([\nabla\cdot(\Psi^*\nabla\Psi) - \nabla\Psi^*\cdot\nabla\Psi] - [\nabla\cdot(\Psi\nabla\Psi^*) - \nabla\Psi\cdot\nabla\Psi^*]\right) \\ &= \frac{i\hbar}{2m}\left[\nabla\cdot(\Psi^*\nabla\Psi) - \nabla\cdot(\Psi\nabla\Psi^*)\right] \\ &= \frac{i\hbar}{2m}\nabla\cdot(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) \end{aligned} \tag{2.20}$$

The continuity equation

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\boldsymbol{J} \tag{2.21}$$

expresses the local conservation of total probability, under the assumption that probability density translates through space continuously as time evolves. If the density of a conserved quantity decreases in a fixed volume of space, then the assumption of continuity implies that a conserved current $\boldsymbol{J}$ has to have flowed across the boundary of the small volume element, where $-\nabla\cdot\boldsymbol{J}$ is equal to the total amount of probability density ρ that has flowed out of the volume element, and thus, across the volume element's boundary, during a small time interval dt.

From equations (2.20) and (2.21) we have

$$\boldsymbol{J} = \frac{i\hbar}{2m}\left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right) \tag{2.22}$$

Plane wave solutions are of the form $\Psi = Ae^{i(kx-\omega t)}$, and give a probability current with magnitude

$$|\boldsymbol{J}| = \frac{\hbar}{2m}\left[\Psi(-ik\Psi^*) - \Psi^*(ik\Psi)\right] \tag{2.23}$$

$$= \frac{\hbar k}{m}|\Psi|^2 \tag{2.24}$$

Unit analysis shows the coefficient $\frac{\hbar k}{m}$ to have units of length divided by time, which is consistent with the fact that a current is often written as a density times a velocity. Then we might think of $\frac{\hbar k}{m}$ as a velocity or analog thereof. With $v = \frac{\hbar k}{m}$, equation (2.22) can be written as

$$\boldsymbol{J} = |\Psi|^2\boldsymbol{v} \tag{2.25}$$

where I've replaced the bold vector notation to be consistent with equation (2.22). This generalization to three-component vector form is justified by the fact that the laws of physics are invariant under rotation.

Probability of transmission

Consider a particle with a wavefunction ψ that a known and nonzero probability amplitude in two classically allowed regions, region I and region III. The probability amplitudes correspond to the coefficients in equations (2.26a) and (2.26c). The classically allowed regions are separated by region II, which is classically forbidden and has an approximated probability amplitude shown in equation (2.26b) . Using general solutions of the WKB approximation from equation (2.13), we can approximate an expression for the wave function inside of the classically forbidden region

$$\psi_I(x) = C_+^I e^{+ikx} + C_-^I e^{-ikx} \tag{2.26a}$$

$$\psi_{II}(x) \approx \frac{C_-^{II}}{\sqrt{|p(x)|}} e^{-\frac{i}{\hbar}\int_{x_0}^{x} p(x)dx} \tag{2.26b}$$

$$\psi_{III}(x) = C_+^{III} e^{+ikx} + C_-^{III} e^{-ikx} \tag{2.26c}$$

where equations (2.26a), (2.26b), and (2.26c) correspond to the wavefunction in region I, II, and III, respectively. The coefficients C_+ and C_- represent the probability amplitude for a wave traveling in the positive and negative x-directions, respectively. Now arbitrarily suppose that in order to go from region I to region III the particle must tunnel in the $+x$-direction. The transmission probability for this tunneling event is

$$T = \frac{|J_+^{III}|}{|J_+^I|} \tag{2.27}$$

where $J_+ I$ and J_+^{II} are the probability currents that correspond to incident and transmitted waves, respectfully. From equation (2.25), for a massless particle ($v = c$) the transmission probability can be written as

$$T = \frac{v_{III}|C_+^{III}|^2}{v_I|C_+^I|^2} = \frac{|C_+^{III}|^2}{|C_+^I|^2} \tag{2.28}$$

For the case in which the transmission probability is small $C_+^I \gg C_+^{III}$, $\frac{|C_+^{III}|}{|C_+^I|}$ is approximately equal to the total decrease in the wavefunction's decaying exponential in the classically forbidden region [10]

$$\frac{|C_+^{III}|}{|C_+^I|} \simeq e^{-\frac{1}{\hbar}\int_{x_0}^{x} |p(x)|dx} \tag{2.29}$$

Note that I've replaced $ip(x)$ with $|p(x)|$ in the exponential. The action is imaginary for the tunneling component of a particle's wavefunction, which effectively transforms the complex exponential into a decaying exponential. I've switched notation for the exponential in equation (2.29) to emphasize the fact that it is a decaying exponential. Equations (2.28) and (2.29) imply that the transmission probability is given by

$$T \simeq e^{-2\frac{i}{\hbar}\int_{x_0}^{x} p(x)dx} \tag{2.30}$$

Chapter 3

Quantum field theory in curved spacetime

3.1 The Klein-Gordon equation

The relativistic dispersion relation is

$$E^2 = p^2 + m^2 \tag{3.1}$$

Combining this with the Einstein and de Broglie relations, $E = \omega$ and $\boldsymbol{p} = \boldsymbol{k}$, from quantum mechanics gives a dispersion relation

$$\omega^2 = k^2 + m^2 \tag{3.2}$$

The associated wave equation operator needs to be able to reproduce equation (3.2) when acting on plane wave solutions $\phi(t, \boldsymbol{r}) = e^{i(\boldsymbol{r}\cdot\boldsymbol{k} - \omega t)}$. From inspection, we can get the right hand side of equation (3.2) by replacing $E \to i\frac{\partial}{\partial t}$ and the left hand side from $p \to -i\nabla$. The imaginary number is included in order to obtain the correct signs after differentiating ϕ. Equation (3.2) becomes

$$-\tfrac{\partial^2}{\partial t^2} = -\nabla^2 + m^2 \tag{3.3}$$

which can be rewritten with indices as

$$\partial_\mu \partial^\mu - m^2 = 0 \tag{3.4}$$

Acting on ϕ with equation (3.4) gives the Klein-Gordon equation [1]

$$\left(\partial_\mu \partial^\mu - m^2\right)\phi = 0 \tag{3.5}$$

An important feature of the Klein-Gordon equation is that it is Lorentz covariant, and is thus compatible with relativity. Lorentz covariance of an equation simply means that a

[1] The Klein-Gordon equation can be written with c and $\hbar$ with the replacement $m \to \frac{mc}{\hbar}$.

change of inertial coordinates $x^\mu \to x'^\mu$ leaves the form of the equation unchanged. To show that equation (3.5) is Lorentz covariant, let $\phi = \phi(x^\mu)$ and $\phi' = \phi(x'^\mu)$ denote the field operator in two distinct inertial reference frames. The coordinates frames are related by a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

$$\begin{aligned}
0 &= \left(\partial'_\mu \partial'^\mu - m^2\right)\phi' && (3.6a)\\
&= \left[\left(\Lambda_\mu{}^\alpha \partial_\alpha\right)\left(\Lambda^\mu{}_\beta \partial^\beta\right) - m^2\right]\phi\\
&= \left(\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta \partial_\alpha \partial^\beta - m^2\right)\phi\\
&= \left(\delta^\alpha_\beta \partial_\alpha \partial^\beta - m^2\right)\phi\\
&= \left(\partial_\beta \partial^\beta - m^2\right)\phi && (3.6b)
\end{aligned}$$

This shows that equation (3.6a) takes the same form as equation (3.6b) after undergoing a Lorentz transformation, implying that the Klein-Gordon equation is Lorentz covariant. Since Lorentz transformations are coordinate transformations between inertial reference frames, the covariance of equations (3.6a) and (3.6b) implies that the Klein-Gordon equation takes the same form in all inertial reference frames. This is important because it makes equation (3.5) compatible with relativistic applications. Note that even though $\phi = \phi(x^\mu)$, the field operator transforms as $\phi(x'^\mu) \to \phi(x^\mu)$ because ϕ is a scalar field.

Additionally, and especially important for Hawking radiation, is that equation (3.2) admits both positive and negative energy solutions,

$$\omega = \pm\sqrt{k^2 + m^2} \tag{3.7}$$

This is in contrast to the Schrödinger equation's dispersion relation $\omega = k^2$, which contains no sign ambiguity on its energy term ω. A derivation of the TISE explaining the reason for its dispersion relation $\omega = k^2$, is given in the appendix .3.

So unlike for non-relativistic physics, the relativistic dispersion relation predicts the existence of both positive and negative energy particles. Although it might be tempting to disregard the negative energy solutions as being "unphysical," this does not eliminate the theory's prediction of negative energy solutions [13, 14]. The theory predicts that processes involving the interaction of a Klein-Gordon field with another system, such as the gravitational field of a black hole for the case of Hawking radiation, will inevitably lead to the field operator's acquisition of negative energy modes [1, 15, 16].

3.2 Deriving the Hamilton-Jacobi equations for a general background metric

The Klein-Gordon equation can be generalized from a flat Minkowski spacetime background to a general background spacetime metric by the method of taking partial derivatives to covariant derivatives. After generalizing the background metric, we will be able to derive the Hamilton-Jacobi equations from the Klein-Gordon equation. From equation (3.5), the Klein-Gordon equation in a flat spacetime background is

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0 \tag{3.8}$$

Taking $\eta^{\mu\nu} \to g^{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - m^2\phi = 0 \tag{3.9}$$

where the covariant derivative ∇_μ can be thought of as a partial derivative with a Christoffel symbol $\nabla_\mu \sim \partial_\mu \pm \Gamma_\mu$, where the sign is $+/-$ when taking the covariant derivative of contravariant/covariant 4-vectors). The Christoffel symbol can intuitively be thought of as accounting for the changes in basis vectors between adjacent points in spacetime. These basis vector changes are caused by the geometry's intrinsic curvature.

The covariant derivative of a scalar function ϕ is equal to the function's partial derivative $\nabla_\mu\phi = \partial_\mu\phi$. This is the case because a scalar field has no coordinate dependence and therefore the Christoffel symbols all evaluate to zero. Equation (3.9) can therefore be written as

$$\begin{aligned} g^{\mu\nu}\nabla_\mu\partial_\nu\phi - m^2\phi &= g^{\mu\nu}\left(\partial_\mu\partial_\nu - \Gamma^\alpha_{\mu\nu}\partial_\alpha\right)\phi - m^2\phi \\ &= g^{\mu\nu}\partial_\mu\partial_\nu\phi - \Gamma^\alpha_{\mu\nu}\partial_\alpha\phi - m^2\phi \\ &= 0 \end{aligned} \tag{3.10}$$

which can be rewritten as [15]

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right) - m^2\phi = 0 \tag{3.11}$$

where g is the metric's determinant. Expanding the partial derivative with the product rule gives

$$\frac{1}{\sqrt{-g}}\left[\partial_\mu\left(g^{\mu\nu}\right)\sqrt{-g}\partial_\nu\phi + g^{\mu\nu}\partial_\mu\left(\sqrt{-g}\right)\partial_\nu\phi + g^{\mu\nu}\sqrt{-g}\partial_\mu\partial_\nu\phi\right] - m^2\phi = 0 \tag{3.12}$$

As a quick check, note that for the special case of a Minkowski spacetime $g_{\mu\nu} \to \eta_{\mu\nu}$, $\sqrt{-\eta} = 1$ and $\partial_\mu\eta^{\mu\nu} = 0$ implies that the first two terms in equation (3.12) evaluate to zero. The third term becomes $\eta^{\mu\nu}\partial_\mu\partial_\nu\phi$. This brings equation (3.12) to

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi = 0 \tag{3.13}$$

which is the Klein-Gordon in the presence of a flat spacetime background.

Going back to the case of a general spacetime that may contain curvature, equation (3.12) can be siimplified by using the fact that the covariant derivative of a metric tensor is always zero. This follows from 1) the fact that any curved surface is locally flat over a small enough region and 2) if a tensor equation is true in one coordinate frame, then it must be true in all coordinate frames.

Since an observer's local metric is Minkowski, there exists a coordinate frame in which the observer's metric tensor is that of a locally inertial frame $g^{\mu\nu} = \eta^{\mu\nu}$. A transformation into the coordinate frame of a locally inertial observer gives

$$\nabla_\sigma g^{\mu\nu} \to \nabla_\sigma \eta^{\mu\nu} = \partial_\sigma \eta^{\mu\nu} = 0 \tag{3.14}$$

Since the covariant derivative of the metric tensor is zero in the coordinate frame of a locally inertial observer, the tensor formulation implies that it must be true in all coordinate frames. Therefore,

$$\nabla_\sigma g^{\mu\nu} = \partial_\sigma g^{\mu\nu} + \Gamma^\mu_{\sigma\tau} g^{\tau\nu} + \Gamma^\nu_{\sigma\tau} g^{\mu\tau} = 0 \tag{3.15}$$

If we now restrict our attention to spacetimes with diagonal metrics, then $\Gamma^\mu_{\sigma\tau} = 0$ for all $\mu \neq \sigma \neq \tau$ [8]. Assuming a diagonal metric and relabeling $\sigma \to \mu$ in equation (3.15),

$$\begin{aligned}\nabla_\sigma g^{\mu\nu} &= \nabla_\mu g^{\mu\nu} \\ &= \partial_\mu g^{\mu\nu} + \Gamma^\mu_{\mu\tau} g^{\tau\nu} + \Gamma^\nu_{\mu\tau} g^{\mu\tau} = 0 \\ &= 0\end{aligned} \tag{3.16}$$

For a diagonal metric it can be shown that [8]

$$\begin{aligned}\Gamma^\mu_{\mu\alpha} &= \partial_\alpha \ln\left(\sqrt{-g}\right) \\ &= \frac{\partial_\alpha \sqrt{-g}}{\sqrt{-g}}\end{aligned} \tag{3.17}$$

Putting this into equation (3.16) we have

$$\nabla_\sigma g^{\mu\nu} = \partial_\sigma g^{\mu\mu} + \frac{\partial_\sigma \sqrt{-g}}{\sqrt{-g}} + \Gamma^\nu_{\mu\tau} g^{\mu\tau} = 0 \tag{3.18}$$

Which can be rearranged to obtain

$$\begin{aligned}\partial_\mu g^{\mu\nu} &= -\Gamma^\mu_{\mu\gamma} g^{\gamma\nu} - \Gamma^\nu_{\mu\tau} g^{\mu\tau} \\ &= -\Gamma^\mu_{\mu\gamma} g^{\gamma\nu} \qquad (3.19a) \\ &= -\frac{\partial_\sigma \sqrt{-g}}{\sqrt{-g}} g^{\sigma\nu} \qquad (3.19b)\end{aligned}$$

where $\Gamma^\nu_{\mu\tau} g^{\mu\tau} = 0$ in equation (3.19a) due to the harmonic coordinate condition [8, 17], and equation (3.19b) is an identity given by [8].

Relabeling the index $\sigma \to \mu$ on the right hand side of equation (3.19b) and making the substitution $\partial_\mu g^{\mu\nu} \to -\frac{\partial_\mu \sqrt{-g}}{\sqrt{-g}} g^{\mu\nu}$ into equation (3.12) gives

$$\begin{aligned}
0 &= \frac{1}{\sqrt{-g}}\left[-\left(\frac{\partial_\mu\left(\sqrt{-g}\right)}{\sqrt{-g}} g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right) + g^{\mu\nu}\partial_\mu\left(\sqrt{-g}\right)\partial_\nu\phi + g^{\mu\nu}\sqrt{-g}\partial_\mu\partial_\nu\phi\right] - m^2\phi \\
&= \frac{1}{\sqrt{-g}}\left[-\left(g^{\mu\nu}\partial_\mu\left(\sqrt{-g}\right)\partial_\nu\phi\right) + g^{\mu\nu}\partial_\mu\left(\sqrt{-g}\right)\partial_\nu\phi + g^{\mu\nu}\sqrt{-g}\partial_\mu\partial_\nu\phi\right] - m^2\phi \\
&= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\partial_\nu\phi\right) - m^2\phi \\
&= g^{\mu\nu}\partial_\mu\partial_\nu\phi - m^2\phi
\end{aligned} \tag{3.20}$$

The Klein-Gordon equation has plane wave solutions of the form

$$\phi \sim e^{iS(x^\mu)/\hbar} \tag{3.21}$$

where I've put $\hbar$ back in because we're going to use it to make a semiclassical approximation. Putting these solutions into equation (3.20) gives,

$$\begin{aligned}
0 = &= g^{\mu\nu}\partial_\mu\left(\partial_\nu e^{iS/\hbar}\right) - m^2 e^{iS/\hbar} \\
&= g^{\mu\nu}\partial_\mu\left[\partial_\nu\left(\frac{i}{\hbar}S\right)e^{iS\hbar}\right] - m^2 e^{iS/\hbar} \\
&= \frac{i}{\hbar}g^{\mu\nu}\partial_\mu\left[\partial_\nu\left(S\right)\right]e^{iS/\hbar} + \partial_\mu\left(\frac{i}{\hbar}S\right)\partial\nu\left(\frac{i}{\hbar}S\right)e^{iS/\hbar} - \left(\frac{mc}{\hbar}\right)^2 e^{iS/\hbar}
\end{aligned} \tag{3.22}$$

where I've replaced $m \to \frac{mc}{\hbar}$ in the last line. The second line in the three lines above comes from the chain rule, and the third line comes from using the product rule. Multiplying both sides of equation (3.22) by $\hbar^2$ and dividing out $e^{iS/\hbar}$ gives

$$i\hbar g^{\mu\nu}\partial_\mu\partial_\nu S - g^{\mu\nu}\partial_\mu S\partial\nu S - (mc)^2 = 0 \tag{3.23}$$

The semiclassical approximation is made by taking the limit $\hbar \to 0$, giving

$$g^{\mu\nu}\partial_\mu S\partial_\nu S + m^2 = 0 \tag{3.24}$$

where I've set $c = 1$ and multiplied both sides by -1. The equations in (3.24) are the Hamilton-Jacobi equations for a background metric that has been generalized to curved spacetimes whose metric is diagonal.

3.2.1 Virtual particles

The vacuum of empty space is not truly empty due to the presence of a residual minimum energy that is required by Heisenberg's uncertainty principle [14, 18]. This energy, called zero-point energy, can take the form of virtual particle pairs, which consists of a positive energy particle and a negative energy antiparticle. A virtual particle might be viewed as a not-fully-developed version of the real "eigen-particle" that it represents. For instance, a virtual particle does not lie on its own mass shell. This means that an electron and positron in a virtual particle pair do not have the actual mass of an electron [19]. As time evolves however, the properties of a virtual particle become more like the properties of the real particle that it "represents."

A result from QED called the Schwinger mechanism predicts that virtual particles can be caused to materialize from a vacuum state that is filled with a strong electric field [20]. Increasing the electric field strength increases the particle's acceleration through the relation

$$\boldsymbol{F} = m\boldsymbol{a} = q\boldsymbol{E} \implies \boldsymbol{a} = \frac{q\boldsymbol{E}}{m} \tag{3.25}$$

The acceleration of each particle bestows it with a Rindler horizon (section 4.3). If the acceleration is large enough the two particles will fall behind each other's future horizons and become mutually causally disconnected. This causal disconnection prevents the virtual particle pair from annihilating, and forces them to materialize into real particles.

Several calculations that make use of the Schwinger mechanism have been used in application to Hawking radiation [21, 22]. The Schwinger mechanism is relevant to Hawking radiation because both processes involve the materialization of virtual particles into real particles as a result of the virtual particles losing causal connection with one another, due to the formation of a particle horizon. This means by which particle horizons are formed will be discussed in section 4.3. In the case of Hawking radiation, one might view the strong electric field as being replaced with a strong gravitational field. This is the source from which the virtual particles are able to extract their energy.

During an event of Hawking radiation emission by a black hole, one of the two virtual particles tunnels across the black hole's event horizon. This causes the two virtual particles to become causally connected and allows them to materialize into real particles. This is an important part of the phenomenology of Hawking radiation and will be discussed in more detail in section 9.1.

Chapter 4

Uniformly accelerating observers and Unruh radiation

The Unruh effect is a prediction of quantum field theory that an observer undergoing uniform acceleration through the vacuum state of a flat spacetime will see a quantum flux of thermal radiation [16, 23–25]. This is in contrast to the observations of an inertial observer in the same flat spacetime vacuum state, who sees no thermal radiation. The Unruh effect has traditionally been studied using the particle interpretation of quantum field theory. However, as recently as 2007 a method was used for calculating the tunneling rates of virtual particles across a uniformly accelerating observer's Rindler horizon [7]. A subsequent approach was able to show that by equating the the particle's transmission probability to a Boltzmann factor, the gravitational WKB approxmation predicts the same temperature as the Unruh effect [8].

The Unruh temperature T_U is the characteristic temperature of the thermal flux of radiation seen by a uniformly accelerating observer in flatspace, and is defined to be

$$T_U = \frac{a}{2\pi} \tag{4.1}$$

where a is the observer's proper acceleration.[1] Interestingly, replacing the observer's proper acceleration with the surface gravity κ at a black hole's event horizon, does not change the form of equation (4.1). Surface gravity can be thought of as the proper acceleration of an observer at the event horizon, after being normalized with a gravitational redshift factor. Replacing $a \to \kappa$ in equation (4.1) gives

$$T_H = \frac{\kappa}{2\pi} \tag{4.2}$$

For this reason, as well as a number of other reasons that are discussed in chapter 9, Unruh radiation is as relevant to Hawking radiation as acceleration is to gravity. Surface gravity is discussed in greater detail in section 6.2.

[1]The Unruh temperature is equal to $T_U = \dfrac{\hbar c^3 a}{2\pi k_B}$ with fundamental constants replaced.

4.1 Constant proper acceleration

Proper acceleration is the acceleration that an object perceives itself to have in its own rest frame. The magnitude (squared) of an observer's proper acceleration is the Lorentz scalar corresponding to the observer's acceleration 4-vector a^μ,

$$a = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_\mu a^\mu} \tag{4.3}$$

In contrast to proper acceleration is the concept of coordinate acceleration. Coordinate acceleration is the acceleration of an object as observed from a chosen and fixed inertial reference frame. The concept of an observer's proper acceleration is essential to constructing the metric for a uniformly accelerating observer. This is because the coordinate acceleration of a uniformly accelerating observer, relative to some chosen and fixed inertial coordinate frame, cannot remain constant indefinitely. Eventually the inertial observer would perceive the uniformly accelerating observer to have a velocity that is faster than the speed of light. However, an accelerating observer *can* perceive his own proper acceleration to be indefinitely uniform, without perceiving himself to ever be traveling faster than the speed of light. This is because the observer perceives himself to be at rest in his one local coordinate frame. It is through this comoving frame, that we are able to relate the accelerating observer's proper acceleration to the coordinate acceleration that is measured by an inertial observer. An observer's proper acceleration is equal to that observer's coordinate acceleration, as it would be measured in an instantaneously comoving inertial reference frame.

The concept of an instantaneously comoving inertial reference frame can be understood as follows: a uniformly accelerating observer has a noninertial coordinate frame. However, at any given instant of time there exists an inertial frame, at rest relative to the accelerating observer's coordinate frame, that can be defined such that its coordinate axes exactly coincide with the axes of the accelerating observer's coordinate frame. The comoving frame is made to be at rest relative to the uniformly accelerating observer by defining it to have a velocity boost (relative to the origin of the global Minkowski coordinate frame) that gives it the same velocity as the uniformly accelerating observer.

There exists a unique comoving frame of this description at each instant of proper time along the uniformly accelerating observer's trajectory. In this sense, an accelerating observer can be thought of as jumping from one inertial frame to the next, as proper time progresses along the observer's worldline. With this picture in mind, we are able to find a proper-time-dependent relation between the coordinates of the uniformly accelerating observer and global Minkowski coordinates. This is done by relating the global Minkowski coordinates to the coordinate frame of an inertial observer who is instantaneously comoving with the uniformly accelerating observer.

The coordinate acceleration measured by an instantaneously comoving inertial observer is defined to be the spatial component of the uniformly accelerating observer's 4-acceleration $a^\mu = (0, a^i)$. From an intuitive perspective, the 4-acceleration's time component is zero because the measurement is defined to take place instantaneously, on a hypersurface of constant time. Mathematically, the velocity vector $\boldsymbol{u} = \dot{\boldsymbol{x}}(\tau)$ and acceleration vector $\boldsymbol{a} =$

$\ddot{\boldsymbol{x}}(\tau)$ are always orthogonal along a parameterized curve $\boldsymbol{x}(\tau)$.[2].

The fact that $a^\mu = (0, a^i)$ must be the case, can be shown by considering two necessary conditions: $u_\mu a^\mu = 0$, and $u^\mu = (1, 0, 0, 0)$ in an observer's local coordinate frame. First,

$$0 = \frac{d}{d\tau}(-1) = \frac{d}{d\tau}(u_\mu u^\mu) = \frac{d}{d\tau}\left(\eta_{\mu\nu} u^\nu u^\mu\right) = \eta_{\mu\nu}\frac{du^\nu}{d\tau}u^\mu + \eta_{\mu\nu}u^\nu\frac{du^\mu}{d\tau} \tag{4.4}$$

where the absence of a third term from the product rule is caused by $\frac{d}{d\tau}\eta_{\mu\nu} = 0$. This is the case because the Minkowski metric only contains constants. Switching index labels $\mu \leftrightarrow \nu$ on the first term gives

$$0 = 2\eta_{\mu\nu}u^\nu\frac{du^\mu}{d\tau} = 2u_\mu a^\mu \tag{4.5}$$

so $u_\mu a^\mu = 0$.

Second, given that $u^\mu = (1, 0, 0, 0)$ in an observer's local coordinate frame, any nonzero component of a^μ must be entirely contained in its spatial component. Otherwise the time-component $u_t a^t = \eta_{tt} u^t a^t = -1 \times a^t$ would contribute a non-zero value to $u_\mu a^\mu$, while making the spatial contribution $u_i a^i = 0 \times a^i = 0$. This would result in $u_\mu a^\mu = a^t + 0 \neq 0$. Therefore we must have $a^\mu = (0, a^i)$.

4.2 Deriving the metric for a uniformly accelerating observer

The spacetime that describes a Minkowski spacetime, as seen from the perspective of a uniformly accelerating observer, is called the Rindler spacetime. In this section we will derive the Rindler spacetime metric. We can construct the Rindler metric by considering a proper-time-dependent Lorentz transformation ${\Lambda^{\mu'}}_\nu = {\Lambda^{\mu'}}_\nu(\tau)$ from the frame of a Minkowski observer to the coordinate frame of an observer who is instantaneously comoving with a uniformly accelerating observer.

Let a' denote the accelerating observer's proper acceleration. u' represents the 3-velocity of the uniformly accelerating observer with respect to the instantaneously comoving inertial coordinate frame. Note that although u' is measured to be zero by the instantaneously comoving observer, its derivative is not zero. Otherwise the uniformly accelerating observer wouldn't be accelerating. Let v denote the relative coordinate velocity of the comoving observer relative to a Minkowski observer. $\gamma = \frac{1}{\sqrt{1-v^2}}$ is the Lorentz factor.

Differentiating the uniformly accelerating observer's 3-velocity, as measured in the co-moving frame, with respect to the Minkowski time coordinate gives

[2]In differential geometry, the velocity and acceleration vectors along a parameterized curve are specifically defined to be first and second order derivatives of the parameterized curve with respect to its parameter. Although the curve's parameter is often taken to bear a time-associated interpretation, it does not have to.

$$\begin{aligned} a' &= \frac{du'}{dt} \\ &= \frac{d(\gamma v)}{dt} \\ &= \frac{d\gamma}{dt} v + \gamma \frac{dv}{dt} \end{aligned} \tag{4.6}$$

From this we see that proper acceleration a' can be related to Minkowski coordinates through Lorentz transformations between the Minkowski observer and the comoving frame. Differentiating γ gives,

$$\begin{aligned} \frac{d\gamma}{dt} &= \frac{d}{dt} \left(1 - v^2\right)^{-1/2} \\ &= \left(1 - v^2\right)^{-3/2} va \\ &= \gamma^3 va \end{aligned} \tag{4.7}$$

where the unprimed $a = \frac{dv}{dt}$ is the coordinate acceleration of the comoving frame relative to the Minkowski observer. Putting equation (4.7) into equation (4.6) gives

$$\begin{aligned} a' &= \left(\gamma^3 va\right) v + \gamma \frac{dv}{dt} \\ &= \gamma a \left(v^2 \gamma^2 + 1\right) \\ &= \gamma a \left(\frac{v^2}{1 - v^2} + 1\right) \\ &= \gamma a \frac{v^2 + 1 - v^2}{1 - v^2} \\ &= \gamma^3 a \\ a' &= \gamma^3 \frac{dv}{dt} \end{aligned} \tag{4.8}$$

Multiplying both sides by the Minkowski time coordinate differential dt and integrating,

$$\begin{aligned} a't &= a' \int_0^t d\widetilde{t} \\ &= \int_0^v \gamma(\widetilde{v})^3 d\widetilde{v} \\ &= \int_0^v \frac{d\widetilde{v}}{\left(1 - \widetilde{v}^2\right)^{3/2}} \end{aligned} \tag{4.9}$$

where the tilde indicates the dummy variable of integration.[3] Let $\widetilde{v} = \sin\theta$, which gives $d\widetilde{v} = \cos\theta d\theta$. The integral can be rewritten as

$$\begin{aligned} a't &= \int \frac{\cos\theta d\theta}{\left(1-\sin^2\theta\right)^{3/2}} \\ &= \int \sec^2\theta d\theta \\ &= \int \frac{d}{d\theta}\tan\theta d\theta \\ &= \tan\theta + C \\ &= \left.\frac{\widetilde{v}}{\sqrt{1-\widetilde{v}^2}}\right|_{\widetilde{v}=0}^{v} \\ a't &= \frac{v}{\sqrt{1-v^2}} \end{aligned} \tag{4.10}$$

Equation (4.10) can be solved for v without much effort to give

$$v = \frac{a't}{\sqrt{1+(a't)^2}} \tag{4.11}$$

Using the relation between Minkowski coordinate time and proper time of the comoving frame (which coincides with that the proper time of the frame with uniform proper acceleration) $t = \gamma\tau$,

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2}} \tag{4.12}$$

Substituting equation (4.11) into equation (4.12) for v gives

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{1}{\sqrt{1-\left(\frac{a't}{\sqrt{1+(a't)^2}}\right)^2}} \\ &= \frac{1}{\sqrt{1+(a't)^2}} \end{aligned} \tag{4.13}$$

Multiplying both sides of equation (4.13) by $a'd\tau$ and integrating with respect to time,

[3]Recall that the velocity parameter v' is normalized to a maximum value of $v' = 1$ by defining the speed of light to be $c = 1$. Otherwise the substitution would have to be $\widetilde{v} = v_0 \sin\theta$.

$$
\begin{aligned}
a'\tau &= a'\int_0^{\tau} d\widetilde{\tau} \\
&= \int_0^{t} \frac{a' d\widetilde{t}}{\sqrt{1+(a'\widetilde{t})^2}} \\
&= \int_0^{a't} \frac{d\widetilde{(a't)}}{\sqrt{1+\widetilde{(a't)}^2}}
\end{aligned}
\tag{4.14}
$$

Using $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$ this can be written as

$$
\begin{aligned}
a'\tau &= \int_0^{a't} \frac{d}{d\widetilde{(a't)}} \sinh^{-1} \widetilde{(a't)} d\widetilde{(a't)} \\
&= \sinh^{-1}(a't)
\end{aligned}
\tag{4.15}
$$

Taking the hyperbolic sine of both sides and then dividing both sides by a' gives

$$
t = \frac{1}{a'} \sinh(a'\tau) \tag{4.16}
$$

We can now find a relation $x = x(\tau, a')$ that gives the spatial coordinate of Minkowski space from the perspective of a uniformly accelerating observer. Equation (4.11) can be written in terms of differentials

$$
v = \frac{dx}{dt} = \frac{a't}{\sqrt{1+(a't)^2}} \tag{4.17}
$$

Which can be put into integral form,

$$
\begin{aligned}
x &= \int_0^{x} d\widetilde{x} \\
&= \int_0^{t} \frac{a'tdt}{\sqrt{1+(a't)^2}}
\end{aligned}
\tag{4.18}
$$

where the second step comes from multiplying both sides of equation (4.17) by dt and using it as a substitution for $d\widetilde{x}$. Using the change of variables $u = 1 + (a't)^2$, which implies $\frac{1}{2a'} du = a'tdt$

$$\begin{aligned}
x &= \frac{1}{2a'}\int u^{-1/2}du \\
&= \frac{1}{a'}u^{1/2} + C \\
&= \frac{1}{a'}\sqrt{1+(a'\tilde{t})^2}\Big|_{\tilde{t}'=0}^{t} \\
&= \frac{1}{a'}\sqrt{1+(a't)^2} - \frac{1}{a'} \qquad (4.19)
\end{aligned}$$

Substituting equation (4.16) in for t gives

$$\begin{aligned}
x &= \frac{1}{a'}\sqrt{1+a'^2\left(\frac{1}{a'}\right)^2\sinh^2(a'\tau)} - \frac{1}{a'} \\
&= \frac{1}{a'}\sqrt{1+\left(\cosh^2(a'\tau)-1\right)} - \frac{1}{a'} \\
&= \frac{1}{a'}\sqrt{\cosh^2(a'\tau)} - \frac{1}{a'} \\
&= \frac{1}{a'}\cosh(a'\tau) - \frac{1}{a'} \qquad (4.20)
\end{aligned}$$

The symmetries of special relativity include coordinate translations, spatial rotations, and velocity boosts along each of the three spatial axes. This means that we can translate our coordinate frame along the x-axis to obtain a new x coordinate

$$x = \frac{1}{a'}\cosh(a'\tau) \qquad (4.21)$$

From this point on I'll unprime the proper acceleration $a' \to a$ because the ambiguity of proper and coordinate accelerations only occurred at the beginning of the derivation. From equations (4.16) and (4.21), the coordinates of Minkowski space expressed in terms a uniformly accelerating observer's proper time are

$$t = x^0 = \frac{1}{a}\sinh(a\tau) \qquad (4.22a)$$

$$x = x^1 = \frac{1}{a}\cosh(a\tau) \qquad (4.22b)$$

$$x^2 = 0 \qquad (4.22c)$$

$$x^3 = 0 \qquad (4.22d)$$

where I've indicated the relation to position 4-vector components for the upcoming generalization to include a term for arbitrary local spacelike coordinate choice within the comoving observer's local frame.

Equation (4.22a) and (4.22b) were derived for the special case of the origin of the local tangent frame. These equations can now be generalized to include the local tangent space sptatial coordinate, along the direction of acceleration.

4.2.1 Generalizing to include arbitrary local spatial position

Let $\boldsymbol{e}_{\mu'} = \boldsymbol{e}_{\mu'}(\tau)$ denote the set of unit basis vectors of the comoving frame's local tangent space. Primed coordinates correspond to the local coordinates of the comoving frame and unprimed coordinates are global Minkowski coordinates. If we assume that the comoving observer is always in his own rest frame, then the comoving observer's local time axis will coincide with the comoving observer's 4-velocity vector $\boldsymbol{e}_{0'} = \boldsymbol{u}$. Additionally, we can align $\boldsymbol{e}_{1'}$ with the direction of the observer's acceleration vector, along the Minkowski observer's x-axis. This implies $\boldsymbol{e}_{1'} = a^{-1}\boldsymbol{a}$, where the coefficient a^{-1} is used to normalize $\boldsymbol{e}_{1'}$ to unity. The additional specification of $\boldsymbol{e}_{2'} = \boldsymbol{e}_2$ and $\boldsymbol{e}_{3'} = \boldsymbol{e}_3$ disallows spatial rotations of the basis vectors relative to the global Minkowski space.

At any given instant of proper time along the Rindler observer's worldline, the basis vectors of the instantaneously comoving inertial frame must be related to the basis vectors of the global Minkowski spacetime coordinates by a Loretnz transformation,

$$\boldsymbol{e}_{\mu'}(\tau) = \Lambda^{\nu}{}_{\mu'}(\tau)\boldsymbol{e}_{\nu} \tag{4.23}$$

The Lorentz transformation

$$\Lambda^{\nu}{}_{\mu'} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.24}$$

corresponds to the proper-time-dependent velocity boost $v(\tau)$ along the Minkowski observer's $\boldsymbol{e}_1$ direction. The velocity boost is defined to be the exact amount that is necessary to bring the instantaneously comoving inertial frame up to the same velocity as the uniformly accelerating observer [26]. Such a Lorentz transformation can be re-expressed as [18, 26, 27]

$$\Lambda^{\nu}{}_{\mu'}(\tau) = \begin{pmatrix} \cosh(a\tau) & -\sinh(a\tau) & 0 & 0 \\ -\sinh(a\tau) & \cosh(a\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.25}$$

with the relation $v = \tanh^{-1}(a\tau)$ [28].

The local tangent space basis vectors can by found using the inverse $\Lambda_{\nu}{}^{\mu'}$ of the Lorentz transformation $\Lambda^{\nu}{}_{\mu'}$ in equation (4.25),

$$[\boldsymbol{e}_{0'}(\tau)]^{\mu} = \Lambda_{\nu}{}^{\mu}(\tau)\,[\boldsymbol{e}_0]^{\nu} = \begin{pmatrix} \cosh(a\tau) & \sinh(a\tau) & 0 & 0 \\ \sinh(a\tau) & \cosh(a\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh(a\tau) \\ \sinh(a\tau) \\ 0 \\ 0 \end{pmatrix} \tag{4.26}$$

where the correct proper time dependences have been explicitly indicated. Repeating this operation for $[\boldsymbol{e}_{i'}(\tau)]^{\mu}$, $i \in \{1, 2, 3\}$ gives us all four tangent space basis vectors, expressed in the global Minkowski coordinates. We find,

$$[\boldsymbol{e}_{0'}]^{\mu} = \begin{pmatrix} \cosh(a\tau) \\ \sinh(a\tau) \\ 0 \\ 0 \end{pmatrix}, \quad [\boldsymbol{e}_{1'}]^{\mu} = \begin{pmatrix} \sinh(a\tau) \\ \cosh(a\tau) \\ 0 \\ 0 \end{pmatrix}, \quad [\boldsymbol{e}_{2'}]^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad [\boldsymbol{e}_{3'}]^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.27}$$

These basis 4-vectors are found by acting on the Minkowski space unit basis 4-vectors

$$\{[\boldsymbol{e}_{\alpha}]^{\nu}\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{4.28}$$

with the inverse matrix of equation (4.25).

This tangent space only covers an infinitesimal neighborhood of the uniformly accelerating observer's spacetime event $x^{\mu'}(\tau)$. However, it can be extended to cover a finite domain that is valid within a proper distance a^{-1} of the uniformly accelerating observer's coordinate origin [26]. As we will see later, in equation (4.104), a^{-1} is the proper distance from the accelerating observer to his Rindler horizon.

At each instant of proper time the uniformly accelerating observer has an exact location at point $P(\tau)$. Additionally, since we defined $\boldsymbol{e}_{0'} = \boldsymbol{u}$, the observer's time coordinate exactly coincides with the proper time along the observer's worldline. Therefore, at each instant of proper time there exists a spacelike hypersurface defined by the point $P(\tau)$ and the three spacelike basis vectors $\boldsymbol{e}_{1'}$, $\boldsymbol{e}_{2'}$, and $\boldsymbol{e}_{3'}$.

Let $\boldsymbol{x}_0(\tau)$ denote the position vector of the uniformly accelerating observer at $P(\tau)$, defined with respect to the origin of the global Minkowski frame. And let $\boldsymbol{x}'$ denote the spacelike separation vector between $P(\tau)$ and an arbitrary point $Q(\tau)$, in the observer's local tangent space. $Q(\tau)$ is located in the comoving frame's local coordinates at $\xi^{\mu} = (\xi^0, \xi^1, \xi^2, \xi^3) = (\tau, \xi^i)$. This implies the spacelike separation vector $\boldsymbol{x}'(\tau) = \xi^{i'} \boldsymbol{e}_{i'}(\tau)$. By vector addition, the position vector $\boldsymbol{x}(\tau)$ of $Q(\tau)$ can be defined with respect to the global Minkowski spacetime origin as

$\xi^{\mu} = (\xi^0, \xi^1, \xi^2, \xi^3) = (\tau, \xi^i)$

$$\begin{aligned} \boldsymbol{x}(\tau) &= \boldsymbol{x}_0(\tau) + \boldsymbol{x}'(\tau) \\ &= \boldsymbol{x}_0(\tau) + \xi^{i'} \boldsymbol{e}_{i'}(\tau) \end{aligned} \tag{4.29}$$

where it's important to note that the separation vector $\boldsymbol{x}'(\tau) = \xi^{i'} \boldsymbol{e}_{i'}(\tau)$ is restricted to a spacelike subspace of the local tangent space. For this reason, the separation vector's time component is equal to zero $[\boldsymbol{x}'(\tau)]^{0'} = 0$. The position 4-vector corresponding to equation (4.29) is

$$x^{\mu}(\tau) = x_0{}^{\mu}(\tau) + \xi^{i'} [\boldsymbol{e}_{i'}(\tau)]^{\mu} \tag{4.30}$$

From equation (4.22), the 4-position of the local tangent space origin is

$$x_0{}^{\mu}(\tau) = \left(a^{-1}\sinh(a\tau), \quad a^{-1}\cosh(a\tau), \quad 0, \quad 0\right) \tag{4.31}$$

And the second term in equation (4.30) is a 4-vector whose components each include a summation over i,

$$\xi^{i'}[\boldsymbol{e}_{i'}(\tau)]^0 = \left(0, \quad \xi^{1'}, \quad \xi^{2'}, \quad \xi^{3'}\right) \cdot \left(\cosh(a\tau), \quad \sinh(a\tau), \quad 0, \quad 0\right) = \xi^{1'}\sinh(a\tau) \tag{4.32a}$$
$$\xi^{i'}[\boldsymbol{e}_{i'}(\tau)]^1 = \left(0, \quad \xi^{1'}, \quad \xi^{2'}, \quad \xi^{3'}\right) \cdot \left(\sinh(a\tau), \quad \cosh(a\tau), \quad 0, \quad 0\right) = \xi^{1'}\cosh(a\tau) \tag{4.32b}$$
$$\xi^{i'}[\boldsymbol{e}_{i'}(\tau)]^2 = \left(0, \quad \xi^{1'}, \quad \xi^{2'}, \quad \xi^{3'}\right) \cdot \left(0, \quad 0, \quad 1, \quad 0\right) = \xi^{2'} \tag{4.32c}$$
$$\xi^{i'}[\boldsymbol{e}_{i'}(\tau)]^3 = \left(0, \quad \xi^{1'}, \quad \xi^{2'}, \quad \xi^{3'}\right) \cdot \left(0, \quad 0, \quad 0, \quad 1\right) = \xi^{3'} \tag{4.32d}$$

From equations (4.31) and (8.38) we have

$$x^0(\xi^{\mu'}) = \left(a^{-1} + \xi^{1'}\right)\sinh(a\xi^{0'}) \tag{4.33a}$$
$$x^1(\xi^{\mu'}) = \left(a^{-1} + \xi^{1'}\right)\cosh(a\xi^{0'}) \tag{4.33b}$$
$$x^2(\xi^{\mu'}) = \xi^{2'} \tag{4.33c}$$
$$x^3(\xi^{\mu'}) = \xi^{3'} \tag{4.33d}$$

where I've temporarily replaced $\tau \to \xi^{0'}$ to emphasize that equation (4.33) expresses the global Minkowski coordinates in terms of the local coordinates of the instantaneously co-moving frame.

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dx^{\mu}dx^{\nu} \\ &= \eta_{\mu'\nu'}d\xi^{\mu'}d\xi^{\nu'} \end{aligned} \tag{4.34}$$

Equations (4.33c), (4.33d), $dx^2 = d\xi^{2'}$ and $dx^3 = d\xi^{3'}$, taken with $\eta_{22}, \eta_{33} = 1$, trivially imply $\eta_{2'2'}, \eta_{3'3'} = 1$. The two remaining metric components can be determined by taking differentials of equations (4.33a) and (4.33b),

$$
\begin{aligned}
-(dx^0)^2 + (dx^1)^2 = &- \left[\left(a^{-1} + \xi^{1'}\right)\cosh(a\xi^{0'})ad\xi^{0'} + \sinh(a\xi^{0'})d\xi^{1'}\right]^2 \\
&+ \left[\left(a^{-1} + \xi^{1'}\right)\sinh(a\xi^{0'})ad\xi^{0'} + \cosh a\xi^{0'}d\xi^{1'}\right]^2 \\
= &- \left(a^{-1} - \xi^{1'}\right)^2 \cosh^2(a\xi^{0'})a^2(d\xi^{0'})^2 + 2\left(a^{-1} + \xi^{1'}\right)\cosh(a\xi^{0'})a\sinh(a\xi^{0'})d\xi^{0'}d\xi^{1'} \\
&- \sinh^2(a\xi^{0'})(d\xi^{1'})^2 + \left(a^{-1} + \xi^{1'}\right)^2 \sinh^2(a\xi^{0'})a^2(d\xi^{0'})^2 \\
&+ 2\left(a^{-1} + \xi^{1'}\right)^2 \sinh^2(a\xi^{0'})a^2\cosh^2(a\xi^{0'})d\xi^{0'}d\xi^{1'} + \cosh^2(a\xi^{0'})(d\xi^{1'})^2 \\
= &- \left(a^{-1} + \xi^{1'}\right)^2 a^2 \left(\cosh^2(a\xi^{0'}) - \sinh^2(a\xi^{0'})\right)(d\xi^{0'})^2 \\
&+ \left(\cosh^2(a\xi^{0'}) - \sinh(a\xi^{0'})\right)(d\xi^{1'})^2 \\
= &- \left(1 + a\xi^{1\prime}\right)^2 (d\xi^{0'})^2 + (d\xi^{1'})^2
\end{aligned}
\tag{4.35}
$$

where I've used the identity $\cosh^2 x - \sinh^2 x = 1$ in the last step. From equation (4.35), the Rindler metric for the (1+1)-dimensional spacetime component is

$$
\eta_{\mu'\nu'} = \begin{pmatrix} -\left(1 + a\xi^{1'}\right)^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.36}
$$

with line element

$$
ds^2 = -\left(1 + a\xi^{1'}\right)^2 (d\xi^{0'})^2 + (d\xi^{1'})^2 \tag{4.37}
$$

For a (3+1)-dimensional spacetime the line element extends to

$$
ds^2 = -\left(1 + a\xi^{1'}\right)^2 (d\xi^{0'})^2 + (d\xi^{1'})^2 + (d\xi^{2'})^2 + (d\xi^{3'})^2 \tag{4.38}
$$

This is the metric for Rindler space.

4.3 Rindler spacetime

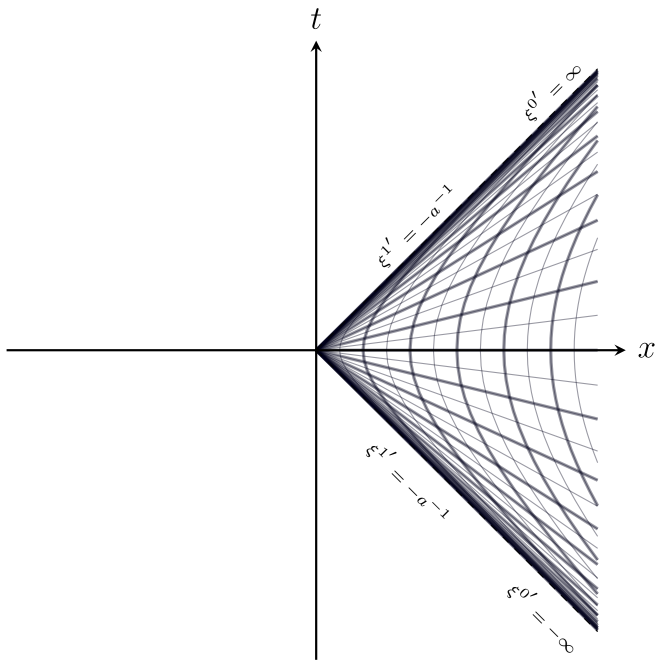

Figure 4.1: Rindler spacetime diagram. Hyperbola correspond to lines of constant $\xi^{1'}$, while rays correspond to lines of constant $\xi^{0'}$. Each hyperbolae represents the trajectory of a Rindler observer, with a specific acceleration a, projected onto Minkowski space. The x, t coordinate axes are the axes of the global Minkowski coordinates.

In the previous section we found that the line element for a uniformly accelerating observer is given by,

$$ds^2 = -\left(1 + a\xi^{1'}\right)^2 (d\xi^{0'})^2 + (d\xi^{1'})^2 + (d\xi^{2'})^2 + (d\xi^{3'})^2 \tag{4.39}$$

The corresponding Rindler spacetime, mapped onto a Minkowski spacetime diagram, is shown in Figure 4.1. Each hyperbola in figures 4.1 and 4.2 represents the worldline of a Rindler observer, and is plotted in parametric Minkowski coordinats,

$$t(\xi^{0'}, \xi^{1'}) = \left(a^{-1} + \xi^{1'}\right) \sinh(a\xi^{0'}) \tag{4.40a}$$

$$x(\xi^{0'}, \xi^{1'}) = \left(a^{-1} + \xi^{1'}\right) \cosh(a\xi^{0'}) \tag{4.40b}$$

for fixed values of $\xi^{1'}$, and parameterized with the Rindler observer's proper time $\xi^{0'} = \tau$. The rays depicted in figures 4.1 and 4.3 correspond to contours of constant Rindler time $\xi^{0'}$. Equations (4.40a) and (4.40b) were found in section 4.2.1.

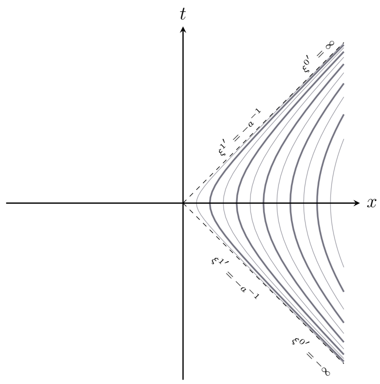

Figure 4.2: Contours of constant $\xi^{1'}$. These contours represent the worldlines of Rindler observers, at rest in their own frames, as they travel along their worldlines through Minkowski space. The Rindler horizon is located at $\xi^{1'} = -a^{-1}$.

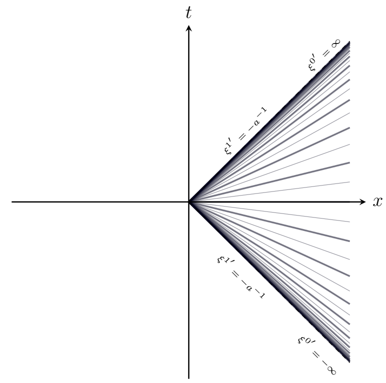

Figure 4.3: Contours of constant $\xi^{0'}$. These contours represent instants of constant time in the local frames of Rindler observers along their worldlines. The past and future Rindler horizons correspond to $\xi^{0'} = -\infty$ and $\xi^{0'} = \infty$, respectfully.

Hyperbolic motion

To see that Rindler observers follow hyperbolic trajectories through Minkowski spacetime, we can consider the invariant interval $x_\mu x^\mu$ that is determined by equations (4.40a) and (4.40b). Since the acceleration occurs along the x-axis, it suffices to consider only the t and x coordinates,

$$\begin{aligned} x_\mu x^\mu &= -t^2 + x^2 \\ &= -\left[\left(a^{-1} + \xi^{1'}\right)\sinh(a\xi^{0'})\right]^2 + \left[\left(a^{-1} + \xi^{1'}\right)\cosh(a\xi^{0'})\right]^2 \\ &= \left(a^{-1} + \xi^{1'}\right)^2 \left[\cosh^2(a\xi^{0'}) - \sinh^2(a\xi^{0'})\right] \\ &= \left(a^{-1} + \xi^{1'}\right)^2 \end{aligned} \tag{4.41}$$

$\xi^{1'} = 0$ at the spatial origin of the Rindler observer's local coordinate frame. Setting $\xi^{1'} = 0$ in equation (4.41) gives

$$x^2 - t^2 + = a^{-2} \tag{4.42}$$

which is the equation for a hyperbola in the Minkowski observer's coordinates.

The image of Rindler spacetime

The Rindler coordinates are incomplete because they only cover one fourth of the Minkowski spacetime. The reason for this can be seen by looking at equations (4.40a) and (4.40b), which show that the Minkowski spacelike coordinate x has a hyperbolic-cosine-dependence on the Rindler time coordinate $\xi^{0'}$, which gives $x(\xi^{0'})$ an even $\xi^{0'}$-dependence. This by itself, restricts the image of the patch $\boldsymbol{x}(\xi^{0'},\xi^{1'}) = \left(t(\xi^{0'},\xi^{1'}), x(\xi^{0'},\xi^{1'})\right)$ to the first and fourth quadrants of the Minkowski spacetime diagram. An additional restriction comes from the fact that $\cosh(y) > \sinh(y)$ for all real values of y. This effectively restricts the Rindler spacetime's image to $x < |t|$, where $t = \pm x$ is the location of the Rindler horizon.

The Rindler horizon

Referring back to the metric in equation (4.39),

$$ds^2 = -\left(1 + a\xi^{1'}\right)^2 \left(d\xi^{0'}\right)^2 + \left(d\xi^{1'}\right)^2 + \left(d\xi^{2'}\right)^2 + \left(d\xi^{3'}\right)^2 \tag{4.43}$$

We can take the metric's determinant to find,

$$g = \det(g_{\mu\nu}) = -\left(1 + a\xi^{1'}\right)^2 \tag{4.44}$$

which is zero at $\xi^{1'} = -a^{-1}$, showing that the metric lacks regularity at this point. This is another way of determining that $\xi^{1'} = -a^{-1}$ is the Rindler horizon's location [8].

Another means of located the Rindler horizon will be discussed in section 4.4.2.

4.4 Lightcone coordinates

Many of the coordinate transformations in relativity involve the use of lightcone coordinates, which correspond to the rest frame of a massless particle. Lightcone coordinates for particles moving along the x-axis are defined as

$$u \equiv t - x \tag{4.45a}$$
$$v \equiv t + x \tag{4.45b}$$

Equations (4.45a) and (4.45b) correspond to the geodesics of massless particles moving in the positive and negative x-directions, respectively. This is easiest to see by rewriting equations (4.45a) and (4.45b) as

$$t(x) = x + u \tag{4.46a}$$
$$t(x) = -x + v \tag{4.46b}$$

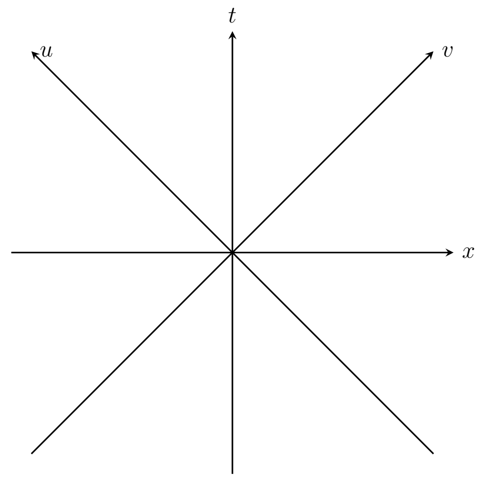

Figure 4.4: Minkowski spacetime diagram with lightcone coordinate axes u, v plotted. x and t are the coordinates of a Minkowski observer. The u, v axes exactly coincide with the lightcone of a Minkowski observer, when projected onto a Minkowski spacetime diagram. This indicates that the u and v axes correspond to the spacetime geodesics of light beams. v and u are generally taken to correspond to the lightcone "time" and "space" coordinates, respectively.

v is generally regarded as the lightcone "time" coordinate and u as the lightcone "spatial" coordinate [29].

The actual physics that takes place in lightcone coordinates can be difficult to interpret, especially when the coordinates are used to describe the trajectories of particles that are not massless. For instance, consider the case of a particle of arbitrary mass moving in the $+x$-direction with velocity v. Given the interpretations of u and v as the lightcone time and space coordinates, we can define lightcone velocity,

$$\frac{dv}{du} = \frac{1 - v'/c}{1 + v'/c} \tag{4.47}$$

which is derived in the appendix (.1). I've put the speed of light c back into equation (4.47) to emphasize that the velocity parameter has a magnitude that is bounded by one. Now consider the case of a particle moving with $v' = c$. Equation (4.47) tells us that this particle has zero lightcone velocity $\frac{dv}{du} = 0$. This is consistent with the interpretation of lightcone coordinates being the rest frame of a light beam moving along the x-axis. In contrast, equation (4.47) tells us that a particle traveling at the speed of light in the $-x$-direction has infinite lightcone velocity $\frac{dv}{du} = \infty$. This means that the lightcone velocity of a particle ranges from zero to infinity, with a massive particle having a velocity somewhere in between.

As this demonstrates, the physics that takes place in lightcone coordinates can be difficult to interpret. Particles can have an infinite velocity and still obey the laws of physics in lightcone coordinates. For this reason, interpretations of the physics that occurs in coordinate transformations that involve lightcone coordinates is often limited. Lightcone coordinates tend to be very useful for understanding causal relations between different observers. Lightcone coordinates are generally not very useful when it comes to interpreting quantities such as relative distances and velocities.

4.4.1 Constant acceleration in lightcone coordinates

Lightcone coordinates with respect to Minkowski coordinates

The derivation of the Rindler metric that was given in section 4.2 might be viewed as the most intuitive approach to deriving the Rindler coordinates. This is because it uses coordinates that have very straightforward interpretations. While it is nice to be able to understand the physics that takes place in the coordinates that we are working with, such coordinates are not always the easiest to work with computationally. In this section I'll derive the Rindler metric using lightcone coordinates. Consider the metric for a (1+1)-dimensional Minkowski spacetime,

$$d\tau^2 = dt^2 - dx^2 \tag{4.48}$$

The worldline $x^\mu(\tau) = (t(\tau), x(\tau))$ of an observer moving through a Minkowski spacetime can be parameterized with the observer's proper time τ along the observer's spacetime trajectory.

We can define lightcone coordinates

$$U \equiv t - x \quad \text{and} \quad V \equiv t + x \tag{4.49}$$

with respect to the Minkowski coordinates t and x. As coordinate differentials,

$$dU = \frac{\partial U}{\partial x^\mu} dx^\mu = \frac{\partial U}{\partial t} dt + \frac{\partial U}{\partial x} dx = dt - dx \tag{4.50a}$$

$$dV = \frac{\partial V}{\partial x^\mu} dx^\mu = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial x} dx = dt + dx \tag{4.50b}$$

Using equations (4.50a) and (4.50b), the line element can expressed in lightcone coordinates

$$\begin{aligned} ds^2 &= -dt^2 + dx^2 \\ &= -(dt - dx)(dt + dx) \\ &= -dU dV \end{aligned} \tag{4.51}$$

The metric for the line element in equation (4.51) can be found by writing the line element in its full form, with zeros for coefficients on vanishing terms,

$$ds^2 = 0 \times dU^2 + \left(-\tfrac{1}{2}\right) dU dV + \left(-\tfrac{1}{2}\right) dV dU + 0 \times dV^2 \tag{4.52}$$

$$g_{\mu\nu}^{LC} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \tag{4.53}$$

Trajectory of an accelerated observer in lightcone coordinates

In lightcone coordinates, the worldline of an observer can be described by a parameterized curve of the form $x^\mu(\tau) = (U(\tau), V(\tau))$. As was explained in section 4.1, the 4-velocity of an inertial frame that is instantaneously comoving with the frame of a uniformly accelerating observer is $u^\mu = (1, 0)$. This follows from the fact that the spatial component of a rest observer's 4-velocity $\dot{x}^i(\tau) = 0$. Additionally, recall that a uniformly accelerating observer can be thought of as jumping from one inertial comoving reference frame to the next, along the observer's worldline. Since the comoving rest observer's frame is inertial, it necessarily satisfies the Lorentz invariant condition $u_\mu u^\mu = -1$ in all inertial coordinate frames.

$$\begin{aligned}
-1 &= u_\mu u^\mu \\
&= \eta_{\mu\nu} u^\nu_M u^\mu_M \\
&= g^{LC}_{\mu\nu} u^\nu_{LC} u^\mu_{LC} \\
&= g_{UU} u^U u^U + g_{UV} u^U u^V + g_{VU} u^V u^U + g_{VV} u^V u^V \\
&= 0 \times \dot{U}^2 + \tfrac{1}{2}\dot{U}\dot{V} + \tfrac{1}{2}\dot{V}\dot{U} + 0 \times \dot{V}^2 \\
&= \dot{U}\dot{V}
\end{aligned} \tag{4.54}$$

where I've used sub/superscript M to denote quantities expressed in Minkowski coordinates and LC for quantities in lightcone coordinates. Second, consider the relation for an observer's 4-acceleration $a^\mu(\tau) = \ddot{x}^\mu(\tau)$. We found earlier, that $a^\mu = (0, a)$ is required by the orthogonality of a^μ to $u^\mu = (1, 0)$. The 4-acceleration has a corresponding Lorentz invariant quantity $a_\mu a^\mu = a^2$. This implies

$$\begin{aligned}
a^2 &= a_\mu a^\mu \\
&= \eta^M_{\mu\nu} a^\nu_M a^\mu_M \\
&= g^{LC}_{\mu\nu} a^\nu_{LC} a^\mu_{LC} \\
&= \ddot{U}(\tau)\ddot{V}(\tau)
\end{aligned} \tag{4.55}$$

From equation (4.54) we can write

$$\dot{V} = -\frac{1}{\dot{U}} \tag{4.56}$$

which implies

$$\ddot{V} = \frac{\ddot{U}}{\dot{U}^2} \tag{4.57}$$

Putting this into equation (4.55) gives

$$\left(\frac{\ddot{U}}{\dot{U}^2}\right)\ddot{U} = a^2 \implies \frac{\ddot{U}}{\dot{U}} = \pm a \tag{4.58}$$

The sign ambiguity can be resolved at the end of the derivation by writing the Minkowski coordinates x, t as functions of lightcone coordinates U, V. It's easily shown that getting the correct sign on $x = x(U, V)$ requires that $-a$ correspond to U and $+a$ correspond to V. To simplify the notation I'll take $-a$ in equation (4.58),

$$\frac{\ddot{U}}{\dot{U}} = -a \tag{4.59}$$

Integrating both sides over the comoving observer's proper time,

$$\begin{aligned} -a\tau &= -a\int_0^\tau d\tau' \\ &= \int_0^\tau \frac{\ddot{U}}{\dot{U}} d\tau' \\ &= \int_0^\tau \frac{d\dot{U}}{\dot{U}} d\tau' \\ &= \ln\left|\dot{U}\right| - \ln\left|\dot{U}_0\right| \end{aligned} \tag{4.60}$$

Which gives

$$\dot{U}(\tau) = \dot{U}_0 e^{-a\tau} \tag{4.61}$$

Integrating over τ to get $U(\tau)$,

$$\begin{aligned} U(\tau) &= \int \dot{U}(\tau) d\tau \\ &= \int dU \\ &= \dot{U}_0 \int e^{-a\tau} d\tau \\ &= -\frac{\dot{U}_0}{a} e^{-a\tau} + C_U \end{aligned} \tag{4.62}$$

From equatiosn (4.54) and (4.61) we have

$$\dot{V} = \frac{1}{\dot{U}} = \frac{1}{\dot{U}_0 e^{-a\tau}} = \dot{U}_0^{-1} e^{+a\tau} \tag{4.63}$$

Integrating both sides to get $V(\tau)$,

$$\begin{aligned} V(\tau) &= \int \dot{V} d\tau \\ &= \int_{V_0}^{V} dV' \\ &= \dot{U}_0^{-1} \int e^{+a\tau} d\tau \\ &= \frac{1}{\dot{U}_0 a} e^{a\tau} + C_V \end{aligned} \tag{4.64}$$

From equations (4.62) and (4.64) we have

$$U = -\frac{\dot{U}_0}{a} e^{-a\tau} + C_U \tag{4.65a}$$

$$V = \frac{1}{\dot{U}_0 a} e^{a\tau} + C_V \tag{4.65b}$$

Since coordinate translations are a symmetry of special relativity we can eliminate the constants of integration in equations (4.65a) and (4.65b) by shifting our coordinate axes to a new inertial frame. Doing so modifies U and V as

$$U = -\frac{\dot{U}_0}{a} e^{-a\tau} \tag{4.66a}$$

$$V = \frac{1}{\dot{U}_0 a} e^{a\tau} \tag{4.66b}$$

Additionally, we can define a coordinate transformation $U = \dot{U}^{-1}U'$ and $V = \dot{U}V'$. This implies

$$ds^2 = -dUdV = -\left(\dot{U}_0^{-1} dU'\right)\left(\dot{U}_0 dV'\right) = -dU'dV' \tag{4.67}$$

The metric is left unchanged which implies that this coordinate change corresponds to a Lorentz transformation [30]. Now taking $U' \to U$ and $V' \to V$ equations (4.66a) and (4.66b) become

$$U = -a^{-1} e^{-a\tau} \tag{4.68a}$$

$$V = a^{-1} e^{a\tau} \tag{4.68b}$$

Substitution of $U = t - x \implies x = t - U$ into $V = t + x = 2t - U$ and $U = t - x \implies t = x + U$ into $V = t + x = 2x + U$ gives

$$t = \frac{1}{2}(V + U) \quad \text{and} \quad x = \frac{1}{2}(V - U) \tag{4.69}$$

Inserting equations (4.68a) and (4.68b) into (4.69) gives

$$t = x^0 = a^{-1}\sinh(a\tau) \tag{4.70a}$$
$$x = x^1 = a^{-1}\cosh(a\tau) \tag{4.70b}$$
$$x^2 = 0 \tag{4.70c}$$
$$x^3 = 0 \tag{4.70d}$$

Giving us the same results as was obtained in the approach without lightcone coordinates in equation (4.22).

4.4.2 The local perspective of the Rindler observer

The Rindler coordinates can be brought to a form that is conformally flat in the time and direction of spatial acceleration component. We can start by assuming the existence of a conformally flat metric for a (1+1)-dimensional spacetime,

$$ds^2 = -f(\zeta^{0'}, \zeta^{1'})^2 \left[(d\zeta^{0'})^2 - (d\zeta^{1'})^2\right] \tag{4.71}$$

In section 4.4.1 we defined lightcone coordinates with respect to the global Minkowski coordinates. Lightcone coordinates can also be defined in terms of the coordinates of an observer's local tangent space. Let u, v denote the lightcone coordinates with respect to the local Cartesian coordinates $\zeta^{\mu'}$ of the instantaneously comoving frame,

$$u = \zeta^{0'} - \zeta^{1'} \tag{4.72a}$$
$$v = \zeta^{0'} + \zeta^{1'} \tag{4.72b}$$

Using the fact that covariant vectors transform as $d\zeta^{\mu'} = \frac{\partial \zeta^{\mu'}}{\partial \zeta^{\nu}} d\zeta^{\nu}$, we can express equations (4.72a) and (4.72b) as differentials,

$$du = d\zeta^{0'} - d\zeta^{1'} \tag{4.73a}$$
$$dv = d\zeta^{0'} + d\zeta^{1'} \tag{4.73b}$$

Inserting equations (4.73a) and (4.73b) into equation (4.71), and re-expressing the scale factor as a function of the local lightcone coordinates $f(\zeta^{0'}, \zeta^{1'}) \to f(u, v)$ gives

$$\begin{aligned} ds^2 &= -f(\zeta^{0'}, \zeta^{1'})^2 \left(d\zeta^{0'} - d\zeta^{1'}\right)\left(d\zeta^{0'} + d\zeta^{1'}\right) \\ &= -f(u, v)^2 du dv \end{aligned} \tag{4.74}$$

If we chose to align the time axis of the comoving observer's local tangent space with his 4-velocity vector $\boldsymbol{e}_{0'} = \boldsymbol{u}$, then he will be at rest in his own frame. Doing this implies $\zeta^{0'} = \tau$ and $\zeta^{1'} = 0$ at the comoving observer's own position. From equations (4.72a) and (4.72b) this means $u, v = \tau$ at the comoving observer's origin. Putting this into equation (4.74) gives

$$\begin{aligned} -ds^2 = d\tau^2 &= f(u,v)^2 du dv \\ &= f(u,v)^2 d\tau^2 \end{aligned} \tag{4.75}$$

which, for the special case of $\zeta^{1'} = 0$, implies

$$f(u,v)^2 = 1 \tag{4.76}$$

From equation (4.49), the global Mikowski lightcone coordinates are

$$U = t - x \tag{4.77a}$$
$$V = t + x \tag{4.77b}$$

which, from the tensor formalism, can be related to the local lightcone coordinates through the line element,

$$ds^2 = -dUdV = -f(u,v)^2 du dv \tag{4.78}$$

which momentarily assumes arbitrary $f(u,v)$ and thus, does not necessarily assume $\zeta^{1'} = 0$. We can relate the lightcone coordinates of the local tangent space u, v to the global Minkowski lightcone coordinates U, V by defining the temporarily unknown coordinate relations $U = U(u,v)$ and $V = V(u,v)$. This implies

$$dU = \frac{\partial U}{\partial u} du + \frac{\partial U}{\partial v} dv \tag{4.79a}$$
$$dV = \frac{\partial V}{\partial u} du + \frac{\partial V}{\partial v} dv \tag{4.79b}$$

Which can now be put into equation (4.78) to obtain a new line element. Momentarily switching the metric signature from spacelike ds^2 to timelike $d\tau^2$ to avoid having to deal with the minus sign gives

$$d\tau^2 = dUdV \tag{4.80a}$$

$$\begin{aligned} &= \left(\frac{\partial U}{\partial u} du + \frac{\partial U}{\partial v} dv\right)\left(\frac{\partial V}{\partial u} du + \frac{\partial V}{\partial v} dv\right) \\ &= \frac{\partial U}{\partial u}\frac{\partial V}{\partial u} du^2 + \left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v} + \frac{\partial U}{\partial v}\frac{\partial V}{\partial u}\right) du dv + \frac{\partial U}{\partial v}\frac{\partial V}{\partial v} dv^2 \end{aligned} \tag{4.80b}$$

The only way for equations (4.80a) and (4.80b) to be consistent is to have U, V each be parameterized by only one parameter u or v. We're also constrained by the condition that U and V cannot both be parameterized by the same parameter, or else the second term in equation (4.80b) will completely disappear. Lets choose,

$$U = U(u) \tag{4.81a}$$
$$V = V(v) \tag{4.81b}$$

Since $u, v = \zeta^{0'} = \tau$ at the origin of the comoving frame, it is also true that $\frac{du}{d\tau} = \frac{dv}{d\tau} = 1$. Then,

$$\frac{dU}{d\tau} = \frac{dU}{du}\frac{du}{d\tau} = \frac{dU}{du} \tag{4.82a}$$
$$\frac{dV}{d\tau} = \frac{dV}{dv}\frac{dv}{d\tau} = \frac{dV}{dv} \tag{4.82b}$$

Recalling equations (4.68a) and (4.68b),

$$U = -a^{-1}e^{-a\tau} \tag{4.83a}$$
$$V = a^{-1}e^{a\tau} \tag{4.83b}$$

and using the property that an exponential function is proportional to its derivative, we can take the proper time derivative of equation (4.83a) to obtain

$$\begin{aligned}\frac{dU}{d\tau} &= e^{-a\tau} \\ &= -a\left(-a^{-1}e^{-a\tau}\right) \\ &= -aU\end{aligned} \tag{4.84}$$

where the last step follows from equation (4.83a). From this, and equation (4.82a) we have

$$\frac{dU}{du} = -aU \implies \int \frac{dU}{U} = -a\int du \tag{4.85}$$

And thus,

$$U(u) = Ae^{-au} \tag{4.86}$$

where A is the coefficient arising from an unspecified constant of integration. We can use equations (4.82b) and (4.83b) to set up and solve the same general problem for a relation between V and v. From equation (4.83b),

$$\begin{aligned}\frac{dV}{d\tau} &= e^{a\tau} \\ &= a\left(a^{-1}e^{a\tau}\right) \\ &= aV \end{aligned} \tag{4.87}$$

and applying equation (4.82b) we obtain

$$\frac{V}{v} = aV \implies \int \frac{dV}{V} = a\int dv \tag{4.88}$$

which has a general solution

$$V(v) = Be^{av} \tag{4.89}$$

for some constant B. Rewriting the line element in terms of $U(u)$ and $V(v)$

$$\begin{aligned} ds^2 &= -dUdV \\ &= +a^2ABe^{a(v-u)}dudv \qquad (4.90a) \\ &= -f(u,v)^2dudv \qquad (4.90b) \end{aligned}$$

where the last line comes from equation (4.78). The plus sign on equation (4.90a) comes from the fact that differentiating equation (4.86) yields a minus sign, which cancels out the minus sign from the line above. Equations (4.90a) and (4.90b) provide a relation between the scale factor $f(u,v)$ and the constants of integration A, B,

$$f(u,v)^2 = -a^2ABe^{a(v-u)} \tag{4.91}$$

Recall from equation (4.76) that $f(u,v)^2 = 1$ for the special case of $\zeta^{1'} = 0$, which corresponds to the origin of the comoving frame. Also recall from equations (4.72a) and (4.72b) that $u, v = \tau$ when $\zeta^{1'} = 0$, which implies that $v - u = \tau - \tau = 0$, which implies $e^{a(v-u)} = 0$. Then at the origin of the comoving frame we have

$$AB = -a^{-2} \tag{4.92}$$

The solutions for A, B are not unique, but choosing $A = -a^{-1}$ and $B = a^{-1}$ is a logical choice because it makes equations (4.86) and (4.89) symmetric to equations (4.68a) and (4.68b), respectively. Note that this arbitrary choice of scaling on U and V simply amounts to a Lorentz transformation [30], and is thus a symmetry of the formalism. Thus, equation (4.86) and (4.89) become

$$\begin{aligned} U &= -a^{-1}e^{-au} \qquad (4.93a) \\ V &= a^{-1}e^{av} \qquad (4.93b) \end{aligned}$$

And inserting $AB = -a^{-2}$ into equation (4.91) gives us the scale factor

$$f(u,v)^2 = e^{a(v-u)} \tag{4.94}$$

and thus, from equation (4.90b)

$$ds^2 = -e^{a(v-u)}dudv \tag{4.95}$$

Equation (4.95) is a conformally flat metric expressed in the lightcone coordinates of the comoving observer's local tangent space. Using the relations

$$u = \zeta^{0'} - \zeta^{1'} \tag{4.96a}$$
$$v = \zeta^{0'} + \zeta^{1'} \tag{4.96b}$$
$$U = x - t \tag{4.96c}$$
$$V = x + t \tag{4.96d}$$

we found, from equation (4.69) that

$$t = \frac{1}{2}(V+U) \quad \text{and} \quad x = \frac{1}{2}(V-U) \tag{4.97}$$

This, with equations (4.93a) and (4.93b) implies

$$\begin{aligned} t &= \frac{1}{2a}\left(e^{av} - e^{-au}\right) \\ &= \frac{1}{2a}\left(e^{a(\zeta^{0'}+\zeta^{1'})} - e^{-a(\zeta^{0'}-\zeta^{1'})}\right) \\ &= a^{-1}e^{a\zeta^{1'}}\frac{\left(e^{a\zeta^{0\prime}} - e^{-a\zeta^{0'}}\right)}{2} \\ &= a^{-1}e^{a\zeta^{1'}}\sinh(a\zeta^{0'}) \end{aligned} \tag{4.98}$$

and similarly for x,

$$\begin{aligned} x &= \frac{1}{2a}\left(e^{av} + e^{-au}\right) \\ &= \frac{1}{2a}\left(e^{a(\zeta^{0'}+\zeta^{1'})} + e^{-a(\zeta^{0'}-\zeta^{1'})}\right) \\ &= a^{-1}e^{a\zeta^{1'}}\frac{\left(e^{a\zeta^{0\prime}} + e^{-a\zeta^{0'}}\right)}{2} \\ &= a^{-1}e^{a\zeta^{1'}}\cosh(a\zeta^{0'}) \end{aligned} \tag{4.99}$$

Equations (4.98) and (4.99) can be written as differentials

$$dt = e^{a\zeta^{1'}} \left[\cosh(a\zeta^{0'})d\zeta^{0'} + \sinh(a\zeta^{0'})d\zeta^{1'}\right] \tag{4.100a}$$

$$dx = e^{a\zeta^{1'}} \left[\sinh(a\zeta^{0'})d\zeta^{0'} + \cosh(a\zeta^{0'})d\zeta^{1'}\right] \tag{4.100b}$$

Putting these differential expressions into $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ gives us a conformally flat metric for the local tangent space of an observer who is instantaneously comoving with the uniformly accelerating observer,

$$ds^2 = -e^{2a\zeta^{1'}} \left[(d\zeta^{0'})^2 - (d\zeta^{1'})^2\right] \tag{4.101}$$

The existence of a local metric that is conformally flat indicates that Rindler space is locally flat [30]. Equation (4.101) is related to Minkowski coordinates by

$$t = a^{-1}e^{a\zeta^{1'}} \sinh(a\zeta^{0'}) \tag{4.102a}$$

$$x = a^{-1}e^{a\zeta^{1'}} \cosh(a\zeta^{0'}) \tag{4.102b}$$

which was shown in equations (4.98) and (4.99).

The Rindler horizon is located at $t = \pm x$ in figure 4.1. As will be shown in section 4.4.2, the local metric for a Rindler observer can be brought to a form that is conformally flat in its relevant (1+1)-component [16, 30],

$$ds^2 = -e^{2a\zeta^{1'}} \left[(d\zeta^{0'})^2 - (d\zeta^{1'})^2\right] \tag{4.103}$$

where the spatial coordinate corresponds to the observer's direction of acceleration. Conformal flatness is made possible by the use of lightcone coordinates, which are explained in section 4.4.

We can find the proper distance of a Rindler observer to his Rindler horizon by integrating over the spacelike interval $-\infty < \zeta^{1'} < 0$. From equation (4.103),

$$\begin{aligned} s &= \int ds \\ &= \int_0^{-\infty} e^{a\zeta^{1'}} d\zeta^{1'} \\ &= a^{-1}e^{a\zeta^{1'}}\Big|_{\zeta^{1'}=0}^{-\infty} \\ &= -a^{-1} \end{aligned} \tag{4.104}$$

This shows that the entire spacelike interval $-\infty < \zeta^{1'} < 0$ only spans a finite proper distance. Note that the scale factor $e^{2a\zeta^{1'}}$ has no $\zeta^{0'}$-dependence, which indicates that this result is the same at all points along the Rindler observer's worldline.

Chapter 5

The Unruh effect as quantum tunneling

Recall the form of the Rindler metric that was derived in section 4.2.1,

$$ds^2 = -\left(1 + a\xi^{1'}\right)^2 \left(d\xi^{0'}\right)^2 + \left(d\xi^{1'}\right)^2 + \left(d\xi^{2'}\right)^2 + \left(d\xi^{3'}\right)^2 \tag{5.1}$$

In section 4.3 we saw that the Rindler metric is singular at $\xi^{1'} = -a^{-1}$, from the fact that the metric's determinant,

$$g = \det(g_{\mu\nu}) = -\left(1 + a\xi^{1'}\right)^2 \tag{5.2}$$

is equal to zero at $\xi^{1'} = -a^{-1}$. However, we also know that this can only be a coordinate singularity because the Rindler metric in equation (4.35) was derived from a global coordinate transformation of Minkowski spacetime, whose geometry is known to be nonsingular [8]. This coordinate singularity at $\xi^{1'} = -a^{-1}$ represents the observer's Rindler horizon. This chapter focuses on a derivation of the Unruh temperature using the gravitational WKB approximation to find the probability of particle tunneling across a uniformly accelerating observer's Rindler horizon. The tunneling probability can then be equated to a Boltzmann factor to find the corresponding Unruh temperature on the observer's Rindler horizon.

5.1 Similarity of Rindler and Schwarzschild metrics

The Rindler metric can be written in a form that is very similar in appearance to the Schwarzschild metric. Recall that the Schwarzschild metric is singular at the event horizon in its radial component. Putting the Rindler metric in a similar form gives us a coordinate singularity to work with for calculating tunneling rates. For simplicity, we can consider only a (1+1)-dimensional Rindler spacetime with no loss of generality. From equation (5.1), the metric is

$$ds^2 = -\left(1 + a\xi^{1'}\right)^2 \left(d\xi^{0'}\right)^2 + \left(d\xi^{1'}\right)^2 \tag{5.3}$$

Consider a local coordinate transformation within the comoving frame $\xi^{\mu'} \to q^\mu$, defined by

$$\xi^{0'} \to q^0 \quad \text{and} \quad \xi^{1'} \to a^{-1}\left(\sqrt{|1 + 2aq^1|} - 1\right) \tag{5.4}$$

which implies, and is motivated by

$$\left(1 + a\xi^{1'}\right) = \sqrt{|1 + 2aq^0|} \tag{5.5}$$

This coordinate transformation will allow us to re-express the metric's time component in a way that redistributes the metric coefficients of $d\xi^{0'}$ and $d\xi^{1'}$ to make equation (5.3) look more like the time-radial component of the Schwarzschild metric. Taking coordinate differentials,

$$\xi^{0'} = dq^0 \tag{5.6a}$$

$$d\xi^{1'} = \frac{dq^1}{\sqrt{|1 + 2aq^1|}} \tag{5.6b}$$

Substituting equations (5.6a) and (5.6b) into equation (5.3) gives

$$\begin{aligned} ds^2 &= -\left(1 + aq^1\right)^2 (dq^0)^2 + (dq^1)^2 \\ &= -\left(1 + 2aq^1\right)(dq^0)^2 + \frac{{dq^1}^2}{1 + 2aq^1} \end{aligned} \tag{5.7}$$

The metric is now singular at $q^1 = -\frac{1}{2a}$, which corresponds to the location of the Rindler horizon in the q^μ coordinate frame,

$$\xi^{1'}\left(q^1 = -\tfrac{1}{2a}\right) = a^{-1}\left(\sqrt{|1 + 2a\left(-\tfrac{1}{2a}\right)|} - 1\right) = -\frac{1}{a} \tag{5.8}$$

The metric in equation (5.7) can be made to take a form that is similar to the time-radial component of the Schwarzschild metric by letting $a \to -\frac{M}{(q^1)^2}$, in which case equation (5.7) becomes

$$ds^2 = -\left(1 - \frac{2M}{q^1}\right)(dq^0)^2 + \frac{(dq^1)^2}{1 - \frac{2M}{q^1}} \tag{5.9}$$

which is of the exact form of the time-radial component of the Schwarzschild metric.

We can re-express the Minkowski coordinates in terms of the coordinate transformations that were enacted in equation (5.4),

$$\begin{aligned} t &= \left(a^{-1} + \xi^{1'}\right)\sinh(a\xi^{0'}) && (5.10a)\\ &= a^{-1}\left(1 + a\xi^{0'}\right)\sinh\left(a\xi^{0'}\right) \\ &= a^{-1}\sqrt{1+2aq^1}\,\sinh\left(aq^0\right) && (5.10b) \end{aligned}$$

$$\begin{aligned} x &= \left(a^{-1} + \xi^{1'}\right)\cosh(a\xi^{0'}) && (5.10c)\\ &= a^{-1}\left(1 + a\xi^{1'}\right)\cosh(a\xi^{0'}) \\ &= a^{-1}\sqrt{1+2aq^1}\,\cosh(aq^0) && (5.10d) \end{aligned}$$

5.2 Unruh radiation as quantum tunneling

The Unruh effect is a prediction of quantum field theory, that a uniformly accelerating observer in the vacuum state of a flat spacetime will observe a thermal flux of radiation, while an inertial observer in the same vacuum state sees no thermal radiation. This thermal radiation has a characteristic temperature called the Unruh temperature T_U, which is determined by the uniformly accelerating observer's proper acceleration. The Unruh effect was explained in greater detail in chapter 4.

The standard WKB approximation is an application of quantum mechanics, and assumes no time evolution during the tunneling event. Quantum field theory's version of the WKB approximation is known as the gravitational WKB problem [8], and does factor in a timelike contribution to the tunneling rate.

Recall from equation (3.24), the Hamilton-Jacobi equations for a scalar field of mass m were found to be

$$g^{\mu\nu}\partial_\mu S \partial_\nu S + m^2 = 0 \tag{5.11}$$

where the classical action S can be separated into distinct time and space components $S(t,x) = \omega t + \bar{S}(x)$, with $\bar{S}(x)$ representing the purely spatial contribution to the action. Let ω denote the energy of the tunneling particle. From equation (5.7), we can express the metric for a (1+1)-dimensional Rindler spacetime as

$$ds^2 = -\left(1+2aq^1\right)(dq^0)^2 + \frac{dq^{1^2}}{1+2aq^1} \tag{5.12}$$

Using this to determine the inverse metric components, equation (5.11) becomes,

$$\begin{aligned} 0 &= g^{00}\left(\partial_0 S\right)^2 g^{11}\left(\partial_1 S\right)^2 + m^2 \\ &= -\left(1+2aq^1\right)^{-1}\left(\partial_0 S\right)^2 + \left(1+2aq^1\right)\left(\partial_1 S\right)^2 + m^2 \end{aligned} \tag{5.13}$$

where script 0 corresponds to the time component and script 1 denotes the spatial coordinate. The action is expressed in the coordinates q^μ as

$$S(q^0, q^1) = \omega q^0 + \bar{S}(q^1) \tag{5.14}$$

which implies

$$\partial_0 S(q^0, q^1) = \frac{\partial S(q^0, q^1)}{\partial q^0} = \omega \tag{5.15a}$$

$$\partial_1 S(q^0, q^1) = \frac{\partial S(q^0, q^1)}{\partial q^1} = \partial_1 \bar{S}(q^1) \tag{5.15b}$$

Putting equations (8.52a) and (8.52b) into (5.13) gives

$$-\frac{\omega^2}{1+2aq^1} + \left(1+2aq^1\right)\left(\partial_1 \bar{S}(q^1)\right)^2 + m^2 = 0 \tag{5.16}$$

Rearranging to solve for $\left(\partial_1 \bar{S}(q^1)\right)^2$,

$$\begin{aligned} \left(\partial_1 \bar{S}(q^1)\right)^2 &= \left(1+2aq^1\right)^{-1}\left(\frac{\omega^2}{1+2aq^1} - m^2\right) \\ &= \frac{\omega^2 - m^2\left(1+2aq^1\right)}{\left(1+2aq^1\right)^2} \\ &= \frac{\omega^2 - m^2\left(1+2aq^1\right)}{(2a)^2\left(q^1+(2a)^{-1}\right)^2} \end{aligned} \tag{5.17}$$

Taking the square root of both sides and integrating over q^1 gives

$$\begin{aligned} \bar{S}(q^1) &= \int_{-\infty}^{\infty} \partial_1 \bar{S}(q^1) dq^1 \\ &= \pm\frac{1}{2a}\int_{-\infty}^{\infty} \frac{\sqrt{\omega^2 - \left(1+2aq^1\right)m^2}}{q^1 - (-2a)^{-1}} dq^1 \end{aligned} \tag{5.18}$$

The sign ambiguity in equation (5.18) arises because ingoing and outgoing‘ particles tunnel in opposite directions across the Rindler horizon. The plus and minus correspond to ingoing and outgoing particles respectively. Notice that since an outgoing particle travels in the negative

q^1-direction, the bounds of integration are interchanged, which makes both contributions positive.

The spatial contribution to the action in equation (5.18) has a simple pole at $q^1 = -\frac{1}{2a}$. Let $\epsilon e^{i\phi} = q^1 + (2a)^{-1}$, which implies $dq^1 = i\epsilon e^{i\phi} d\phi$. For the case of ingoing positive energy particles, equation (5.18) can be written as

$$\begin{aligned}\bar{S} &= \lim_{\epsilon \to 0} \frac{1}{2a} \int_{\pi}^{2\pi} \frac{\sqrt{\omega^2 - \left(1 + 2a\left(\epsilon e^{i\phi} - (2a)^{-1}\right)\right) m^2}}{\epsilon e^{i\phi}} \left(i\epsilon e^{i\phi} d\phi\right) \\ &= \frac{i}{2a} \int_{\pi}^{2\pi} \sqrt{\omega^2 - \left(1 + 2a(2a)^{-1}\right) m^2} d\phi \\ &= \frac{i\omega}{2a} \int_{\pi}^{2\pi} d\phi \\ &= \frac{i\pi\omega}{2a} \end{aligned} \tag{5.19}$$

A similar integral is calculated for the case of outgoing particles to give the same result as equation (5.19). The total spatial contribution to the action is had by adding the contributions of ingoing and outgoing particles,

$$S_{0,Total} = \frac{i\pi\omega}{2a} + \frac{i\pi\omega}{2a} = \frac{i\pi\omega}{a} \tag{5.20}$$

Calculating the time component's contribution to tunneling

The time component's contribution to the action is associated with the rotation of axes that occurs when the metric switches its signature. The point along the x-axis at which this occurs can be found by looking at the invariant interval,

$$x^2 - t^2 = a^{-2} \implies t(x) = \pm\sqrt{x^2 - a^{-2}} \tag{5.21}$$

It is easy to see from equation (5.21) that the coordinate axes undergo rotation into the complex plane at about the instant when $x = a^-1$. Therefore, $x = a^{-1}$ can be used to define the radius of rotation. We can define an analytic continuation of the time coordinate to calculate the amount of time that occurs during this rotation.

$$t \equiv t_0 e^{i\phi} \tag{5.22}$$

which implies $dt = it_0 e^{i\phi} d\phi$. Since the axes rotate with a radius $x = a^{-1}$, it makes sense that $t_0 = a^{-1}$. Two time coordinate translations occur when a particle crosses the horizon, one for equation (5.10a) and one for equation (5.10c). The axis rotations are caused by an imaginary time translation $\xi^{0'} \to \xi^{0'} - \frac{i\pi}{2}$,

$$\omega \Delta t = \omega \int dt = i\omega a^{-1} \int_{\pi/2}^{0} d\phi = -\frac{i\pi\omega}{2a} \tag{5.23}$$

Since both t in equation (5.10a) and x in equation (5.10c) are parameterized with the Rindler observer's time coordinate, each contributes a factor of $-\frac{i\pi\omega}{2a}$ to the tunneling rate. The total contribution from the time component is then

$$\omega\Delta t = -\frac{i\pi\omega}{a} \tag{5.24}$$

The Unruh temperature

The tunneling rate predicted by the gravitational WKB approximation is given by [8]

$$\Gamma \sim e^{-\frac{1}{\hbar}\left[\mathrm{Im}\left(\oint p_x dx\right) - \omega \mathrm{Im}(\Delta t)\right]} \tag{5.25}$$

where Γ denotes the tunneling rate. The temperature of the Unruh radiation can be found by assuming a Boltzmann distribution of energy states

$$\Gamma \sim e^{-\omega/T} \tag{5.26}$$

Equation (5.26) expresses the probability that the tunneling particle is in the energy state ω, given that the thermal flux has a temperature T. To relate the tunneling rate of Unruh radiation to the Unruh temperature we can set the exponents of equations (5.25) and (5.26) equal to each other,

$$\omega/T = \tfrac{1}{\hbar}\left[\mathrm{Im}\left(\oint p_x dx\right) - \omega \mathrm{Im}\left(\Delta t\right)\right] \tag{5.27}$$

which can be rearranged to obtain a formula for the Unruh temperature,

$$T_U = \frac{\omega}{\mathrm{Im}\left(\oint p_x dx\right) - \omega \mathrm{Im}\left(\Delta t\right)} \tag{5.28}$$

where I've set $\hbar = 1$. From equations (5.20) and (5.24),

$$\mathrm{Im}\left(\oint p_x dx\right) = \frac{\pi\omega}{a} \qquad \text{and} \qquad \mathrm{Im}\left(\Delta t\right) = \frac{\pi\omega}{a} \tag{5.29}$$

which, together with equation (5.28), gives the Unruh temperature

$$T_U = \frac{\omega}{\frac{\pi\omega}{a} - \left(-\frac{\pi\omega}{a}\right)} = \frac{a}{2\pi} \tag{5.30}$$

Chapter 6

Proper acceleration in the Schwarzschild geometry and surface gravity

6.1 Proper acceleration of an observer in the Schwarzschild geometry

The 4-acceleration of an object in arbitrary coordinates is defined as

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{d\tau} \tag{6.1}$$

where as a summation $\boldsymbol{a} = a^{\mu}\boldsymbol{e}_{\mu}$.

The 4-acceleration of an object traveling along its spacetime geodesic has a scalar magnitude $a = \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}} = 0$. Important to note, is that general relativity does not view the gravitational influence on an object's motion as an external force acting on that object. Rather, general relativity interprets gravity as an intrinsic property of spacetime, that influences the observer's trajectory by curving the spacetime manifold. This means that an object traveling along its spacetime geodesic has a trajectory that is characterized by

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{d\tau} = 0 \tag{6.2}$$

This equation can be shown to be exactly equivalent to the geodesic equation [18].

If the object has a force other than the influence of gravity acting on it, then the object's 4-acceleration has a nonzero scalar value,

$$\boldsymbol{a} = \frac{d\boldsymbol{u}}{d\tau} \neq 0 \tag{6.3}$$

A force acting on the object prevents the object from following its natural spacetime geodesic.

Consider a static observer situated at some arbitrary r in the Schwarzschild geometry. In order to remain at r the observer needs to be acted on by a force that is directed radially away from the black hole.

$$\begin{aligned}\boldsymbol{a} = \frac{d\boldsymbol{u}}{d\tau} &= \left(\nabla_\mu u^\nu\right) \boldsymbol{e}_\nu \frac{dx^\mu}{d\tau} \\ &= \left[\partial_\mu u^\nu + \Gamma^\nu_{\mu\alpha} u^\alpha\right] \boldsymbol{e}_\nu u^\mu \end{aligned} \tag{6.4}$$

where the terms with Christoffel symbols can be thought of as the influence of gravity on the observer's 4-acceleration.

Now consider the case of an observer that remains stationary at some Schwarschild radial coordinate r. We can construct a local tangent space for this observer, with a basis $\boldsymbol{e}_{\mu'}$. I'm using primed coordinates to represent the coordinates of the observer's local tangent space and the unprimed coordinates are Schwarzschild coordinates. We can also choose to align the observer's local time axis $\boldsymbol{e}_{t'}$ with the Schwarzschild time axis $\boldsymbol{e}_t$. Since the observer is stationary in the Schwarzschild coordinates, the only nonzero component of the observer's 4-velocity is the time component,

$$u^\mu = \left(u^t, 0, 0, 0\right) \tag{6.5}$$

To avoid potential confusion, let me remark that u^μ represents the observer's 4-velocity in Schwarzschild coordinates, and $u^{\mu'}$ represents the observer's 4-velocity in the coordinates of the observer's local tangent space. Since $\boldsymbol{u}\cdot\boldsymbol{u} = -1$ is a scalar invariant, it is the same in all coordinate systems. Therefore we can write

$$\begin{aligned}-1 &= \boldsymbol{u}\cdot\boldsymbol{u} \\ &= \left(u^\mu \boldsymbol{e}_\mu\right)\cdot\left(u^\nu \boldsymbol{e}_\nu\right) \\ &= \left(\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu\right) u^\mu u^\nu \\ &= g_{\mu\nu} u^\mu u^\nu \end{aligned} \tag{6.6}$$

As is shown in equation (6.5), the time component of u^μ is its only nonzero component. This means that the summation over μ and ν in equation (6.6) simplifies to

$$\begin{aligned}-1 &= g_{tt}\left(u^t\right)^2 \\ &= -\left(1 - \frac{2M}{r}\right)\left(u^t\right)^2 \end{aligned} \tag{6.7}$$

which implies

$$u^t = \left(1 - \frac{2M}{r}\right)^{-1/2} \tag{6.8}$$

Therefore, the 4-velocity of the stationary observer at r is expressed in Schwarzschild coordinates as

$$u^\mu = \left(\left(1 - \frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right) \tag{6.9}$$

Since the only nonzero component of u^μ is its time component, equation (6.4) can be dramatically simplified by setting $\mu, \alpha = t$,

$$\boldsymbol{a} = \left[\partial_t u^\nu + \Gamma^\nu_{tt} u^t\right] \boldsymbol{e}_\nu u^t \tag{6.10}$$

The partial derivative $\partial_t u^\nu = 0$ vanishes for all ν because equation (6.9) is independent of t. This simplifies equation (6.10) to

$$\boldsymbol{a} = \Gamma^\nu_{tt} \boldsymbol{e}_\nu \left(u^t\right)^2 \tag{6.11}$$

Using a general formula for Christoffel symbols in terms of the metric tensor [18]

$$\Gamma^\nu_{\mu\alpha} = \frac{1}{2} g^{\nu\beta} \left[\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\beta\mu} - \partial_\beta g_{\mu\alpha}\right] \tag{6.12}$$

Adapting this to equation (6.11), we can specify $\mu, \alpha \to t$. And noting that the Schwarzschild metric is diagonal allows $\beta \to \nu$. Therefore equation (6.12) simplifies to

$$\Gamma^\nu_{tt} = \frac{1}{2} g^{\nu\nu} \left[\partial_t g_{t\nu} + \partial_t g_{\nu t} - \partial_\nu g_{tt}\right] \tag{6.13}$$

For $\nu = t$ this becomes

$$\Gamma^t_{tt} = \frac{1}{2} g^{tt} \partial_t g_{tt} = 0 \tag{6.14}$$

which follows from the fact that the metric is stationary

$$\partial_t g_{tt} = \frac{\partial}{\partial t}\left(1 - \frac{2M}{r}\right) = 0 \tag{6.15}$$

For $\nu = r$

$$\Gamma^r_{tt} = \frac{1}{2} g^{rr} \left[\partial_t g_{tr} + \partial_t g_{rt} - \partial_r g_{tt}\right] = -\frac{1}{2} g^{rr} \partial_r g_{tt} \tag{6.16}$$

where the first two terms evaluate to zero as a result of the metric being diagonal. From equation (**??**) for the Schwarzschild metric, $g^{rr} = g_{tt} = \left(1 - \frac{2M}{r}\right)$. Putting this into equation (6.16),

$$\Gamma^r_{tt} = \frac{1}{2}\left(1 - \tfrac{2M}{r}\right) \frac{\partial}{\partial_r}\left(1 - \tfrac{2M}{r}\right) = \frac{M}{r^2}\left(1 - \tfrac{2M}{r}\right) \tag{6.17}$$

For $\nu = \theta$ equation (6.13) gives,

$$\Gamma^{\theta}_{tt} = \frac{1}{2} g^{\theta\theta} \left[\partial_t g_{t\theta} + \partial_t g_{\theta t} - \partial_\theta g_{tt} \right] = 0 \tag{6.18}$$

The first two terms evaluate to zero because the metric is diagonal, and the third term is zero because $g_{tt} = \left(1 - \frac{2M}{r}\right)$ is independent of θ. Similarly, for $\nu = \phi$ equation (6.13) gives

$$\Gamma^{\phi}_{tt} = \frac{1}{2} g^{\phi\phi} \left[\partial_t g_{t\phi} + \partial_t g_{\phi t} - \partial_\phi g_{tt} \right] = 0 \tag{6.19}$$

with all three terms evaluating to zero for the same reasons as $\nu = \theta$. Since the only nonzero Christoffel symbol is Γ^r_{tt}, from equations (6.9), (6.11), and (6.17), we have

$$\begin{aligned} \boldsymbol{a} &= \Gamma^r_{tt} \left(u^t\right)^2 \boldsymbol{e}_r \\ &= \frac{M}{r^2} \left(1 - \tfrac{2M}{r}\right) \left(1 - \frac{2M}{r}\right)^{-1} \boldsymbol{e}_r \\ &= \frac{M}{r^2} \boldsymbol{e}_r \end{aligned} \tag{6.20}$$

From this we can find the corresponding scalar invariant,

$$\begin{aligned} a &= \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} \\ &= \left[\left(a^\mu \boldsymbol{e}_\mu\right) \left(a^\nu \boldsymbol{e}_\nu\right) \right]^{1/2} \\ &= \left[\left(\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\mu\right) a^\mu a^\nu \right]^{1/2} \\ &= \left[g_{\mu\nu} a^\mu a^\nu \right]^{1/2} \\ &= \sqrt{g_{rr}} a^r \\ &= \frac{M}{r^2 \sqrt{1 - \frac{2M}{r}}} \end{aligned} \tag{6.21}$$

The magnitude of the stationary observer's 4-acceleration a is a coordinate-independent quantity, which means that its value is the same regardless of what coordinate system we use to calculate it. a represents the observer's proper acceleration, and can thus be viewed as the acceleration that an observer at r would actually need in order to remain stationary. Just as was the case for the proper acceleration used to derive the Rindler coordinates, the magnitude of $\boldsymbol{a}$ is the same as would be measured by a comoving observer that is instantaneously at rest at r. This is because the comoving observer's proper time coincides with the proper time of a stationary and uniformly accelerating observer at r.

Note equation (6.21) becomes infinite at $r = 2M$,

$$a = \left. \frac{M}{r^2 \sqrt{1 - \frac{2M}{r}}} \right|_{r=2M} = \frac{1}{4M \times 0} = \infty \tag{6.22}$$

which reflects the fact that once an observer has fallen into a black hole they cannot escape. The fact that the observer's proper acceleration is no longer defined once we reach the neighborhood of $r = 2M$ means that it is no longer good measure of acceleration. It is for this reason that the concept of surface gravity κ was developed.

6.2 Surface gravity κ

As was explained in the previous section, a stationary observer in the Schwarzschild geometry has proper acceleration that diverges near the event horizon. Surface gravity κ can be thought of as a measure of acceleration due to gravity that has been normalized by gravitational redshifting. Because of this normalization, surface gravity remains finite near the event horizon. This section contains a derivation of surface gravity.

Suppose that an object of unit mass is located at r, and is accelerated upward by a force F an infinitesimal distance δr. The amount of work δW done by the force is

$$\delta W = F(r)\delta r = ma(r)\delta r|_{m=1} = a(r)\delta r \tag{6.23}$$

From equation (6.21), $a(r) = \frac{M}{r^2\sqrt{1-2M/r}}$ is the proper acceleration required for an observer at r to remain stationary in the Schwarzschild geometry. Substituting proper acceleration $a(r)$ into equation (6.23) gives,

$$\delta W = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}\delta r \tag{6.24}$$

Now suppose that the observer at r were to convert this work into radiation with 100% efficiency, and then transmit the radiation to an observer at infinity where the radiation is intercepted and measured. The work δW can be represented as a single high energy photon of angular frequency $\delta\omega$. Using this high energy photon $\delta\omega$, we can re-express equation (6.24) as

$$\delta\omega = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}\delta r \tag{6.25}$$

where I've just made the replacement $\delta W \rightarrow \delta\omega$.

An observer at infinity who intercepts this photon perceives it to be of much lower energy than the observer who transmitted it. The high energy photon has been gravitationally redshifted. To find the amount of gravitational redshifting, consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2 \tag{6.26}$$

Setting $dr, d\Omega = 0$ in equation (7.3), recalling that $d\tau^2 = -ds^2$, and taking the square root of both sides gives

$$d\tau = \sqrt{1 - \frac{2M}{r}} dt \tag{6.27}$$

From this equation it is easy to see that the proper time for a rest observer at infinity is equal to the Schwarzschild time coordinate $d\tau_\infty = dt$. We can then rewrite equation (6.27) as

$$d\tau_r = \sqrt{1 - \frac{2M}{r}} d\tau_\infty \tag{6.28}$$

where τ_r is the proper time of a stationary rest observer at r. If we take the proper frequency of the high-energy photon to be $\delta\omega = 1/d\tau$, then the proper frequency of the photon at r is related to the frequency of the photon at infinity by

$$\delta\omega_\infty = \sqrt{1 - \frac{2M}{r}} \delta\omega_r \tag{6.29}$$

where I've taken the inverse of both sides of equation (6.28), replaced $1/d\tau$ with $\delta\omega$, and multiplied both sides by $\sqrt{1 - 2M/r}$. Replacing $\delta\omega_r$ with the right hand side of equation (6.25) gives

$$\delta\omega_\infty = \frac{M}{r^2} \delta r \tag{6.30}$$

Finally, note that the terms in equation (6.30) are energy, which has units of [force][distance]. Dividing both sides of equation (6.30) by δr gives us a force,

$$\delta F_\infty = \frac{M}{r^2} \tag{6.31}$$

Since this force is acting on a unit mass it can also be interpreted as an acceleration,

$$\kappa(r) \equiv \frac{M}{r^2} \tag{6.32}$$

This measure of acceleration is called surface gravity. Specifically, it is a measure of the acceleration that must be exerted on a stationary rest observer at r in order to prevent them from falling into the black hole, normalized to the perspective of an observer at infinity by assuming that the acceleration is "gravitationally redshifted." In the case of black holes, surface gravity is generally calculated at the event horizon, in which case equation (6.32) gives

$$\kappa(r = 2M) = \frac{1}{4M} \tag{6.33}$$

Alternative approaches to deriving surface gravity can be found in [22, 31].

Chapter 7

Metrics of the Schwarzschild geometry

7.1 The Gullstrand-Painléve metric

The original WKB calculation for Hawking radiation as a quantum tunneling process, by Parikh and Wilczek, made use of the Gullstrand-Painlevé metric [2]. Among the Gullstrand-Painléve metric's nice features are that it is regular at the event horizon, the metric is stationary, and its time coordinate has the very straightforward interpretation of being the proper time of an observer who free falls from rest at infinity.

This free falling observer's proper time t_{ff} can be written as a function of the Schwarzschild coordinate time t_S in the following form [2],

$$t_{ff} = t_S + 2\sqrt{2Mr} + 2M\ln\left|\frac{\sqrt{r}-\sqrt{2M}}{\sqrt{r}+\sqrt{2M}}\right| \tag{7.1}$$

From this relation, the Gullstrand-Painléve coordinates can be had by rewriting the Schwarzschild metric in terms of t_{ff}, and solving for a line element that eliminates the Schwarzschild coordinate time t_s from the metric. After simplifying the notation with the replacement $t_{ff} \to t$, the Gullstrand-Painléve line element takes the form of

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2 \tag{7.2}$$

where $d\Omega^2 = d\theta^2 + \sin\theta d\phi^2$ is the metric's angular component.

7.1.1 Derivation of the Gullstrand-Painléve coordinates

The derivation of the Gullstrand-Painléve metric performed in this section takes a different approach than the approach that was previously mentioned. This derivation starts from the Schwarzschild metric and assumes a time translation $t_S = t + f(r)$ that makes hypersurfaces of constant time flat. The Schwarzschild metric is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2 \tag{7.3}$$

The time coordinate for any observer in a spherically symmetric spacetime can be related to the proper time of an observer at infinity through a function of the Schwarzschild radial coordinate.

$$t_S = t + f(r) \tag{7.4}$$

whose squared differential expression is

$$dt_S = dt + f'(r)dr \implies dt_S^2 = dt^2 + 2f'(r)dtdr + f'(r)^2dr^2 \tag{7.5}$$

Substitution of equation (7.5) into the Schwarzschild metric gives

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)\left(dt^2 + 2f'(r)dtdr + f'(r)^2dr^2\right) + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2d\Omega^2 \\ &= -\left(1-\frac{2M}{r}\right)dt^2 - 2\left(1-\frac{2M}{r}\right)f'(r)dtdr + \left(\frac{1}{1-\frac{2M}{r}} - \left(1-\frac{2M}{r}\right)f'(r)^2\right)dr^2 \\ &+ r^2d\Omega^2 \end{aligned} \tag{7.6}$$

If we require hypersurfaces of constant time to be flat, then the coefficient of dr^2 must equal one.

$$1 = \frac{1}{1-\frac{2M}{r}} - \left(1-\frac{2M}{r}\right)f'(r)^2 \implies f'(r) = \sqrt{\frac{2M}{r}}\left(1-\frac{2M}{r}\right)^{-1} \tag{7.7}$$

whre the positive sign choice is given by [32].

Replacing $f'(r)$ in equation (7.1.1) with its relation in equation (7.7) gives the Gullstrand-Painléve metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2d\Omega^2 \tag{7.8}$$

7.1.2 Radial lightlike geodesics

Lightlike geodesics are characterized by $ds^2 = 0$. The reason for this is most easily seen from the line element for a (1+1)-dimensional Minkowski spacetime,

$$ds^2 = -dt^2c^2 + dx \tag{7.9}$$

where I've replaced the speed of light c to illustrate a point. Suppose that the distance dx is traversed by a light beam during a time interval dt. Since light invariably travels at speed $v = c$, the light beam traveled a total distance $dx = cdt$ through space. Putting this into equation (7.9) gives

$$ds^2 = -dt^2c^2 + dt^2c^2 = 0 \tag{7.10}$$

Therefore $ds^2 = 0$ for lightlike geodesics. The same proof can be given for any diagonal metric with arbitrary components by placing the metric components in front of their corresponding differentials.

Setting $ds^2 = 0$ to make the geodesics lightlike, and setting $d\Omega = 0$ to make the geodesics radial, we can write equation (7.8) as,

$$0 = -\left(1 - \frac{2M}{r}\right) dt^2 + 2\sqrt{\frac{2M}{r}} dt dr + dr^2 \tag{7.11}$$

Dividing Equation (7.11) by dt^2 gives us a function that is quadratic in $\frac{dr}{dt}$,

$$\left(\frac{dr}{dt}\right)^2 + 2\sqrt{\frac{2M}{r}}\frac{dr}{dt} - \left(1 - \frac{2M}{r}\right) = 0 \tag{7.12}$$

which can be solved using the quadratic formula to give

$$\frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}} \tag{7.13}$$

with $+/-$ corresponding to the outgoing/ingoing radial null geodesics. This correspondence is easy to see by putting c back into equation (7.13),

$$\frac{dr}{dt} = \pm c - \sqrt{\frac{2M}{r}} \tag{7.14}$$

This replacement makes it easy to see that the ± 1 term in equation (7.13) corresponds to the contribution to $\frac{dr}{dt}$ that comes from light's inherent property of propagating through space at speed c. The second term $-\sqrt{2M/r}$ in equation (7.14) is the contribution to $\frac{dr}{dt}$ from the black hole's curvature effects on the spacetime geometry.

7.2 The Eddington-Finkelstein coordinates

Like the Gullstrand-Painléve coordinates, the Eddington-Finkelstein coordinates are non-singular at the event horizon. And as is shown in section 9.1.2, the calculation for Hawking radiation tunneling rates are slightly more straightforward in the Eddington-Finkelstein coordinates than in the Gullstrand-Painléve coordinates.

7.2.1 Deriving the Eddington-Finkelstein coordinates

The tortoise coordinate

The tortoise coordinate $r^* = r^*(r)$ is defined non-uniquely as a coordinate r^* that satisfies

$$dr^* = \frac{dr}{1 - \frac{2M}{r}} \tag{7.15}$$

where r is the Schwarzschild radial coordinate. Equation (7.15) indicates that $dr^* \to dr$ in the limit of large r. So the tortoise coordinate is approximately equal to the Schwarzschild radial coordinate at distances far from the black hole. Equation (7.15) also indicates that dr^* blows up near $r = 2M$. Integrating both sides of equation (7.15),

$$\begin{aligned} r^* &= \int_*^{r^*} dr^{*\prime} \\ &= \int \frac{dr}{1-\frac{2M}{r}} \\ &= \int \frac{rdr}{r-2M} \\ &= \int \frac{r-2M+2M}{r-2M} dr \\ &= \int \left(1+\frac{2M}{r-2M}\right) dr \\ &= r + 2M \ln\left|\frac{r-2M}{2M}\right| + C \end{aligned} \tag{7.16}$$

Note that the natural logarithm is undefined for $r \leq 2M$, which means that the tortoise coordinate only describes the spacetime geometry outside of the black hole. This gives $r^* = r^*(r)$ a domain $r \in (2M, \infty)$, which means that the tortoise coordinate ranges over $r^* \in (-\infty, \infty)$. As was mentioned before, the tortoise coordinate is defined non-uniquely. And since r^* can take on any value on the real line, we are free to make C, in equation (7.16), equal to any real number that we want. The most common choice is to choose $C = 0$,

$$r^* = r + 2M \ln\left|\frac{r-2M}{2M}\right| \tag{7.17}$$

The reason for defining r^* such that it satisfies equation (7.15) is because it can then be used to put the Schwarzschild metric's time-radial component in a form that is conformally flat. Squaring both sides of equation (7.15) allows us to rewrite the Schwarzschild metric's radial component as,

$$g_{rr} = \frac{dr^2}{1-\frac{2M}{r}} = \left(1-\frac{2M}{r}\right) dr^{*2} \tag{7.18}$$

Substituting this into the Schwarzschild metric in equation (7.3) gives

$$ds^2 = -\left(1-\frac{2M}{r}\right)\left(dt^2 - dr^{*2}\right) + r^2 d\Omega^2 \tag{7.19}$$

Defining the Eddington-Finkelstein coordinates

The Eddington-Finkelstein coordinates u, v are lightcone coordinates defined with respect to Schwarzschild coordinate time t and tortoise coordinate r^*,

$$u = t - r^* \tag{7.20a}$$
$$v = t + r^* \tag{7.20b}$$

with u and v corresponding to outgoing and ingoing radial lightlike geodesics, respectfully. A general description of lightcone coordinates is given in section 4.4. Taking coordinate differentials of equations (7.20a) and (7.20b),

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial r^*}dr^* = dt - dr^* \tag{7.21a}$$
$$dv = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial r^*}dr^* = dt + dr^* \tag{7.21b}$$

Equation (7.21b) implies $dt = dv - dr^*$. Squaring both sides gives

$$dt^2 = dv^2 - 2dvdr^* + dr^{*2} \tag{7.22}$$

We can now substitute equation (7.22) into the metric in equation (7.19) to re-express its time-radial component as

$$\begin{aligned} -\left(1 - \frac{2M}{r}\right)\left(dt^2 - dr^{*2}\right) &= -\left(1 - \frac{2M}{r}\right)\left(dv^2 - 2dvdr^*\right) \\ &= -\left(1 - \frac{2M}{r}\right)dv^2 + 2\left(1 - \frac{2M}{r}\right)dvdr^* \\ &= -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr \end{aligned} \tag{7.23}$$

where the last step follows from $dr^* = (1 - \frac{2M}{r})^{-1}dr$. Adding the angular component back in gives

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2 \tag{7.24}$$

Equation (7.24) is the line element for the ingoing Eddington-Finkelstein coordinates. The same procedure can be applied for $du = dt - dr^*$ to give

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2dudr + r^2 d\Omega^2 \tag{7.25}$$

which is the line element for the outgoing Eddington-Finkelstein coordinates.

Radial lightlike geodesics in the ingoing and outgoing Eddington-Finkelstein coordinates

Lightlike geodesics are characterized by $ds^2 = 0$, so to find the radial lightlike geodesics of the ingoing Eddington-Finkelstein coordinates, we can start by setting the time-radial component of equation (7.24) equal to zero,

$$0 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr \tag{7.26}$$

Ingoing radial null geodesics are well represented through an equation of the form $\frac{dr}{dv}$. Equation (7.26) is quadratic in dv, which suggests dividing equation by dr^2 to solve for the inverse function $\frac{dv}{dr}$. Dividing both sides of equation (7.26) by $-dr^2$ gives,

$$\left(1 - \frac{2M}{r}\right)\left(\frac{dv}{dr}\right)^2 - 2\frac{dv}{dr} = 0 \tag{7.27}$$

Equation (7.27) can be solved for $\frac{dv}{dr}$ through the quadratic equation, which gives

$$\frac{dv}{dr} = 2\left(1 - \frac{2M}{r}\right)^{-1} \tag{7.28}$$

Inverting equation (7.28) gives us the radial null geodesics of the ingoing Eddington-Finkelstein coordinates,

$$\frac{dr}{dv} = +\frac{1}{2}\left(1 - \frac{2M}{r}\right) \tag{7.29}$$

The same procedure applied for the outgoing Eddington-Finkelstein coordinate u gives

$$\frac{dr}{du} = -\frac{1}{2}\left(1 - \frac{2M}{r}\right) \tag{7.30}$$

7.3 The Kruskal-Szekeres Coordinates

Among the many appearances of the Kruskal-Szekeres metric is a form that is conformally hyperbolic in its time-radial component,

$$ds^2 = -\Lambda(R)^2\left(R^2 d\omega^2 - dR^2\right) + r^2 d\Omega^2 \tag{7.31}$$

where

$$\Lambda(R)^2 = \frac{(4M)^2}{R^2}\left(1 - \frac{2M}{r}\right) \tag{7.32a}$$

$$R = Me^{r^*/4M} \tag{7.32b}$$

As equation (7.31) shows, the Kruskal-Szekeres spacetime is very similar to a Rindler spacetime. The two metrics have a very similar interpretation as well: both metrics can be viewed as describing an observer undergoing uniform acceleration. The difference comes from the fact that a Kruskal-Szekeres observer is accelerating in the presence of a black hole, while a Rindler is accelerating in ordinary flatspace. Because of this, a Kruskal-Szekeres observer corresponding to a line of constant acceleration, can be thought of as accelerating just enough to offset the pull of gravity, in order to hover above the black hole.

The metric for a spherically symmetric black hole can be written in the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \tag{7.33}$$

where

$$f(r) = \left(1 - \frac{2M}{r}\right) \qquad \text{(Schwarzschild)} \tag{7.34}$$

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \qquad \text{(Reissner-Nordström)} \tag{7.35}$$

Taking the time-radial component of equation (7.33), and multiplying the radial term by $\frac{f(r)}{f(r)}$ allows us to pull out $f(r)$ as a scale factor. The metric and then be rearranged to be written in terms of the lightlike Eddington-Finkelstein coordinates,

$$\begin{aligned} ds^2 &= -f\left[dt^2 - \frac{dr^2}{f^2}\right] \\ &= -f\left[dt^2 - dr^{*2}\right] \\ &= -f\left(dt - dr^*\right)\left(dt + dr^*\right) \\ &= -f du dv \end{aligned} \tag{7.36}$$

where r^* is the tortoise coordinate from equation (7.17), and u, v are Eddington-Finkelstein coordinates from equations (7.20a) and (7.20b). Let's now define U, V with the condition that they satisfy

$$du \equiv -\frac{1}{\kappa} d\ln U = \frac{dU}{\kappa U} = dt - dr^* \tag{7.37a}$$

$$dv \equiv = \frac{1}{\kappa} d\ln V = \frac{dV}{\kappa V} = dt + dr^* \tag{7.37b}$$

where κ is a temporarily unknown constant. Multiplying both sides of equation (7.37a) by $-\kappa$ and taking an indefinite integral,

$$\begin{aligned}\ln U &= \int d\ln U \\ &= -\kappa \int du \\ &= -\kappa\,(u + C_U) \\ &= -\kappa\,(t - r^* + C_U)\end{aligned} \tag{7.38}$$

Exponentiating equation (7.38) gives,

$$U = e^{-\kappa(u+C_U)} = e^{-\kappa(t-r^*+C_U)} \tag{7.39}$$

Similarly for V,

$$\begin{aligned}\ln V &= \int d\ln V \\ &= \kappa \int dv \\ &= \kappa\,(v + C_V) \\ &= \kappa\,(t + r^* + C_V)\end{aligned} \tag{7.40}$$

Exponentiating gives

$$V = e^{\kappa(v+C_V)} = e^{\kappa(t+r^*+C_V)} \tag{7.41}$$

Putting equations (7.37a) and (7.37b) into (7.36) gives

$$\begin{aligned}ds^2 &= -f\,du dv \\ &= +\frac{f}{\kappa^2} d\ln U\, d\ln V \\ &= \frac{f}{\kappa^2}\frac{dU}{U}\frac{dV}{V} \\ &= \frac{f}{\kappa^2}\frac{dU}{\exp\left[-\kappa\,(t - r^* + C_U)\right]}\frac{dV}{\exp\left[\kappa\,(t + r^* + C_V)\right]} \\ &= \frac{Cf}{\kappa^2} e^{-2\kappa r^*} dU dV\end{aligned} \tag{7.42}$$

where the t coordinates canceled and $C = e^{\kappa(C_U - C_V)}$. The second to last step comes from replacing U and V with equatons (7.39) and (7.41), respectively. Substitution of

$$f(r) = 1 - \frac{2M}{r} \quad \text{and} \quad r^* = r + 2M \ln\left|\frac{r-2M}{2M}\right| \tag{7.43}$$

into equation (7.42) gives

$$\begin{aligned} ds^2 &= \frac{1-\frac{2M}{r}}{\kappa^2} C \exp\left[-2\kappa\left(r + 2M\ln\left|\frac{r-2M}{2M}\right|\right)\right] dUdV \\ &= \frac{C\left(r-2M\right)}{r\kappa^2} e^{-2\kappa r}\left(\frac{r-2M}{2M}\right)^{4M\kappa} dUdV \\ &= \frac{C\left(2M\right)^{4M\kappa} e^{-2\kappa r}}{\kappa^2 r}\left(r-2M\right)^{1-4M\kappa} dUdV \end{aligned} \tag{7.44}$$

This metric is singular at the event horizon unless $(r-2M)^{1-4M\kappa}$ except for the special case that $\kappa = \frac{1}{4M}$. This is interesting, because it just so happens that the surface gravity $\kappa(r = 2M) = \frac{1}{4M}$ for a Schwarzschild black hole. In order for the metric to be regular at $r = 2M$ we must define κ in equations (7.37a) and (7.37b) to be equal to the black hole's surface gravity.

It's shown in [22] that equations (7.37a) and (7.37b) can be solved to give,

$$U = \kappa^{-1}e^{-\kappa u} \quad \text{and} \quad V = \kappa^{-1}e^{+\kappa v} \tag{7.45}$$

with

$$\frac{U}{V} = e^{2\kappa t} \tag{7.46}$$

Chapter 8

Calculating the Hawking temperature in the near horizon approximation

In this chapter we use the connection between Rindler space and the local Schwarzschild geometry to calculate the Hawking temperature using the gravitational WKB approximation, in the coordinates of the near horizon approximation. This derivation involves a sequence of coordinate transformations, and draws upon a lot of material from previous chapters. The first step is to start with the Schwarzschild metric and derive the near horizon approximation. Second, we use the fact that the Rindler metric is conformally flat, and thus can be related to a local set of Minkowski coordinates. Third, we find a new set of local Rindler coordinates, which are easier to work with than the ones given to us by the near horizon approximation. The fourth step is to use these new local Rindler coordinates to calculate the Hawking temperature, in the same way that the Unruh temperature was calculated in section 5.2, with proper acceleration replaced by surface gravity.

8.1 The near horizon approximation

The near horizon approximation describes the Schwarzschild geometry in the local inertial frame of an observer near the event horizon. The near horizon approximation's significance, is that locally it converges to the Rindler metric, illustrating the near equivalence between gravity and acceleration.

We can derive the near horizon approximation by making approximations for a local inertial observer, to each component of the Schwarzschild metric individually. Starting from the Schwarzschild metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2 \tag{8.1}$$

An infinitesimal element of proper radial distance, as measured by an observer at $r > 2M$, is found by setting $dt, d\Omega = 0$ and taking the square root of both sides of equation (8.1),

$$ds = \frac{dr}{\sqrt{1 - \frac{2M}{r}}} \tag{8.2}$$

Let the proper distance ρ be the proper radial distance from r to $2M$, such that $r > 2M$. Integrating equation (8.2),

$$\begin{aligned} \rho &= \int_0^\rho ds \\ &= \int_{2M}^{r} \frac{dr'}{\sqrt{1 - \frac{2M}{r'}}} \\ &= 2M \sinh^{-1}\left(\sqrt{\frac{r}{2M} - 1}\right) + \sqrt{r(r - 2M)} \end{aligned} \tag{8.3}$$

The integral shown in equation (8.3) has a long calculation that can be found in appendix section .4.1. Since $d\rho = ds$ it follows from equation (8.2) that

$$ds = \frac{dr}{\sqrt{1 - \frac{2M}{r}}} \implies d\rho^2 = \frac{dr^2}{1 - \frac{2M}{r}} \tag{8.4}$$

which can be used to rewrite the Schwarzschild metric as a function of proper radial distance to the event horizon from arbitrary $r > 2M$,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + d\rho^2 + r^2 d\Omega^2 \tag{8.5}$$

where $r = r(\rho)$.

We can start by simplifying the expression for $\rho(r)$ in equation (8.3). The argument of $\sinh^{-1}\left(\sqrt{\frac{r}{2M} - 1}\right)$ approaches zero in the neighborhood of $r = 2M$. This makes a first order Taylor expansion of $\sinh^{-1}\left(\sqrt{\frac{r}{2M} - 1}\right)$ about $r = 2M$ a good approximation near the horizon. Using

$$\sinh^{-1} x = x - \tfrac{1}{6}x^3 + ... \tag{8.6}$$

the first term in equation (8.3) can be approximated as

$$\begin{aligned} 2M \sinh^{-1}\left(\sqrt{\frac{r}{2M} - 1}\right) &\approx 2M\sqrt{\frac{r}{2M} - 1} \\ &= \sqrt{2M(r - 2M)}, \quad \text{near } r = 2M \end{aligned} \tag{8.7}$$

Taylor expanding the argument of the second term in equation (8.3) at $r = 2M$ gives

$$r(r - 2M)|_{at\ r=2M} \approx 2M(r - 2M) \tag{8.8}$$

With this the second term in equation (8.3) can be approximated as

$$\sqrt{r(r-2M)} \approx \sqrt{2M(r-2M)} \tag{8.9}$$

So by using equations (8.7) and (8.9) to respectfully approximate the first and second terms of equation (8.3), in the neighborhood of $r = 2M$, equation (8.3) for proper distance to the event horizon can be approximated locally near the event horizon as

$$\rho(r) = 2\sqrt{2M(r-2M)}, \quad \text{near } r = 2M \tag{8.10}$$

The inverse $r(\rho) = \frac{p^2}{8M} + 2M$ can now be put into equation (8.5) to rewrite the metric's time component g_{tt} as a function of ρ,

$$\begin{aligned} g_{tt}(\rho) &= -\left(1 - \frac{2M}{\frac{p^2}{8M} + 2M}\right) dt^2 \\ &= -\left(\frac{\frac{p^2}{8M} + 2M - 2M}{\frac{p^2}{8M} + 2M}\right) dt^2 \\ &= -\left(\frac{\frac{p^2}{8M}}{\frac{p^2}{8M} + 2M}\right) dt^2 \\ &= -\rho^2 \left(\frac{1}{p^2 + 16M^2}\right) dt^2 \\ &= -\rho^2 \left(\frac{1}{\left(\frac{p}{4M}\right)^2 + 1}\right) \left(\frac{dt}{4M}\right)^2 \\ &\approx -\rho^2 \left(\frac{dt}{4M}\right)^2 \end{aligned} \tag{8.11}$$

Important to note is that this approximation assumes that the observer is very close to the horizon, and therefore $\rho \ll 4M \implies \left(\frac{\rho}{4M}\right)^2 \ll 1$. Replacing the metric's time component in equation (8.5) with equation (8.11) yields

$$ds^2 = -\rho^2 \left(\frac{dt}{4M}\right)^2 + d\rho^2 + r^2 d\Omega^2 \tag{8.12}$$

We can rescale the Schwarzschild time coordinate by defining

$$\omega \equiv \frac{t}{4M} \tag{8.13}$$

which allows us to write the metric as

$$ds^2 = -\rho^2 d\omega^2 + d\rho^2 + r^2 d\Omega^2 \tag{8.14}$$

Lastly, we come to the metric's angular component. As previously mentioned, the near horizon approximation assumes that its coordinate frame is in the neighborhood of $r = 2M$. This implies

$$\begin{aligned} r^2 d\Omega^2 &\approx (2M)^2 d\Omega^2 \\ &= (2M)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{aligned} \tag{8.15}$$

$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ represents an infinitesimal displacement that takes place on the surface of a sphere, rather than on a flat two-dimensional plane. This is the most significant difference between the near horizon approximation and the Rindler metric.

To approximate the angular component as lying in a plane, we can approximate $d\Omega$ to correspond to a disc instead of a sphere.

Taking the standard relations between Cartesian and spherical coordinates,

$$x = \sin\theta\cos\phi \tag{8.16a}$$

$$y = \sin\theta\sin\phi \tag{8.16b}$$

These coordinates can be turned into squared differentials (the full calculation shown in appendix section .4.2), which gives

$$\begin{aligned} dx^2 + dy^2 &= \cos^2\theta d\theta^2 + \sin^2\theta d\phi^2 \\ &\neq d\theta^2 + \sin^2\theta d\phi^2 = d\Omega^2, \quad \text{unless } \theta \equiv 0 \end{aligned} \tag{8.17}$$

The geometry of a Schwarzschild black hole is invariant under rotation. This allows us to arbitrarily center our coordinate frame at $\theta = 0$. Unfortunately however, fixing our coordinates at $\theta = 0$ reduces Ω to having just one degree of freedom $\Omega(\theta, \phi) \to \Omega(\phi)$, which does not provide us with two degrees of freedom that we need to approximate the two-dimension spatial component of Rindler space that is orthogonal to the direction of acceleration. This problem can be solved if we allow θ to vary about zero by a small amount, restricting our attention to a region with small angles. Small θ implies $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and thus

$$\begin{aligned} dx^2 + dy^2 &\approx d\theta^2 + \theta^2 d\phi^2 \\ &\approx d\Omega^2 \end{aligned} \tag{8.18}$$

This result together with equation (8.14) gives

$$ds^2 \approx -\rho^2 d\omega^2 + d\rho^2 + dx^2 + dy^2 \tag{8.19}$$

The significance of the final result, equation (8.19), is that it implies that the local geometry at the event horizon of a Schwarzschild black hole is approximately equivalent to the hyperbolic spacetime of a uniformly accelerating observer in flatspace.

8.2 Relating near horizon coordinates to a (1+1) Minkowski space metric

Starting from the locally hyperbolic metric of the near horizon approximation,

$$ds^2 \approx -\rho^2 d\omega^2 + d\rho^2 + dx^2 + dy^2 \tag{8.20}$$

The fact that the Schwarzschild geometry has a metric that is conformally flat in it's time-radial component means that ρ and ω are equivalent to the local radial and hyperbolic angle coordinates of Minkowski space [33]. From these hyperbolic coordinates we can define a local Minkowski space,

$$T = \rho \sinh \omega \tag{8.21a}$$
$$X = \rho \cosh \omega \tag{8.21b}$$
$$Y = y \tag{8.21c}$$
$$Z = z \tag{8.21d}$$

Recall from equation (8.13) in the previous section, that ω was defined as,

$$\omega \equiv \frac{t}{4M} \tag{8.22}$$

where t is Schwarzschild time. It was shown in section 6.2 that the surface gravity for the Schwarzschild metric is given by

$$\kappa(r) \equiv \frac{M}{r^2} \tag{8.23}$$

Equation (8.23) can be Taylor expanded about $r = r_0$ using,

$$\kappa(r = 2M) = \frac{1}{4M} \tag{8.24}$$

For the present time, lets take κ to be the black hole's surface gravity $\kappa \equiv \frac{1}{4M}$. This implies $\omega = \kappa t$. Therefore, we can rewrite equations (8.21a) and (8.21b) as

$$T = \rho \sinh(\kappa t) \tag{8.25a}$$
$$X = \rho \cosh(\kappa t) \tag{8.25b}$$

Equations (8.25a) and (8.25b) define the coordinates of a local (1+1)-dimensional Minkowski space, for an observer just above the event horizon. The Hawking temperature is defined as the black holes proper temperature for an observer at infinity, rather than for an observer located near $r = 2M$. Note that both the time t and surface gravity κ in equations (8.25a) and (8.25b) are defined as proper quantities for an observer at infinity. Since the Hawking temperature of a black hole is also defined for an observer at infinity, we need to rescale the local coordinates T and X with a conformal scale factor in order to redefine them as distances measured using the coordinates of an observer at infinity. The time-radial component of the Schwarzschild metric, whose time and radial coordinates, t and r, bear the interpretations of the proper time and proper radial distance of an observer located at infinity.

From equation (8.10)

$$\rho(r) = 2\sqrt{2M(r-2M)}, \quad \text{near } r = 2M \tag{8.26}$$

Multiplying equation (8.26) by the gravitational redshift normalization factor $f(r) = \left(1 - \frac{2M}{r}\right)^{-1/2}$, gives

$$\begin{aligned} f(r)\rho(r) &= \frac{1}{\sqrt{1-\frac{2M}{r}}} \times 2\sqrt{2M(r-2M)} \\ &= 2\sqrt{\frac{2M(r-2M)}{1-\frac{2M}{r}}} \\ &= 2\sqrt{\frac{2Mr(r-2M)}{r-2M}} \\ &= 2\sqrt{2Mr} \end{aligned} \tag{8.27}$$

In the neighborhood of $r = 2M$ equation (8.27) becomes

$$\begin{aligned} f(r \approx 2M)\rho(r \approx 2M) &\approx 2\sqrt{2M \times 2M} \\ &= 4M \\ &= \kappa^{-1} \end{aligned} \tag{8.28}$$

Therefore, $\rho \approx \kappa^{-1}$ in the neighborhood of $r = 2M$. This allows us to rewrite equations (8.25a) and (8.25b) as

$$T \approx \kappa^{-1} \sinh(\kappa t) \tag{8.29a}$$

$$X \approx \kappa^{-1} \cosh(\kappa t) \tag{8.29b}$$

8.3 Generalizing the coordinate relations to arbitrary position within the local tangent space

Equations (8.29a) and (8.29b) define the coordinate frame S for a locally inertial rest observer, located just above the event horizon. To put equations (8.29a) and (8.29b) into a form that is compatible with the metric that was used to calculate the Unruh temperature, in section 5.2, consider the case of a second observer in reference frame S'. If we drop this second observer from the origin of S, and let S' fall into the black hole, the second frame S' will have a proper acceleration a with respect to S, that can be normalized to κ with respect to the coordinates of an observer at infinity. We can define a reference frame that is instantaneously comoving with S' in the same way that was done in section 4.2.1, which will allow us to generalize equations (8.29a) and (8.29b) to an arbitrary local position along the spatial dimension, with respect to the origin of S.

To define a frame that is instantaneously comoving with S', let us define a set of basis vectors $\boldsymbol{e}_{\mu'} = \boldsymbol{e}_{\mu'}(\tau)$ for the comoving frame. The primed coordinates correspond to the local coordinates of the frame that is comoving with S'. As was explained in section 4.2.1, we can assume that the comoving observer is always at rest in his own coordinate frame by choosing to align his local time axis with his 4-velocity vector $\boldsymbol{e}_{0'} = \boldsymbol{u}$. Additionally, we can align $\boldsymbol{e}_{1'}$ with the direction of the observer's acceleration vector, which coincides with the X-axis of coordinate frame S. Choosing $\boldsymbol{e}_{2'} = \boldsymbol{e}_2$ and $\boldsymbol{e}_{3'} = \boldsymbol{e}_3$ disables spatial rotations.

The local instantaneously comoving basis vectors must be related to the basis vectors of S by a proper-time-dependent Loretnz transformation,

$$\boldsymbol{e}_{\mu'}(\tau) = \Lambda^{\nu}{}_{\mu'}(\tau)\boldsymbol{e}_{\nu} \tag{8.30}$$

The Lorentz transformation

$$\Lambda^{\nu}{}_{\mu'} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.31}$$

corresponds to the proper-time-dependent velocity boost $v(\tau)$ along the X-axis of S. The velocity boost is defined to be the exact velocity that is necessary to bring the instantaneously comoving inertial frame up to the same the velocity of S' [26]. Such a Lorentz transformation can be re-expressed as [18, 26, 27]

$$\Lambda^{\nu}{}_{\mu'}(\tau) = \begin{pmatrix} \cosh(\kappa\tau) & -\sinh(\kappa\tau) & 0 & 0 \\ -\sinh(\kappa\tau) & \cosh(\kappa\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.32}$$

with the relation $v = \tanh^{-1}(\kappa\tau)$ [28].

The local tangent space basis vectors can by found using the inverse $\Lambda_{\nu}{}^{\mu'}$ of the Lorentz transformation $\Lambda^{\nu}{}_{\mu'}$ in equation (8.32),

$$[\boldsymbol{e}_{0'}(\tau)]^{\mu} = \Lambda_{\nu}{}^{\mu}(\tau)\,[\boldsymbol{e}_0]^{\nu} = \begin{pmatrix} \cosh(\kappa\tau) & \sinh(\kappa\tau) & 0 & 0 \\ \sinh(\kappa\tau) & \cosh(\kappa\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh(\kappa\tau) \\ \sinh(\kappa\tau) \\ 0 \\ 0 \end{pmatrix} \tag{8.33}$$

Repeating this operation for $[\boldsymbol{e}_{i'}(\tau)]^\mu$, $i \in \{1,2,3\}$ gives us all four tangent space basis vectors, expressed in the global Minkowski coordinates,

$$[\boldsymbol{e}_{0'}]^\mu = \begin{pmatrix} \cosh(\kappa\tau) \\ \sinh(\kappa\tau) \\ 0 \\ 0 \end{pmatrix}, \quad [\boldsymbol{e}_{1'}]^\mu = \begin{pmatrix} \sinh(\kappa\tau) \\ \cosh(\kappa\tau) \\ 0 \\ 0 \end{pmatrix}, \quad [\boldsymbol{e}_{2'}]^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad [\boldsymbol{e}_{3'}]^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{8.34}$$

Technically, this tangent space only covers the infinitesimal neighborhood of the origin of S'. However, we can extend it to cover a finite domain of radius κ^{-1}, as explained in section 4.2.1.

Let $P(\tau)$ denote the origin of S' at proper time τ along the frames worldline. At each instant of proper time there exists a spacelike hypersurface defined by the point $P(\tau)$ and the three spacelike basis vectors $\boldsymbol{e}_{1'}$, $\boldsymbol{e}_{2'}$, and $\boldsymbol{e}_{3'}$. Let $\boldsymbol{x}_0(\tau)$ denote the position vector of $P(\tau)$, defined with respect to the origin of S. And let $\boldsymbol{x}'$ denote the spacelike separation vector between $P(\tau)$ and an arbitrary point $Q(\tau)$, in S'. $Q(\tau)$ is denoted in the coordinates of S' as $\xi^\mu = (\xi^0, \xi^1, \xi^2, \xi^3) = (\tau, \xi^i)$. This implies the spacelike separation vector $\boldsymbol{x}'(\tau) = \xi^{i'} \boldsymbol{e}_{i'}(\tau)$. By vector addition, the position vector $\boldsymbol{x}(\tau)$ of $Q(\tau)$ can be defined with respect to the origin of S

$$\begin{aligned} \boldsymbol{x}(\tau) &= \boldsymbol{x}_0(\tau) + \boldsymbol{x}'(\tau) \\ &= \boldsymbol{x}_0(\tau) + \xi^{i'} \boldsymbol{e}_{i'}(\tau) \end{aligned} \tag{8.35}$$

The position 4-vector corresponding to equation (4.29) is

$$x^\mu(\tau) = x_0{}^\mu(\tau) + \xi^{i'} [\boldsymbol{e}_{i'}(\tau)]^\mu \tag{8.36}$$

From equations (8.29a), the 4-position of the local tangent space origin is

$$x_0{}^\mu(\tau) = \big(\kappa^{-1}\sinh(\kappa\tau), \quad \kappa^{-1}\cosh(\kappa\tau), \quad 0, \quad 0\big) \tag{8.37}$$

And the second term in equation (4.30) is a 4-vector whose components each include a summation over i,

$$\xi^{i'}[\boldsymbol{e}_{i'}(\tau)]^0 = \big(0, \quad \xi^{1'}, \quad \xi^{2'}, \quad \xi^{3'}\big) \cdot \big(\cosh(\kappa\tau), \quad \sinh(\kappa\tau), \quad 0, \quad 0\big) = \xi^{1'}\sinh(\kappa\tau) \tag{8.38a}$$

$$\xi^{i'}[\boldsymbol{e}_{i'}(\tau)]^1 = \big(0, \quad \xi^{1'}, \quad \xi^{2'}, \quad \xi^{3'}\big) \cdot \big(\sinh(\kappa\tau), \quad \cosh(\kappa\tau), \quad 0, \quad 0\big) = \xi^{1'}\cosh(\kappa\tau) \tag{8.38b}$$

$$\xi^{i'}[\boldsymbol{e}_{i'}(\tau)]^2 = \big(0, \quad \xi^{1'}, \quad \xi^{2'}, \quad \xi^{3'}\big) \cdot \big(0, \quad 0, \quad 1, \quad 0\big) = \xi^{2'} \tag{8.38c}$$

$$\xi^{i'}[\boldsymbol{e}_{i'}(\tau)]^3 = \big(0, \quad \xi^{1'}, \quad \xi^{2'}, \quad \xi^{3'}\big) \cdot \big(0, \quad 0, \quad 0, \quad 1\big) = \xi^{3'} \tag{8.38d}$$

From equations (4.31) and (8.38) we have

$$T(\xi^{\mu'}) = \left(\kappa^{-1} + \xi^{1'}\right) \sinh(\kappa \xi^{0'}) \tag{8.39a}$$

$$X(\xi^{\mu'}) = \left(\kappa^{-1} + \xi^{1'}\right) \cosh(\kappa \xi^{0'}) \tag{8.39b}$$

$$Y(\xi^{\mu'}) = \xi^{2'} \tag{8.39c}$$

$$Z(\xi^{\mu'}) = \xi^{3'} \tag{8.39d}$$

where I've replaced $\tau \to \xi^{0'}$ to emphasize that equation (8.39) expresses the coordinates of S in terms of the coordinates of the frame that is instantaneously comoving with S'. As was done in section 4.2.1, we can obtain a new metric for the local coordinates of S' by taking differentials of equations putting equation (8.39) into the local metric, which is Minkowski. The full calculation is the same as is shown in section 4.2.1. The line element for S' is found to be

$$ds^2 = -\left(1 + \kappa \xi^{1'}\right)^2 \left(d\xi^{0'}\right)^2 + \left(d\xi^{1'}\right)^2 + \left(d\xi^{2'}\right)^2 + \left(d\xi^{3'}\right)^2 \tag{8.40}$$

8.4 Using the gravitational WKB approximation to calculate Hawking Temperature

Taking the (1+1)-dimensional component of equation (8.40) that includes $\xi^{0'}$ and $\xi^{1'}$,

$$ds^2 = -\left(1 + \kappa \xi^{1'}\right)^2 \left(d\xi^{0'}\right)^2 + \left(d\xi^{1'}\right)^2 \tag{8.41}$$

Consider a local coordinate transformation within the comoving frame $\xi^{\mu'} \to q^{\mu}$, defined by

$$\xi^{0'} \to q^0 \quad \text{and} \quad \xi^{1'} \to \kappa^{-1}\left(\sqrt{|1 + 2\kappa q^1|} - 1\right) \tag{8.42}$$

which is the same coordinate transformation that was used in section 4.2.1. Equation (8.42) implies, and is motivated by

$$\left(1 + \kappa \xi^{1'}\right) = \sqrt{|1 + 2\kappa q^0|} \tag{8.43}$$

Taking coordinate differentials of equation (8.42),

$$\xi^{0'} = dq^0 \tag{8.44a}$$

$$d\xi^{1'} = \frac{dq^1}{\sqrt{|1 + 2\kappa q^1|}} \tag{8.44b}$$

Substituting equations (8.44a) and (8.44b) into equation (8.41) gives

$$\begin{aligned} ds^2 &= -\left(1+\kappa q^1\right)^2 (dq^0)^2 + (dq^1)^2 \\ &= -\left(1+2\kappa q^1\right)(dq^0)^2 + \frac{dq^{1\,2}}{1+2\kappa q^1} \end{aligned} \tag{8.45}$$

The metric is now singular at $q^1 = -\frac{1}{2\kappa}$, which corresponds to the location of the Rindler horizon in the q^μ coordinate frame,

$$\xi^{1'}\left(q^1 = -\tfrac{1}{2\kappa}\right) = \kappa^{-1}\left(\sqrt{|1+2\kappa\left(-\tfrac{1}{2\kappa}\right)|} - 1\right) = -\frac{1}{\kappa} \tag{8.46}$$

We can re-express the Minkowski coordinates in terms of the coordinate transformations that were enacted in equation (8.42),

$$T = \left(\kappa^{-1} + \xi^{1'}\right)\sinh(\kappa\xi^{0'}) \tag{8.47a}$$

$$\begin{aligned} &= \kappa^{-1}\left(1+\kappa\xi^{0'}\right)\sinh\left(\kappa\xi^{0'}\right) \\ &= \kappa^{-1}\sqrt{1+2\kappa q^1}\sinh\left(\kappa q^0\right) \end{aligned} \tag{8.47b}$$

$$X = \left(\kappa^{-1} + \xi^{1'}\right)\cosh(\kappa\xi^{0'}) \tag{8.47c}$$

$$\begin{aligned} &= \kappa^{-1}\left(1+\kappa\xi^{1'}\right)\cosh(\kappa\xi^{0'}) \\ &= \kappa^{-1}\sqrt{1+2\kappa q^1}\cosh(\kappa q^0) \end{aligned} \tag{8.47d}$$

From equation (3.24), the Hamilton-Jacobi equations for a scalar field of mass m are

$$g^{\mu\nu}\partial_\mu S \partial_\nu S + m^2 = 0 \tag{8.48}$$

where the classical action S can be separated into distinct time and space components $S(q^0, q^1) = \omega q^0 + \bar{S}(q^1)$, with $\bar{S}(q^1)$ representing the purely spatial contribution to the action. Let ω denote the energy of the tunneling particle. From equation (8.45), we can express the metric for a (1+1)-dimensional Rindler spacetime as

$$ds^2 = -\left(1+2\kappa q^1\right)(dq^0)^2 + \frac{dq^{1\,2}}{1+2\kappa q^1} \tag{8.49}$$

We can use this to determine the inverse metric components, equation (8.48) becomes,

$$\begin{aligned} 0 &= g^{00}\left(\partial_0 S\right)^2 g^{11}\left(\partial_1 S\right)^2 + m^2 \\ &= -\left(1+2\kappa q^1\right)^{-1}\left(\partial_0 S\right)^2 + \left(1+2\kappa q^1\right)\left(\partial_1 S\right)^2 + m^2 \end{aligned} \tag{8.50}$$

The action is expressed in the coordinates q^μ as

$$S(q^0, q^1) = \omega q^0 + \bar{S}(q^1) \tag{8.51}$$

which implies

$$\partial_0 S(q^0, q^1) = \frac{\partial S(q^0, q^1)}{\partial q^0} = \omega \tag{8.52a}$$

$$\partial_1 S(q^0, q^1) = \frac{\partial S(q^0, q^1)}{\partial q^1} = \partial_1 \bar{S}(q^1) \tag{8.52b}$$

Putting equations (8.52a) and (8.52b) into (5.13) gives

$$-\frac{\omega^2}{1 + 2\kappa q^1} + \left(1 + 2\kappa q^1\right)\left(\partial_1 \bar{S}(q^1)\right)^2 + m^2 = 0 \tag{8.53}$$

Rearranging to solve for $\left(\partial_1 \bar{S}(q^1)\right)^2$,

$$\begin{aligned}
\left(\partial_1 \bar{S}(q^1)\right)^2 &= \left(1 + 2\kappa q^1\right)^{-1}\left(\frac{\omega^2}{1 + 2\kappa q^1} - m^2\right) \\
&= \frac{\omega^2 - m^2\left(1 + 2\kappa q^1\right)}{\left(1 + 2\kappa q^1\right)^2} \\
&= \frac{\omega^2 - m^2\left(1 + 2\kappa q^1\right)}{(2\kappa)^2\left(q^1 + (2\kappa)^{-1}\right)^2}
\end{aligned} \tag{8.54}$$

Taking the square root of both sides and integrating over q^1 gives

$$\begin{aligned}
\bar{S}(q^1) &= \int_{-\infty}^{\infty} \partial_1 \bar{S}(q^1) dq^1 \\
&= \pm\frac{1}{2\kappa}\int_{-\infty}^{\infty} \frac{\sqrt{\omega^2 - \left(1 + 2\kappa q^1\right)m^2}}{q^1 - (-2\kappa)^{-1}} dq^1
\end{aligned} \tag{8.55}$$

As was mentioned in section 5.2, the sign ambiguity in equation (8.55) arises because ingoing and outgoing particles tunnel in opposite directions across the Rindler horizon. The plus and minus correspond to ingoing and outgoing particles respectively. Since an outgoing particle travels in the negative q^1-direction, the bounds of integration are interchanged, which makes both contributions positive.

The spatial contribution to the action in equation (8.55) has a simple pole at $q^1 = -\frac{1}{2\kappa}$. Let $\epsilon e^{i\phi} = q^1 + (2\kappa)^{-1}$, which implies $dq^1 = i\epsilon e^{i\phi} d\phi$. For the case of ingoing positive energy particles, equation (8.55) can be written as

$$\begin{aligned}
\bar{S} &= \lim_{\epsilon \to 0} \frac{1}{2\kappa} \int_{\pi}^{2\pi} \frac{\sqrt{\omega^2 - \left(1 + 2\kappa \left(\epsilon e^{i\phi} - (2\kappa)^{-1}\right)\right) m^2}}{\epsilon e^{i\phi}} \left(i\epsilon e^{i\phi} d\phi\right) \\
&= \frac{i}{2\kappa} \int_{\pi}^{2\pi} \sqrt{\omega^2 - \left(1 + 2\kappa (2\kappa)^{-1}\right) m^2} d\phi \\
&= \frac{i\omega}{2\kappa} \int_{\pi}^{2\pi} d\phi \\
&= \frac{i\pi\omega}{2\kappa}
\end{aligned} \tag{8.56}$$

A similar integral is calculated for the case of outgoing particles to give the same result as equation (8.56). The total spatial contribution to the action is had by adding the contributions of ingoing and outgoing particles,

$$S_{0,Total} = \frac{i\pi\omega}{2\kappa} + \frac{i\pi\omega}{2\kappa} = \frac{i\pi\omega}{\kappa} \tag{8.57}$$

Calculating the time component's contribution to tunneling

The time component's contribution to the action is associated with the rotation of axes that occurs when the metric switches its signature. The point along the X-axis at which this occurs can be found by looking at the invariant interval,

$$X^2 - T^2 = \kappa^{-2} \implies T(X) = \pm\sqrt{X^2 - \kappa^{-2}} \tag{8.58}$$

It is easy to see from equation (8.58) that the coordinate axes undergo rotation into the complex plane at about the instant when $X = \kappa^{-}1$. Therefore, $X = \kappa^{-1}$ can be used to define the radius of rotation. We can define an analytic continuation of the time coordinate to calculate the amount of time that occurs during this rotation.

$$T \equiv T_0 e^{i\phi} \tag{8.59}$$

which implies $dT = iT_0 e^{i\phi} d\phi$. Since the axes rotate with a radius $X = \kappa^{-1}$, it makes sense that $T_0 = \kappa^{-1}$. Two time coordinate translations occur when a particle crosses the horizon, one for equation (8.47a) and one for equation (8.47c). The axis rotations are caused by an imaginary time translation $\xi^{0'} \to \xi^{0'} - \frac{i\pi}{2}$,

$$\omega \Delta T = \omega \int dT = i\omega\kappa^{-1} \int_{\pi/2}^{0} d\phi = -\frac{i\pi\omega}{2\kappa} \tag{8.60}$$

Since both T in equation (8.47a) and X in equation (8.47c) are parameterized with the comoving observer's time coordinate, each contributes a factor of $-\frac{i\pi\omega}{2\kappa}$ to the tunneling rate. The total contribution from the time component is then

$$\omega \Delta T = -\frac{i\pi\omega}{\kappa} \tag{8.61}$$

The Hawking temperature

The tunneling rate predicted by the gravitational WKB approximation is given by [8]

$$\Gamma \sim e^{-\frac{1}{\hbar}\left[\mathrm{Im}\left(\oint p_x dx\right) - \omega \mathrm{Im}(\Delta t)\right]} \tag{8.62}$$

where Γ denotes the tunneling rate. The temperature of the Hawking radiation can be found by assuming a Boltzmann distribution of energy states

$$\Gamma \sim e^{-\omega/T_H} \tag{8.63}$$

Equation (8.63) expresses the probability that the tunneling particle is in the energy state ω, given that the thermal flux has a temperature T_H. To relate the tunneling rate of Hawking radiation to the Hawking temperature we can set the exponents of equations (8.62) and (8.63) equal to each other,

$$\omega/T_H = \tfrac{1}{\hbar}\left[\mathrm{Im}\left(\oint p_x dx\right) - \omega \mathrm{Im}\left(\Delta t\right)\right] \tag{8.64}$$

which can be rearranged to obtain a formula for the Unruh temperature,

$$T_H = \frac{\omega}{\mathrm{Im}\left(\oint p_x dx\right) - \omega \mathrm{Im}\left(\Delta t\right)} \tag{8.65}$$

where I've set $\hbar = 1$. From equations (**??**) and (8.61),

$$\mathrm{Im}\left(\oint p_x dx\right) = \frac{\pi\omega}{\kappa} \qquad \text{and} \qquad \mathrm{Im}\left(\Delta t\right) = \frac{\pi\omega}{\kappa} \tag{8.66}$$

which, together with equation (8.65), gives the Hawking temperature

$$T_H = \frac{\omega}{\frac{\pi\omega}{\kappa} - \left(-\frac{\pi\omega}{\kappa}\right)} = \frac{\kappa}{2\pi} \tag{8.67}$$

which is the Hawking temperature.

Hawking radiation inside of a black hole

We can replace the surface gravity in equations (8.39a) and (8.39b) with its expression from equation (8.23), $\kappa(r = 2m) \rightarrow \kappa(r) = \frac{M}{r^2}$, generalizing equations (8.39a) and (8.39b) to

$$T = \left(1 + \frac{M}{r^2}\bigg|_{r=r_0} \xi^{1'}\right) \sinh\left(\kappa \xi^{0'}\right) \tag{8.68a}$$

$$X = \left(1 + \frac{M}{r^2}\bigg|_{r=r_0} \xi^{1'}\right) \cosh\left(\kappa \xi^{0'}\right) \tag{8.68b}$$

where the vertical bars with subscript $r = r_0$ are used to indicate that κ is meant to be treated as a constant with respect to integration during the tunneling calculation. We effectively choose what radius to calculate tunneling rates at by choosing a particular r-value to put into $\kappa(r)$. And since κ is treated as a constant during the calculation of tunneling rates, we're able to write a general expression,

$$T_H(r) = \frac{\kappa(r)}{2\pi} = \frac{M}{2\pi r^2} \tag{8.69}$$

The determinant of the metric in equation (8.40) is singular at $\xi^{1'} = -\frac{1}{\kappa}$,

$$g = -\left(1 + \kappa\xi^{1'}\right)^2 \tag{8.70}$$

which means that $\xi^{1'} = \kappa^{-1}$ is a coordinate singularity within the local coordinate frame, because the only geometric singularity in the Schwarzschild geometry is at $r = 0$.

Since there is nothing geometrically special about $r = 2M$, equation (8.69) is valid for all other inertial frames of the Schwarzschild geometry as well. Perhaps the most interesting finding in all of this, is the following: in order for an object to hit the singularity of a black hole, it has to also hit its own Rindler horizon.

This follows from the inverse relation between surface gravity and the distance of a Rindler observer to his Rindler horizon $\xi^{1'} = -\frac{1}{\kappa}$.

Chapter 9

Hawking radiation as quantum tunneling

9.1 The original Parikh-Wilczek calculation

The original tunneling model of Hawking radiation, by Parikh and Wilczek [2], cleverly model the event horizon as the tunneling object, rather than the particle. In doing so, they implicitly assumed a time-dependent Hamiltonian with a scalar interpretation as the total rest energy of the black hole. Aside from this, the original tunneling model did not pay much attention to the phenomenology of the tunneling process. The present model's pheneomenological assumptions will be explained for the calculation made in the Gullstrand-Painléve coordinates, but applies equally well to all other coordinate choices as well.

Outgoing positive energy particles

Recall the final form of the action from equation (2.16),

$$S = \int_{r_i}^{r_f} p_r dr \tag{9.1}$$

A decrease in a black hole's mass from $M \to M - \omega$ is necessarily accompanied by a corresponding decrease in its Schwarzschild radius, from $R_s = 2M \to 2(M - \omega)$. Consistent with this total change in radius, the present model treats the outgoing particle as initially occupying the finite and well-defined region of space between $r_{in} = 2M - \omega$ and $r_{out} = 2M$, in an s-wave configuration. The classically forbidden region is taken to be the spherical surface at $r = 2(M - \omega)$, thus having an infinitesimal width. It was shown by Kraus and others in [34], that the outgoing positive energy particle feels an effective gravitational force from a black hole of mass $M - \omega$, and thus travels in the spacetime geodesics of a spacetime containing a Schwarzschild black hole of mass $M - \omega$.

For computational convenience we can treat the particle as a rigid object, which undergoes an infinitesimal rigid motion during integration. Treating the particle as a rigid object makes sense computationally, because if one point on the particle moves, all of the rest of the points on the particle have to move with it. $r = 2(M - \omega) - \epsilon$ is smallest r-value occupied by the outgoing particle before undergoing a rigid motion. When the particle is treated as a spherical shell of width 2ω, this is the largest r coordinate the particle can be located at while still objectively be contained inside of the black hole. During integration, this point is treated as the particle's location, and the particle is translated via rigid motion from $r = 2(M - \omega) - \epsilon$ to $r = 2(M - \omega) + \epsilon$. The assumed steps of the tunneling process are explained below.

Phenomenology of the tunneling process

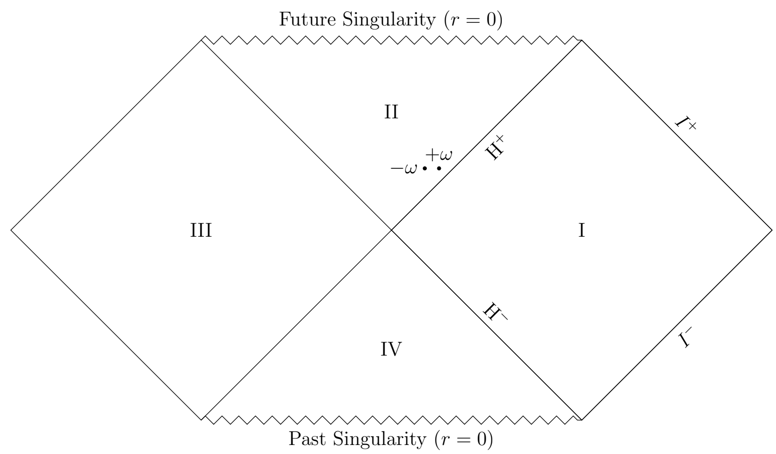

Figure 9.1: Penrose diagram for a Schwarzschild black hole. H^- and H^+ correspond to the past and future horizons, I^- and I^+ correspond to past and future null infinity, and $-\omega$ and $+\omega$ correspond to the negative and positive energy particles. This diagram shows the initial starting positions of the positive and negative energy particles, denoted $\pm\omega$. Immediately after pair creation, both particles find themselves behind the past horizon, and thus out of causal contact with observers above the event horizon.

Event 1: A virtual particle pair with energies $\pm\omega$ is created inside of a black hole of mass M. The particles are denoted as $\pm\omega$ in figure 9.1. The positive energy particle is assumed to occupy a finite region of space, with its initial position starting at $r_{in} = 2(M - \omega) - \epsilon$. The exact initial position of its negative energy partner is not important for this calculation, so long as it is at some r that is below the event horizon. At this point in time, observers at $r > 2M$ perceive the gravitational effects of a black hole of mass M.

Event 2: The positive energy particle, denoted $+\omega$ in figure 9.1, tunnels an infinitesimal distance across the effective event horizon, from $r_{in} = 2(M - \omega) - \epsilon$ to $r_{out} = 2(M - \omega) + \epsilon$. In the limit that $\epsilon \to 0$, the classically forbidden region converges to a surface of infinitesimal width, located at $r = 2(M - \omega)$. If we assume that the spherical mass shell occupies a finite amount of space, and treat adjacent points on the s-wave as maintaining their relative positions, then any part of the spherical mass shell only has to move an infinitesimal distance, to greater r, before part of the shell will be in causal contact with observers above the event horizon. The relative positions of the two particles after the tunneling event are indicated in figure 9.2.

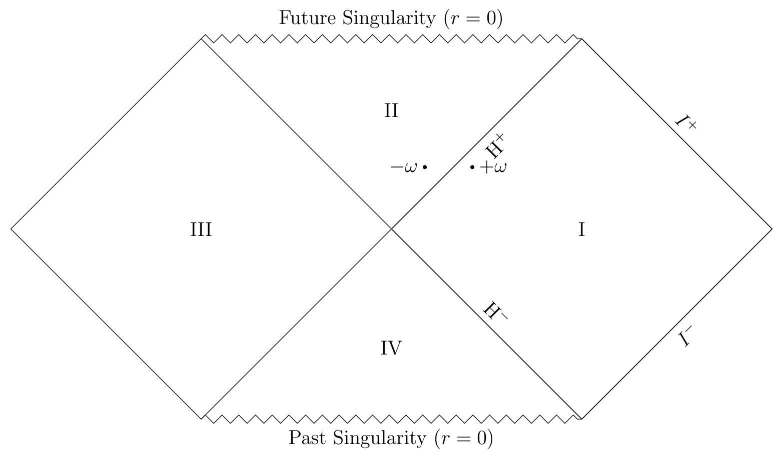

Figure 9.2: Penrose diagram for a Schwarzschild black hole. H^- and H^+ correspond to the past and future horizons, I^- and I^+ correspond to past and future null infinity, and $-\omega$ and $+\omega$ correspond to the negative and positive energy particles. The figure above indicates that after the tunneling event, the negative energy particle $-\omega$ is now behind the future horizon of $+\omega$. This means that $-\omega$ has no causal influence over $+\omega$. This loss of causal influence allows $+\omega$ to materialize into a real particle and travel off to future lightlike infinity.

Event 3: The spacetime diagram above shows that after the tunneling event, the negative energy particle is now behind the positive energy particle's past horizon. For all the positive energy particle knows, its negative energy partner might as well no longer exists. This causal disconnection allows the simultaneous occurrence of two events:
1) The positive energy particle materializes into a real particle
2) The event horizon contracts from $r = 2M$ to $r = 2(M - \omega)$

Event 4: The positive energy particle is now free to escape to future lightlike infinity. Observers at future infinity detect a particle that has been emitted by a black hole of mass $M - \omega$, since from [34], the particle was traveling in the geodesics of a spacetime containing a black hole of mass $M - \omega$.

As explained for Event 1, the objects in this spacetime include a black hole of mass M, and a virtual particle pair of energies $\pm\omega$. This means that total ADM mass of the spacetime is $(M - \omega) + \omega = M$. In contrast to the approach taken by Parikh and Wilczek [2], the present model takes the Hamiltonian's scalar value to be equal to the spacetime's ADM mass $H = M$. This approach has the advantage of giving us a time-independent Hamiltonian. This assumption is important to point out because the model by Parikh and Wilczek assumed $dH \neq 0$, while the present model does assume $dH = 0$.

Another difference between the approach taken here and the approach taken in [2] is the phenomenological interpretation of the pair creation process. In this model I'm assuming a Dirac-sea-like interpretation, in the sense that the positive energy particle's formation is assumed to create a hole in the black hole's mass. Assuming this phenomenology of pair creation has the advantage of making the kinetic and potential energy components of the Hamiltonian very straightforward.

Since virtual particles are created in pairs, the outgoing positive energy particle can be treated as traveling along geodesics in a spacetime that contains a black hole of mass $M - \omega$. In effect, the negative energy particle is treated as a decrease in the black hole's mass $M \to M - \omega$. This change in the black hole's effective mass is equivalent to treating the positive energy particle as self-gravitating, as is done in the original derivation [2]. Both interpretations lead to the same final result for the calculation of tunneling rates.

We can begin the calculation with equation (2.16) as a starting point.

$$\begin{aligned} S &= \int_{r_{in}}^{r_{out}} p_r dr \\ &= \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r' dr \end{aligned} \tag{9.2}$$

where $r_{in} = 2(M - \omega) - \epsilon$ and $r_{out} = 2(M - \omega) + \epsilon$ correspond to the radial point along the s-wave's radial width that has the smallest r-value (i.e. the inner most radial coordinate of the outgoing particle). At this point in the original calculation, a substitution for dp_r is used to rewrite the interior integral in equation (9.2) as an integral over H [2]. Treating H as a time-independent quantity, as is done in the present approach, requires a slightly different approach to setting up the integral over r in equation (9.2). Hamilton's equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} \qquad \text{and} \qquad \dot{p_r} = -\frac{\partial H}{\partial r} \tag{9.3}$$

Keeping in mind the Hamiltonian's role as a generating function, partial derivatives act on H to generate the equations of motion, but do not generally give us complete information about H itself. A total derivative is required if we want to know how the value of H itself is changing. This means that $\dot{r} = \frac{\partial H}{\partial p_r}$ cannot necessarily be rearranged to obtain total differentials dp_r and dH. To see why this happens for the present case, consider the action of $\frac{\partial}{\partial p_r}$ as an operator acting on H. The operator is only a partial derivative, and is effectively only operating on part of H,

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{\partial}{\partial p_r} \left(T(p_r) + U(r) \right) = \frac{\partial T}{\partial p_r} \tag{9.4}$$

which follows from $\frac{\partial U}{\partial p_r} = 0$. Equation (9.4) indicates that $\frac{\partial H}{\partial p_r}$ only represents the partial change in H that is associated with explicit changes in the particle's momentum p_r.

Since H is conserved, the total differential $dH = 0$, and $dp_r \neq \frac{dH}{\dot{r}}$. $\frac{\partial T}{\partial p_r} = \frac{dT}{dp_r}$ because the potential energy is independent of p_r. This means that we can rearrange equation (9.4) for $dp_r = \frac{dT}{\dot{r}}$. The Hamiltonian can be expressed in terms of the system's scalar quantities as

$$\begin{aligned} H &= T + U \\ &= \omega + (M - \omega) \end{aligned} \tag{9.5}$$

Using the Dirac-sea-like interpretation of pair creation, the ω in $U = M - \omega$, should be thought of as a correction to the black hole's mass that is extracted by the positive energy particle's creation.

With this in mind, equation (9.5) tells us that the kinetic energy of the system is $T = \omega \implies dT = d\omega$ is associated entirely with the energy of the outgoing particle. This means $dp_r = \frac{d\omega}{\dot{r}}$, which can be substituted into equation (9.2) to give

$$S = \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{d\omega'}{\dot{r}} dr \tag{9.6}$$

The outgoing particle's geodesics are characterized by the + case of equation (7.13),

$$\dot{r} = +1 - \sqrt{\frac{2M}{r}} \tag{9.7}$$

This equation gives us the radial lightlike geodesics for an outgoing positive energy particle in the presence of a black hole of mass M. However, equation (9.5) indicates that the mass term in equation (9.7) needs to be modified as $M \to M - \omega$, in order to account for the effective decrease in the black hole's mass that results from the positive energy particle's creation. Equation (9.6) can be modified with $M \to M - \omega$,

$$\dot{r} = +1 - \sqrt{\frac{2(M-\omega)}{r}} \tag{9.8}$$

Putting equation (9.8) into (9.6),

$$\begin{aligned} S &= \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{d\omega'}{1 - \sqrt{\frac{2(M-\omega')}{r}}} dr \\ &= \lim_{\epsilon \to 0^+} \int_{2(M-\omega)-\epsilon}^{2(M-\omega)+\epsilon} \int_0^{\omega} \frac{d\omega'}{1 - \sqrt{\frac{2(M-\omega')}{r}}} dr \end{aligned} \tag{9.9}$$

The reason for these bounds of integration are explained above. Notice that the integrand in equation (9.9) is singular at $r = 2(M - \omega)$,

$$\left. \frac{1}{1 - \sqrt{\frac{2(M-\omega)}{r}}} \right|_{r=2(M-\omega)} = \frac{1}{1-1} = \frac{1}{0} = \infty \tag{9.10}$$

Also notice that the bounds of integration over r are separated by an infinitesimal distance 2ϵ. The fact that r is integrated over an infinitesimal distance provides justification for ω to be treated as a constant with respect to integration over r. This subtle fact is important because it allows us to interchange the order of integration in equation (9.9) without having to worry about ω's appearance in the bounds of integration over r. Equation (9.10) can therefore be re-expressed as

$$\begin{aligned} S &= \int_0^{\omega} \left(\lim_{\epsilon \to 0^+} \int_{2(M-\omega)-\epsilon}^{2(M-\omega)+\epsilon} \frac{dr}{1 - \sqrt{\frac{2(M-\omega')}{r}}} \right) d\omega' \\ &= \int_0^{\omega} \left(\lim_{\epsilon \to 0^+} \int_{\sqrt{2(M-\omega)-\epsilon}}^{\sqrt{2(M-\omega)+\epsilon}} \frac{2u^2 du}{u - \sqrt{2(M-\omega')}} \right) d\omega' \qquad (9.11) \end{aligned}$$

with the substitution $r = u^2$. Equation (9.11) has a simple pole at $u = \sqrt{2(M-\omega')}$. Letting $u - \sqrt{2(M-\omega')} = \epsilon e^{i\phi}$, which implies $du = i\epsilon e^{i\phi} d\phi$, allows us to make a contour deformation. It can be done either as a contour integral over an open semicircule deformed into the lower quadrants of the complex plane, or alternatively, we can to deform the contour into a closed and right-handed semicircle extending into the upper quadrants of the complex plane. Taking the open semicircular contour approach, the integral over u in equation (9.11) can now be re-expressed as

$$\begin{aligned} \int_{u_{in}}^{u_{out}} \frac{2u^2 du}{u - \sqrt{2(M-\omega')}} &= \lim_{\epsilon \to 0^+} 2 \int_{\pi}^{2\pi} \frac{\left(\epsilon e^{i\phi} + \sqrt{2(M-\omega')}\right)^2}{\epsilon e^{i\phi}} \left(i\epsilon e^{i\phi} d\phi\right) \\ &= \lim_{\epsilon \to 0^+} 2i \int_{\pi}^{2\pi} \left(\epsilon e^{i\phi} + \sqrt{2(M-\omega')}\right)^2 d\phi \\ &= 2i \int_{\pi}^{2\pi} \left(\sqrt{2(M-\omega')}\right)^2 d\phi \\ &= 4i(M-\omega') \int_{\pi}^{2\pi} d\phi \\ &= i4\pi(M-\omega') \qquad (9.12) \end{aligned}$$

With this, equation (9.11) becomes,

$$\begin{aligned} S &= \int_0^{\omega} i4\pi(M-\omega') d\omega' \\ &= i4\pi \left(M\omega - \frac{\omega^2}{2} \right) \qquad (9.13) \end{aligned}$$

So the imaginary component of the action is given by

$$\text{Im } S = 4\pi\omega(M - \frac{\omega}{2}) \tag{9.14}$$

which is consistent with the result found by Parikh and Wilczek [2].

which gives a transmission coefficient

$$T \simeq e^{-2 \text{ Im } S} = e^{-8\pi\omega(M - \frac{\omega}{2})} \tag{9.15}$$

Calculating S with Feynman's method for contour deformation

The contour integral that was used in equation (9.12) gives the same result as a calculation that uses Feynman's approach for deforming contours. Feynman's method displaces the particle's energy ω, rather than displacing the path of integration. In this approach, a replacement of $\omega' \to \omega' - i\epsilon$ is made into equation (9.11). The rule is, for $c\Delta t > 0$, which corresponds to positive energy particles, the contour is deformed by $\omega \to \omega - i\epsilon$. And for $c\Delta t < 0$, which corresponds to a negative energy particle that is treated as positive energy particles propagating backwards in time, the contour is deformed with $\omega \to \omega + i\epsilon$ [14, 35]. Making the replacement $\omega' \to \omega' - i\epsilon$ into the left-hand side of equation (9.12) gives

$$\int_{u_{in}}^{u_{out}} \frac{2u^2 du}{u - \sqrt{2(M - \omega' + i\epsilon)}} \tag{9.16}$$

where $u_{in} = 2(M - \omega) - \epsilon$ and $u_{out} = 2(M - \omega) + \epsilon$. We can make the simple pole in equation (9.16) easier to visualize by noting that $\sqrt{2(M - \omega' + i\epsilon)} \sim \sqrt{2(M - \omega')} + i\epsilon$ up to a rescaling on ϵ.

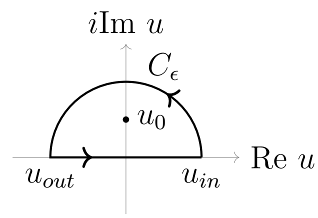

Figure 9.3: The integral for the contour shown above is given in equation (9.17). This integral over the closed semi-circular contour can be broken apart into an integral from u_{in} to u_{out} on the real line, and a right-handed semicircle C_ϵ of radius ϵ. The contour extends a radial distance ϵ into the upper complex plane, and encloses a simple pole at $u_0 = \sqrt{2(M - \omega')} + i\epsilon$.

The deformation made by $\omega' \to \omega' - i\epsilon$ allows equation (9.16) to be solved using the semicircular contour that is shown in figure 9.3. The radius $\epsilon > 0$ of the semicircle asymptotes to zero, which effectively treats the classically forbidden region as a surface at $r = 2(M - \omega)$. The integral corresponding to the contour in figure 9.3 is given by

$$\oint_C \frac{2u^2 du}{u - u_0} = \left(\int_{u_{in}}^{u_{out}} + \int_{C_\epsilon} \right) \frac{2u^2 du}{u - u_0} = 2\pi i \text{Res} \left(\frac{2u^2}{u - u_0}, u_0 \right) \tag{9.17}$$

which implies

$$\int_{u_{in}}^{u_{out}} \frac{2u^2du}{u-u_0} = 2\pi i \text{Res}\left(\frac{2u^2}{u-u_0}, u_0\right) - \int_{C_\epsilon} \frac{2u^2du}{u-u_0} \tag{9.18}$$

The residue for the simple pole at $u_0 = \sqrt{2(M-\omega')}$ is calculated as,

$$\begin{aligned} \text{Res}\left(\frac{2u^2}{u-u_0}, u_0\right) &= \lim_{u\to u_0} (u-u_0)\frac{2u^2}{u-u_0} \\ &= 2u_0^2 \\ &= 2\left(\sqrt{2(M-\omega')} + i\epsilon\right)^2 \end{aligned} \tag{9.19}$$

$$\lim_{\epsilon\to 0} \text{Res}\left(\frac{2u^2}{u-u_0}, u_0\right) = 4(M-\omega') \tag{9.20}$$

The integral over C_ϵ in equation (9.18) is equivalent to the integral that was calculated in equation (9.12). The two contours differ by a rotation through an angle π, but they give the same result because both contours have the same orientation. Using the result of equation (9.12) as the value of the integral over C_ϵ,

$$\int_{C_\epsilon} \frac{2u^2du}{u-u_0} = i4\pi(M-\omega') \tag{9.21}$$

So from equations (9.18), (9.20), and (9.21),

$$\int_{u_{in}}^{u_{out}} \frac{2u^2du}{u-u_0} = i8\pi(M-\omega') - i4\pi(M-\omega') = i4\pi(M-\omega') \tag{9.22}$$

This is the same result as before, which means that both calculations give

$$\text{Im } S = 4\pi\omega(M - \frac{\omega}{2}) \tag{9.23}$$

which is consistent with the calculation made by Wilczek and Parikh in [2].

9.1.1 Ingoing negative energy particle

A negative energy particle propagating forward in time is equivalent to a positive energy particle propagating backward in time [14, 36]. The calculation for an ingoing negative energy particle can be made either way.

The bounds of integration over r for an ingoing particle are interchanged with respect to the bounds of integration for an outgoing particle. The ingoing particle starts at $r_{out} = 2M+\epsilon$ and tunnels an infinitesimal distance 2ϵ, across the classically forbidden surface at $r = 2M$, to $r_{in} = 2M-\epsilon$. A quick calculation for the negative energy ingoing particle can be made as follows: Let p_r^- represent the negative energy particle and p_r^+ represent the positive energy particle. The action is

$$S = \int_{r_{out}}^{r_{in}} p_r^- dr = \int_{r_{out}}^{r_{in}} -p_r^+ dr = \int_{r_{in}}^{r_{out}} p_r^+ dr = i4\pi\omega\left(M - \frac{\omega}{2}\right) \tag{9.24}$$

A more detailed calculation for ingoing negative energy particles can be worked out in two different ways.

Negative-energy particle propagating forward in time

In order for the negative energy particle to lose causal contact with its positive energy partner, every point on a finite amount of the space that the s-wave occupies must be either at or below $r = 2M$. This can also be stated as saying that the lowest $r - value$ of points on the s-wave, must reach $r \leq 2(M - \omega)$ in order for the tunneling event to occur. This turns out to be a very convenient requirement to define the radial coordinate at which tunneling events occur during the calculation.

The order of events for the case of an ingoing negative energy particle is guided by the following order of events:

Event 1: A virtual particle pair with energies $\pm\omega$ is created just above the event horizon of a black hole of mass M. This can be represented as $H = M \rightarrow M + \omega - \omega$. The negative energy particle (which is represented by the smallest r-value that is occupied by the s-wave) starts at $r_{out} = 2(M - \omega) + \epsilon$ and tunnels to $r_{in} = 2(M - \omega) - \epsilon$.

Event 2: The negative energy particle tunnels an infinitesimal distance across the classically forbidden surface at $r = 2(M - \epsilon)$, to $r = 2(M - \omega) - \epsilon$.

Event 3: The negative energy particle is now behind the positive energy particle's future horizon. For all the positive energy particle knows, its negative energy partner no longer exists. The positive energy particle is now free to materialize and travel off to future lightlike infinity.

Event 4: An observer at future infinity measure's the positive energy particle to correspond to the characteristic temperature of a black hole of mass $M - \omega$.

A calculation for tunneling rates of an ingoing negative energy particle can start from the action integral of the WKB approximation.

$$S = \int_{r_{out}}^{r_{in}} p_r dr \tag{9.25}$$

where $r_{out} = 2(M - \omega) + \epsilon$ and $r_{in} = 2(M - \omega) - \epsilon$. Using the same trick that Parikh and Wilczek employed in [2], we can rewrite the particle's momentum as an integral.

$$S = \int_{r_{out}}^{r_{in}} \int_0^{p_r} dp_r' dr \tag{9.26}$$

Just as was done for the outgoing positive energy particle, we can make use of the particle's generalized velocity relation,

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{dT}{dp_r} \implies dp_r = \frac{dT}{\dot{r}} = \frac{d\omega}{\dot{r}} \tag{9.27}$$

where $dT = d\omega$ is true because the particle is assumed to be massless. We can use this relation to rewrite equation (9.26) as an integral over the particle's energy,

$$S = \int_{r_{out}}^{r_{in}} \int_0^{-\omega} \frac{d\omega'}{\dot{r}} dr \tag{9.28}$$

The $\dot{r}$ term in the denominator of equation (9.28) can be replaced with the equation for radially-inward null geodesics, which is given by

$$\dot{r} = -1 + \sqrt{\frac{2M}{r}} \tag{9.29}$$

A self-gravitating negative energy particle in a spacetime containing a Schwarzschild black hole of mass M, follows the geodesics of a spacetime containing a black hole of mass $M + \omega$ [2]. This requires a modification $M \to M + \omega$ on equation (7.13), before substituting it into equation (9.28). From this, and equation (9.29), we can rewrite equation (9.28) as

$$S = \int_{r_{out}}^{r_{in}} \int_0^{-\omega} \frac{d\omega'}{-1 + \sqrt{\frac{2(M+\omega')}{r}}} dr \tag{9.30}$$

Since the particle effectively tunnels across a surface, and thus only an infinitesimal distance, we can safely treat the parameter ω' that appears in integrand as being fixed as the particle tunnels over r. This allows us to interchange the order of integration in equation (9.30). This is important to note, since ω appears in the bounds of integration over both r and ω'.

$$\begin{aligned} S &= \int_0^{-\omega} \int_{r_{out}}^{r_{in}} \frac{dr}{-1 + \sqrt{\frac{2(M+\omega')}{r}}} d\omega' \\ &= \int_0^{-\omega} \int_{u_{out}}^{u_{in}} \frac{-2u^2 du}{u - \sqrt{2(M+\omega')}} d\omega' \end{aligned} \tag{9.31}$$

where $u_{out} = \sqrt{2(M+\omega') + \epsilon}$ and $u_{in} = \sqrt{2(M+\omega') - \epsilon}$, where $\epsilon > 0$ asymptotes to zero. To evaluate the integral over u, let $u - \sqrt{2(M+\omega')} = \epsilon e^{i\phi}$. With this substitution, the integral over u in equation (**??**) becomes

$$\int_{u_{out}}^{u_{in}} \frac{-2u^2 du}{u - \sqrt{2(M+\omega')}} = \lim_{\epsilon \to 0} \int_0^{\pi} \frac{-2\left(\epsilon e^{i\phi} + \sqrt{2(M+\omega')}\right)^2}{\epsilon e^{i\phi}} \left(i\epsilon e^{i\phi} d\phi\right) \tag{9.32a}$$

$$= \lim_{\epsilon \to 0} \ -2i \int_0^{\pi} \left(\epsilon e^{i\phi} + \sqrt{2(M+\omega')}\right)^2 d\phi \tag{9.32b}$$

$$= -2i \int_0^{\pi} 2(M+\omega') d\phi \tag{9.32c}$$

$$= -i4\pi(M+\omega') \tag{9.32d}$$

Replacing the integral over u in equation (9.31) with the result shown in equation (9.32d) gives

$$\begin{aligned} S &= \int_0^{-\omega} -i4\pi(M+\omega')d\omega' \\ &= -i4\pi \left(M(-\omega) + \frac{(-\omega)^2}{2}\right) \\ &= i4\pi\omega\left(M - \frac{\omega}{2}\right) \end{aligned} \tag{9.33}$$

$$\text{Im } S = 4\pi\omega\left(M - \frac{\omega}{2}\right) \tag{9.34}$$

Which is in agreement with [2].

Positive-energy particle propagating backward in time

The main idea of this calculation is the same as for the positive energy outgoing particle. Consider the case of an ingoing negative energy particle propagating backward in time. This time, starting from equation (2.14), the action is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)(-dt)^2 + 2\sqrt{\frac{2M}{r}}(-dt)dr + dr^2 + r^2 d\Omega^2 \tag{9.35}$$

Setting $ds^2 = 0$, dividing both sides by dt^2 to get an equation that is quadratic in $\dot{r}$, and then using the quadratic formula to solve for $\dot{r}$ gives,

$$\dot{r} = \pm 1 + \sqrt{\frac{2M}{r}} \tag{9.36}$$

where $+/-$ corresponds to outgoing/ingoing radial null geodesics. Equation (9.36) differs from equation (7.13), for positive energy geodesics, through a sign change $-\sqrt{\frac{2M}{r}} \to +\sqrt{\frac{2M}{r}}$. This time, starting from equation (2.14), the action is given by

$$\begin{aligned} S &= \int_{r_{out}}^{r_{in}} p_r dr \\ &= \int_{r_{out}}^{r_{in}} \int_0^{p_r} dp'_r dr \end{aligned} \tag{9.37}$$

where $r_{out} = 2M + \epsilon$ and $r_{in} = 2M - \epsilon$. Using the canonical relation $\dot{r} = \frac{\partial H}{\partial p_r} \implies dp_r = \frac{dH}{\dot{r}}$ allows us to rewrite equation (9.37) in terms of the particle's geodesics

$$S = \int_{r_{out}}^{r_{in}} \int_M^{M-\omega} \frac{dH}{\dot{r}} dr \tag{9.38}$$

where dH can be associated with the change in gravitational potential energy that the particle feels as it tunnels an infinitesimal distance dr through the classically forbidden region. This decrease in potential energy corresponds to a decrease in the black hole's mass, such suggests the change of notation $dH \to dM'$. And just as before, we can treat the gravitational potential energy as constant because the classically forbidden region is just a surface for a negative energy particle that starts at $r_{out} = 2M + \epsilon$.

H should be treated as constant while the particle tunnels through the surface. However, the particle notices a difference in the gravitational potential energy once it reaches $r_{in} = 2M - \epsilon$, which corresponds from a decrease in the black hole's mass $M \to M - \omega$.

$$S = \int_{r_{out}}^{r_{in}} \int_M^{M-\omega} \frac{dM'}{\dot{r}} dr \tag{9.39}$$

Replacing $\dot{r}$ with the ingoing case of equation (9.36) gives

$$\begin{aligned} S &= \int_{r_{out}}^{r_{in}} \int_M^{M-\omega} \frac{dM'}{-1 + \sqrt{\frac{2M'}{r}}} dr \\ &= \int_{r_{in}}^{r_{out}} \int_M^{M-\omega} \frac{dM'}{1 - \sqrt{\frac{2M'}{r}}} dr \\ &= \int_M^{M-\omega} \int_{u_{in}}^{u_{out}} \frac{2u^2 du}{u - \sqrt{2M'}} dM' \end{aligned} \tag{9.40}$$

Equation (9.40) is the same as the integral for the outgoing positive energy particle. Note that equation (9.40) has no time-dependence and assumes a positive energy particle. This means that its calculation uses the same contour deformation as for the outgoing case, $\omega \to \omega - i\epsilon \implies M' \to M' - i\epsilon$. This gives,

$$S = i4\pi\omega \left(M - \frac{\omega}{2} \right) \tag{9.41}$$

9.1.2 Outgoing tunneling rate in the Eddington-Finkelstein coordinates

The calculation for outgoing tunneling rates in the Eddington-Finkelstein coordinates is analogous to the calculation in the Gullstrand-Painléve coordinates (section 9.1). Starting from the action,

$$
\begin{aligned}
S &= \int_{r_{in}}^{r_{out}} p_r dr \\
&= \int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r' dr
\end{aligned}
\tag{9.42}
$$

where $r_{in} = 2(M - \omega - \epsilon)$ and $r_{out} = 2(M - \omega + \epsilon)$. dp_r' can be replaced with the total differential of the system's kinetic energy divided by $\dot{r}$ by rearranging Hamilton's generalized velocity equation,

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{dT}{dp_r} = \frac{d\omega}{dp_r} \implies dp_r = \frac{d\omega}{\dot{r}} \tag{9.43}$$

Substituting $dp_r' = \frac{d\omega}{\dot{r}}$ into equation (9.1.2) replaces the integral over the particle's momentum with an integral over its kinetic energy,

$$S = \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{d\omega'}{\dot{r}} dr \tag{9.44}$$

Outgoing null geodesics in the Eddington-Finkelstein were calculated in section 7.2.1. From equation (7.29), radial null geodesics in the outgoing Eddington-Finkelstein coordinates can be expressed as

$$\dot{r} = \frac{1}{2}\left(1 - \frac{2M}{r}\right) \tag{9.45}$$

However, the outgoing particle feels the gravitational attraction from a black hole of mass $M - \omega$. This is represented mathematically by modifying equation (9.45) as $M \to M - \omega$. Making this correction and rewriting the action in terms of the particle's geodesics gives

$$S = \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{d\omega'}{\frac{1}{2}\left(1 - \frac{2(M-\omega')}{r}\right)} dr \tag{9.46a}$$

$$= \int_{r_{in}}^{r_{out}} \int_0^{\omega} \frac{2r d\omega'}{r - 2(M - \omega')} dr \tag{9.46b}$$

As was the case for the integrals calculated in the Gullstrand-Painléve coordinates, the bounds of integration over r are located on either side of a classically forbidden surface at

$r = 2(M - \omega)$. This means that the integral over r in equation (9.46b) is taken over an infinitesimal distance $r_{out} - r_{in} = \epsilon$. However, we can still expect a finite integral because the integrand of equation (9.46b) is singular at $r = 2(M - \omega)$, giving it δ-function-like character.

The causal disconnection described in section 9.1, between the positive- and negative energy particles, does not occur until *after* the tunneling event. This physical condition is significant because it implies that the correct order of integration in equation (9.46b) is first over r, and then over ω'. The two virtual particles must be out of causal contact before the positive energy particle materializes into a real particle. Interchanging the order of integration gives,

$$S = \int_0^{\omega} \int_{r_{in}}^{r_{out}} \frac{2r dr}{r - 2(M - \omega')} d\omega' \tag{9.47}$$

Letting $r - 2(M - \omega') = \epsilon e^{i\phi}$ allows us to rewrite the action as

$$\begin{aligned}
S &= \int_0^{\omega} \left(\lim_{\epsilon \to 0} \int_0^{\pi} \frac{2\left(\epsilon e^{i\phi} + \sqrt{2(M + \omega')}\right)^2}{\epsilon e^{i\phi}} \left(i\epsilon e^{i\phi} d\phi\right) \right) d\omega' \\
&= 2i \int_0^{\omega} \left(\lim_{\epsilon \to 0} \int_0^{\pi} \left(\epsilon e^{i\phi} + \sqrt{2(M - \omega')}\right)^2 d\phi \right) d\omega' \\
&= 2i \int_0^{\omega} \int_0^{\pi} 2(M - \omega') d\phi \omega' \\
&= i4\pi \int_0^{\omega} (M - \omega') d\omega' \\
&= i4\pi\omega \left(M - \frac{\omega}{2}\right)
\end{aligned} \tag{9.48}$$

Which is the same as the calculation made in the Gullstrand-Painléve coordinates.

9.2 Rewriting the action as a function of surface gravity

A more recent approach to calculating Hawking radiation tunneling rates was used by Zinkoo Yun [22]. His approach involves rewriting the action in terms of surface gravity κ. Although Yun's approach to the calculation is very useful, he did not account for the existence of a negative energy partner for the outgoing photon, and as a result did not obtain the same result as Parikh and Wilczek in [2]. In this section I will show that Yun's method of rewriting the action in terms of surface gravity gives the same results as Parikh and Wilczek obtained in their original calculation.

Recall the Gullstrand-Painléve metric from equation (7.2) in section 7.1.1

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2 \tag{9.49}$$

$$S = \int_{r_{in}}^{r_{out}} p_r dr = \int_{t_1}^{t_2} p_r \dot{r} dt = \int_{t_1}^{t_2} \omega dt \tag{9.50}$$

where t is the Gullstrand-Painléve time coordinate. Recall that a virtual particle does not lie on its mass shell, and should not be treated as a materialized particle of energy ω until the outgoing virtual particle has materialized as a real particle. To represent this mathematically I'm rewriting ω in an unevaluated integral form $\omega = \int_0^\omega d\omega'$,

$$S = \int_{t_1}^{t_2} \int_0^\omega d\omega' dt \tag{9.51}$$

From equation (7.13) for outgoing radial lightlike geodesics in the Gullstrand-Painléve coordinates, the time differential can be rewritten as a function of r

$$\dot{r} = 1 - \sqrt{\frac{2M}{r}} \implies dt = \frac{dr}{1 - \sqrt{\frac{2M}{r}}} \tag{9.52}$$

which must be modified as $M \to M - \omega'$ account for the decrease in M that occurs in response to the creation of ω (this decrease in M is equivalent to the production of a negative energy virtual particle). The time differential becomes

$$dt = \frac{dr}{1 - \sqrt{\frac{2(M-\omega')}{r}}} \tag{9.53}$$

which can be put into equation (9.50) to give

$$\begin{aligned} S &= \int_{r_{in}}^{r_{out}} \int_0^\omega \frac{1}{1 - \sqrt{\frac{2(M-\omega')}{r}}} d\omega' dr \\ &= \int_0^\omega \int_{r_{in}}^{r_{out}} \frac{dr}{1 - \sqrt{\frac{2(M-\omega')}{r}}} d\omega' \end{aligned} \tag{9.54}$$

where interchanging the order of integration is done with the physical justification that the virtual particle must tunnel across the horizon before it can lose causal contact with its negative energy partner.

To a first order Taylor expansion about $r = 2(M - \omega')$

$$1 - \sqrt{\frac{2(M-\omega')}{r}}\Bigg|_{\text{at } r=2(M-\omega')} = (1-1) + \frac{\sqrt{2(M-\omega')}}{2(2(M-\omega'))^{3/2}}(r - 2(M-\omega')) + ... \tag{9.55a}$$

$$\approx \frac{1}{4(M-\omega')}(r - 2(M-\omega')) \tag{9.55b}$$

For $f(r) \equiv \left(1 - \frac{2M}{r}\right)$, the surface gravity $\kappa = \frac{1}{2}f'(2M) = \frac{1}{4M}$. However, the modification $2M \to 2(M - \omega') \implies \frac{1}{4M} \to \frac{1}{4(M-\omega')}$ is necessary in order to account for conservation of energy. So $\kappa = \frac{1}{4(M-\omega')}$. Equation (9.55b) becomes

$$1 - \sqrt{\frac{2(M-\omega')}{r}}\Bigg|_{\text{at } r=2(M-\omega')} \approx \kappa\left[r - 2(M - \omega')\right] \tag{9.56}$$

Replacing the denominator of equation (9.54) with equation (9.56) gives

$$S = \int_0^{\omega}\int_{r_{in}}^{r_{out}} \frac{dr}{\kappa\left[r - 2(M-\omega')\right]} d\omega' \tag{9.57}$$

As with the previous cases, the bounds of integration over r are $r_{in} = 2(M - \omega') - \epsilon$ and $r_{out} = 2(M - \omega') + \epsilon$, with $\epsilon \to 0$. Letting $\epsilon e^{i\phi} = r - 2(M - \omega')$,

$$\begin{aligned} S &= \int_0^{\omega}\left(\frac{1}{\kappa(\omega)}\int_{\pi}^{2\pi}\frac{(i\epsilon e^{i\phi}d\phi)}{\epsilon e^{i\phi}}\right)d\omega' \\ &= \int_0^{\omega}\left(\frac{i}{\kappa(\omega)}\int_{\pi}^{2\pi} d\phi\right)d\omega' \\ &= i\pi\int_0^{\omega}\frac{1}{\kappa(\omega)}d\omega' \\ &= i\pi\int_0^{\omega} 4(M-\omega')d\omega' \\ &= i4\pi\omega\left(M - \frac{\omega}{2}\right) \end{aligned} \tag{9.58}$$

which gives a transmission coefficient

$$T \simeq e^{-2 \text{ Im } S} = e^{-8\pi\omega(M-\frac{\omega}{2})} \tag{9.59}$$

Conclusion

Much of this thesis was focused on the Rindler metric and its application to the local inertial reference frames of observers in the Schwarzschild geometry. The Rindler metric bears a physical description that many of us have a reasonable intuition for, which is uniform acceleration. Thanks to Einstein's equivalence principle, we are able to try to understand the unfamiliar environment of highly curved spacetime from the perspective of a uniformly accelerating observer. The Rindler coordinates simplify the situation both mathematically and intuitively. The most significant finding in this thesis comes from section 8, where I showed that a correct calculation of the Hawking temperature can be obtained using the gravitational WKB approximation in a set of modified coordinates of the near horizon approximation. I also showed that the near horizon tunneling model can be extended to $r < 2M$, which led to a very interesting finding: In order for an object to hit the event horizon of a black hole, it must also come into contact with its own Rindler horizon. And in order for an object to hit the geometric singularity of a Schwarzschild black hole, it must come into contact with the Rindler horizon analog that is determined by replacing proper acceleration a with surface gravity κ.

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.1 Lightcone velocity

The lightcone time v and space u coordinates are defined as

$$u = t - x \tag{.1.1a}$$
$$v = t + x \tag{.1.1b}$$

Consider the case of a particle of arbitrary mass traveling in the $+x$-direction with velocity v'. This implies a position function in Minkowski coordinates, given by

$$x(t) = v't \tag{.1.2}$$

From equation (.1.1a) it follows that $x = t - u$, which can be put into equation (.1.1b) to replace x giving

$$v = 2t - u \implies t = \frac{1}{2}(v + u) \tag{.1.3}$$

Additionally, we can substitute equation (.1.2) into equation (.1.1b) to obtain

$$\begin{aligned} v &= t + x = t + v't \\ &= (1 + v')\,t \end{aligned} \tag{.1.4}$$

which implies

$$t = \frac{v}{1 + v'} \tag{.1.5}$$

Substituting equation (.1.5) into equation (.1.3) gives

$$v = \frac{1}{2}(v + u)(1 + v') \tag{.1.6}$$

Multiplying both sides by $\frac{2}{v(1+v')}$ and subtracting 1 from both sides gives

$$\begin{aligned} \frac{u}{v} &= \frac{2}{1 + v'} - 1 \\ &= \frac{1 - v'}{1 + v'} \end{aligned} \tag{.1.7}$$

which implies

$$u(v) = \frac{1 - v'}{1 + v'}v \tag{.1.8}$$

Differentiating $u(v)$ gives the lightcone velocity

$$\frac{du}{dv} = \frac{1 - v'}{1 + v'} \tag{.1.9}$$

.2 Proper time at rest at infinity

An expression for the proper time of a rest observer at arbitrary r is found by first setting $dr, d\Omega = 0$ in equation (7.3), recalling that $d\tau^2 = -ds^2$, and taking the square root of both sides,

$$d\tau = \sqrt{1 - \frac{2M}{r}} dt_S \tag{.2.1}$$

Equation (.2.1) has no sign ambiguity because by definition, a time coordinates only moves in one direction. Equation (.2.1) shows that Schwarzschild time is the same as the proper time of an observer at rest at infinity is $d\tau = dt_S$.

.3 The Schrödinger equation

The time-independent Schrödinger equation is solved to find the probability distribution for a wavefunction that is confined to some region specified by it's boundary conditions, in the presence of a potential energy function $V(x)$. Just like any other wave equation, the Schrödinger equation is characterized by it's dispersion relation. The dispersion relation, itself, has the significance of describing the relation between the space and time derivatives that govern the behavior of a wave.

An intuitive derivation of the Schrödinger equation can be given as follows:

Consider a particle of mass m and total E, that is moving in some direction momentum p. This particle can be described classically by

$$E = \frac{p^2}{2m} \tag{.3.1}$$

Note that whether or not the particle has some form of internal energy, which amounts to an mc^2-like term, is not important becuase only differences in energy have any real, measurable physical meaning. Therefore, we can drop any terms associated with the particles internal energy by adjusting the axis for E.

The energy in equation (.3.1) is the energy for the point-like representation of a particle. To turn equation (.3.1) into an equation for the wave representation of a particle, we can replace $E \to \hbar\omega$ and $p \to \hbar k$, which represent the difference in energy and momentum levels of a quantum mechanical harmonic oscillator. After making these replacements, equation (.3.1) becomes,

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} \tag{.3.2}$$

From unit analysis, we can make the associations $\omega \sim i\frac{\partial}{\partial t}$ and $k \sim i\frac{\partial}{\partial x}$. Equation (.3.2) can then be rewritten as

$$i\hbar\frac{\partial}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{.3.3}$$

And sure enough, the identifications $i\hbar\frac{\partial}{\partial t} \rightarrow \hat{H}$ and $i\hbar\frac{\partial}{\partial x} \rightarrow \hat{p}$ are consistent with the quantum mechanical operators that represent them. Making the replacement $\hat{H} \rightarrow i\hbar\frac{\partial}{\partial t}$ and then assuming conservation of energy to get $\hat{H} = E$ allows us to put E back into (.3.3) through the replacement $E \rightarrow i\hbar\frac{\partial}{\partial t}$ giving,

$$E = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{.3.4}$$

If we now introduce a potential energy function $V(x)$ that interacts with the particle in some way, equation (.3.4) becomes

$$E = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \tag{.3.5}$$

Letting the quantities in equation (.3.5) act on the wavefunction ψ gives

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi \tag{.3.6}$$

which is the familiar time-independent Schrödinger equation. Note that we could have easily obtained the time-dependent Schrödinger equation by assuming that energy is not conserved, in which case $\hat{H} \neq E$. After including the potential energy $V(x)$ this derivation would have given,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi \tag{.3.7}$$

which is the time-dependent Schrödinger equation in one spatial dimension.

.4 Near horizon approximation calculations

.4.1 Proper distance from $r > 2M$ to $r = 2M$

$$\begin{aligned}\rho &= \int_0^\rho ds \\ &= \int_{2M}^r \frac{dr'}{\sqrt{1-\frac{2M}{r'}}} \\ &= \int_{2M}^r \frac{\sqrt{r'}dr'}{\sqrt{r'-2M}}\end{aligned} \tag{.4.1}$$

Let $u^2 = r'$, which implies $2udu = dr'$. This substitution gives

$$\rho = \int \frac{2u^2du}{\sqrt{u^2-2M}} \tag{.4.2}$$

Equation (.4.2) can be solved using trig substitution. Let $u = \sqrt{2M}\sec t$, $du = \sqrt{2M}\sec t \tan t dt$. This gives,

$$\begin{aligned}\rho &= \int \frac{2\left[(\sqrt{2M})^2 \sec^2 t\right]\left[\sqrt{2M}\sec t \tan t dt\right]}{\sqrt{2M}\sqrt{\sec^2 t - 1}} \\ &= 4M\int \sec^3 t dt \qquad (.4.3)\end{aligned}$$

Where the identity $\sqrt{\sec^2 x - 1} = \tan x$ was used in the denominator. Temporarily setting $4M = 1$ to simplify things,

$$\begin{aligned}\int \sec^3 t dt &= \int \sec t \sec^2 t dt \\ &= \int \sec t \left(1 + \tan^2 t\right) dt \\ &= \int \sec t dt + \int \sec t \tan^2 t dt \qquad (.4.4)\end{aligned}$$

The first integral in equation (.4.4), $\int \sec t dt$, can be solved by letting $l = \sec t + \tan t$, which gives

$$dl = \sec t\,(\tan t + \sec t)\,dt = l \sec t dt \implies \sec t dt = \tfrac{dl}{l} \qquad (.4.5)$$

so that

$$\begin{aligned}\int \sec t dt &= \int \frac{dl}{l} \\ &= \ln|l| + C \\ &= \ln|\sec t + \tan t| + C \qquad (.4.6)\end{aligned}$$

The second integral in equation (.4.4) can be solved by making use of the product rule,

$$\begin{aligned}\int \sec t \tan^2 t dt &= \int \tan t\,(\tan t \sec t)\,dt \\ &= \int \tan t \frac{d}{dt}(\sec t)\,dt \\ &= \int \frac{d}{dt}(\tan t \sec t)\,dt - \int \frac{d}{dt}(\tan t)\sec t dt \\ &= \tan t \sec t\Big|_{t_1}^{t_2} - \int \sec^3 t dt \qquad (.4.7)\end{aligned}$$

Putting equations (.4.6) and (.4.7) into (.4.4) gives

$$\int \sec^3 t dt = \ln|\sec t + \tan t| + \tan t \sec t \Big|_{t_1}^{t_2} - \int \sec^3 t dt + C \tag{.4.8}$$

Adding $\int \sec^3 dt$ to both sides and dividing by 2 gives

$$\int \sec^3 t dt = \frac{1}{2}\left(\ln|\sec t + \tan t| + \tan t \sec t \Big|_{t_1}^{t_2}\right) \tag{.4.9}$$

Making the appropriate substitutions with $\sec t = \frac{u}{\sqrt{2M}}$ and $\tan t = \sqrt{\frac{u^2-2M}{2M}}$, equation (.4.9) can now be written as

$$\rho = 2M\left[\ln\left|\frac{u}{\sqrt{2M}} + \sqrt{\frac{u^2-2M}{2M}}\right| + \sqrt{\frac{u^2-2M}{2M}}\frac{u}{\sqrt{2M}}\Big|_{u_1}^{u_2}\right] \tag{.4.10}$$

Making the replacement $u = \sqrt{r}$,

$$\begin{aligned}\rho(r) &= 2M\left[\ln\left|\sqrt{\frac{r}{2M}} + \sqrt{\frac{r-2M}{2M}}\right| + \sqrt{\frac{r'-2M}{2M}}\sqrt{\frac{r'}{2M}}\Big|_{r_1=2M}^{r_2=r}\right] \\ &= 2M\left[\ln\left(\sqrt{\frac{r}{2M}} + \sqrt{\frac{r}{2M}-1}\right) + \sqrt{\frac{r}{2M}-1}\sqrt{\frac{r}{2M}}\right]\end{aligned} \tag{.4.11}$$

Now comes the tricky part,

$$\begin{aligned}\ln\left(\sqrt{\frac{r}{2M}} + \sqrt{\frac{r}{2M}-1}\right) &= \ln\left(\sqrt{1+(\frac{r}{2M}-1)} + \sqrt{\frac{r}{2M}-1}\right) \\ &= \ln\left(\sqrt{1+\left(\sqrt{\frac{r}{2M}-1}\right)^2} + \sqrt{\frac{r}{2M}-1}\right)\end{aligned} \tag{.4.12}$$

Using the relation $\sinh^{-1} x = \ln\left(x + \sqrt{1+x^2}\right)$ with $x = \sqrt{\frac{r}{2M}-1}$ this becomes

$$\ln\left(\sqrt{\frac{r}{2M}} + \sqrt{\frac{r}{2M}-1}\right) = \sinh^{-1}\left(\sqrt{\frac{r}{2M}-1}\right) \tag{.4.13}$$

which can be put into equation (.4.11) to give

$$\begin{aligned}\rho(r) &= 2M\left[\sinh^{-1}\left(\sqrt{\frac{r}{2M}-1}\right) + \frac{1}{2M}\sqrt{r(r-1)}\right] \\ &= 2M\sinh^{-1}\left(\sqrt{\frac{r}{2M}-1}\right) + \sqrt{r(r-2M)}\end{aligned} \tag{.4.14}$$

.4.2 Angular component

$$
\begin{aligned}
dx^2 + dy^2 &= (\cos\theta\cos\phi d\theta - \sin\theta\sin\phi d\phi)^2 + (\cos\theta\sin\phi d\theta - \sin\theta\cos\phi d\phi)^2 \\
&= \left(\cos^2\theta\cos^2\phi d\theta^2 - 2\cos\theta\cos\phi\sin\theta\sin\phi d\phi d\theta + \sin^2\theta\sin^2\phi d\phi^2\right) \\
&+ \left(\cos^2\theta\sin^2\phi d\theta^2 + 2\cos\theta\cos\phi\sin\theta\sin\phi d\phi d\theta + \sin^2\theta\cos^2\phi d\phi^2\right) \\
&= \cos^2\theta\left(\sin^2\phi + \cos^2\phi\right) d\theta^2 + \sin^2\theta\left(\sin^2\phi + \cos^2\phi\right) d\phi^2 \\
&= \cos^2 d\theta^2 + \sin^2\theta d\phi^2 \qquad (.4.15)
\end{aligned}
$$