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# Improving Chiral Perturbation Theory with Lattice QCD Determined $\pi$ - $\pi$ Scattering

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**Improving Chiral Perturbation Theory with Lattice QCD  
Determined  $\pi - \pi$  Scattering**

by

**Hu Zhao**

University of Colorado Boulder

A thesis submitted to the faculty of the  
University of Colorado in partial fulfillment  
of the requirements for the degree of  
Bachelor of Science  
Department of Physics

2017

This thesis entitled:  
Improving Chiral Perturbation Theory with Lattice QCD Determined  $\pi - \pi$  Scattering  
written by Hu Zhao  
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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Zhao, Hu (B.S., Physics)

Improving Chiral Perturbation Theory with Lattice QCD Determined  $\pi - \pi$  Scattering

Thesis directed by Daniel R. Bolton, Department of Physics

The first principle calculation of Lattice QCD is complicated and usually requires significant computational power to perform the calculation. By applying the Chiral Perturbation Theory (ChPT), the calculation becomes much simpler. This project is to find a set of parameters called low energy constants (LECs) in the ChPT calculation that gives results that agree with the Lattice QCD calculations. Once this is done, ChPT can be used to give a better prediction on experimental results.

## Dedication

I dedicate this thesis to my cousin and my friends for all of their unwavering support.

## Acknowledgements

Thanks to Dr. Daniel Bolton, Adam Ellzey, and Issac Kim for their collaboration on this project.

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# Chapter 1

## Introduction

This project is mainly about building an effective model for  $\pi - \pi$  scattering based on ChPT. When two pions scatter, depending on their initial energy, different results can be observed. They will either scatter off or form a resonance ( $\rho$  at I=1 channel,  $\sigma$  at I=0 channel). The scattering process can be described as scattering amplitude. Furthermore, as a function of scattering amplitude, the T-matrix and phase shift also give valuable information about the scattering process.

The first principle theory for the scattering process that involves the strong force is QCD. QCD is a quantum field theory that describes strong interaction. One of the core concepts of QFT, which is the base of QCD, is the principle of least action. Properties of quantum systems can be computed according to this principle using a path integral approach. The key ingredient in a path integral calculation is the action. It is the time integral of the Lagrangian. However, it is almost impossible to solve for analytic solutions of this path integral at low energy due to QCD's non-linear nature. In order to solve this issue, Lattice QCD is developed. Lattice QCD treats the space and time as discrete intervals so that the path integral can be simplified as a summation of a huge quantity of numerical integrals. Although Lattice QCD calculations are possible, it takes significant computational power to get the result. So far, these calculations can only be done on supercomputers.

A significantly faster means to obtain similar results is by taking a perturbative approach to QCD which is ChPT. The reason ChPT is perturbative is that when looking at the interaction, it zooms out and describes only hadronic interactions. However, the low energy constants estimated

decades ago in ChPT [7] are not accurate enough when compared with the Lattice QCD spectrum which is the first principle calculation. Therefore, this research project's goal is to optimize the parameters in ChPT by fitting them with the most recent Lattice QCD data.

There are many settings for the fit. It turns out that every setting has succeeded under certain condition 7.1. For future research, all scenarios are included. One may choose which set of parameters to use based on the purpose.

## Chapter 2

### Theoretical Background

This chapter provides foundational theories and methodologies necessary to understand the procedure of improving low energy constants (LECs) in chiral perturbation theory (ChPT).

#### 2.1 $\pi - \pi$ Scattering

Pion ( $\pi$ ), also known as pi meson, is a hadron with a quark and an anti-quark. There are three kinds of pions:  $\pi^+(u\bar{d})$ ,  $\pi^-(d\bar{u})$  and  $\pi^0(u\bar{u}$  or  $d\bar{d})$ .  $\pi^+$  has an isospin  $I_3 = 1$ ,  $\pi^-$  has an isospin  $I_3 = -1$  and  $\pi^0$  has an isospin  $I_3 = 0$ . The  $\pi - \pi$  scattering process happens when two pions collide together [18] [10].

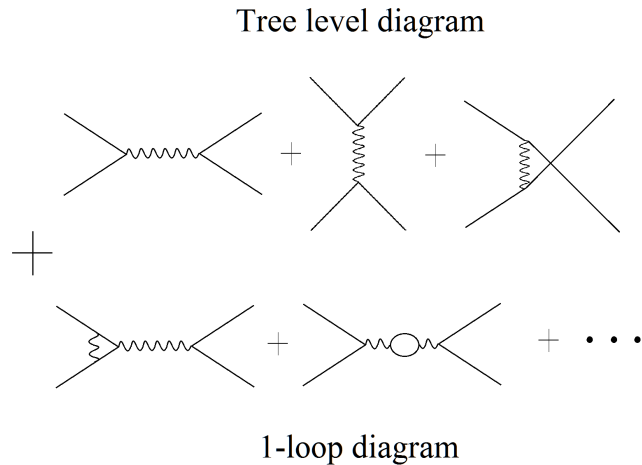


Figure 2.1: Feynman diagram for  $\pi - \pi$  scattering up to 1-loop [8].

There are many ways to describe the scattering process; one of them is the phase shift. It gives valuable information about the scattering process by observing the outgoing wave function [12].

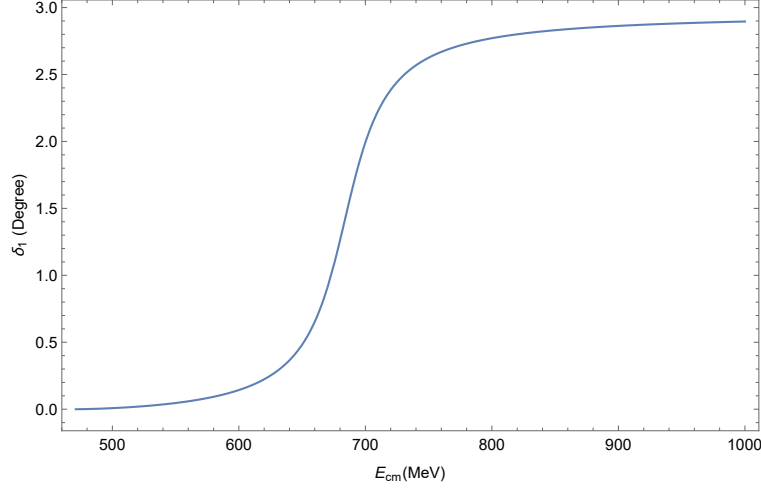


Figure 2.2: An example of isovector  $\pi - \pi$  scattering phase shift. When the initial energy is below the resonance threshold ( $\rho$  at isovector channel), the phase of the outgoing pion wave function will be slightly shifted. When the initial energy exceeds the lower threshold, the phase of the outgoing pion wave function will be significantly shifted.

In the  $\pi - \pi$  scattering process, in terms of the initial center of mass frame energy of the system, there is a lowest energy threshold, the  $\pi\pi$  threshold, where the lowest center of mass energy equals to the energy of 2 rest pions. There is also an upper threshold, which is the  $K\bar{K}$  threshold [12], where the initial center of mass energy is high enough that a kaon and anti-kaon pair will be generated. The kaon mass is 493 or 497 MeV, depending on its isospin. So, the  $K\bar{K}$  threshold is about 1000 MeV. But at such high energy, ChPT becomes less effective. Therefore, this project mainly studies the  $\pi - \pi$  scattering process with total isospin being 0 and 1, and with a center of mass energy in the range from  $\pi\pi$  threshold to  $K\bar{K}$  threshold. In terms of the natural unit, the initial energy is in the range of 480-1000 MeV.

## 2.2 QCD and Lattice QCD

The force that is related to the  $\pi - \pi$  interaction is mainly strong force. It is one of four fundamental forces in the universe. The other three are electromagnetic force, weak force and gravitational force. The theory that describes strong interaction is QCD. The QCD Lagrangian is given by [11]

$$\mathcal{L}_{QCD} = \bar{\psi}_i \left( i (\gamma^\mu D_\mu)_{ij} - m \delta_{ij} \right) \psi_i - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} \quad (2.1)$$

where  $\psi$  is the quark field,  $\gamma$  is the Dirac matrix and  $G$  is the gluon field strength tensor. Due to the non-linear and non-perturbative nature of QCD at low energy, it is almost impossible to obtain an analytical solution. Thus, Lattice QCD was developed to solve QCD computations numerically by treating space and time as points on a lattice. Lattice QCD also makes 2 more approximations which are finite volume interaction and non-physical quark mass. By making the lattice spacing small enough, it will be very close to the continuous space. Thus, the approximations on space and volume do not make a large difference from the exact solution. However, because the quark mass is non-physical, and the quantity of numerical integrals over discrete space and time intervals is immense, it is not easy to perform the Lattice QCD calculation. Despite the difficulty, the Lattice QCD spectrum of  $\pi - \pi$  scattering in both isoscalar and isovector channels has been recently obtained by the Hadron Spectrum Collaboration [3]. In this project, their recent results are used to optimize the parameters for ChPT.

## 2.3 ChPT and Chiral Symmetry

QCD has two important properties, which are asymptotic freedom and color confinement. Color confinement is the phenomenon that quarks do not show up individually; instead, they appear in groups and form hadrons. Asymptotic freedom means that at high energy regimes, quarks and gluons interact weakly and form a plasma. The asymptotic freedom property becomes significant only at high energy and small-length scale. Between the  $\pi\pi$  threshold and  $K\bar{K}$  threshold, the color confinement property is the dominant one. ChPT uses the color confinement property at low

energy regimes. Instead of starting from the very basic structure and performing the first principle calculation of quark and gluon level interaction, it assumes the existence of chiral symmetry and describes the scattering process with hadron level interaction, ignoring the quark level structure. The chiral symmetry is the direct product of  $SU(3)_L \times SU(3)_R$ , which is based on the  $SU(3)$  group. The  $SU(3)$  group plays an important role in the context of the strong interaction [chpt primer]. However, chiral symmetry is not a real symmetry because its broken due to the fact that up, down and strange quarks are massive. By assuming chiral symmetry, the quark mass terms vanish and the whole interaction can be described as hadron interactions [9] [13] [20]. Therefore, although QCD itself is not perturbative, ChPT is an effective field theory of QCD at low energy regimes that utilizes the perturbative nature of hadron interactions. By taking the chiral limit, which treats quarks as massless particles, it is possible to construct a Lagrangian that is consistent with QCD where the assumption of chiral symmetry is made [15] [21]. Enlightened by QCD Lagrangian, where the chiral symmetry is assumed, the ChPT Lagrangian at  $O(p^4)$  is given by [8]

$$\begin{aligned}
\mathcal{L}_{ChPT}^4 = & l_1(\nabla^\mu U^T \nabla_\mu U)^2 + l_2(\nabla^\mu U^T \nabla^\nu U)(\nabla_\mu U^T \nabla_\nu U) \\
& + l_3(\chi^T U)^2 + l_4(\nabla^\mu \chi^T \nabla_\mu U) + l_5(U^T F^{\mu\nu} F_{\mu\nu} U) \\
& + l_6(\nabla^\mu U^T F_{\mu\nu} \nabla^\nu U) + l_7(\chi^T \chi)^2 + \dots
\end{aligned} \tag{2.2}$$

the low energy constants (LECs)  $l_i$  can be written in terms of renormalized LECs  $l_i^r$ , which are the parameters being fitted in this project [8]

$$l_i = l_i^r + \gamma_i \lambda \quad i = 1, 2, 3, \dots \tag{2.3}$$

Detailed explanations of mathematical notations can be found in [8].

## 2.4 Lüscher's Method and Inverse Amplitude Method (IAM)

Because of the nature of Lattice QCD, it gives an energy spectrum in a finite volume as the result instead of scattering amplitude in infinite space and time. For the purpose of fitting ChPT with Lattice QCD, the same physical observable is required for comparison. Thus, a special



technique will be used to connect this finite volume energy spectrum with the phase shift calculated from ChPT model. This technic is called Lüscher's method. By setting an appropriate quantization condition in a finite volume, it is possible to subtract the information of energy spectrum from unitarized projected amplitude [14] [16].

$$\text{Re}(t_{uIJ})^{-1} = \text{quantization condition} \quad (2.4)$$

where  $t_{uIJ}$  is the unitarized projected amplitude at particular isospin channel  $I$  with initial angular momentum  $J$ . The quantization condition is obtained by Lüscher's method given a finite volume.

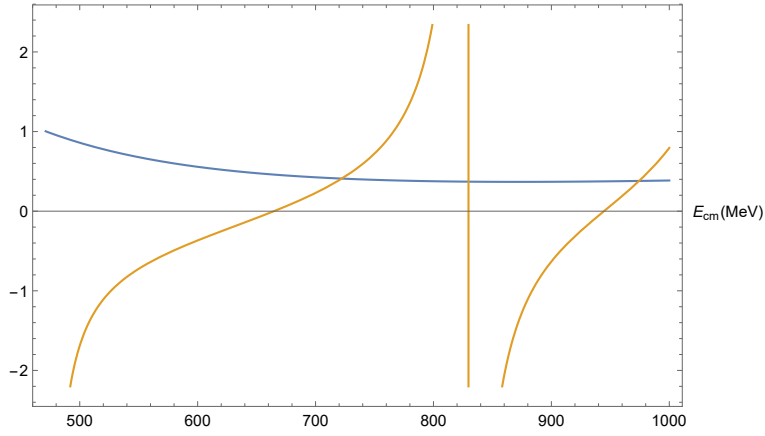


Figure 2.3: Visualization of an example of Equation 2.4. The intersection of  $\text{Re}(t_{uIJ})^{-1}$  (Blue) and the quantization condition function (Orange) is the energy spectrum in a finite volume corresponding to the scattering amplitude. Note that the vertical line is the pole of the quantization condition function. Thus this plot gives two energy spectrum lines.

When dealing with strong interactions, it is usual to project the amplitudes in partial waves with definite angular momentum and isospin [4]. The result is the projected amplitude. It is obtained by projecting scattering amplitude to the isospin channel then to the angular momentum channel. Then it can be unitarized by using inversed amplitude method. The unitarized projected amplitude  $t_{uIJ}$  is given by [4]

$$t_{uIJ} = \frac{t_{lOIJ}^2}{t_{lOIJ} - t_{nlOIJ}} \quad (2.5)$$

where  $t_{lOIJ}$  and  $t_{nlOIJ}$  are the leading order and next leading order terms of projected amplitude

$t_{IJ}$ .  $t_{IJ}$  is given by

$$t_{IJ}(s) = \frac{1}{64\pi} \int_{-1}^1 d(\cos\theta) P_J(\cos\theta) T_I(s, t) \quad (2.6)$$

where  $\theta$  is the scattering angle,  $P_J$  is the  $J^{\text{th}}$  term in Legendre polynomials and  $T_I$  is the isospin amplitude, which is the projection of ChPT scattering amplitude at isospin channel  $I$ .  $T_I$  is given by

$$\begin{aligned} T_0(s, t, u) &= 3A(s, t, u) + A(t, s, u) + A(u, t, s) \\ T_1(s, t, u) &= A(t, s, u) - A(u, t, s) \end{aligned} \quad (2.7)$$

where the ChPT scattering amplitude  $A$  and variables  $s, t$  and  $u$  can be found in [7]. The ChPT scattering amplitude  $A$  for  $\pi - \pi$  scattering includes some of the LECs. Explicitly, they are  $l_1, l_2, l_3$  and  $l_4$ . If one treats  $A$  as a function of these parameters, it is not hard to see that  $t_{uIJ}$  and the energy spectrum obtained from Equation 2.4 can also be treated as a function of these parameters.

## 2.5 $\chi^2$ Fit and the Renormalized LECs

As shown previously, there are many parameters called low-energy constants (LECs) in ChPT Lagrangian that have not been well determined yet. In this project, a part of renormalized LECs will be optimized. From that, it's easy to obtain  $\bar{l}_i$ , which is the scale independent version of renormalized LECs  $l_i^r$  and the original LECs  $l_i$ . The tool that will be used to optimize the parameters is the  $\chi^2$  fit.  $\chi^2$  is a statistical quantity that describe how a model agrees with the real data. In general, it is given by

$$\chi^2 = \sum_{i=1}^n \left( \frac{\delta X_i}{\sigma_i} \right)^2, \quad \delta X = X_{model} - X_{data} \quad (2.8)$$

where  $X$  is the quantity that is being compared.  $n$  is the number of data points.  $\sigma$  is the uncertainty of data. In this case, the ChPT energy spectrum generated by Lüscher's method is the model that will be compared with Lattice QCD spectrum. However, the Lattice QCD calculation has a special property that the energy spectrums have in correlation with each other. If this correlation is not included in the  $\chi^2$  computation, then the discrepancy will be counted multiple times. It will create

a significant error. To include this correlation, the  $\chi^2$  formula with covariance matrix will be used, it is given by

$$\chi^2 = \sum_{i=1, j=1}^n \delta X_i C_{ij}^{-1} \delta X_j \quad (2.9)$$

where  $C$  is the covariance matrix. The relationship between covariance matrix and correlation matrix is given by

$$C_{ij} = R_{ij} \sigma_i \sigma_j \quad (2.10)$$

where  $R$  is the correlation matrix. The correlation matrix is provided by the Hadron Spectrum Collaboration, which computed the Lattice QCD spectrum.

The procedure of fitting for the ChPT parameters is to minimize the  $\chi^2$  with Lattice QCD spectrum at both I=0 and I=1 channel. Although mathematically the renormalized LECs  $l_3^r$  and  $l_4^r$  contributes to the  $\chi^2$ , it is a fact that physically ChPT assumes the chiral symmetry where all quarks are massless and l3 and l4 are the parameters that relate to the quark mass term in ChPT Lagrangian. Also, they can be straightly calculated in theory. Therefore, these two parameters should not affect the fit significantly. According to this fact, it is reasonable to treat  $l_3^r$  and  $l_4^r$  as constants and only fit for  $l_1^r$  and  $l_2^r$ . In this project, for the completeness, settings of both two parameters fit and four parameters fit will be included. The list of settings is

- Fitting  $l_1^r, l_2^r, l_3^r, l_4^r$  with Lattice QCD spectrum at both isoscalar and isovector channel.
- Fitting  $l_1^r, l_2^r$  with Lattice QCD spectrum at both isoscalar and isovector channel.
- Fitting  $l_1^r, l_2^r, l_3^r, l_4^r$  with Lattice QCD spectrum at isoscalar channel.
- Fitting  $l_1^r, l_2^r$  with Lattice QCD spectrum at isoscalar channel.
- Fitting  $-2l_1^r + l_2^r, l_4^r$  with Lattice QCD spectrum at isovector channel.
- Fitting  $-2l_1^r + l_2^r$  with Lattice QCD spectrum at isovector channel.

The use of linear combination is for a purpose. Detailed explanation is in the next chapter.

## Chapter 3

### Isvector Fit

At the isospin  $I = 1$  (also known as isovector) channel, the isospin amplitude is given by Equation 2.7. To simplify the calculation, a set of parameters will be introduced. They are called the fitting parameters to distinguish from LECs in ChPT Lagrangian. With these fitting parameters, it is much easier to compare the results at different isospin channels than just use the renormalized LECs. The fitting parameters are defined as

$$\begin{aligned}\alpha_1 &= -2l_1^r + l_2^r \\ \alpha_2 &= l_2^r \\ \alpha_3 &= l_3^r \\ \alpha_4 &= l_4^r\end{aligned}\tag{3.1}$$

This definition will be used throughout the project regardless of the isospin channel. Based on [7], the leading order terms of the projected amplitude  $t_1$  is given by

$$tlo_1 = \frac{1}{64\pi^2} \int_{-1}^1 d(\cos\theta) \cos\theta \frac{t-u}{F_\pi^2}\tag{3.2}$$

where  $F_\pi$  is a constant given in [7]. At the isovector channel, the  $l_3^r$  term cancelled out in Eqn.2.7. Therefore, there are only two fitting parameters relative to the isovector channel. The next leading order terms of the projected amplitude  $t_1$  can be written as a linear combination of fitting parameters which is

$$tnlo_1 = M_0 + M_1\alpha_1 + M_4\alpha_4\tag{3.3}$$

The coefficients are given by

$$\begin{aligned}
M_0 &= \frac{1}{64\pi^2} \int_{-1}^1 d\cos(\theta) \cos(\theta) \left[ \frac{t-u}{96\pi^2 F_\pi^2} \left( \frac{s-12m_\pi^2}{F_\pi^2} - \frac{12m_{\pi phys}^2}{F_\pi^2} \ln\left(\frac{m_{\pi phys}^2}{\mu^2}\right) \right) \right. \\
&\quad \left. + B(t, s, u) - B(u, t, s) \right] \\
M_1 &= \frac{1}{64\pi^2} \int_{-1}^1 d\cos(\theta) \cos(\theta) \frac{s(t-u)}{F_\pi^4} \\
M_4 &= \frac{1}{64\pi^2} \int_{-1}^1 d\cos(\theta) \cos(\theta) \frac{2m_{\pi phys}^2(t-u)}{F_\pi^4}
\end{aligned} \tag{3.4}$$

where  $m_{\pi phys}$  is the physical pion mass.  $\mu$  is the scale factor which is 770 in this project. The definition of function  $B$  can be found in [7]. Then the unitarized projected amplitude  $tu$  as well as  $\chi^2$  can be treated as a function of fitting parameters [17]. When  $\chi^2(\alpha_1, \alpha_4)$  reaches global minimum, the corresponding fitting parameters will be at the best fit value.

Because of the property of  $l_3^r$  and  $l_4^r$  mentioned in Sec.[2.5], depending on whether  $l_3^r$  and  $l_4^r$  are set to be constant, there are two options of fitting which are fitting for only  $\alpha_1$  and fitting for both  $\alpha_1$  and  $\alpha_4$ .

### 3.1 One Parameter Fit

The most recent estimated value for  $\bar{l}_3$  and  $\bar{l}_4$  can be found in Ref.[Flag]. Under the condition that the number of flavors  $N_f = 2 + 1$ , they are given by

$$\bar{l}_3 = 3.05, \quad \bar{l}_4 = 4.02 \tag{3.5}$$

They are connected with the renormalized LECs in a way that

$$\begin{aligned}
l_i^r &= \frac{\gamma_i}{32\pi^2} \left[ \bar{l}_i + \ln\left(\frac{m_{\pi phys}^2}{\mu^2}\right) \right] \\
\gamma_3 &= -\frac{1}{2}, \quad \gamma_4 = 2
\end{aligned} \tag{3.6}$$

Therefore, the renormalized version of 3.5 is

$$l_3^r = -0.0011, l_4^r = 0.0105 \tag{3.7}$$

The one parameter  $\chi^2$  fit at isovector channel gives a result that

$$\alpha_1 = 0.0120 \pm 0.0001 \tag{3.8}$$

The minimum of  $\chi^2$  is 91.8729. Based on the number of spectrum lines which is 24, the  $\chi^2$  per degree of freedom is 3.83. This result is relatively large. There are some facts that might reduce the quality of fit. I will talk about it in Chapter 7.

### 3.2 Two Parameters Fit

In the two parameters fit at isovector channel, the relative fitting parameters are  $\alpha_1$  and  $\alpha_4$ .  $\chi^2$  fit gives a result that

$$\alpha_1 = 0.0147 \pm 0.0004, \quad \alpha_4 = -0.0331 \pm 0.0063 \quad (3.9)$$

The minimum of  $\chi^2$  is 26.8485. The  $\chi^2$  per d.o.f. is 1.12.

### 3.3 Comparison with Lattice QCD Spectrum

The phase shift plots calculated from these best fit parameters are

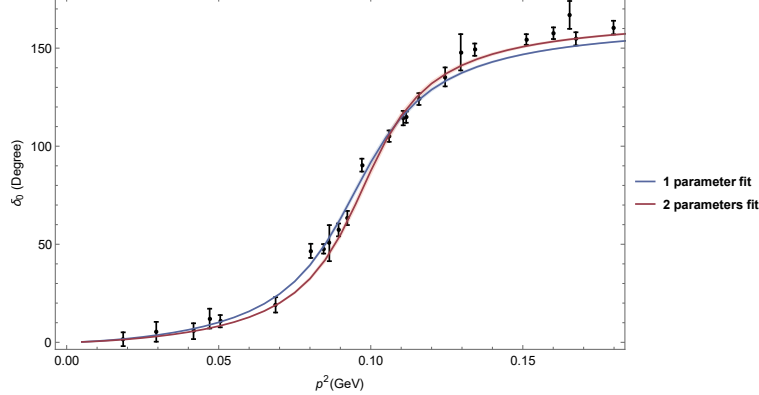


Figure 3.1: Isovector Phase shift with best fit parameters. Black data points are the Lattice QCD spectrum computed by Hadron Spectrum Collaboration [2].

This plot shows that both one parameter and two parameters single isospin channel fit agree with the Lattice QCD spectrum. Although the  $\chi^2$  of one parameter fit is relatively large, it behaves pretty well at low energy region.

## Chapter 4

### Isoscalar Fit

At the isospin  $I = 0$  (also known as isoscalar) channel, the leading order and the next leading order terms of the projected amplitude  $t_0$  can be written in the form

$$\begin{aligned} tlo_0 &= \int_{-1}^1 d\cos(\theta) \frac{3s + t + u - 5m_\pi^2}{F_\pi^2} \\ tnlo_0 &= N_0 + N_1\alpha_1 + N_2\alpha_2 + N_3\alpha_3 + N_4\alpha_4 \end{aligned} \quad (4.1)$$

where the coefficients are

$$\begin{aligned} N_0 &= \int_{-1}^1 d\cos(\theta) [3B(s, t, u) + B(t, s, u) + B(u, t, s) \\ &\quad - \frac{1}{96\pi^2 F_\pi^4} (20s^2 + 20t^2 + 20st - 56m_\pi^2 s - 80m_\pi^2 t + 93m_\pi^4) \ln\left(\frac{m_{\pi phys}^2}{\mu^2}\right) \\ &\quad - \frac{1}{144\pi^2 F_\pi^4} (31s^2 + 28t^2 + 28st - 88m_\pi^2 s - 112m_\pi^2 t + 139.5m_\pi^4)] \\ N_1 &= -\frac{22s^2 - 80m_\pi^2 s + 88m_\pi^4}{3F_\pi^4} \\ N_2 &= \frac{50s^2 - 160m_\pi^2 s + 200m_\pi^4}{3F_\pi^4} \\ N_3 &= \frac{20m_\pi^4}{F_\pi^4} \\ N_4 &= \frac{8m_\pi^2 s - 4m_\pi^4}{F_\pi^4} \end{aligned} \quad (4.2)$$

#### 4.1 Two Parameters Fit

In the two parameters fit at isoscalar channel, the relative fitting parameters are  $\alpha_1$  and  $\alpha_2$ .  $\chi^2$  fit gives a result that

$$\alpha_1 = -0.0704 \pm 0.0153, \quad \alpha_2 = -0.0290 \pm 0.0064 \quad (4.3)$$

In terms of the renormalized LECs, the result is

$$l_1^r = 0.0207 \pm 0.0044, \quad l_2^r = -0.0290 \pm 0.0064 \quad (4.4)$$

The minimum of  $\chi^2$  is 18.9345. Based on the number of spectrum lines which is 13, the  $\chi^2$  per d.o.f. is 1.46.

## 4.2 Four Parameters Fit

In the four parameters fit at isoscalar channel, the relative fitting parameters are  $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ .  $\chi^2$  fit gives a result that

$$\alpha_1 = -0.0865 \pm 0.0264, \quad \alpha_2 = -0.0332 \pm 0.0109, \quad \alpha_3 = 0.1231 \pm 0.0683, \quad \alpha_4 = -0.0598 \pm 0.0412 \quad (4.5)$$

In terms of the renormalized LECs, the result is

$$l_1^r = 0.0266 \pm 0.0078, \quad l_2^r = -0.0332 \pm 0.0109, \quad l_3^r = 0.1231 \pm 0.0683, \quad l_4^r = -0.0598 \pm 0.0412 \quad (4.6)$$

The minimum of  $\chi^2$  is 15.2187. The  $\chi^2$  per d.o.f. is 1.17. Although 4 parameters fit gives a small  $\chi^2$ , there are many global minimums. With the same  $\chi^2$ , the corresponding parameters falls in a wide range. The reason might be that too many parameters being fitted. There is no unique set of best fit parameters. The result given in 4.5 is a median of many set of best fit parameters.

## 4.3 Comparison with Lattice QCD Spectrum

The phase shift plots calculated from these best fit parameters are



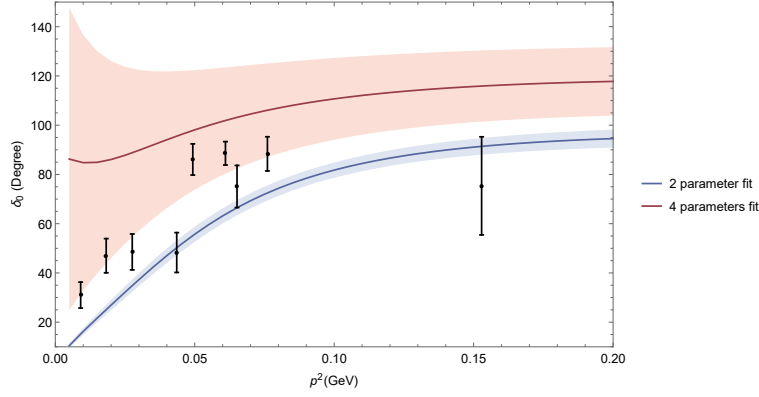


Figure 4.1: Isoscalar Phase shift with best fit parameters. Black data points are the Lattice QCD spectrum computed by Hadron Spectrum Collaboration [3].

As mentioned in 4.2, the 4 parameters fit at isoscalar channel is very unstable. The wide error band represents such instability. The upper boundary of error band is even at the bound state regime. One way to interpret is that there are too many degrees of freedom in fitting parameters. Another interpretation is that there are too less constraints. Once extra data points are added to the fit, 4 parameters fit behaves much better 5.2.

## Chapter 5

### Overall Fit

Because the Lattice QCD calculated spectrum at both isoscalar and isovector channel are equally reliable, the overall chi-square is given by

$$\chi^2_{total} = \chi^2_{I=0} + \chi^2_{I=1} \quad (5.1)$$

The two parameters  $\chi^2$  fit based on both isovector and isoscalar channel gives

$$\alpha_1 = 0.0119 \pm 0.0001, \alpha_2 = 0.0046 \pm 0.0002 \quad (5.2)$$

In terms of the renormalized LECs, the result is

$$l_1^r = -0.0037 \pm 0.0002, \quad l_2^r = 0.0046 \pm 0.0002 \quad (5.3)$$

The minimum of  $\chi^2$  is 187.817 Based on the number of spectrum lines which is 37, the  $\chi^2$  per d.o.f. is 5.08. The four parameters  $\chi^2$  fit based on both isovector and isoscalar channel gives

$$\alpha_1 = 0.0148 \pm 0.0003, \alpha_2 = 0.0088 \pm 0.0004, \alpha_3 = 0.0134 \pm 0.0169, \alpha_4 = -0.0346 \pm 0.0047 \quad (5.4)$$

In terms of the renormalized LECs, the result is

$$l_1^r = -0.0030 \pm 0.0007, \quad l_2^r = 0.0088 \pm 0.0004, \quad l_3^r = 0.0134 \pm 0.0169, \quad l_4^r = -0.0346 \pm 0.0047 \quad (5.5)$$

The minimum of  $\chi^2$  is 44.82, and the  $\chi^2$  per d.o.f. is 1.21.

## 5.1 Isovector Channel

The isovector phase shift plots calculated from these best fit parameters are given below 5.1.

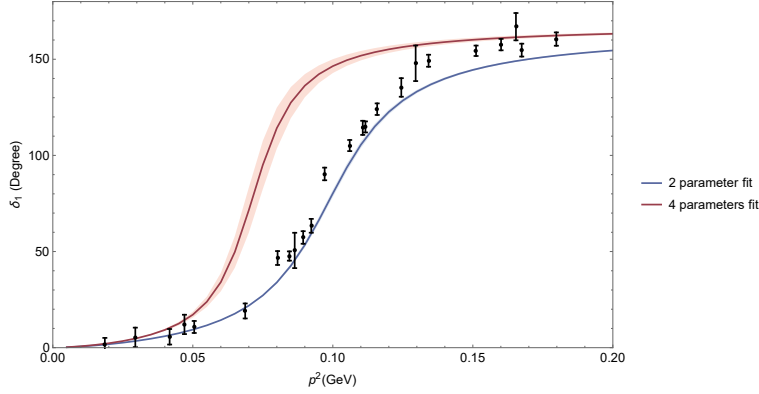


Figure 5.1: Isovector Phase shift with best fit parameters obtained by fitting at both channel. Black data points are the Lattice QCD spectrum computed by Hadron Spectrum Collaboration [2].

This plot shows that the two parameters overall fit gives better agreement with Lattice QCD spectrum than the four parameters overall fit. The reason is that four parameters fit has more degree of freedom. It is easier to be dragged away while fitting at isoscalar channel. At the isovector channel, the two parameters fit result is better.

## 5.2 Isoscalar Channel

The isoscalar phase shift plots calculated from these best fit parameters are given below 5.2. This plot shows that each fitting option gives agreement with a part of Lattice QCD spectrum. It is difficult to claim which one is better. One thing that is worthy to point out is the four parameters fit phase shift has a significantly large error band. As mentioned in 4.2, four parameters fit has no unique set of best fit parameters. This fact contributes to a large uncertainty which makes four parameters fit less reliable.

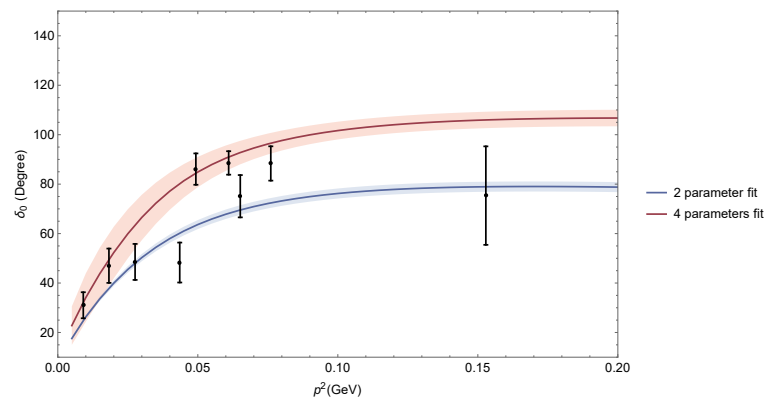


Figure 5.2: Isoscalar Phase shift with best fit parameters obtained by fitting at both channel. Black data points are the Lattice QCD spectrum computed by Hadron Spectrum Collaboration [3].

## Chapter 6

### Comparison with the Experimental Data

The best way to test a model is to compare it with the physical experimental data. An effective model should be able to reproduce experimental observations. In this project, since the data being fitted with is Lattice QCD spectrum, the pion mass is set to a non-physical value which is 236 MeV. It is valuable to see how the parameters obtained with a virtual pion mass perform in a physical pion mass scenario.

#### 6.1 Isovector $\pi - \pi$ Scattering Experiment

The phase shift plots at isovector channel with physical pion mass and parameters obtained in different settings is given below. The isovector channel  $\pi - \pi$  scattering experimental data used here can be found in [6] and [19]. At the isovector channel, all sets of best fit parameters give good predictions to the experiment result. The error band is very narrow as well. Considering the result in 3.3, it is safe to draw the conclusion that ChPT is an effective model for the isovector channel  $\pi - \pi$  scattering.

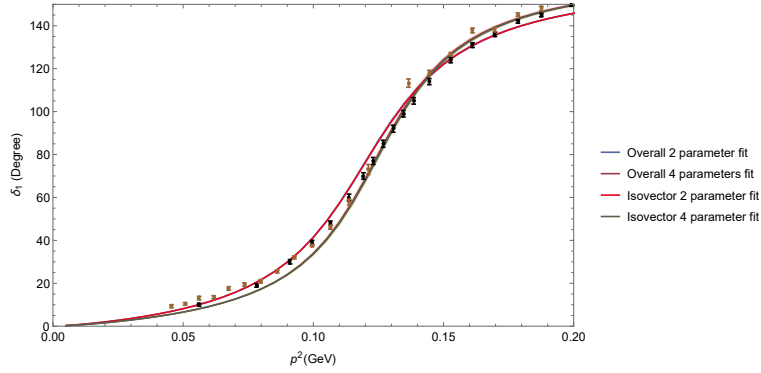


Figure 6.1: Experiment data with Phase shift plot at isoscalar channel using different fitting options. Black data points are from [19]. Brown data points are from [6]. Except for the red phase shift plot, other three phase shift plots overlap on each other.

## 6.2 Isoscalar $\pi - \pi$ Scattering Experiment

The phase shift plots at isoscalar channel with physical pion mass and parameters obtained in different settings is given below 6.2. The isovector channel  $\pi - \pi$  scattering experimental data used here can be found in [5].

The phase shift plot of the experiment is much flatter than the model. It is most likely due to the large variation of isoscalar Lattice QCD spectrum at low energy that affects the performance of ChPT model at that energy range. Relatively, two parameters fit in general gives a better post-diction at isoscalar channel.

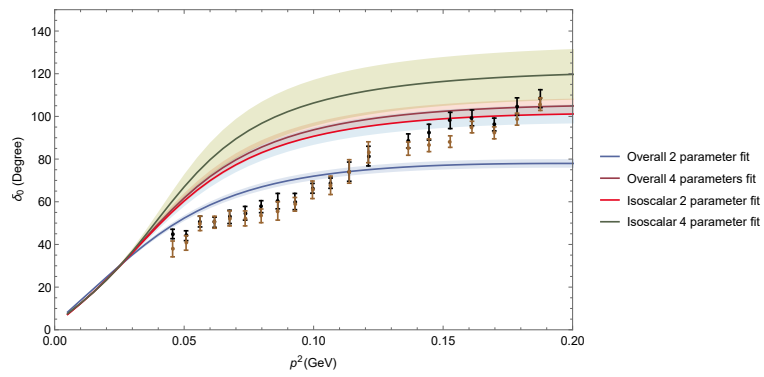


Figure 6.2: Experiment data with Phase shift plot at isoscalar channel using different fitting options. Black data points are at s channel. Brown data points are at t channel [5].

### 6.3 Summary

It turns out that depending on the isospin channel, the agreements with experimental data are different. There are some reasons for the difference between isospin channels. Firstly, the isovector experimental data has small uncertainty. That means the experimental data is very reliable. Secondly, the isovector experimental data is close to the Lattice QCD spectrum which is being fitted with. Thus, the small difference between phase shift plots calculated from fitting parameters and experimental data is reasonable. Thirdly, the isoscalar Lattice QCD spectrum has large uncertainty. Also, it has a different shape from the isoscalar experimental data. Moreover, since these parameters are determined with non-physical pion mass, it is not surprising that there is a discrepancy from the real experimental data. To improve the quality of prediction and post-diction, one can fit for Lattice QCD spectrum calculated with different virtual pion mass and find out the dependency of pion mass in ChPT model. Then it is possible to exclude the disturbance from differences between virtual and physical pion mass.



## Chapter 7

### Conclusion

In this thesis, I have demonstrated the mathematical tool I used to fit for the low energy constants. By setting different scenarios, many sets of best fit parameters are obtained.

Table 7.1: Best fit parameters obtained with different settings

Settings <sup>†</sup>	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$	$\chi^2$	$\chi^2/d.o.f.$
Isvector-1	0.0120	/	/	*	91.87	3.83
Isvector-2	0.0147	/	/	-0.0331	26.85	1.12
Isoscalar-2	-0.0704	-0.0290	*	*	18.93	1.46
Isoscalar-4	-0.0865	-0.0332	0.1231	-0.0598	15.22	1.17
Overall-2	0.0119	0.0046	*	*	187.82	5.08
Overall-4	0.0148	0.0088	0.0134	-0.0346	44.82	1.21

<sup>†</sup> The settings are in the form: Isospin channel - Number of parameters being fitted.

\* The value of  $\alpha_3$  and  $\alpha_4$  are given in 3.5 .

The result shows that in general, fittings with more free parameters give smaller  $\chi^2$ . However,  $l_3^r$  and  $l_4^r$  obtained from the fits have a large difference from the theory estimated values. There is a possibility that the estimated values for  $l_3^r$  and  $l_4^r$  are not accurate enough. After all, the computation for these LECs is not first principle calculation [1]. Regardless of the possible discrepancy, it is still reasonable to use the two parameters fit result. The phase shift plots shows that the two parameters fit result also agrees with the Lattice QCD spectrum and the experimental data. Depending on the needs, one may choose whether to set  $l_3^r$  and  $l_4^r$  as theoretical values.

Compared to the ideal  $\chi^2$  per *d.o.f.* which is 1.0, some fit settings have relatively large  $\chi^2$  per *d.o.f.*. There are many facts that possibly generate errors in the fit that I was not able to include

in the calculation, which will make the  $\chi^2$  larger. The most fundamental one is the incompleteness in ChPT Lagrangian. In this project, I used  $SU(2)$  ChPT instead of  $SU(3)$  which is the exact symmetry group for chiral symmetry. The purpose is to make the calculation simpler. Otherwise, there will be numerous terms to handle and the ChPT calculation will be nearly impossible to make. Theoretically, if the exact  $SU(3)$  chiral symmetry group is used in ChPT model, the kaon contribution in pion terms will be separated and form an individual term that has its own low energy constants. Then the ChPT model will be more accurate for  $\pi - \pi$  scattering. Another fact is that mathematic tool I used, the inverse amplitude method. In the inverse amplitude method, the unitarized projected amplitude is divided to a series of infinite terms. The typical cut-off point is at the next leading order. By introducing the next next leading order terms and probably more terms after that, it can give a more accurate approximation. Also, there are other systematic errors that are not included in this project. For example, the Lattice QCD estimated pion mass and some constants in ChPT model like  $F_\pi$  have uncertainties.

In summary, all settings of fit relatively agree with Lattice QCD spectrum and experimental data. Depending on the needs, one may choose different set of parameters. The results in this thesis can be improved after the isospin-2 channel Lattice QCD spectrum calculation is finished. However, with the observations in this thesis, one may expect there is the possibility that the overall fit including isospin-2 channel would not give a global best set of parameters. If at some point the strategy of fitting is limited to the single isospin channel, then the ChPT model itself or the computational methods need to be modified.

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## Appendix A

### Mathematica Code

# Definitions

```

Clear["Global`*"]
fpi =  $\frac{922}{10}$ ;
mpi =  $\frac{3928 * 10^{10}}{166761338157}$ ;
mpiphys =  $\frac{13957}{100}$ ;
(*lr1 = -54/10000;
lr2 = 567/100000;
lr3 = 82/10000;
lr4 = 56/10000; *)
 $\gamma_1 = 1/3$ ;
 $\gamma_2 = 2/3$ ;
 $\gamma_3 = -1/2$ ;
 $\gamma_4 = 2$ ;
 $\mu = 770$ ;
newl3bar = 3.05;
newl4bar = 4.02;
newl3 =  $\left( \text{newl3bar} + \text{Log}\left[\left(\frac{\text{mpi}}{\mu}\right)^2\right] \right) * \frac{\gamma_3}{32 \text{ Pi}^2}$ ;
newl4 =  $\left( \text{newl4bar} + \text{Log}\left[\left(\frac{\text{mpi}}{\mu}\right)^2\right] \right) * \frac{\gamma_4}{32 \text{ Pi}^2}$ ;
s[e_?NumberQ, x_] := e2
t[e_?NumberQ, x_] := -  $\left(\frac{e^2}{2} - \text{mpi}^2\right) - \left(\frac{e^2}{2} - \text{mpi}^2\right) + 2 \left(\left(\frac{e^2}{2} - \text{mpi}^2\right) \left(\frac{e^2}{2} - \text{mpi}^2\right)\right)^{1/2} * x$ 
u[e_?NumberQ, x_] := -  $\left(\frac{e^2}{2} - \text{mpi}^2\right) - \left(\frac{e^2}{2} - \text{mpi}^2\right) - 2 \left(\left(\frac{e^2}{2} - \text{mpi}^2\right) \left(\frac{e^2}{2} - \text{mpi}^2\right)\right)^{1/2} * x$ 
 $\sigma[s_] := (1 - 4 \text{ mpi}^2 / s)^{1/2}$ ;
jbar[s_] :=  $(16 \text{ Pi}^2)^{-1} * \text{If}[s < \text{mpi}^2, \sigma[s] * (\text{Log}[\sigma[s] - 1] - \text{Log}[\sigma[s] + 1]) + 2,$ 
 $\text{If}[s > 4 * \text{mpi}^2, \sigma[s] * (\text{Log}[1 - \sigma[s]] - \text{Log}[\sigma[s] + 1] + \text{I} * \text{Pi}) + 2,$ 
 $2 - 2 * \left(\frac{4 * \text{mpi}^2}{s} - 1\right)^{1/2} * \text{ArcCot}\left[\left(\frac{4 * \text{mpi}^2}{s} - 1\right)^{1/2}\right]]];$ 
b[s_, t_, u_, paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
 $(6 \text{ fpi}^4)^{-1} * (3 (s^2 - \text{mpi}^4) * \text{jbar}[s] + (t (t - u) - 2 \text{ mpi}^2 * t + 4 \text{ mpi}^2 * u - 2 \text{ mpi}^4) * \text{jbar}[t] +$ 
 $(u (u - t) - 2 \text{ mpi}^2 * u + 4 \text{ mpi}^2 * t - 2 \text{ mpi}^4) * \text{jbar}[u])$ 
c[s_, t_, u_, paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
 $(96 \text{ Pi}^2 * \text{fpi}^4)^{-1} \left( 2 \left( \frac{32 \text{ Pi}^2 * \frac{1}{2} (\text{parab} - \text{paraa})}{\gamma_1} - \text{Log}\left[\left(\frac{\text{mpi}}{\mu}\right)^2\right] - \frac{4}{3} \right) (s - 2 \text{ mpi}^2)^2 + \right.$ 

```

$$\left( \frac{32 \text{ Pi}^2 * \text{parab}}{\gamma_2} - \text{Log}\left[\left(\frac{\text{mpi}}{\mu}\right)^2\right] - \frac{5}{6} \right) (s^2 + (t-u)^2) - 12 \text{ mpi}^2 * s + 15 \text{ mpi}^4$$

alo[s\_, t\_, u\_, paraa\_?NumberQ, parab\_?NumberQ, parac\_?NumberQ, parad\_?NumberQ] :=

$$\frac{1}{\text{fpi}^2} * s - \frac{\text{mpi}^2}{\text{fpi}^2}$$

anlo[s\_, t\_, u\_, paraa\_?NumberQ, parab\_?NumberQ, parac\_?NumberQ, parad\_?NumberQ] :=

$$\begin{aligned} & \frac{1}{\text{fpi}^2} * \frac{1}{8 \text{ Pi}^2} * \frac{\text{mpi}_{\text{phys}}^2}{\text{fpi}^2} * \left( \frac{32 \text{ Pi}^2 * \text{parad}}{\gamma_4} - \text{Log}\left[\left(\frac{\text{mpi}_{\text{phys}}}{\mu}\right)^2\right] \right) * s - \\ & \frac{\text{mpi}^2}{\text{fpi}^2} * \frac{1}{32 \text{ Pi}^2} * \frac{\text{mpi}^2}{\text{fpi}^2} * \left( \frac{32 \text{ Pi}^2 * \text{parac}}{\gamma_3} - \text{Log}\left[\left(\frac{\text{mpi}_{\text{phys}}}{\mu}\right)^2\right] \right) - \\ & \frac{\text{mpi}^2}{\text{fpi}^2} * \frac{1}{8 \text{ Pi}^2} * \frac{\text{mpi}_{\text{phys}}^2}{\text{fpi}^2} * \left( \frac{32 \text{ Pi}^2 * \text{parad}}{\gamma_4} - \text{Log}\left[\left(\frac{\text{mpi}_{\text{phys}}}{\mu}\right)^2\right] \right) + \end{aligned}$$

b[s, t, u, paraa, parab, parac, parad] + c[s, t, u, paraa, parab, parac, parad]

alo[e\_?NumberQ, x\_, paraa\_?NumberQ, parab\_?NumberQ, parac\_?NumberQ,

parad\_?NumberQ] := 3 alo[s[e, x], t[e, x], u[e, x], paraa, parab, parac, parad] +  
alo[t[e, x], s[e, x], u[e, x], paraa, parab, parac, parad] +  
alo[u[e, x], t[e, x], s[e, x], paraa, parab, parac, parad]

anlo[e\_?NumberQ, x\_, paraa\_?NumberQ, parab\_?NumberQ, parac\_?NumberQ,

parad\_?NumberQ] := 3 anlo[s[e, x], t[e, x], u[e, x], paraa, parab, parac, parad] +  
anlo[t[e, x], s[e, x], u[e, x], paraa, parab, parac, parad] +  
anlo[u[e, x], t[e, x], s[e, x], paraa, parab, parac, parad]

tlo[e\_?NumberQ, paraa\_?NumberQ, parab\_?NumberQ, parac\_?NumberQ, parad\_?NumberQ] :=

$$\frac{1}{64 \text{ Pi}} * \text{NIntegrate}[\text{alo}_0[\text{e}, \text{x}, \text{paraa}, \text{parab}, \text{parac}, \text{parad}],$$

{x, -1, 1}, MaxRecursion -> 12, AccuracyGoal -> 20]

tnlo[e\_?NumberQ, paraa\_?NumberQ, parab\_?NumberQ, parac\_?NumberQ,

$$\text{parad_?NumberQ}] := \frac{1}{64 \text{ Pi}} * \text{NIntegrate}[\text{anlo}_0[\text{e}, \text{x}, \text{paraa}, \text{parab}, \text{parac}, \text{parad}],$$

{x, -1, 1}, MaxRecursion -> 12, AccuracyGoal -> 20]

tu[e\_?NumberQ, paraa\_?NumberQ, parab\_?NumberQ, parac\_?NumberQ, parad\_?NumberQ] :=

$$\frac{(\text{tlo}_0[\text{e}, \text{paraa}, \text{parab}, \text{parac}, \text{parad}])^2}{(\text{tlo}_0[\text{e}, \text{paraa}, \text{parab}, \text{parac}, \text{parad}] - \text{tnlo}_0[\text{e}, \text{paraa}, \text{parab}, \text{parac}, \text{parad}])}$$

$$\sigma[\text{e}_] := \frac{2 \text{ Sqrt}\left[\frac{\text{e}^2}{4} - \text{mpi}^2\right]}{\text{e}}$$

δ[e\_?NumberQ, paraa\_?NumberQ,

parab\_?NumberQ, parac\_?NumberQ, parad\_?NumberQ] :=

$$\frac{1}{2} \text{If}[\text{Arg}[1 + 2 \text{ I} * \sigma[\text{e}] * \text{tu}_0[\text{e}, \text{paraa}, \text{parab}, \text{parac}, \text{parad}]] > 0,$$

Arg[1 + 2 I \* σ[e] \* tu<sub>0</sub>[e, paraa, parab, parac, parad]],

2 Pi + Arg[1 + 2 I \* σ[e] \* tu<sub>0</sub>[e, paraa, parab, parac, parad]]]

b1[s\_, t\_, u\_] := (6 fpi<sup>4</sup>)<sup>-1</sup> \* (3 (s<sup>2</sup> - mpi<sup>4</sup>) \* j<sub>bar</sub>[s] +

(t (t - u) - 2 mpi<sup>2</sup> \* t + 4 mpi<sup>2</sup> \* u - 2 mpi<sup>4</sup>) \* j<sub>bar</sub>[t] +

```

(u (u - t) - 2 mpi^2 * u + 4 mpi^2 * t - 2 mpi^4) * jbar[u]
c1[s_, t_, u_] := (96 Pi^2 * fpi^4)^-1 (2 ( (32 Pi^2 * l_r1) / \gamma_1 - Log[(mpi/\mu)^2] - 4/3) (s - 2 mpi^2)^2 +
( (32 Pi^2 * l_r2) / \gamma_2 - Log[(mpi/\mu)^2] - 5/6) (s^2 + (t - u)^2) - 12 mpi^2 * s + 15 mpi^4)
alo1[s_, t_, u_] := 1/fpi^2 * s (*fLO*) - mpi^2/fpi^2 (*mLOfLO*)
anlo1[s_, t_, u_] := 1/fpi^2 * 1/(8 Pi^2) * mpi_phys^2/fpi^2 * ( (32 Pi^2 * l_r4) / \gamma_4 - Log[(mpi_phys/\mu)^2] ) * s
(*fNLO*) - mpi^2/fpi^2 * 1/(32 Pi^2) * mpi^2/fpi^2 * ( (32 Pi^2 * l_r3) / \gamma_3 - Log[(mpi_phys/\mu)^2] )
(*mNLOfLO*) - mpi^2/fpi^2 * 1/(8 Pi^2) * mpi_phys^2/fpi^2 * ( (32 Pi^2 * l_r4) / \gamma_4 - Log[(mpi_phys/\mu)^2] )
(*mLOfNLO*) + b1[s, t, u] + c1[s, t, u]
alo11[e_, x_] := alo1[t[e, x], s[e, x], u[e, x]] - alo1[u[e, x], t[e, x], s[e, x]]
anlo11[e_, x_] := anlo1[t[e, x], s[e, x], u[e, x]] - anlo1[u[e, x], t[e, x], s[e, x]]
tlo1[e_?NumberQ, paraa_?NumberQ, parad_?NumberQ] :=
1/(64 Pi) * NIntegrate[x * alo11[e, x], {x, -1, 1}, MaxRecursion -> 12, AccuracyGoal -> 20]
tnlo1[e_?NumberQ, paraa_?NumberQ, parad_?NumberQ] :=
1/(64 Pi) * NIntegrate[x * (anlo11[e, x] /. {l_r1 -> 0, l_r2 -> paraa, l_r4 -> parad}),
{x, -1, 1}, MaxRecursion -> 12, AccuracyGoal -> 20]
tu1[e_?NumberQ, paraa_?NumberQ, parad_?NumberQ] :=
(tlo1[e, paraa, parad])^2 / (tlo1[e, paraa, parad] - tnlo1[e, paraa, parad])
tlo0a[e_?NumberQ] := NIntegrate[ (3 s[e, x] + t[e, x] + u[e, x] - 5 mpi^2) / fpi^2,
{x, -1, 1}, AccuracyGoal -> Infinity]
tnlo0s[e_?NumberQ] := NIntegrate[3 b[s[e, x], t[e, x], u[e, x], 0, 0, 0, 0] +
b[t[e, x], s[e, x], u[e, x], 0, 0, 0, 0] + b[u[e, x], t[e, x], s[e, x], 0, 0, 0, 0] -
1/(96 Pi^2 * fpi^4) (20 s[e, x]^2 + 20 t[e, x]^2 + 20 s[e, x] * t[e, x] -
56 mpi^2 * s[e, x] - 80 mpi^2 * t[e, x] + 93 mpi^4) * Log[(mpi/\mu)^2] -
1/(144 Pi^2 * fpi^4) (31 s[e, x]^2 + 28 t[e, x]^2 + 28 s[e, x] * t[e, x] - 88 mpi^2 * s[e, x] -
112 mpi^2 * t[e, x] + 279/2 mpi^4), {x, -1, 1}, AccuracyGoal -> Infinity]
delta1[e_?NumberQ, paraa_?NumberQ, parad_?NumberQ] :=
1/2 If[Arg[1 + 2 I * sigma[e] * tu1[e, paraa, parad]] > 0,
Arg[1 + 2 I * sigma[e] * tu1[e, paraa, parad]],
2 Pi + Arg[1 + 2 I * sigma[e] * tu1[e, paraa, parad]]]

```



# Quantization Condition and Lattice QCD Spectrum

```

SetDirectory["*****"];
myreTinvPts0A = ToExpression[Import["QCL320pts.dat", "Table"]];
myreTinvPts0B = ToExpression[Import["QCL321pts.dat", "Table"]];
myreTinvPts0C = ToExpression[Import["QCL322pts.dat", "Table"]];
myreTinvPts0D = ToExpression[Import["QCL323pts.dat", "Table"]];
myreTinvPts0E = ToExpression[Import["QCL324pts.dat", "Table"]];
reTinvFns0TempA = Interpolation[myreTinvPts0A];
reTinvFns0TempB = Interpolation[myreTinvPts0B];
reTinvFns0TempC = Interpolation[myreTinvPts0C];
reTinvFns0TempD = Interpolation[myreTinvPts0D];
reTinvFns0TempE = Interpolation[myreTinvPts0E];
reTinvFns0 = {reTinvFns0TempA, reTinvFns0TempB,
  reTinvFns0TempC, reTinvFns0TempD, reTinvFns0TempE};
rhsOfqc01[atEcm_] := reTinvFns0[[1]][atEcm]
rhsOfqc02[atEcm_] := reTinvFns0[[2]][atEcm]
rhsOfqc03[atEcm_] := reTinvFns0[[3]][atEcm]
rhsOfqc04[atEcm_] := reTinvFns0[[4]][atEcm]
rhsOfqc05[atEcm_] := reTinvFns0[[5]][atEcm]
myreTinvPts1 = ToExpression[Import["myreTinvPts32.txt", "Data"]];
myreTinvPts12 = Table[Table[
  {myreTinvPts1[[i]][[j]][[1]], If[Chop[myreTinvPts1[[i]][[j]][[2]]] > 100, 100,
    If[Chop[myreTinvPts1[[i]][[j]][[2]]] < -50, -50, Chop[myreTinvPts1[[i]][[j]][[2]]]}], {j, 1, Length[myreTinvPts1[[i]]]}, {i, 1, Length[myreTinvPts1]};
reTinvFns1Temp = Interpolation /@ myreTinvPts12;
reTinvFns1 = {reTinvFns1Temp[[1]], reTinvFns1Temp[[1]], reTinvFns1Temp[[2]],
  reTinvFns1Temp[[2]], reTinvFns1Temp[[2]], reTinvFns1Temp[[3]],
  reTinvFns1Temp[[3]], reTinvFns1Temp[[4]], reTinvFns1Temp[[4]],
  reTinvFns1Temp[[4]], reTinvFns1Temp[[5]], reTinvFns1Temp[[5]],
  reTinvFns1Temp[[5]], reTinvFns1Temp[[6]], reTinvFns1Temp[[6]],
  reTinvFns1Temp[[7]], reTinvFns1Temp[[7]], reTinvFns1Temp[[7]],
  reTinvFns1Temp[[8]], reTinvFns1Temp[[8]], reTinvFns1Temp[[9]],
  reTinvFns1Temp[[9]], reTinvFns1Temp[[10]], reTinvFns1Temp[[10]]};
rhsOfqc1[atEcm_] := reTinvFns1[[1]][atEcm]
at = 166 761 338 157 / 1015;
ξ = 34 534 / 10 000;
LQCDstates0 = {{0, 0, 0}, A1, L32, 0.115947, 0.00142602},
  {{1, 0, 0}, A1, L32, 0.0848943, 0.000877511}, {{1, 0, 0}, A1, L32,

```

```

0.121063, 0.00115686}, {{1, 1, 0}, A1, L32, 0.0906976, 0.00115657},
{{1, 1, 0}, A1, L32, 0.108073, 0.00105681}, {{1, 1, 0}, A1, L32,
0.132887, 0.0013674}, {{1, 1, 0}, A1, L32, 0.159622, 0.000791565},
{{1, 1, 1}, A1, L32, 0.096255, 0.00121419}, {{1, 1, 1}, A1, L32,
0.113939, 0.000790731}, {{1, 1, 1}, A1, L32, 0.152313, 0.00332102},
{{2, 0, 0}, A1, L32, 0.105065, 0.00134643}, {{2, 0, 0}, A1, L32,
0.124077, 0.00156035}, {{2, 0, 0}, A1, L32, 0.147224, 0.00264803}};
LQCDstates1 = {{{0, 0, 0}, T1, 32, 0.124876532196`, 0.000401563202305`},
{{0, 0, 0}, T1, 32, 0.145334926652`, 0.000520338746667`},
{{0, 0, 1}, A1, 32, 0.0908986393585`, 0.000586482665072`},
{{0, 0, 1}, A1, 32, 0.130361681351`, 0.000547012291416`},
{{0, 0, 1}, A1, 32, 0.151688479581`, 0.000451381960166`},
{{0, 0, 1}, E2, 32, 0.127045955533`, 0.000544248373249`},
{{0, 0, 1}, E2, 32, 0.154926989947`, 0.000494288739937`},
{{1, 1, 0}, A1, 32, 0.097326974844`, 0.000841641908606`},
{{1, 1, 0}, A1, 32, 0.134126256082`, 0.000482944684762`},
{{1, 1, 0}, A1, 32, 0.156823655244`, 0.00118300544542`},
{{1, 1, 0}, B1, 32, 0.108686603849`, 0.000520025908684`},
{{1, 1, 0}, B1, 32, 0.13604068184`, 0.000612023007148`},
{{1, 1, 0}, B1, 32, 0.161859449921`, 0.000577075714187`},
{{1, 1, 0}, B2, 32, 0.128305260412`, 0.000599940698685`},
{{1, 1, 0}, B2, 32, 0.157552411196`, 0.000557074900846`},
{{1, 1, 1}, A1, 32, 0.104094074151`, 0.000672088298386`},
{{1, 1, 1}, A1, 32, 0.122979511613`, 0.000601204241731`},
{{1, 1, 1}, A1, 32, 0.138133652805`, 0.000500717725465`},
{{1, 1, 1}, E2, 32, 0.117633799824`, 0.000649386168558`},
{{1, 1, 1}, E2, 32, 0.141559319897`, 0.000804362062953`},
{{0, 0, 2}, A1, 32, 0.106887826676`, 0.000838665019393`},
{{0, 0, 2}, A1, 32, 0.136433977868`, 0.000477181177098`},
{{0, 0, 2}, E2, 32, 0.125713358936`, 0.00153256226506`},
{{0, 0, 2}, E2, 32, 0.143558938688`, 0.00154096829379`}};

```

## Interpolating Functions

```

tlo0table = Table[{e, tlo0a[e]}, {e, 500, 1000, 0.5}];
tnlo0tables = Table[{e, tnlo0s[e]}, {e, 500, 1000, 0.5}];
tlo0int = Interpolation[tlo0table];
tnlo0ints = Interpolation[tnlo0tables];
tnlo0inta[e_] := -  $\frac{22 (e^2)^2 - 80 \text{mpi}^2 * e^2 + 88 \text{mpi}^4}{3 \text{fpi}^4}$ ;
tnlo0intb[e_] :=  $\frac{50 (e^2)^2 - 160 \text{mpi}^2 * e^2 + 200 \text{mpi}^4}{3 \text{fpi}^4}$ ;
tnlo0intc[e_] :=  $\frac{20 \text{mpi}^4}{\text{fpi}^4}$ ;
tnlo0intd[e_] :=  $\frac{8 \text{mpi}^2 * e^2 - 4 \text{mpi}^4}{\text{fpi}^4}$ ;
tuint0[e_?NumberQ, paraa_?NumberQ,
  parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
  (tlo0int[e])^2 / (tlo0int[e] - (tnlo0inta[e] * paraa + tnlo0intb[e] * parab +
    tnlo0intc[e] * parac + tnlo0intd[e] * parad + tnlo0ints[e]))

```

```

tlo1[e_?NumberQ] :=
  
$$\frac{1}{64 \text{ Pi}} * \text{NIntegrate}\left[x * \left(\frac{t[e, x] - u[e, x]}{fpi^2}\right), \{x, -1, 1\}, \text{AccuracyGoal} \rightarrow \text{Infinity}\right]$$

tnlola[e_?NumberQ] :=  $\frac{1}{64 \text{ Pi}} * \text{NIntegrate}\left[x * \left(\frac{t[e, x] - u[e, x]}{96 \text{ Pi}^2 * fpi^2} * \left(\frac{s[e, x] - 12 \text{ mpi}^2}{fpi^2} - 12 * \frac{\text{mpi}_{\text{phys}}^2}{fpi^2} * \text{Log}\left[\left(\frac{\text{mpi}_{\text{phys}}}{\mu}\right)^2\right]\right) + \right.$ 
  
$$\left. b1[t[e, x], s[e, x], u[e, x]] - b1[u[e, x], t[e, x], s[e, x]]\right), \{x, -1, 1\}, \text{AccuracyGoal} \rightarrow \text{Infinity}\right]$$

tnlolb[e_?NumberQ] :=  $\frac{1}{64 \text{ Pi}} * \text{NIntegrate}\left[x * \left(\frac{s[e, x] * (t[e, x] - u[e, x])}{fpi^4}\right), \{x, -1, 1\}, \text{AccuracyGoal} \rightarrow \text{Infinity}\right]$ 
tnlolc[e_?NumberQ] :=  $\frac{1}{64 \text{ Pi}} * \text{NIntegrate}\left[x * \left(2 * \frac{\text{mpi}_{\text{phys}}^2 * (t[e, x] - u[e, x])}{fpi^4}\right), \{x, -1, 1\}, \text{AccuracyGoal} \rightarrow \text{Infinity}\right]$ 
tlo1table = Table[{e, tlo1[e]}, {e, 2.001 mpi, 1000, 0.5}];
tnlolatable = Table[{e, tnlola[e]}, {e, 2.01 mpi, 1000, 0.5}];
tnlolbtable = Table[{e, tnlolb[e]}, {e, 2.01 mpi, 1000, 0.5}];
tnlolcetable = Table[{e, tnlolc[e]}, {e, 2.01 mpi, 1000, 0.5}];
tlo1int = Interpolation[tlo1table];
tnlolaint = Interpolation[tnlolatable];
tnlolbint = Interpolation[tnlolbtable];
tnlolcint = Interpolation[tnlolcetable];
tuintl[e_?NumberQ, paraa_?NumberQ, parad_?NumberQ] := (tlo1int[e])^2 /
  (tlo1int[e] - (tnlolaint[e] + tnlolbint[e] * paraa + tnlolcint[e] * parad))

```

## $\chi^2$ with Covariance Matrix

```

guesslist0 = {722.2137791170269`, 514.648744128735`, 773.4538235804388`,
  554.579394830871`, 653.746085138525`, 845.3395112878601`, 947.245983433109`,
  582.893384859294`, 693.6167252847172`, 1009.5401309471729`,
  626.4648286749675`, 774.3333003351929`, 922.3842530032591`};
emodell[i_, paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
  e /. FindRoot[
    Re[ $\left(\frac{1}{64 \text{ Pi}} * \text{tuintl}[e, \text{paraa}, \text{parab}, \text{parac}, \text{parad}]\right)^{-1}$ ] == reTinvFns0[[1]][at * e],
    {e, guesslist0[[i]]}, AccuracyGoal -> Infinity, PrecisionGoal -> 100]
emodel2[i_, paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
  e /. FindRoot[
    Re[ $\left(\frac{1}{64 \text{ Pi}} * \text{tuintl}[e, \text{paraa}, \text{parab}, \text{parac}, \text{parad}]\right)^{-1}$ ] == reTinvFns0[[2]][at * e],

```

```

    {e, guesslist0[[i]]}, AccuracyGoal → Infinity, PrecisionGoal → 100]
emodel3[i_, paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
e /. FindRoot[
  Re[ $\left(\frac{1}{64 \text{ Pi}} * \text{tuint0}[e, \text{paraa}, \text{parab}, \text{parac}, \text{parad}]\right)^{-1}$ ] == reTinvFns0[[3]][at * e],
  {e, guesslist0[[i]]}, AccuracyGoal → Infinity, PrecisionGoal → 100]
emodel4[i_, paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
e /. FindRoot[
  Re[ $\left(\frac{1}{64 \text{ Pi}} * \text{tuint0}[e, \text{paraa}, \text{parab}, \text{parac}, \text{parad}]\right)^{-1}$ ] == reTinvFns0[[4]][at * e],
  {e, guesslist0[[i]]}, AccuracyGoal → Infinity, PrecisionGoal → 100]
emodel5[i_, paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
e /. FindRoot[
  Re[ $\left(\frac{1}{64 \text{ Pi}} * \text{tuint0}[e, \text{paraa}, \text{parab}, \text{parac}, \text{parad}]\right)^{-1}$ ] == reTinvFns0[[5]][at * e],
  {e, guesslist0[[i]]}, AccuracyGoal → Infinity, PrecisionGoal → 100]
uncertainty0[i_] := LQCDstates0[[i]][[5]]/at
corMatrix0 = {
  {1.000, 0.040, 0.137, 0.090, 0.080, -0.015, 0.041, 0.067,
    0.054, 0.133, 0.131, 0.077, 0.027}, {0.040, 1.000, 0.514, 0.150,
    0.049, 0.194, -0.082, 0.179, 0.214, 0.103, 0.179, 0.291, 0.093},
  {0.137, 0.514, 1.000, 0.099, 0.055, 0.032, -0.063, 0.108, 0.212, 0.128,
    0.111, 0.251, 0.054}, {0.090, 0.150, 0.099, 1.000, 0.577, 0.143, 0.033,
    0.202, 0.048, 0.174, 0.328, 0.350, -0.009}, {0.080, 0.049, 0.055, 0.577,
    1.000, 0.216, 0.113, 0.209, -0.036, 0.140, 0.206, 0.226, -0.010},
  {-0.015, 0.194, 0.032, 0.143, 0.216, 1.000, 0.068, 0.125, 0.004, 0.133,
    0.049, 0.135, 0.183}, {0.041, -0.082, -0.063, 0.033, 0.113, 0.068,
    1.000, 0.059, -0.038, 0.002, 0.103, 0.113, 0.049}, {0.067, 0.179, 0.108,
    0.202, 0.209, 0.125, 0.059, 1.000, 0.023, 0.436, 0.183, 0.259, 0.148},
  {0.054, 0.214, 0.212, 0.048, -0.036, 0.004, -0.038, 0.023, 1.000, 0.224,
    0.044, 0.068, -0.061}, {0.133, 0.103, 0.128, 0.174, 0.140, 0.133,
    0.002, 0.436, 0.224, 1.000, 0.111, 0.135, 0.033}, {0.131, 0.179, 0.111,
    0.328, 0.206, 0.049, 0.103, 0.183, 0.044, 0.111, 1.000, 0.658, 0.230},
  {0.077, 0.291, 0.251, 0.350, 0.226, 0.135, 0.113, 0.259, 0.068, 0.135,
    0.658, 1.000, 0.335}, {0.027, 0.093, 0.054, -0.009, -0.010,
    0.183, 0.049, 0.148, -0.061, 0.033, 0.230, 0.335, 1.000}};
sd0[i_] := LQCDstates0[[i]][[5]]/at
covMatrix0 =
  Table[corMatrix0[[i]][[j]] * sd0[i] * sd0[j], {i, 1, 13, 1}, {j, 1, 13, 1}];
δE0list1[paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] := Table[
  emodel1[i, paraa, parab, parac, parad] - LQCDstates0[[i]][[4]]/at, {i, 1, 1, 1}]
δE0list2[paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] := Table[
  emodel2[i, paraa, parab, parac, parad] - LQCDstates0[[i]][[4]]/at, {i, 2, 3, 1}]

```

```

δE0list3[paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] := Table[
  emodel3[i, paraa, parab, parac, parad] - LQCDstates0[[i]][[4]]/at, {i, 4, 7, 1}]
δE0list4[paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] := Table[
  emodel4[i, paraa, parab, parac, parad] - LQCDstates0[[i]][[4]]/at, {i, 8, 10, 1}]
δE0list5[paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] := Table[
  emodel5[i, paraa, parab, parac, parad] - LQCDstates0[[i]][[4]]/at, {i, 11, 13, 1}]
δE0[paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
  Join[δE0list1[paraa, parab, parac, parad],
    δE0list2[paraa, parab, parac, parad], δE0list3[paraa, parab, parac, parad],
    δE0list4[paraa, parab, parac, parad], δE0list5[paraa, parab, parac, parad]];
chisquare0[paraa_?NumberQ, parab_?NumberQ, parac_?NumberQ, parad_?NumberQ] :=
  δE0[paraa, parab, parac, parad].Inverse[covMatrix0].δE0[paraa, parab, parac, parad]

guesslist1 = {683, 853, 541, 709, 901, 687, 916, 582, 723, 941,
  635, 743, 967, 689, 943, 610, 720, 764, 663, 794, 630, 745, 685, 847};
guess1[i_] := guesslist1[[i]]
emodeli1[i_, paraa_?NumberQ, parad_?NumberQ] := e /.
  FindRoot[Re[(tuint1[e, paraa, parad])^-1] == reTinvFns1[[i]][at*e], {e, guess1[i]}]
uncertainty1[i_] := LQCDstates1[[i]][[5]]/at
corMatrix1 = ToExpression[Import["corr32.txt", "Data"]];
covMatrix1 = Table[corMatrix1[[i]][[j]] * uncertainty1[i] * uncertainty1[j],
  {i, 1, 24, 1}, {j, 1, 24, 1}];
δE1[paraa_?NumberQ, parad_?NumberQ] :=
  Table[emodeli1[i, paraa, parad] - LQCDstates1[[i]][[4]]/at, {i, 1, 24, 1}];
chisquare1[paraa_?NumberQ, parad_?NumberQ] :=
  δE1[paraa, parad].Inverse[covMatrix1].δE1[paraa, parad]
(*caisquare[paraa_?NumberQ, parad_?NumberQ] :=
  Sum[(emodeli1[i, paraa, parad] - LQCDstates1[[i]][[4]]/at)^2, {i, 1, 24, 1}] *)
  (uncertainty1[i])^2

```

## I = 0, 4 - parameters fit

```

Quiet[Table[{FindMinimum[chisquare0[paraa, parab, parac, parad],
  {paraa, i}, {parab, i}, {parac, i}, {parad, i}], i}, {i, -0.2, 0.2, 0.01}]]
{{{631.756,
  {paraa → -0.251497, parab → -0.082178, parac → -0.200152, parad → -0.199109}},
-0.2}, {{330.015, {paraa → -0.23752, parab → -0.0795259,
  parac → -0.189595, parad → -0.187753}}, -0.19}, {{276.731,
  {paraa → -0.224871, parab → -0.0769607, parac → -0.179967, parad → -0.178838}},
-0.18}, {{230.503, {paraa → -0.210011, parab → -0.0744633,
  parac → -0.16831, parad → -0.165662}}, -0.17}, {{22077.8,
  {paraa → -0.250397, parab → 0.0479128, parac → -0.158509, parad → -0.157569}},
-0.16}, {{15.2187, {paraa → -0.128861, parab → -0.0518312,

```

```

    parac → 0.126566, parad → -0.0428322}}, -0.15},
  {{15.2187, {paraa → -0.121773, parab → -0.0487129, parac → 0.125996,
    parad → -0.0456648}}, -0.14}, {{15.2187,
    {paraa → -0.114708, parab → -0.0456045, parac → 0.125424, parad → -0.0484871}},
  -0.13}, {{15.2187, {paraa → -0.107649, parab → -0.042499,
    parac → 0.124839, parad → -0.0512963}}, -0.12},
  {{24859.5, {paraa → -0.110125, parab → -0.109711, parac → -0.11, parad → -0.109995}},
  -0.11}, {{19743.4, {paraa → -0.100803, parab → -0.0981531,
    parac → -0.0999984, parad → -0.0999755}}, -0.1},
  {{15.2187, {paraa → -0.0864559, parab → -0.0331738, parac → 0.123146,
    parad → -0.0597768}}, -0.09}, {{15.2187,
    {paraa → -0.07939, parab → -0.0300644, parac → 0.1226, parad → -0.062615}}, -0.08},
  {{15.2187, {paraa → -0.0723292, parab → -0.0269574,
    parac → 0.122044, parad → -0.0654448}}, -0.07},
  {{1337.38, {paraa → -0.0615454, parab → -0.0563894,
    parac → -0.0599951, parad → -0.0598929}}, -0.06},
  {{1069.49, {paraa → -0.0506779, parab → -0.0484087,
    parac → -0.0499942, parad → -0.0499456}}, -0.05},
  {{3213.61, {paraa → -0.0407225, parab → -0.0383227, parac → -0.0399984,
    parad → -0.0399617}}, -0.04}, {{15.2187,
    {paraa → -0.0440732, parab → -0.0145252, parac → 0.119766, parad → -0.0767361}},
  -0.03}, {{15.2187, {paraa → -0.0370134, parab → -0.0114187,
    parac → 0.119213, parad → -0.0795659}}, -0.02},
  {{15.2187, {paraa → -0.0299499, parab → -0.0083107, parac → 0.118646,
    parad → -0.0823917}}, -0.01}, {{15.2187,
    {paraa → -0.022889, parab → -0.00520375, parac → 0.118085, parad → -0.085219}},
  0.}, {{15.2187, {paraa → -0.0158284, parab → -0.00209705,
    parac → 0.117524, parad → -0.0880452}}, 0.01},
  {{15.2187, {paraa → -0.0172879, parab → -0.00273926, parac → 0.117639,
    parad → -0.0874605}}, 0.02}, {{10556.6,
    {paraa → 0.0299987, parab → 0.0300004, parac → 0.0300004, parad → 0.0300001}},
  0.03}, {{4581.38, {paraa → 0.0400002, parab → 0.0399962,
    parac → 0.0399992, parad → 0.0400008}}, 0.04}, {{4416.49,
    {paraa → 0.0537627, parab → 0.0694156, parac → 0.0450045, parad → 0.0446827}},
  0.05}, {{3656.93, {paraa → 0.0577139, parab → 0.0628784,
    parac → 0.0575644, parad → 0.0570567}}, 0.06}, {{5828.12,
    {paraa → 0.0696717, parab → 0.0707586, parac → 0.0699874, parad → 0.0700164}},
  0.07}, {{14708.1, {paraa → 0.0799327, parab → 0.0801494,
    parac → 0.0799975, parad → 0.0800001}}, 0.08},
  {{3617.69, {paraa → 0.09, parab → 0.09, parac → 0.09, parad → 0.09}}, 0.09},
  {{25118.8, {paraa → 0.230374, parab → -0.323663,
    parac → -0.0625909, parad → -0.0645505}}, 0.1}, {{7471.27,
    {paraa → 1.10832, parab → -2.38569, parac → -0.626067, parad → 0.0417856}}, 0.11},

```

```

{{3140.93, {paraa → 0.120585, parab → 0.116934, parac → 0.118558, parad → 0.118767}},
 0.12},
{{4184.86, {paraa → 0.137962, parab → 0.0766211, parac → 0.100406, parad → 0.103993}},
 0.13}, {{10063.7,
  {paraa → 0.139991, parab → 0.140013, parac → 0.139998, parad → 0.139999}}, 0.14},
{{14972.4, {paraa → 0.149986, parab → 0.150028, parac → 0.15, parad → 0.149997}},
 0.15}, {{2647.63,
  {paraa → 0.159209, parab → 0.161946, parac → 0.160058, parad → 0.160159}}, 0.16},
{{13774.7, {paraa → 0.192994, parab → 0.11961, parac → 0.167743, parad → 0.16018}},
 0.17}, {{5353.78,
  {paraa → 0.837125, parab → 0.00675816, parac → 0.0256457, parad → 1.50907}}, 0.18},
{{14580.8, {paraa → 0.189913, parab → 0.190149, parac → 0.189949, parad → 0.189955}},
 0.19}, {{8361.98,
  {paraa → 0.168363, parab → 0.260005, parac → 0.190976, parad → 0.173269}}, 0.2}}

```

## $l = 0, 2$ - parameters fit

```

Quiet[Table[
  {FindMinimum[chisquare0[paraa, parab, newl3, newl4], {paraa, i}, {parab, i}], i},
  {i, -0.2, 0.2, 0.01}]]

```



```

{{{6608.15, {paraa → -0.558558, parab → -0.544021}}, -0.2},
 {{10743.5, {paraa → -0.276578, parab → 0.00842102}}, -0.19},
 {{1413.34, {paraa → -0.19072, parab → -0.155325}}, -0.18},
 {{1556.68, {paraa → -0.17878, parab → -0.149694}}, -0.17},
 {{5375.19, {paraa → -0.16, parab → -0.16}}, -0.16},
 {{1326.21, {paraa → -0.169958, parab → -0.103792}}, -0.15},
 {{8764.82, {paraa → -0.140159, parab → -0.139633}}, -0.14},
 {{10593.8, {paraa → -0.200508, parab → 0.0274954}}, -0.13},
 {{4518.71, {paraa → -0.121129, parab → -0.117389}}, -0.12},
 {{18.9999, {paraa → -0.0781883, parab → -0.0324694}}, -0.11},
 {{18.9999, {paraa → -0.0781889, parab → -0.0324698}}, -0.1},
 {{1078.94, {paraa → -0.0981304, parab → -0.0712583}}, -0.09},
 {{1049.29, {paraa → -0.0869484, parab → -0.0638838}}, -0.08},
 {{1460.7, {paraa → -0.0708607, parab → -0.0680132}}, -0.07},
 {{1101.48, {paraa → -0.0610764, parab → -0.0574904}}, -0.06},
 {{1579.99, {paraa → -0.0522163, parab → -0.0448668}}, -0.05},
 {{1582.62, {paraa → -0.0401033, parab → -0.039763}}, -0.04},
 {{18.9999, {paraa → -0.0781891, parab → -0.0324698}}, -0.03},
 {{18.9999, {paraa → -0.0781895, parab → -0.03247}}, -0.02},
 {{18.9999, {paraa → -0.0781896, parab → -0.03247}}, -0.01},
 {{18.9999, {paraa → -0.0781892, parab → -0.0324699}}, 0.},
 {{18.9999, {paraa → -0.0781887, parab → -0.0324696}}, 0.01},
 {{18.9999, {paraa → -0.0781799, parab → -0.0324658}}, 0.02},
 {{8403.77, {paraa → 0.0289001, parab → 0.0329343}}, 0.03},
 {{8231.78, {paraa → 0.0754519, parab → 0.0690084}}, 0.04},
 {{10335.7, {paraa → 0.05, parab → 0.05}}, 0.05},
 {{10010., {paraa → 0.06, parab → 0.06}}, 0.06},
 {{8184.43, {paraa → 0.0699947, parab → 0.070043}}, 0.07},
 {{9231.37, {paraa → -0.0533785, parab → 0.0732905}}, 0.08},
 {{7331.97, {paraa → 0.0823415, parab → 0.0921589}}, 0.09},
 {{4695.43, {paraa → 0.100658, parab → 0.100377}}, 0.1},
 {{8185.55, {paraa → 0.11, parab → 0.11}}, 0.11},
 {{18.9999, {paraa → -0.0781896, parab → -0.03247}}, 0.12},
 {{2596.26, {paraa → 0.193162, parab → 0.127428}}, 0.13},
 {{6913.46, {paraa → 0.166207, parab → 0.158846}}, 0.14},
 {{7004.17, {paraa → 0.149979, parab → 0.150003}}, 0.15},
 {{936.175, {paraa → 0.177646, parab → 0.0621615}}, 0.16},
 {{9648.63, {paraa → 0.170545, parab → 0.169505}}, 0.17},
 {{59618.7, {paraa → 0.18, parab → 0.180001}}, 0.18},
 {{51992.5, {paraa → 0.188883, parab → 0.19669}}, 0.19},
 {{50474., {paraa → 0.199983, parab → 0.200047}}, 0.2}}

```

# $l = 1, 2$ - parameters fit

```
Quiet[Table[{FindMinimum[chisquare1[paraa, parad], {paraa, i}, {parad, i}], i},  
  {i, -0.2, 0.2, 0.01}]]
```

```

{{{166539., {paraa → 0.299963, parad → -0.193737}}, -0.2},
{{58259.3, {paraa → 0.0768579, parad → -0.165604}}, -0.19},
{{1974.27, {paraa → 0.0607864, parad → -0.654284}}, -0.18},
{{202205., {paraa → 1.05241, parad → -0.158422}}, -0.17},
{{161422., {paraa → 0.282424, parad → -0.178299}}, -0.16},
{{172586., {paraa → 0.351704, parad → -0.146995}}, -0.15},
{{145531., {paraa → 0.193566, parad → -0.135314}}, -0.14},
{{177323., {paraa → 0.467716, parad → -0.102852}}, -0.13},
{{176666., {paraa → 0.379946, parad → -0.112639}}, -0.12},
{{179522., {paraa → 0.787993, parad → -0.0163032}}, -0.11},
{{154880., {paraa → 0.232901, parad → -0.0948287}}, -0.1},
{{146601., {paraa → 0.246404, parad → -0.0764416}}, -0.09},
{{189741., {paraa → 0.947603, parad → -0.83397}}, -0.08},
{{190654., {paraa → 0.538802, parad → -0.0329038}}, -0.07},
{{218317., {paraa → 1.20998, parad → -0.0990439}}, -0.06},
{{172736., {paraa → 0.745485, parad → -0.00992701}}, -0.05},
{{178498., {paraa → 0.455285, parad → 0.0206652}}, -0.04},
{{149525., {paraa → 0.207225, parad → -0.012741}}, -0.03},
{{161375., {paraa → 0.361805, parad → -0.0740949}}, -0.02},
{{136850., {paraa → 0.641135, parad → -2.61444}}, -0.01},
{{118590., {paraa → 0.139446, parad → 0.00608957}}, 0.},
{{26.8485, {paraa → 0.0146992, parad → -0.0330938}}, 0.01},
{{26.8485, {paraa → 0.0146992, parad → -0.0330938}}, 0.02},
{{26.8485, {paraa → 0.0146992, parad → -0.0330938}}, 0.03},
{{26.8485, {paraa → 0.0146992, parad → -0.0330938}}, 0.04},
{{26.8485, {paraa → 0.0146992, parad → -0.0330938}}, 0.05},
{{77110.7, {paraa → 0.0600161, parad → 0.0600156}}, 0.06},
{{98945.4, {paraa → 0.0700002, parad → 0.0700002}}, 0.07},
{{109197., {paraa → 0.0800132, parad → 0.0799975}}, 0.08},
{{78374.6, {paraa → 0.0370188, parad → 0.0987548}}, 0.09},
{{103734., {paraa → 0.100025, parad → 0.100028}}, 0.1},
{{78869.2, {paraa → 0.0548589, parad → 0.109607}}, 0.11},
{{76031.5, {paraa → 0.0391588, parad → 0.156192}}, 0.12},
{{26.8485, {paraa → 0.0146992, parad → -0.0330938}}, 0.13},
{{26.8485, {paraa → 0.0146992, parad → -0.0330938}}, 0.14},
{{138247., {paraa → 0.15, parad → 0.15}}, 0.15},
{{148861., {paraa → 0.164317, parad → 0.161233}}, 0.16},
{{142825., {paraa → 0.170012, parad → 0.169977}}, 0.17},
{{151513., {paraa → 0.18, parad → 0.18}}, 0.18},
{{154705., {paraa → 0.19, parad → 0.19}}, 0.19},
{{162107., {paraa → 0.200003, parad → 0.200003}}, 0.2}}

```

## $l = |, |$ - parameter fit

```

Quiet[
  Table[FindMinimum[chisquare1[paraa, newl4], {paraa, i}], i], {i, -0.2, 0.2, 0.01}]
{{{181220., {paraa → 0.440637}}, -0.2},
 {{175201., {paraa → 0.311613}}, -0.19}, {{39926.9, {paraa → 0.0388732}}, -0.18},
 {{241810., {paraa → 2.33}}, -0.17}, {{105.591, {paraa → 0.0118555}}, -0.16},
 {{105.591, {paraa → 0.0118555}}, -0.15}, {{240545., {paraa → 1.45739}}, -0.14},
 {{105.591, {paraa → 0.0118555}}, -0.13}, {{184296., {paraa → 0.380082}}, -0.12},
 {{178649., {paraa → 0.420565}}, -0.11}, {{175279., {paraa → 0.44207}}, -0.1},
 {{191991., {paraa → 0.498569}}, -0.09}, {{166031., {paraa → 0.317011}}, -0.08},
 {{105.591, {paraa → 0.0118555}}, -0.07}, {{181105., {paraa → 0.440001}}, -0.06},
 {{177440., {paraa → 0.450082}}, -0.05}, {{173284., {paraa → 0.327472}}, -0.04},
 {{105.591, {paraa → 0.0118555}}, -0.03}, {{251181., {paraa → 2.48001}}, -0.02},
 {{208630., {paraa → 0.604234}}, -0.01}, {{105.591, {paraa → 0.0118555}}, 0.},
 {{105.591, {paraa → 0.0118555}}, 0.01}, {{105.591, {paraa → 0.0118555}}, 0.02},
 {{105.591, {paraa → 0.0118555}}, 0.03}, {{105.591, {paraa → 0.0118555}}, 0.04},
 {{36714.9, {paraa → 0.0447919}}, 0.05}, {{76135.3, {paraa → 0.06}}, 0.06},
 {{105.591, {paraa → 0.0118555}}, 0.07}, {{109768., {paraa → 0.0800001}}, 0.08},
 {{38001.7, {paraa → 0.0448219}}, 0.09}, {{113817., {paraa → 0.0988408}}, 0.1},
 {{97331.9, {paraa → 0.0562564}}, 0.11}, {{119836., {paraa → 0.123664}}, 0.12},
 {{121335., {paraa → 0.132038}}, 0.13}, {{116079., {paraa → 0.140039}}, 0.14},
 {{74964.3, {paraa → 0.053098}}, 0.15}, {{82036.1, {paraa → 0.068574}}, 0.16},
 {{112740., {paraa → 0.116094}}, 0.17}, {{146555., {paraa → 0.18}}, 0.18},
 {{135914., {paraa → 0.193236}}, 0.19}, {{39450.9, {paraa → 0.0392874}}, 0.2}]

```

## Overall, 4 - parameters fit

```

Quiet[Table[
  FindMinimum[(chisquare0[paraa, parab, parac, parad] + chisquare1[paraa, parad]),
    {paraa, i}, {parab, i}, {parac, i}, {parad, i}], i], {i, -0.2, 0.2, 0.01}]
{{{133543.,
  {paraa → 0.10692, parab → -0.0394653, parac → -0.199841, parad → -0.174536}},
 -0.2}, {{170779., {paraa → 0.297236, parab → -0.044287,
  parac → -0.18979, parad → -0.175698}}, -0.19}, {{147619.,
  {paraa → 0.148138, parab → -0.0814158, parac → -0.179914, parad → -0.173886}},
 -0.18}, {{224787., {paraa → 0.309154, parab → -0.0273243,
  parac → -0.170648, parad → -0.162495}}, -0.17},
 {{245927., {paraa → 2.23423, parab → 0.558003, parac → -0.15938, parad → -0.113898}},

```

```

-0.16}, {{167 356., {paraa → 0.161299, parab → -0.0577812,
  parac → -0.149906, parad → -0.143889}}, -0.15}, {{184 554.,
  {paraa → 0.478735, parab → 0.0459114, parac → -0.143676, parad → -0.128904}},
-0.14}, {{19 216.2, {paraa → 0.0205317, parab → -0.0669542,
  parac → -0.129946, parad → -0.12564}}, -0.13},
{{192 697., {paraa → 0.376597, parab → -0.0624052, parac → -0.119924,
  parad → -0.111382}}, -0.12}, {{180 949.,
  {paraa → 0.44811, parab → -0.0521149, parac → -0.10992, parad → -0.114066}},
-0.11}, {{195 169., {paraa → 0.473706, parab → -0.0434046,
  parac → -0.0999317, parad → -0.0897954}}, -0.1}, {{201 414.,
  {paraa → 0.459244, parab → -0.0164338, parac → -0.0898774, parad → -0.0645711}},
-0.09}, {{97 151.8, {paraa → 0.0722873, parab → -0.0641699,
  parac → -0.0799822, parad → -0.0767024}}, -0.08},
{{189 797., {paraa → 0.473113, parab → -0.0356281, parac → -0.12672,
  parad → -0.253647}}, -0.07}, {{107 335.,
  {paraa → 0.137843, parab → -0.120452, parac → -0.0679535, parad → -0.137156}},
-0.06}, {{14 585.8, {paraa → 0.0161613, parab → -0.0489953,
  parac → -0.0499986, parad → -0.0486087}}, -0.05},
{{4381.51, {paraa → 0.0150202, parab → -0.0353738,
  parac → -0.0399967, parad → -0.0378433}}, -0.04},
{{88 666., {paraa → 0.0670868, parab → -0.0261216, parac → -0.0300063,
  parad → -0.0277529}}, -0.03}, {{927.476,
  {paraa → 0.0130258, parab → -0.00530835, parac → -0.0200148, parad → -0.019508}},
-0.02}, {{138 849., {paraa → 0.149442, parab → -0.00770354,
  parac → -0.00998691, parad → -0.00575967}}, -0.01},
{{44 808.2, {paraa → 0.0414054, parab → 0.00234134,
  parac → -0.0000175633, parad → 0.00106674}}, 0.},
{{44.8217, {paraa → 0.0147882, parab → 0.00882046, parac → 0.0134014,
  parad → -0.0345738}}, 0.01}, {{7528.15,
  {paraa → 0.0115764, parab → 0.0205282, parac → 0.0211338, parad → 0.0195482}},
0.02}, {{8708.11, {paraa → 0.0115441, parab → 0.0352205,
  parac → 0.0301714, parad → 0.0254952}}, 0.03}, {{7541.38,
  {paraa → 0.0103954, parab → 0.0292813, parac → 0.0376163, parad → 0.0385897}},
0.04}, {{100 096., {paraa → 0.0501688, parab → 0.0494634,
  parac → 0.0498067, parad → 0.0497225}}, 0.05},
{{93 330.2, {paraa → 0.0600133, parab → 0.06, parac → 0.06, parad → 0.0600258}}, 0.06},
{{105 466., {paraa → 0.07, parab → 0.07, parac → 0.07, parad → 0.07}}, 0.07},
{{123 943., {paraa → 0.08, parab → 0.08, parac → 0.08, parad → 0.08}}, 0.08},
{{8728.86, {paraa → 0.00652908, parab → 0.0906805,
  parac → 0.0895166, parad → 0.103084}}, 0.09},
{{150 976., {paraa → 0.1, parab → 0.1, parac → 0.1, parad → 0.1}}, 0.1},
{{106 181., {paraa → 0.0867269, parab → 0.110001,
  parac → 0.110003, parad → 0.110037}}, 0.11}, {{131 602.,

```

```

{paraa → 0.0895664, parab → 0.119964, parac → 0.119984, parad → 0.133637}}, 0.12},
{{7176.89, {paraa → 0.0164107, parab → 0.110165, parac → 0.118865,
parad → -0.0247528}}, 0.13}, {{91328.3,
{paraa → 0.0594391, parab → 0.140035, parac → 0.139984, parad → 0.0891392}}, 0.14},
{{153220., {paraa → 0.15, parab → 0.15, parac → 0.15, parad → 0.15}}, 0.15},
{{150680., {paraa → 0.160335, parab → 0.16, parac → 0.16, parad → 0.160096}}, 0.16},
{{132042.,
{paraa → 0.222277, parab → 0.170455, parac → 0.170094, parad → -0.388314}}, 0.17},
{{166254., {paraa → 0.18, parab → 0.18, parac → 0.18, parad → 0.18}}, 0.18},
{{169325., {paraa → 0.19, parab → 0.19, parac → 0.19, parad → 0.19}}, 0.19},
{{175558., {paraa → 0.200002, parab → 0.2, parac → 0.2, parad → 0.200002}}, 0.2}}

```

## Overall, 2 - parameters fit

```

Quiet[Table[
  {FindMinimum[(chisquare0[paraa, parab, newl3, newl4] + chisquare1[paraa, newl4]),
    {paraa, i}, {parab, i}], i}, {i, -0.2, 0.2, 0.01}]

```

```

{{{363853., {paraa → -0.2, parab → -0.2}}, -0.2},
 {{177654., {paraa → 0.422716, parab → 0.215093}}, -0.19},
 {{140282., {paraa → 0.146011, parab → -0.110498}}, -0.18},
 {{87738.7, {paraa → 0.0524483, parab → -0.16094}}, -0.17},
 {{225577., {paraa → 0.357387, parab → -0.148045}}, -0.16},
 {{13312., {paraa → 0.0132831, parab → -0.143677}}, -0.15},
 {{156402., {paraa → 0.194433, parab → -0.0840281}}, -0.14},
 {{229479., {paraa → 0.365507, parab → -0.0631225}}, -0.13},
 {{77253.5, {paraa → 0.0376251, parab → -0.107499}}, -0.12},
 {{189487., {paraa → 0.512045, parab → 0.19551}}, -0.11},
 {{239517., {paraa → -0.0996391, parab → -0.0999649}}, -0.1},
 {{81227.9, {paraa → 0.0662345, parab → 0.0214503}}, -0.09},
 {{222079., {paraa → 0.383659, parab → 0.0124657}}, -0.08},
 {{256874., {paraa → 0.429994, parab → -0.067639}}, -0.07},
 {{85751.9, {paraa → 0.0694173, parab → -0.0594119}}, -0.06},
 {{1581.47, {paraa → 0.0119213, parab → -0.0496222}}, -0.05},
 {{91062.9, {paraa → 0.0688373, parab → -0.0383262}}, -0.04},
 {{89019.1, {paraa → 0.080621, parab → -0.0258565}}, -0.03},
 {{193735., {paraa → 0.31279, parab → -0.0171438}}, -0.02},
 {{37415.6, {paraa → 0.0426604, parab → -0.00844376}}, -0.01},
 {{81614.5, {paraa → 0.067299, parab → -0.000113593}}, 0.},
 {{187.817, {paraa → 0.0118511, parab → 0.00458442}}, 0.01},
 {{187.817, {paraa → 0.0118511, parab → 0.00458442}}, 0.02},
 {{9653.13, {paraa → 0.015826, parab → 0.0301898}}, 0.03},
 {{43436.5, {paraa → 0.0399493, parab → 0.0400259}}, 0.04},
 {{93143.3, {paraa → 0.05, parab → 0.05}}, 0.05},
 {{45897.4, {paraa → 0.0152045, parab → -0.121267}}, 0.06},
 {{91606.8, {paraa → 0.07, parab → 0.07}}, 0.07},
 {{116147., {paraa → 0.101372, parab → 0.0799457}}, 0.08},
 {{18446.6, {paraa → 0.0118684, parab → 0.090265}}, 0.09},
 {{121329., {paraa → 0.0993145, parab → 0.1}}, 0.1},
 {{94803.6, {paraa → 0.101673, parab → 0.110286}}, 0.11},
 {{127143., {paraa → 0.122221, parab → 0.119908}}, 0.12},
 {{131269., {paraa → 0.130001, parab → 0.13}}, 0.13},
 {{126706., {paraa → 0.140005, parab → 0.14}}, 0.14},
 {{141823., {paraa → 0.15, parab → 0.15}}, 0.15},
 {{88780.4, {paraa → 0.0696342, parab → 0.104644}}, 0.16},
 {{153520., {paraa → 0.19504, parab → 0.16829}}, 0.17},
 {{199563., {paraa → 0.225193, parab → 0.203035}}, 0.18},
 {{198370., {paraa → 0.511109, parab → 0.251491}}, 0.19},
 {{161265., {paraa → 0.12841, parab → 0.200972}}, 0.2}}

```