Effective Potentials and Infinite Nuclear Matter

Sam Bein
University of Colorado Boulder

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Effective Potentials and Infinite Nuclear Matter

By Sam Bein

April 11, 2011

Advisor: Jim Shepard

Committee Members: Jim Shepard, John Cumalat, Robert Parson
1. Introduction

An old problem in theoretical physics survives today: how to the properly characterize the atomic nucleus? Such a characterization should have predictive power as well as descriptive. Nuclear properties such as the binding energy and saturation density are among the predictions with which such a model can be tested. To approach this question it is first necessary to understand what smaller particles make up the nucleus, and how these particles interact with one another.

A nucleus can be thought of as a Fermi sea made up of protons and neutrons interacting with each other via complicated and uncertain force laws. Both the strong and electromagnetic forces play a role in this interaction, as do the weak force and the Pauli exclusion principle. In addition there are spin and angular momentum dynamics to consider, as well as isospin.

The complexity of these interactions does not manifest merely as an aesthetic problem for calculations. With difficulties arising from renormalization, large quark coupling constants, and massive exchange particles, the Standard model fails to supply a nucleon potential field that is smooth enough to be used in a perturbation or mean field theory calculation. This omission renders the task of modeling the nucleus on nucleon-nucleon (NN) interactions unapproachable—without some cleverness. To overcome these challenges, nuclear physicists have adopted the use of effective NN interactions in an effort to understand atomic nuclei [1][2][3][4]. Effective potentials are constructed to match experimental NN scattering data, and can then be used in Hartree-Fock or similar calculations[1]. Such calculations can predict quantities of interest like binding energy and saturation of nuclear matter [1][5][6]. Selecting such a potential and building a model from it is the problem at hand.

A formulation and application of smooth effective NN potentials to account for nuclear properties constitutes part of the research of my mentor, James Shepard, and it is the research in which I have participated over the last year. This paper describes a method for building and testing a model of nuclear matter. The heart of
the analysis was conveyed to me through personal correspondence with my mentor. Rather than an initial review of previous related literature, this paper's discussion of literature will appear in each section as it is relevant.

2. The nucleon-nucleon interaction.

Before the many-nucleon system that is the nucleus can be understood and modeled, it is first necessary to understand the interactions of just two nucleons. Considerable study has been done on this front, as well as the higher dimensional problem of three- and few-nucleon interactions, all of which in principle need proper treatment [1]. However, this paper focuses on the two particle interaction, which is the most fundamental building block of the nucleus model.

The end goal of studying the NN interaction is to obtain an effective potential that best describes it. Before we look directly at the role of the potential, however, we need to understand a bit about the conditions under which two nucleons are observed to interact, the scattering process. Much of the information about the way nucleons interact comes from experiments in which two energetic nucleons are scattered off each other, and debris are examined at a later time.

2.1 Scattering

Part of the analysis in this and the next section parallels work of introductory QM textbooks. For a companion to these sections I recommend Schiff's Quantum Mechanics chapter 19 [7].

In a quantum scattering model, the incident beam of nucleons is treated as a plane wave that encounters the potential field of a target nucleon, producing an asymptotically spherical outgoing wave. The total asymptotic wavefunction is then a superposition of the plane and spherical waves:

$$\psi (r \to \infty) = e^{ikz} + f \frac{e^{ikr}}{r}$$

(0)
A uniform spherical wave takes the form $\psi_{\text{spherical and uniform}} = \frac{e^{ikr}}{r}$, but in general a scattered wave has angular dependence modulated by $f = f(\Omega) = f(k)$, the scattering amplitude. For my purposes, the scattering amplitude contains most of the useful information about the NN interaction that can be extracted from the scattering process, including information about the scattering phase shifts. Phase shifts are an insight into the innerworkings of the NN interaction, and the primary information by which effective NN potentials are constructed.

**Phase Shifts**

If you compare the outgoing spherical wave with the incident plane wave, you will notice that the two share the same wave number $k$, and thus the same energy and wavelength. But they are different in one crucial way: they carry different phase. The picture below illustrates the phase shift between incoming and outgoing waves.

Figure 1.1: Scattering phase shifts; outgoing wave is phase shifted from the incident wave. The phase shift is apparent when the waves are projected to overlap. The reason for the factor of 2 is that the wave is considered to be phase shifted once as it enters the potential, and once again as it leaves.
The potential itself is unknown, and the shape of the wavefunction inside the range of the potential is not measurable. However, NN phase shifts are experimentally well-documented for incident energies up to 350 MeV. The phase shifts carry information about the nature of the interaction that took place within the potential range (in some sense, they are a coding of how much the potential "messes up" the wave as it went by), although they do not uniquely define the potential that was seen. In other words, myriad potential fields can be chosen that reproduce correct phase shifts over certain energy ranges. Since the "true" potential of the interaction is unknown, we are at liberty (or at the mercy of) picking a potential with an obliging form, and adjusting it to best reproduce the measured phase shifts. How to relate the potential to the phase shifts is the primary topic of section 2.

In the initial stages of my research, I wrote a program to numerically compute phase shifts associated with any spherically symmetric potential. Since it is not at the moment entirely relevant, I will not describe this process in detail; but in short, I solicited the Numerov method to numerically solve the Schrodinger equation in 1-dimension (the spherical symmetry reduces the problem to 1D). For any spherically symmetric potential, the solution at large r could be compared to an unshifted sinewave to determine the phase shift. While this brute force method gave me a way to see the correct phase shifts come out of any potential in configuration space, it did not allow me to optimize non-local potentials or potentials specified in momentum space.

There is, however, a more theoretical, analytic mapping between the potential and the phase shifts, involving a connection with the scattering amplitude \( f(k) \). In the next two sections I show how the Schrodinger equation can be manipulated into a form that relates the potential to the scattering amplitude, and that the scattering amplitude holds a one-to-one correspondence with the phase shifts. This will put me in business to fit a potential function directly to experimental phase shifts, as long as the potential is selected appropriately. I show this process now.

### 2.2 From the potential to the scattering amplitude

#### Schrödinger in Integral Form

The time-independent Schrödinger Equation in three dimensions is
\[
\left(-\frac{\hbar^2}{2\mu} \nabla^2 + V\right)\psi = E\psi,
\]

which can be written

\[
\left(\nabla^2 + k^2\right)\psi = \frac{2\mu}{\hbar^2} V\psi,
\]

where \( k = \left(\frac{2\mu E}{\hbar^2}\right)^{1/2} \), and \( \mu \) is the reduced mass of the two-nucleon system.

By the method of Green's functions, Schrödinger is equivalently expressed in integral form as

\[
\psi(r) = \int d^3 r_o G(r-r_o, k) \frac{2\mu}{\hbar^2} V(r_o)\psi(r_o),
\]

where \( G(r, k) \) is a Green's function propagator. The canonical solution is \( G(r, k) = \frac{-e^{ikr}}{4\pi r} \). While the derivation of this solution is not provided here, I will demonstrate that (3) and (1) are equivalent under this particular Green's function. Start by substituting \( G(r, k) \) into (3):

\[
\psi(r) = \int d^3 r_o G(r-r_o, k) \frac{2\mu}{\hbar^2} V(r_o)\psi(r_o)
\]

\[
= \int d^3 r_o \frac{e^{ik|r-r_o|}}{4\pi|r-r_o|} \frac{2\mu}{\hbar^2} V(r_o)\psi(r_o).
\]

Acting \( (\nabla^2 + k^2) \) on both sides yeilds

\[
(\nabla^2 + k^2)\psi(r) = (\nabla^2 + k^2)\int d^3 r_o \frac{e^{ik|r-r_o|}}{4\pi|r-r_o|} \frac{2\mu}{\hbar^2} V(r_o)\psi(r_o)
\]

\[
= -\int d^3 r_o (\nabla^2 + k^2) \frac{e^{ik|r-r_o|}}{4\pi|r-r_o|} \frac{2\mu}{\hbar^2} V(r_o)\psi(r_o).
\]

When the Laplacian acts on the Green's function, \( (\nabla^2 \frac{e^{ik|r-r_o|}}{r-r_o}) \) expands into

\[
(\nabla^2 \frac{e^{ik|r-r_o|}}{r-r_o}) = -k^2 \frac{e^{ik|r-r_o|}}{|r-r_o|} - 4\pi \frac{e^{ik|r-r_o|}}{|r-r_o|} \delta^3(r-r_o),
\]

where straightforward intermediate steps have been omitted. Substituting (6) into (5) gives
\[
(\nabla^2 + k^2) \psi(r) = \int d^3 r_o e^{i k |r-r_o|} \delta^3(r-r_o) \frac{2 \mu}{\hbar^2} V(r_o) \psi(r_o) \\
= \frac{2 \mu}{\hbar^2} V(r) \psi(r),
\]

which is identical to equation (2). Indeed, the integral equation (3) is entirely equivalent to the time-independent Schrödinger equation.

Equation 4 would be a closed form expression for the wavefunction if it weren't for the \( \psi \) inside the RHS integral. Given this, it is tempting to guess a reasonable solution for \( \psi \), plug it into the integrand, and obtain a "new and improved" \( \psi \) by performing the integral. This intuition is in alignment with the Born approximation, which crudely "guesses" that \( \psi \) is just the untouched incident plane wave \( \psi = e^{ikx} \), plugs this plane wave into the RHS \( \psi \), and carries out the integral. The result is a corrected \( \psi \) on the left hand side, no longer a mere plane wave, but a slightly "disturbed" plane wave. The spirit of this approximation is at the heart of the analysis in the next section. Reiteration of this process yields higher order corrections, and the generalization to all orders will be made shortly.

**Spotting the Scattering Amplitude**

Any function \( \phi \) can be added to (4) for which \( (\nabla^2 + k^2)\phi = 0 \), and the above analysis holds. This is true as long as \( \phi \) is some combination of plane waves, so we are free to add in the "extracted" incident plane wave traveling in the z direction with momentum \( p \):

\[
\psi(r) = e^{ipz} - \int d^3 r_o \frac{e^{ik|r-r_o|}}{4\pi |r-r_o|} \frac{2 \mu}{\hbar^2} V(r_o) \psi(r_o). \tag{8}
\]

With some slight of hand, equation (8) can be manipulated into a form identical to equation (0). We want to now take the large \( r \) limit. For \( r >> r_o \), \( |r-r_o| \) expands:

\[
|r-r_o| \approx \sqrt{r^2 - 2 r \cdot r_o} \approx r(1 - \hat{r} \cdot r_o/r + ...). \tag{9}
\]

and similarly

\[
\frac{1}{|r-r_o|} \approx \frac{1}{r} + \frac{\hat{r} \cdot r_o}{r^2} + .... \tag{10}
\]
Keeping the first two terms in (9) and only the first term in (10) \(^1\) allows for the following replacement:

\[
\frac{e^{i|\mathbf{k}| \mathbf{r}}}{|\mathbf{r}|} \approx \left( \frac{1}{r} \right) e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_o)} = \frac{e^{i\mathbf{k} \cdot \mathbf{r}_o}}{r} e^{-i\mathbf{k} \cdot \mathbf{r}_o}
\]  

(11)

here \( \mathbf{k} = k\hat{\mathbf{r}} \) is the wave vector, pointing radially when \( r \) is large.

Substituting this result into (8) produces a very nice form of the integral Schrödinger equation:

\[
\psi (r \to \infty) = e^{ipz} - \int d^3 r_o \frac{e^{i\mathbf{k} \cdot \mathbf{r}_o}}{4\pi} e^{-i\mathbf{k} \cdot \mathbf{r}_o} \frac{2\mu}{\hbar^2} V (r_o) \psi (r_o)
\]

\[
= e^{ipz} - \frac{e^{i\mathbf{k} \cdot \mathbf{r}_o}}{r} \int d^3 r_o \frac{e^{-i\mathbf{k} \cdot \mathbf{r}_o}}{4\pi} \frac{2\mu}{\hbar^2} V (r_o) \psi (r_o).
\]

This is exactly the form of equation (9) in the large \( r \) limit, so the scattering amplitude \( f \) is evidently

\[
f = -\int d^3 r_o \frac{e^{-i\mathbf{k} \cdot \mathbf{r}_o}}{4\pi} \frac{2\mu}{\hbar^2} V (r_o) \psi (r_o).
\]

Apparently \( f \) is the inverse Fourier transform of the function \(-\frac{2\mu}{4\pi \hbar^2} V (r) \psi (r)\):

\[
f (\mathbf{k}) = -\frac{2\mu}{4\pi \hbar^2} \langle \mathbf{k} | V \psi \rangle.
\]

(13)

Here at last is a closed-form expression for the scattering amplitude, but \( f (\mathbf{k}) \) is not yet determined because we don't know \( \psi \). To redeem this deficiency, we need a "Born approximation-like" version of equation (13). This is the Lippmann-Schwinger Equation.

**The Lippmann-Schwinger Equation**

It is possible to force the odd expression for the scattering amplitude in (13) to appear directly in the Schrödinger equation. Between equations (5) and (7), it was shown that

\[
-\int d^3 r_o \frac{e^{i|\mathbf{k}| \mathbf{r}}}{4\pi |\mathbf{r} - \mathbf{r}_o|} \frac{2\mu}{\hbar^2} V (r_o) \psi (r_o) = \frac{1}{(\mathbf{p}^2 + \mathbf{k}^2)} \frac{2\mu}{\hbar^2} V (r_o) \psi (r_o).
\]

This identity allows our beautiful equation (12) to be rewritten

\[
\psi (\mathbf{r}) = e^{ipz} + \frac{1}{(\mathbf{p}^2 + \mathbf{k}^2)} \frac{2\mu}{\hbar^2} V (\mathbf{r}) \psi (\mathbf{r}),
\]

(14)
which is verifiably still the Schrödinger equation. The fastest way to make our scattering amplitude visible is to multiply each side by $\frac{-2\mu}{4\pi \hbar^2} V$, and project into momentum space:

$$- \frac{2\mu}{4\pi \hbar^2} \langle k \mid V \psi \rangle = - \frac{2\mu}{4\pi \hbar^2} \langle k \mid V e^{i p z} \rangle - \frac{2\mu}{\hbar^2} \left( \frac{1}{(\nabla^2 + k^2)} V \right) \langle k \mid \psi \rangle. \quad (15)$$

Now the LHS is exactly the scattering amplitude according to equation (13). The conspicuous product $\frac{2\mu}{4\pi \hbar^2} V \psi$ inside the right-most integral is nearly the scattering amplitude, but it is not directly projected into momentum space. It can be turned into the scattering amplitude by inserting a complete set of outgoing spherical plane wave states $\mid q \rangle = \frac{e^{i p r}}{r}$:

$$f(k) = - \frac{2\mu}{4\pi \hbar^2} \langle k \mid V e^{i p z} - \frac{2\mu}{\hbar^2} \int \frac{d^3q}{(2\pi)^3} \left( k \mid V \mid q \rangle - \frac{1}{(\nabla^2 + k^2)} \right) \langle q \mid V \rangle \frac{2\mu}{4\pi \hbar^2} \langle q \mid \psi \rangle. \quad (16)$$

These spherical waves $\langle q \mid$ are eigenstates of the Laplacian with eigenvalue $-q^2$, so $\nabla^2 \rightarrow -q^2$ in the denominator of the right-most term. The exponential $e^{i p z}$ in the middle term is just the plane wave state $\mid p_o \rangle$:

$$f(k) = - \frac{2\mu}{4\pi \hbar^2} \langle k \mid V \mid p_o \rangle + \frac{2\mu}{\hbar^2} \int \frac{d^3q}{(2\pi)^3} \langle k \mid V \mid q \rangle \frac{1}{(-q^2 + k^2)} f(q). \quad (17)$$

This is the Lippmann-Schwinger equation for the two-body T-matrix, up to an overall constant. It can be viewed as either a Born approximation equivalent of the scattering amplitude or a perturbation theory for the perturbed incident plane wave, where the integral over $q$ sums the contributions of all possible virtual excitations.

It will be shown that if the potential can be written as a product of separable terms in momentum space, or in other words if the potential takes the form

$$\langle k \mid V \mid k' \rangle = g(k) \lambda g(k'), \quad (18)$$

then (17) can be solved to all orders in a closed form, where $\lambda$ is a coupling constant and $g$ is a unitless form factor. The first limitation of requiring this of the potential is that we are ostensibly ignoring many possibly adequate non-separable potentials that cannot be written in the form of (18). However, it has been shown and
reviewed by Harms [8] and Shepard/McNiel[9], respectively, that any potential can be expanded as a series of separable terms. Additionally, Tabakin shows that a single separable potential term can include both attractive and repulsive elements [2]. The second limitation is that equation 18 is only valid for S-waves because it does not include a centrifugal barrier. The partial wave decomposition and the case of arbitrary angular momentum is discussed at the bottom of this section.

**Infinite Order Born Approximation**

With this (l=0) separable potential in place, the scattering amplitude (from 17) becomes

\[
f(k) = -\frac{2\mu}{4\pi\hbar^2} \sum \frac{g(k) \lambda g(p_o) + g(k) \lambda^2 g(p_o)}{(2\pi)^3} \sum \frac{2\mu}{\hbar^2} \sum \frac{1}{(q^2 + k^2)^2} f(q)
\]

\[
= -\frac{2\mu}{4\pi\hbar^2} \sum \frac{g(k) \lambda g(p_o) + g(k) \lambda^2 g(p_o)}{(2\pi)^3} \sum \frac{2\mu}{\hbar^2} \sum \frac{1}{(q^2 + k^2)^2} f(q)
\]

(19)

Since \(f\) appears on both sides of the equation, it is interesting to plug the entire RHS into \(f\) inside the integral.

This can be done repeatedly ad infinitum, but with just a few iterations, the resulting expression is:

\[
f(k) = -\frac{2\mu}{4\pi\hbar^2} \left( \sum g(k) \lambda g(p_o) + \frac{g(k) \lambda^2 g(p_o)}{(2\pi)^3} \sum \frac{2\mu}{\hbar^2} \sum \frac{1}{(q^2 + k^2)^2} g(q) + \right.
\]

\[
\left. g(k) \lambda^3 g(p_o) \sum \frac{2\mu}{(2\pi)^3} \frac{1}{(q^2 + k^2)^2} g(q) + \frac{1}{(q^2 + k^2)^2} g(q) \sum \frac{2\mu}{(2\pi)^3} \frac{1}{(q^2 + k^2)^2} g(q') + \right.
\]

\[
\left. \ldots \right)
\]

With the substitution

\[
x = \int \frac{d^3q}{(2\pi)^3} \frac{2\mu}{\hbar^2} g(q)^2 \frac{1}{(q^2 + k^2)^2},
\]

(20)

it can be seen that \(f(k)\) forms the geometric series

\[
f(k) = -\frac{2\mu}{4\pi\hbar^2} g(k) \lambda g(p_o) \left( 1 + \lambda x + (\lambda x)^2 + (\lambda x)^3 + \ldots \right) = -\frac{2\mu}{4\pi\hbar^2} g(k) \lambda g(p_o) \left( \frac{1}{1 - \lambda x} \right)
\]
which converges as long as \( x \) is less than 1. \( p_i \) and \( k \) are the incident and final momenta, respectively, which are the same in magnitude. Thus,

\[
    f(k) = \frac{-2 \mu}{4 \pi \hbar^2} \frac{g(k)^2 \lambda}{1 - \lambda x}.
\]  

(21)

This, finally, is the exact form of the \( l=0 \) scattering amplitude with Born corrections of all orders. This expression will allow us to relate the phase shifts to any potential that can be written in the form of equation (18). These phase shifts and their relation to \( f \) is the topic of the next section, but there are two outstanding objects that first need discussing.

**Non-zero Angular Momentum**

First, I should at least make mention of the other angular momentum channels. For arbitrary angular momentum, the potential (18) and scattering amplitude (21) generalize to

\[
    \langle k \mid V \mid p \rangle = g(k) \lambda g(p) (2 l + 1) P_l(\cos \theta) \quad \text{and} \quad f(k, l) = \frac{-2 \mu}{4 \pi \hbar^2} \frac{g(k)^2 \lambda}{1 - \lambda x} (2 l + 1) P_l(\cos \theta),
\]

respectively, where \( x = \int \frac{d^3 q}{(2 \pi)^3} \frac{2 \mu}{\hbar^2} \frac{g(q)^2}{(-q^2 + k^2)} \). \( P_l \) is the \( l \)th Legendre polynomial, and \( (2l + 1) \) is a degeneracy factor. The total scattering amplitude, \( \mathcal{F}(k) \), is then the sum over all \( l \) of \( f(k,l) \): \( \mathcal{F}(k) = \sum_l f(k,l) \). The total potential is summed similarly.

It is important to ask how many angular momentum channels need to be considered in the NN interaction. According to Tabakin, eigenstates involving S, P, D, G, F, and I channels should all be considered[1]. For now, I will just state that higher angular momentum channels play increasingly smaller roles in fundamentally low-energy phenomena, though I will not be equipped to justify this reasoning until the form of the potential is specified, which brings me to the second object that needs discussing, the form factor \( g(q) \).

**Potential Form Factor**

We are now in a position to state the criteria by which the unitless potential form factor \( g(q) \) is chosen.
In one sense, \( g(q) \) defines the shape of the effective potential. In another sense, \( g(q) \) cuts off contributions from excitations above a certain range of momenta, enforcing the assumption that low energy phenomena do not depend on the high energy physics \([9][5]\). This is essentially the criteria put in place to resolve quantum field renormalization issues. In a typical perturbation theory, large virtual excitations are suppressed by the off-shell energy denominator, which in our case is \((k^2 - q^2)\). The more off-shell the excitation, the larger this denominator becomes, so very large excitations are suppressed as expected. In the case of equation (20), however, \( x \) would diverge on its own, even with the mentioned suppression effects. Physically, we know \( x \) can't diverge, because scattering cross sections are observed to be non-zero, so a form factor \( g(q) \) that further suppresses large excitations is affixed to the potential. In short, \( g(q) \) must be selected to ensure the convergence of \( x \) as defined by equation (20), which assures the scattering amplitude in (21) is non-zero. To achieve this end, \( g(q) \) must be well-behaved at the origin\(^3\), and it must converge to zero at a sufficiently rapid rate.

Moszkowski and Scott (M&S) used a potential with one repulsive and one attractive Dirac delta function placed adjacent to each other, followed by a finite square well. This repulsive core/attractive tale combination affords adequate phase shift replicability in the low energy regime \([3]\); however M&S suggest that a more realistic, Yukawa-shaped potential be used for more detailed calculations. Based on this suggestion, two "realistic" effective potential form factors are presented. The first is the Yukawa Klein-Gordon propagator and the second is an alternative form factor:
Looking at figures 2a and 2b, it is apparent that these form factors do not diverge at the origin, they peak at low momentum, and they sharply fall off toward zero for large q. It will be seen that they do indeed ensure the convergence of x. Note that I have added a negative sign to the plots of g(q), stressing the fact that the effective potential is attractive.

We are now in a position to address the question of how many angular momentum channels need to be considered. The presence of angular momentum adds a centrifugal barrier \( \frac{\hbar (l+1)}{r^2} \) to the NN Hamiltonian, which dominates the potential in (18) at low energies and large distances. Only higher energy incident waves can penetrate the centrifugal barrier and reach the range of the effective potential, but the higher momenta q are screened off by the form factors g(q), so one can expect higher angular momentum channels to play a decreasing role in the effective interaction. The S-wave provides the largest contribution to the scattering amplitude, and it is hoped that only a few of the lowest channels need to be considered. This is why I have been and will continue to focus the analysis on the \( l = 0 \) channel, but periodically quote the generalized momentum version alongside.
The S-wave channel is also unique because it is the only channel for which there exists a bound state of the NN system, known as the deuteron. An inspection of this bound state will be made in section 2.3.

As for the coupling constant $\lambda$ and cut-off parameter $\beta$, they determine the depth and range of the potential. Their values are to be determined by the phase shift information, the topic of the next section.

1 The second order term in (9) falls off like the first order term of (10). We keep orders up to $1/r$.

2 Note that when $l$ is assumed to be zero, the "$l$ " in the argument off $f$ is suppressed.

3 At the origin, $g(k)^2$ must not diverge more rapidly than $k^{-2l+3}$, if at all. We focus on the no divergence case.

2.3 From Scattering Amplitude to Phase Shifts

Some Scattering Theory

Scattering theory shows that the scattering amplitude and the phase shifts are related by the following expression:

$$f(k, l) = \frac{1}{2i k} (2l + 1) (e^{2i\delta} - 1) P_l(cos\theta)$$

(see Schiff equation 19.11) [7].

Reorganizing (23) and taking the inverse gives

$$f(k, l) = (2l + 1) \frac{1}{2i k} (e^{i\delta} - e^{-i\delta}) e^{i\delta} P_l(cos\theta) = (2l + 1) \frac{1}{k} (\sin \delta) e^{i\delta} P_l(cos\theta)$$

$$= (2l + 1) \frac{1}{k} \frac{\sin \delta}{\cos \delta - i \sin \delta} P_l(cos\theta)$$

$$f(k, l)^{-1} = \frac{1}{P_l(cos\theta)(2l+1)} (k \cot \delta - i k).$$

(24)

Letting $l=0$, we have
Equating the real part of (25) with the real part of the inverse of (21) relates the phase shifts directly to the form factors:

$$\text{Re}[f(k)^{-1}] = \text{Re}[k \cot \delta - i k] = k \cot \delta = \text{Re} \left[ \frac{-4 \pi \hbar^2}{2 \mu} \frac{(1 - x \lambda)}{g(k)^2 \lambda} \right].$$  \hfill (26)

We find that this expression can be fit to experimental phase shifts, optimizing over \(\lambda\) and \(\beta\). After the parameters have been determined, our expression for the potential will be complete. We will look at these phase shifts now.

**Nijmegen Phase Angles**

Below are plots of the 1S0 and 3S1 NN scattering phase angles for energies up to 300 MeV.

![Phase Angle (degrees)](image1)

2.6 a. 3S0 channel NN  
2.6 b. 3S1 channel

Figure 2.6) Nijmegen scattering angles for both S-wave channels up to 300 MeV in the lab frame. Credit U. Nijmegen II [11].

In order to directly compare these phase shifts to equation (26), these plots need to be transformed from \(\delta\) to \(k \cot \delta\), and the energies need to be transformed into the lab frame (the energy transformation is simply the division by 2). Transformed phase shifts are shown in figure 2.7 below:
These phase shifts and equation (26) are now compatible for a best fit of the parameters, but a couple of observations about the plots in figure 2.7 can first be made. Scattering theory shows that $k\cot\delta$ can be expanded in powers of $k^2$, in a series known as the effective range expansion:

$$k\cot\delta = -1/a + (1/2\ r_o)\ k^2 + \nu\ k^4 + ...$$

where "a" in the constant term is the scattering length. The scattering length can thus be read off the plots in figure 2.7 as the negative reciprocal of the y-intercept. Apparently the 1S0 channel has a close to infinite scattering length, while 3S1 has a small, positive scattering length. A positive scattering length implies a bound state, so the 3S1 spin triplet (isospin singlet) is expected to exhibit binding at some energy. This is consistent with the known properties of the deuteron (proton-neutron bound state), which is the isospin singlet of the NN system. This bound state will be further examined after the phase shift fits have been made.

**Fitting to the Phase**

Taking $g(k)$ to be the Yukawa form factor, $x$ is found to be

$$x = \int \frac{d^3q}{(2\pi)^3} \frac{2\mu}{\hbar^2} \frac{\beta^4}{(\beta^2+q^2)} \frac{1}{(-q^2+k^2)} = \frac{\beta^3\mu}{4\pi (k+i\beta)^2} \frac{1}{\hbar^2}.$$ 

Inserting this $x$ into the RHS of (26), taking the real part, and collecting terms gives:
\[
(k \cot \delta)_{Yuk} = \Re \left[ -\frac{4 \pi \hbar^2}{2 \mu} \frac{(k^2 + \beta^2)^2 (4 \pi (k + i \beta)^2 \hbar^2 - \beta^3 \mu \lambda)}{\beta^4 4 \pi (k + i \beta)^2 \hbar^2} \right]
\]

\[
= -\left( \frac{\beta}{2} + \frac{2 \pi \hbar^2}{\lambda \mu} \right) + k^2 \left( \frac{1}{2 \beta} - \frac{4 \pi \hbar^2}{\beta^2 \lambda \mu} \right) - k^4 \frac{2 \pi \hbar^2}{\beta^4 \lambda \mu}
\]

This equation relates \( k \cot \delta \) directly to the Yukawa-form potential's parameters. The alternative potential is related similarly:

\[
(k \cot \delta)_{Alt} = -\frac{3 (5 \beta \lambda \mu + 64 \pi \hbar^2)}{16 \lambda \mu} - \frac{1}{k^2} \frac{\beta^4 (5 \beta \lambda \mu + 32 \pi \hbar^2)}{16 \lambda \mu} - \frac{1}{k^2} \frac{\beta^2 (5 \beta \lambda \mu + 128 \pi \hbar^2)}{16 \lambda \mu} - k^2 \frac{(-5 \beta \lambda \mu + 128 \pi \hbar^2)}{16 \beta^2 \lambda \mu} - k^4 \frac{2 \pi \hbar^2}{\beta^4 \lambda \mu}
\]

A graphical look at \( k \cot \delta \) is often more illuminating than an analytic expression. A best fit of \( (k \cot \delta)_{Yuk} \) and \( (k \cot \delta)_{Alt} \) to the experimental phase shifts \( (k \cot \delta)_{Exp} \) up to 2.2 \( \text{fm}^{-2} \) is shown below for the 1S0 and 3S1 channels, respectively. The reduced mass \( \mu \) used was 469.5 MeV/\( c^2 \), and every \( \lambda \) absorbed a \( 1 / \hbar^2 \).

![Figure 2.8: Singlet 1S0 channel kcotδ fit by the two sample potentials individually up to 180MeV (lab). Nijmegen phase shifts are in black.](image-url)
Although chi-squared values have not yet been computed, we can see that the potentials give fairly adequate fits at energies above .5 fm$^{-2}$. At low energy neither potential's fit appears entirely faithful to the experimental phases, though the Yukawa-based potential is noticeably truer. For a first test of these potentials, we can examine the 3S1 channel for the deuteron bound state.

By "Yukawa" potential, I mean the potential $V = g\lambda g$, where $g$ is the Yukawa function given in eq (22).

### 2.3 Testing the Potential

#### Finding the Deuteron

The theory of scattering tells us that the scattering amplitude will exhibit a pole at the energy of a bound state. Using this criteria to hunt for the deuteron bound state, we can examine either the case in which (21) diverges or (25) vanishes. Both are straightforward, but the latter is a little easier. For the Yukawa-based potential, we need to find some $k$ for which equation 27 (minus $ik$) goes to zero:
Although chi-squared values have not yet been computed, we can see that the potentials give fairly adequate fits at energies above 0.5 fm\(^{-2}\). At low energy neither potential's fit appears entirely faithful to the experimental phases, though the Yukawa-based potential is noticeably truer.

For a first test of these potentials, we can examine the \(3S_1\) channel for the deuteron bound state.

By "Yukawa" potential, I mean the potential \(V = gV_{\text{Yukawa}}\), where \(g\) is the Yukawa function given in eq (22).

\[
-\left(\frac{\beta}{2} + \frac{2\pi}{\lambda \mu}\right) + k^2 \left(\frac{1}{2\beta} - \frac{4\pi}{\beta^2 \lambda \mu}\right) - k^4 \frac{2\pi}{\beta^4 \lambda \mu} - ik = 0
\] (28)

where \(1/\hbar^2\) has been absorbed into every \(\lambda\). The only solutions to (28) with the parameters listed in figure 2.9 are extraneously large in absolute value, around -0.59 fm\(^{-2}\), corresponding to a binding energy of around 25MeV. What's more, with slightly different energy range fits, the new solutions differ sporadically from this number. This looks very bleak; however, it is found that if the fit is restrained to energies of 50MeV(lab) and lower, the solutions are binding energies consistently in the range of the known deuteron binding energy, \(-2.224\text{ MeV} (-0.0536324\text{ fm}^{-2})\). A low energy fit is shown below, as well as a plot of the scattering amplitude for negative energies.

Figure 2.10: A low energy fit of the Yukawa-form potential up to 50 MeV (lab) in the spin triplet channel.
The pole of $f$ in the 3S1 channel is the deuteron bound state, and it is represented by the x-intercept in figure 2.11.

The bound state calculation is robust to energy range modifications within the 50 MeV upper limit. Calculations in this energy range lie within a few percent of the known deuteron binding energy. For higher energy fits to agree with experimental binding energies, a higher rank potential can be used, with a greater number of fitting parameters, but these findings demonstrate the merit of a simple 2-parameter potential. We are now in a position to apply our potentials to a calculation of infinite nuclear matter, staying focused on the S-wave contributions.

3. Infinite Nuclear Matter

Assumptions of Nuclear Matter

Consider a nucleus, infinite in spacial extent, composed of an equal and infinite in number protons and
neutrons, held at absolute zero temperature. This degenerate Fermi sea is a model known as infinite nuclear matter. Though highly idealized, it provides insights into the stability of nucleon systems. Using phase-fit potentials, Tabakin performs a calculation of energy per particle of infinite nuclear matter, as well as a calculation of the Fermi energy using a first order perturbation theory in Fock space[1]. In this section, I will follow Tabakin's general procedure, but using the Yukawa-form fit potential developed in section 2.

The first benefit of considering infinite nuclear matter is that its infinite size and uniform density imply spacial invariance, which gives an eigenbasis in k-space. This means the unperturbed eigenstates are plane waves, and the unperturbed degenerate Fermi gas is a free Fermi gas. The perturbation, in our case, is governed by the two-particle interaction.

The second benefit of Since all states in the degenerate Fermi sea are occupied, there are only two allowed interactions for a pair of nucleons: they can interact and swap their quantum states, or they can interact and resume their original states; other possibilities are prevented by Pauli blocking. The first order interaction of the system is simplified to the aggregation of all possible 2-body exchange contributions. This aggregation is given by the second term of the following expression, which is derived in a similar fashion to Fetter & Walecka's equation 3.34 [10] as well as Tabakin's equation 3.4 [1]:

\[
E = \frac{3}{5} \frac{\hbar^2 k_f^2}{2m} N + \text{Volume} \sum_p \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \langle q | V | q \rangle \Theta(k_f - |p_1|) \Theta(k_f - |p_1 + 2q|).
\]  

(29)

The first term is the unperturbed free energy \( E(0) \), and the second term is the Hartree-Fock correction of a perturbed free Fermi gas \( E(1) \), with the usual definition of the momentum separation, \( 2q = p_2 - p_1 \). \( \Theta \) is the step function:

\[
\Theta(x) = \begin{cases} 
1 & \text{for } x > 0 \\
0 & \text{otherwise}, 
\end{cases}
\]

which enforces the requirement that both particles be inside the Fermi Sea. The volume term can be written in terms of \( \frac{N}{\rho} \), where \( \rho \) is the number density of nuclear matter, and Tabakin shows that the density can be written in
terms of the Fermi momentum:

\[ \rho = \frac{2 k_f^3}{3 \pi^3}. \]  

(30)

\( \eta \) is the degeneracy index,

\[ \sum_{\eta} \rightarrow \sum_{T,J} (2 T + 1) (2 J + 1) \rightarrow \sum_{T,S} (2 T + 1) (2 S + 1) \rightarrow \sum_{S} 3. \]  

(31)

Where \( J=S+L=S \) for the S-wave. Combining these results and dividing by \( N \), the energy per particle is written,

\[ \frac{E}{N} = \frac{3}{5} \frac{\hbar^2 k_f^2}{2 m} - \frac{3 \pi^2}{2 k_f^3} \sum_{S} 3 \int \frac{d^3 p_1}{(2 \pi)^3} \frac{d^3 q}{(2 \pi)^3} \langle q \mid V \mid q \rangle \Theta(k_f - |p_1|) \Theta(k_f - |p_1 + 2 q|). \]

(32)

Defining the combined total momentum \( P = p_1 + p_2 \) and substituting allows for the overlap function to be "symmetrized":

\[ \frac{E}{N} = \frac{3}{5} \frac{\hbar^2 k_f^2}{2 m} - \frac{3 \pi^2}{2 k_f^3} \sum_{S} 3 \int \frac{d^3 p}{(2 \pi)^3} \frac{d^3 q}{(2 \pi)^3} \langle q \mid V \mid q \rangle \Theta(2 k_f - |P - 2 q|) \Theta(2 k_f - |P + 2 q|). \]

Now the integral over \( P \) is simply the intersection volume of two spheres of radius \( 2k_f \) that are spaced \( 4q \) apart.

This is pictured below as the projection of a hyper Venn diagram.

---

Figure 3.1: Overlap volume of two Fermi spheres is the volume over which \(|P+2q|\) and \(|P-2q|\) are both less than twice the Fermi
The overlap volume of these spheres is given by 
\[ V = \frac{32\pi k_f^3}{3} \left( 1 - \frac{3q}{2k_f} + \frac{q^3}{2k_f^3} \right), \]
as long as \( q \) is less than the Fermi radius \( (q < k_f) \). Thus, our expression for the Energy becomes

\[ \frac{E}{N} = \frac{3}{5} \frac{\hbar^2 k_f^2}{2\mu} + \frac{3\beta^2}{2k_f^3} \frac{1}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \langle q | V | q \rangle \frac{32\pi k_f^3}{3} \left( 1 - \frac{3q}{2k_f} + \frac{q^3}{2k_f^3} \right) \Theta(k_f - q) \] (33)

The Yukawa-form potential is inserted into the integral, \( \frac{1}{\hbar^2} \) is once again extracted from \( \lambda^6 \), and the right term bubbles out the following expression:

\[ \frac{E(1)}{N} = \sum_S 3 \frac{\hbar^2 \lambda}{\pi^2} \int d^3q \frac{\beta^4}{(\beta^2 + q^2)^2} \left( 1 - \frac{3q}{2k_f} + \frac{q^3}{2k_f^3} \right) \Theta(k_f - q) = \]

\[ \sum_S 3 \frac{\hbar^2 \lambda}{\pi^2} \frac{k_f^3\beta^3}{4k_f^3} \left( \frac{2}{\beta^2 + k_f^2} \arctan \left( \frac{k_f}{\beta} \right) + \beta \left( 2 - 3 \log \left( 1 + \frac{k_f^2}{\beta^2} \right) \right) + 2 \beta^6 \log \left( \frac{\beta^2}{k_f^2 + \beta^2} \right) \right) \] (34)

The sum includes the 3S1 and 1S0 contributions, each of which is shown below for low fits of the Yukawa potential.

Figure 3.2: Energy/particle contributions of both S-wave eigenstates are shown for low-energy fits of the Yukawa-form potential. The 3S1 channel is more negative and thus more attractive, as expected from this bearer of the bound state.
Figure 3.3: The binding energy per nucleon vs the Fermi momentum with S-wave contributions. The saturation (minimum) is predicted at 3 fm\(^{-1}\) at 40 MeV per particle.

For comparison, Tabakin calculated a Fermi momentum of around 1.8 fm\(^{-1}\) with energy per particle in the range of -8 to -14 MeV, while Krewald and Epelbaum calculate saturation around 1.3 fm\(^{-1}\) and -16 MeV per particle using an effective field theory involving regulators [6]. Their calculations consider several higher angular momentum channels, so an analysis of P and D wave contributions is certainly the next step foreword. The fact that the S-waves give such a high saturation is due to their very attractive nature. This leads one to suspect that higher angular momentum channels give repulsive contributions on average. The most encouraging aspect of this picture is that saturation of nuclear matter is predicted at all--that Fig 3.3 has a relative minimum for some Fermi momentum. Grygorov states, the "absence of saturation is one of the main problems in calculations of nuclear matter" [5].

\(^6\) As indicated above fig 2.8, \(1/\hbar^2\) was absorbed by \(\lambda\) for ease of fitting purposes.
Final Conclusion

Low energy phase fits of the 2-parameter Yukawa potential predicted satisfactory quantitative binding energy of the deuteron, and satisfactory qualitative pictures of nuclear matter saturation. The qualitative accuracy of this procedure is encouraging, especially considering that only the l=0 channel was accounted for. The lack of precision of these final calculations strongly suggest that more channels need consideration. We would do well to examine the P and D wave contributions, as well as develop a procedure for a higher rank potential.

Other future work that I am interested in includes calculation of the symmetry energy of nuclear matter, as well as a Hartree-Fock calculation of the Shell-model spectrum, as Tabakin suggested. Finally, a configuration-space wavefunction of the deuteron would be a fun calculation because it would marry the programs I wrote to calculate wavefunctions numerically with the work I have come to enjoy more recently.
Works Cited

Data:
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Acknowledgement: It is a pleasure to thank James R. Shepard for his extensive help and guidance.