Improvements to a Strontium Optical Lattice Clock

Rees Mcnally
Rees.Mcnally@Colorado.EDU

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Improvements to a Strontium Optical Lattice Clock

by

R. L. McNally

A thesis submitted to the
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Improvements to a Strontium Optical Lattice Clock
written by R. L. McNally
has been approved for the Department of Physics

Prof. Jun Ye

Prof. Harvey Segur

Date ________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
In 2013, for the first time ever, a many-atom optical lattice clock surpassed single ion clocks as the most stable, and accurate frequency standard in existence. Since this first result, continued improvements to the clock’s accuracy and stability have been made. This work will focus on two of these major improvements. Firstly the frequency uncertainty was brought down to $2.5 \times 10^{-18}$ [1], thanks in part to an accurate determination of the black body radiation shift, based on better characterization of the level structure of $^{87}\text{Sr}$. Secondly, a control modification will be demonstrated which provides 20% reduction in the standard deviation for the clock’s frequency under differential measurement schemes. This represents a new record for clock accuracy, pushing performance towards the $10^{-19}$ level.
Dedication

To my co-workers for their help, my friends for their support, and my family for their love.
Acknowledgements

Some of my fondest memories of childhood are working in the garage with my dad on whatever zany project we could think of. These projects covered everything from a catapult, to RC cars, to a home built forge, and even a rather ill conceived go-cart. Many projects were successful, many were not, and several resulted in large burn marks spread across the property. It was not until I was much older that I realized most kids didn’t spend their weekends living out mad scientist fantasies, but I would not have had it any other way. I am grateful to my parents for the environment of discovery they created, and it is the largest factor to which I attribute my success.

Over my time at CU I have been incredibly lucky to meet some amazing people, and work on some amazing projects. In period of three years I was able to see not one, but two satellites I worked on slip the bonds of gravity, though one of the only escaped for 15 minutes before the heavens sent it back. These projects, and the Colorado Space Grant Consortium in general, was an amazing experience.

My next adventure at CU took place under the guidance of Dr. Jun Ye, working on his optical lattice clock (the subject of this thesis). Working with Jun was my first experience in a physics lab, and I was nervous that the work would be to theoretical for my interests. As it turns out I have never been so wrong in my life. Jun’s intense concentration on technical challenges has given me a great appreciation that before great science, comes great engineering. This is what convinced me that experimental physics is exactly where I want to be.
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Chapter 1

Introduction

1.1 Frequency Standards

The idea of a frequency standard is to take a natural phenomenon we can accurately measure, and use it to define a period of time. From this period you can then define the more usual notions of time such as the second. The first frequency standard used by humanity is quite likely the rising and setting of the sun in the sky, defining the period of time we call a day. Since then technological improvements have enabled engineers and scientists to use better suited phenomenon as the basis for the standardized period. Thinking about what would make a phenomena a good standard, one can see there are two primary requirements; stability and accuracy. Stability is needed, because the standard must not vary as a function of time, like the length of the day varies with the season. Accuracy is needed because everyone must agree on its value for it to be useful, unlike the length of a day which varies with latitude. The frequency standard currently agreed upon by the scientific community is based on a microwave hyperfine transition in cesium, at 9,192,631,770 Hz. This means that what we call a second is defined as 9,192,631,770 oscillations of a microwave field, tightly locked to an atomic transition. However, this technology has matured to the point that further improvements are not likely. To continue improvements to the accuracy of frequency standards, this work focuses on a potential replacement based on a much higher frequency optical transition in $^{87}\text{Sr}$ at 429,228,004,229,874 Hz [2]. This clock is the most accurate atomic clock to
date [3] and this work represents a new record in accuracy.

One of the key reasons for this improvement lies in the difference between the two transition frequencies. If each transition is known to the one Hz level, then the relative uncertainty will be a thousand times lower for the optical clock. This is analogous to why a ruler with finer spacing between measurement marks will be able to make more accurate measurements. However, the advantage gained by moving to optical frequencies comes with its own set of challenges. This work will focus on the accurate determination of one shift to the transition frequency, and an improvement to the scheme currently used to lock the clock laser to the transition. Techniques for how to implement and distribute an optical frequency standard are based on frequency combs which are beyond the scope of this thesis, but for which many excellent review articles exist [4]. The work discussed here can be broken down into three main ideas, how the $^{87}$Sr lattice clock works, how thermal radiation effects the clock transition, and how we can keep the clock laser and the transition as tightly locked as possible. Before we begin, a demonstration for why an improved frequency standard is a powerful tool will be provided.

1.2 Motivation

One of the wonderful things about clocks, is their fascinating applications across a huge range of subjects. These range from the purely technical, to some of the most fundamental experiments in physics. These fundamental experiments stem from the fact that measurements of transition frequencies in atomic clocks are among the most precise measurements made in any field, and thus are sensitive to things not normally resolvable. As a brief introduction, a selection of applications for improved frequency standards will be introduced, starting with the most practical, and ending with the most fundamental.
1.2.1 Timing Systems Improvements

Any network of interacting digital devices (ie. computers) must be synchronized in some way to make communication possible. In fact any application which is timing sensitive must somehow be referenced to a stable timing source. One of the more advanced application that has been proposed for optical clocks, is the generation a network of entangled clocks, serving as an incredibly accurate timing network across the globe. When optical clocks are robust enough to be launched in to orbit, such a network can serve to provide a synchronized definition of time at an unprecedented level [5]. This would enable any number of advancements, from ultra high speed data transfer, to better synchronization of far separated experiments.

Another application of precision time keeping commonly used today, is global positioning satellites (GPS). GPS is based on an array of satellites, that each transmit time in a synchronized manner. Based on the time a receiver sees a certain satellites signal arrive, relative to the other satellites, the receiver’s position on earth can be directly determined. These satellites currently carry cesium atomic clocks on board, as the precision to which a location can be measured is directly based on how well each satellite tells time. If the next generation of GPS systems is to continue advances in position determination, improvements to these clocks must be made, and switching to optical clocks would do just that. These are two applications where the applicability of improved clocks is obvious, but more subtle uses can be engineered.

1.2.2 Quantum Sensors

Because optical clock transitions are so precisely known, small perturbations to these transitions can be measured. This sort of dependence, from a pure clock stand point, is an annoyance and is something we look to minimize. However if one looks at this problem the other direction, clocks can be used to measure these small perturbations directly. Due to the accuracy of these
clocks, they are sensitive to things you would never imagine. One such application, is the use of optical clocks in precision geodesy [6]. As clocks currently operate, they are capable of resolving height differences of less than one meter. That is to say the red shift associated with the earth gravitational field perturbs the energy levels of the clock’s transition enough that it can be experimentally resolved. As the performance of clocks improves, and more robust implementations develop, optical clocks are set to become an invaluable tool for geodesy, and geophysics as a whole. Measuring time accurately enough can be used to measure how far from the earth’s center you are.

1.2.3 Fundamental Physics

The last use, and in my opinion the most fascinating, is the application of optical lattice clocks to fundamental measurements. Similar to the justification behind why clocks make good quantum sensors, high measurement accuracy means small perturbations are resolvable. Now instead of something like gravitational fields, let us look to something more fundamental. Because different implementations of optical clocks are based on different atomic species, their transition frequencies have different functional dependencies. Specifically it was noticed that different clock species had different dependencies on the fine structure constant. Therefore by bounding how little the ratio of these different clock transitions vary as a function of time, one can limit the size of variations in the fine structure constant. So far this work has given some of the most accurate bounds to time variations of the fine structure constant, and further improvements have been described which are capable of sensitivity approaching predictions from string theory [7]. This means that improvements to clocks could be capable of the first experimental observation of string theory, or as a tool to further restrict predictions.

Another fascinating application in this domain, comes from the analysis of how clock frequencies change over the course of a year. Because the distance from the sun changes over the course of the year, the magnitude of the gravitational field seen on earth also changes. This is shown in
By analyzing clock performance over the course of the year, the current best limitation for the coupling between gravity and the fine structure constant was measured [8]. This is shown on the rightmost figure above, where the data thus far shows no dependence on gravity. This black line represents the largest possible dependency, based on the experimental data. This shows how clocks can be used as tools for very practical applications like GPS, next generation tools like gravitational geodesy, and incredibly fundamental measurements such as those just discussed. With motivation for why optical clocks are a powerful tool, let us move to discussion of the particular clock of interest, JILA’s strontium optical lattice clock.
In this section the complete operation of the clock will be considered. This will begin with an introduction of the relevant physics, then a walk through of a clock measurement sequence will be provided. This will include several derivations that will be referenced in later sections.

2.1 Laser Manipulation of Atoms

One the key developments that enables optical lattice clocks, and the entire field of atomic molecular and optical physics, is the precision manipulation of atomic species with light. For this work there are three main categories of light-atom interactions; laser cooling, optical dipole trapping, and Rabi spectroscopy. Each of these general classes will be introduced, then their application to the strontium optical lattice clock will be explained. We will begin with the introduction of each method.

2.1.1 Laser Cooling

The basis of laser cooling atoms lies with the fact that photons carry momentum. Consider an atom traveling with some initial momentum \( \vec{P} \). If this atom absorbs a photon, it’s momentum is altered, based on the wave vector of the photon, resulting in a new momentum, \( \vec{P} + \hbar \vec{k} \), where
\[ |k| = 2\pi/\lambda. \] From this basic concept one can see that if you were able to construct a situation where \( \vec{k} \) and \( \vec{P} \) were anti-parallel, this would result in a reduction of the atom’s speed. This is possible through clever exploitation of the Doppler shift, caused by the atom’s velocity relative to the laser light. Consider again an atom traveling in some direction, but this time confined to one dimension. Now imagine this atom is illuminated by a laser red-detuned from a transition in the atom. If the atom is at rest or moving parallel to the laser light the detuning results in the atom not absorbing any photons, and it’s momentum remains unchanged. However, if the atom is traveling anti-parallel to the laser light, it experiences the laser in its now moving frame, as blue shifted. If the degree of detuning is properly set, the Doppler shift from this motion will cancel the detuning, and bring the atom into resonance with the laser. This will result in absorption of the photon by the atom, and a transfer of momentum occurs as shown below.

![Figure 2.1: Atom undergoing one cycle of cooling. Blue lines represent emission and absorption, black lines represent momentum resulting from interactions](image)

As the atom relaxes back to its ground state (the rightmost picture), another photon of the same frequency (and therefore momentum) will be emitted. However, unlike the initial absorption, this emission is isotropic so the momentum transfer will be in a randomized direction. Repeated cycles of this directional absorption, followed by omni-directional emission, result in a net force slowing the atom down, which is equivalent to cooling it. This same idea, can be expanded to three dimensions with two counter propagating lasers along each Cartesian dimension. One can see that
this would result in a cooling of any atoms in the center of these six intersecting beams.

Using the technique described above results in slowing of the atoms, but does nothing to actually contain the atoms. To make the atoms easier to work with, and to keep them in the cooling beams long enough to reach low temperatures, confinement is also required. Stated another way, in addition to velocity-selective absorption, we need to generate position-dependent absorption. To achieve this we look to the Zeeman shift, as shown in the figure below.

Figure 2.2: Zeeman shifts for a 2 \(m_j\) levels in a B-field gradient, with incident circularly polarized light. Adapted from [9]

If a magnetic field is applied in such a way that it is zero in the center of the trap and increasing in magnitude as you leave the center of the trap, atoms will see a position-dependent Zeeman shift, with the sign of the Zeeman shift depending on the \(m_j\) level of the excited state. In addition to this \(m_j\) dependent shift we can apply selections rules to see that light with polarization \(\sigma^+\) is only capable of driving them from the ground state into the \(m_j = 1\) state, and light polarization \(\sigma^-\) can only drive them into the \(m_j = -1\) state. This is because spin must be conserved during absorption, and light polarization \(\sigma^+\) is defined as spin +1 (\(\sigma^-\) defined as spin -1). With this state and position selective absorption in mind, consider what happens when a red-detuned laser is applied to the atoms. This detuning means only atoms with energy levels shifted down by the magnetic field can be absorbed. Now looking just at the \(\sigma^+\) light as it travels across the region...
we see it goes into resonance with the $m_j = 1$ transition, resulting in absorption. As the $\sigma^+$ light crosses into the left region, we see it come into resonance with the $m_j = -1$ transition, but this transition is forbidden so no absorption occurs. An identical analysis can be carried out for the $\sigma^-$ light, reaching a similar conclusion. This results in a preferential absorption for photons with momentum vectors towards the center of the trap. Similar to the case of atom cooling, this process is repeated many times, and radiation pressure is generated towards the trap's center. Because the schemes for positional and velocity-dependent absorption both depend on red-detuned light, they can be implemented using the same laser, and this is referred to as a magneto-optical trap (MOT). For a more detailed derivation of the relevant physics, the reader is referred to [10]

2.1.2 Optical Dipole Trapping

As compared to atomic cooling, optical dipole traps are not based on the absorption of photons by an atom, but rather on the effects of light far detuned from any transitions. We will begin with a brief derivation of the optical trapping, with the introduction of an atomic physics quantity referred to as the atomic polarizability. This quantity will be derived in detail in chapter 3, but for simplicity’s sake let us call it the proportionality constant (as a function of the frequency of the oscillating electric field) between the amplitude of an applied electric field, and the dipole-moment of an atom. Note that we only consider the real portion of the atomic polarizability for the duration of this thesis, though this is an excellent approximation when used in each case considered (as will be shown later). We begin with the expression below.

$$P = \alpha(\omega)E$$

This is an equation for the dipole-moment induced on an atom, by an electric field of magnitude $E$ oscillating at frequency $\omega$. Note that when $\omega$ goes to zero, we are left with the DC polarizability of the atom, responsible for the DC Stark shift. So now that we have an expression for the dipole-moment induced by an applied electric field, we can immediately write down the potential energy
between this induced dipole, and the electric field which induced it in the first place. Here constants are neglected as only a qualitative description is important.

\[ U_{\text{dipole}} \propto -\langle PE \rangle \]

Where angular brackets denote a time average over one cycle (or any integer number of cycles) of the electric field. Recalling that the intensity of an electric field scales as the amplitude squared, we arrive at an expression relating the intensity of a radiation field (I), the polarizability of the atom, and the potential seen by the atom.

\[ U_{\text{dipole}} \propto (-\alpha(\omega)I) \]

From this potential we can derive a force, by directly taking the gradient, using the fact that this force is conservative.

\[ F_{\text{dipole}} \propto \alpha(\omega)\nabla I \]

A more detailed review of this technique can be found in [11] and [12]. With this expression for the force applied to an atom from a far-detuned light field let us introduce the specific situation used in our experiment, a one-dimensional standing wave optical lattice. Consider laser light in a high reflectivity cavity. This results in a one-dimensional standing wave of light, with local maximum and minimum spacing based on the wavelength of the laser used. Remembering from an intro E&M course that light can be thought of as a time varying electric field, it becomes clear that this standing optical wave can be thought of as a positionally dependent electric potential. Now looking at the expression for the force induced on an atom from a time-varying electric field derived above, we see the utility of this technique. Because we have generated a standing sinusoid of electric field intensity, we find a series of local minimums in the potential, or equivalently a restorative force keeping each atoms confined to one local minimum. Each local minimum are referred to as a lattice site. When an atomic sample is cooled to a sufficient degree and is exposed to this standing wave potential, atoms will naturally congregate into each of these lattice sites, as shown in the figure below.
This allows confinement of atomic samples without any resonant interactions, which results in extremely low loss rates, and allows for a variety of interesting science. The particular application used in optical lattice atomic clocks comes when a lattice with extremely high intensity is used. Consider the picture above, then slowly ramp up the degree of confinement each atom sees. At a certain point thinking of these atoms in a classical sense will begin to break down, and their description will become quantum in nature. Looking at a single lattice site, with simplification that the potential is quadratic (true for low lying states) this becomes the picture of an atom confined in a quantum harmonic oscillator. As a reminder, this picture is shown below.

![Eigenstates of a Quantum Harmonic Oscillator](image)

Figure 2.4: Optical dipole potential seen by an atom tightly confined in a lattice. The Lambe-Dicke regime occurs when energy spacing is much greater then the momentum transfer from spectroscopy. This freezes atoms in the state they occupy, disallowing changes in position.
As you can see, this results in discrete allowable states the atoms can have in their respective lattice sites. When atoms are confined in such a way, it is referred to as the Lambe-Dicke regime [13] [14]. What this allows precision measurement, is that if the energy levels are sufficiently split then the atoms cannot recoil when they are interrogated. This is because the momentum transfer is to little to jump to the next highest level. This means that any motional degrees of freedom are frozen out of the system, allowing measurement to be made free from Doppler broadening, and other motional based concerns. This is exactly the regime in which the clock is operated, and is why it is called an optical LATTICE clock.

2.1.3 Rabi Spectroscopy

Rabi spectroscopy is a subset of the more general class of coherent spectroscopy techniques. The general way one should think of these schemes is that over the time the atoms and the light are interacting, the laser maintains phase coherence. As compared to other techniques this enables some interesting dynamics. Rabi spectroscopy is truly the core of the clock operations, as it is how we measure the $^{1}S_{0}$ to $^{3}P_{0}$ clock transition, so it will derived in full detail. Let us begin by considering strontium atoms as a two-level system, labeling the two states as $|e>$ and $|g>$, which are the excited and ground state respectively. The Hamiltonian that describes the evolution of these two states, is given by

$$H = \frac{-\hbar}{2} \begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix}$$

where $\Omega$ is the Rabi frequency, or the frequency of oscillation between the ground and excited state, and $\Delta$ is the angular frequency detuning from the transition frequency, $\Delta = \omega - \omega_{0}$. With this in hand we now wish to derive an expression for the excitation probability as a function of the Hamiltonian above. To do this we will use the Lindblad master equation, in the absence of spontaneous decay. This is an appropriate simplification because the clock state has a lifetime on the order of 100’s of seconds, and we will only interrogate the atoms for hundreds of milliseconds,
making spontaneous decay negligible.

\[ \frac{dI}{dt} = -i \frac{\hbar}{h} [H, I] \]

Where the density matrix \( I \) is defined by

\[
I = \begin{pmatrix}
\rho_{ee} & \rho_{eg} \\
\rho_{ge} & \rho_{gg}
\end{pmatrix}
\]

This notation essentially means that diagonal elements are pure states of either the ground or the excited state, and off-diagonal terms are superpositions. Additionally it is important to note that normalization between the pure states must hold. With this system in hand, we continue our analysis with the introduction of the Bloch vector, defined as

\[
R = \begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
\]

and the individual elements of this matrix \((u,v,w)\) are defined in such a way as to aid in solving the problem.

\[
u = \frac{1}{2}(\rho_{eg} + \rho_{ge}) \quad v = \frac{1}{2i}(\rho_{eg} - \rho_{ge}) \quad w = \frac{1}{2}(\rho_{ee} - \rho_{gg})
\]

Re-expressing the Lindblad equation in terms of this new Bloch vector notation, we obtain an equation which is much more manageable to solve, as it becomes a system of 3 coupled ODE’s.

\[
\frac{d}{dt}R = \begin{pmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & -\Omega \\ 0 & \Omega & 0 \end{pmatrix} R
\]

This coupled system can solved analytically, and because we are only interested in how the excited population changes as a function of time, only \(w(t)\) will be given.

\[
w(t) = -\frac{\Delta^2 + \Omega^2 \text{Cosh}(\sqrt{(\Delta^2 + \Omega^2)t})}{2(\Delta^2 + \Omega^2)}
\]

From here we can solve for excitation fraction, as \(\rho_{ee} = w(t) + .5\). This is a very powerful expression, and a lot of the design goals for clock performance can easily demonstrated with it. One key
feature, is that width of the resonance is related to how long the clock laser interacts with the atoms. This is true until the length of the pulse begins to approach the natural lifetime of the transition \((\approx 100\,\text{seconds})\), which is well beyond current laser coherence times. Below this Rabi line-shape (excitation as a function of laser detuning) is shown for 4 different pulse durations.

\[\text{Rabi Lineshape as a Function of Pulse Duration}\]

\[\text{Laser Detuning (Hz)} \quad \text{Excitation Fraction}\]

![Figure 2.5: Line-shapes for 100 ms, 400 ms, 700 ms, and 1 second pulse durations](image)

Note that the \(\Omega\) needs to be adjusted for each of these pulse times, to maintain \(\pi\) pulses. Here a \(\pi\) pulse just refers to a pulse which is capable of driving the atoms entirely to the excited state, with zero laser detuning. How this is implemented experimentally, is by adjusting the laser intensity for different pulse lengths. With this in mind you can see why clock laser design is so critical to clock performance. The longer the coherence time of the laser, the longer pulses can be applied and the narrower the transition effectively becomes. For strontium, clock lasers would have to improve by two orders of magnitude in order to take full advantage of the natural transition width. Additionally, as will be discussed in detail in chapter four, when pulses get longer the time the laser has to drift away also increases. This can lead to decreased stability, if the laser drifts significantly over the duration of the pulse. This gives another limit to how long a pulse we can use. Now with each of the relevant light-matter interactions derived, we will move on to how they are used in clock operation.
2.2 Clock Implementation

With these basic concepts of light-matter interactions explained, we will now demonstrate their value to the strontium optical lattice clock. The sequence begins with an oven heated to 625 degrees Celsius (above the vapor point of strontium), which generates a thermal beam of strontium atoms. These atoms exit the oven through a small aperture, where the beam is collimated by the application of 4 laser beams at 461 nm, the wavelength associated with the $^1S_0$ to $^1P_1$ transition, perpendicular to the thermal beam’s propagation direction. The now collimated, but still hot beam, then enters a Zeeman slower. A Zeeman slower is essentially a length of vacuum tube with a magnetic field gradient over its length, and a counter propagating laser beam near a resonance in the atom. As the laser light slows the atoms, the Doppler shift seen by the atoms begins to change, and this drives the laser off resonance from the atoms, decreasing the efficiency of the deceleration. The key feature is that the magnetic field gradient applied along the length of the Zeeman slower, is setup to cancel this changing Doppler shift with an applied Zeeman shift. This keeps the atoms on resonance with the laser for the entire length, increasing cooling efficiency. The atoms then undergo two stages of MOT cooling, first based on the $^1S_0$ to $^1P_1$ cooling the ensemble to a theoretical lower bound temperature of 770 $\mu$K. The atoms are then loaded into a second MOT based on the much narrower $^1S_0$ to $^3P_1$ transition. While the first stage of cooling is very well described in the description in section 2.1.1. However, the second stage is more complicated due to the transition’s narrow line-width. Despite these complications, the second stage of cooling is capable of much lower temperatures, with a theoretical lower bound of 200 nK. Further discussion of the narrow line width MOT can be found here[9]. With the atoms now cooled, we are ready to load them into an optical lattice in preparation for clock spectroscopy.
2.2.1 Magic Wavelength Lattice

Remembering back, the goal of all this work is to make a very accurate measurement of the $^1S_0$ to $^3P_0$ transition in strontium, when the atom is free from any perturbations. With this in mind, it would seem the last thing you would want to do is trap these atoms in an intense laser field, as this would cause an AC Stark shift to the transition. However, there is a special wavelength where you do not have to worry about the lattice shifting the frequency, known as the ‘magic wavelength’. The basic idea, is that the lattice will shift both the ground and excited state energy levels, but if you pick the right wavelength these two energy shift are equal and the actual transition energy will remain unchanged. This is shown graphically below, and for strontium occurs at approximately 813 nm.
Figure 2.6: Atomic polarizability for the ground, and excited state of $^{87}\text{Sr}$. Bottom figure shows frequencies close to the magic wavelength, where the differential polarizability between the two states goes to zero. Taken from [9]

This allows us to confine our atoms deeply in the Lambe-Dicke regime, as discussed in the previous section, without shifting the frequency of the clock transition.
2.2.2 Rabi Spectroscopy and Read Out

Once the atoms are confined in the lattice, we are ready to measure the $^1S_0$ to $^3P_0$ transition. We begin, by taking the clock laser and applying a $\pi$ pulse to the atoms. The terminology here refers back to Bloch vector formalism shown in the derivation of Rabi spectroscopy. After the clock pulse the atoms are in some super-position between the excited state and the ground state. In order to measure the excitation fraction, or the percentage of the atoms in the excited state compared to the ground state, we undergo the sequence outlined below.

As shown above, we begin by applying light at 461 nm to the atoms, and recording the fluorescence this causes with a Photo Multiplier Tube (PMT). This is a direct measurement of the number of atoms in the ground state, because only the ground state atoms are resonant with the 461 nm light. Additionally this light induces enough heating to remove all the atoms in the ground state from the lattice. This leaves us with only excited atoms in the lattice. We then transfer the atoms back to the ground state, by optically pumping them out of the $^3P_0$ state which has a very long lifetime, to the $^3P_1$ state which has a shorter lifetime, and decays quickly to ground. Now that the excited state population has been transferred to the ground state we again apply the 461 nm light and watch for florescence. This gives us a direct measurement of the excitation fraction, which uses the same detection for each state. How this helps us know the transition frequency is not quite so obvious because any excitation fraction measurement gives no information about...
which side of the transition the laser’s frequency is on. How we resolve this is with a differential measurement scheme illustrated below.

![Differential measurement scheme](image)

Figure 2.8: Differential measurement scheme to obtain difference between the atomic transition frequency, and the clock laser’s frequency.

We start by taking the frequency of the clock laser, lowering it slightly, then measuring the excitation fraction. We then raise the clock frequency but the same amount, and measure the excitation fraction again. The difference in excitation fraction between the two measurements tells us both the sign and the magnitude of the frequency difference between the clock laser and the atomic transition. In this way we are able to determine where the laser is relative to the transition in the atom. This generates an error signal that we can act upon to lock the clock laser to the atomic transition. At this point clock operation leaves the world of atomic physics, and enters the realm of control systems, as will be discussed in chapter 4.
2.2.3 Quantum Projection Noise

Up until this point no motivation as to why optical lattice clock are worth this trouble has been given. Not bothering with many atoms, and just trapping a single atom, would seem to be a much easier experiment to run, and single atom clocks do exist (see [15] [16] for current ion clock performance). The key here is that looking back to figure 2.8, the clock’s error signal is based on the excitation fraction (which is the probability that an atom is in the excited state). The probabilistic nature of this signal introduces noise to practical measurements of the excitation fraction which is entirely fundamental in nature. To gain intuition on this, consider a clock sequence where we end up at a .5 excitation fraction. This means that for each individual atom interrogated there is a fifty percent chance that when it is measured, it will be in the excited state. This is identical to saying that each atom just flips a coin and decides which state it is in based on the outcome. When we flip only one coin, we can not really tell anything about the likelihood of its two states, we need flip the same coin many times in order to tell that it is truly fair. In fact looking to probability, this is described by the binomial distribution.

\[ P(x = k) = \binom{n}{k} p^k (1 - p)^{n-k} \]

where \( n \) is the total number of atoms interrogated, \( k \) is the number of atoms measured to be in the excited state, and \( p \) is the true excitation fraction. From this connection we realize that if one wants to estimate the excitation fraction, the uncertainty in this measurement scales as \( 1/\sqrt{n} \) where \( n \) is the number of atoms interrogated. It is clear then that if one wants to improve the measurement of the error signal, more atoms are always better. This clock is referred to as quantum projection noise (QPN) limited to 2000 atoms as the noise limits in our measurements are equal to the QPN from 2000 atoms. This means that each measurement of the clock transition in this experiment yields as much information as 2000 measurements in a single ion clock, making the improvement obvious.
Chapter 3

Accuracy Improvement: The BBR Shift

The current limitation in the accuracy to which the clock transition in $^{87}\text{Sr}$ is known lies in the perturbation to the transition caused by thermal blackbody radiation incident on the atoms. In our system this thermal radiation comes from the vacuum chamber, which is sitting at room temperature. The BBR shift will be derived from first principles, then a background context for experimental progress will be given. This will conclude with an introduction of our measurement to further reduce the uncertainty in the BBR correction and the results it achieved. We begin with the derivation of the atomic polarizability of an atom, then show how this can be used to calculate the BBR shift.

3.1 Theoretical Derivation

We will start with a derivation of the atomic polarizability of an atom. This will then motivate the connection between the Stark shift, and the BBR shift. We will conclude with a derivation for all of the relevant atomic physics needed for an improved measurement of the BBR shift.
3.1.1 Atomic Polarizability

We begin by deriving the atomic polarizability of the atoms, keeping only dipole order terms, with the assumption that the incident electric field is homogeneous over the size of the atom itself. This assumption is valid, as the radiation we concerned with has a wavelength on the order of microns. The derivation will focus on a two-level system, and then be generalized out to an n-level system. We start with the wave functions for a two-level atomic system.

\[ \Psi_1(r, t) = e^{-iE_1 t} \phi_1(r) \]
\[ \Psi_2(r, t) = e^{-iE_2 t} \phi_2(r) \]

Where the spatial portions of the wave functions (\( \phi \)) satisfy the expression

\[ H_0 \psi_n(r) = E_n \psi_n(r) \]

meaning that they are eigenfunctions of the unperturbed Hamiltonian \( H_0 \), with eigenvalues \( E_n \) respectively. Now let us add a time-dependent perturbation to this Hamiltonian, defined by

\[ H = H_0 + H'(t) \]

where \( H'(t) \) is caused by some time varying electric field at a fixed frequency. Keeping only dipole order terms, we find this interaction Hamiltonian to be the total dipole-moment of the system, times the instantaneous electric field. Remembering back to the discussion of optical dipole trapping, this is exactly the same concept.

\[ H'(t) = -p_a(r) \varepsilon \cos(\omega t) \]

With these preliminaries setup, we now define a composite wave function, made from both levels of this two-level system, defined as

\[ \Psi(r, t) = c_1(t) \Psi_1(r, t) + c_2(t) \Psi_2(r, t) \]

where the coefficients \( c_1 \) and \( c_2 \) are variable with time, and subject to the usual normalization conditions. We now take this composite wave function, and insert it directly into the time-dependent
Schroedinger equation, using the specific wave functions defined above.

\[ H\Psi(r,t) = i\hbar \frac{d}{dt}\Psi(r,t) \]

After cranking through this substitution, and eliminating any terms common to both side, we obtain an expression for the evolution of the coefficients as a function of the perturbing Hamiltonian. Arguments of the wave functions and coefficients will be dropped for clarity.

\[ H'(c_1\Psi_1 + c_2\Psi_2) = i\hbar(c_1\frac{d}{dt}c_1 + \Psi_2\frac{d}{dt}c_2) \]

With this in hand we wish to express the problem as a system of ODEs, then solve them explicitly. Additionally we do not want any spatial dependence in our final answer, so a means of canceling it out would be very beneficial. This can be done by first left multiplying by \( \Psi_1^* \), then \( \Psi_2^* \), giving us the first and second equation in the system respectively. There are two key things to notice when doing this. The first is that the spatial component of these wave functions are defined to be orthonormal, so

\[ <\phi_i|\phi_j> = \delta_{ij} \]

The second is that we define what is commonly referred to as the dipole matrix element to be

\[ P_{ij} = <\phi_i|p_a|\phi_j> \]

Due to the nature of some of these wave functions, this so called dipole element cannot be calculated to the level of precision required, and will have to be determined experimentally. This will be discussed in a later section. Utilizing these two relations we arrive at a coupled system of equations for the time-evolution of the coefficients. This is similar to the Rabi equation already discussed.

\[ \frac{d}{dt}c_1(t) = c_2(t)i\hbar P_{12}\varepsilon \cos(\omega t)e^{-i(E_2 - E_1)t} \]

\[ \frac{d}{dt}c_2(t) = c_1(t)i\hbar P_{21}\varepsilon \cos(\omega t)e^{i(E_2 - E_1)t} \]

At this point it is helpful to define a new variable to simplify the remaining work, so let

\[ \omega_0 = \frac{E_2 - E_1}{\hbar} \]
which is the angular frequency representation of the difference between the energy levels in our
two-level system. Additionally, we specify state 1 is the ground state and state 2 is the excited
state ($\omega_0 > 0$). We are also neglecting any natural decay to from the excited state to ground.
For our work, this is a fair approximation, and physically means that there is no damping of the
resonant peaks in the atomic polarizability. This is fine because all wavelengths of radiation we
are concerned with in the BBR calculation are extremely far from any resonances in our atom’s
ground or excited state. Now with this in hand, we can solve it in a more general form then given
in section 2.3.

The general process used, described verbally, is to make an initial estimate for $c_1(t)$ and $c_2(t)$
then plug them into the right side of the system of equations above. For our purposes only the first
non-trivial solution will be required. Given that $c_1$ is the ground state, $c_2$ is the excited state, and
the driving radiation is far from resonance, we will initially assume that the atom is entirely in the
ground state.

$$c_1(t) = 1 \quad c_2(t) = 0$$

We then plug these expressions into the right side of the equations above and solve the now uncou-
pled differential equations. In doing this we find that $c_1(t) = 1$, so there is no first order solution
for $c_1$. This is not true for the other equation, and we obtain the expression

$$\frac{d}{dt}c_2(t) = \frac{i}{\hbar} P_{21} \varepsilon \cos(\omega t) e^{i(E_2 - E_1)t}$$

Solving this ODE, we obtain an expression for the first order evolution of $c_2(t)$, where we take the
$\cos(\omega t)$, expand it in complex notation, then pull it into the other exponentials

$$c_2(t) = \frac{P_{21} \varepsilon}{2\hbar} \left( \frac{e^{i(\omega_0 + \omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0 - \omega)t} - 1}{\omega_0 - \omega} \right)$$

Now, because the experiment involves integrations on a time scale much much longer then the
period of oscillation of the relevant radiation fields, we want to take the limit where atoms have
been exposed to the H’ perturbation for infinitely long. This can also of thought of, as wanting
to take the expectation value of the dipole-moment, over all the possible spatial variations, similar
to the time-averaging done in the discussion of optical dipole trapping. This gives us a new, more physically relevant expression for the dipole-moment of the atom

\[ p(t) = \langle \Psi | p_a | \Psi \rangle \]

Plugging in the defined expressions for \( \Psi \) into this equation, we find the expression

\[ p(t) = c_1^* c_2 P_{12} e^{-i\omega_0 t} + c_1 c_2^* P_{21} e^{i\omega_0 t} \]

Previously this expression, while technically correct, would not have been useful. However, we have developed expressions for the time-evolution of the coefficients to first order, so we can now utilize that knowledge to get a nice expression for the how the dipole-moment of the atom varies in time.

\[ p(t) = \frac{P_{12} P_{21}}{2\hbar} \varepsilon \left( \frac{e^{i\omega t} + e^{-i\omega t}}{\omega_0 + \omega} + \frac{e^{i\omega t} + e^{-i\omega t}}{\omega_0 - \omega} \right) \]

From this expression we can define the atomic polarizability, to be

\[ \alpha(\omega) = |d_{in}| \left( \frac{1}{\omega_0 + \omega} + \frac{1}{\omega_0 - \omega} \right) \]

This derivation has been based upon a two-level system, but can be expanded to account for an n-level system, by individually adding up the contributions to the polarizability of the ith state, from every other state.

\[ \alpha_i(\omega) = \sum_{n \neq i} \frac{|d_{in}|}{\hbar} \left( \frac{1}{\omega_{in} + \omega} + \frac{1}{\omega_{in} - \omega} \right) \]

Additionally it is worth noting, that the expression \( P_{in} P_{ni} \) is typically written in dipole matrix notation, as \( |d_{in}|^2 \), but these are totally equivalent expressions. Lastly, this allows us to write down the AC Stark shift. Here it has been averaged in time to remove time-dependence, and expressed in terms of angular frequency.

\[ \omega_{Stark} = -\frac{1}{4\hbar} \alpha(\omega) \varepsilon^2 \]

### 3.1.2 BBR Shift

Now that we have derived atomic polarizability, we can use this to calculate the energy shift induced on the clock transition by thermal radiation from its surroundings, the BBR shift. One
key feature in calculating the BBR shift is the realization that the BBR spectra is incoherent and isotropic, so it can be treated as a superposition of plane waves with spectral components specified by the Planck distribution. This allows us to integrate up each component’s contribution to the overall BBR shift, without worrying about interference effects. So we can begin with an expression for the spectral radiance of a perfect blackbody, given by Planck’s law. This essentially tells us how much radiation is emitted by a black body in thermal equilibrium, as a function of the radiation’s frequency, and the temperature of the radiating body.

\[ B_\omega(T) = \frac{\hbar \omega^3}{4\pi^3 c^2 \varepsilon^2} e^{\frac{\omega}{\hbar k_b T}} \frac{\omega^3}{4\pi^3 c^2} \left( \frac{1}{\omega_{in} + \omega} + \frac{1}{\omega_{in} - \omega} \right) e^{\frac{\hbar \omega}{(\hbar k_b T)}} - 1 \]

Additionally we know from E&M that the peak electric field magnitude is related to the intensity of a plane wave by the expression

\[ I = \frac{\varepsilon_0 c^2}{2} \varepsilon^2 \]

where \( \varepsilon \) is the peak electric field strength, and \( \varepsilon_0 \) is the vacuum permittivity constant. We also must take into account the fact that the atoms are experiencing the radiation from all sides, which when integrated over gives us an additional \( 4\pi \). This allows us to immediately write down an expression for the BBR shift, in terms of an integrated expression for the AC Stark shift weighted by the spectral distribution of the BBR radiation.

\[ \Delta \omega_{BBR} = -\frac{1}{\varepsilon_0 c} \int_0^\infty \omega_{Stark}(\omega) I_{BBR}(\omega) d\omega \]

In order to keep track of constants when the expression is written in full, the coefficients introduced by each new term will remain isolated from one another, with the \( 4\pi \) being absorbed into the BBR coefficient. Writing this all out for the shift on the \( i \)th level for the general \( n \)-level problem we find the expression

\[ \Delta \omega^i_{BBR} = -\frac{1}{\varepsilon_0 c} \frac{\hbar}{\pi^3 c^2} \sum_{i \neq n} |d_{in}|^2 \int_0^\infty \left( \frac{1}{\omega_{in} + \omega} + \frac{1}{\omega_{in} - \omega} \right) e^{\frac{\hbar \omega}{(\hbar k_b T)}} - 1 \]

This integral can be simplified, using the change of variables \( x = \frac{\hbar \omega}{\hbar k_b T} y = \frac{\hbar \omega_{in}}{\hbar k_b T} \). This is helpful (as you will see later) because \( x << y \). Pulling out constants, but again keeping them isolated, this
substitution gives us the expression.
\[
\Delta \omega_{i}^{BBR} = \frac{-1}{\varepsilon_0 c} \frac{\hbar}{\pi^2 c^2} \frac{1}{4\hbar} \left( \frac{k_B T}{\hbar} \right)^3 \sum_{i \neq n} |d_{in}|^2 \int_{0}^{\infty} \left( \frac{1}{y + x} + \frac{1}{y - x} \right) x^3 e^x - 1 dx
\]

This expression is not directly calculable, because the integral is not known. However, given that
the transition frequencies \( \omega_{in} \) are much higher then the radiation’s frequency \( \omega \), we can expand the
expression in the parenthesis of the integral. This is possible, because the BBR spectra falls off so
rapidly, that resonant features in the integral are suppressed (as will be shown later)
\[
\left( \frac{1}{y + x} + \frac{1}{y - x} \right) \approx \frac{2}{y} + \frac{2x^2}{y^3} + O\left( \frac{x}{y} \right)^4
\]

This expansion splits the expression for the general BBR into 2 portions (neglecting higher order
terms). The first is commonly called that static BBR shift, and the later is called the dynamic
BBR shift. Continuing the analysis the use of this approximation casts the integral into an exactly
known form.
\[
\int_{0}^{\infty} \left( \frac{2}{y} + \frac{2x^2}{y^3} \right) x^3 e^x - 1 dx = \frac{2\pi^4}{15y} + \frac{16\pi^6}{63y^3}
\]

Here we see why resonant features are not a problem in this integral, as it has a closed form solution.

For ease of understanding the remainder of the analysis will be done separating the static, and the
dynamic corrections. We will begin with the static correction, utilizing the integral identity given
above, and obtaining the expression
\[
\Delta \omega_{static}^{i} = \frac{-1}{\varepsilon_0 c} \frac{\hbar}{\pi^2 c^2} \frac{1}{4\hbar} \left( \frac{k_B T}{\hbar} \right)^4 \left( \frac{2\pi^4}{15y} \right) \sum_{i \neq n} |d_{in}|^2 \omega_{in}
\]

The sum in the expression is exactly the DC polarizability of the atom which can be measured
very accurately by applying a DC electric field to the atoms and measuring the shift in the clock
transition frequency. This works extremely well because kilovolt per centimeter fields can easily
be applied, and the electric field from BBR, is on the order of 1 volt per centimeter. All other
constants are also known accurately leaving the major contribution to this terms uncertainty as
the uncertainty in temperature of the emitting black body. We now move on and write out the
dynamic polarizability of the atom.
\[
\Delta \omega_{dynamic}^{i} = \frac{-1}{\varepsilon_0 c} \frac{\hbar}{\pi^2 c^2} \frac{1}{4\hbar} \left( \frac{k_B T}{\hbar} \right)^6 \left( \frac{16\pi^6}{63} \right) \sum_{i \neq n} |d_{in}|^2 \omega_{in}^3
\]
Here we see that last term is NOT the DC polarazability, but rather each individual dipole matrix
element weighted by the inverse cube of the transition frequency. This term cannot be measured
accurately enough in a single experiment, so the technique used is to measure each of the major
contributions to this sum, and neglect terms smaller then your desired accuracy. Because the
transition frequency for each of the relevant transitions is known well enough through previous
work, the major contributors are the uncertainty in the dipole matrix elements. These numbers
can be inferred from measurements of the relevant decay lifetimes, through techniques introduced
in 3.1.4.

3.1.3 Deviations from an Ideal Blackbody

The results derived above hold absolutely true for the case when the atoms are bathed in a
perfect BBR environment. However this is not necessarily true in our experiment, due to optical
view-port access, and non-homogeneous heat distribution on the chamber walls. Therefore it is
critical that we get a sense for how deviations from the ideal case will impact our measurement of
the shift. Let me begin by expressing the BBR shift some other more general way, with the same
expansion of the polarizability as used before.

$$\Delta \Omega_{BBR} = c_1 \int_0^\infty I(\omega)d\omega + c_2 \int_0^\infty \omega^2 I(\omega)d\omega$$

where $I(\omega)$ is some general radiation distribution, and $c_1$ and $c_2$ are some constants which are at
this point not relevant. This expression is true for any radiation distribution, and is known as the
general heat shift. Focusing on the first term, and considering the case where the heat environment
is similar at least in form (which seems very reasonable given there is no major source of temperature
variation in the chamber), we can write down the relationship between the integrated power, and
the measured temperature.

$$T_{\text{sense}}^4 = k \int_0^\infty I(\omega)d\omega$$
This is expression is given with a unknown constant $k$, which is determined through calibration of the thermometer used. This means that the static correction, will actually be entirely accurate. Stated another way, the quantity relevant to the static correction, is exactly the quantity that thermometer actually measures, so no assumptions about the spectrum are needed. Moving on to the second term.

$$\Delta \omega_{\text{dyn}} = c_2 \int_0^\infty \omega^2 I(\omega) d\omega$$

From previous work, we can re-write this expression as

$$c_2 \int_0^\infty \omega^2 I(\omega) d\omega = \omega_{\text{dyn}} T_{\text{eff}}^6$$

Here $\omega_{\text{dyn}}$ is just some constant. Note that I have not written this expression in terms of $T_{\text{sens}}$, because this expression for a general spectrum is NOT the definition of temperature, as we are not simply integrating over the power spectrum, there is some weighting factor in front. This is not a trivial problem, but modeling has shown that in order for this to introduce an uncertainty on the $10^{-18}$ level, temperature fluctuations around the chamber on the order of 10 K would be required, which is certainly not true for the system once it has reached equilibrium. This means we can assume, after accounting for some added uncertainty in the assumption, that we can treat them equivalently

$$T_{\text{eff}} \approx T_{\text{sens}}$$

We now move on to derivation of the relationship between dipole matrix elements, and decay lifetimes.

### 3.1.4 Experimental Determination Of Dipole Matrix Elements

Given that the current clock uncertainty is dominated by the uncertainty in the dynamic correction coefficient, this is the obvious place to focus improvements. Remembering from the derivation of the dynamic correction, the term dependent on these dipole matrix elements is given
by the expression
\[ \sum_{i \neq n} \frac{|d_{in}|^2}{\omega_{in}^2} \]
Note that for the static BBR shift, individual contributions to the DC polarizability were irrelevant, as only the sum of each term is required, and this sum can be directly measured through the application of a DC electric field. One thing to notice is that the denominator greatly suppresses transitions at higher frequencies, so it is reasonable to think that only a few of the lowest energy transitions contribute the bulk of the total. In fact it will be shown in section 3.1.5, that 98% of this term is contained in only one element, so improved measurement of this element will drive the uncertainty lower. So how exactly does one experimentally determine a dipole matrix element?

We begin, with Einstein A and B coefficients, originally introduced by Albert Einstein as a way to relate spontaneous decay, and absorption.

\[ B_{ni} = \frac{4\pi^2}{3\hbar^2} \frac{1}{4\pi\varepsilon_0} |d_{in}|^2 \]

This is known as the Einstein B coefficient, which is proportionality constant that relates the population transfer from state n to state i or vice versa from either absorption or stimulated emission respectively. Combining this with the expression for spontaneous decay based on the so far undetermined Einstein A coefficients, we can obtain a total expression accounting for both spontaneous and stimulated decay. Here \( N_i \) is the number of atoms, in the ith state.

\[ \frac{d}{dt} N_i = A_{ni} N_i + B_{ni} N_i I(\omega_{ni}) \]

Einstein’s major contribution to this work, was the realization that the A and B coefficients would have to be some how related. The reason for this, is that when equilibrium has been reached and that the expression above is equal to zero, thermal equilibrium must also have been reached. When thermal equilibrium is reached we know \( I(\omega_{ni}) \) is is given by the Planck distribution already defined above. Following the logic laid out above, you come to the realization that the B coefficients for stimulated emission and absorption are identical, and related to the Einstein A coefficient, as

\[ A_{ni} = \frac{\hbar \omega_{in}^3}{\pi^2 c^3} B_{ni} \]
Because we already have an expression for the Einstein B coefficient (in terms of the dipole matrix elements) we can now relate the A coefficients to the dipole elements. How this helps us experimentally, comes from re-analysis of the state evolution given before. All of the transitions that are of interest to us are optical, while BBR radiation is approximately on the order of 10 microns. This means that $I(\omega_{ni})$ is essentially zero for our work, because there is negligible amounts of stimulated emission from this thermal radiation. This means that the equation simplifies to

$$\frac{d}{dt} N_{ni} = A_{ni} N_i$$

So the state lifetime has a decay constant given exactly by the Einstein A coefficient. So if we are interested in measuring a specific dipole matrix element, all we need to do is measure the decay lifetime, then go back through the derivation given above to recover the matrix element. This allows us to directly measure the value of matrix dipole elements.

### 3.1.5 Overview of Previous Experimental Work

Here I will write brief overviews of the previous experimental work that has gone into the BBR shift in strontium, and the related alkaline earth metal, ytterbium. This will help provide context to the current measurement, as well as highlight some difference between the current and past work. This previous work is best summarized by three papers, a high accuracy measurement of the static BBR shift in strontium by a group at Physikalisch-Technische Bundesanstalt (PTB) [17], a theoretical derivation of both the static, and dynamic polarizability by a theorist at Joint Quantum Institute (JQI) [18], and a measurement of the $^3D_1$ lifetime in ytterbium [19]. This final paper is qualitatively similar to our experiment.

**PTB: BBR Static Measurement**

PTBs work[17] focused on an accurate determination of the static BBR correction by directly measuring the DC polarizability of strontium. To take this measurement, a sample of strontium atoms is cooled, loaded into a lattice which is mounted on a translation stage, then the atoms
are transferred to the center of a carefully designed capacitor. This capacitor is calibrated, so the separation between the plates, the applied voltage, and the uniformity of the generated field are each known very precisely. Then a variable voltage, and therefore electric field, is applied to the atoms and the shift in the clock transition is measured as a function of the applied field. These data are then fit to a quadratic curve, and the DC polarizability is directly recovered. Any biases in electric field from the capacitor were teased out by varying the sign of the applied voltage, and occasionally measuring the transition with no applied voltage at all. The final result of this work is that the static correction coefficient is determined down to the 10 $\mu$Hz level, which is a fractional uncertainty at the $10^{-20}$ level. This removes the static BBR coefficient entirely from the uncertainty budget. Note that there is still uncertainty in the temperature which contributes to the total BBR uncertainty so the static correction term in general is not entirely negligible, but the coefficient itself is now known to a high enough degree. This is why we are currently only concerned about improvements to the dynamic correction.

**JQI: BBR Theory**

JQIs contribution [18] consists of a full theoretical calculation of the static and dynamic BBR shifts in strontium, based on all previously performed measurements that could be levied towards the calculation. It is particularly interesting that essentially every measurement of a state lifetime in strontium was used, as well as features already introduced such as the magic wavelength. This work’s key result is the determination that the $^3D_1$ lifetime contributes 98.2 % of the total dynamic correction, which is the reason we are interested in a better measurement of it in the first place. The derivation begins with entirely theoretical calculations for the transition energy, and dipole matrix elements relevant to the shift. These are then compared with the current best measured values, and good agreement is found in every case. The paper goes on to take three experimentally determined numbers, and combine them to determine the best recommended values for several dipole matrix elements of interest. Specifically, the lifetime of $^1P_1$ state, the DC Stark shift, and the magic wavelength. The paper lays out the process used in 5 steps.
• The $^1P_0$ lifetime is used to get an experimental values of the $^1S_0$ to $^1P_1$ matrix element, based on the work previously laid out with Einstein A coefficients.

• This matrix element dominates for both the static and dynamic shifts of the ground state at the magic wavelength. This enables us to calculate the ground states static, and AC polarizability, as 197.14 a.u. and 286.0 a.u. respectively.

• With this knowledge in hand we also know the AC polarizability of the excited clock state, because it is exactly equal to that of the ground state, at the magic wavelength.

• Again looking at theoretical derivations, we can see that there are three dipole matrix elements that dominantly contribute to this AC polarizability we know very accurately at the magic wavelength. Additionally the paper has previously made calculations of each of these element. This means that these theoretical calculations, can be adjusted slightly, such that they add to the correct value. Note that these adjustments were small, on the order of 1%, which is entirely within the uncertainty of such calculations.

• With these adjusted values, we can now get a corrected measurement for the DC polarizability, a quantity that is experimentally determined to a very high degree [17]

This process allows the theoretical values for the relevant dipole matrix elements to be fine tuned based on experimentally known combinations. The final result is an improved value and uncertainty for the dipole elements, which corresponds to a 1% uncertainty in the dynamic correction. The paper also strongly advocates for an improved measurement of the $^3D_1$ lifetime, as an improvement in this measurement from the current 7% level, to .5%, would reduce the overall BBR uncertainty by a factor of 2 therefore improving the clock, because the dynamic BBR shift is dominant in the uncertainty budget.

**NIST: ytterbium $^3D_1$ Lifetime Measurement**

The final work is a measurement performed by the time and frequency division of National Institute
of Standards and Technology (NIST) [19]. Similar to strontium, ytterbium is used in optical lattice clocks, and also has a BBR shift that is relevant in the uncertainty budget. In fact, due to shared structure of the two elements (both have 2 valence electrons) the dynamic correction is also dominated by the same dipole matrix element. To combat this the NIST team took an accurate measurement of the state lifetime, giving a final measurement of the total BBR shift to .05% precision. Due to the nature of strontium, that is better then we can obtain, but the process is similar in principle. An interesting side note on this measurement is that for ytterbium the theoretical and measured value for this lifetime vary by 3 \( \sigma \), so there is a bit of controversy in the agreed upon value. Because of this, the paper includes a theoretical derivation based on other measured values, which is extremely similar in form the that outlined in [18].

For the actual measurement made, it is qualitatively similar the the measurement needed to improve the BBR uncertainty in strontium. Because of this it is worth discussing the systematics in their measurement. The first systematic that they investigated, was the effect of a longer then impulse excitation of the \(^3D_1\) state. As will be shown, the functional fit is based on the assumption the the atoms all jump to the excited state instantaneously. Realistically this is not possible as there is finite laser power to drive this transition. This will be discussed in detail when our measurement in strontium is introduced. The next systematic they measured was the effect of atomic density on the state lifetime. They found a pronounced effect, on both of the state lifetimes measured, one caused by super radiance, and the other caused by an increase in the optical depth of the atomic cloud at higher densities. These effects are both fully explored in our version of this experiment. Their work here, though it not directly applicable, is very useful as it provides a route one can follow to make similar measurements in strontium.
3.2 $^3D_1$ Lifetime Measurement

We will now introduce our measurement of the $^3D_1$ state lifetime, beginning with an introduction to the relevant level structure in strontium.

![Energy level diagram for $^{87}$Sr](image)

Figure 3.1: Energy level diagram for $^{87}$Sr.

At first glance, one could see that direct measurement of the $^3D_1$ states decay the the excited clock state ($^3P_0$) would be the simplest experiment. However, because this transition occurs at 2.6 microns direct detection is not very practical. This is because detectors in this range typically have very low quantum efficiency, and we will need to measure individual photons. Instead we make our measurement based on the secondary fluorescence from $^3P_1$ to the $^1S_0$ ground state. This is qualitatively identical to the work already discussed in ytterbium. To achieve this, we operate the clock as usual, then excite the atoms as completely as we can into the excited state, then we stop the sequence. Now instead of performing a read out sequence as we would for clock operation, we apply a short pulse of 2.6 micron laser light to the atoms, exciting the atoms in the clock state to the $^3D_1$ level. Because the lifetime of the excited clock state is on the order of 100 seconds, we do not need to worry about leaving some of the atoms in excited state, as they will simply stay there.
for the duration of the measurement. With the atoms now in the $^3D_1$ state, they will begin to decay into the $^3P$ manifold. However, the life time of the $^3P_1$ and $^3P_2$ states are on the order of hundreds of seconds. This means that all the photons emitted over the course of the measurement, will be from the $^3P_1$ states decay. These photons will be collected in a photo multiplier tube (PMT) which will serve as the work horse from a data collection standpoint. Raw data for this data collection scheme is shown below.

![Figure 3.2: Measured florescence for $^3D_1$ lifetime measurement](image)

As you can see, result of this is a PMT signal consists of the sum of two exponentials, as there is an exponential decay from the $^3D_1$ to the $^3P_1$ state, which in turns will decay exponentially to ground. This gives us a fitting function, with the approximation of an infinitely short $^3D_1$ excitation pulse, given as

$$y(t) = A(-e^{-(t-t_0)/\tau_a} + e^{-(t-t_0)/\tau_b}) + B$$

Where $\tau_a$ is the lifetime of the $^3D_1$ state, $\tau_b$ is the lifetime of the $^3P_1$ state, and $t_0$ is used to specifying the start time to the decay. It is worth noting here that a lot of details in this measurement, including excitation efficiency, PMT efficiency, and atom fluctuations will only effect the amplitude
of the signal, A. This means the measurement is insensitive to many of these features, leaving the only systematics of concern to be deviations from the fit function provided (finite pulse duration), and effects which serve to change decay lifetime (density). Thankfully, as compared to ytterbium, no density dependence was found [1]. The function above was used as the basis for the data processing, and consists of five fitted parameters, for the decay start time, the two decay constants, the background level, and the decay amplitude. The reason that the $^3P_1$ was included as a fit parameter, is this work also represents the most accurate measurement of its value. Fortunately, it turned out that no density dependence was measured, and the are compelling reasons for why that is [1]. Another issue to note here is that this does not represent a situation where the noise each measurement is a constant. Instead the noise is proportional to the size number of counts received in a certain bin, this requires weights to be included for data fitting.

3.2.1 Pulse Duration Effects

In order to account for effects based on finite pulse width, two techniques were used. The first is that we minimized the duration of the $^3D_1$ excitation pulse, finding that 200 ns was the shortest pulse we were capable of using, in a stable manner. The second technique was implemented as part of the data processing. Because any effects from pulse lifetime would be limited to the beginning portion of the data (when the pulse occurred), cuts were made to the beginning of the data. This is to say, that initial portions of the data were neglected and the effect this has on the fitted values was investigated. This is shown below, where the value of the decay of interest is plotted (with fitted uncertainty) as a function of the depth of the cut in micro-seconds.
As you can see for too shallow of cuts, the decay value is dependent on the cut depth. There is however a region which is essentially insensitive to changes, and this is the region we are interested in. A cut depth of 21.56 micro seconds was decided upon, because it is on the left most side of this region, so it throws as little data away as possible (this decreases the fit uncertainty). This behavior is common for all five free parameters.

### 3.2.2 Final Results

With these systematics accounted for we are able to come to a final value for the decay measurement, giving a value of 2.182 micro-seconds, with a fractional uncertainty of .46%. It is worth noting that the added weights to account for non uniform noise actually inflate the error, so it is the more conservative technique. With this improved measurement in hand we can now move on to the how this measurement reduces our uncertainty. Accounting for this improved measurement, a, we get a final uncertainty of $1.9 \times 10^{-18}$ for the dynamic BBR shift. The final fit is shown below, using only data that passes the cut.
Figure 3.4: Fitted $^3D_1$ lifetime

3.3 Current Accuracy Table

All of the work discussed to this point all relates to the BBR uncertainty because that was the measurement to which I was able to contribute. Beyond this single measurement, many other systematics were greatly improved since the last reported clock evaluation[3]. The table of these uncertainties is shown below.
Clock Uncertainty Table, in Units of $10^{-18}$ Fractional Frequencies

<table>
<thead>
<tr>
<th>Systematic Effect</th>
<th>Shift</th>
<th>Old Uncertainty</th>
<th>Improved Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>BBR static</td>
<td>-4962.9</td>
<td>1.8</td>
<td>0.7</td>
</tr>
<tr>
<td>BBR dynamic</td>
<td>-345.7</td>
<td>3.7</td>
<td>1.6</td>
</tr>
<tr>
<td>Density shift</td>
<td>-4.7</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Lattice Stark</td>
<td>-461.5</td>
<td>3.7</td>
<td>0.5</td>
</tr>
<tr>
<td>Probe Stark</td>
<td>-1.6</td>
<td>0.13</td>
<td>0.7</td>
</tr>
<tr>
<td>1st-order Zeeman</td>
<td>-0.1</td>
<td>1.1</td>
<td>0.3</td>
</tr>
<tr>
<td>2nd-order Zeeman</td>
<td>-51.7</td>
<td>1.2</td>
<td>0.3</td>
</tr>
<tr>
<td>Lattice vector shift</td>
<td>0.0</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Line pulling + tunneling</td>
<td>0.0</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>DC Stark (vertical)</td>
<td>0.0</td>
<td>2.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Background gas collisions</td>
<td>0.0</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>AOM phase chirp</td>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>2nd-order Doppler</td>
<td>0.0</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Servo error</td>
<td>0.4</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Totals</td>
<td>-5827.2</td>
<td>6.4</td>
<td>2.5</td>
</tr>
</tbody>
</table>

For clarity, a very brief description of each improvement in uncertainty will be given below.

- **BBR static**: The frequency shifted associated with DC electric fields from black body radiation. Improvement was made thanks to reduced temperature uncertainty, by using a NIST calibrated temperature sensor.

- **BBR dynamic**: Shift associated with the time varying electric fields from black body radiation. Improvement was made using improved thermometry, and an improved measurement of the atomic structure of $^{87}\text{Sr}$. This was the work described above.

- **Lattice Stark**: Shift associated with the intensity of the confining lattice. Though we use a so called ‘magic wavelength’ lattice, we still need to experimentally confirm there is no shift induced on the atoms. This systematic was improved by varying lattice intensity, then
extrapolating to zero shift. Because the clock cannot operate with zero lattice intensity,
this error is then minimized by running with as shallow a lattice as can be maintained.

This represents another record breaking advance in optical clock accuracy, and drives the cutting
gedge ever closer to the $10^{-19}$ level.
Chapter 4

Stability Improvement: Control Law Analysis

Remembering back, there are two requirements for a natural phenomena to be a good frequency standard. It must be measured to a high degree of precision in a repeatable way, and it must not vary as a function of time. This means the pristine transition measured in strontium, must be mapped directly to a frequency that can be used in other applications. We will now discuss how the differential excitation fraction can be used to keep the clock laser locked to the transition. This section will begin with a basic introduction to digital control theory, proceed to a model for clock laser noise power spectral density (PSD), then a full model for the control implementation for the clock will be introduced. This model will serve as a platform to test improvements to the control law implementation, and will be used to show how an adaptive feed-forward technique is capable of a 20% reduction in the standard deviation of differential clock measurements.

4.1 Introduction to Clock Control

Before a model for the clock can be derived, some background in control theory must be introduced. We will begin with an introduction to discrete controls systems, then two measures for quantifying control loop performance will be introduced.
4.1.1 Discrete Control Theory

Due to the discrete nature of differential excitation measurements (the error signal used in the clock) any control action must be digital in nature. The controller utilized in the current implementation, is known as a proportional integral derivative (PID) controller. This is to say, that the control signal sent to the clock laser (to change its frequency), is based on a term proportional to measured frequency difference, a term based on the integral of this frequency error, and the derivative of the clock laser error. However, how this is implemented in a discrete nature is slightly more complex, and can expressed below.

\[ C(n) = kp \times e(n) + ki \times \sum_{i=0}^{i=n} e_i + kd \times (e(n) - e(n-1)) \]

Here we are denoting the nth control signal sent to the clock as \( C(n) \), \( k_x \) is the gain associated with each term, and \( e(n) \) is the error from the nth frequency measurement. The first thing we notice here, is that the proportional and integral term are perfectly logical, just as discrete implementations of their continuous analogs. The derivative however, would seem untrustworthy, as it is only based on 2 points and therefore is very susceptible to noise. In general derivatives are susceptible to noise, and the discrete case is even worse. This will be one of the key features we will improve upon for the feed-forward implementation.

4.1.2 Measures of Control Quality

In order for a quantitative discussion of control quality, we need to introduce several metrics used for evaluation. One of these are commonly used, and the other is more specific to experiments which require differential measurement schemes. The need for differential measurement stems from the clock’s superior accuracy compared to the SI second. Because of this accuracy, there is nothing to directly compare the clock frequency to, everything must be measured relative to itself. This is a concern which only arises when accuracy is extended beyond the definition of SI units. We will begin with the introduction of the Allen deviation of a signal.
4.1.2.1 Allan deviation

One of the most common quantities to describe a measurements consistency is the variance. This quantity gives a measure for far a signal varies from its mean. Allan deviation, which is the square root of Allen variance, extends this concept beyond this simple point by point comparison. It is enlightening, to first introduce the formula for Allen deviation, then discuss its interpretation. Note that while Allen deviation is a generally useful quantity, it is most commonly used to describe the stability of oscillators, hence why it is relevant to this work.

\[
\sigma(\tau) = \sqrt{\frac{1}{2(M-1)} \sum_{i=1}^{M-1} (y_{i+1} - y_i)^2}
\]

where \(\tau\) is time between frequency measurements, \(M\) is the total number of frequency measurements taken, and the individual frequency offsets measured are the \(y_i\). How this expression is used is by changing \(\tau\) over a range of values and averaging data points included in the tau period. This can be thought of as binning the data into larger and larger bins, corresponding to longer averaging times, then comparing the standard deviation of each of these bins. In the end this gives us a tool to describe how quickly noise in a signal averages down and give us some idea of the nature of the noise. In particular, noise corresponding to different processes will have different slopes as a function of averaging time. Flicker phase noise averages down as \(1/\tau\), white noise as \(1/\sqrt{\tau}\) and Brownian motion actually increases as a function of averaging time. What is referred to as a clock’s stability is the rate at which this noise averages down, and because QPN and the Dick effect contribute white noise, we know the error will average down as \(1/\sqrt{\tau}\). The key advance in optical lattice clocks is how fast these errors average down, enabling more time efficient evaluation of systematics.
4.1.2.2 Standard Deviation under Differential Measurement

Unlike Allen deviation discussed above, the standard deviation under differential measurement is unique to differential measurement schemes. We begin with a discussion of what is meant by a self-comparison. As discussed below, measurement at this accuracy requires differential self-comparisons. An example of this would be running the clock using two different lattice powers, to study the effect of lattice intensity on the clock transition. What is meant by 'two different lattice powers' is not obvious, and relies on the control implementation to work. This statement, that the clock is undergoing a self-comparison, means two separate loops, each operating independently of each other. We would then set up the experiment, such that whenever the frequency of clock X was being measured, we would use one lattice intensity, and when clock Y was being measured, we would use another. In this way the control frequency associated with these two loops can be directly compared, giving us a frequency shift associated with each operating condition. The standard deviation of the difference between these two control signals us what is meant, by the deviation under differential measurement. This essentially gives us a measure of how tightly the laser is locked to the transition, in an instantaneous way (compared to the time averaged way described by Allen deviation). While Allen variance is traditionally the only measure used for clock stability, this is an incomplete picture, as the instantaneous lock quality is also of concern for many applications. We will now introduce the model used in investigation of the control law.

4.2 Precision Metrology in the Frequency Plain

We will begin with a discussion of how laser noise effects clock operation. Then an introduce a model for the clock laser noise will be given. These concepts will then be combined, and a numerical tool to investigate the concept of the clock’s frequency response will be shown. This model for the full clock’s frequency response will include not only the effects of Rabi spectroscopy, but also the impacts that dead time has on the measurement. This is a powerful concept, and
provides additional insight into clock operation from a perspective not commonly considered.

4.2.1 Laser Noise Model

In order to fully model the clock, an accurate model for the laser noise must be generated. This is because this laser noise is the dominate source in the system. Thankfully, this work was carried out previously for our clock laser [20]. The basis of this work is what is referred to as a sensitivity function. This function describes how a particular spectroscopy sequence is affected by laser noise. Note that this is not the clock’s performance, but just the effect of a certain type of spectroscopy directly. To take advantage of this concept, the sensitivity function for several spectroscopy sequences was derived, then the clock was ran with these different sequences in place of Rabi spectroscopy. Then based on the noise properties from the clock running each of the sequence, a model was developed. It was found that the dominant type of noise in the laser, is 1/f noise, associated with thermal fluctuations. This is exactly as expected, but what was not expected, was the additional of several resonant peaks in the noise spectrum. The largest of these peaks in terms of integrated area, is centered on 60 Hz, and is likly caused by electrical noise. The modeled Power Spectral Density (PSD) is shown below.
This model will be used for the remainder of the model. Further details on this technique, and the specifics of how it was implemented, can be found in [20].

4.2.2 Excitation Fraction Noise

What we are interested in here is that assuming I have a time series representation of the clock laser’s frequency (accounting for noise), how does this effect excitation fraction under normal clock operations. The first thing we need, is a means of calculating excitation fraction that accounts for laser noise. Looking back to our derivation, we see exactly the expression we are interested in, given below

\[
\frac{d}{dt} R = \begin{pmatrix}
0 & -\Delta & 0 \\
\Delta & 0 & -\Omega \\
0 & \Omega & 0
\end{pmatrix} R
\]

Looking at this expression we see the deviation from the ideal clock frequency (\(\Delta\)) entering as a coefficient in an system of ODEs to calculate excitation fraction. Moving away from treating
this problem as a constant coefficient system (as done previously) let us generalize this expression by letting the laser frequency vary as function in time. This leaves a variable coefficient system of ODEs to convert a time series of laser noise to excitation fraction. Now to get a feel for the frequency dependence of clock operations to laser noise, we generate a set of time series data, for a range of frequencies then solve for the effect of this single frequency laser noise has on the clock system when it is locked. This can be thought of as developing a frequency domain transfer function for the clock. The results of this are shown below.

![Rabi Spectroscopy Frequency Response](image)

Figure 4.2: Transfer function for Rabi Spectroscopy, 160 ms pulse duration, 840 ms dead time

It is important to note again that this is all operating while the clock frequency is locked, and therefore is different from the sensitivity function already introduced. The fitting function used here is that of a third order low pass filter and shows how high frequency noise in the laser is suppressed in the spectroscopy sequence. The corner frequency associated with the fit function is 5.9 Hz (approximately $1/\tau$), but is not the most interesting feature we see. The dips you see above, occur at .5 Hz, 1.5 Hz, 2.5 Hz etc; frequencies associated with the 1 second duty cycle of the experiment. This shows how the duty cycle effects the noise spectrum we see, and this
noise getting mixed down to low frequencies is the source of the Dick effect. With this insight into the performance of the spectroscopy sequence in the frequency plane, we now move on to an introduction of the full model.

4.3 Modeling an Optical Clock

With the model now in hand for the Power Spectral Density (PSD) of the clock laser, we wish to develop a model to simulate clock operations. This will be done entirely in the time domain, to enable further modifications for which expression in the frequency domain would be extremely difficult. In addition to this, we wish to utilize the numerical technique for excitation fraction measurement introduced above. We will begin with a conversion from this noise PSD, to a time series noise. This is done numerically, based on discretization of the PSD, with randomized phases to preserve the noise like properties.

\[ x(t_n) = PSD(\omega_m)^2 \cos(\omega_m t_n + \Phi_m) \]

where \( n \) is the index of the time series, \( m \) is the index of the discretized selections for frequencies from the PSD. When this is implemented, we find time series noise shown below, and confirm it has the desired distribution, via PSD estimation algorithms.
At first glance the blue estimation does not appear to be a great match with the model, however this sort of behavior is very typical for PSD estimation. Now that we have this time series noise, we can utilize the numerical technique derived above. This enables us to get the effect of this directly measured laser noise back into the excitation fraction. With this in hand we now have everything we need to simulate the clock locking performance, based on the measured clock laser PSD. Everything will be implemented as a differential measurement, so the self-comparison discussion given above will hold.

4.3.1 Pseudo Code Implementation

In order to best capture how each element of the model was implemented psuedo code will be shown. While the model matches current clock operation as closely as possible, exact treatment of laser noise is important to show. This algorithm will takes laser noise as a time series, spectroscopy time, dead time, and servo gains, and returns control signal and error signal after each measurement, for X loop and Y loop respectively.
Data: Noise, $\tau_{\text{spec}}, \tau_{\text{dead}}, \text{Gains}$

Result: $CTRL_X, \text{ERR}_x, CTRL_Y, \text{ERR}_Y$

1. Initialize CTRL for each loop, offset .2 Hz from transition (simulate starting transient);
2. Establish flag, to switch between loops X and Y;
3. Initialize $i=0$ and $j=0$, to track the X and Y loops respectively;
4. Initialize Wall time to be zero;
5. while Wall time < Max time do
   6. if flag = X then
      7. Laser Noise = Noise[Wall time + $\tau_{\text{dead}}$, Wall time + $\tau_{\text{dead}} + \tau_{\text{spec}}$];
      8. Laser frequency = Laser Noise +$CTRL_X(i)$;
      9. Calculate $ERR_X(i)$ from laser frequency;
     10. Add QPN, based on atom number;
     11. $CTRL_X(i+1) = Gains \times (ERR_X(i) + (ERR_X(i) - ERR_X(i-1)) + \sum_{p=1}^{i} ERR_X(p))$;
     12. Wall time = Wall time + Wall time + $\tau_{\text{dead}} + \tau_{\text{spec}}$;
     13. $i = i+1$;
     14. Flag = Y;
   6. end
   7. if flag = Y then
      8. Laser Noise = Noise[Wall time + $\tau_{\text{dead}}$, Wall time + $\tau_{\text{dead}} + \tau_{\text{spec}}$];
      9. Laser frequency = Laser Noise +$CTRL_Y(j)$;
     10. Calculate $ERR_Y(j)$ from laser frequency;
     11. Add QPN, based on atom number;
     12. $CTRL_Y(j+1) = Gains \times (ERR_Y(j) + (ERR_Y(i) - ERR_Y(j-1)) + \sum_{p=1}^{j} ERR_Y(p))$;
     13. Wall time = Wall time + Wall time + $\tau_{\text{dead}} + \tau_{\text{spec}}$;
     14. $j = j+1$;
     15. Flag = X;
   7. end
5. end

Algorithm 1: Clock Simulation Model
In this way we are able to simulate the performance of 2 independent clocks, exactly how the system operates under self-comparison. Looking at this, we can see how neither clock is communicating with the other, and most of the noise process simulated is unused. This is exactly what the Dick effect is, because the system can not tell what the laser is doing during the dead time in the cycle.

4.3.2 Model Results

With this model now derived we wish to verify it against experimental performance. The best way to make these comparison, is to optimize the gains used in the model, then compare the Allen deviation obtained, with what we see on the system. This is shown below.

![Figure 4.5: Allen deviation, with and without laser noise introduced to the model. Solid lines represent theoretical estimates.](image)

As you can see the Allen deviation for the case with laser noise matches up well with the theoretical limits, as well as for the case when just QPN noise is considered. For the QPN case,
the laser noise was set to zero for the duration of the experiment, meaning the only source of error was coming from the added QPN noise. This provides good support that the model is an accurate vehicle for further tests of clock performance.

4.4 Improvements beyond PID

We now have a playground to experiment with modifications to the control law, far more efficiently then could be achieved on the system alone. It turns out that improvements can be made, based on the addition of an adaptive feed-forward to control signal. Pseudo code is presented below. The additional code will be bold to emphasize the difference.

4.4.1 pseudo code

Here, FF is an abbreviation for feed forward, and the function extrapolate(X,N,M), takes the last N points from the vector X, then extrapolates forward at order N, with N=1 being linear, N=2 being quadratic etc.
Data: Noise, $\tau_{spec}$, $\tau_{dead}$, Gains, FFGain, FFCount, FFOrder

Result: $CTRL_X$, $ERR_x$, $CTRL_Y$, $ERR_y$

1 while Wall time < Max time do
   2 if flag = X then
      3 Laser Noise = Noise[Wall time + $\tau_{dead}$, Wall time + $\tau_{dead} + \tau_{spec}$];
      4 Laser frequency = Laser Noise + $CTRL_X(i)$;
      5 Calculate $ERR_X(i)$ from laser frequency;
      6 Add QPN, based on atom number;
      7 $CTRL_X(i+1) = Gains \times (ERR_X(i) + (ERR_X(i) - ERR_X(i-1)) + \sum_{p=1}^{i} ERR_X(p))$;
      8 $FF = \text{Extrapolate}(CTRL_X, \text{FFCount, FFOrder})$;
      9 $CTRL_X(i+1) = CTRL_X(i+1) + \text{FFGain} \times FF$;
      10 Wall time = Wall time + Wall time + $\tau_{dead} + \tau_{spec}$;
      11 i = i+1;
      12 Flag = Y;
   13 end
   14 if flag = Y then
      15 Laser Noise = Noise[Wall time + $\tau_{dead}$, Wall time + $\tau_{dead} + \tau_{spec}$];
      16 Laser frequency = Laser Noise + $CTRL_Y(j)$;
      17 Calculate $ERR_Y(j)$ from laser frequency;
      18 Add QPN, based on atom number;
      19 $CTRL_Y(j+1) = Gains \times (ERR_Y(j) + (ERR_Y(i) - ERR_Y(j-1)) + \sum_{p=1}^{i} ERR_Y(p))$;
      20 $FF = \text{Extrapolate}(CTRL_Y, \text{FFCount, FFOrder})$;
      21 $CTRL_Y(j+1) = CTRL_Y(j+1) + \text{FFGain} \times FF$;
      22 Wall time = Wall time + Wall time + $\tau_{dead} + \tau_{spec}$;
      23 j = j+1;
      24 Flag = X;
   25 end
   26 end

Algorithm 2: Adaptive Feed Forward Control Implementation
In words, this feed forward operates by looking at trends in the control signal, and predicting that they will continue. Another way to think of this, is that the regular PID implementation is given another derivative term that is not based on the two latest data points, but rather the last 20 data points. This provides a derivative estimate that is more noise insensitive, and as such can support higher gain. This improves the tightness of the lock, but leads to many free variables to control in order to ensure the gains are real. The gains used in the PID control will greatly impact the lock quality, so these are this first variables considered. To automate the process of PID tuning, the clock is simulated using 10 different noise sequences, over a range of gains. The best gain is then selected, based on the tightness of the lock. The gains are selected, such that several possibilities lie within the noise of the signal to avoid missing optimal values. In this way we can ensure that gains are selected in an optimal way. After that are range of feed forward gains and the number of data points used in the extrapolation were investigated, and optimal values found. It was determined based on the model, that 20 data points, and a weighting of .3 was capable of a 20% reduction in the standard deviation under differential measurements. This is to say, that the difference between the control signal \((CTRL_X - CTRL_Y)\) is reduced, indicating a tighter lock has been produced. This improvement was very consistent over 10 different noise samples, and is currently awaiting testing on the main system. The results of feed forward are shown below
As you can see, the reduction in differential variance means the first point is driven downwards. This is accompanied by a bulge outwards after this initial improvement. This new technique has been implemented on the real system, but testing has not yet been possible. Moving forward, there are several open questions on this issue. The differential variance seems to indicate a tighter instantaneous lock, so can longer pulses be used and still maintain this lock? Sadly the model is not helpful in this question, as comparable lock breaking performance to the real system could not be achieved. Additionally, the bulge outward for short averaging times may indicate something detrimental occurring, but how this effects the clock is not known. In order for this technique to prove useful in clock operation, these questions must be answered using the real system.
Bibliography


