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Mechanisms for Fracton Phases

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Mechanisms for Fracton phases

by

Han Ma

B.S., Fudan University, 2011
M.S., Fudan University, 2014

A thesis submitted to the
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written by Han Ma
has been approved for the Department of Physics

Prof. Michael Hermele

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Date ________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
Strongly correlated many-body systems provide a platform for novel phases of matter where constituent particles organize themselves in a variety of ways. At low temperature, these particles interact quantum mechanically and generate entanglement building up exotic quantum phases, such as topological order, where there can be emergent excitations which cannot be created locally. Such excitations, if gapped, are also called topological excitations.

Fracton is one of such gapped point-like topological excitation in three dimensional system. Different from conventional topological excitation, it is immobile and was firstly discovered in exact solvable models exhibiting fracton topological order. This new order has sub-extensive topological ground-state degeneracy and generically also possesses other mobile excitations restricted to move in sub-dimensional spaces. Also, it has been noticed that the charges of symmetric-tensor $U(1)$ gauge theories can be fractons. Its immobility is due to the existence of multiple conservation laws.

In this thesis, I will present the relation among the gapped fracton topological order, gapless fracton phase described by $U(1)$ symmetric tensor gauge theories, and ordinary topological ordered phases. Particularly, the fracton topological order exhibited in an exact solvable model called X-cube model can be constructed by coupling toric code layers. The mechanism leads to fracton topological orders is dubbed “p-string condensation” or “p-membrane condensation,” in which strings or membranes built up from particle excitations from layers of topological orders condense. This allows the fusion properties, braiding statistics, and ground-state degeneracy of the resulting fracton order to be easily understood in terms of more familiar degrees of freedom. And the fracton topological order in the X-cube model can also be obtained from a particular rank-2 symmetric tensor gauge theory called scalar charge theory by a partial confinement transition followed by Higgs mechanism removing the gapless photon modes.
Dedication

To my parents.
Acknowledgements

I am indebted to my teachers, colleagues, friends and family for all their help and support.

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5.4 The EBC quantum critical point between two conventional Bose condensates gives rise to a finite temperature EBC phase. For small nonzero $|\kappa|$, the EBC exists as an intermediate phase between the superfluid and disordered phases.

5.5 Terms in the critical Hamiltonian of boson $e^{i\phi}$ on the honeycomb lattice. $\hat{\tau} = \frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y}$ and $\hat{\tau} = -\frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y}$.

5.6 The boson field $\phi$ lives on the sites of the honeycomb lattice. The three diagonal components of the tensor field $E_{ij}$ live on the sites of the triangular lattice (center of hexagons of the honeycomb lattice), while the off-diagonal components live on the links of the triangular lattice (links of the honeycomb lattice).

5.7 Possible charge configurations created by $E_{ij} \rightarrow E_{ij} + 1$. Two vector charges are created by a diagonal element while four charges are created by an off-diagonal element.

5.8 Three rhombus terms represent the Gauss law in the vector charge theory.

5.9 The gauge invariant is a summation of twelve $A_{ij}$ variables including six off-diagonal components and six diagonal components. The minus sign in front of the variable indicates its sign in the summation for the gauge invariant.
Chapter 1

Introduction

1.1 Phases of matter

In twentieth century, the understanding of phases of matter gains a qualitative leap since the proposal of Landau’s symmetry breaking paradigm[46]. It was proposed by Landau that phases of matter can be classified by their symmetries. Between two phases with different symmetries, there must be a phase transition. If the symmetry groups of the phases have group-subgroup relation, the transition between them is predicted to be continuous transition. Otherwise, the transition is expected to be of first order.

A prominent example for Landau’s theory is the Ising model[72][64]. It is a classical model of $Z_2$ spins on the sites of a $d$ dimensional lattice. At high temperature, the system is in a disordered state due to large thermal fluctuation. While at low temperature, all spins tend to align spontaneously to reduce the energy cost. As the temperature decreases, from a symmetric phase, the system goes into a symmetry breaking phase with $Z_2$ symmetry spontaneously broken. It is a continuous phase transition. Local order parameter characterizes such a transition. If there exists a magnetic field breaking the $Z_2$ symmetry explicitly, then there is no such transition. Similar analysis can be applied to other system with more complicated symmetry group. Without knowing any microscopic detail of the system, the nature of the phases and phase transitions can be grasped quickly and correctly as long as the information of the symmetry of each phase is accessible. Landau’s theory is remarkably powerful so that phases of matter are demystified gradually, until the discovery of a new type of phase of matter[117].
It was revolutionary that a phase beyond Landau’s symmetry breaking paradigm, called fractional quantum Halls[104] is discovered. It is a symmetric and gapped phase. However, it gives a richer story far more than our imagination about a disordered and featureless phase. In 1982, Tsui, Stormer, and Gossard studied 2D electron gas in the extreme quantum limit[106]. The spin-polarized Landau level is found to be partially filled resulting in the plateaus of Hall resistivity[106] at fractional Landau-level filling factor resulting in the theoretical proposal of the excitation carrying fractional charge[47, 32, 130, 131]. Also, the statistics obeyed by those excitations is neither Bose-Einstein nor Fermi-Dirac statistics. The non-trivial phase factor generated by exchange of two fractional particles is $e^{i\theta}$ with $0 < \theta < \pi$. This leads to the concept of anyons with interesting braiding and fusion properties[119, 68], which is studied extensively later and open up an avenue in the condensed matter physics.

1.2 Gauge theory and topological phases

The order in the fractional quantum Hall is later named topological order[117]. Topological order is a new characterization of quantum phase of matter. This concept provides a concrete scheme of emergent phenomena, which claims that in some sense the physics of a system is more than the sum of each ingredient, showing the complexity in the strongly interacting many-body systems. Because of the emergent phenomena, in a locally interacting bosonic system, fermions, such as electrons and quarks, and gauge bosons, such as photons and gluons, can emerge as quasiparticles even though these particles are not elementary particles in the system.

Quantum phases of matter in $d$ spatial dimensions are said to have topological order[113, 114, 116, 118] when they have an excitation gap, exhibit degenerate ground states on the torus (or other topologically nontrivial manifolds) that cannot be distinguished by local measurements, and support excitations that can be localized in space but cannot be created by a local process.\footnote{More precisely, we are describing non-invertible topological orders, a class that excludes symmetry-protected topological phases such as the $S = 1$ Haldane chain and topological band insulators, as well as integer topological phases such as the integer quantum Hall liquids. We note that in some cases the topological excitations are not point objects, but can be localized to a $c$-dimensional subspace where $c < d.$} Above
the ground state, there are gapped excitations. Among them, fractionalized excitations are also signatures of topological orders. These excitations are distinct from local elementary excitations. Their fusion and braiding give a complete characterization of the underlying topological order.

In $d = 2$, there is by now a good understanding of topological order, ranging from its realization in fractional quantum Hall liquids[106, 47] and in bosonic models,[85, 86, 115, 87, 2, 93, 60, 59, 1, 42, 50] to topological quantum field theories such as Chern-Simons theory,[120] to the general framework of modular tensor category theory that describes topological orders in bosonic systems.[43]

1.2.1 $Z_2$ topological order

The simplest topological order is the $Z_2$ topological order in two dimensions. A simple exact solvable model called the toric code would give us all the physics of this order[42]. It is a $Z_2$ spin model defined on the links of two dimensional square lattice, for simplicity. The Hamiltonian of this model is

$$H_{TC} = - \sum_v A_v - \sum_p B_p = - \sum_v \prod_{l \in v} Z_l - \sum_p \prod_{l \in p} X_l, \quad (1.1)$$

where, as shown in Fig. 1.1, $B_p$ is the product of the four $X$ operators surrounding the plaquette $p$, and $A_v$ is the product of the four $Z$ operators connected to vertex $v$. $X$ and $Z$ operators are usual Pauli operators with commutation relation $[X_l, Z_{l'}] = 2iY_l \delta_{ll'}$.

It is obvious that every $A_v$ commutes with every $B_p$. So it is easy to determine that a ground state $|\psi\rangle$ satisfying $A_v|\psi\rangle = B_p|\psi\rangle = |\psi\rangle$. Immediately, we can point out that $|\psi\rangle$ cannot have any classical spin order because of this condition. Nevertheless, this disordered $|\psi\rangle$ has non-trivial topological properties.

If the system is on a plane, there is no other operator commuting with the Hamiltonian, besides those which are products of $A_v$ and $B_p$. However, this is no longer true if we put the system on a torus, which is topologically non-trivial. As shown in Fig. 1.2, there are non-contractible loop operators $L_{O,j}$ with $O = X, Z$ and $j = 1, 2$ which are products of $X$ or $Z$ operators along the loops on the torus[21]. These non-local operators commute with the Hamiltonian. Being eigenstate of
Figure 1.1: Toric code model on the square lattice. Red dots are spin degrees of freedom on each link while blue links forming plaquette and vertex represent the plaquette terms ($B_p$) and vertex terms ($A_v$).

$L_{Z,j}$, on this closed manifold, we can have four possible $|\psi\rangle$ with $L_{Z,j}|\psi\rangle = \pm|\psi\rangle$. This indicates the four fold ground state degeneracy on the torus. This is a sign of topological order.

Still, if we write $|\psi\rangle$ as an eigenstate of any $Z$ operator, then for a particular vertex, there are eight possible configurations according to the value of spins on the links connecting to this vertex as shown in Fig. 1.3. For any $v$, since $A_v$ would have eigenvalue $+1$, there are always even number of links with spin $+1$ (also spin $-1$). By connecting all these links with spin $+1$, we would obtain configurations of closed loops. Starting from one of the allowed spin configurations in $|\psi\rangle$, acting $B_p$ operators locally modifies the closed loop configuration but it doesn’t break any loop. Therefore, we can conclude that $|\psi\rangle$ is a superposition of closed loop configuration. Equivalently, we can say that the ground state of this topological order is a loop condensate[50].

Above the ground state, there are various types of excitations. By acting one $X$ ($Z$) operator, excitations called $m$ ($e$) particles are created. It is straightforward to see that these excitations are always created in pair meaning each of these excitations cannot be created by local operations. A more interesting fact is that although $e$ and $m$ particles are bosons since the braiding between themselves generates a trivial topological phase factor, the braiding between $e$ and $m$ particle gives a non-trivial topological phase factor $e^{i\pi}$ indicating they have mutual semionic statistics. This never happens for the elementary particles in the system, which are local bosons. We thus also call the $e$
and $m$ particles anyons, like what we have in the fractional quantum Hall effect. The existence of anyons is an emergent phenomenon called fractionalization. We can check there are only four types of excitations in this system: $1$, $e$, $m$ and $\epsilon = em$, where $1$ is the local bosonic excitation braiding any other excitation trivially. Those anyons have simple fusion rules: $e \times e = m \times m = \epsilon \times \epsilon = 1$ and $e \times m = \epsilon$ leading to $\epsilon \times e = m$ and $\epsilon \times m = e$. Together these properties of excitations with the ground state degeneracy on the torus, this particular topological order is called the $Z_2$ topological order.

This model and the $Z_2$ topological order can be generalized in various different ways. For example, the underlying lattice of the model can be changed to arbitrary graph in two dimensions without changing any low energy physics. In addition, we can also construct the same model on three dimensional cubic lattice. Its ground state exhibits $Z_2$ topological order as well, but in three dimensions. The ground state degeneracy is certainly increased if the system is put on the 3-torus due to the extra non-contractible loop operators. Generally speaking, the degeneracy depends on the topology of the manifold. On a manifold with $q$ genus, the ground state degeneracy of $Z_2$ topological order can be expressed as $4^q$[42]. From another perspective, the wave function of two dimensional topological order can also be generalized from closed loop condensation to the string-net condensation[50, 49]. By considering arbitrary string-net, we can obtain many other topological
orders with anyons characterized by different fusion and braiding. These bosonic topological orders can have a gapped edge and are classified by unitary modular tensor categories[118].

1.2.2 Topological entanglement entropy

In a parallel line of development, ideas from quantum information have been transplanted to the field of correlated systems, providing potent methods with which to characterize and study complex quantum many-body systems. Foremost among these new tools is the use of quantum entanglement – specifically, its characterization in terms of entanglement entropy — in developing the understanding of topological order (see e.g. Ref. [27] for a review). For two-dimensional gapped phases, the entanglement entropy contains a universal subleading ‘constant’ term [44, 49]. This is intimately related to the TQFT description of the topological phase and provides a partial characterization of the nature of topological order.

We here review some essential facts about entanglement entropy, focusing in particular on topological contributions and how to extract them. We will consider a system in a topological phase, with $|\Psi\rangle$ one of its degenerate ground states. We divide the system into a small subregion $A$ and ‘everything else’ ($B$), and construct the reduced density matrix $\rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi|$. The entanglement entropy is then defined to be the von Neumann entropy of $\rho_A$,

$$S = -\text{Tr}_A \rho_A \log_2 \rho_A,$$

(1.2)

(Here and throughout, we measure logarithms in base 2 to remove unwieldy factors of ln 2.) In a gapped phase — such as the fracton phases considered in this paper — the entanglement entropy of a ground state is expected to follow an ‘area law’ i.e. to be proportional at leading order to the

Figure 1.3: The eight possible vertex configuration of a ground state. Solid lines represent +1 eigenvalues of $Z$ at those links while dash lines represent −1 eigenvalues.
surface area of the subregion $A$. If $A$ has linear size $R$, then we may expand in powers of $R$, viz.

$$S = A_1 R^{d-1} + A_2 R^{d-2} + \ldots$$  \hspace{1cm} (1.3)

where $d$ is the spatial dimension. The leading term is non-universal and dominated by short-distance physics. The ‘topological’ information is contained in the subleading corrections; the challenge is to extract it.

For concreteness, let us review how this works in $d = 2$, where we have

$$S = A_1 R - c\gamma + \ldots,$$  \hspace{1cm} (1.4)

and the topological contribution is the constant piece, $S_{\text{topo}} = -c\gamma$, with $c$ the number of connected components of the boundary of $A$. In principle it appears as though extracting $\gamma$ should be a straightforward exercise: we should simply compute $S$, and extract its constant contribution and identify it as topological using the scaling form Eq. (1.4). In practice, however, most systems of interest and all those we are concerned with in this paper are defined on a lattice. Then subregions often have sharp corners that can lead to non-universal constant contributions in Eq. (1.4), assuming the entanglement cut is a sequence of edges forming a continuous path. This complicates a direct identification of $S_{\text{topo}}$ from the scaling of the bipartite entanglement entropy in Eq. (1.4).

![Illustration of the two types of prescriptions used to obtain the topological entanglement entropy. An ABC prescription is illustrated in (a), while (b) illustrates a PQWT prescription.](image)

Figure 1.4: Illustration of the two types of prescriptions used to obtain the topological entanglement entropy. An ABC prescription is illustrated in (a), while (b) illustrates a PQWT prescription.

However, there are prescriptions [44, 49] to extract topological entanglement by suitably
combining the results for a variety of bipartitions. Two such prescriptions in two dimensions are illustrated in Fig. 1.4. We refer to the type of prescription illustrated in Fig. 1.4(a) as an ABC prescription, where $A$, $B$ and $C$ are three disjoint regions, and the topological entanglement entropy is given by

$$ S_{ABC}^{\text{topo}} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}, \quad (1.5) $$

where $AB \equiv A \cup B$ and so on. An alternate prescription is shown in Fig. 1.4(b) and referred to as a PQWT prescription. In this case, the regions $P$, $Q$, $W$ and $T$ have the properties that $P = Q \cup W$ and $T = Q \cap W$, and the topological entanglement entropy is given by

$$ S_{PQWT}^{\text{topo}} = S_P - S_Q - S_W + S_T. \quad (1.6) $$

In both cases, these linear combinations of entropies are chosen to ensure that the dependence on local contributions from boundaries, including corner contributions, cancels out. These topological entanglement entropies are related to Eq. (1.4) by $S_{ABC}^{\text{topo}} = -\gamma$ and $S_{PQWT}^{\text{topo}} = -2\gamma$, which can be understood by counting the number of connected components in the boundaries of the various regions involved.

Both ABC type and PQWT type prescriptions have been generalized [12, 26] to $d = 3$, and we will make use of these generalizations in this paper. While different geometries and topologies of the regions are possible in these generalizations, in ABC type prescriptions we require the regions $A$, $B$ and $C$ be disjoint. In contrast, in the PQWT type prescriptions we employ, the set theoretic properties $P = Q \cup W$ and $T = Q \cap W$ will always be satisfied.

Much progress has been made in understanding topological order and topological entanglement in $d = 2$ by linking these ideas to TQFTs. For instance, we may understand the dependence of the topological contribution on the number of connected components of the boundary by recognizing that $S_{\text{topo}}$ reflects the additional information obtained by counting field lines of an gauge field constrained by a lattice analog of Gauss’s law. Entanglement entropy in turn can provide an important tool for extracting TQFT data [27], such as the braiding and statistics of the fractionalized excitations [132].
The $\mathbb{Z}_2$ topological order is effectively described by $\mathbb{Z}_2$ gauge theory in the low energy limit\cite{117, 21}. Starting from the $\mathbb{Z}_2$ lattice gauge theory, we can see that the toric code is the exact solvable point of its deconfined phase. Being well-studied, the lattice gauge theory has $\mathbb{Z}_2$ variables defined on the links of two dimensional square lattice. The corresponding Hamiltonian is

$$H = -g \sum_l Z_l - \frac{1}{g} \sum_p \prod_{l \in p} X_l$$

where the second term is the same plaquette term as we have in the toric code. The first term gives quantum fluctuation to the system. The Hamiltonian is invariant under gauge transformation

$$X_l \to \tau_{v_1}^x X_l \tau_{v_2}^x$$

where $\tau_v^x$ is a $\mathbb{Z}_2$ degree of freedom defined on the vertices. $l$ is the link connecting two vertices $v_1$ and $v_2$. Based on the study above, we know that the transformation by operator $\tau_v^x$ is generated by the vertex operator $A_v$. In other words, the states of this theory $|\phi\rangle$ is gauge invariant in the following form

$$A_v|\phi\rangle = \prod_{l \in v} Z_l |\phi\rangle = |\phi\rangle$$

This defines the Gauss law of this gauge theory. $A_v$ is the charge operator defined on vertices.

This theory has two phases. When $g > g_c$, all the spins tend to align along $+Z$ direction. The ground state is a featureless paramagnet. While when $g < g_c$, the ground state is the $\mathbb{Z}_2$ topological order. A limiting case is at $g \to 0$. In this case, if we relax the Gauss law and allow states with non-zero charge in the Hilbert space, we can reproduce the toric code model. All the analysis of this phase follow the discussion of toric code identifying the $e$ particle as the electric charge and the $m$ particle as the magnetic flux of the $\mathbb{Z}_2$ gauge theory.

It is straightforward to generalize the $\mathbb{Z}_2$ gauge theory to $\mathbb{Z}_N$ gauge theory by replacing the $\mathbb{Z}_2$ variables with $\mathbb{Z}_N$ variables. The resulting gauge theory has exact solvable point modeled as $\mathbb{Z}_N$ toric code. One can then figure out all the excitations and their fusion and braiding easily.
1.2.4 U(1) gauge theory

In fact, the U(1) gauge theory with continuous U(1) gauge group is more realistic, although the U(1) gauge theory does not give gapped topological ordered phase. Instead, the phase is gapless if the fractionalized excitations are deconfined due to the existence of gapless photon modes [60]. Being a topic of general interest, this gauge theory can describe the physics of many systems. For example, it was proposed to be the ground state of spin-1/2 Heisenberg antiferromagnet on the pyrochlore lattice in certain parameter region [36]. The U(1) lattice gauge theory is easily generalized from the $\mathbb{Z}_2$ lattice gauge theory we studied above by just replacing the $\mathbb{Z}_2$ variable with the U(1) variable $E_\mu$ and $A_\mu$ on each link. Besides the U(1) charge excitation, there is photon mode due to the continuous U(1) group. Here, we are interested in compact U(1) gauge theory where $A_\mu$ is 2$\pi$-periodic phase variable and $E_\mu$ is an integral value ranging from 0 to $\infty$ due to the commutation relation $[E_\mu(r), A_\nu(r')] = i\delta_{\mu\nu}\delta r r'$. The compactness of the theory results in the existence of magnetic charge called U(1) monopole. If we suppress the monopole, the theory is reduced to the non-compact U(1) gauge theory, which is equivalently the familiar electromagnetism. Without particular mentioning, the U(1) gauge theory here always refer to the compact one.

Due to the existence of monopole, the physics of U(1) gauge theory is different from the $\mathbb{Z}_2$ gauge theory. In terms of $E_\mu$ and $A_\mu$, the Hamiltonian of the theory on the general dimensional hypercubic lattice can be expressed as

$$H = \frac{g}{2} \sum_{\mathbf{r}, \mu} E^2_\mu(\mathbf{r}) - \frac{1}{g} \sum_{\mu, \nu, \mathbf{r}} \cos F_{\mu\nu}$$

(1.10)

where $F_{\mu\nu} = \Delta_\mu A_\nu - \Delta_\nu A_\mu$ gives the Maxwell term. The Hamiltonian is invariant under gauge transformation generated by the Gauss law $Q_\mathbf{r} = \Delta_\mu E_\mu$. In the strong coupling limit, i.e., when $g$ is very large, the ground state of this theory is a confining state and requires all $E_\mu = 0$, which is an analog of paramagnet in the $\mathbb{Z}_2$ gauge theory. Actually, in (2 + 1) or lower dimensions, previous study by Polyakov says that the pure compact U(1) gauge theory is always in a confining phase for all values of the coupling $g > 0$.

But in the presence of the matter field, the U(1) gauge field is possible to get Higgsed
resulting in a deconfined phase of discrete gauge theory.\textsuperscript{[95, 21]} The simplest example is called charge-2 condensation. When the bound states of two U(1) charge condense, the gauge group is discretized to $Z_2$ – the U(1) gauge charge reduces to the $Z_2$ gauge charge; the U(1) flux loop reduces to the $Z_2$ flux loop. Below, we formulate this process on the $d = 2$ cubic lattice where the $U(1)$ charge $e^{i\theta(r)}$ defined on vertex at $r$. Then, we have Hamiltonian

$$H = -K \sum_{r,\mu} \cos [\theta(r) - \theta(r + \mu) - A_\mu(r)] - \frac{1}{g} \sum_{r,\mu,\nu} \cos [A_\mu(r) + A_\nu(r + \mu) - A_\mu(r + \nu) - A_\nu(r)]$$

$$+ \frac{1}{\lambda} \sum_r n_\theta^2(r) + \frac{g}{2} \sum_{r,\mu} E_\mu^2(r) \quad (1.11)$$

where, compared with the Hamiltonian of the pure gauge theory, we have additional term – the first term as the hopping term of the matter field $\theta(r)$ and the third term describing the energy cost due to charge number occupation $n_\theta(r)$. Within this theory, we have additional commutation relations

$$[\theta(r), n_\theta(r')] = i\delta_{r,r'} \quad (1.12)$$

Suppose we add new terms

$$H_{2e} = -\Delta \sum_r \cos [\Theta(r) - 2\theta(r)] - K' \sum_{r,\mu} \cos [\Theta(r) - \Theta(r + \mu) - 2A_\mu(r)] \quad (1.13)$$

where the first cosine identifies the new charge $e^{i\Theta}$ as two charges $e^{i\theta}$ and the second cosine generates hopping process for the new charges. As we tune $K'$ and $\Delta$ very large, charge-2 would be condensed. In the $K', \Delta \to \infty$ limit, we can treat the cosine terms as constraints. New $Z_2$ variables can be defined as following. The first cosine gives $Z_2$ charge $\tau^x \equiv \exp(i\eta)$, where

$$\eta(r) = \frac{1}{2} \Theta(r) - \theta(r) = 0, \pi \quad (1.14)$$

The second cosine gives

$$\alpha_\mu(r) = \frac{1}{2} \Theta(r) - \frac{1}{2} \Theta(r + \mu) - A_\mu(r) \quad (1.15)$$

where $\alpha_\mu = 0, \pi$. Therefore, we can define a $Z_2$ vector potential $X_\mu = \exp(i\alpha_\mu)$. With the conjugate variables $\tau^z$ and $Z$, we have the new Hamiltonian through Higgs mechanism in terms of new $Z_2$
variables given by

\[ H_{Higgs} = -K \sum_{r, \mu} \tau^z_{r, \mu} X^z_{r, \mu} - \frac{1}{g} \sum_p \prod_{l \in p} X_l - \frac{1}{\lambda} \sum_r \tau^x(r) - \frac{g}{2} \sum_{r, \mu} Z_{\mu}(r) \] (1.16)

This is the $Z_2$ gauge theory on the same $d = 2$ cubic lattice with $X(Z)$ living on the links while $\tau^{x(z)}$ living on the sites. Choose a unitary gauge where $\tau^z|\phi\rangle = |\phi\rangle$ for physical state $|\phi\rangle$. Then, the Hamiltonian can be rewritten as

\[ H_{Higgs} = -K \sum_{r, \mu} X^z_{r, \mu} - \frac{1}{g} \sum_p \prod_{l \in p} X_l - \frac{1}{\lambda} \sum_r \prod_{l \in v_r} Z_l - \frac{g}{2} \sum_{r, \mu} Z_{\mu}(r) \] (1.17)

This is the pure $Z_2$ gauge theory which turns exactly to the toric code model in the limit of $\frac{1}{\lambda} \gg K, g \rightarrow 0$.

Therefore, by condensing charge pairs in the $U(1)$ gauge theory, we break down this $U(1)$ gauge symmetry to $Z_2$ and obtain the $Z_2$ gauge theory where there is a deconfining phase.

### 1.3 Fracton phases

It was discovered that there are other exotic new types of topological-like order exist in three spatial dimensions.[14, 11, 28, 9, 127, 10, 29, 30, 109, 110, 81, 79] A number of exactly solvable quantum spin models have been proposed with non-trivial ground states. Putting on topological non-trivial manifold, there is non-trivial topological degeneracy. Above them, there would be a gap to local excitations. And there are fractionalized excitations that cannot be created locally. These properties are shared with familiar two-dimensional topological orders and with three-dimensional discrete gauge theories, including twisted (Dijkgraaf-Witten) gauge theories.[18] Unlike those examples, however, these new discovered exotic three dimensional states have point-like fractionalized excitations that are confined to move in zero, one or two-dimensional subspaces. We refer to such excitations as zero-, one- and two-dimensional particles, respectively. The zero-dimensional particles are dubbed fractons,[109] and are fundamentally immobile in the sense that they cannot be created at ends of one-dimensional string operators. Instead, in some models fractons are created at corners of two-dimensional membrane operators.[14, 109, 110] while in other
models they are created at corners of fractal operators.\cite{28, 127} In either case, a process destroying a single isolated fracton must create more than one fracton elsewhere in space, so that individual fractons cannot simply move from one point to another on their own.

### 1.3.1 $\mathbb{Z}_2$ fracton topological order

#### 1.3.1.1 X-cube model

The simplest model possess this feature is call the X-cube model\cite{110}. It is a three dimensional exact solvable model on the cubic lattice. $\mathbb{Z}_2$ spins live on the link of the lattice as shown in Fig. 1.5. The Hamiltonian is

$$H_{XC} = - \sum_C A_C - \sum_{\mu=x,y,z} \sum_{v_{\mu}} B^\mu_v = - \sum_C l \in C X_l - \sum_{\mu=x,y,z} \sum_{v_{\mu}} l \in v_{\mu} Z_l$$

(1.18)

where the first term is the cubic term while the second one includes the vertex terms on $xy$, $yz$ and $xz$ planes. The model is exact solvable, so it is to get the ground state $|\psi\rangle$ satisfying

Figure 1.5: The cubic term (left) and vertex term (right) in the Hamiltonian of the X-cube model.

$A_C |\psi\rangle = B^\mu_v |\psi\rangle = |\psi\rangle$.

Although there is no degeneracy for the system on the topological trivial manifold, on the 3-torus, previous calculation gave the ground state degeneracy which is system size dependent expressed as

$$\log_2(GSD)_{XC} = 6L - 3$$

(1.19)

if the system lives on a $L \times L \times L$ cubic lattice. According to the study of toric code, we know that the model would have $6L - 3$ non-local operator commuting with the Hamiltonian on the 3-torus. This is different from what we have for ordinary topological order whose ground state degeneracy
is independent of system size. Nevertheless, this results indicates that this system has topological property. Therefore, we call the ground state fracton topological order.

Above the ground state, there are exotic excitations. Acting single $X$ or $Z$ operator creates excitations, as shown in Fig. 1.6. Particularly, acting an $X$ operator onto the ground state would lead to state $|\psi'\rangle = X_l |\psi\rangle$ satisfying $A_C |\psi'\rangle = |\psi'\rangle$ for four cubic terms. This means four excitations, called cubic excitations, are created at the same time. These four excitations can be separated apart by acting a membrane operator of $X$ operator among them without cost any energy, which means they are deconfined excitations. In the absence of string operator, single cubic excitation cannot move without moving or creating other cubic excitations. But the bound state of two cubic excitations (in purple in Fig. 1.6 (Left)) is mobile since the membrane operator can be regarded as a thick string operator. If we act a $Z$ operator onto $|\psi\rangle$, two excitations are created. These excitations

![Figure 1.6: Excitations in the X-cube model. (Left) The cubic excitations created in four. Two purple excitations can form a mobile bound state. (Right) The one dimensional excitation created in two. The string operator between them is rigid with new excitation always created at turning point.](image_url)

can only move in one dimensional subspace determined by the link where the $Z$ operator stays. In other words, although there is a string operator connecting two excitations, the string operator is rigid and it cannot turn without creating excitation at the turning point and cost energy, as shown in Fig. 1.6 (Right).

The fracton and the one dimensional excitation can have non-trivial braiding. We can wind the bound state of two fractons around an one-dimensional excitation. This generates $\pi$ phase factor which means they have mutual semionic statistics. This is new because two point-like excitation
cannot have anyonic statistics in three dimensions. Now, with the constraint on the motion of the excitation, we can have exotic statistics other than bosonic or fermionic ones. This type of fractionalization drive us to think more about the nature of the fracton topological order related to ordinary topological order.

There are also other models possessing the similar properties, e.g., the checkerboard model which turns out to be two copies of the X-cube model\cite{110, 99}, several majorana models\cite{109, 37}, Chamon’s octahedron model\cite{15}, the FCC model\cite{55} and so on. All of those three dimensional $\mathbb{Z}_2$ models have sub-dimensional excitations and sub-extensive ground state degeneracy as a function of system size on the 3-torus.

### 1.3.1.2 Haah’s cubic code

Another exotic three dimensional model exhibiting similar phenomena is the Haah’s cubic code\cite{28}. It is also a $\mathbb{Z}_2$ spin model constructed on the three dimensional cubic lattice. Each site has two spins. The Hamiltonian is

$$H = -\sum_{c} (A_c + B_c)$$

Each term is a cubic term which is a product of $X$ or $Z$ operator according to Fig. 1.7. Same as before, this model has ground state as a mutual eigenstate of $A_c$ and $B_c$. It also has sub-extensive ground state degeneracy on 3-torus. But the system size dependence of the degeneracy is rather complicated than we have for the X-cube model\cite{29}. This model, to some extent, is different from...
the model we discussed above due to the absence of any mobile fractionalized excitations. In other
words, this model only has fractons as its excitations. We can create point-like excitations at
four corners of a tetrahedron by acting single $X$ or $Z$ operator, as shown in Fig. 1.8. They can
move away from each other simultaneously by acting fractal operator among them. Thus, these
excitations are deconfined fractons. Their bound states are also fractons.

![Figure 1.8: The excitations created by acting $X$ or $Z$ operator on one site. Black dots represent
the locations of excitations, which are also the center of cubes of original cubic lattice.](image)

According to the properties of the excitations, we can classify those fracton model to two
types. The models like X-model in the previous subsection belong to type I model while the Haah’s
code is one of the type II model. So far, the physics of the type I models are gradually understood
from various perspective. The main purpose of this thesis is to illustrate the mechanisms for those
fracton topological orders in the type I models. However, type II models still remain mysterious.
We hope there can be appropriate approaches in the near future to deal with those models.

1.3.2 U(1) symmetric tensor gauge theories

As the topological order described by gauge theory in the low energy limit, the fracton
topological order is expected to have effective theory involving gauge degrees of freedom. A break-
through occurs when the fractons are found to appear in the higher rank tensor symmetric U(1)
gauge theories studied in Ref. [79, 81].

In a vector gauge theory, Gauss’ law leads to the conservation of total charge; once a positive-
negative charge pair is created, each of the charges can move freely in space without violating
charge conservation. For a higher rank gauge theory, the situation can be very different. A modified
Gauss’s law can lead to the conservation of not only the total charge, but also the conservation of dipole moment, quadrupole moment, etc. Because of the extra conservation laws, the charge excitations cannot move freely any more – they become fractons (or sub-dimensional particles). We also note several earlier studies of higher-rank U(1) gauge theories, although the restricted mobility of charged excitations was not pointed out.[122, 121, 70, 126, 124, 84]

It is worth mentioning that unlike the higher form gauge theories with antisymmetric tensors as their gauge fields, which do not give us new topological orders in three dimensions, the higher rank gauge fields are symmetric tensors. This difference leads to deconfined phases with sub-dimensional charges in \( d \geq 3 \).

Without dynamical matter and in the continuum, we can formulate this theory as following. The electric field \( E_{\mu\nu} \) and vector potential \( A_{\mu\nu} \) are both symmetric tensors, with the Greek indices running over spatial directions, i.e. \( \mu, \nu = x, y, z \). Eigenvalues of \( E_{\mu\nu} \) and \( A_{\mu\nu} \) are real numbers, and we have the commutation relations

\[
[A_{\mu\nu}(\mathbf{r}), E_{\lambda\sigma}(\mathbf{r}')] = -i(\delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\lambda})\delta(\mathbf{r} - \mathbf{r}'),
\]

where \( \mathbf{r}, \mathbf{r}' \) are positions in \( d = 3 \) space.

Such rank-2 gauge theories can be classified according to their Gauss laws and the nature of the corresponding charges. For example, if the Gauss law contracts all the indices of the tensor electric field and gives scalar charge, we have a scalar charge theory. On the other hand, if the Gauss law gives a charge transforming as a vector, then we have a vector charge theory.

We consider the scalar charge theory, where the Gauss law is \( \partial_\mu \partial_\nu E_{\mu\nu} = \rho \), where repeated indices are summed over. This Gauss law leads to invariance under gauge transformations \( A_{\mu\nu} \rightarrow A_{\mu\nu} + \partial_\mu \partial_\nu f \), where \( f \) is an arbitrary function of spatial position. From \( A_{\mu\nu} \) we can construct the gauge-invariant magnetic field tensor \( B_{\mu\nu} = \epsilon_{\mu\lambda\sigma} \partial_\lambda A_{\sigma\nu} \), which is traceless but not symmetric, and satisfies \( \partial_\mu B_{\mu\nu} = 0 \) in the non-compact theory. Deferring until later a discussion of dynamical matter degrees of freedom, the Hamiltonian density is \( \mathcal{H} = \frac{1}{2} E_{\mu\nu} E_{\mu\nu} + \frac{1}{4} B_{\mu\nu} B_{\mu\nu} \).

The unconventional Gauss law leads to a conservation of both electric charge and dipole...
moment. Consider some bounded spatial region $V$, with boundary $\partial V$. The total charge in $V$ is given by

$$Q = \int_V \rho \, d^3 \mathbf{r} = \int_{\partial V} \partial_\mu E_{\mu \nu} n_\mu \, dS, \quad (1.22)$$

where $n_\mu$ is a unit vector field normal to $\partial V$ and $dS$ is the surface area element. Because the right-hand side is a boundary term, this implies that it is impossible to locally create electric charges; of course, this is familiar from vector gauge theory. Here it is also true that the dipole moment $d = \int_V \mathbf{r} \, \rho \, d^3 \mathbf{r}$ can be written as an integral of the electric field over $\partial V$, so it is also impossible to locally create dipole moments. One dramatic consequence of this dipole conservation is that single electric charges are immobile, because moving an electric charge changes the dipole moment.

![Figure 1.9: Rank-2 U(1) gauge theory defined on the cubic lattice. The off-diagonal elements of $E_{\mu \nu}$ and $A_{\mu \nu}$ live on plaquettes, while diagonal elements reside on sites. Gauge charges $n_\mathbf{r}$, with conjugate phase $\theta_\mathbf{r}$, also reside on sites, which are labeled by $\mathbf{r}$.](image)

Here, we discuss how to put the rank-2 scalar charge theory on the simple cubic lattice, starting from the continuum theory. As shown in Fig. 1.9, the off-diagonal elements $E_{\mu \nu} = E_{\nu \mu}$ (also $A_{\mu \nu} = A_{\nu \mu}$) with $\mu \neq \nu$ are defined on the plaquettes in the $\mu - \nu$ plane, while each site hosts all three diagonal elements $E_{\mu \mu}$ (also $A_{\mu \mu}$). Each conjugate pair $A_{\mu \nu}, E_{\mu \nu}$ is an $O(2)$ quantum rotor, with $A_{\mu \nu}$ a $2\pi$-periodic phase variable; this makes the theory compact. The continuum form of the commutation relations implies that $[A_{\mu \nu}, E_{\mu \nu}] = -i$, with $\mu \neq \nu$, when the two variables lie on the same plaquette. For the diagonal components we have $[A_{\mu \mu}, E_{\mu \mu}] = -2i$ (no sum on $\mu$). This implies that off-diagonal elements of $E$ have integer eigenvalues, while diagonal elements have
even integer eigenvalues. This distinction is somewhat undesirable; we will see how to correct it below.

On the lattice, the Gauss law is given by

$$\Delta_\mu \Delta_\nu E_{\mu\nu} = n_r$$  \hspace{1cm} (1.23)

where $\Delta_\mu$ is a finite-difference operator, and $n_r$ is the charge at site $r$. In more detail, Gauss law can be written as

$$2(\Delta_x \Delta_y E_{xy} + \Delta_y \Delta_z E_{yz} + \Delta_z \Delta_x E_{xz})$$
$$+ (\Delta_x \Delta_x E_{xx} + \Delta_y \Delta_y E_{yy} + \Delta_z \Delta_z E_{zz}) = n_r.$$  \hspace{1cm} (1.24)

Here, similar to the difference in commutation relations, there is a factor of two difference in how the diagonal and off-diagonal components of $E$ appear in Gauss’ law. We address these two undesirable features by defining $E'_{\mu\mu} = E_{\mu\mu}/2$, so that $E'_{\mu\mu}$ takes integer eigenvalues and $[A_{\mu\nu}, E'_{\mu\mu}] = -i$. Putting this into Gauss’ law, we see that $n_r$ is now restricted to be an even integer, so we define the integer-valued charge $n'_r = n_r/2$. Dropping the primes, we then have the commutation relations $[A_{\mu\nu}, E_{\mu\nu}] = -i$, and the Gauss law

$$(\Delta_x \Delta_y E_{xy} + \Delta_y \Delta_z E_{yz} + \Delta_z \Delta_x E_{xz})$$
$$+ (\Delta_x \Delta_x E_{xx} + \Delta_y \Delta_y E_{yy} + \Delta_z \Delta_z E_{zz}) = n_r.$$  \hspace{1cm} (1.25)

The Gauss’ law directly determines those charge configurations that can be created locally, which are illustrated in Fig. (1.10). All these “locally-creatable” charge configurations have vanishing dipole moment.

We now include dynamical electrically charged matter degrees of freedom, and at the same time describe the lattice Hamiltonian. The matter fields are O(2) quantum rotors placed on the cubic lattice sites $r$, with number $n_r$ and phase $\theta_r$, satisfying $[\theta_r, n_r] = i\delta_{r,r'}$. The Hamiltonian is

$$H = U \sum_{r, \mu \leq \nu} E_{\mu\nu}^2 - K \sum_{r, \mu, \nu} \cos(B_{\mu\nu})$$
$$+ u \sum_r n_r^2 - J \sum_{r, \mu \leq \nu} \cos(\Delta_\mu \Delta_\nu - A_{\mu\nu}).$$  \hspace{1cm} (1.26)
Figure 1.10: Two possible electric charge configurations in the scalar charge theory are shown in (a) and (b). Any configurations related to these by cubic symmetry can also appear.

Here the lattice magnetic field is $B_{\mu\nu} = \epsilon_{\mu\lambda\sigma} \Delta_{\lambda} A_{\sigma\nu}$, which can be viewed as a traceless but not symmetric tensor field defined on the dual cubic lattice, where diagonal components reside on dual sites (cube centers of the original lattice), and off-diagonal components reside on dual plaquettes. The Gauss law is given by Eq. (1.25), which leads to gauge transformations

$$A_{\mu\nu} \rightarrow A_{\mu\nu} + \Delta_{\mu} \Delta_{\nu} f$$

$$\theta_{r} \rightarrow \theta_{r} + f_{r}.$$  

When $u$ and $K$ are the largest energy scales, the charged matter is gapped, and there is a stable deconfined phase that can be described by expanding the $-K \cos(B_{\mu\nu})$ terms to leading (quadratic) order[81]. This phase has a gapless, linearly dispersing photon mode with five polarizations and gapped charge excitations. There are also gapped magnetic monopole particle excitations, described in Sec. 4.5.1.

There are also various types of symmetric tensor gauge theory besides the scalar charge theory. Different theory is defined by its particular Gauss law. Previous works considered the related rank-2 vector charge theory, also on the simple cubic lattice [122, 121, 124].

1.4 Outline of this thesis

After introducing different phases of matter involving fracton excitations, including the fracton topological order and the gapless fracton phase described by symmetric tensor gauge theory,
we can ask: what is the relation among those fracton phases and ordinary topological order? Can we obtain one of these phases from other phases through particular transition? This thesis is aimed at answering these questions, giving the mechanisms for the fracton phases as concluded in the Fig. 1.11.

Figure 1.11: The relation among ordinary topological order, fracton topological order and gapless fracton phase.

In Chapter 2, we discuss how to get the three dimensional fracton topological order by coupling ordinary topological ordered layers. The coupling induces a transition characterized by condensation of loops of excitations from the original topological ordered layers, which we dubbed particle-loop condensation. Particularly, we construct the X-cube model introduced above by coupling toric code layers[55]. This approach make it possible to understand the physics of fracton model, such as the X-cube model in terms of degrees of freedom in the ordinary topological order. From the perspective of topological entanglement entropy, we again verify the connection between topological layer and fracton topological order due to the extra linear term contributing to the non-local part of the entanglement entropy[57]. The detailed calculation is in Chapter 3.

The relation among U(1) fracton phase, topological order, fracton topological order is discussed in Chapter 4. Surprisingly, the Higgs mechanism cannot directly give us fracton topological order from the U(1) fracton phase, unlike what happened to the ordinary vector U(1) gauge theory[54]. Actually, the fractons lose their immobility after Higgsing the U(1) gauge group to
discrete ones. And the Higgs mechanism leads to copies of ordinary topological orders. So we need extra transition to get the fracton topological order. This transition is driven by the condensation of particular excitations and partially confines the rest of excitations in the topological orders. So it is called partial confinement transition. Partial confinement transition can also happen in the symmetric tensor gauge theory. In this case, the transition is caused by the condensation of monopoles.

Being discussed extensively, it is still an question if fracton phases, or just sub-dimensional excitations, can be found in reality. In Chapter 5, we explore this question in a two dimensional critical system in the absence of Lorentz invariance. The system is at Lifshitz transition. A simple mapping shows that the critical theory is equivalent to a symmetric tensor theory with vector charges\[56\]. Those vector charges, if deconfined, are well defined excitations which can only move along one dimensional subspace. Detailed analysis in Chapter 5 shows that those excitations are logarithmically interacting at the critical point at zero temperature. However, at finite temperature, those excitations becomes deconfined above critical temperature through a BKT-like transition due to the thermal fluctuation. This critical point can be stabilized into a phase if we impose sub-dimensional symmetry.

Finally, in the chapter 6, we give a summary and discussion about this thesis.
Chapter 2

Fracton topological order via coupled layers

2.1 Introduction

Many basic questions about fracton topological orders remain open. One such question is whether some fracton topological orders can be related to and understood in terms of more familiar quantum phases of matter and their degrees of freedom. For example, the existence of two-dimensional particles could originate from a weakly coupled stack of $d = 2$ topologically ordered layers. Along the same lines, composites of excitations in two intersecting layers can be confined to move in one dimension, along the intersection line of the layers, and composites of excitations in three intersecting layers are completely immobile. Not all the features of fracton topological orders can be explained by simply stacking two dimensional topological orders. However, since some properties are similar to those of simple stacks, it is natural to ask whether we can take decoupled $d = 2$ topologically ordered layers, and couple them so as to obtain $d = 3$ fracton topological orders.

In this chapter, we show that some fracton topological orders can be understood by suitably coupling layers of familiar $d = 2$ topologically ordered systems. The coupling can be understood as a condensation of one-dimensional extended objects formed from particle excitations of the $d = 2$ layers, which we dub particle strings or “p-strings.” This provides a simple understanding of the properties of excitations, and ground state degeneracy, of the resulting fracton state. We also take this idea one step further, and couple together $d = 3$ fracton topological orders to obtain new $d = 3$ fracton topological orders by condensing two-dimensional membranes built from point particles, dubbed p-membranes.
This chapter is organized as follows. In Sec. 2.2, we show how to obtain a certain type of fracton topological order, realized in the X-cube model of Ref. [110], by coupling together layers of toric codes [42] covering the simple cubic lattice. The coupling is a $ZZ$ exchange interaction, and in the strong coupling limit, we reproduce the Hamiltonian of the X-cube model at sixth order in perturbation theory. We also consider $XX$ coupling, where we obtain the usual $d = 3$ toric code model in the strong coupling limit.

The strong coupling analysis suggests that the X-cube topological order can be understood in terms of the degrees of freedom of $d = 2$ toric code layers. Section 2.3.1 examines this relationship through the lens of p-string condensation at intermediate coupling strengths. The X-cube model supports one-dimensional “electric” particle excitations and zero-dimensional “magnetic” fractons.

We show that the one-dimensional particles are formed from pairs of toric code $e$-particles, and that fractons are created at the ends of open p-strings. $m$-particles survive in the X-cube model as bound states of two fractons. The resulting perspective also allows us to easily calculate the ground state degeneracy (GSD) of the X-cube model, which we do in Sec. 2.3.2. In Sec. 2.3.3, we apply a similar intermediate coupling picture to the case of $d = 2$ toric codes coupled by $XX$ coupling, where condensation of composites of two $e$-particles leads to a standard $d = 3$ toric code phase.

In Sec. 2.4, we introduce a new type of fracton topological order obtained via p-membrane condensation in a system of four interpenetrating X-cube models. We dub the resulting exactly solvable model the “Four Color Cube (FCC)” model. The FCC model has an electric-magnetic self-duality, and all its excitations can be obtained as composites of immobile fractons. We describe the properties of the FCC model’s excitations, and calculate its GSD on the three-torus, which we find to be $\log_2 GSD_{FCC} = 32L - 24$.

In Sec. 2.5 we construct a semionic version of the X-cube model, obtained by coupling together stacks of models with doubled semion topological order. The main new feature in the semionic version of the X-cube model is the addition of nontrivial “braiding statistics” between the one-dimensional excitations. We expect that our method of obtaining the semionic X-cube model
can be readily extended to construct fracton phases from coupled stacks of more general types of topological order.

The chapter concludes in Sec. 2.6, where we discuss future directions and open questions raised by our work. Technical details are given in two appendices.

### 2.2 X-cube model from toric code layers: strong coupling

Our starting point is a model of coupled layers of \( d = 2 \) toric codes. In this section, we consider the limit of strong coupling, and show that our model reduces exactly to the X-cube Hamiltonian in this limit, using standard techniques of degenerate perturbation theory. We also show that, in the same model but with a different form of coupling, upon taking the strong coupling limit we obtain the conventional \( d = 3 \) toric code model.

We begin by describing the system in the decoupled limit. We consider three independent stacks of square-lattice toric codes along the \( \mu = x, y, z \) directions of the cubic lattice. The toric codes cover the cubic lattice in such a way that two Ising spins reside on every cubic link. For example, a link oriented in the \( x \)-direction is contained in one \( xy \) plane and one \( xz \) plane. One of the spins on the link is part of an \( xy \)-plane toric code, while the other participates in a \( xz \)-plane toric code.

Before describing how to couple the toric codes, we establish some notation. We label cubic lattice vertices by \( i, j \), links by \( \ell \), square plaquettes by \( p \), and \{100\}, \{010\} and \{001\} lattice planes by \( P \). Lattice directions are indicated by \( \mu = x, y, z \), we write \( \ell = (i, \mu) \) for the link extending from \( i \) in the \(+\mu\) direction, and we refer to links as \( \mu \)-links when we want to indicate their direction. Sometimes it is convenient to indicate links by nearest-neighbor pairs of sites \( ij \). Each plaquette \( p \) has an orientation denoted \( o(p) = x, y, z \), which specifies the direction normal to \( p \). Similarly, the orientation of the plane \( P \) is written \( o(P) \).

On each link we place two Ising spins. \( Z \) and \( X \) Pauli operators for the spins on the link \( \ell = (i, \mu) \) are written \( Z_{(i,\mu)}^{\nu}, X_{(i,\mu)}^{\nu} \). The superscript \( \nu = x, y, z \) \((\nu \neq \mu)\) indicates the orientation of the toric code plane in which the spin participates. That is, each spin is a member of a \( d = 2 \) toric
code on plane \( P \) that contains \( \ell \), and with \( \nu = o(P) \).

The toric code Hamiltonian for plane \( P \) is written

\[
H^{TC}_P = -\sum_{i\in P} A^{o(P)}_i - \sum_{p\in P} B_p, \tag{2.1}
\]

where we introduced the usual vertex operators

\[
A^\mu_i = \prod_{ij\perp \mu} Z^\mu_{ij}, \tag{2.2}
\]

Here, the product is over the four links touching \( i \) and perpendicular to the direction \( \mu \). We also introduced plaquette operators

\[
B_p = \prod_{\ell \in p} X^{o(p)}_\ell. \tag{2.3}
\]

We note that \( B_p \) does not carry a superscript indicating its orientation, as this is already implicit upon specifying \( p \).

Now we couple together the toric code layers with the Hamiltonian

\[
H = \sum_P H^{TC}_P - J_z \sum_\ell Z^\mu_1 Z^\mu_2. \tag{2.4}
\]

Here, \( \mu_1 \) and \( \mu_2 \) are the two lattice directions perpendicular to \( \ell \). The two spins on each link now interact via a \( ZZ \) exchange interaction with coefficient \( J_z > 0 \). When \( J_z = 0 \) we have decoupled toric code layers.

We now take the limit \( J_z \to \infty \) and treat \( H^{TC} = \sum_P H^{TC}_P \) as a perturbation. First ignoring the perturbation, we have an extensively degenerate ground state space, where any spin configuration in the \( Z \) basis satisfying the constraint \( Z^\mu_1 = Z^\mu_2 \) is a ground state. The ground space on link \( \ell \) is that of an Ising spin with Pauli operators

\[
Z_\ell \equiv Z^\mu_1 Z^\mu_2, \tag{2.5}
\]

\[
X_\ell \equiv X^\mu_1 X^\mu_2, \tag{2.6}
\]

which commute with the \( J_z \) coupling term. Any operator acting within the many-body ground space can be written in terms of these operators.
The ground state space is split by an effective Hamiltonian we obtain using Brillouin-Wigner degenerate perturbation theory. The details are described in Appendix 2.7; it is necessary to take care of some technical issues in order to go beyond leading order. We obtain the X-cube Hamiltonian at sixth order in perturbation theory:

\[
H_{XC} = -\sum_i \sum_{\mu=x,y,z} A_{i}^{\mu} - K \sum_c B_c. \tag{2.7}
\]

where the relation between terms in the X-cube model and toric code layers is shown in Fig. (5.5).

Here, we have dropped constant terms, and we defined

\[
A_{i}^{\mu} = \prod_{ij \perp \mu} Z_{ij}. \tag{2.8}
\]

Moreover, \(c\) labels elementary cubes of the lattice, and

\[
B_c = \prod_{\ell \in c} X_{\ell}, \tag{2.9}
\]

where the product is over the 12 edges of the cube \(c\). The coupling of the cube term is \(K = C/J_z^5\), where \(C > 0\) is a constant factor that can be computed if desired following the discussion of Appendix 2.7. We have not computed \(C\) because its value plays no role in our discussion.

![Figure 2.1: Illustration of cube and vertex terms in the X-cube model related to plaquette and vertex terms from toric code layers.](image)

To summarize the perturbation theory analysis, the vertex term of \(H^{TC}\) contributes at first order and gives the vertex term (first term) of the X-cube Hamiltonian. This is simple to understand, as projection to the ground state space simply replaces \(Z_{\ell}^{\mu}\) with \(Z_{\ell}\). The cube term is a
product of six $B_p$’s, where each $B_p$ operator anticommutes with the $J_z$ term on the four links in the perimeter of $p$. To obtain a non-trivial operator within the ground state space as a product of $B_p$’s, we have to take a product over plaquettes forming a closed surface. The smallest such surface is a single cube with six faces, so the cube term is the lowest-order such contribution arising in perturbation theory. This is described in more detail in Appendix 2.7, where it is also shown that no other terms arise between first and sixth order.

We have shown that $H$ interpolates between decoupled $d = 2$ toric codes when $J_z = 0$, and the X-cube Hamiltonian when $J_z \to \infty$. This suggests that the topological order of the X-cube model can be understood in terms of degrees of freedom of the decoupled toric codes. To develop this idea, we need to consider the effect of the $J_z$ term at weak and intermediate coupling, which is done in Sec. 2.3.

We conclude this section with a brief discussion of the effect of replacing the $ZZ$ coupling with $XX$ coupling, specifically adopting the Hamiltonian

$$H_{XX} = \sum_{\ell} X_{\ell}^\mu X_{\ell}^\nu.$$  \hspace{1cm} (2.10)

Again considering the $J_x \to +\infty$ limit, the single-site ground space is that of an Ising spin with Pauli operators $X_{\ell} \equiv X_{\ell}^\mu = X_{\ell}^\nu$ and $Z_{\ell} = Z_{\ell}^\mu Z_{\ell}^\nu$. Degenerate perturbation theory results in the usual $d = 3$ toric code model,

$$H_{3dTC} = -\sum_{p} B_p - \tilde{K} \sum_{i} A_i,$$  \hspace{1cm} (2.11)

where $B_p = \prod_{\ell \in p} X_{\ell}$, and $A_i = \prod_{j} Z_{ij}$, with the latter product over the six links touching $i$. The coefficient of the vertex term satisfies $\tilde{K} \propto J_x^{-2}$. The fact that we obtain the $d = 3$ toric code model when $J_x \to \infty$ can be understood coming from the limit of weak $J_x$ as a condensation of bound states $e_1 e_2$, where $e_1, e_2$ are $e$-particles in two intersecting toric code layers. This is described in Sec. 2.3.3.
2.3 X-cube model from toric code layers: intermediate coupling

While it is suggestive, the fact that we obtain the X-cube model upon strongly coupling toric code layers does not directly relate the properties of the X-cube model to the properties of decoupled toric codes. Motivated by the strong coupling analysis, here we consider small and intermediate values of $J_z$, and show that the topological order of the X-cube model can be obtained from decoupled toric codes by condensation of extended one-dimensional objects we dub “p-strings.” This allows us to describe properties of the X-cube model in terms of the degrees of freedom of decoupled toric code layers. In particular, we use this condensation picture to recover the properties of the X-cube model’s topological excitations, and to give a simple computation of the ground state degeneracy on a three-torus. In Sec. 2.3.3, we consider a similar picture for toric code layers with $XX$ coupling, where condensation of bound pairs of two $e$ particles leads to a description of $d = 3$ toric code topological order.
Figure 2.3: A larger p-string, obtained by acting with $Z^{\mu_1}_\ell Z^{\mu_2}_\ell$ operators along the links orthogonal to a rectangular membrane (marked in green).

### 2.3.1 Fracton topological order from p-string condensation

Since the $J_z$ coupling term does not commute with the $B_p$ term in the original toric code Hamiltonian, acting with it creates toric code $m$ particles, which occur on plaquettes that violate the $B_p$ terms in $H^{TC}_P$. In particular, acting with the coupling operator $Z^{\mu_1}_\ell Z^{\mu_2}_\ell$ creates two pairs of $m$ particles on the four plaquettes touching the link $\ell$ (Fig 2.2).

If we represent each $m$ particle by a line segment normal to the plane in which it moves, the line segments for the four $m$ particles created by $Z^{\mu_1}_\ell Z^{\mu_2}_\ell$ can be joined into a closed string. We refer to this string as a p-string, where “p” stands for particle, as it is built out toric code particle excitations. An elementary p-string created by the application of a single $Z^{\mu_1}_\ell Z^{\mu_2}_\ell$ operator is shown in Fig. 2.2. Acting with a collection of $Z^{\mu_1}_\ell Z^{\mu_2}_\ell$ operators on links orthogonal to a rectangular membrane creates larger p-strings, as shown in Fig. 2.3.

As we increase $J_z$ from zero, at some point we expect to induce a condensation of the p-strings created by the $J_z$ exchange interaction. Upon condensation, p-strings appear at all sizes and propagate freely through the system, driving a confinement transition in the electric sectors of the toric code layers. In particular, we will see that this condensation process leads to the confinement of individual $e$ particles, while bound pairs of $e$ particles on intersecting planes survive the condensation process.

In order to examine what happens to the toric code $e$ particles under the p-string conden-
sation, we will examine how $e_P$ particles braid with the p-string condensate, where $e_P$ denotes a plane-$P$ toric code $e$ particle (a violation of the $A^{(P)}_i$ term in $H_{P}^{TC}$). The kind of braiding process we need to look at is one in which a p-string winds around a single $e_{P^0}$ particle in a particular plane $P^0_\mu$ with $o(P^\mu_0) = \mu$, as illustrated in Fig. 2.4.

![Figure 2.4: A braiding process between a p-string in the condensate and an $e_{P^0}$ particle. The process is drawn in a “continuum limit,” where we do not show the individual $m$ particles making up the p-string. During the braiding process the right side of the p-string is held fixed, while the left side sweeps out the motion indicated by the gray arrow.](image)

During the braiding process, we move the p-string around the $e_{P^0}$ excitation by keeping the right part of the loop fixed and moving the left part around the $e_{P^0}$ excitation as shown by the path of the gray arrow in Figure 2.4. Since p-strings are composed of $m$ particles, each intersection of the loop with a plane $P$ defines the location of an $m_P$ particle in that plane. We set the location of the $e_{P^0}$ excitation as the origin of our coordinate system, and label the three planes containing the origin by $P^\nu_0$ for $\nu = x, y, z$, and $o(P^\nu_0) = \nu$. In particular, the $xy$ plane containing the origin is $P^z_0$. Tracking the intersection of the p-string with $P^z_0$, we see that moving the p-string induces a braiding of an $m_P$ particle with $e_{P^0}$, which contributes a phase of $\theta_{e_{P^0}, m_P} = \pi \delta_{\mu, z}$ to the braiding of the p-string with $e_{P^0}$. Intersections of the p-string with other $xy$ planes $P$ not containing the origin ($o(P) = z$ but $P \neq P^z_0$) do not contribute, because in that case $\theta_{e_{P^0}, m_P} = 0$.

Similar arguments apply to $P^x_0$ and $P^y_0$, whose intersections with the p-string are $m_{P^x}$ and $m_{P^y}$ particles. These particles are braided around $e_{P^0}$ during the p-string braiding process,
contributing phases of $\theta_{e_{P_0^\mu},m_{P_0^x}} = \pi \delta_{\mu,x}$ and $\theta_{e_{P_0^\mu},m_{P_0^y}} = \pi \delta_{\mu,y}$, respectively. Putting everything together, we see that

$$\theta_{e_{P_0^\mu},\text{p-string}} = \sum_{\nu=x,y,z} \theta_{e_{P_0^\mu},m_{P_0^\nu}} = \pi.$$  \hfill (2.12)

Because this braiding phase is non-trivial, all individual $e_{P}$ particles become confined after inducing the p-string condensation.

However, the condensation process does not completely confine the electric sector excitations of the original decoupled $J_z = 0$ theory. Instead, it allows bound-state pairs of $e$ particles on intersecting planes to remain deconfined. Indeed, consider the bound state of two distinct $e_{P}$ excitations $e_{P_i} e_{Q_i}$, with $P \neq Q$, $o(P) \neq o(Q)$ so that the planes intersect, and where $e_{P_i}$ denotes an $e$ excitation located on vertex $i$ in plane $P$. These bound states have trivial braiding with the p-string condensate, so these composites of two $e$ particles are deconfined even in the $J_z \to \infty$ limit\(^1\). We denote these bound-state excitations as

$$\epsilon^i_{\mu} = e_{P_i^\mu} e_{P_i^\lambda},$$  \hfill (2.13)

where $\mu, \nu, \lambda$ are three distinct directions, and $P_i^\mu$ denotes the plane containing the site $i$ and normal to the $\mu$ direction. We will often drop the $i$ superscript in $\epsilon^i_{\mu}$ when it will cause no confusion.

An $\epsilon^i_{\mu}$ particle is able to move only along the $\mu$ direction, and so is a fundamentally one-dimensional particle. The $\epsilon_{\mu}$ excitations are precisely the 1d particles of Ref. [110]. We can use the fusion rule $e_{P} \times e_{P} = 1$ of the toric code to derive the fusion rules of the $\epsilon_{\mu}$ fractons:

$$\epsilon^i_{\mu} \times \epsilon^i_{\nu} = \begin{cases} \epsilon^i_{o(\mu,\nu)}, & \mu \neq \nu \\ 1, & \mu = \nu \end{cases},$$  \hfill (2.14)

where $o(\mu, \nu)$ is the direction normal to both $\hat{\mu}$ and $\hat{\nu}$.

Having discussed the electric excitations of the X-cube model, we now turn to an analysis of the magnetic excitations, which arise at ends of open p-strings. Working first in the decoupled limit, we consider the “stack” of $m_{P}$ particles in $xy$ planes shown in Fig. 2.5a. The following discussion

\(^1\) Pairs like $e_{P_e Q}$ with $o(P) = o(Q)$ are also confined, since p-strings can pass between the two $e_{P}$ excitations, giving a $\pi$ braiding phase that confines them.
Figure 2.5: (a) An open p-string (dark blue), created at the edge of a series of $xy$ plane $m$-string operators (which act on the red links) stacked in the $z$-direction. The ends of the $m$-string operators are $m$-particles, and are marked with black crosses. (b) Acting with $Z^x_\ell Z^z_\ell$ on the green link creates a short $m$-string in the $yz$ plane and deforms the p-string.

holds more generally, but we focus on the particular geometry of Fig. 2.5 to simplify the notation. Each $m_P$ particle is created at the end of an $m$-string operator

$$S_m(\gamma) = \prod_{\ell | \ell \cap \gamma \neq 0} Z^z_\ell,$$

(2.15)

where $\gamma$ is a path lying in a $xy$ plane that intersects links transversely, and the product is over all links $\ell$ intersecting $\gamma$.

The stack of $m_P$ particles is taken to have finite extent in the $z$-direction, and can be represented as an open p-string, as shown in Fig. 2.5 (a). Acting on the string with the coupling term $Z^{\mu_1}_\ell Z^{\mu_2}_\ell$ moves it around (Fig. 2.5 (b)), but keeps the locations of the p-string endpoint fixed. This is because acting with $Z^{\mu_1}_\ell Z^{\mu_2}_\ell$ creates loops of p-string and therefore acting with $Z^{\mu_1}_\ell Z^{\mu_2}_\ell$ cannot change the $\mathbb{Z}_2$ flux of p-strings through any given cube. Therefore, upon p-string condensation, the fixed endpoints of p-strings remain as excitations, but the one-dimensional “bulks” of the strings become tensionless, allowing them to fluctuate at all length scales without incurring any energetic penalty. This results in a condensate of p-strings, and the one-dimensional “bulks” of the strings cease to be physically observable objects. The endpoints of the p-strings reside in cubes of the simple cubic lattice, and are the fracton excitations of the X-cube model. We will denote a fracton excitation supported at the cube $c$ by $m_c$. From the $\mathbb{Z}_2$ fusion rule of $m$ particles in the toric code
(i.e. the fusion rule $m_P \times m_P = 1$), we see that the $\mathfrak{m}_c$ fractons also obey $\mathbb{Z}_2$ fusion rules.

Figure 2.6 shows two stacks of $m_P$ particles, created by the operator

$$M_\sigma = \prod_{\ell : \ell \cap \sigma \neq 0} Z_\ell,$$  \hspace{1cm} (2.16)

where $\sigma$ is a rectangular membrane, and the product is over links $\ell$ cutting $\sigma$ transversely (drawn in red in Figure 2.6). This operator can be viewed as a stack of $m$-strings, $M_\sigma = \prod_{\gamma \in \sigma} S_m(\gamma)$, with each string creating two $m_P$ particles on opposite sides of the membrane. Upon condensing $p$-strings, the operator $M_\sigma$ creates four fracton excitations at its corners. Indeed, in the strong coupling limit we can replace $Z_{\ell}^{\mu_1(\mu_2)}$ by $Z_\ell \equiv Z_{\ell}^{\mu_1} = Z_{\ell}^{\mu_2}$ and

$$M_\sigma \rightarrow M_\sigma = \prod_{\ell : \ell \cap \sigma \neq 0} Z_\ell,$$  \hspace{1cm} (2.17)

which is simply a membrane operator of the X-cube model that creates fractons at its corners, as discussed in Ref. [110].

It is instructive to consider the case where $\sigma$ contains only a single $m$-string, so that the membrane operator in the decoupled limit creates a pair of $m_P$ particles in a single layer, and $M_\sigma = S_m(\gamma)$. Taking the strong coupling limit, this allows us to define an $m$-string operator in the X-cube model by

$$S_m(\gamma) = \prod_{\{\ell \cap \gamma \neq 0\}} Z_\ell.$$  \hspace{1cm} (2.18)

In general, $\gamma$ is some path lying in a single $\{100\}$ plane that cuts links transversely, as shown in Fig. 2.9. In the X-cube model, the $m$-string $S_m(\gamma)$ creates two $\mathfrak{m}_c$ fracton excitations at each end as shown in Fig. 2.7. Therefore, $m$-particles survive in the X-cube topological order as bound states of two $\mathfrak{m}_c$ fractons.

In summary, our procedure of obtaining the X-cube model showcases two different ways of restricting the movement (or reducing the “dimensionality”) of particles in topological phases. One mechanism, which occurs in the electric sector, is to bind together two particles which are free to move in two different (but intersecting) planes, resulting in a composite excitation free to move only
in one dimension. The other mechanism, which occurs in the magnetic sector, is to fractionalize particles by “breaking them apart” into pairs of immobile particles.

Our perspective on the X-cube model allows for a simple understanding of the statistical properties of e and m excitations. Without trying to give a general definition of statistical processes of particles moving in restricted dimensionalities, we note that any statistical process involving only e excitations must be trivial, because these excitations originate from a collection of toric code e particles, which have trivial self and mutual statistics. The same statement holds for any statistical process involving only m particles.

However, there are non-trivial statistical processes involving both e and m excitations. Consider a rectangular prism, and let $\sigma_1, \ldots, \sigma_4$ be four of its faces, excluding the two faces lying in $xy$ planes. Then the membrane operator

$$M_{\text{prism}} = M_{\sigma_1} \cdots M_{\sigma_4}$$

(2.19)

does not create any excitations in the X-cube model, as the excitations created by each $M_{\sigma_i}$ cancel out. The prism membrane operator counts the total number of $e_x$ and $e_y$ excitations inside the prism, modulo two, as can be seen from

$$M_{\text{prism}} = \prod_{i \in \text{prism}} A_i^\varepsilon.$$  

(2.20)
Figure 2.7: A single m-string (dashed line) creates two \( m_c \) excitations (purple cubes) at each of its ends, illustrating the fact that \( m \)-particles (black crosses) survive in the X-cube topological order as bound states of two \( m_c \) fractons.

Since \( A_z^i = -1 \) acting on a state with an \( e_x^i \) or \( e_y^i \) excitation, the eigenvalue of \( M_{\text{prism}} \) is 1 when an even number of such excitations are inside the prism, and \(-1\) for an odd number.

The operator \( M_{\text{prism}} \) can be viewed as effecting a process where two \( m \) fractons are brought around the perimeter of the top and bottom faces of the prism, as shown in Fig. 2.8a. Thinking in terms of the underlying toric code degrees of freedom, this corresponds to braiding a stack of \( xy \)-plane \( m_P \) particles around the prism (Fig. 2.8b), which results in a statistical phase of \( \pi \) with any \( xy \)-plane \( e_P \) particles contained inside. Each \( e_x \) and \( e_y \) particle is a bound state of an \( xy \)-plane \( e_P \) particle with another \( e_P \) particle, and contributes a phase of \( \pi \). On the other hand, \( e_z \) excitations are bound states of \( yz \)-plane and \( xz \)-plane \( e_P \) particles, and do not contribute to the statistical phase. Therefore this picture recovers the properties of \( M_{\text{prism}} \) in the X-cube model deduced above.

2.3.2 Logical operators and ground state degeneracy

We now proceed to derive the ground state degeneracy (GSD) of the X-cube model on an \( L \times L \times L \) three-torus \( T^3 \) (the calculation for other spatial topologies proceeds in a similar way). This result has been obtained previously by more rigorous methods for odd \( L \).[110] Our approach relates the ground state degeneracy of the X-cube model to the underlying toric code degrees of freedom.

To determine the ground state degeneracy, we need to count the number of independent logical operators in the theory. We first review this for the \( d = 2 \) toric code on the two-torus \( T^2 \),
where logical operators correspond to distinct ways to thread excitations around non-contractible cycles. We consider threading \( m \) particles; we could just as well thread \( e \) particles instead. We let \( \gamma_x \) and \( \gamma_y \) be paths winding around the different cycles of \( T^2 \), then \( S_m(\gamma_x) \) and \( S_m(\gamma_y) \) are string operators threading \( m \) particles around the torus. It is simple to show that they are linearly independent. Thus, these two operators form a complete set of independent, commuting logical operators, and their eigenvalues completely label the ground state space, which has degeneracy \( \text{GSD} = 2^2 = 4 \).

Now we consider the X-cube model, with \( T^3 \) topology obtained by enforcing periodic boundary conditions along each direction of the stack of toric codes, so that each plane in the stack has the topology of \( T^2 \). We count the number of distinct ways to thread magnetic \( M_\sigma \) membranes through the non-contractible cycles of \( T^3 \). (We could just as well construct logical operators by threading \( e \) particles around non-contractible cycles.)

A generic membrane operator \( M_\sigma \) can be constructed out of a stack of \( S_m(\gamma) \) string operators, where each path \( \gamma \) lies in a single \( \{100\} \) plane, which has the topology of a 2-torus. Therefore, in order to count independent logical operators, it is enough to consider the operators \( S_m(\gamma) \), where \( \gamma \) winds around a non-contractible loop in its associated plane. Therefore we are led to consider logical operators that correspond to threading \( m \) particles, which are really bound states of two \( m \) fractons, around the torus. Each plane in the stack has two such independent logical operators, one
for each of the independent non-contractible loops on the 2-torus, giving a contribution of $2^2$ to the GSD. Since there are a total of $3L$ planes in the stack, we obtain a total of $6L$ logical operators.

However, these $6L$ logical operators are not all independent. To see this, let $(\mu, n)$ denote the plane normal to the $\mu$ direction and with $\mu$-coordinate $n$, and let $\gamma'_(\mu, n)$ denote the path lying within the plane $(\mu, n)$ and passing around the non-contractible loop in the $\nu \neq \mu$ direction. Then we observe that for all $\mu \neq \nu$, we have the relation

$$
\prod_{0 \leq n < L} S_m(\gamma'_{(\mu, n)}) = \prod_{0 \leq n < L} S_m(\gamma'_{(\nu, n)}) = M_{\mu\nu},
$$

(2.21)

where $M_{\mu\nu}$ is a membrane operator where the surface $\sigma$ covers an entire $\mu\nu$-plane. We have thus found one relation among the $6L$ logical operators for each (unordered) pair $\mu, \nu$ with $\mu \neq \nu$. There are three such pairs, and these three pairs exhaust the linear relations among the $6L$ logical operators meaning the number of independent logical operators is $6L - 3$, a result we verified numerically using the methods of Appendix 2.8. This results in a GSD of

$$
\log_2(\text{GSD}_X) = 6L - 3,
$$

(2.22)

in agreement with the result obtained by commutative algebra methods in Ref. [110] for odd $L$.

The sub-extensive $6L$ contribution is the GSD of $3L$ decoupled toric codes, and it is somewhat remarkable that the only modification of the GSD caused by the $J_z$ coupling term is a constant subleading correction which is independent of the number of toric codes in the stack. The analysis here is less rigorous than that of Ref. [110] because, in principle, we could have missed additional
logical operators independent from the ones already listed. However, it is straightforward to check these results numerically for reasonably small values of $L$ as described in Appendix 2.8. Numerical counting of the X-cube model gives the same number of independent stabilizers as above. Therefore, above $6L - 3$ logical operators indeed form a complete set.

2.3.3 3d toric code from interlayer pair charge condensation of toric code layers

As discussed at the end of Sec. 2.2, layers of toric codes interacting via $XX$ coupling result in the $d = 3$ toric code model in the strong coupling limit ($J_x \to \infty$). Just like the case of $ZZ$ coupling, where the X-cube model results at strong coupling, there is a corresponding intermediate coupling picture that allows us to understand the excitations of the $d = 3$ toric code in terms of the excitations of the $d = 2$ toric code layers.

Acting with the coupling term $X_{\ell}^{\mu_1} X_{\ell}^{\mu_2}$ on the link $\ell$ creates a pair $e_1e_2$ of $e$-particles at each end of $\ell$. Here $e_1$ and $e_2$ are the two-dimensional $e$-particles residing in the two perpendicular layers containing $\ell$. The composite $e_1e_2$ is a one-dimensional particle that is expected to condense above a critical value of $J_x$ (recall that a $d$-dimensional particle is one that is constrained to move along a $d$-dimensional manifold).

In the presence of such a condensate, when $e_1$ approaches the intersection of the two planes, it can be converted into $e_2$ and move into the other plane. Therefore any two-dimensional $e$ particle in the lattice can convert to any other two-dimensional $e$ particle. Only a single type of $e$ particle remains as an independent excitation, and it is a three-dimensional particle. This is the point charge excitation (violation of the vertex term) of the $d = 3$ toric code.

It is easy to see that the $e_1e_2$ condensate confines single two-dimensional $m$ excitations. However, closed p-string configurations of $m$ particles have trivial statistics with the condensate. This follows from the analysis of Sec. 2.3.1, upon noting that the $e_1e_2$ particles forming the condensate here are the same as the one-dimensional $e$ excitations of the X-cube model, which also arise as $e_1e_2$ composites. Those excitations are shown in Sec. 2.3.1 to have trivial statistics with closed p-strings. Here, this means that while single $m$ excitations are confined, closed p-strings of $m$ particles remain
as deconfined excitations, and form the flux line excitations of the $d = 3$ toric code.

2.4 P-Membrane condensation: FCC model from coupled X-cube models

Above, we obtained the fracton topological order of the X-cube model from toric code layers via p-string condensation. This raises the possibility of related condensation mechanisms that also lead to interesting fracton topological orders. One option is to condense two-dimensional membranes built from particle excitations, or p-membranes. Here, we describe a realization of p-membrane condensation in a system of four coupled X-cube models. We obtain a new exactly solvable model dubbed the Four Color Cube (FCC) model, which also happens to have a face centered cubic (fcc) lattice structure. The p-membranes forming the condensate are composed of the one-dimensional particle excitations of the underlying X-cube models. After discussing the coupling of X-cube models to obtain the FCC model, we discuss its properties and show that it possesses fracton topological order distinct from that of the X-cube model.

![Figure 2.10: Lattice geometry of the coupled X-cube and FCC models, which are defined on four interpenetrating simple cubic lattices that we label by the colors black, red, green and blue. Dots, squares, up triangles and down triangles indicate the vertices of the four simple cubic lattices. Simple cubic lattice links intersect in mutually perpendicular triples of three different colors; for example, red, green and blue links intersect at centers of black cubes, as shown.](image)

The model is defined on four interpenetrating simple cubic lattices that we label by the colors black ($k$), red ($r$), green ($g$) and blue ($b$). The geometry is shown in Fig. 2.10. One way to
understand the lattice geometry is to start with the black cubic lattice, and observe that there are 
three different orientations of plaquettes. For each orientation, the plaquette centers form a simple 
cubic lattice, and these lattices are colored red ($xy$ plaquettes), green ($xz$ plaquettes) and blue ($yz$
plaquettes). Taken together, all the vertices form a fcc lattice.

We place an Ising spin on each link of a given color, for a total of 12 spins per simple cubic 
unit cell. We use the term site (as opposed to vertex) to refer to the locations where spins reside. 
Each site is an intersection of three mutually perpendicular links of three different colors, and we 
label sites by $\ell$. There are thus three spins located at $\ell$, with Pauli operators $X^{w}_\ell$, $Z^{w}_\ell$, where $w$ is 
any of the three colors located at $\ell$. Sites with colors $w_1, w_2, w_3$ are located at cube centers of the 
$w_4$-colored lattice. The sites also form a fcc lattice, so we can view our model as a fcc lattice spin 
system with three Ising spins per site.

The Hamiltonian of coupled X-cube models is

$$H_{CXC} = \sum_{w=k,r,g,b} H^w_{XC} - h \sum_{\ell} X^{w_1}_\ell X^{w_2}_\ell X^{w_3}_\ell,$$  \hspace{1cm} (2.23)

where the first term is simply four decoupled X-cube Hamiltonians on the four cubic lattices, and 
the second term couples the different colors, where $w_1, w_2$ and $w_3$ are the three different colors at 
$\ell$. The Hamiltonian of the color $w$ X-cube model is written

$$H^w_{XC} = -\sum_{i \in w} \sum_{\mu=x,y,z} A^\mu_i - \sum_{c \in w} B_c,$$  \hspace{1cm} (2.24)

where we have taken the coefficients of the two terms to be equal, the first sum is over all vertices 
in the color $w$ cubic lattice, and the second sum is over all cubes in the color $w$ lattice. Since the 
notation we are using here differs slightly from the previous sections, we again give the form of the 
stabilizers. The Z-stabilizers are

$$A^\mu_i = \prod_{ij \perp \mu} Z_{ij},$$  \hspace{1cm} (2.25)

where $j$ is a vertex adjacent to $i$ in the lattice of the same color, and the product is over links $ij$
perpendicular to $\mu$. The X-stabilizers are

$$B_c = \prod_{\ell \in c} X^w_\ell,$$  \hspace{1cm} (2.26)
where the product is over edges of the cube $c$, and the color $w$ is specified by the choice of $c$.

The Hamiltonian $H_{CXC}$ has the full translational symmetry of the fcc lattice, provided translations are accompanied by certain spin rotations that correspond to permuting the colors, i.e. translation acts in a “spin-orbit coupled” manner. Under translation by a Bravais lattice basis vector $a_1, a_2, a_3$, we make the following permutations of colors

$$a_1 : \ k \leftrightarrow b, \ r \leftrightarrow g$$
$$a_2 : \ k \leftrightarrow g, \ r \leftrightarrow b$$
$$a_3 : \ k \leftrightarrow r, \ g \leftrightarrow b,$$

where the basis vectors are

$$a_1 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$$
$$a_2 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$
$$a_3 = \left(\frac{1}{2}, \frac{1}{2}, 0\right).$$

Here and elsewhere we set the distance between neighboring vertices of the same color to one.

Before proceeding to analyze $H_{CXC}$, we first motivate the form of the coupling. The coupling term creates two-dimensional p-membrane objects that are expected to condense for $h$ sufficiently large. To see this, we consider the effect of acting with $H_{\ell}^\text{int} = X_{\ell}^w X_{\ell}^{w_2} X_{\ell}^{w_3}$ in the decoupled ($h = 0$) limit. Each $X_{\ell}^w$ operator creates a pair of one-dimensional $e$ particles at the two endpoints of the link $\ell$, in the color $w$ X-cube model. Therefore, $H_{\ell}^\text{int}$ creates six such excitations located at the vertices of an octahedron centered at the site $\ell$ (Fig. 2.11). It is natural to view each $e$ particle as a square plaquette, as illustrated in Fig. 2.11, where the plane of the square represents the directions in which $e$ cannot move. The six squares join together to form a closed cube surrounding $\ell$, so that we can view $H_{\ell}^\text{int}$ as creating a small closed p-membrane. It is then natural to conjecture that the physics of the large-$h$ limit can be understood in terms of condensation of p-membranes. This is directly analogous to $p$-string condensation, where we represented two-dimensional $m$ particles as line segments indicating the direction in which the particle cannot move, and these line segments
join together into closed p-loops. In both cases, the dimension of the object condensing (p-string or p-membrane) is the co-dimension of the space in which the constituent particle excitations move.

Figure 2.11: The p-membrane condensation in FCC model. The big black dot denotes one of the cubes in the black X-cube model. It is surrounded by plaquettes of the other three colors representing the planes perpendicular to the motion of \( d = 1 \) particles, generated at vertices of different cubic lattices denoted by different shapes as shown in Fig. (2.10).

We proceed with our analysis of \( H_{CXC} \) by considering the strong coupling (\( h \to \infty \)) limit, where we obtain the exactly solvable FCC model in degenerate perturbation theory. For a single site, in the \( h \to \infty \) limit we have the constraint

\[
X_{w_1}^\ell X_{w_2}^\ell X_{w_3}^\ell = 1, \quad (2.33)
\]

which defines a low-energy Hilbert space of two effective Ising spins. Rather than solving the constraint, e.g. by eliminating one of the \( X_w^\ell \) operators, we find it convenient to work in the constrained Hilbert space of three spins. Because the constraint is on-site, this is purely a matter of convenience. An arbitrary single-site operator is built from sums and products of \( X_{w}^\ell \) and \( Z_{w_1}^\ell Z_{w_2}^\ell \); single \( Z_w^\ell \) operators do not commute with the constraint in Eq. (2.33).

To understand the large-\( h \) limit, we need to carry out degenerate perturbation theory. As discussed in Appendix 2.7, the necessary calculation is essentially the same as that in the large-\( J_z \) limit of coupled toric codes described in Sec. 2.2 and in Appendix 2.7. At sixth order in perturbation theory we obtain the FCC model,

\[
H_{FCC} = - \sum_c B_c - K \sum_c A_c, \quad (2.34)
\]
where $K$ is a positive constant proportional to $1/h^5$. The sums are over all cubes $c$, in all four cubic lattices. All the terms in $H_{FCC}$ are stabilizers, or products of stabilizers, of the underlying X-cube models, so that any two terms commute and the model is exactly solvable.

The first term in $H_{FCC}$ is simply the projection of the $X$-stabilizer terms of the X-cube models into the low-energy subspace. Since these terms commute with the constraint, their form is unaffected, although we do need to keep in mind that the underlying $X^w_\ell$ operators now obey the constraint Eq. (2.33).

The operators $A_c$ in the second term, which are the $Z$-stabilizers of the FCC model, are obtained as the smallest non-constant products of X-cube $Z$-stabilizers ($A^\mu_i$'s) that commute with the constraint (Eq. (2.33)). To understand the form of $A_c$, we observe that each vertex term $A^\mu_i$ in one of the underlying X-cube models lies on a face $f$ of a cube $c$, where $c$ and the vertex $i$ have different colors, as shown in Fig. 2.12. We thus write $A_f \equiv A^\mu_i$. Then we obtain $A_c$ by taking a product over the six faces of $c$,

$$A_c = \prod_{f \in c} A_f. \quad (2.35)$$

As illustrated in Fig. 2.12, $A_c$ has two $Z^w_\ell$ operators on each edge of $c$, so it commutes with the constraint.

---

Figure 2.12: Illustration of a FCC model $A_c$ operator on a black cube. The operator is a product of $A^\mu_i$ over the cube’s six faces shown by the red, green and blue links, whose vertices are represented by corresponding shapes shown in Fig. (2.10). For each edge of the cube, two $Z^w_\ell$ operators contribute to $A_c$. 


The FCC model obeys an electric-magnetic self-duality that we will exploit in our analysis. To expose the duality, we define new Pauli operators $Z^w_\ell$ and $X^w_\ell$ by

$$X^{w_1}_\ell = Z^{w_2}_\ell Z^{w_3}_\ell$$

$$Z^{w_1}_\ell Z^{w_2}_\ell = X^{w_3}_\ell,$$

where $w_1, w_2, w_3$ are the three distinct colors at $\ell$. These new operators obey the same algebraic relations as the constraint $X^{w_1}_\ell X^{w_2}_\ell X^{w_3}_\ell = 1$. If we set $K = 1$ in $H_{FCC}$, this change of variables becomes a symmetry of the model, where the $A_c$ and $B_c$ stabilizers are exchanged.

The electric-magnetic self-duality appears quite surprising, given that the starting point of coupled X-cube models has no such property. However, it can be rationalized by going back to the construction of the underlying X-cube models from toric code layers. In the limit of decoupled toric codes, there is of course an electric-magnetic self-duality. To obtain X-cube models, we condense p-strings of $m$ particles, and then obtain the FCC model by condensing p-membranes of $e$ particles. Since condensations occur in both the electric and magnetic sectors, it is reasonable that electric-magnetic self-duality can be restored in the FCC model. This suggests that there might be a manifestly self-dual route directly from toric code layers to the FCC model, bypassing the intermediate step of X-cube models; we leave exploration of this possibility to future work.

Now we turn to an analysis of the FCC model. The first property to establish is that the model has topological order. That is, we would like to argue that there is a non-trivial ground state degeneracy on the 3-torus, and that the degenerate ground states cannot be distinguished by local measurements. We argue that this is the case in Appendix 2.8, where we find that the ground state degeneracy GSD on a $L \times L \times L$ torus satisfies

$$\log_2 \text{GSD} = 32L - 24.$$  \hspace{1cm} (2.38)

We note that this is not simply four times the result for a single X-cube model, which is $4 \times (6L - 3) = 24L - 12$.

Next, we discuss the excitations of the FCC model. As illustrated in Fig. 2.13a, acting with $X^{w_1}_\ell$ on a ground state creates eight $A_c = -1$ excitations. $X^{w_1}_\ell$ is dual to $Z^{w_2}_\ell Z^{w_3}_\ell$, which also
creates eight $B_c = -1$ excitations, illustrating the self-duality (Fig. 2.13b). Each $Z^w_\ell$ operator can be thought of as creating four fractons in the underlying color-$w$ X-cube model, which are created at corners of a membrane perpendicular to the color-$w$ link $\ell$. Therefore the eight excitations created by $Z^w_\ell Z^w_\ell$ can be viewed as created at corners of two perpendicular membrane operators, as shown in Fig. 2.13c. By self-duality, the same picture holds for the eight $A_c = -1$ excitations created by $X^{w_1}_\ell$.

![Diagram of excitations and membraes](image)

**Figure 2.13:** (a) Acting with $X^k_\ell$ on the thick black link (pointing in the $z$ direction) creates eight $A_c = -1$ excitations, four of which are blue (open squares) and four green (solid squares). (b) $X^k_\ell$ is dual to $Z^b_\ell Z^b_\ell$, where each $Z$ operator creates four $B_c = -1$ excitations of the same color, again for a total of eight excitations, which illustrates the self-duality. The four excitations created by each $Z^w_\ell$ are four fractons in one of the underlying X-cube models. (c) In both cases, the eight excitations can be viewed as created at corners of two perpendicular membrane operators.

Acting with a product of $X^w_\ell$ along a straight line creates one-dimensional particle excitations at ends of the line, as illustrated in Fig. 2.14a. These particles are made up of four $A_c = -1$ excitations, and are remnants of the one-dimensional $c$ excitations of the underlying X-cube models. Electric-magnetic self-duality shows there are corresponding one-dimensional particles made up of four $B_c = -1$ excitations, as shown in Fig. 2.14b.

Isolated $A_c = -1$ excitations can be created by a “skyscraper operator,” which is a product of $X^w_\ell$ over a pattern resembling a skyscraper, which we illustrate with a particular example. The
skyscraper is formed from a stack of (001) planes, where on each plane we act with a product of \( X^w_\ell \) over a diamond pattern of black and red links pointing in the \( z \)-direction, as shown in Fig. 2.15. The skyscraper operator thus consists of black and red string operators for the one-dimensional particles shown in Fig. 2.14a, with the strings forming a diamond pattern when the skyscraper is viewed from “above” in the \( z \)-direction. At the top and bottom of the skyscraper, \( A_c = -1 \) excitations are created at the corners as shown in Fig. 2.15. Electric-magnetic self-duality implies that isolated \( B_c = -1 \) excitations can be created in a corresponding manner.

![Figure 2.14](image)

Figure 2.14: (a) Acting with a product of \( X^k_\ell \) along a straight line oriented along the \( z \)-axis creates one-dimensional particle excitations at the end of the line, each composed of four \( A_c = -1 \) excitations (open/solid squares). These excitations are the remnants of one-dimensional \( \epsilon \) excitations in the black X-cube model. (b) Electric-magnetic duality shows that acting with a product of \( Z^g_\ell Z^b_\ell \) along a line in the \( z \)-direction also creates one-dimensional particle excitations. These one-dimensional excitations can be thought of as bound states of two-fracton two-dimensional particles from green and blue X-cube models. The blue bound state moves in a \( yz \) plane, while the green bound state moves in a \( xz \) plane, so the bound state of both of them is constrained to move in the \( z \)-direction.

If we act with the skyscraper operator in the limit of decoupled X-cube models, we create \( p \)-membranes of black and red \( \epsilon \) particles at the top and bottom of the skyscraper, in the pattern indicated by filled circles in Fig. 2.15a. In the FCC model, the only excitations lie at the corners, which illustrates that the two-dimensional “bulk” of the \( p \)-membrane, and its one-dimensional edges, are condensed.

Naively, the skyscraper construction suggests that isolated \( A_c = -1 \) excitations are created
at corners of a **volume** operator, with support over the interior of a solid three-dimensional region. However, we can act with $X$-stabilizers to “hollow out” the inside of the skyscraper. Specifically, we act with products of black and red $B_c$ operators over non-overlapping vertical columns, which cancel out the string operators on the inside of the skyscraper, and introduce new $X^{w}_c$ operators on the top and bottom faces. This shows that $A_c = -1$ excitations are created at corners of a membrane operator supported over a two-dimensional region, the boundary of the skyscraper. However, the resulting object appears complicated geometrically, and so far we have found the skyscraper construction more useful for visualization purposes.

We now argue that isolated $A_c = -1$ and $B_c = -1$ excitations are immobile fractons, using statistical properties of these excitations. Self-duality allows us to focus on $B_c = -1$ excitations for convenience. Suppose that a single isolated $B_c = -1$ excitation is contained in some volume. We can detect this excitation by acting with the operator

$$B = \prod_{c' \in \text{prism}} B_{c'},$$

where “prism” is a rectangular prism containing $c$, and includes only cubes of the same color as $c$. $B$ has eigenvalue 1 acting on the ground state, and eigenvalue $-1$ on the state with the excitation.
Most of the $X^w_\ell$ operators in the product defining $B$ cancel out, and, indeed, $B$ is a product of $X^w_\ell$ over the edges of the rectangular prism. That is, $B$ is comprised of string operators for the one-dimensional particles of Fig. 2.14a on the edges of the prism.

![Figure 2.16](image)

Figure 2.16: Acting with the operator $B$ can be viewed as implementing a process where three one-dimensional particles are created at each corner of a rectangular prism, and these particles are then moved along edges to annihilate. This process has a non-trivial statistical phase of $\pi$ when a single $B_c = -1$ excitation is present inside the prism, where $c$ and the edges of the prism all belong to the same color cubic lattice.

Acting with $B$ effects a process where three one-dimensional particles are created at each corner of the prism, and are then brought together along edges to annihilate, as shown in Fig. 2.16. This process can be used to remotely detect the $B_c = -1$ excitation contained inside the prism. This is only consistent if this excitation is a fracton, meaning that it cannot be moved by acting with any string operator. Indeed, if it could be transported in this way, acting with the string operator could move it out of the prism through one of the faces. Since the size and shape of the $B$ prism operator is arbitrary (as long as it contains the $B_c = -1$ excitation), $B$ can always be chosen so that any putative string operator moving the excitation does not intersect the edges of $B$, guaranteeing that the excitation can be trivially moved out of the prism. This is inconsistent with the fact that $B$ can be used to remotely detect the excitation, and so the excitation must be a fracton.

### 2.5 Semionic X-cube model

Interpreting the X-cube model in terms of a coupled layer construction allows us to generalize the model. In this section, we show how to generalize the construction to coupled layers of the double semion model so that the resulting “semionic” X-cube model has “semionic” 1D particles.
In this section, in contrast with the discussion of the coupled-layer construction of the X-cube model, we will not be as concerned with the fine details of degenerate perturbation theory in the strong coupling limit. In particular, we will not try to establish the absence of perturbations to the semionic X-cube model in sixth order perturbation theory. Answering this question will not be important for our main purpose, which is to discover the semionic X-cube model, to understand some of its properties, and to understand how its excitations are related to those of the underlying double semion layers.

To construct the “semionic” version of the model, we will need to start from trivalent 2D lattices instead of the square lattices as discussed in previous sections. First, we demonstrate how to obtain from this starting point a model with the same topological order as the X-cube model. Consider the decorated square lattice as shown in Fig.2.17 (a). A small diamond shape is added at each vertex of the square lattice so that in the new lattice each vertex has degree three.

![Figure 2.17](image)

Figure 2.17: (a) A trivalent 2D lattice from decorating 2D square lattice with diamond shapes at each vertex. (b) Stacks of such trivalent lattices in $x$ (solid lines), $y$ (dash lines), and $z$ (dotted lines) planes; the edges in $x$, $y$ and $z$ directions overlap in pairs.

To define a toric code on the decorated lattice, we put spins on all the edges and impose Hamiltonian terms as

$$
\tilde{H}^{TC} = - \sum \prod_{\ell \in \land} Z_{\ell} - \sum \prod_{\ell \in \Diamond} X_{\ell} - \sum \prod_{\ell \in \Box} X_{\ell}.
$$

(2.40)

The topological order of this model remains the same as the square lattice toric code model (2.1), with the only difference coming from the fact that there can be extra plaquette excitations in the
diamonds which correspond to the same type of anyon as the octagon (originally square) plaquette excitations.

Now we take three stacks of such 2D models in the $x$, $y$ and $z$ planes and couple them as shown in Fig. 2.17 (b). The planes are positioned in such a way that edges in the $x$, $y$ and $z$ directions overlap in pairs and share two spins. The diagonal edges do not overlap. For overlapping edges, we denote Pauli operators by $X^\mu_\ell$ and $Z^\mu_\ell$, where $\mu$ gives the normal direction of the toric code layer containing the spin. We couple the layers by adding a $Z \otimes Z$ term on each pair of overlapping edges, as in our coupled-layer construction of the original X-cube model. The total Hamiltonian becomes

$$\tilde{H} = \sum_P \tilde{H}_P^{TC} - J_z \sum_{\ell \parallel x,y,z} Z^\mu_\ell Z^{\mu_2}_\ell. \quad (2.41)$$

where the latter sum is over edges aligned in the $x$, $y$ and $z$ directions.

Among the original Hamiltonian terms, the vertex terms and the plaquette terms on the diamonds commute with the coupling term while the plaquette terms on the octagons anticommute with the coupling. Therefore, if we pass to the strong coupling $J_z \to \infty$ limit, the octagon terms need to reorganize and combine into cubes. In the strong coupling limit, the two overlapping edges combine into one and there is effectively one Ising spin per edge. As in Sec. 2.2, we introduce $Z_\ell$ and $X_\ell$ Pauli operators for the effective Ising spins in the $J_z \to \infty$ limit, and the effective Hamiltonian becomes

$$\tilde{H}_{XC} = - \sum_{\ell \parallel x,y,z} \prod_{\ell \in \ell} Z_\ell - \sum_{\ell \in \diamond} \prod_{\ell \in \diamond} X_\ell - \sum_{\ell \in \bigotimes} \prod_{\ell \in \bigotimes} X_\ell. \quad (2.42)$$

Now we are in a position to examine the excitations of this model. If we apply $Z_\ell$ for $\ell$ an edge in the $x$, $y$ and $z$ direction, we create four cube excitations, which can be separated to four corners of a membrane by subsequent action of $Z_\ell$ operators on edges perpendicular to the membrane. These are the fracton excitations of the modified X-cube model. If we apply $Z_\ell$ to one of the edges of the diamond plaquettes, we create two cube excitations together with a diamond plaquette excitation. This is saying that two fractons are equivalent to a diamond plaquette excitation. As the combination of two fractons can move freely in a two dimensional plane, so can the diamond
plaquette excitation. This can be seen from Fig.2.18 (a) where a string of $Z_\ell$ operators can move a diamond plaquette excitation around.

If we apply $X_\ell$ to one of the edges of the diamond plaquettes, we create one vertex excitation at each end of $\ell$. If we want to move these vertex excitations, we need to apply $X_\ell$ to the edges in $x$, $y$ and $z$ directions, which creates two vertex excitations at each end of $\ell$. Similarly to the original X-cube model, vertex excitations on intersecting planes move together, which restricts their motion to the intersection line. The only small difference is that when a pair of vertex excitations passes through a diamond plaquette, their path separates onto two planes before merging again, as shown in Fig.2.18 (b).

![Figure 2.18: Illustration of the properties of excitations in the modified X-cube model on a decorated lattice. (a) Excitations in diamond plaquettes can move freely in 2D with a string of $Z$ operators as shown; (b) Excitations at vertices can move in 1D with a string of $X$ operators as shown.](image)

In this way, by starting from toric code layers on trivalent 2D lattices, we obtain a generalized X-cube model with the same topological content.

To obtain the semion version of the X-cube model, we start with the double semion model\cite{50} on the decorated square lattice with Hamiltonian

$$\tilde{H}^{DS} = -\sum_\ell \prod_{\ell \in \ell} Z_\ell - \sum_\ell P_v \left( \prod_{\ell \in \ell} X_\ell \prod_{\ell \in \ell} S_\ell \right) - \sum_\ell P_v \left( \prod_{\ell \in \ell} X_\ell \prod_{\ell \in \ell} S_\ell \right).$$

\hfill (2.43)

where the second product in each plaquette term is over $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ on all the legs pointing outward from the plaquette, and $P_v$ is a projector onto configurations satisfying the vertex term
for all the vertices contained in the plaquette. This model is known to have the double semion topological order.\footnote{The lattice geometry differs from that considered in Ref. \cite{50}. However, because the lattice is still trivalent, the analysis of Ref. \cite{50} goes through, and the model supports double semion topological order.} In particular, there are semion string operators which anti-commute with each other when they intersect.

Now we couple the layers, using the same coupling term as in the toric code case:

$$\tilde{H}^s = \sum_P \tilde{H}^P_{DS} - J_z \sum_{\ell} Z^{\mu_1}_\ell Z^{\mu_2}_\ell. \quad (2.44)$$

In the strong coupling limit, overlapping edges combine into one. The vertex and diamond plaquette terms commute with the $J_z$ term, and lead to terms of the same form in first-order perturbation theory. The octagon plaquette terms do not commute with the $J_z$ term, and a product of six such terms over the faces of a cube appears at 6th order in perturbation theory. It is possible that other terms appear in the perturbation series, but we ignore them (see remarks at the beginning of this section). The resulting Hamiltonian takes a form very similar to Eq. (2.42), except that the diamond plaquette terms and the octagon cube terms are supplemented with a product of $S$ operators over outward pointing edges, and with projectors onto vertex-term-satisfying configurations. It can be checked that all the terms in the Hamiltonian commute with one another.

What is the excitation structure of this semionic version of X-cube model? Fractons can be created exactly as in the (trivalent) toric code version of the X-cube model and have the same properties. The 1D particles (pairs of vertex excitations) can no longer be created simply with an $X$ string as in Fig. 2.18b, as this operator has nontrivial commutation with various plaquette and cube terms along the path of the string. Instead, to create a 1D particle, we need to combine two semion string operators from two intersecting planes along the intersection line. That is we need to supplement the $X$ string operator with some extra phase factors depending on the $Z$ configuration along the string.

As a result of the structure of the string, the 1D particles have nontrivial “braiding” statistics with each other. For example, consider a 1D particle moving in the $x$ direction, which is a composite of a semion in the $xy$ plane and a semion in the $zx$ plane. Suppose that the line on which this
1D particle moves intersects that of another 1D particle moving in the $y$ direction, which is a composite of a semion in $yz$ plane and a semion in the $xy$ plane. The string operators of the two 1D particles anti-commute with each other, because the string operator of one semion on the $xy$ plane anti-commutes with that of another semion on the same plane, while string operators of semions on different planes commute with each other. This anti-commutation of string operators is related to the fact that two such 1D particles can undergo a full braid, so their mutual statistics is well defined. In this case, the two 1D particles have mutual statistics $\theta = \pi$, which contrasts with trivial mutual statistics $\theta = 0$ in the original X-cube model.

In making the above statements, we need to account for the fact that there are different types of 1D particles moving along a given line. For instance, given a 1D particle moving in the $x$ direction, we can attach to it a 2D particle in an $xy$ plane, and obtain a new 1D particle moving in the $x$ direction. Such attachments can change the statistics, so some care is needed to be sure the statistics we find in the semionic X-cube model is really different from that in the original X-cube model.

This can be addressed by demanding that the 1D particles satisfy a certain fusion condition. In particular, we require that three 1D particles moving in $x$, $y$, and $z$ directions fuse to a trivial excitation when they meet at a point. This amounts to making a certain natural choice of 1D particle excitations. When this fusion condition is satisfied, 1D particles moving in orthogonal directions indeed have $\theta = 0$ mutual statistics in the original X-cube model, while they have $\theta = \pi$ statistics in the semionic X-cube model.

2.6 Discussion

In this work, we showed how several different fracton topological orders could be realized through coupled layer constructions, both by forming coupled stacks of conventional 2d topologically ordered phases, and by “stacking” and coupling a finite number of 3d models with fracton topological order. This perspective allowed us to shed light on the physics of the X-cube model of Ref.[110], which we demonstrated could be obtained from a three-dimensional stack of coupled toric
codes in the strong coupling limit. These results can be understood through mechanisms we have
dubbed “p-string condensation” and “p-membrane condensation,” in which either one-dimensional
strings of particles or two-dimensional membranes of particles are driven to condense. Using these
ideas, we constructed two new models of fracton topological order: a semionic generalization of
the X-cube model and a phase obtained by inducing p-membrane condensation in a system of four
interpenetrating X-cube models, which we dubbed the “Four Color Cube model.”

We now turn to a discussion of questions raised by our work and potential further avenues of
study. A natural first question to ask is whether other known examples of fracton topological order,
like the checkerboard model of Ref. [110] or the fractal topological order of Haah’s code [28], admit
a coupled-layer description similar to the ones presented here. More ambitiously, we can consider
asking whether or not all fracton topological phases can be realized by a coupled layer construction,
and if not, whether it is possible to complete a classification of the ones that do. Making progress
in this direction would be useful for determining the extent to which fracton topological phases lie
“beyond” quantum field theory, and may help us to construct more interesting examples of such
phases. Along these lines, if possible it would also be interesting to obtain a continuum-picture field-
theoretic understanding of p-string and p-membrane condensation, which could further elucidate
the degree of the relationship between fracton topological phases and more familiar topological
quantum field theories.

Another natural direction for future work is to perform our p-string condensation procedure
for stacks of two-dimensional topological phases other than the toric code and doubled semion
examples considered in this work. We expect the extension to more general types of Abelian
topological orders realized in commuting projector models to be fairly straightforward, and can
likely be done by following the framework developed in Sec. 2.5. For example, performing our
analysis for coupled layers of non-Abelian topological phases has been studied [Refs], and some
new features are found. We might also expect other exotic physics in more general settings.

Our ability to understand certain fracton topological phases from within the general frame-
work of the theory of two-dimensional topological phases raises the possibility of easily studying
other properties of fracton phases, like how their classification is enriched by the presence of sym-
metries, how symmetry fractionalization and anomalies are classified in fracton phases, and how
such phases’ edge theories can be constructed. As there already exist a large number of theoretical
techniques for studying these properties in conventional topological phases, one could anticipate us-
ing our coupled-layer approach to reduce the study of these questions in fracton phases to problems
involving conventional topological phases, which could then be solved using existing methods.

Finally, this is a rather phenomenological view towards the study of the fusion and braiding
properties of the excitations in fracton topological phases, addressing the fusion and statistical
properties of each model on a case-by-case basis. It would be useful to develop the fusion and
braiding theory of fracton phases on a more general level and to see how such a picture fits in with
the commutative algebra methods developed in Ref.[110]; we plan to pursue this in future work.
Appendix

2.7 X-cube Hamiltonian from degenerate perturbation theory

Here we apply Brillouin-Wigner perturbation theory to derive the X-cube model in the limit of strongly coupled toric code layers. As briefly discussed at the end of this appendix, essentially same analysis holds in the strong coupling limit of the coupled X-cube model, where we obtain the FCC model. The perturbation theory used to obtain the semionic X-cube model has the same general structure, but some statements in the analysis here assume that the Hamiltonian is a sum of terms that square to unity, which is not true in that case.

We write the Hamiltonian as $H = H_0 + H_1$, where $H_0$ is the $J_z$ coupling term. $H_1 = H_{1v} + H_{1p}$ is the Hamiltonian of decoupled toric code layers, where $H_{1v}$ is the sum of all toric code vertex terms, and $H_{1p}$ the sum of all plaquette terms. Let $|\psi\rangle$ be an energy eigenstate with energy $E$ that lies in the ground state space of $H_0$ if the perturbation is turned off. We write $|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle$, where $|\psi_0\rangle$ is chosen to be normalized and lies in the ground state space, and $|\psi_1\rangle$ lies in the orthogonal complement of the ground state space. We let $\mathcal{P}$ project onto the ground state space, and $(1 - \mathcal{P})$ projects onto the orthogonal complement. Note that $\mathcal{P}$ commutes with $H_0$.

The Schrödinger equation can be written in the form

$$|\psi\rangle = |\psi_0\rangle + (E - H_0)^{-1}(1 - \mathcal{P})H_1|\psi\rangle,$$  \hspace{1cm} (2.45)

and iterated to find the formal solution

$$|\psi\rangle = \sum_{n=0}^{\infty} \left[(E - H_0)^{-1}(1 - \mathcal{P})H_1\right]^n |\psi_0\rangle.$$  \hspace{1cm} (2.46)
This is not a closed-form solution because $E$ is the unperturbed energy. This does not matter at leading order, and it is common to stop at leading order in Brillouin-Wigner perturbation theory, but we need to go beyond leading order here. Fortunately, the commuting projector nature of $H_1$ will allow us to simplify the perturbation series.

Acting with $\mathcal{P}H$ on both sides of Eq. (2.46), we have

$$\mathcal{P}H|\psi\rangle = E_0|\psi_0\rangle + \mathcal{P}H_1 \sum_{n=0}^{\infty} [(E - H_0)^{-1}(1 - \mathcal{P})H_1]^n \mathcal{P}|\psi_0\rangle,$$

where we inserted a factor of $\mathcal{P}$ in front of the $|\psi_0\rangle$ on the right. We also have

$$\mathcal{P}H|\psi\rangle = E|\psi_0\rangle. \quad (2.47)$$

Comparing these two expressions we see that

$$H_{\text{eff}}|\psi_0\rangle = (E - E_0)|\psi_0\rangle, \quad (2.48)$$

we have defined the effective Hamiltonian $H_{\text{eff}}$ to be

$$H_{\text{eff}} = \mathcal{P}H_1 \sum_{n=1}^{\infty} [(E - H_0)^{-1}(1 - \mathcal{P})H_1]^{n-1} \mathcal{P} \equiv \sum_{n=1}^{\infty} \tilde{H}_{\text{eff}}^{(n)}, \quad (2.49)$$

where $\tilde{H}_{\text{eff}}^{(n)}$ is the $n$th term in the series.

This expression for the effective Hamiltonian depends on the energy $E$ of the eigenstate. In particular, this means that $\tilde{H}_{\text{eff}}^{(n)}$ is not purely of $n$th order in the perturbation. This is an undesirable property that we shall eliminate perturbatively, expanding in corrections to $E$ to get an ordinary Hamiltonian that does not depend on the energy. We will put the effective Hamiltonian in the form

$$H_{\text{eff}} = \sum_{n=1}^{\infty} H_{\text{eff}}^{(n)}, \quad (2.50)$$

where the tilde has been dropped to signify that $H_{\text{eff}}^{(n)}$ is truly of $n$th order in the perturbation and does not depend on $E$.

The leading-order contribution is

$$\tilde{H}_{\text{eff}}^{(1)} = \mathcal{P} H_1 \mathcal{P} = H_{\text{eff}}^{(1)}, \quad (2.51)$$
because $H_{1p}$ has vanishing projection onto the ground state space. This term is of first order in the perturbation and does not depend on $E$, so it happens that $\tilde{H}_{\text{eff}}^{(1)} = H_{\text{eff}}^{(1)}$.

The behavior at second order is more generic, in that $\tilde{H}_{\text{eff}}^{(2)} \neq H_{\text{eff}}^{(2)}$. We have

$$\tilde{H}_{\text{eff}}^{(2)} = \mathcal{P}H_{1p} \frac{1 - \mathcal{P}}{E - H_0} H_{1p} \mathcal{P}. \quad (2.52)$$

In the above expression there is no $H_{1v}$ to the right of the $(1 - \mathcal{P})$ projector, since $H_{1v}$ cannot take states in the ground state space out of the ground state space and since $(1 - \mathcal{P})$ annihilates states in the ground state space. Similarly, there is no $H_{1v}$ to the left of the $(1 - \mathcal{P})$ term. To make the notation more compact we define

$$D \equiv \frac{1 - \mathcal{P}}{E - H_0}, \quad (2.53)$$

so that $\tilde{H}_{\text{eff}}^{(2)} = \mathcal{P}H_{1p}DH_{1p} \mathcal{P}$. It is important to keep in mind that $D$ depends on $E$.

To eliminate the $E$-dependence in $\tilde{H}_{\text{eff}}^{(2)}$, we write $E = E_0 + E^{(1)}$, where $E^{(1)}$ is the first-order correction to the energy. We will not need to include higher-order corrections, because we will see that all such corrections vanish up through fifth order, except for constant corrections that we drop.

Now suppose $\tilde{H}_{\text{eff}}^{(2)}$ acts on an eigenstate of $H_{\text{eff}}^{(1)}$. We expand in powers of $E^{(1)}$, and bring all factors of $E^{(1)}$ to the right of the expression, but still inside the rightmost projector $\mathcal{P}$. Each factor of $E^{(1)}$ can then be replaced by $H_{1v}$, as a consequence of the form of $H_{\text{eff}}^{(1)}$ and the fact that we are acting on an eigenstate of $H_{\text{eff}}^{(1)}$. We can then return $H_{1v}$ to the position where $E^{(1)}$ originally appeared in the expression, because, using the commuting projector structure of $H_1$, $H_{1v}$ commutes with $H_0$, $\mathcal{P}$ and $H_{1p}$. Since this holds for any eigenstate of $H_{\text{eff}}^{(1)}$, it holds for all states, and we can expand $D$ by

$$D = \left( \sum_{n=0}^{\infty} (-1)^n (D_0 H_{1v})^n \right) D_0, \quad (2.54)$$

where we define $D_0 = (1 - \mathcal{P})/(E_0 - H_0)$. Up to this point, we have only used the properties that $H_1$ is a sum of commuting terms, and that $H_{1v}$ commutes with $H_0$; we note that these properties also hold in the coupled-layer construction of the semionic X-cube model.
Applying this to our expression for $\tilde{H}_{\text{eff}}^{(2)}$ we obtain

$$\tilde{H}_{\text{eff}}^{(2)} = \mathcal{P}H_{1p}D_{0}H_{1p}\mathcal{P} - \mathcal{P}H_{1p}D_{0}H_{1v}D_{0}H_{1p}\mathcal{P} + \sum_{n=2}^{4} (-1)^{n}\mathcal{P}H_{1p} \left(D_{0}H_{1v}\right)^{n}D_{0}H_{1p}\mathcal{P} + \cdots.$$  

The second order contribution of original Hamiltonian to the effective Hamiltonian, $\mathcal{P}H_{1p}D_{0}H_{1p}\mathcal{P}$, is easily seen to be a constant. We therefore drop it and put $H_{\text{eff}}^{(2)} = 0$. This is convenient because then we then have $E^{(2)} = 0$, which simplifies going to higher orders. The higher order contributions will contribute to $H_{\text{eff}}^{(3)}$, and so on.

At third order we have

$$\tilde{H}_{\text{eff}}^{(3)} = \mathcal{P}H_{1p}D_{0}H_{1v}D_{0}H_{1p}\mathcal{P}, \quad (2.55)$$

where only this arrangement of $H_{1v}$ and $H_{1p}$ contributes. Expanding to sixth order, we obtain

$$\tilde{H}_{\text{eff}}^{(3)} = \mathcal{P}H_{1p} \left( \sum_{n=1}^{4} n(-1)^{n+1}(D_{0}H_{1v})^{n} \right) D_{0}H_{1p}\mathcal{P}. \quad (2.56)$$

To determine $H_{\text{eff}}^{(3)}$, we add the $n = 1$ term here with the same order of $H$ contributed by $\tilde{H}_{\text{eff}}^{(2)}$. Then we get $H_{\text{eff}}^{(3)} = 0$; similar cancelations will occur at higher order.

We now give expressions for $\tilde{H}_{\text{eff}}^{(n)}$ for $n = 4, 5, 6$, expanding each up through sixth order. The fourth-order term is

$$\tilde{H}_{\text{eff}}^{(4)} = \mathcal{P} \left(H_{1p}D_{0}\right)^{3}H_{1p}\mathcal{P} + \mathcal{P}H_{1p}(D_{0}H_{1v})^{2}D_{0}H_{1p}\mathcal{P} \equiv \tilde{H}_{\text{eff}}^{(4),a} + \tilde{H}_{\text{eff}}^{(4),b}. \quad (2.57)$$

Expanding $\tilde{H}_{\text{eff}}^{(4),a}$ up through 6th order we have

$$\tilde{H}_{\text{eff}}^{(4),a} = \mathcal{P} \left(H_{1p}D_{0}\right)^{3}H_{1p}\mathcal{P} + \mathcal{P}H_{1p} \left( \sum_{n=1}^{2} (-1)^{n} \sum_{\text{perm}} (D_{0}H_{1p})^{n} \right) D_{0}H_{1p}\mathcal{P},$$

where perm$([A]^{k},[B]^{\ell})$ is a term of the form $A^{n_{1}}B^{m_{1}}A^{n_{2}}B^{m_{2}}\cdots$, with $n_{i}, m_{i}$ non-negative integers satisfying $\sum_{i} n_{i} = k$ and $\sum_{i} m_{i} = \ell$, and the sum runs over all such distinct terms. For example,

$$\sum_{\text{perm}} \text{perm}([A]^{2},[B]^{2}) = A^{2}B^{2} + AB^{2}A + B^{2}A^{2} + ABAB + BA^{2}B + BABABA. \quad (2.58)$$

In general, perm$([A]^{k},[B]^{\ell})$ runs over $(k + \ell)!/(k!\ell!)$ different terms.
The lowest order term in $H_{1p}$ in the (2.57) is a constant that we drop. The attentive reader will notice that this term contains a super-extensive contribution to the energy, which we expect would have been canceled had we kept the extensive constant contribution at second order.

For $\tilde{H}_{1p}^{(4),b}$, we have

$$\tilde{H}_{1p}^{(4),b} = \mathcal{P}H_{1p} \left( (\mathcal{D}_0 H_{1v})^2 - 3(\mathcal{D}_0 H_{1v})^3 + 6(\mathcal{D}_0 H_{1v})^4 \right) \mathcal{D}_0 H_{1p} \mathcal{P}. \quad (2.59)$$

For higher orders, it is helpful to use the more general expression

$$\tilde{H}_{1p}^{(n)} = \mathcal{P}H_{1p} \left[ \frac{[n/2-1]}{k=0} \sum_{\text{perm}} \left( [\mathcal{D}H_{1p}]^{2k}, [\mathcal{D}H_{1v}]^{n-2k-2} \right) \right] \mathcal{D}H_{1p} \mathcal{P}. \quad (2.60)$$

which holds for $n > 1$. Setting $n = 5$ for fifth order, we obtain

$$\tilde{H}_{1p}^{(5)} \mathcal{P}H_{1p} \left( \sum_{\text{perm}} \left( [\mathcal{D}H_{1p}]^{2}, [\mathcal{D}H_{1v}]^{1} \right) \right) \mathcal{D}H_{1p} \mathcal{P} + \mathcal{P}H_{1p} \left( \mathcal{D}H_{1v} \right)^{3} \mathcal{D}H_{1p} \mathcal{P} \quad (2.61)$$

Expanding each term up to sixth order gives us

$$\tilde{H}_{1p}^{(5),a} = \mathcal{P}H_{1p} \left( \sum_{\text{perm}} \left( [\mathcal{D}H_{1p}]^{2}, [\mathcal{D}H_{1v}]^{1} \right) \right) \mathcal{D}H_{1p} \mathcal{P}, \quad (2.62)$$

$$\tilde{H}_{1p}^{(5),b} = \mathcal{P}H_{1p} \left( (\mathcal{D}_0 H_{1v})^3 - 4(\mathcal{D}_0 H_{1v})^4 \right) \mathcal{D}_0 H_{1p} \mathcal{P}. \quad (2.63)$$

Finally, we consider the sixth order term:

$$\tilde{H}_{1p}^{(6)} = \mathcal{P}H_{1p} \left( \sum_{\text{perm}} \left( [\mathcal{D}H_{1p}]^{2}, [\mathcal{D}H_{1v}]^{2} \right) \right) \mathcal{D}H_{1p} \mathcal{P} + \mathcal{P}H_{1p} \left( \mathcal{D}H_{1v} \right)^{4} \mathcal{D}H_{1p} \mathcal{P} + \mathcal{P} \left( \mathcal{H}_{1p} \mathcal{D} \right)^{5} \mathcal{H}_{1p} \mathcal{P} \quad (2.64)$$

$$\tilde{H}_{1p}^{(6),a} = \mathcal{P}H_{1p} \left( \sum_{\text{perm}} \left( [\mathcal{D}H_{1p}]^{2}, [\mathcal{D}H_{1v}]^{2} \right) \right) \mathcal{D}H_{1p} \mathcal{P}, \quad (2.65)$$

$$\tilde{H}_{1p}^{(6),b} = \mathcal{P}H_{1p} \left( \mathcal{D}_0 H_{1v} \right)^{4} \mathcal{D}_0 H_{1p} \mathcal{P}, \quad (2.66)$$

Since we will not go beyond sixth order, we can replace all the $\mathcal{D}$’s with $\mathcal{D}_0$’s, and write

$$\tilde{H}_{1p}^{(6),a} = \mathcal{P}H_{1p} \left( \sum_{\text{perm}} \left( [\mathcal{D}_0 H_{1p}]^{2}, [\mathcal{D}_0 H_{1v}]^{2} \right) \right) \mathcal{D}_0 H_{1p} \mathcal{P}, \quad (2.67)$$

$$\tilde{H}_{1p}^{(6),b} = \mathcal{P}H_{1p} \left( \mathcal{D}_0 H_{1v} \right)^{4} \mathcal{D}_0 H_{1p} \mathcal{P}, \quad (2.68)$$

$$\tilde{H}_{1p}^{(6),c} = \mathcal{P} \left( \mathcal{H}_{1p} \mathcal{D}_0 \right)^{5} \mathcal{H}_{1p} \mathcal{P}. \quad (2.69)$$
To compute $H_{\text{eff}}^{(n)}$ for $n = 4, 5, 6$, we simply collect terms. We find $H_{\text{eff}}^{(4)} = H_{\text{eff}}^{(5)} = 0$, and

$$H_{\text{eff}}^{(6)} = \mathcal{P} \left( H_{1p} D_0 \right)^5 H_{1p} \mathcal{P}.$$  \hspace{1cm} (2.70)

In this term, the only combination of six $H_{1p}$ operators which is not a constant, and which survives the leftmost projection onto the ground state space, is the cube operator $B_c$ defined in the main text. There are also constant contributions, which we drop.

Thus, up to sixth order the effective Hamiltonian is given by

$$H_{\text{eff}} = \mathcal{P} H_{1v} \mathcal{P} + \mathcal{P} \left( H_{1p} D_0 \right)^5 H_{1p} \mathcal{P}. \hspace{1cm} (2.71)$$

Ignoring constant contributions, this is identical to $H_{XC}$ in Eq. (2.7). The $\mathcal{P} H_{1v} \mathcal{P}$ term corresponds to the $A_i^\mu$ term, and the sixth order of $H_{1p}$ builds up to the $B_c$ term. The coefficient of the latter is proportional to $1/J_z^5$, because each denominator $D_0$ carries a factor of $J_z^{-1}$.

The analysis here applies essentially without modification to the coupled X-cube model described in Sec. 2.4, where we obtain the FCC model in the strong coupling limit. There, the $A_i^\mu$ terms play the role of $H_{1p}$, and the $B_c$ terms play the role of $H_{1v}$. The same analysis applies because the lowest-order non-constant term formed by taking a product product of $A_i^\mu$ operators also appears at sixth order, where the product is over the faces of a cube as described in Sec. 2.4.

### 2.8 Topological order and ground state degeneracy of the FCC model

Here, we argue that the FCC model has topological order, in the sense that it has a non-trivial ground state degeneracy on the 3-torus, and that the degenerate ground states cannot be distinguished by local measurements. As a byproduct of this discussion, we compute the ground state degeneracy on a finite $L \times L \times L$ torus, where $L$ is the linear system size, setting the spacing between neighboring vertices of the same color to one.

Our strategy is to first construct a complete set of commuting observables (CSCO), and then to use the properties of the CSCO to argue that the model has topological order. We begin by reviewing how this works for the $d = 2$ toric code on a $L \times L$ torus, which contains a total of
2L^2 Ising spins. The CSCO consists of stabilizers and logical operators. The stabilizers are the $A_i$ vertex and $B_p$ plaquette operators; there are $L$ operators of each type, but they are not all independent, because $\prod_p B_p = 1$ and $\prod_i A_i = 1$. Therefore there are a total of $2L^2 - 2$ independent stabilizers. The stabilizers appear in the Hamiltonian and have eigenvalue 1 in a ground state, so it follows that $\log_2 \text{GSD} = 2$. It also follows that the CSCO also includes two logical operators to be complete and label the ground space. One choice for the logical operators is described in Sec. 2.3.2, where they are taken to be string operators $S_m(\gamma_x)$ and $S_m(\gamma_y)$ threading $m$ particles around two independent cycles of the torus, along paths $\gamma_x$ and $\gamma_y$. Only the topological character (more precisely, the homology class) of $\gamma_x$ and $\gamma_y$ is important, because the exact geometry of the paths can be altered by multiplying the string operators by $A_i$ stabilizers.

To argue that the model has topological order, we need to show that a basis for the ground space $\{|\psi_i\rangle\}$ ($i = 1, \ldots, 4$) satisfies the property

$$\langle \psi_i | O | \psi_j \rangle = c_O \delta_{ij} \quad (2.72)$$

with $c_O$ a constant independent of $i$ and $j$. This equation holds when $O$ is any local operator, or more generally a multi-point correlation function of a finite number of local operators. We take the $|\psi_i\rangle$ to be eigenstates of the CSCO. Without loss of generality, we can take $O$ to be a product of $X$ and $Z$ Pauli operators, so that $O$ either commutes or anticommutes with all the operators in our complete commuting set. We should assume that $O$ commutes with all the stabilizers, since otherwise $c_O = 0$ and the property holds trivially. Then suppose $O$ fails to commute with one of the logical operators. In this case, we can “move the logical operator over” by multiplying with stabilizers, to get a new logical operator that commutes with $O$, as $O$ is assumed to be local. This is only consistent if $O$ anticommutes with some stabilizers, which contradicts our assumptions. Therefore, $O$ commutes with all the logical operators. This shows the right-hand side of Eq. (2.72) vanishes when $i \neq j$.

To complete the argument, we have to consider Eq. (2.72) when $i = j$, and show the proportionality constant $c_O = \langle \psi_i | O | \psi_i \rangle$ is independent of $i$. To do this, we observe that products of $X$
along closed curves winding around the torus can be used to flip logical operators and connect any
ground state to any other ground state. These products are simply string operators that thread e
particles around the torus, and can also be “moved out of the way of \( \mathcal{O} \)” by multiplying with \( B_p \)
stabilizers. Suppose we call such an operator \( \chi \), so that \( |\psi_j\rangle = \chi |\psi_i\rangle \) and \( \chi \mathcal{O} = \mathcal{O} \chi \). Then we have

\[
\langle \psi_i | \mathcal{O} | \psi_i \rangle = \langle \psi_i | \mathcal{O} \chi \chi | \psi_i \rangle \\
= \langle \psi_i | \chi \mathcal{O} \chi | \psi_i \rangle = \langle \psi_j | \mathcal{O} | \psi_j \rangle ,
\]

which establishes the desired result.

With this review out of the way, we now describe the application of a similar strategy to
the FCC model on a \( L \times L \times L \) torus. The analysis turns out to be simpler for \( L \) odd, so we
focus on that case. As an intermediate step, we first count the independent \( X \)-stabilizers \( (B_c \) cube
operators) in the X-cube model. There are \( L^3 \) such operators, but they are not all independent. If
\( P \) is some \( \{100\} \) lattice plane, then we have in total \( 3L \) constraints among the stabilizers

\[
C^X_P \equiv \prod_{c \in P} B_c = 1.
\]

But now not all those constraints are independent. We have relation

\[
\prod_{P \in \{xy \ planes\}} C^X_P = \prod_{P \in \{xz \ planes\}} C^X_P = \prod_{P \in \{yz \ planes\}} C^X_P .
\]

Each expression above is thus actually the same constraint, and we find \( 3L - 2 \) independent con-
straints, for a total of \( L^3 - 3L + 2 \) independent \( X \)-stabilizers.

This conclusion, and other similar counting problems below, can be checked numerically
for reasonably small values of \( L \). This is important because the reasoning employed here is not
rigorous; in principle, some dependency could have been missed. Our numerical approach is based
on mapping the counting of independent stabilizers to a problem in linear algebra over the two-
element field \( \mathbb{F}_2 \). Any product of \( X \) Pauli operators can be thought of as an element of the \( \mathbb{F}_2 \) vector
space \( V_X \simeq (\mathbb{F}_2)^{3L^3} \), where vector addition corresponds to operator multiplication. \( X \)-stabilizers
comprise a subspace \( S_X \subset V_X \), and the \( B_c \) stabilizers make up a spanning set for \( S_X \) with \( L^3 \)}
elements. Viewing the spanning set as a $L^3 \times 3L^3$ matrix, the dimension of $S_X$ is the rank of this matrix, which can be determined via row reduction. We used this method to check the counting of independent $X$-stabilizers in the X-cube model for $L = 2, \ldots, 8$. Our results agree with those obtained for arbitrary odd $L$ in Ref. [110] by rigorous algebraic methods.

We now count independent $X$-stabilizers in the FCC model. Naively, not taking any constraints into account, there are $L^3$ such operators for each underlying X-cube model, for a total of $4L^3$. For each color, there are $3L - 2$ constraints involving that color alone, for a total of $4(3L - 2)$ constraints. In addition, there are constraints that couple all four colors together. Letting $B_0$ be some $X$-stabilizer, we have a constraint

$$C_X = \prod_{n_1, n_2} T(n_1a_1 + n_2a_2)B_0T(n_1a_1 + n_2a_2)^{-1} = 1,$$  (2.76)

where $T(R)$ is the unitary operator realizing translation by a Bravais lattice vector $R$. The primitive fcc lattice vectors $a_1, a_2, a_3$ are defined in Sec. 2.4. In general, the translation $T(R)$ acts in a “spin-orbit coupled” manner where the colors are permuted, as described in Sec. 2.4. Here, this implies that $C_X$ contains contributions from $B_c$ stabilizers of all four colors. It is not obvious a priori that $C_X = 1$, but this can be shown by a tedious calculation. The product in Eq. (2.76) is over a $\{111\}$ plane, and by symmetry the same constraint holds for any such plane. Naively this gives $4L$ constraints, since there are four orientations of $\{111\}$ planes, and $L$ different planes for a given orientation. The actual number of such constraints is $4L - 4$, because taking a product over all $\{111\}$ planes of the same orientation gives a product over all $X$-stabilizers of all four colors, which is not an independent constraint as it can be obtained by taking products of single-color constraints on $\{100\}$ planes. The total number $N_X$ of independent $X$-stabilizers is thus

$$N_X = 4L^3 - 4(3L - 2) - (4L - 4) = 4L^3 - 16L + 12.$$  (2.77)

This result has been checked numerically for $L = 2, \ldots, 7$, including even values of $L$.

By electric-magnetic self duality, the number of independent $Z$-stabilizers is equal to the number of independent $X$-stabilizers, $N_Z = N_X$. Therefore the total number of independent...
stabilizers is
\[ N_S = N_Z + N_X = 8L^3 - 32L + 24. \tag{2.78} \]
Taking into account the on-site \( X^u_\ell X^w_\ell X^u_\ell = 1 \) constraint, there are eight Ising spins per simple cubic unit cell, for a total of \( 8L^3 \) Ising spins. We thus infer the ground state degeneracy
\[ \log_2 \text{GSD} = 32L - 24. \tag{2.79} \]

The counting of stabilizers implies that we need to find \( 32L - 24 \) independent logical operators, to complete our CSCO. We choose to work with \( X \)-logical operators (XLOs), i.e. those built from products of \( X \) operators. Using the \( \mathbb{F}_2 \) vector space notation described above, we introduce a subspace \( C_X \) satisfying \( S_X \subset C_X \subset V_X \), which is defined to contain all products of \( X \)'s that commute with every \( Z \)-stabilizer. Then \( C_X \) is spanned by the union of all \( X \)-stabilizers and all XLOs. Two XLOs are considered equivalent if they are related by multiplication of \( X \)-stabilizers; therefore, independent XLOs are associated with elements of the quotient space \( L_X \equiv C_X/S_X \).

The number of independent XLOs is then given by
\[ N_{\text{XLO}} = \dim L_X = \dim C_X - \dim S_X. \tag{2.80} \]
Since we expect \( N_{\text{XLO}} = 32L - 24 \), we can use this to complete the CSCO. We choose a putative spanning set for \( C_X \) consisting of all \( X \)-stabilizers and a conjectured generating set of XLOs. If, using this set and Eq. (2.80), we find \( N_{\text{XLO}} = 32L - 24 \), then we have found a CSCO that consists of all \( X \) and \( Z \) stabilizers, and the generating set of XLOs.

We choose a generating set of XLOs that includes two types of operators. Type 1 XLOs are products of \( X^u_\ell \) of a single color, along a closed straight line (winding once around the torus). We include all such XLOs in our generating set. Even though there are \( 12L^2 \) type 1 XLOs, we find numerically for \( L = 3, 5, 7 \) that they do not form a complete generating set; that is, when we only include type 1 XLOs in the generating set, we find \( \dim C_X - \dim S_X < 32L - 24 \). Evidently, the type 1 XLOs in the generating set are not all independent.

Type 2 XLOs are string operators that run along a \( \langle 110 \rangle \) direction; an example is shown in Fig. 2.19. These strings involve links of two colors that cut transversely to the \( \langle 001 \rangle \) plane in which
the string lies. We include in the generating set all XLOs running within one (001) plane with arbitrary normal coordinate $z$, and similarly for one (100) and one (010) plane.\textsuperscript{3} For $L = 3, 5, 7$, we find that this generating set including both type 1 and type 2 XLOs is enough to generate $32L - 24$ independent XLOs.

To finish arguing for topological order of the FCC model, we have to establish the local indistinguishability of ground states as expressed in Eq. (2.72). We can follow essentially the same argument for the toric code, noting first that type 1 XLOs can be "moved away from $O$" by multiplying by $X$-stabilizers. It is not clear how to move type 2 XLOs, but the three planes in which these string operators run can be chosen arbitrarily in such a way that the region on which the local operator $O$ is supported is avoided. Finally, electric-magnetic self-duality implies that there is a generating set of $Z$ logical operators (ZLOs) with the same properties as the generating set of XLOs, and these ZLOs can be used to connect different ground states that are eigenstates of the XLOs.

\textsuperscript{3} In more detail, for XLOs running in [110] and [11] directions, we consider all red-black XLOs running within one (001) plane with normal coordinate $z$, and all green-blue XLOs running within a neighboring (001) plane with normal coordinate $z + 1/2$. 

Figure 2.19: Illustration of a type 2 XLO running along the [110] direction. A product is taken over a line of thick-shaded black and red links as shown. The string extends in the [110] direction, while the links point in the $z$-direction.
Chapter 3

Topological Entanglement Entropy of Fracton Stabilizer Codes

3.1 Introduction

As introduced, the entanglement entropy is an important characterization of the quantum state. Thus, it is natural to ask, what insights are afforded by studying entanglement in fracton phases of matter? A previous work [98] has introduced certain bounds on the non-local part of the entanglement entropy in fracton models.

In this chapter, we discuss the entanglement entropy in fracton phases. We begin by reviewing the formalism developed to study the entanglement structure of stabilizer codes. We then explicitly calculate the entanglement entropy for several $d = 3$ models, including two paradigmatic fracton phases: the ‘X-cube model’ and ‘Haah’s code’. Both fracton phases are found to exhibit a term of non-local origin in the entanglement entropy that scales linearly with subsystem size, with a coefficient that we calculate. We refer to this term as topological entanglement entropy.

We also point out that the existence of topological entanglement entropy indicates that on the boundary of the subregion, certain non-local constraints act on the ground state wavefunction. We explicitly identify these constraints for the X cube model. This provides an argument that the topological entanglement entropy is robust under arbitrary local perturbations of the Hamiltonian. Consider a perturbation $H' = \lambda V$, where $\lambda$ is a control parameter, and $V$ is of unit norm and contains perturbations local in real space which do not commute with $H(\lambda = 0)$. When $\lambda \neq 0$, the stabilizers are no longer eigenoperators of the groundstate. However, for $\lambda$ sufficiently small that it does not induce a phase transition as increased from zero, the new ground state is related to the
\( \lambda = 0 \) ground state by a unitary transformation

\[
|\Psi(\lambda)\rangle = U|\Psi(\lambda = 0)\rangle,
\]

where \( U \) is a local unitary operator which can in principle be constructed by adapting the method of Schrieffer-Wolff transformations [8] (see Appendix 3.5 for details). Non-local constraints on the wavefunction cannot be altered by local unitary transformations; thus the topological entanglement entropy (being of non-local origin) must be invariant under local perturbations.

Finally we generalize this argument to excited states of disordered fracton models, which we argue can display localization protected fracton order, characterized by topological entanglement entropy.

### 3.2 Topological entanglement entropy in stabilizer codes

#### 3.2.1 Entanglement entropy for stabilizer codes

We begin our analysis by reviewing the method developed in Ref. [134, 33, 20] for calculating the entanglement entropy for the ground states of stabilizer codes, and discussing the straightforward extension to arbitrary eigenstates. This extension will be used in Sec. 3.6 when we discuss localization protected fracton topological order in excited states. We will illustrate the use of the stabilizer formalism on the toric code before applying these techniques to fracton models. Readers familiar with the stabilizer formalism for ground-state entanglement computations may wish to skip ahead to Sec. 3.4.2.

Throughout, we will consider systems of spin-1/2 degrees of freedom ('qubits') that reside on either vertices or links of \( d \)-dimensional hypercubic lattices. The Hamiltonians is built solely of terms containing either \( X \) type stabilizer operators that are products of Pauli matrices \( X_i \), or \( Z \) type stabilizer operators that are products of Pauli matrices \( Z_i \). The stabilizer operators all mutually commute, so that the Hamiltonian is exactly solvable. By construction, both \( X \)- and \( Z \)-type stabilizers square to the identity and hence have eigenvalues \( \pm 1 \).
Let us take $\mathcal{S} = \{O_s\}$ to be a set of mutually commuting stabilizers. Then, a state $|\psi\rangle$ is stabilized by $\mathcal{S}$ if it satisfies

$$O_s |\psi\rangle = |\psi\rangle,$$

for all $O_s \in \mathcal{S}$. The set of states stabilized by $\mathcal{S}$ is called a stabilizer code and is the ground state manifold of the Hamiltonian, $H_{\text{stab}} = -\sum_s J_s O_s$ with $J_s > 0$. Here, we are assuming that the $O_s$ operators are local, but, below, we relax this assumption.

It is useful to consider the Abelian group $G$ that is multiplicatively generated by the stabilizers in $\mathcal{S}$. We assume that all the stabilizers in $\mathcal{S}$ are independent and collectively act on exactly $N$ spins-1/2. For open boundary conditions, this is generally expected to be the case for a maximal set of independent local stabilizers. For periodic boundary conditions, we usually need to include some non-local stabilizers that, for instance, wrap non-contractible loops. Elements $g \in G$ may be labeled by a binary vector $\vec{n} = \{n_1, n_2, \ldots, n_{|\mathcal{S}|}\}$, with $n_i \in \{0, 1\}$ via

$$g(\vec{n}) = O_1^{n_1} O_2^{n_2} \ldots O_{|\mathcal{S}|}^{n_{|\mathcal{S}|}},$$

where $|\mathcal{S}|$ is the number of stabilizers in $\mathcal{S}$. Note that $\mathcal{S}$ is a generating set. Because $G$ is Abelian, we may label states in the Hilbert space by their eigenvalues of the group elements $g \in G$.

We note that $G$ can be viewed as a vector space over the two-element field $\mathbb{F}_2$, a fact that will be useful in our approach to numerical calculation of entanglement entropy. This statement holds for any group, like $G$, that is isomorphic to a product of $\mathbb{Z}_2$ factors. Vector addition corresponds to multiplication of stabilizers. The zero vector corresponds to the group identity. Scalar multiplication is trivial; multiplication by $1 \in \mathbb{F}_2$ is the identity operation, and multiplication by $0 \in \mathbb{F}_2$ sends any element of $g \in G$ to the identity of $G$. We will use both vector space and group language to describe operations in $G$, and will do this without comment when the meaning is clear from the context. We note that, in vector space language, the set $\mathcal{S}$ is a basis.

It remains to determine the size of the group $G$, that we denote $|G|$. We further suppose that there is a unique eigenstate $|\psi\rangle$ of eigenvalue +1 for all $g \in G$; if this is not the case, then we may add elements to $\mathcal{S}$ until this is so. Then, $G$ cannot contain any pure scalar element $\eta \neq 1$ since
such an element must have an eigenvalue that is not 1 (a scalar element is an element proportional to the identity). We may then write the projector onto $|\psi\rangle$ as

$$
|\psi\rangle\langle\psi| = \frac{1}{|G|} \sum_{\vec{n}} g(\vec{n}).
$$

(To see this, observe using group properties that the RHS squares to itself, i.e. is a projector, and acts as the identity on $|\psi\rangle$, whence the result Eq. (3.4) follows by the uniqueness of the ground state.) Taking the trace on both sides, we find

$$
1 = \frac{1}{|G|} \sum_{\vec{n}} \text{Tr} g(\vec{n}) = \frac{1}{|G|} \text{Tr} I = \frac{2^N}{|G|},
$$

(3.5)

where we use the uniqueness of the identity $I \in G$ and the fact that any non-identity element in $G$ is traceless; the final step follows simply from the fact that the Hilbert space is the tensor product of $N$ spins-1/2. Thus, we see that the size of $G$ is the full dimension of the Hilbert space, $|G| = 2^N$.

From this and Eq. (3.3) we conclude that there are $N$ independent stabilizers in $\mathcal{S}$, i.e. $|\mathcal{S}| = N$. Furthermore, we see that the group $G$ must be isomorphic to $\mathbb{Z}_2^{\otimes N}$, the group of spin-flips on $N$ spins (intuitively, we may think of the ‘spin’ as the eigenvalue of the stabilizer $O_s$.) Therefore, we may completely label eigenstates of $H_{\text{stab}}$ in terms of irreducible representations of the spin-flip group [33]. The irreps are one-dimensional and are labeled by a binary string $\vec{k}$ of length $N$, and defined by the map $\rho_{\vec{k}}[g(\vec{n})] = (-1)^{\vec{k} \cdot \vec{n}}$; in other words, the eigenvalue of a group element $g$ in state $\vec{k}$ is given by $(-1)^{\vec{k} \cdot \vec{n}}$. It then follows from standard orthogonality relations that the density matrix of state $|\vec{k}\rangle$ can be written as

$$
|\vec{k}\rangle\langle\vec{k}| = \mathcal{P}(\vec{k}) = \frac{1}{|G|} \sum_{\vec{n}} (-1)^{\vec{k} \cdot \vec{n}} g(\vec{n}).
$$

(3.6)

In this notation, the ground state $|\psi\rangle$ corresponds to the vector $\vec{k} = (0,0,\ldots)$.

We will now demonstrate how to compute the entanglement entropy of a subregion for any such eigenstate $|\vec{k}\rangle$ of $H_{\text{stab}}$. Consider any bipartition $(A,B)$ as described above. Then, we see that the reduced density matrix of subregion $A$ in state $|\vec{k}\rangle$ is

$$
\rho_A = \text{Tr}_B|\vec{k}\rangle\langle\vec{k}| = \frac{1}{|G|} \sum_{\vec{n}} (-1)^{\vec{k} \cdot \vec{n}} \text{Tr}_B g(\vec{n}).
$$

(3.7)
Any element $g$ that is \textit{not} equal to the identity on $B$ must contain at least one $X$ or $Z$ operator acting in $B$ and consequently will have vanishing trace in $B$. Thus, the only non-zero contributions to the sum in Eq. (3.7) are from operators supported only on $A$, i.e. equal to the identity on $B$. Operators in the sum supported only on $A$ form a subgroup $G_A$, with elements $g(\vec{n}_A)$, and irreps labeled by $\vec{k}_A$, where we have made the obvious generalizations of notation. Since the identity on $B$ has trace $\text{Tr} I_B = 2^{N_B}$ where $N_B$ is the number of spins in $B$, we have (using $|G| = 2^N = 2^{N_A+N_B}$) that

$$\rho_A = \frac{2^{N_B}}{|G|} \sum_{\vec{n}_A} (-1)^{\vec{k}_A \cdot \vec{n}_A} g(\vec{n}_A) = \frac{|G_A|}{2^{N_A}} P_A(\vec{k}_A). \quad (3.8)$$

Note that the group $G_A$ only includes \textit{complete} stabilizers in $A$, since stabilizers that ‘dangle’ across the entanglement cut are not equal to the identity on $B$. Since Eq. (3.8) expresses $\rho_A$ as a projector, its entanglement entropy follows straightforwardly:

$$S_A = N_A - \log_2 |G_A|. \quad (3.9)$$

Thus, the computation of entanglement entropy of a subregion $A$ reduces to that of determining the size of the subgroup $G_A \subset G$ that consists of stabilizers contained entirely within $A$. In vector space language, $\log_2 |G_A| = \dim G_A$, so

$$S_A = N_A - \dim G_A. \quad (3.10)$$

Viewed as an $\mathbb{F}_2$ vector space, $G_A = G_A^Z \oplus G_A^X$, where $G_A^Z$ and $G_A^X$ are vector spaces of $Z$ and $X$ stabilizers, respectively. Therefore

$$\dim G_A = \dim G_A^Z + \dim G_A^X, \quad (3.11)$$

so that $Z$ and $X$ stabilizers can be treated separately in computing $S_A$.

\textbf{3.2.2 Local and non-local stabilizers}

Here, we briefly discuss some further properties of the stabilizer group $G_A$ in connection with entanglement entropy and topological entanglement entropy. These properties are then used in
Figure 3.1: Two-dimensional toric code on the square lattice. The region enclosed by red the line is the subregion $A$, which has size $2 \times 2$ measured in edges of the lattice. Spins on links cut by the red line lie outside $A$. The vertex and plaquette terms $A_v$ and $B_p$ are also shown.

the numerical procedure for computing entanglement entropy described in Sec. 3.2.3. Moreover, we show that $S_{\text{topo}}^{PQWT}$ is entirely determined by counting non-local stabilizers, while in many cases $S_{\text{topo}}^{ABC}$ can be determined by counting local stabilizers.

Given a stabilizer Hamiltonian, we obtain a set $S_{\text{loc}}^{Z}$ of local $Z$ stabilizers, which are just the terms of the Hamiltonian. (For simplicity of discussion we focus for the moment on $Z$ stabilizers; identical statements hold for $X$ stabilizers.) Local $Z$-stabilizers supported entirely in $A$ generate a subgroup $G_{A,\text{loc}}^{Z} \subset G_{A}^{Z}$. If $G_{A}^{Z} = G_{A,\text{loc}}^{Z}$, we say that $G_{A}^{Z}$ is \textbf{locally generated}. In this case, $\dim G_{A}^{Z}$ can be obtained by simple counting, accounting for possible constraint equations satisfied by the local stabilizers. (A constraint is a product of local stabilizers which evaluates to the identity, i.e. $\prod_{O_s \in F} O_s = I$ for some subset $F \subseteq S$.) In general, $G_{A,\text{loc}}^{Z} \neq G_{A}^{Z}$. If $g \in G_{A}^{Z}$ but $g \notin G_{A,\text{loc}}^{Z}$, we call $g$ a \textbf{non-local stabilizer}. In addition to $\dim G_{A}^{Z}$, we will be interested in the number of independent non-local stabilizers, which is defined by

$\Omega_{A}^{Z} \equiv \dim G_{A}^{Z}/G_{A,\text{loc}}^{Z}$. \hspace{1cm} (3.12)

By taking the quotient, we are counting non-local stabilizers up to multiplication by local stabilizers. That is, two non-local stabilizers in $G_{A}^{Z}$ that are related by a product of local stabilizers in $G_{A,\text{loc}}^{Z}$
are not considered independent in this counting. We have

$$\dim G_A^Z = \dim G_{A,\text{loc}}^Z + \Omega_A^Z. \quad (3.13)$$

Now we consider topological entanglement entropy $S_{\text{topo}}^{ABC}$, obtained in an ABC type prescription as discussed in Sec. 1.2.2. We assume that the stabilizer groups $G_A, G_B, G_C, G_{AB}$, and so on, are all locally generated. Moreover, we assume the generators of these groups are drawn from a set of local stabilizers that do not obey any local constraints. This holds trivially for the $d = 2$ toric code and Haah’s code, where the stabilizers do not obey any local constraints. For the X-cube model, the vertex stabilizers do obey local constraints, but for suitable regions $A, B, C$ it is possible to use only $xy$ plane and $xz$ plane vertex stabilizers, which do not obey local constraints, as shown in Appendix 3.9.1. Under these assumptions, $-S_{\text{topo}}$ is simply the number of independent local stabilizers that have non-trivial support in each of the disjoint regions $A, B$ and $C$. The contributions of other local stabilizers to $S_{\text{topo}}^{ABC}$ cancel out. For instance, suppose some local stabilizer is supported entirely in $A$. Then it is also contained in $AB, AC$ and $ABC$, and it contributes +1 to each of $\dim G_A, \dim G_{AB}, \dim G_{AC}$ and $\dim G_{ABC}$. These contributions cancel in $S_{\text{topo}}$. Similarly, if a local stabilizer has non-trivial support in both $A$ and $B$, but not $C$, then it is contained in both $AB$ and $ABC$, and its contribution to $S_{\text{topo}}$ cancels. A more careful argument for this result is given in Appendix 3.8.

In contrast, $S_{\text{topo}}^{PQWT}$, the topological entanglement entropy obtained via a PQWT type prescription is determined in many cases entirely by non-local stabilizers. Here we give a rough argument that all contributions of local stabilizers cancel out; a more complete treatment is given in Appendix 3.8. First recall that $P = Q \cup W = Q \cup W \cup T$, so it is enough to consider different types of local stabilizers supported entirely on $P$. Suppose a local stabilizer is supported entirely in $T$, then it is contained in all four regions, and its contribution to $S_{\text{topo}}^{PQWT}$ cancels. Now suppose a stabilizer is contained in $Q$ but not in $T$, then it is also contained in $P$ but not in $W$, so its contribution cancels. This covers all the possibilities for local stabilizers. Because $N_P + N_T = N_Q + N_W$,
we have the result
\[ S_{\text{topo}}^{PQWT} = -\Omega_P + \Omega_Q + \Omega_W - \Omega_T + \Delta_{PQWT}, \quad (3.14) \]
where \( \Omega_P = \Omega^Z_P + \Omega^X_P \) is the total number of non-local stabilizers in \( P \), and similarly for the other regions. \( \Delta_{PQWT} \) is a correction associated with non-local constraints among local stabilizers. In Appendix 3.8, we derive Eq. (3.14), and show that \( \Delta_{PQWT} = 0 \) for all the models discussed in this work except the \( d = 3 \) toric code.

### 3.2.3 Calculating the entanglement entropy

In many cases, it is possible to determine \( |G_A| \) and hence the entanglement entropy \( S_A \) analytically. However, this is not always straightforward, and numerical calculation is useful as a check on other methods and sometimes as a primary means of determining the entanglement entropy. Here, we describe a numerical procedure to determine \( \dim G_A = \log_2 |G_A| \). We are always interested in the case where \( A \) is a subset of a thermodynamically large region. Under a suitable assumption discussed below, which can be verified for particular models of interest, \( \dim G_A \) does not depend on global properties of the large region containing \( A \).

To proceed, we choose a finite enclosing region \( B \) with \( A \subset B \). It is obvious that \( G^Z_A \subset G^Z_B \). We make the assumption that it is always possible to choose \( B \) so that \( G^Z_B \) is locally generated. This assumption implies \( G^Z_A \subset G^Z_{B,\text{loc}} \); that is, all stabilizers in \( A \) are products of local stabilizers in \( B \). We show in Appendix 3.9 that this assumption holds for the stabilizer codes studied in this work.

Now we introduce \( F^Z_{B,\text{loc}} \), the group of formal products of local \( Z \) stabilizers supported on \( B \). We let \( M \) be the number of local \( Z \) stabilizers supported entirely in \( B \), and denote these operators by \( O_s \) (\( s = 1, \ldots, M \)). A general product of these stabilizers is \( (O_1)^{n_1} \cdots (O_M)^{n_M} \), where \( n_i = 0, 1 \). There are \( 2^M \) such products, and in the group \( F^Z_{B,\text{loc}} \), we treat them all as distinct elements, so that \( F^Z_{B,\text{loc}} \simeq 2 \otimes^M \). In general, two formal products in \( F^Z_{B,\text{loc}} \) can correspond to the same operator,
because there can be constraints among the local stabilizers. There is a linear map

$$\phi_Z : F^Z_{B,\text{loc}} \rightarrow P^Z_B,$$  

(3.15)

where $P^Z_B$ is the group (or $\mathbb{F}_2$ vector space) generated by all $Z$ Pauli operators supported on $B$. The map $\phi_Z$ is defined by replacing the $O_s$ in a formal product with their expressions in terms of Pauli operators. $G^Z_{B,\text{loc}}$ is a subspace of $P^Z_B$, and moreover $G^Z_{B,\text{loc}} = \text{Im}\phi_Z$. If $f \in \text{Ker}\phi_Z$, then $f$ is a formal product of stabilizers that evaluates to the identity operator. This happens when the stabilizers obey some constraint equations. Indeed, the number of independent such constraints is precisely $\dim \text{Ker}\phi_Z$.

We note that $P^Z_B = P^Z_{B-A} \oplus P^Z_A$, where $B - A$ is the complement of $A$ in $B$. We thus have the projection map

$$\pi_{B-A} : P^Z_B \rightarrow P^Z_{B-A},$$  

(3.16)

defined by

$$\pi_{B-A}(g) = \begin{cases} 
1 & g \in P^Z_A \\
g & g \in P^Z_{B-A} 
\end{cases}.$$  

(3.17)

Elements in the kernel of $\pi_{B-A}$ are products of $Z$ Pauli operators supported entirely on $A$.

The last step is to consider the composition $\pi_{B-A} \circ \phi_Z$. Suppose $f \in \text{Ker}(\pi_{B-A} \circ \phi_Z)$, but $f \notin \text{Ker}\phi_Z$. This means that $f$ corresponds to a non-trivial element of $G^Z_A$. We are finally able to express the number of stabilizers in $A$ as

$$\dim G^Z_A = \dim \text{Ker} \pi_{B-A} \circ \phi_Z - \dim \text{Ker}\phi_Z.$$  

(3.18)

Here, the last term is subtracted to avoid incorrectly counting constraints among local stabilizers as non-trivial elements of $G^Z_A$. The linear maps $\phi_Z$ and $\pi_{B-A}$ can be constructed explicitly as matrices, and numerical linear algebra methods can then be used to compute the dimensions of the kernels. Our calculations were done using routines for linear algebra over $\mathbb{F}_2$ in Mathematica.

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1 To prove this result, we let $\{a_1, \ldots, a_n\}$ be a basis for $\text{Ker}\phi_Z$ and extend it to a basis $\{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ for $\text{Ker} \pi_{B-A} \circ \phi_Z$. It can then be checked that $\{\phi_Z(b_1), \ldots, \phi_Z(b_m)\}$ is a basis for $G^Z_A$, and the result follows.
There is a minor modification of the above approach that significantly reduces the computational effort required. We consider the subspace $F^Z_{A,loc} \subset F^Z_{B,loc}$, which consists of formal products of stabilizers supported entirely in $A$. Then, we have $F^Z_{B,loc} = F^Z_{A,loc} \oplus F$ where $F$ consists of formal products of stabilizers in $B$ that either lie completely outside $A$ or are not fully contained in $A$. Taking $f \in F$, if $f \notin \text{Ker} \pi_{B-A} \circ \phi_Z$, then $\phi_Z(f)$ is a non-trivial stabilizer in $G^Z_A$. The number of such stabilizers is

$$\tilde{\Omega}^Z_A = \dim \text{Ker} \pi_{B-A} \circ \phi_Z|_F - \dim \text{Ker} \phi_Z|_F.$$  \hspace{1cm} (3.19)

It can happen that $\phi_Z(f)$ is a local stabilizer in $G^Z_A$; this can happen when the local stabilizers obey some local constraints, so that some local stabilizers in $G^Z_A$ can be written as a product of local stabilizers not supported entirely in $A$. We let $K^Z_{A,loc} \subset G^Z_{A,loc}$ be the subspace generated by such local stabilizers in $A$. If the local stabilizers generating $G^Z_{B,loc}$ obey no local constraints, then $\tilde{\Omega}^Z_A = \Omega^Z_A$. More generally,

$$\Omega^Z_A = \tilde{\Omega}^Z_A - \dim K^Z_{A,loc},$$

which determines $\dim G^Z_A$ via Eq. (3.13).

We use this method to compute topological entanglement entropy for Haah’s code as discussed in Sec. 3.4.3. In addition, we employ the same method to check our results for the X-cube model in Sec. 3.4.2.

3.2.4 Simple example: topological entanglement entropy of $d = 2$ toric code

To illustrate the use of the stabilizer formalism, we now use it to compute the entanglement entropy of the $d = 2$ toric code model [42]. One qubit resides on each link of the square lattice, and the Hamiltonian is

$$H_{TC} = - \sum_v A_v - \sum_p B_p,$$

where $B_p$ is the product of the four $Z$ operators surrounding the plaquette $p$, and $A_v$ is the product of the four $X$ operators connected to vertex $v$. 
We compute the entanglement entropy of a subsystem $A$ of size $R \times R$, shown in Fig. 3.1 for $R = 2$. This region contains $N_A = 2(R + 1)R$ spins. Additionally, there are $R^2$ plaquette terms and $(R - 1)^2$ vertex terms confined entirely within subregion $A$; these are local stabilizers and they are all independent. Using the fact that $A$ is simply connected, it can be seen easily that $G_A$ is generated by the plaquette and vertex stabilizers supported entirely in $A$. Moreover, these stabilizers obey no local constraints. Therefore, $|G_A| = 2^{R^2+(R-1)^2}$.

Using Eq. (3.9), we find that the entanglement entropy is

$$S_A = 2(R + 1)R - R^2 - (R - 1)^2 = 4R - 1.$$  \hspace{1cm} (3.22)

Since the boundary has length $|\partial A| = 4R$ measured in lattice edges, we have $S_A = |\partial A| + S_{\text{topo}}$, where $S_{\text{topo}} = -1$ is the well-known topological entanglement entropy of the two dimensional toric code. This is something of an accident; this region has sharp corners, but the corner contributions happen to vanish. To actually obtain $S_{\text{topo}}$, we can break region $A$ into three subregions $A, B, C$ as in Fig. 1.4a, and apply the ABC prescription to obtain $S_{\text{topo}}^{ABC}$ as in Eq. (1.5).

To illustrate features not arising in the above simple example, we calculate $S_P$ for an annulus-shaped region $P$ shown in Fig. (1.4b). There, due to the non-trivial topology, $G_P$ is no longer generated by local stabilizers supported entirely on $P$. Using the terminology of Sec. 3.2.2, $G_P$ is generated by the local stabilizers in $P$, together with two non-local stabilizers. These are products of plaquette and vertex stabilizers, respectively, over the hole in the annulus. Taking the linear size of the hole to be $R$ and that of the exterior edges to be $3R$, we obtain $S_P = 16R - 6 = |\partial P| - 4 - 2$, where the constant $-4$ is non-universal and contributed by vertex terms at four concave corners of the subsystem while the $-2$ results from the number of non-local plaquette and vertex stabilizers acting on this subsystem. Although there are non-universal parts in this entropy due to the detailed geometry, the topological part of the entropy can be extracted by canceling all of those boundary contributions out via the PQWT prescription in Eq. (1.6).

A different perspective on the topological entanglement entropy, that further clarifies its robustness, is afforded by an understanding of the ground state wave function of the toric code as a
condensate of closed loops of $\mathbb{Z}_2$ ‘electric’ field lines. Each connected component of the entanglement surface intersects any loop an even number of times; this topological fact provides exactly one bit of extra information about the ground state, thereby reducing the entanglement entropy by a universal correction of $-1$ for each connected component of the entanglement surface. Similar ideas can be used to clarify the topological entanglement entropy of other topological orders in two dimensions by constructing their ground states as string-net condensates [50, 49].

3.3 Topological Entanglement Entropy in $d = 3$

As we have discussed, extracting the topological contribution to the entanglement entropy generally requires employing a prescription designed to cancel local contributions. This becomes even more important in $d = 3$, where Ref. [26] argued that even for a subregion $A$ with a smooth boundary, in the absence of parity and continuous rotation symmetry, every term in the expansion of $S_A$ in powers of the inverse linear size $R^{-1}$ acquires a local, non-universal contribution. Continuous rotation symmetry is absent in fracton models, and, more seriously, the dynamics of fracton excitations leads to an expectation that fracton topological orders lack a continuum description with continuous rotation symmetry. Discrete symmetries like parity may or may not be present in a given solvable fracton model, but certainly need not be present upon perturbing such a model to make it generic.

Therefore, we rely on $d = 3$ generalizations of the ABC and PQWT prescriptions discussed in Sec. 1.2.2. We use two different PQWT type prescriptions [12], one is illustrated in Fig. 3.3, the other in Fig. 3.6. A naïve extension of the $d = 2$ ABC prescription (Fig. 1.4(a)) is shown in Fig. 3.2. As noted in Ref. [26], this prescription fails to cancel local contributions from the two points where regions $A$, $B$ and $C$ all meet at the top and bottom boundaries. This implies that $S_{\text{topo}}^{ABC}$ is contaminated by non-universal contributions that are constant in $R$. However, we will still employ this prescription, because in fracton models we will find a contribution to $S_{\text{topo}}$ proportional to $R$, which is unaffected by the uncanceled constant local contributions. While we do not use them in this work, we note that Ref. [26] introduced different $d = 3$ ABC prescriptions that do not suffer
To illustrate our approach in $d = 3$, we briefly discuss the calculation of topological entanglement entropy for the $d = 3$ toric code, using the PQWT prescription illustrated in Fig. 3.3. The Hamiltonian is given by Eq. (3.21) but on the cubic lattice, so that the vertex terms involve six spins, and there are plaquette terms for each face of a cubic unit cell. As found in Eq. 3.14, $S_{\text{PQWT}}^{\text{topo}}$ is given entirely by counting non-local stabilizers in each region. Region $P$ has a single non-local stabilizer, which can be constructed by taking a product of local plaquette stabilizers over an $xy$ plane surface that cuts through the inner cube (i.e. the “hole” at the center of $P$), as illustrated in Fig. 3.4. Regions $Q$, $W$ and $T$ have no non-local stabilizers, so $S_{\text{topo}}^{\text{PQWT}} = -\Omega_P = -1$. This is consistent with the fact that the ground state wave function is a loop condensate, with loops cutting the entanglement surface an even number of times. Ref. [26] computed the topological entanglement entropy of the $d = 3$ toric code using a different prescription, which produces the same result.

### 3.4 Topological entanglement entropy of fracton models

In this section, we compute the topological entanglement entropy of fracton models. We begin (Sec. 3.4.1) with a simple warm-up example, a stack of decoupled two-dimensional toric code
layers, which is related to the X-cube model by a coupled-layer construction. Then we proceed to consider the X-cube model (Sec. 3.4.2) and Haah’s code (Sec. 3.4.3).

### 3.4.1 Decoupled layers of $d = 2$ toric codes

We consider a stack of $d = 2$ toric codes, with layers spaced uniformly and arranged normal to the $z$ axis. We consider the “square torus” PQWT prescription shown in Fig. 3.3, with topological entanglement entropy defined by Eq. (1.6). The intersection of each layer with $P$ is the annulus-shaped region discussed in Sec. 3.2.4, which has two non-local stabilizers. The regions $Q$, $W$ and $T$ do not support non-local stabilizers, so the topological entanglement entropy is two bits per layer. Therefore, choosing the unit of length to be the layer spacing, and choosing the regions $P$, $Q$, $W$, $T$ to contain precisely $R$ toric code layers, we obtain

$$S_{\text{topo}}^{PQWT} = -2R.$$  \hspace{1cm} (3.23)

In this example of decoupled layers, it is no surprise that we obtain a $R$-linear term in $S_{\text{topo}}$. This term is of interest because it also appears in fracton models.

We remark that slightly changing the detailed specification of the regions can alter the
Figure 3.4: Nonlocal stabilizer (red square) for $d = 3$ toric code, which is a product of plaquette terms in the $B = \bar{P}$ subregion and those on the boundary but only acts nontrivially on the $P$ subsystem.

constant term in $S_{\text{topo}}^{\text{PQWT}}$. For instance, instead of stating the regions contain precisely $R$ layers, we could define them to extend a distance $R$ along the $z$-axis between toric code layers on the “bottom” and “top” surfaces, and to contain these surface layers. Then each region intersects $R + 1$ layers, and we obtain $S_{\text{topo}}^{\text{PQWT}} = -2R - 2$. Note that this changes the constant term by an even integer; it is possible that this term does have a robust meaning modulo two. In a generic system in the same phase as the decoupled stack we are considering, we would not have precise control over the number of layers intersected by these regions, so that only the $R$-linear term is clearly meaningful. This illustrates a general point that, in $d = 3$, one should be cautious in ascribing any meaning to the constant term in $S_{\text{topo}}$ when an $R$-linear term is present.

We also consider the topological entanglement entropy using the $d = 3$ ABC prescription shown in Fig. 3.2, with the preferred axis along the $z$-axis. Considering a single layer, this reduces to the $d = 2$ ABC prescription of Fig. 1.4, giving a topological entanglement entropy of one bit per layer. Again choosing the regions to intersect precisely $R$ layers, we have

$$S_{\text{topo}}^{\text{ABC}} = -R.$$  \hfill (3.24)

The example of decoupled layers of $d = 2$ toric codes is also instructive in that it illustrates the answer can depend on the orientation of the entanglement cut. Indeed, different results would be obtained for either the ABC or PQWT prescriptions, if we choose the preferred axis to lie in an arbitrary direction, since the number of intersecting layers would be different. Although this
somewhat complicates the interpretation of $S_{\text{topo}}$, it has the advantage of providing a means to identify the “natural” axes for entanglement in this system, by rotating the orientation of the preferred axis so as to obtain a maximal answer. In fracton models, this could potentially help to discover new coupled-layer constructions along the lines of Refs. [107, 55].

We also consider a stacking of decoupled toric code layers along the $x$, $y$ and $z$ directions simultaneously, as in the coupled-layer construction of the X-cube model.[107, 55] In this case, we obtain the same results for topological entropy using the two prescriptions employed above, because these prescriptions do not capture the topological entanglement of the layers normal to the $x$ and $y$ axes. We are not aware of a single ABC or PQWT type prescription that captures all of the topological entanglement in this system in one shot. Instead, it seems to be necessary to compute $S_{\text{topo}}$ for different sets of regions to obtain a full picture of the non-local entanglement.

### 3.4.2 X-cube model

We now apply the stabilizer formalism to compute our first new result: namely, the entanglement entropy of the X-cube model, an archetypal example of a ‘type I’ fracton phase [110]. The model is defined on a cubic lattice with a spin $1/2$ variable on each link, with Hamiltonian

$$H_{\text{XC}} = - \sum_v \left( A_v^{(xy)} + A_v^{(yz)} + A_v^{(zx)} \right) - \sum_c B_c, \tag{3.25}$$

where the $A$-type stabilizers involve a product of four $Z_i$ operators that surround a vertex in one of three orthogonal planes, and the $B$-type stabilizers involve a product of twelve $X_i$ operators around a elementary cube.

To compute the topological entanglement entropy, we employ the construction Eq. (1.6), using the regions shown in Fig. 3.3, and taking the preferred axis to be the $z$-axis. As discussed in Sec. 3.2.2, $S^{\text{PQWT}}_{\text{topo}}$ is given by counting non-local stabilizers via Eq. (3.14). It follows from Appendix 3.9.1 that regions $Q$, $W$ and $T$ have only local stabilizers (i.e. their stabilizer groups are locally generated), so we have $S^{\text{PQWT}}_{\text{topo}} = -\Omega_P$, with $\Omega_P$ the number of non-local stabilizers in $P$.

To determine $\Omega_P = \Omega_P^X + \Omega_P^Z$, we first note that a product of cube terms over an $xy$ plane
layer cutting through the “hole” in the center of $P$ produces a non-local $X$ stabilizer, as shown in Fig. 3.5. There are $R$ different layers, and we expect that the resulting $R$ non-local stabilizers are independent in the sense that they cannot be deformed into one another by taking products with local stabilizers, so that $\Omega_P^X = R$. The reason for this expectation is that each of these non-local stabilizers is a closed-loop string operator for a distinct non-trivial quasi-particle excitation confined to move in the corresponding $xy$ plane.[107, 55] It should not be possible to change the particle type of a string operator by multiplying it with local operators. We also verified that $\Omega_P^X = R$ using the numerical method of Sec. 3.2.3, for $R = 2, \ldots, 10$. Similarly, taking a product of $xy$-plane “vertex type” stabilizers over the hole in $P$ also gives a non-local stabilizer. There are $R + 1$ such $xy$-plane layers, giving $\Omega_P^Z = R + 1$, which we again verified numerically for $R = 2, \ldots, 8$. Therefore we find $\Omega_P = 2R + 1$ and

$$S_{\text{topo}}^{PQWT} = -2R - 1. \quad (3.26)$$

We note that the $R$-linear term in this result is identical to that obtained in a stack of decoupled $d = 2$ toric codes along $x$, $y$ and $z$ axes; as discussed in Sec. 3.4.1, only the layers normal to the $z$-axis contribute to the topological entanglement entropy for this choice of the regions $P$, $Q$, $W$, $T$.

As in the case of the toric code, the linear term of this result can be understood within a loop condensate picture. In the $Z$ basis, configurations satisfying the vertex terms of the Hamiltonian
can be viewed in terms of strings of flipped links $\ell$ with $Z_\ell = -1$, where in every \{100\} plane the strings form closed loops. Each \{100\} plane thus gives a non-local contribution of $-c$ to $S_A$, where $c$ is the number of connected components in the intersection of the plane with the boundary of $A$. Applying this simple rule to $S_{\text{PQWT}}^{\text{topo}}$, we find that each $xy$ plane contributes $-2$, while $yz$ and $xz$ planes do not contribute; this reproduces the $-2R$ term obtained above. We note that this is the same loop condensate picture as for a stack of decoupled toric codes. In the X-cube model, the layer-by-layer loop constraints in the ground state wave function are not truly independent; a more detailed analysis taking this into account would presumably also reproduce the constant term in $S_{\text{PQWT}}^{\text{topo}}$.

It is evident that, as for the case of decoupled layers of $d = 2$ toric codes, the number of topological constraints (and hence the topological entanglement entropy) will depend on the orientation of the entanglement surface. Our discussion here is for an entanglement surface aligned with the symmetry axes of the problem. The entanglement entropy for arbitrary orientations could be evaluated using analogous methods, but we do not discuss it further here.

We also compute the topological entanglement entropy using the ABC prescription, with regions shown in Fig. 3.2. Because regions $A$, $B$, $C$, $AB$ and so on all have locally generated stabilizer groups generated by cube stabilizers and $xy$ and $xz$ plane vertex stabilizers (Appendix 3.9.1), following Sec. 3.2.2 and Appendix 3.8, $S_{\text{topo}}^{ABC}$ is determined by counting local stabilizers whose support is split among all of $A$, $B$ and $C$. The detailed geometry can be chosen so that this only occurs for cube stabilizers, and the number of these cube stabilizers is $R$. Therefore we find

$$S_{\text{topo}}^{ABC} = -R, \quad (3.27)$$

where the vanishing of the constant term is presumably unimportant, because this prescription does not cancel all constant local contributions. We observe that the coefficients of the topological entanglement entropy using both ABC and PQWT prescriptions in the X-cube model are exactly as in the corresponding system of decoupled toric codes. Evidently, the linear term in the topological entanglement entropy is insensitive to the $m$-string condensation that occurs going from the
decoupled toric codes to X-cube phase.[55]

### 3.4.3 Haah’s code

We now turn to Haah’s code, the archetype of a ‘type-II’ fracton model [110]. This model is introduced in Sec. 1.3.1.2.

To compute the topological entanglement entropy, we first employ the ABC prescription with regions as shown in Fig. 3.2. It is shown in Appendix 3.9.2 that the stabilizer groups for regions $A$, $B$, $C$, $AB$, and so on are all locally generated. Therefore, following the discussion of Sec. 3.2.2, $-S_{topo}^{ABC}$ is the number of local stabilizers contained in $ABC$ that have support split among all three regions $A, B, C$. These stabilizers reside on cubes along the axis where the three regions meet, and there are $2(R-1)$ of them, where the factor of 2 accounts for counting both $X$ and $Z$ stabilizers. Therefore,

$$S_{topo}^{ABC} = -2R + 2,$$  \hspace{1cm} (3.28)

where only the $R$-linear term is expected to have any universal meaning.

We also consider the topological entanglement entropy captured by two different PQWT prescriptions, determined using the numerical method of Sec. 3.2.3. We first simplify the problem using the spatial inversion symmetry of Haah’s code, which acts non-trivially on the spins, sending $X \rightarrow Z$ and $Z \rightarrow -X$, and also exchanging the two qubits on each site. If a region $A$ is inversion-symmetric, then $\Omega_Z^A = \Omega_X^A$. On the other hand, if two regions $A$ and $B$ are related to one another by inversion, then $\Omega_Z^A = \Omega_X^B$ and $\Omega_X^A = \Omega_Z^B$. In the PQWT prescription of Fig. 3.3, regions $P$ and $T$ are inversion-symmetric, while inversion exchanges $Q$ and $W$. This implies

$$S_{topo}^{PQWT} = -2\Omega_Z^P + 2\Omega_Z^Q + 2\Omega_Z^W - 2\Omega_Z^T.$$  \hspace{1cm} (3.29)

Our numerical calculations lead to the conclusion that $S_{topo}^{PQWT} = 0$ for the regions of Fig. 3.3. (In more detail, we show in Appendix 3.9.2 that the stabilizer groups for regions $Q$, $W$ and $T$ are all locally generated, i.e. $\Omega_Z^Q = \Omega_Z^W = \Omega_Z^T = 0$. Numerically, we find that $\Omega_Z^P = 0$ for $R = 4, \ldots, 11$. While $\Omega_Z^P = 2$ for $R = 2$ and $\Omega_Z^P = 1$ for $R = 3$, this seems to be a finite-size effect.) This result
is strikingly different from $S_{topo}^{ABC}$, while the linear term in these two entropies only differed by a factor of two for the X-cube model. The contrast with the X-cube model suggests that Haah’s code may not have a coupled-layer description where the layers lie in \{100\} planes.

To find a different PQWT prescription that does capture some of the non-local entanglement in Haah’s code, we note that we should not expect the non-local stabilizers of Haah’s code to be one-dimensional objects, as they are in the X-cube model. This expectation is based on the fact that none of the topologically charged excitations in Haah’s code can be transported by string operators, so we should expect that any non-local stabilizers are higher-dimensional objects. Moreover, this expectation is further substantiated by the fact that $\Omega_{Z}^{P} = 0$ for the solid torus region of Fig. 3.3. This motivates us to employ the PQWT prescription of Ref. [12], with regions shown in Fig. 3.6. Here, the region $P$ is more isotropic, allowing for non-local stabilizers wrapping entirely around the interior cube.

The regions of Fig. 3.6 indeed give a non-zero result for $S_{topo}^{PQWT}$. Our numerical results are summarized in Table 3.1, and we find

$$S_{topo}^{PQWT} = -4R + 12,$$

based on numerical calculations up through $R = 11$. This topological entanglement entropy also has a $R$-linear term. It is interesting to remark that, while Haah’s code has a well-known intricate dependence of the ground state degeneracy on system size [29], the behavior of the topological entanglement entropy is much simpler.

Table 3.1: The number of non-local stabilizers $\Omega_{A}^{Z}$ as a function of $R$ for regions $A = P, Q, W, T$ shown in Fig. 3.6. The functional forms shown are exact from $R = 4$ up through $R = 11$ (the largest value of $R$ for which calculations were done). These results determine $S_{topo}^{PQWT}$ via Eq. (3.29).

<table>
<thead>
<tr>
<th>$\Omega_{P}^{Z}$</th>
<th>$\Omega_{Q}^{Z}$</th>
<th>$\Omega_{W}^{Z}$</th>
<th>$\Omega_{T}^{Z}$</th>
<th>$S_{topo}^{PQWT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6R - 7$</td>
<td>$2R$</td>
<td>$2R - 1$</td>
<td>$0$</td>
<td>$-4R + 12$</td>
</tr>
</tbody>
</table>

At this point it is natural to ask if the topological entanglement entropy in Haah’s code may also be given a geometrical interpretation in terms of constraints on the structure of the ground
state wavefunction, like the loop condensate picture for toric code and X-cube models. Because isolated fractons in Haah’s code are created at corners of fractal operators, it seems likely that the ground state can be viewed as some kind of condensate of fractal objects. At present, to our knowledge there is no clearer meaning that can be given to this picture, or whether it can shed light on non-local entanglement.

3.5 Schrieffer-Wolff perturbation theory

In this section we discuss the construction of dressed stabilizers upon perturbation of the stabilizer Hamiltonian. The derivation is a variation on the standard method of Schrieffer-Wolff transformations, most closely related to the method of Wegner-Wilson flow discussed in Ref. [73]. We outline it here mainly in the interests of completeness.

Let the eigenvectors of the unperturbed Hamiltonian ($H_0$) be $\{|n\rangle\}$ with eigenvalues $\{E_n\}$. Let the new Hamiltonian be written as $H = H_0 + \lambda V$, where $V$ is some Hermitian operator and $\lambda$ is any real number. Also let the eigenvectors of this Hamiltonian be $\{|n'\rangle\}$ with eigenvalues $\{E_n'\}$.
The idea is to find the unitary operator $U(\lambda)$ such that

$$|n'\rangle = U(\lambda) |n\rangle.$$  
(3.31)

If we assume that $U(\lambda)$ is an analytic function, then there exists a Hermitian operator-valued function $F(\lambda)$ such that

$$U(\lambda) = \exp (iF(\lambda)).$$  
(3.32)

We may then expand $F(\lambda)$ as a power series in $\lambda$. To find a relation for $U_{mn} = \langle m|n'\rangle$, note that

$$\langle m|H_0 + \lambda V|n'\rangle = E_m \langle m|n'\rangle + \lambda \langle m|V|n'\rangle = E'_n \langle m|n'\rangle.$$  
(3.33)

We may also expand the $V$ term in the original eigenbasis, to obtain the self-consistent equation

$$U_{mn}(\lambda) = \lambda \sum_k \frac{V_{mk}U_{kn}(\lambda)}{E'_n(\lambda) - E_m}.$$  
(3.34)

We now make the standard assumption that $V$ does not act within degenerate subspaces of $H_0$ (any portion of $V$ that does so act should be absorbed into our definition of $H_0$), so we only have to worry about ‘off diagonal’ matrix elements of $V$. If we define

$$A(\lambda) = \sum_k \frac{V_{mk}U_{kn}(\lambda)}{E'_n(\lambda) - E_m} |m\rangle \langle n|,$$  
(3.35)

we can use (3.34) to express the expansion coefficients as

$$\left( \frac{\partial U}{\partial \lambda^j} \right)_{\lambda=0} = j \left( \frac{\partial^{j-1} A}{\partial \lambda^{j-1}} \right)_{\lambda=0},$$  
(3.36)

assuming the derivatives of $A$ are well behaved at $\lambda = 0$. At first order, we have

$$\left( \frac{\partial U}{\partial \lambda} \right)_{\lambda=0} = i \left( \frac{\partial F}{\partial \lambda} \right)_{\lambda=0} = A_{\lambda=0} = iL,$$  
(3.37)

where $L$ can be expressed in terms of its matrix elements in the old eigenbasis,

$$L_{mn} = i \frac{V_{mn}}{E_m - E_n + 0},$$  
(3.38)

or $[H_0, L] = iV$. One can continue the expansion to any desired order. Note that if one is concerned only with the ground state, and the system is gapped within a topological sector, then...
the denominator has a non-zero lower bound. It may be shown using standard techniques that if the original Hamiltonian has a conserved quantity, i.e. \([H_0, \hat{S}] = 0\), then one can find a conserved quantity for the perturbed Hamiltonian, \(\hat{S}\) by solving

\[
\frac{\partial \hat{S}}{\partial \lambda} = i[L, \hat{S}],
\]

(3.39)

The expert reader will recognize this as the equation of motion for Wegner-Wilson flow\[73\]. It is well known that this sort of flow equation preserves locality of the integrals of motion both for gapped systems and for localized systems (see e.g. Refs. \[40, 73\] for recent discussions), where ‘local’ means ‘local up to an exponentially decaying tail.’

### 3.6 Localization-protected fracton order in excited states

Thus far, we have focused exclusively on ground states. However, for stabilizer Hamiltonians, the entire spectrum shares the same entanglement entropy properties as the ground state (excited states differ only in their eigenvalues under stabilizer operators). The difference between ground states and excited states only manifests itself when the Hamiltonian is perturbed away from the stabilizer form. The entanglement structure of ground states is then ‘protected’ from perturbations by the energy gap (as discussed above), whereas the excited states lack such protection. Upon perturbation, the excited states of translationally invariant fracton models are thus expected to thermalize\[75\] to volume law entanglement, in the process losing their topological order. Indeed, if one tries to construct the unitary \(U\) of Eq. (3.1) using Schrieffer-Wolff perturbation theory outside the ground space, one finds that the perturbation theory diverges, suggesting that \(U\) may not be a local unitary. Since our discussion thus far relied on the existence of a local unitary \(U\), it has nothing further to say about the translationally invariant case.

However, as noted by Ref. \[39\], this scenario could change dramatically once we break translational invariance by introducing quenched disorder: the topological order can be protected even in excited states by localization, in a manner that we now sketch. Consider a disordered fracton
Hamiltonian of the form

$$H = -\sum_i J_i^A A_i - \sum_j J_j^B B_j - \lambda H_{\text{int}},$$

(3.40)

where the $J_i^A$ and $J_j^B$ are random numbers drawn from some distribution of width $W$. While the excitations are frozen (non-propagating) for any $W$ at the stabilizer point, once we add small perturbations they will be able to propagate for small $W$, but for sufficiently large $W$ the system can enter a many body localized phase [67], where excitations cannot propagate freely. In this many-body localized phase, topological order (including fracton order) can persist at non-zero energy densities — i.e., even in highly excited states. The challenge is how to detect this topological order.

‘Excited state degeneracy’ cannot serve as a diagnostic, since the many body level spacing in the middle of the spectrum is exponentially small in the volume of the system and thus there is no longer a distinction between topological and other degeneracies in the thermodynamic limit. In Ref. [39], non-local correlation functions — related to the ‘Fredenhagen-Marcu’ order parameters familiar to lattice gauge theorists — were argued to be good diagnostics of topological order. These diagnostics were generalized to certain fractonic models in Ref. [17], but nevertheless such non-local correlation functions can be challenging to compute. Here, we will demonstrate that topological entanglement can diagnose fracton topological order in excited states.

The fact that excited states in the localized regime can support fracton topological order follows straightforwardly from our preceding discussions. First note that at $\lambda = 0$, excited states share the entanglement properties of the ground state (including topological entanglement), since the excited states are also eigenstates of the stabilizer operators. Now note that, in the localized phase, the unitary transformation $U$ is local, with at most exponential tails. This follows because of the ‘mobility gap’ in many body localized systems [66] i.e. the Schrieffer-Wolff perturbation theory has matrix elements in the numerator, and matrix elements vanish between near degenerate eigenstates. Again, the topological entanglement entropy is non-local, and is expected to be unaffected by a local unitary transformation, subject to the same caveats discussed in the previous section.

We also note that the dressed stabilizers are simply the local integrals of motion or ‘lbits’ of
the localized system [97, 38], and these must be localized by postulate. For a detailed construction of dressed integrals of motion via Schrieffer-Wolff perturbation theory, showing that these are local in the MBL regime, see Ref. [40]. (For the cognoscenti, we note that our argument here parallels more closely the construction of l-bits via Wegner-Wilson flow in Ref. [73]).

We note that thus far we have assumed that a many body localized phase can exist in three dimensional lattice models. There is some debate about whether many body localization can arise in spatial dimensions $d > 1$ with random short range correlated disorder[16], because of ‘thermalizing avalanches’ triggered by rare regions. For truly random short range correlated disorder, our discussion applies to systems that are small enough to lack the relevant rare regions, and perhaps also in the thermodynamic limit, if the argument from Ref. [16, 53, 74] can be somehow circumvented. However, the problem may also be sidestepped by making the disorder long range correlated or quasiperiodic, such that the ‘rare region obstruction’ identified in Ref. [16] does not apply.

We therefore conclude that in disordered fracton models, fracton topological order can arise even in highly excited states, where it may be diagnosed through a ‘topological entanglement entropy’ linear in the size of the subsystem.

### 3.7 Conclusions

We have explicitly computed the entanglement entropy of two archetypal fracton models — the X-cube model and Haah’s code — and have demonstrated the existence of a topological contribution to the entanglement entropy that is linear in the size of the subsystem. At a minimum, this provides a coarse characterization of fracton topological phases, in that for a given system, two states with distinct topological entanglement must be in distinct phases. There is also an obvious extension of this diagnostic to anisotropic models, wherein one separately considers the scaling of entanglement entropy with the size of subregion $A$ in the $x$, $y$ and $z$ directions respectively, thereby characterizing the topological entanglement with three indices. (More carefully, one would separately consider the topological entanglement entropy for a torus of thickness $R$ oriented in three
orthogonal planes). In general the topological entanglement can depend not just on the size of the region, but also on its orientation, which may provide a useful means of diagnosing the symmetry axes of a phase by rotating the entanglement surface to obtain a maximal answer.

What more information could be extracted from a study of entanglement? In two-dimensional topologically ordered phases, a careful analysis of the action of symmetries such as rotation and reflection within the manifold of degenerate ground states can provide insights into the fractionalized statistics of quasiparticle excitations in the phase; whether such manipulations can shed additional light on the properties of fracton excitations (that do not admit a quasiparticle description) remains an open question. It would also be interesting to study the entanglement spectrum [51], as this may contain more information than is encapsulated in entanglement entropy. Finally, the dynamics of entanglement has in other contexts (see, e.g. Refs. [133, 4]) provided much insight into the nature of thermalization and the approach to or avoidance of equilibrium. We leave investigation of these issues to future work.
Appendix

3.8 Simplified expressions for the topological entanglement entropies

We derive the formula Eq. (3.14) for \( S_{\text{topo}}^{PQWT} \) in terms of the number of non-local stabilizers in each region. Treating \( X \) and \( Z \) stabilizers together, we pick a basis \( B_{T,\text{loc}} \) for \( G_{T,\text{loc}} \). Because \( T \) is a subset of all the other regions, we can extend this to bases \( B_{Q,\text{loc}}, B_{W,\text{loc}} \) and \( B_{P,\text{loc}} \) for \( G_{Q,\text{loc}}, G_{W,\text{loc}} \) and \( G_{P,\text{loc}} \). Each of these bases can then be extended to a basis for the full stabilizer group in the corresponding region, including non-local stabilizers. The topological entanglement entropy is

\[
S_{\text{topo}}^{PQWT} = -\Omega_P + \Omega_Q + \Omega_W - \Omega_T - |B_{P,\text{loc}}| + |B_{Q,\text{loc}}| + |B_{W,\text{loc}}| - |B_{T,\text{loc}}|. \tag{3.41}
\]

To simplify this expression, we first note that

\[
B_{T,\text{loc}} = B_{Q,\text{loc}} \cap B_{W,\text{loc}}. \tag{3.42}
\]

This holds because the local stabilizers added to extend \( B_{T,\text{loc}} \) to \( B_{Q,\text{loc}} \) are contained only in \( Q \) and not in \( W \), and vice versa. Next we consider the union \( B_{Q,\text{loc}} \cup B_{W,\text{loc}} \). This set spans \( G_{P,\text{loc}} \), because there is a basis for \( G_{P,\text{loc}} \) where the basis elements are local operators that are thus fully contained either in \( Q \) or \( W \). However it can happen that this set is not linearly independent. Therefore we have

\[
|B_{P,\text{loc}}| = |B_{Q,\text{loc}} \cup B_{W,\text{loc}}| - \Delta_{PQWT}, \tag{3.43}
\]

where \( \Delta_{PQWT} \) can be thought of as the number of constraint equations obeyed by elements of \( B_{Q,\text{loc}} \cup B_{W,\text{loc}} \). These constraints must be non-local, involving stabilizers contained in \( Q \) but not
in $W$, and vice versa. It follows that

$$S_{\text{topo}}^{PQWT} = -\Omega_P + \Omega_Q + \Omega_W - \Omega_T + \Delta_{PQWT}. \quad (3.44)$$

We now discuss $\Delta_{PQWT}$ in more detail; among the models we consider, it can be non-zero only for the $d = 3$ toric code. Each constraint equation contributing to $\Delta_{PQWT}$ is a product of local stabilizers in $P$ that equals the identity operator. Such a constraint can always be obtained as a product of local constraints, involving stabilizers that may lie in a larger region containing $P$. This can occur for the plaquette stabilizers $d = 3$ toric code. For example, using the regions $P, Q, W, T$ shown in Fig. 3.6, a product of plaquette stabilizers in $P$ over a surface $S$ enclosing the inner cube is the identity operator. The plaquette stabilizers satisfy the local constraint that a product of stabilizers over the faces of an elementary cube is unity, and the non-local constraint in $P$ can be obtained by from these local constraints by taking a product over all cubes inside $S$.

However, there are no such non-local constraints in the other models we consider, where $\Delta_{PQWT} = 0$. In the $d = 2$ toric code and Haah’s code, the stabilizers obey no local constraints. This is also true for the cube stabilizers of the X-cube model. The vertex stabilizers of the X-cube model do obey local constraints, but because each vertex stabilizer participates in exactly one local constraint, it is not possible to obtain non-local constraints contributing to $\Delta_{PQWT}$ by taking products of local ones.

Now we give a similar discussion of $S_{\text{topo}}^{ABC}$, to obtain the result that it is determined by stabilizers whose support is split among all of $A$, $B$ and $C$. Here, we consider only models and regions where $A$, $B$ and $C$ are locally generated by a basis of local stabilizers that obey no local constraints. This property is established for the X-cube model in Appendix 3.9.1 using a basis of cube stabilizers, and $xy$ and $xz$ plane vertex stabilizers. For Haah’s code and the $d = 2$ toric code, the stabilizers obey no local constraints. We choose bases $B_A$, $B_B$ and $B_C$ for the stabilizer groups $B_A$, $B_B$ and $B_C$, respectively.

To obtain a basis for the pairwise unions $AB$ and so on, we consider the set $b_{AB} = B_A \cup B_B$. This set is clearly linearly independent, except possibly for the vertex stabilizers of the X-cube
model. Any linear relation would have to involve basis stabilizers in both $A$ and $B$. It cannot be a non-local constraint as discussed above. It also cannot be a local constraint near the boundary of $A$ and $B$, because the stabilizers in $B_A$ and $B_B$ are drawn from a subset of local stabilizers that do not obey any local constraints. Therefore $b_{AB}$ is linearly independent. In order to obtain a basis $B_{AB}$ for $G_{AB}$, we extend $b_{AB}$ by adding stabilizers $\delta_{AB}$ whose support is split between $A$ and $B$.

Finally we consider the union $ABC$. To find a basis, we first consider the set $b_{ABC} = B_A \cup B_B \cup B_C \cup \delta_{AB} \cup \delta_{BC} \cup \delta_{AC}$. Again, this set is linearly independent, and we extend it to a basis $B_{ABC}$ by adding a set of stabilizers $\delta_{ABC}$, whose support is split among the three regions.

We then obtain for the topological entanglement entropy

$$S_{\text{topo}}^{ABC} = -|B_A| - |B_B| - |B_C| + |B_{AB}| + |B_{BC}| + |B_{AC}| - |B_{ABC}| = -|\delta_{ABC}|, \quad (3.45)$$

the desired result.

### 3.9 Regions with locally generated $G_A$

Here, we consider the X-cube model and Haah’s code, and show that the group of stabilizers $G_A$ is locally generated for certain regions $A$ that we characterize. Recall that in Sec. 3.2.3, we defined $G_A$ to be locally generated when $G_A = G_{A,\text{loc}}$, i.e. when every stabilizer in $A$ is a product of local stabilizers.

#### 3.9.1 X-cube model

In the X-cube model, we show that $G_A$ is locally generated for regions $A$ obtained by taking a simply connected region in the $xy$ plane, and stacking this region along the $z$-axis. (The choice of plane and normal stacking direction is of course arbitrary, due to cubic symmetry.) We assume the boundary of each $xy$ plane slice is a sequence of edges that are contained in $A$ and form a path in the cubic lattice. Two examples of such regions for the $d = 2$ square lattice are shown in Fig. 3.7. We further assume that acute corners of the slice (see Fig. 3.7) are sufficiently isolated from other points on the boundary to carry out the cleaning procedure for $Z$ stabilizers discussed
below. A more precise statement of this assumption is given below; essentially, we are assuming the boundary of the slice is not too rough. This condition allows the slice shown in Fig 3.7a, but rules out that in Fig. 3.7b. We refer to cubic lattice links oriented along the $x$, $y$ and $z$ axes as $x$, $y$ and $z$ links, respectively. In addition to spins residing on $x$ and $y$ links contained in each slice, the region $A$ contains all $z$-links joining two adjacent slices.

In fact, under the assumptions given, we show an even stronger property: $G_A$ is locally generated using only $xy$-plane and $xz$-plane vertex stabilizers, and cube stabilizers. (Alternatively, we can use $xy$-plane and $yz$-plane vertex stabilizers.) Because there are no local constraints among $xy$ and $xz$ plane vertex stabilizers, using this as a generating set allows us to simply establish Eq. (1.5) in Appendix 3.8.

We first consider some operator $X_0$ supported on $A$, which we take to be an $X$-stabilizer. By definition, this means that $X_0$ commutes with all local $Z$-stabilizers, including those not entirely supported on $A$. We would like to show that $X_0$ is a product of local $X$-stabilizers supported on $A$, which can be accomplished by a cleaning procedure, where we successively multiply $X_0$ by such stabilizers until we obtain the identity operator. We denote the $X$-stabilizer obtained from $X_0$ at the current stage of the cleaning procedure by $X'$.

We begin with the bottom layer (i.e. smallest $z$ coordinate) of $X_0$. In this slice, commutation between $xy$ plane vertex stabilizers is exactly as in the $d = 2$ toric code. Using the fact that the stacked region is simply connected, this implies that $X'$ restricted to this layer is a product of plaquette operators $\prod_{\ell \in p} X_\ell$, where $p$ is a square plaquette in the $xy$ plane. We can thus clean this layer by multiplying $X_0$ with a suitable product of local $X$ stabilizers whose cube centers lie just above the slice; this works because the restriction of these stabilizers to the slice are plaquette operators.

It appears this cleaning step may leave dangling $z$-links lying just above the bottom slice, where $X'|_\ell = X$ on such links. The notation $X'|_\ell$ means the restriction of $X'$ to the link $\ell$, i.e. $X'$ is a product over links of Pauli operators, and $X'|_\ell$ is the Pauli operator (1 or $X$) at the link $\ell$ appearing in the product. By considering commutation of $X'$ with $xz$- and $yz$-plane vertex stabilizers whose
centers lie in the original bottom layer, we see that $\mathcal{X}|_\ell = 1$ for all these dangling links.

These steps reduce the height of the stack by one unit cell, resulting in a new region of the
with the same properties as the one we started with. Therefore, we can continue this procedure
until $\mathcal{X}$ is only supported on a single slice. We can show $\mathcal{X} = 1$ by considering commutation with
$xz$ and $yz$ plane vertex stabilizers whose centers lie in the same slice. Consider $x$-links in the slice
of interest with some fixed $y$ coordinate. Starting at large negative values of $x$, and moving in the
positive $x$ direction, find the first link $\ell$ with $\mathcal{X}|_\ell = X$. Then $\mathcal{X}$ anticommutes with the $xz$ vertex
stabilizer whose center lies adjacent to this link in the negative $x$ direction. This is a contradiction,
and implies $\mathcal{X}|_\ell = 1$ for all $x$-links, and similarly for all $y$-links. Therefore $\mathcal{X} = 1$ and we have
reached the end of the cleaning process.

Now we consider a $Z$ stabilizer $Z_0$ supported on the same region $A$, and clean it via multipli-
cation with vertex stabilizers. Again we start with the bottom slice, where commutation between
$Z_0$ and cube stabilizers whose centers lie just below the slice is the same as in the $d = 2$ toric code.
Therefore, $Z_0$ restricted to this slice is a product of $xy$ plane vertex stabilizers, which we clean
off. This may leave dangling $z$-links just above the bottom slice, where $Z|_\ell = Z$. First consider a
$z$-link $\ell$ just above an acute corner. We assume there is a cube stabilizer that contains this link,
but contains no other links of $A$; this is a more precise statement of our assumption that the acute
corners are sufficiently isolated. This cube stabilizer anticommutes with $Z$ unless $Z|_\ell = 1$. All
dangling $z$-links not above an acute corner can be cleaned by multiplying with a $xz$ plane vertex stabilizer, whose vertex lies just above the link to be cleaned.

This cleaning process can be continued until $Z$ is supported on a bilayer consisting of the top slice, the slice just below it, and $z$-links joining these slices. Both the top and bottom slice can be cleaned off as above, leaving only the $z$-links. Considering the set of $z$ links with $Z|\ell = Z$, we go to the leftmost column of this layer (i.e. smallest $x$), and find the $z$ link in this column with smallest $y$ coordinate. We see that $Z$ anticommutes with the cube stabilizer whose center lies in the same layer, and lies diagonally adjacent to the selected link in the negative $x$, negative $y$ direction. This is a contradiction, so we must have $Z = 1$, and the cleaning procedure is complete.

### 3.9.2 Haah’s code

In Haah’s code, we first show that $G_A$ is locally generated for $A$ a rectangular prism, i.e. a region containing all sites with $x$ coordinate satisfying $x_{\text{min}} \leq x \leq x_{\text{max}}$, and similarly for $y$ and $z$. Axes are chosen as in Fig. 1.7. Next, we generalize this to show that $G_A$ is locally generated for certain “L-shaped” regions shown in Fig. 3.9. In all cases, it is enough to concentrate on $Z$ stabilizers: The rectangular prism regions are inversion-symmetric, so the corresponding result for $X$ stabilizers follow from inversion symmetry. The L-shaped regions are not inversion-symmetric, but each region of one type as shown in Fig. 3.9 is related to a region of a different type under inversion, so that if all four types of regions have locally generated $G^Z_A$, the corresponding statement for $X$ stabilizers follows.

Let $A$ be a rectangular prism region, and choose a $Z$ stabilizer $Z$ supported in $A$. We write

$$Z = \prod_{\vec{r} \in A} \prod_{i=0,1} (Z_{\vec{r},i})^{n(\vec{r},i)}, \quad (3.46)$$

where $Z_{\vec{r},i}$ is the $Z$ Pauli operator for the $i$th qubit at position $\vec{r} = (x, y, z)$, and where the choice of operator is specified by the binary numbers $n(\vec{r}, i) = 0, 1$. The restriction $Z|_{\vec{r}}$ is given by

$$Z|_{\vec{r}} = (Z_{\vec{r},0})^{n(\vec{r},0)}(Z_{\vec{r},1})^{n(\vec{r},1)}, \quad (3.47)$$
We often suppress some of the indices when using this notation, so for instance if \( n(\vec{r}, 0) = n(\vec{r}, 1) = 1 \), we write

\[
Z|_{\vec{r}} = ZZ.
\]  

(3.48)

In order for \( Z \) to be a stabilizer, it must commute with all the local \( X \) stabilizers. We label local stabilizers \( A_c \) and \( B_c \) by the position of the corner with smallest values of the coordinates \( x, y, z \), thus writing \( A_c = A_{\vec{r}} \) and \( B_c = B_{\vec{r}} \). Then the condition \( ZB_{\vec{r}} = B_{\vec{r}}Z \) can be written

\[
n(\vec{r}, 0) + n(\vec{r}, 1) + n(\vec{r} + \hat{x}, 1) + n(\vec{r} + \hat{y}, 1) + n(\vec{r} + \hat{z}, 1)
+ n(\vec{r} + \hat{x} + \hat{y}, 0) + n(\vec{r} + \hat{x} + \hat{z}, 0) + n(\vec{r} + \hat{y} + \hat{z}, 0) = 0 \mod 2. 
\]  

(3.49)

As discussed above for the X-cube model, we will carry out a cleaning process where we successively multiply \( Z \) by local \( Z \) stabilizers contained in \( A \) until \( Z = 1 \).

Equation (3.49) looks complicated, but it simplifies if we choose \( \vec{r} \rightarrow \vec{r}_0 \) to be the corner of \( A \) with largest \( x, y \) and \( z \) coordinates (see Fig. 3.8), where it reduces to

\[
n(\vec{r}, 0) + n(\vec{r}, 1) = 0 \mod 2.
\]  

(3.50)

This implies that either \( Z|_{\vec{r}_0} = 1 \) or \( Z|_{\vec{r}_0} = ZZ \). Supposing the latter case, we multiply \( Z \) by \( A_{\vec{r}_0 - (\hat{x} + \hat{y} + \hat{z})} \), to obtain a new \( Z \) with \( Z|_{\vec{r}_0} = 1 \).

Figure 3.8: Rectangular prism region \( A \) in Haah’s code, with some lattice sites (circles) labeled to facilitate the discussion of the cleaning procedure in the text.

We repeat this procedure, moving down in the \( z \)-direction, to clean operators in the column below \( \vec{r}_0 \) (red dots in the figure) until \( \vec{r}_1 \) is reached. \( Z|_{\vec{r}_1} \) cannot be cleaned in the same way,
because the $Z$ stabilizer we would need to act with lies outside $A$. However, again we have $Z|\vec{r}_1 = 1$ or $Z|\vec{r}_1 = ZZ$. Considering commutation of $Z$ with $B_{\vec{r}_1 - \hat{z}}$, we find $n(\vec{r}_1, 1) = 0 \mod 2$. Therefore we have $Z|\vec{r}_1 = 1$, and no cleaning is needed.

So far we have cleaned the vertical column below $\vec{r}_0$. To proceed, we move to $\vec{r}_2$, and repeat the same procedure to clean the vertical column below $\vec{r}_2$. Proceeding in this way we can clean until we reach the position $\vec{r}_3$. By the same reasoning as before, either $Z|\vec{r}_3 = 1$ or $Z|\vec{r}_3 = ZZ$. We cannot apply the same cleaning procedure for $\vec{r}_3$, because the local $Z$ stabilizer we would need to multiply lies outside of $A$. However, commutation with $B_{\vec{r}_3 - \hat{x}}$ implies either $Z|\vec{r}_3 = 1$ or $Z|\vec{r}_3 = ZI$. Therefore $Z|\vec{r}_3 = 1$ and no cleaning is needed. The same argument applies to all the sites vertically below $\vec{r}_3$, and therefore we have cleaned off the entire $+y$ face of $Z$.

We can repeat the same steps to clean almost all of $Z$ until position $\vec{r}_4$ is reached. Again we have either $Z|\vec{r}_4 = 1$ or $Z|\vec{r}_4 = ZZ$. Commutation with $B_{\vec{r}_4 - \hat{y}}$ implies either $Z|\vec{r}_4 = 1$ or $Z|\vec{r}_4 = ZI$. Therefore $Z|\vec{r}_4 = 1$ and no cleaning is needed. We proceed vertically below $\vec{r}_4$, and

![Figure 3.9: Four types of L-shaped regions in Haah’s code with axes shown. Each region is a stack along the z-axis of the an xy plane slice, with the top slice is shown for each region. Regions of types 1, . . . , 4 must be considered separately, because Haah’s code lacks four-fold rotation symmetry about the z-axis. The solid and dashed circled subregions, and the site label $\vec{r}_0$ for the type 3 and 4 regions, are referred to in the text.](image)
then on to position $r_5$, and so on, to see that $Z|_{r} = 1$ everywhere on the remaining $-y$ face, and no further cleaning is needed. This completes the argument, and we have shown $G_A$ is locally generated for $A$ a rectangular prism.

Now we consider L-shaped regions of four types, as shown in Fig. 3.9, and show that $G_A$ is locally generated for such regions. Regions of this geometry appear in the computation of $S_{\text{topo}}$ via the ABC prescription (see Sec. 3.4.3). Each region is a stack along the $z$-axis for $z_{\text{min}} \leq z \leq z_{\text{max}}$ of an $xy$ plane slice. The figure shows the top such slice ($z = z_{\text{max}}$) for each region. Similar regions that are stacks along $x$ and $y$ axes can be obtained from these by three-fold rotational symmetry about the [111] axis, and do not need to be considered separately.

![Four types of C-shaped regions](image)

Figure 3.10: Four types of C-shaped regions in Haah’s code with axes shown. Each region is a stack along the $z$-axis of the an $xy$ plane slice, with the top slice is shown for each region.

For each type of region, we again consider a stabilizer $Z$ supported in the region. In each case, the entire subregion including and below the solid circled regions in the figure can be cleaned off by following the cleaning procedure described above for a rectangular prism region. For type 1 and 2 regions, this results in a stabilizer $Z$ supported within a rectangular prism, which we have
already shown is a product of local $Z$ stabilizers. For type 3 and 4 regions, it remains to consider
the subregion including and below the dashed ovals. Starting with the type 3 region, we consider
the site $\vec{r}_0$ in the top layer (see figure). Considering commutation of $Z$ with $B_{\vec{r}_0 - \hat{x}}$ and $B_{\vec{r}_0 - \hat{x} - \hat{y}}$
implies $Z|_{\vec{r}_0} = 1$. The same reasoning shows that $Z|_{\vec{r}} = 1$ for all sites in the column below $\vec{r}_0$, and
in the whole subregion containing and below the dashed oval. This again reduces the problem to
the already solved rectangular prism case. The argument for type 4 regions proceeds essentially
the same way, except that we consider commutation of $Z$ with $B_{\vec{r}_0}$ and $B_{\vec{r}_0 - \hat{y}}$.

Similarly, we can show that in the C-shaped regions involved in PQWT constructions like
Fig. (3.3), $G_A$ is also locally generated. As shown in Fig. (3.10), there are four possible C-shaped
regions. For each type of region, we still consider a stabilizer $Z$ supported in that region. In each
case, the entire subregion including and below the solid circled regions in the figure can be cleaned
off by following the cleaning procedure described above for a rectangular prism region. For type 1
region, this results in a stabilizer $Z$ supported within a rectangular prism, which we have already
shown is a product of local $Z$ stabilizers. For type 2 and 4 regions, the solid and dashed oval
regions can be cleaned off in the same way as we clean the L-shaped regions. Then what is left are
type 2 and 1 L-shaped regions respectively, shown in Fig. 3.9, which can be cleaned using the same
procedure. For the type 3 region, it is more convenient to start from the lower left corner, i.e. $\vec{r}_0$.
The commutation of $Z$ with $B_{\vec{r}_0 - \hat{y}}$ leads to either $Z_{\vec{r}_0} = 1$ or $Z_{\vec{r}_0} = ZI$. Suppose the latter one is
true, then we can multiply by $A_{\vec{r}_0 - \hat{x} - \hat{z}}$ to clean this site. All sites below $\vec{r}_0$ (with the same $x$ and $y$
coordinates) can be cleaned by a similar procedure, until one arrives at the bottom site $\vec{r}_1$, which
has $z = z_{\text{min}}$. The commutation of $Z$ with both $B_{\vec{r}_1 - \hat{y}}$ and $B_{\vec{r}_1 - \hat{y} - \hat{z}}$ requires $Z_{\vec{r}_1} = 1$. The next
column beginning with $\vec{r}_2$ can be cleaned off in the same way. Next, we can move to $\vec{r}_3$. Notice the
commutation of $Z$ with both $B_{\vec{r}_3 - \hat{y}}$ and $B_{\vec{r}_3 - \hat{x} - \hat{y}}$ implies that $Z_{\vec{r}_3} = 1$. In a similar fashion we can
proceed to clean the whole subregion in the dashed circle, resulting in a type 4 L-shaped region
which we have already be able to clean as discussed above.
Chapter 4

Fracton topological order from Higgs and partial confinement mechanisms
of rank-two gauge theory

4.1 Introduction

Fracton topological order has been studied from a number of different perspectives, and several physical mechanisms leading to it have been proposed. Some fracton phases can be obtained via coupled-layer constructions, as we discussed before in chapter 2, where the appearance of fractons is driven by ‘particle loop condensation,’ starting from a system of decoupled two-dimensional topological states, obtained by coupling one-dimensional chains. Ref. introduced parton theories of fracton states, providing a route to construct variational wave functions in more physically realistic models.

A closely related development, as we introduced in Sec. 1.3.2, fractons also appear in higher rank U(1) gauge theories, i.e. those where the electric field and vector potential are symmetric tensors of rank two or higher. Higher rank U(1) gauge theories, being studied extensively recently, are different from fracton topological phases as they exhibit gapless photon modes. It is possible to remove the gapless modes and make a connection with the gapped fracton models mentioned above. A natural way to remove gapless modes in a continuous gauge theory is to ‘Higgs’ the gauge field and reduce the gauge group to a discrete one. As scratched in Sec. 1.2.4, when the normal vector U(1) gauge theory is Higgsed down to $Z_2$, we get a $Z_2$ gauge theory. Similarly, if we Higgs a gapless higher rank U(1) gauge theory, we get a gapped higher rank $Z_N$ gauge theory, which seems to give a natural way to generate gapped fracton topological
Surprisingly, as we show in this work, the fracton nature of the charge excitations may be lost via the Higgs mechanism. This conclusion was also addressed in another study.[102] Here, we show in detail that while the scalar charge rank-2 $U(1)$ gauge theory contains fractons, its Higgsed version does not and is equivalent to several copies of a discrete vector gauge theory. To arrive at a gapped fracton phase via the Higgs mechanism, something more is needed.

In particular, we discuss two paths that lead from the scalar charge rank-2 $U(1)$ gauge theory to the fracton topological order of the X-cube model, a gapped fracton model introduced in Ref. [110]. As shown in Fig.4.1, in one of the paths we can first Higgs the rank-2 $U(1)$ gauge theory by condensing a charge-two matter field, arriving at four copies of $Z_2$ vector gauge theory (the toric code). Then we can condense certain flux loops to partially confine the gauge fields and arrive at the X-cube topological order.

The second path to the X-cube topological order begins by first condensing certain monopole excitations of the rank-2 $U(1)$ gauge theory, to arrive at a distinct ‘hollow’ rank-2 $U(1)$ theory, whose field tensors only have the off-diagonal components. This theory was studied previously in Ref. [126], where it was shown that it is unstable to confinement arising from proliferation of instantons. Nonetheless, upon condensing charge-two matter in this theory, we again obtain the X-cube fracton topological order.

Similar to the X-cube model, the checkerboard fracton model[110] can also be interpreted as a rank-2 $Z_2$ scalar charge theory. An important difference is that two different forms of Gauss’ law are alternately enforced on the even/odd layers of the system.

We would like to emphasize that in this work we are not concerned with the critical prop-

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1 The ground state degeneracy on a 3-torus of the exactly solvable X-cube model is stable under local perturbation. This can be verified using degenerate perturbation theory. If we add a local perturbation $V = \lambda \sum O_{\text{loc}}$ to the Hamiltonian, the matrix elements in the resulting effective Hamiltonian for the degenerate ground state space are proportional to $\lambda^L$, where $\lambda \propto \lambda$ is a constant, and $L$ is the linear size of the system. This holds because only logical operators supported on a region of size $L$ or larger have non-vanishing matrix elements within the ground state subspace. Since the degenerate subspace has dimension $\sim c^L$ for some constant $c$, the matrix Frobenius norm is bounded by $(c\lambda)^L$, which in turn bounds all the eigenvalues. Therefore, as long as $\lambda$ is below some finite threshold, the splitting of the ground state subspace is exponentially small and approaches to zero in the thermodynamic limit. A very similar argument applies to many other gapped fracton models.
erties of transitions between phases, or even whether continuous transitions exist. Instead, we are interested in physical mechanisms by which one phase can be driven into another phase, by condensation of some excitations. Such mechanisms can be considered independent of critical phenomena, and indeed are still relevant even in cases where a transition is driven first order by fluctuations. All transitions between phase in this work are discussed in this spirit, leaving the study of critical phenomena for future work.

This chapter is organized as follows. We begin our discussion by describing a special rank-2 U(1) gauge theory, the hollow theory, with scalar charge in Sec. 4.2, which has only off-diagonal elements in the electric field and gauge field tensors. Sec. 4.3 studies the effect of Higgsing the conservation laws of higher-rank gauge theories, and the resulting predictions for the mobility of charges in the corresponding gapped states. Sec. 4.4 begins the left-hand branch of Fig. 4.1; we couple the rank-2 U(1) scalar charge theory to charge-2 matter, and condense the charge-2 matter to obtain the rank-2 $\mathbb{Z}_2$ scalar charge theory. This theory does not support fractons and is found in Sec. 4.4.2 to be equivalent to four copies of the $d = 3$ toric code, with some further details given in Appendix 4.8. We complete the left-hand branch of Fig. 4.1 in Sec. 4.4.3, where we show that condensing flux loops in a selective manner results in the X-cube model. Appendix 4.9 studies a two-dimensional version of the rank-2 $\mathbb{Z}_2$ scalar charge theory on the square lattice, showing it is

Figure 4.1: Two paths from the rank-2 U(1) scalar charge theory to the X-cube model.
equivalent to three copies of the $d = 2$ toric code.

The right-hand branch of Fig. 4.1 is taken in Sec. 4.5. First, in Sec. 4.5.1, we implement an electric-magnetic duality transformation to describe the gapped magnetic excitations of the rank-2 $U(1)$ scalar charge theory. These excitations are point-like magnetic monopoles that move in two-dimensional planes, and we show that condensing these excitations in their planes of motion leads to the hollow $U(1)$ gauge theory. Sec. 4.5.2 then shows that condensing charge-2 matter in the hollow $U(1)$ gauge theory leads to the X-cube model. Finally, in Sec. 4.6 we show that the checkerboard fracton model can be interpreted as a rank-2 $\mathbb{Z}_2$ gauge theory, with two different forms of Gauss’ law on different lattice sites. The chapter concludes with a discussion in Sec. 4.7.

4.2 Hollow symmetric rank-2 tensor $U(1)$ gauge theory

We also consider a variant of the rank-2 scalar charge theory, where $E_{\mu\nu}$ and $A_{\mu\nu}$ remain symmetric but have only off-diagonal elements. We refer to this theory as the “hollow” rank-2 scalar charge theory, because if we write $E$ and $A$ as $3 \times 3$ matrices, the diagonal elements are zero.

The Gauss law constraint is

$$\Delta_x \Delta_y E_{xy} + \Delta_y \Delta_z E_{yz} + \Delta_x \Delta_z E_{xz} = n_r,$$

which leads to the same form of gauge transformations as in Eqs. (1.27, 1.28). The Gauss law can also be written

$$\frac{1}{2} \Delta_{\mu} \Delta_{\nu} E_{\mu\nu} = n_r,$$

where we take the diagonal components of $E$ to be zero.

This Gauss law implies that the charge on every $\{100\}$ lattice plane is conserved. Let $R$ be a bounded subset of some lattice plane $p$. Without loss of generality, we take $p$ to be an $xy$ plane. We have

$$Q_p = \sum_{r \in R} n_r = \frac{1}{2} \sum_{r \in R} \Delta_{\mu} \Delta_{\nu} E_{\mu\nu}.$$

Each term in the summand contains at least one of $\Delta_x$ or $\Delta_y$, so the sum reduces to a boundary term on $\partial R$, which implies that $Q_p$ is conserved.
If we try to define a magnetic field tensor using the same expression $B_{\mu\nu} = \epsilon_{\mu\lambda\sigma} \Delta_{\lambda} A_{\sigma\nu}$, but setting terms with diagonal components of $A$ to zero, we find that off-diagonal elements of $B$ are no longer gauge-invariant. Therefore we have only the diagonal elements $B_{\mu\mu}$ (no sum on $\mu$), which satisfy a tracelessness constraint $\sum_\mu B_{\mu\mu} = 0$, where the sum is over the three diagonal elements residing on the same dual lattice site. This gives us two independent elements of the magnetic field tensor, which is the correct number of degrees of freedom, as there are three independent elements of $A_{\mu\nu}$ and one unphysical gauge degree of freedom.

The Hamiltonian for the hollow gauge theory is

$$H_{\text{hollow}} = U \sum_{r,\mu<\nu} E_{\mu\nu}^2 - K \sum_{r,\mu} \cos(B_{\mu\mu}) + u \sum_r n_r^2 - J \sum_{r,\mu<\nu} \cos \left[ \Delta_{\mu} \Delta_{\nu} \theta - A_{\mu\nu} \right]. \quad (4.4)$$

This theory was studied previously in Ref. [126], where it was shown that a putative deconfined phase with gapped matter is unstable to confinement via proliferation of instantons. Even though this theory does not have a deconfined phase (at least with gapped matter), it will still be of interest to us, because it can be Higgsed to obtain the same quantum phase of matter as the X-cube model.

The hollow gauge theory is not just a variant of the ordinary rank-2 scalar charge theory, but it can be obtained from it by a partial confinement transition, as we show in Sec. 4.5.1. Starting from the scalar charge theory, we can make $U$ different for the diagonal and off-diagonal elements of $E$, so that the $E^2$ part of the Hamiltonian becomes

$$H_{\text{electric}} = U_d \sum_{r,\mu} E_{\mu\mu}^2 + U_{od} \sum_{r,\mu<\nu} E_{\mu\nu}^2. \quad (4.5)$$

Making $U_d$ large results in a state where $E_{\mu\mu} \approx 0$. This is precisely the hollow gauge theory, which is clear because the Gauss law of the scalar charge theory becomes that of the hollow gauge theory in this limit. We expect there should be a phase transition as $U_d$ is increased, from the deconfined phase of the scalar charge theory to the (necessarily) confined hollow gauge theory. Indeed, in Sec. 4.5.1, we show that condensing certain monopoles in the scalar charge theory leads to the hollow gauge theory.
4.3 Higgsing the conservation laws

As we mentioned before, the motion of U(1) charge is subject to extra conservation laws in higher rank gauge theories, such as dipole moment conservation. When the U(1) gauge theory is Higgsed, we can understand the effects on mobility of charges by directly studying how the conservation laws are modified, which can then be confirmed by a more detailed analysis starting from a Hamiltonian or functional integral formulation. We thus refer to Higgsing of the conservation laws themselves. In this section, we study the effect of a condensate of charge-$N$ matter fields, which breaks the gauge group down to $\mathbb{Z}_N$. We discuss charge conservation, dipolar conservation, and the planar conservation law obtained by Higgsing the hollow U(1) gauge theory.

Below, all the gauge theories studied have scalar charges living on the cubic lattice sites $\mathbf{r} = (x, y, z)$, with lattice spacing $a = 1$.

4.3.1 Charge conservation

We begin this section with the charge conservation law as a warm up. In a U(1) gauge theory, the U(1) charge is conserved, meaning, as discussed in Sec. 1.3.2, that the total charge $Q$ in a region cannot be changed by acting with local operators within that region. Upon Higgsing the theory to $\mathbb{Z}_N$, charge-$N$ objects can appear from and be absorbed into the condensate. Therefore $Q$ is now well-defined only modulo $N$, and becomes a conserved $\mathbb{Z}_N$ charge. This of course is familiar from vector gauge theory. The conservation of $\mathbb{Z}_N$ charge puts no constraints on the mobility of charges.

4.3.2 Dipole conservation

Now we consider the conservation of dipole moment that arises in the rank-2 scalar charge theory. On the lattice, the dipole moment $\mathbf{d}$ of some region $V$ is given by $\mathbf{d} = \sum_{\mathbf{r} \in V} \mathbf{r} n_{\mathbf{r}}$, and is conserved in the sense that it cannot be changed locally.

We now suppose we have a condensate of $N$-charge objects, which results in $\mathbb{Z}_N$ charges. By analogy with charge conservation, we can write down the Higgsed form of dipole conservation,
which is simply that $d \mod N$ is conserved, i.e. each component of $d$ is separately conserved modulo $N$. Here it is important that we set the lattice constant to one; then charge $\pm N$ objects appearing from the condensate can change each component of $d$ by integer multiples of $N$.

To understand the effects of the Higgsed dipole conservation law, it is useful first to consider configurations of charges lying on the $x$-axis, i.e. with $r_y = r_z = 0$. We describe such a charge configuration by a set of ordered pairs $\{(q_1, r_{x1}), (q_2, r_{x2}), \ldots \}$, where $q_i$ is the value of the charge at $r_x = r_{xi}$, and the charge is zero for points not listed explicitly.

Dipole conservation tells us that $\{(+1, 0), (-1, r_x)\}$ can be locally created only when $r_x$ is a multiple of $N$. Moreover, it is instructive to note that the locally creatable configuration $\{(+1, 0), (-1, N)\}$ can be obtained starting from a charge configuration with vanishing U(1) charge and dipole moment, and then exploiting the condensate to add and remove charge-$N$ objects. We consider the configuration $\{(+1, 0), (-N, i), (N, i + 1), (-1, N)\}$, which is easily seen to have vanishing charge and dipole moment. The charges at site $i$ and $i + 1$ can be absorbed into the condensate, and this configuration is thus equivalent to $\{(+1, 0), (-1, N)\}$.

On the other hand, the configuration $\{(+1, 0), (-1, k)\}$ with $0 < k < N$ has non-vanishing $d \mod N$ and is not locally creatable. Moreover, a configuration with a $+1$ charge at the origin and a $-1$ charge at $r = (r_x, r_y, r_z)$ is easily seen to have vanishing dipole moment modulo $N$, and thus be locally creatable, only when each of $r_x$, $r_y$ and $r_z$ are integer multiplies of $N$. From these observations we see that, although the charge in the rank-2 U(1) gauge theory is immobile, charges in the Higgsed rank-2 $\mathbb{Z}_N$ gauge theories can hop freely in any direction, but can only hop $N$ lattice spacings at a time along the $x$, $y$ and $z$ directions. This suggests that the rank-2 $\mathbb{Z}_N$ gauge theories obtained in this way are equivalent to the usual vector $\mathbb{Z}_N$ gauge theories. Indeed, we will show in Sec. 4.4.2 that the rank-2 $\mathbb{Z}_2$ scalar charge theory on the cubic lattice is equivalent to four copies of vector $\mathbb{Z}_2$ gauge theories (toric codes). With increasing $N$, the number of distinct particle species is expected to increase. In the limit of $N \to \infty$, the charge can only hop an infinite number of lattice spacings at a time, and we go back to the U(1) case where charge excitations are immobile.
4.3.3 Planar conservation law

In the U(1) scalar charge theory, the charge and dipole conservation laws impose strong constraints on the motion of excitations. However, after Higgsing, the $\mathbb{Z}_N$ charges turn out to be fully mobile. In order to further restrict the motion of $\mathbb{Z}_N$ charges to be sub-dimensional, we obviously need stronger constraints resulting from more powerful conservation laws.

If the charge on every lattice plane is separately conserved, then clearly single charges will not be able to move. This is true even for a discrete gauge theory with $\mathbb{Z}_N$ charges, and indeed occurs in the X-cube model (see Sec. 4.4.3 and Sec. 4.5.2). In other words, this strong conservation law would lead to fractons and sub-dimensional excitations in discrete gauge theory.

In Sec. 4.2, we showed that the hollow U(1) gauge theory indeed has conservation of charge on each \{100\} lattice plane. Upon Higgsing this conservation law by condensing charge-$N$ objects, we obtain $\mathbb{Z}_N$ charges that are conserved separately in each plane, which are thus immobile fractons. At $N = 2$, this is exactly the same conservation law as in the X-cube model, and we expect that condensing charge-2 objects in the hollow U(1) gauge theory will lead to the X-cube topological order. This expectation is verified in Sec. 4.5.2.

4.4 X-cube model via $\mathbb{Z}_2$ scalar charge theory

4.4.1 Charge-2 condensation

The results on the mobility of $\mathbb{Z}_N$ charges predicted above by Higgsing the conservation laws can be obtained explicitly by coupling the lattice model to a charge-$N$ matter field. We do this here for the rank-2 U(1) scalar charge theory. Starting with the Hamiltonian in Eq. 1.26, we add a charge-2 matter field on sites $r$ with number $N_r$ and phase $\Theta_r$. We add the following terms to the Hamiltonian in Eq. 1.26

$$H_{2\epsilon} = u_2 \sum_r N_r^2 - \Delta \sum_r \cos [\Theta_r - 2\theta_r] - J_2 \sum_{r, \mu \leq \nu} \cos \left[ \Delta_{\mu} \Delta_{\nu} \Theta - 2A_{\mu\nu}(r) \right].$$  \hspace{1cm} (4.6)
The Gauss law in Eq. (1.25) is modified to become
\[
(\Delta_x \Delta_y E_{xy} + \cdots) + (\Delta_y \Delta_x E_{xx} + \cdots) = n_r + 2N_r. \tag{4.7}
\]

Tuning \(J_2\) to be large results in a condensation of the charge-2 field \(e^{i\Theta}\). We also take \(\Delta\) large for convenience, and we treat the \(\Delta\) and \(J_2\) cosine terms as constraints. To solve the constraint imposed by \(\Delta \to \infty\), we define a \(\mathbb{Z}_2\) charge creation operator \(\tau^z_r \equiv e^{i\zeta_r}\), where
\[
\zeta_r = \frac{1}{2} \Theta_r - \theta_r = 0, \pi. \tag{4.8}
\]
The operator \(\tau^z_r\) anticommutes with \(\tau^x_r \equiv \exp \left\{ i \pi n_r \right\}\), which justifies the notation suggestive of Pauli matrices.

The second cosine leads to a rank-2 \(\mathbb{Z}_2\) gauge field \(Z_{\mu \nu} \equiv \exp(i\eta_{\mu \nu})\) on plaquettes \((\mu \neq \nu)\) and sites \((\mu = \nu)\), where
\[
\eta_{\mu \nu} = \frac{1}{2} \Delta_{\mu} \Delta_{\nu} \Theta - A_{\mu \nu}(r) = 0, \pi. \tag{4.9}
\]
The operator \(Z_{\mu \nu}\) anticommutes with the rank-2 \(\mathbb{Z}_2\) electric field \(X_{\mu \nu} \equiv \exp(i\pi E_{\mu \nu})\). The \(\mathbb{Z}_2\) magnetic flux is given by \(F_{\mu \nu} = \exp(iB_{\mu \nu}) = \exp(i\epsilon_{\mu \lambda \sigma} \Delta_{\lambda} \eta_{\lambda \nu})\). Two elements of \(F_{\mu \nu}\) are shown in Fig. (4.2).

Figure 4.2: Two elements of the \(\mathbb{Z}_2\) magnetic flux tensor \(F_{\mu \nu}\) are shown as products of \(\mathbb{Z}_2\) gauge field operators \(Z_{\mu \nu}\).

We can now write down the effective Hamiltonian describing the system upon condensing \(e^{i\Theta}\), in terms of the new \(\mathbb{Z}_2\) variables, which is a rank-2 \(\mathbb{Z}_2\) scalar charge gauge theory. We have
\[
H_{\mathbb{Z}_2} = -J \sum_{r, \mu \leq \nu} \exp(i\Delta_{\mu} \Delta_{\nu} \zeta_r) Z_{\mu \nu} - K \sum_{r, \mu, \nu} F_{\mu \nu} - u \sum_r \tau^x_r - U \sum_{r, \mu \leq \nu} X_{\mu \nu}. \tag{4.10}
\]
Here we have replaced the $n_r^2$ and $E_{\mu\nu}^2$ terms with $-\tau_r^x$ and $-X_{\mu\nu}$, respectively. These are the simplest terms in the $\mathbb{Z}_2$ variables that penalize non-zero charge and electric field, respectively.

The Gauss’ law is obtained by exponentiating that of the $U(1)$ theory, and is

$$\exp \left( i\pi (\Delta x \Delta y E_{xx} + \cdots) + (\Delta x \Delta y E_{xy} + \cdots) \right) = \tau_r^x. \quad (4.11)$$

The left-hand side, which we denote $G_r$, is a product of 18 $X_{\mu\nu}$ operators on the edges and vertices of an octahedron, as illustrated in Fig. 4.3.

We are interested in the deconfined phase of this theory, where magnetic flux is suppressed and where the matter fields are gapped. An extreme limit of the deconfined phase obtains for $J = U = 0$, where the model becomes exactly solvable. The $U$ term gives a tension to non-trivial electric field configurations, and, equivalently, leads to fluctuations of the magnetic field. For large enough $U$, we expect the theory to become confining. The $J$ term is a kinetic energy for the $\mathbb{Z}_2$ charges, which we expect to condense for large enough $J$.

We will analyze the deconfined phase at its exactly solvable point. To do this, it will be convenient to exploit a mapping to a local bosonic model with a tensor product Hilbert space (i.e., not a gauge theory). This mapping works in the same way as the familiar mapping between vector $\mathbb{Z}_2$ gauge theory and the $\mathbb{Z}_2$ toric code.[42] The degrees of freedom in the bosonic model are Pauli operators $\tilde{Z}_{\mu\nu}$ and $\tilde{X}_{\mu\nu}$. These are related to the gauge theory degrees of freedom by

$$\tilde{X}_{\mu\nu} = X_{\mu\nu} \quad (4.12)$$

$$\tilde{Z}_{\mu\nu} = \exp (i\Delta_{\mu}\Delta_{\nu}\zeta) Z_{\mu\nu}. \quad (4.13)$$

Using Gauss’ law to express $\tau_r^x$ in terms of $G_r$, the Hamiltonian for the bosonic model is

$$\tilde{H}_{\mathbb{Z}_2} = -K \sum_{r,\mu,\nu} \tilde{F}_{\mu\nu} - u \sum_r \tilde{G}_r - J \sum_{r,\mu} \tilde{Z}_{\mu\nu} - U \sum_{r,\mu,\nu} \tilde{X}_{\mu\nu}, \quad (4.14)$$

where $\tilde{F}_{\mu\nu}$ and $\tilde{G}_r$ are products of $\tilde{Z}_{\mu\nu}$ and $\tilde{X}_{\mu\nu}$, respectively, given by the same expressions as $F_{\mu\nu}$ and $G_r$. 
Figure 4.3: The octahedron term $G_r$ at $r$ involves eighteen Pauli operators denoted as solid dots. Six of them are at sites (red). The other ones are at plaquettes adjacent to $r$ in $xy$ (yellow), $yz$ (blue) and $xz$ (green) planes. The open circle shows the location of the charge created by violating this $G_r$ term.

4.4.2 Lattice $\mathbb{Z}_2$ scalar charge theory

We now study the deconfined phase of the $\mathbb{Z}_2$ scalar charge theory at its exactly solvable point. We work with the local bosonic model of Eq. (4.14), set $U = J = 0$, and drop the tildes that distinguish between operators acting in the gauge theory and tensor product Hilbert spaces. We will show that this model is equivalent to four copies of the three dimensional toric code (equivalently, four copies of $\mathbb{Z}_2$ vector gauge theory). In the next section, we start from this model and describe how to obtain the X-cube model via a confinement transition.

First, in Appendix 4.8, we establish that the ground state degeneracy GSD on a $L \times L \times L$ torus is a constant: $\log_2(\text{GSD}) = 12$, independent of system size. Moreover, the degenerate ground states cannot be locally distinguished, and the model is topologically ordered. This is consistent with the model being equivalent to four copies of the $d = 3$ toric code.

Here, we describe the excited states of the model, which can be labeled by the eigenvalues of the commuting operators $G_r$ and $F_{\mu\nu}$. First, we consider “electric” excitations where $G_r = -1$ for some sites $r$ contained in a bounded region, and where $F_{\mu\nu} = 1$. We will see that any such excitation can be built up from four independent types of fully mobile $\mathbb{Z}_2$ point charges.
Figure 4.4: Four types of $Z_2$ gauge charges $\tau_{1,2,3,4}$ live on different lattice sites as shown. Charges on other sites are equivalent to one of these four types. The large open black circle is a choice of origin, and by definition charges of type $\tau_1$ reside at the origin. These charges can be transported by string operators and are freely mobile. A string operator creating the $\tau_4$ charge is shown; this operator is a product of $Z_{xx}$ over every other site on a line in the $x$-direction.

Acting with a $Z_{\mu\nu}$ operator on a ground state creates point-like charges at sites by flipping the sign of some $G_r$ eigenvalues. Depending on whether $\mu = \nu$ or $\mu \neq \nu$, two charge configurations are possible, and are the same as in Fig. 1.10, but with $Z_2$ charges. In the linear charge configuration of Fig. 1.10b, obtained by acting with $Z_{\mu\mu}$, there are now only two nontrivial charges, as the middle one vanishes into the condensate. Because of this, single charges are able to hop from site to site by an even number of lattice spacings. Therefore, by acting repeatedly with $Z_{\mu\mu}$ operators, we can transform a general electric excitation into one supported on a cube of eight sites $r = (r_x, r_y, r_z)$, where $r_x, r_y, r_z = 0, 1$, and where we take the origin to be one corner of the cube.

At this point it might appear that there are eight types of independent charges, corresponding to the eight vertices of the cube. However, acting with off-diagonal $Z_{\mu\nu}$ operators on the faces of the cube, any charge configuration can be brought to one with excitations at only four of the eight vertices, as shown in Fig. 4.4. There are thus four types of $Z_2$ charges, corresponding to $G_r = -1$ at only a single one of the four vertices of Fig. 4.4. We label the charge types by $\tau_{1,2,3,4}$, as shown in the figure.

The existence of four types of $Z_2$ charge is consistent with equivalence to four copies of the
Figure 4.5: Four types of $yz$-plane loop excitations in the rank-2 $\mathbb{Z}_2$ scalar charge theory. In each panel, red dots show the locations of the $X_{\mu\nu}$ operators creating the loop, and the black open circle shows the coordinate origin defined in Fig. 4.4. (a) shows loops of type 1 and 2, created by acting with $X_{xx}$ within different $yz$-plane layers with even and odd $x$ coordinate. (b) and (c) show two different loop excitations created by acting with a product of $X_{xx}$ together with $X_{xz}$ or $X_{xy}$, respectively. These four loops are distinct excitations because they have different statistical interactions with the four different types of $\mathbb{Z}_2$ gauge charge.

$d = 3$ toric code. To go further, we should also establish the existence of magnetic loop excitations that have non-trivial braiding with the charges. For a single copy of the $d = 3$ toric code, there is a statistical phase $\theta = \pi$ when a point charge braids around a loop. Here, upon braiding any composite of the elementary charges $\tau_{1,2,3,4}^z$, we expect a statistical phase of $\theta = 0$ or $\theta = \pi$.

We need to show that there exist four different types of magnetic loops, such that if we braid a single point-like excitation with each of the four loops, the pattern of statistical phases uniquely determines the charge type. A set of such loop excitations is shown in Fig. 4.5; in each case the loop and the membrane operator creating it lies in a $yz$ plane. The statistics of each of the four elementary charges with a given loop is then easily determined by noting whether the string operator transporting the charge in the $x$-direction, which is a product of $Z_{xx}$ operators, commutes or anti-commutes with the loop’s membrane operator. This information is shown in Table 4.1, and it is straightforward to verify that the charge type is fully resolved by braiding with the four different loops.

Based on the ground state degeneracy and properties of excitations and logical operators, we conclude that the $\mathbb{Z}_2$ scalar charge model has the same topological order as four copies of the $d = 3$
<table>
<thead>
<tr>
<th>Loop type</th>
<th>Elementary charges with $\theta = \pi$ statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\tau_2^z$</td>
</tr>
<tr>
<td>2</td>
<td>$\tau_1^z, \tau_3^z, \tau_4^z$</td>
</tr>
<tr>
<td>3</td>
<td>$\tau_1^\pm, \tau_2^\pm, \tau_3^\pm$</td>
</tr>
<tr>
<td>4</td>
<td>$\tau_1^\pm, \tau_2^\pm, \tau_4^\pm$</td>
</tr>
</tbody>
</table>

Table 4.1: In this table, for each type of magnetic loop as shown in Fig. 4.5, we list the elementary charges $\tau_{1,2,3,4}^z$ that acquire a statistical phase $\theta = \pi$ when braided around the loop indicated. The statistical phase is $\theta = 0$ for elementary charges not listed.

toric code. There are no sub-dimensional particle excitations in this system, which is consistent with our prediction from Higgsing the conservation law.

We note that a similar analysis can be carried out in two dimensions, starting with the rank-2 U(1) scalar charge theory on the square lattice, and Higgsing it by coupling to charge-2 matter. There, the resulting $\mathbb{Z}_2$ scalar charge theory is equivalent to three copies of the $d = 2$ toric code. This analysis is presented in Appendix 4.9.

4.4.3 X-cube model from selective flux loop condensation

Here we will describe how to get the X-cube fracton topological order from the rank-2 $\mathbb{Z}_2$ scalar charge theory, by condensing certain flux loops. Before doing this, we discuss a simpler but similar transition from the cubic-lattice $d = 3$ $\mathbb{Z}_2$ toric code to a stack of decoupled $d = 2$ toric code layers.

We consider a perturbed $d = 3$ $\mathbb{Z}_2$ toric code, which has a single qubit on each nearest-neighbor link $\ell$ of the simple cubic lattice. The Hamiltonian is

$$H_{3dTC} = -\sum_p B_p - \sum_r A_r - h \sum_{\ell \parallel z} X_\ell$$  \hspace{1cm} (4.15)$$

where $p$ labels square plaquettes, $r$ labels cubic lattice sites, and we take $h \geq 0$. The sum in the last term is over links parallel to the $z$-axis. The plaquette term is given by $B_p = \prod_{\ell \in p} Z_\ell$, while the vertex term is $A_r = \prod_{\ell \sim r} X_\ell$, where latter product is over the six links touching the site $r$. When $h = 0$, we have the usual exactly solvable $d = 3$ toric code, which has point charge excitations where $A_r = -1$, and flux excitations that are loops along which $B_p = -1$. 
Figure 4.6: Loop condensation (red) between \( z = L \) and \( z = L \pm 1 \) layers in the \( d = 3 \) toric code. The horizontal segments of the green loop disappear into the condensate, while the intersection points with the \( z = L \) layer (yellow plaquettes) become gapped point-like excitations that can move freely within the \( z = L \) plane.

Now we consider the effect of the perturbation, which tends to freeze the degrees of freedom between \( xy \) plane layers. In the strong coupling \( h \to \infty \) limit, we set \( X_\ell = 1 \) for all \( \ell \parallel z \). The remaining terms can be treated in first-order degenerate perturbation theory. Projecting the Hamiltonian into the degenerate subspace, each \( A_r \) term becomes a product of \( X_\ell \) over the four touching \( r \) and lying in an \( xy \) plane. Plaquette terms \( B_p \) where \( p \) lies in an \( xy \) plane survive the projection unchanged, while other plaquette terms have vanishing projection into the degenerate subspace. The resulting Hamiltonian is simply that of a stack of decoupled \( d = 2 \) toric code layers.

The transition from \( d = 3 \) toric code to \( d = 2 \) toric code layers can be interpreted in terms of a condensation of certain flux loops. Considering first small \( h \), we see that acting with \( X_\ell \) for \( \ell \parallel z \) creates a small flux loop lying within an \( xy \) plane between two square lattice layers. For intermediate \( h \), many such \( xy \)-plane flux loops are created, and eventually we expect these loops will condense at infinite \( h \). The point charge excitations of the \( d = 3 \) toric code can no longer move in the \( z \)-direction, because they have a statistical interaction with the loop condensate. However, the charges can still move freely within \( xy \)-plane square lattice layers, where the loop condensate does not penetrate. In addition, a flux loop that intersects a single square lattice layer at two points, as shown in Fig. 4.6, reduces to two point-like flux excitations at the intersection points, in the presence of the condensate.
Now we return to the rank-2 $\mathbb{Z}_2$ scalar charge theory discussed in the previous section, from which we obtain the X-cube topological order via a similar selective flux loop condensation. We add the following term to the Hamiltonian

$$H' = -h \sum_{\mathbf{r}} \sum_{\mu=x,y,z} X_{\mu\mu}(\mathbf{r}),$$

where we recall that the diagonal components $X_{\mu\mu}$ reside at lattice sites $\mathbf{r}$. In the strong coupling limit, $h \to \infty$, $X_{\mu\mu}(\mathbf{r})$ is restricted to be 1. As in the $d=3$ toric code case, we treat the remaining terms in first-order degenerate perturbation theory. Projecting the $G_{\mathbf{r}}$ term into the degenerate subspace with the projector $P$ results in

$$P\tilde{G}_{\mathbf{r}}P = \prod_{p\sim \mathbf{r}} X_{\mu\nu}(p),$$

where the product is over the 12 plaquettes $p$ with a corner at the site $\mathbf{r}$. The off-diagonal $F_{\mu\nu}$ terms have vanishing projection into the degenerate subspace, while the diagonal terms $F_{\mu\mu}$ survive unchanged. The resulting model has only off-diagonal ($\mu \neq \nu$) Pauli operators $X_{\mu\nu}$ and $Z_{\mu\nu}$ residing on plaquettes. Replacing the cubic lattice with its dual, the plaquette variables become link variables $X_\ell$ and $Z_\ell$, where $\ell$ labels dual lattice links. It is straightforward to see that

$$PG_{\mathbf{r}}P = \prod_{\ell \in \mathbf{c}} X_\ell,$$

where $\mathbf{c}$ is the dual lattice cube centered at $\mathbf{r}$, and the product is over the 12 edges of the cube. This is precisely the cube term of the X-cube model. Moreover,

$$F_{\mu\mu} = \prod_{\ell\sim \mathbf{s}, \ell \perp \mu} Z_\ell,$$

where the product is over the four links touching the dual lattice site $\mathbf{s}$ that are perpendicular to the $\mu$-direction; this is a vertex term of the X-cube model. Therefore, the Hamiltonian projected to the degenerate subspace is precisely the X-cube model.

Again, we can develop a physical picture based on flux loop condensation by considering small and intermediate $h$. Acting with the perturbation creates certain flux loops within $xy$, $yz$ and $xz$ planes, as shown in Fig. 4.7. When these loops condense, the charge excitations of the rank-2 $\mathbb{Z}_2$
Figure 4.7: The red loops depict the selective flux loop condensate in the rank-2 $\mathbb{Z}_2$ scalar charge with $h$ sufficiently large. In the presence of the loop condensate, acting with a string of $X_{xz}$ operators over a line extending in the $y$-direction creates two gapped excitations (open circles) at the ends of the string. Acting with the same operator in the deconfined phase of the rank-2 gauge theory (i.e. with $h$ small) also creates excitations along the string depicted by the orange and green loops. When the loops are condensed, these excitations disappear into the condensate, with only the gapped excitations at the ends of the string remaining.

scalar charge theory are confined, as they have a statistical interaction with the loops. However, bound states of two charges separated along the $x$, $y$ or $z$ axis can propagate freely (see Fig. 4.8); string operators transporting such bound states are a product of off-diagonal $Z_{\mu\nu}$ operators, which commute with the perturbation. We thus arrive at the conclusion that individual charges are now created at the corners of a rectangular membrane operator, and become fractons.

We can also obtain the one-dimensional particles of the X-cube model from flux loops in this picture. Acting on a plaquette $p$ with an off-diagonal operator such as $X_{xz}(p)$ in the rank-2 $\mathbb{Z}_2$ scalar charge theory creates an excitation where eight different $F_{\mu\nu}$ operators (crosses in Fig. 4.7) flip sign. Four of these are off-diagonal operators with $\mu \neq \nu$, and upon condensing loops these excitations disappear into the condensate. The other four operators are the $F_{xx}$ and $F_{zz}$ terms adjacent to the plaquette $p$, which remain as gapped excitations. By acting with a product of $X_{xz}$ over a stack of $xz$-plane plaquettes (see Fig. 4.7), these two excitations can be separated along a line. These are the one-dimensional particle excitations of the X-cube model.
4.5 X-cube model via hollow U(1) scalar charge theory

Now we study the right-hand path of Fig. 4.1, which is a different route to the X-cube model from the rank-2 U(1) scalar charge theory. In Sec. 4.2, we already noted that adding $U_d \sum_{r,\mu} E_{\mu}^2$ to the Hamiltonian of the scalar charge theory, and making $U_d$ large, freezes out the diagonal degrees of freedom, resulting precisely in the hollow U(1) gauge theory. Evidently this is some kind of partial confinement transition, but by what mechanism does it proceed? Here, in Sec. 4.5.1, we carry out an electric-magnetic duality transformation and show that increasing $U_d$ drives the condensation of certain point-like magnetic monopoles, and that the hollow U(1) gauge theory describes the system in the presence of the monopole condensate. The same hollow U(1) scalar charge theory was studied before[126], and it was found that it does not have a stable deconfined phase. Nonetheless, the effect of condensing charge-2 matter can still be studied as a mechanism to drive the system into another phase, and in Sec. 4.5.2 we show that this results in the X-cube model. Because of the confinement in the hollow U(1) gauge theory, we can interpret the condensation described in Sec. 4.5.2 as a mechanism for a transition between a confined phase and a phase with the topological order of the X-cube model.

Figure 4.8: The green shaded plaquettes represent an operator transporting a bound state of two charges aligned along the $z$-axis from one location (orange squares) to another (red squares). The operator is a product of $Z_{\mu \nu}$ over the shaded plaquettes. This operator can be viewed as a string operator transporting these bound states within an $xy$-plane, or as a membrane operator creating individual fracton excitations at the corners.
4.5.1 Magnetic monopole condensation in the rank-2 U(1) scalar charge theory

In order to understand the magnetic sector of the rank-2 U(1) scalar charge theory, we go to its dual theory. We will work with a discrete-time Euclidean action that can be derived from the lattice Hamiltonian of Eq. (1.26). Since we are interested only in the magnetic sector, we drop the coupling to matter fields and consider a pure gauge theory. The Euclidean action takes the form

\[ S = i \sum_{\tau, r} \sum_{\mu \leq \nu} E_{\mu \nu} \Delta_{\tau} A_{\mu \nu} - i \sum_{\tau, r} \phi(\tau, r) \sum_{\mu \leq \nu} \Delta_{\mu} \Delta_{\nu} E_{\mu \nu} + \sum_{\tau, r} \left[ u_d \sum_{\mu} E_{\mu \mu}^2 + u_{od} \sum_{\mu < \nu} E_{\mu \nu}^2 \right] - k \sum_{\tau, r, \mu, \nu} \cos(B_{\mu \nu}). \] (4.20)

Here, pairs \( \tau, r \) label space-time points, \( E_{\mu \nu} \) is an integer-valued field residing on sites \( (\mu = \nu) \) and plaquettes \( (\mu \neq \nu) \) of the spatial lattice, and \( A_{\mu \nu} \) is a \( 2\pi \)-periodic variable residing in the same locations. The magnetic field tensor \( B_{\mu \nu} \) was defined in terms of \( A_{\mu \nu} \) in Sec. 1.3.2, is not symmetric but is traceless, and resides on sites and plaquettes of the spatial dual cubic lattice. The field \( \phi(\tau, r) \) is \( 2\pi \)-periodic Lagrange multiplier that imposes the Gauss’ law constraint.

Integrating out \( \phi(\tau, \tau) \), the Gauss’ law constraint is solved by writing \( E_{\mu \nu} \) in terms of a dual gauge field \( \alpha_{\mu \nu} \) taking values in \( 2\pi \mathbb{Z} \):

\[ E_{\mu \mu} = \frac{1}{2\pi} \epsilon_{\mu \lambda \sigma} \Delta_{\lambda} \alpha_{\sigma \mu} \quad \text{(no sum over } \mu) \] (4.21)

\[ E_{\mu \nu} = \frac{1}{2\pi} \left( \epsilon_{\mu \lambda \sigma} \Delta_{\lambda} \alpha_{\sigma \nu} + \epsilon_{\nu \lambda \sigma} \Delta_{\lambda} \alpha_{\sigma \mu} \right) \quad (\mu \neq \nu). \] (4.22)

Here, \( \alpha_{\mu \nu} \) lives on dual lattice sites if \( \mu = \nu \) and on dual plaquettes if \( \mu \neq \nu \). The field \( \alpha_{\mu \nu} \) transforms under dual gauge transformations by

\[ \alpha_{\mu \nu} \rightarrow \alpha_{\mu \nu} + \Delta_{\mu} \lambda_{\nu} + \delta_{\mu \nu} f, \] (4.23)

where \( \lambda_{\mu} \) lives on the dual lattice links and \( f \) lives on dual sites, and both \( \lambda_{\mu} \) and \( f \) take values in \( 2\pi \mathbb{Z} \).

Next, we proceed to the Villain representation for the cosine terms in the action, i.e. we replace

\[ -k \cos(B_{\mu \nu}) \rightarrow \frac{1}{2k} D_{\mu \nu}^2 + iD_{\mu \nu} B_{\mu \nu} \] (4.24)
where $D_{\mu\nu}$ is a new integer valued field that is summed over. Similarly to $\alpha_{\mu\nu}$, $D_{\mu\nu}$ is a tensor field living on the dual lattice sites and plaquettes.

Now we can integrate out $A_{\mu\nu}$ field, which results in a constraint on $D_{\mu\nu}$, that is solved by

$$D_{\mu\nu} = \frac{1}{2\pi}(\Delta_\tau \alpha_{\mu\nu} - \Delta_\mu \psi_\nu - \delta_{\mu\nu}\theta). \quad (4.25)$$

Here, invariance of $D_{\mu\nu}$ under dual gauge transformations requires introducing $2\pi\mathbb{Z}$-valued fields $\psi_\mu$ and $\theta$ living on the links and sites of the dual lattice, respectively. These fields transform under dual gauge transformations by

$$\psi_\mu \to \psi_\mu + \Delta_\tau \lambda_\mu,$$
$$\theta \to \theta + \Delta_\tau f. \quad (4.26)$$

We thus obtain the dual action

$$S_{\text{dual}} = \frac{1}{4\pi^2} \sum_{\tau, r} \left[ u_d \sum_{\mu} (\epsilon_{\mu\lambda\sigma} \Delta_\lambda \alpha_{\sigma\mu})^2 + u_{od} \sum_{\mu<\nu} (\epsilon_{\mu\lambda\sigma} \Delta_\lambda \alpha_{\sigma\nu} + \epsilon_{\nu\lambda\sigma} \Delta_\lambda \alpha_{\sigma\mu})^2 \right]$$
$$+ \frac{1}{8\pi^2 k} \sum_{\tau, r} \sum_{\mu, \nu} (\Delta_\tau \alpha_{\mu\nu} - \Delta_\mu \psi_\nu - \delta_{\mu\nu}\theta)^2. \quad (4.27)$$

To proceed, we promote the discrete fields to real-valued fields and introduce cosine terms to softly restore the discreteness constraint. It is convenient to allow for real-valued dual gauge transformations; in order to do this and keep the cosine terms gauge invariant, we need to introduce new compact fields $\phi_\mu$ on dual links and $\gamma$ on dual sites. These fields transform as $\phi_\mu \to \phi_\mu + \lambda_\mu$ and $\gamma \to \gamma + f$ under dual gauge transformations. We add the cosine terms

$$S' = -t_m \sum_{\tau, r} \sum_{\mu, \nu} \cos(\Delta_\mu \phi_\nu - \alpha_{\mu\nu} - \gamma \delta_{\mu\nu}) - t_m' \sum_{\tau, r} \sum_{\mu, \nu} \cos(\Delta_\tau \phi_\mu - \psi_\mu)$$
$$- t_\theta \sum_{\tau, r} \cos(\Delta_\tau \gamma - \theta). \quad (4.28)$$

The discreteness of $\theta$ will not play an important role in our discussion, so we take $t_\theta = 0$ and integrate out $\gamma$ to obtain the action

$$S'_{\text{dual}} = S_{\text{dual}} - t_m \sum_{\tau, r} \sum_{\mu \neq \nu} \cos(\Delta_\mu \phi_\nu - \alpha_{\mu\nu}) - t_m' \sum_{\tau, r} \sum_{\mu, \nu} \cos(\Delta_\mu \phi_\mu - \Delta_\nu \phi_\nu - \alpha_{\mu\nu} + \alpha_{\nu\mu})$$
$$- t_m'' \sum_{\tau, r} \sum_{\mu} \cos(\Delta_\tau \phi_\mu - \psi_\mu). \quad (4.29)$$
Here, we see that $\phi_\mu$ is a matter field coupled to the dual gauge field. We therefore expect it to represent gapped magnetic excitations, as we shall see below.

Next, we decouple the second term in Eq. (4.27) with a Hubbard-Stratonovich transformation,

$$\frac{1}{8\pi^2 k}(\Delta_\tau \alpha_{\mu\nu} - \Delta_\mu \psi_\nu - \delta_{\mu\nu}\theta)^2 \to 2\pi^2 k \beta_{\mu\nu}^2 - i\beta_{\mu\nu}(\Delta_\tau \alpha_{\mu\nu} - \Delta_\mu \psi_\nu - \delta_{\mu\nu}\theta).$$ (4.30)

Here, $\beta_{\mu\nu}$ should be interpreted as the magnetic field; that is, $\beta_{\mu\nu} \sim B_{\mu\nu}$. We note that integrating out $\theta$ gives the constraint $\sum_\mu \beta_{\mu\mu} = 0$. Next, we go to a Villain representation for the last term in Eq. (4.29), i.e.

$$t_m'' \cos (\Delta_\tau \phi_\mu - \psi_\mu) \to \frac{1}{t_m} n_\mu^2 + i n_\mu (\Delta_\tau \phi_\mu - \psi_\mu)$$ (4.31)

This introduces the “number operator” $n_\mu$ conjugate to the phase $\phi_\mu$ of the dual matter field.

Integrating out $\psi_\mu$ gives the dual Gauss’ law,

$$\Delta_\mu \beta_{\mu\nu} = n_\nu,$$ (4.32)

which shows that $n_\mu$ is a point-like (in space) vector magnetic charge, or magnetic monopole, created by $e^{i\phi_\mu}$. From this constraint, it is straightforward to show that the quantity $\sum_\mu r_\mu n_\mu$ is conserved. Conservation of this quantity allows a $n_\mu$ monopole to hop freely in a $d = 2$ plane normal to $\mu$, but forbids the monopole to move independently in the $\mu$-direction. Therefore, these monopoles are two-dimensional particles. This is born out by inspecting the the first two terms in Eq. (4.29), which describe two different dynamical processes of monopoles. The $t_m$ term is a hopping of $n_\mu$ monopoles in a plane perpendicular to $\mu$. The $t_m'$ term is a cooperative two-monopole hopping process.

We now consider the effect of condensing monopoles within their $d = 2$ planes of motion, by making $t_m$ and $t_m''$ large, while keeping $t_m'$ fixed. We will see that, because we are keeping $t_m'$ fixed, this does not fully Higgs out the dual gauge field. To understand why this is the partial confinement transition we expect when $U_d$ (and hence $u_d$) becomes large, we note that the $t_m$ cosine originates from the discreteness constraint on off-diagonal elements of $\alpha_{\mu\nu}$. Now, diagonal elements of $E_{\mu\nu}$
only depend on off-diagonal elements of $a_{\mu\nu}$, in contrast to off-diagonal elements of $E_{\mu\nu}$, which depend on both diagonal and off-diagonal elements of $a_{\mu\nu}$ [see Eqs. 4.21 and 4.22]. Therefore, as $U_d$ is increased in the original Hamiltonian, the discreteness of diagonal elements $E_{\mu\mu}$ becomes more important, and we expect $t_m$ to increase in this effective dual description.

To analyze the large $t_m$ and $t''_m$ limit, we make the changes of variables

$$
\begin{align*}
\alpha_{\mu\nu} &\to \alpha_{\mu\nu} + \Delta_{\mu} \phi_{\nu} \\
\psi_{\mu} &\to \psi_{\mu} + \Delta_{\tau} \phi_{\mu}.
\end{align*}
$$

We thus obtain constraints $\alpha_{\mu\nu} = 0$ (for $\mu \neq \nu$), and $\psi_{\mu} = 0$. The first of these constraints implies $E_{\mu\mu} = 0$, as expected in the large $U_d$ limit. We define $h_{\mu} \equiv \alpha_{\mu\mu}$, and write the resulting action as

$$
S_{\text{hollow dual}} = \frac{u_{od}}{4\pi^2} \sum_{\tau, \rho} \left[ (\Delta_{\tau} h_{x} - \Delta_{\rho} h_{y})^2 + (\Delta_{\tau} h_{y} - \Delta_{\rho} h_{z})^2 + (\Delta_{\tau} h_{z} - \Delta_{\rho} h_{x})^2 \right] + \frac{1}{8\pi^2 k} \sum_{\tau, \rho} \sum_{\mu} (\Delta_{\tau} h_{\mu} - \theta)^2 - t'_m \sum_{\tau, \rho} \sum_{\mu < \nu} \cos(h_{\mu} - h_{\nu}).
$$

which is invariant under the gauge transformations

$$
\begin{align*}
h_{\mu} &\to h_{\mu} + f \\
\theta &\to \theta + \Delta_{\tau} f,
\end{align*}
$$

It was shown previously by Xu and Wu that this theory is a dual description of the hollow U(1) scalar charge theory,[126] so that this theory indeed describes the system upon condensing monopoles within their $d = 2$ planes of motion. Moreover, Xu and Wu also showed that the cosine term in the above action is relevant, corresponding to a proliferation of instantons in space-time, that results in a gapped confined phase.[126]

### 4.5.2 X-cube model from lattice hollow U(1) scalar charge theory by Higgs mechanism

Here, we couple the hollow U(1) rank-2 scalar charge theory to charge-2 matter, and show that condensing this matter results in the X-cube model. The details of the analysis are very
similar to that carried out in Sec. 4.4.1, so we only summarize the key points. Starting with the Hamiltonian Eq. (4.4), we add a charge-2 matter field with number \( N_r \) and conjugate phase \( \Theta_r \).

We add the following terms to the Hamiltonian

\[
H_2 = u \sum_r N_r^2 - \Delta \sum_r \cos[\Theta_r - 2\theta_r] - J_2 \sum_{r,\mu<\nu} \cos[\Delta_\mu \Delta_\nu \Theta - 2A_{\mu\nu}(r)].
\]  

(4.38)

The charge-2 matter condenses for sufficiently large \( J_2 \). As in Sec. 4.4.1, we take both \( J_2 \) and \( \Delta \) to be large, which allows us to treat the corresponding cosine terms as constraints.

We obtain a model defined in terms of symmetric-tensor \( Z_{\mu\nu} \), \( X_{\mu\nu} \) Pauli operators with \( \mu \neq \nu \) living on plaquettes, and \( \tau^x_r, \tau^z_r \) Pauli operators living on sites. As in Sec. 4.4.1, \( X_{\mu\nu} \) is a rank-2 \( \mathbb{Z}_2 \) electric field, \( Z_{\mu\nu} \) the conjugate rank-2 \( \mathbb{Z}_2 \) gauge field, and \( \tau^x_r \) is the \( \mathbb{Z}_2 \) gauge charge of the un-condensed charge-1 matter. The Hamiltonian is

\[
H_{\text{hollow}}^{\mathbb{Z}_2} = -J \sum_{p,\mu<\nu} \tau^z_{r_1} \tau^z_{r_2} \tau^z_{r_3} \tau^z_{r_4} Z_{\mu\nu}(p) - K \sum_{r,\mu} F_{\mu\mu} - u \sum_r \tau^x_r - U \sum_{r,\mu<\nu} X_{\mu\nu}.
\]  

(4.39)

Here, the sum in the first term is over plaquettes \( p \), and the sites \( r_1, \ldots, r_4 \) lie at the corners of \( p \). \( F_{\mu\mu} \) is defined in Sec. 4.4.1; only the diagonal elements play a role here, because only the diagonal elements \( B_{\mu\mu} \) appear in the U(1) hollow gauge theory. The Gauss’ law is

\[
G^h_r \equiv \prod_{p \sim r} X_{\mu\nu}(p) = \tau^x_r,
\]  

(4.40)

where the product is over the 12 plaquettes sharing a corner with \( r \).

This theory can be viewed as a hollow rank-2 \( \mathbb{Z}_2 \) scalar charge theory on the cubic lattice. To see it is equivalent to the X-cube model, we exploit the same mapping to a local bosonic model discussed in Sec. 4.4.1. The resulting Hamiltonian is

\[
\tilde{H}_{\text{hollow}}^{\mathbb{Z}_2} = -J \sum_{p,\mu<\nu} \tilde{Z}_{\mu\nu}(p) - K \sum_{r,\mu} \tilde{F}_{\mu\mu} - u \sum_r \tilde{G}^h_r - U \sum_{r,\mu<\nu} \tilde{X}_{\mu\nu}.
\]  

(4.41)

Passing to the dual lattice so that the plaquette variables become link variables, the \( \tilde{F}_{\mu\mu} \) term becomes precisely the vertex term of the X-cube model, while the \( \tilde{G}^h \) term becomes the cube term (see Fig. 4.9). Therefore, we have obtained precisely the X-cube model, perturbed by the \( J \) and \( U \) terms, which give dynamics to the fractons and one-dimensional particle excitations, respectively.
Figure 4.9: (a) The operator $G_r^h$ is a product of $X_{\mu\nu}$ over the plaquettes touching $r$ (shaded plaquettes). Viewing $X_{\mu\nu}$ as a link variable on the dual lattice, $G_r^h$ is a product over the edges of the dual lattice cube surrounding $r$ (dashed lines). (b) Diagonal components of $F_{\mu\nu}$, such as $F_{zz}$ illustrated here, are products of $Z_{\mu\nu}$ over four of the faces surrounding a cube. On the dual lattice, this becomes a product over four links touching a dual site, as shown.

4.6 Checkerboard model as a $\mathbb{Z}_2$ scalar charge model

All the rank-2 gauge theories considered thus far have a Gauss’ law of the same form at every point in space. Here, we construct a rank-2 $\mathbb{Z}_2$ gauge theory with two different Gauss’ laws at different vertices of the cubic lattice, and show that this theory is equivalent to the checkerboard fracton model.[110]

We work on the cubic lattice, and place qubits on $xz$ and $yz$ plaquettes, with no degrees of freedom residing on $xy$ plaquettes. We denote the corresponding Pauli operators by $X_{xz(yz)}$ and $Z_{xz(yz)}$. It is convenient to work with electric field $E_{\mu\nu}$ and gauge field $A_{\mu\nu}$ variables introduced by defining $X_{\mu\nu} = \exp(i\pi E_{\mu\nu})$, and $Z_{\mu\nu} = \exp(i\pi A_{\mu\nu})$, where $E_{\mu\nu}$ and $A_{\mu\nu}$ take values in \{0,1\}. We write

$$
E_{\mu\nu} = \begin{bmatrix}
0 & E_{xz} & E_{yz} \\
E_{zx} & 0 & 0 \\
E_{yz} & 0 & 0
\end{bmatrix}
A_{\mu\nu} = \begin{bmatrix}
0 & A_{xz} & A_{yz} \\
A_{zx} & 0 & 0 \\
A_{yz} & 0 & 0
\end{bmatrix}
$$

We impose two Gauss’ laws. On layers with even $z$ coordinate, we impose $\Delta_x \Delta_z E_{xz} + \Delta_y \Delta_z E_{yz} = n_r$. The corresponding charge resides at even-$z$ vertices of the cubic lattice, as shown in Fig. 4.10a. On odd-$z$ layers, we impose $\Delta_y \Delta_z E_{xz} + \Delta_x \Delta_z E_{yz} = n_{r'}$, where $r'$ is the center of an
xy plaquette, so that charge resides on xy plaquettes (see Fig. 4.10b). Viewing these Gauss’ laws in terms of the finer cubic lattice, we see that they are defined on a “checkerboard” of edge-sharing cubes whose centers (where charges reside) form a FCC lattice.

From the Gauss law, we can obtain the gauge transformation of $A_{\mu\nu}$, e.g. $A_{yz}(r) \rightarrow A_{yz}(r) + \Delta_y f(r - \frac{1}{2} \hat{z}) + \Delta_z f(r + \frac{1}{2} \hat{z})$ as shown in Fig. 4.10(d). The simplest gauge-invariant combinations of the $A_{\mu\nu}$ are sums of eight variables that take precisely the same form as the sums in the two Gauss’ laws. Passing to a local bosonic model as above, the Hamiltonian consists of a term imposing the Gauss’ laws, and the gauge-invariant terms built from $A_{\mu\nu}$. In terms of Pauli matrices, we have

$$H_{chk} = - \sum_{C} \prod_{i \in C} X_i - \sum_{C} \prod_{i \in C} Z_i$$

where $C$ denotes the edge-sharing cubes of the checkerboard lattice. This is precisely the Hamiltonian of the checkerboard fracton model.

![Figure 4.10](image.png)

Figure 4.10: (a) The Gauss law for the even layers involves eight electric field variables forming a cube surrounding an even-z site of the cubic lattice (black circle). (b) The Gauss law for the odd layers is a sum of the electric field over the eight vertices of a cube surrounding the center of an xy-plane plaquette whose center has odd z coordinate. (c) Gauge transformation of $A_{yz}$. The black and blue circles show the locations of the nearby checkerboard cube centers where the function $f$ is defined.

### 4.7 Discussion

In this work, we showed that Higgs and partial confinement mechanisms provide relationships among gapped fracton phases and U(1) symmetric-tensor gauge theories. These relationships also
encompass more conventional topological orders, e.g. the rank-2 $\mathbb{Z}_2$ scalar charge theory that is obtained from the rank-2 U(1) scalar charge theory by condensing charge-2 matter, and which becomes the X-cube fracton topological order upon selective loop condensation. Our results are a starting point to investigate quantum critical phenomena at transitions between different fracton states. At a more basic level, the mechanisms we discuss give insight into fracton phases, by allowing us to understand the degrees of freedom in terms of another proximate phase.

In this work, we focused on the scalar charge theory, but there are other rank-2 U(1) gauge theories\cite{81} with different forms of Gauss’ law, e.g. the vector charge theory and traceless scalar charge theories. It will be interesting to investigate what types of fracton topological order can arise from more general higher-rank gauge theories, via Higgs mechanisms and partial confinement transitions. In addition, it may be fruitful to explore whether new types of fracton topological order can arise as gauge theories with different forms of Gauss’ law on different lattice sites, along the lines of our presentation of the checkerboard fracton model as a rank-2 gauge theory.

Another open question is whether any “type II” fracton phases like that in Haah’s code, where all non-trivial excitations are fractons created at the corners of fractal operators, are related to U(1) symmetric-tensor gauge theory in a manner similar to the examples discussed here. In a certain sense, it is obvious that Haah’s code can be viewed as a gauge theory, where one type of term in the Hamiltonian (either the $X$ or $Z$ term) is viewed as implementing a Gauss’ law constraint energetically. The question then becomes whether this Gauss’ law can emerge from simpler or more familiar theories by some sequence of Higgs and/or partial confinement transitions, which could shed light on type II fracton topological order beyond the realm of exactly solvable models.
Appendix

4.8 Ground state degeneracy and logical operators of the $\mathbb{Z}_2$ scalar charge theory

In this section, we calculate the ground state degeneracy in the deconfined phase of the rank-2 $\mathbb{Z}_2$ gauge theory on the cubic lattice. We also discuss the logical operators and argue that the degenerate ground states are locally indistinguishable, i.e. the ground state degeneracy is topological in nature. We work at the exactly solvable point and use the description in terms of a bosonic model (i.e. one with a tensor product Hilbert space), whose Hamiltonian is given by Eq. (4.14) with $U = J = 0$.

In a $L \times L \times L$ system, there are $L^3$ unit cells with four distinct sites within each unit cell. Three sites lie on the plaquettes, with one qubit on each plaquette. The other site lies on the vertex and has three qubits. In total, the number of qubits is $6L^3$. We always take $L$ to be even.

We count the number of independent stabilizers. There is one octahedron term $G_r$ per vertex, thus we have $L^3$ of these terms in total. However, these terms are not all independent and satisfy some constraints. We separate the vertices into eight groups, according to the parity $p_\mu = r_\mu \mod 2$ of the coordinates ($\mu = x, y, z$), labeling the groups by $(p_x, p_y, p_z)$. The product $G_{(p_x, p_y, p_z)} = \prod_{r \in (p_x, p_y, p_z)} G_r$ is a product of $X_{xy}$ over $xy$ planes with $z \mod 2 = p_z$, $X_{yz}$ over $yz$ planes with $x \mod 2 = p_x$, and $X_{zx}$ over $zx$ planes with $y \mod 2 = p_y$. We can think of the operators $G_{(p_x, p_y, p_z)}$ as lying at the vertices of a cube, and it is easy to check that the product of these operators over any face of the cube is unity. This gives four independent constraints, and the number of independent vertex terms is $L^3 - 4$. 

The magnetic field terms are more complicated, and to count them we resort to a numerical method employed previously in Ref. [55], which we also use to check the counting of $X$ stabilizers above. A product of $X$ or $Z$ Pauli operators is viewed as an element of the $\mathbb{F}_2$ vector space $V_{X(Z)} \cong (\mathbb{F}_2)^{6L^3}$, with vector addition corresponding to operator multiplication. The set of all $X$ ($Z$) stabilizers is a subspace $S_{X(Z)} \subset V_{X(Z)}$, and the number of independent stabilizers is the dimension of this subspace. Any $X$ stabilizer is a product of the $L^3$ operators $G_r$; in linear algebra language, we can say that $G_r$ gives a spanning set for $S_X$. This spanning set can be represented as a $L^3 \times 6L^3$ matrix, and its rank, which can be determined via row reduction, gives the dimension of $S_X$. The same method can be applied to $Z$ stabilizers, where $F_{\mu\nu}$ constitute a generating set. We used this method for even $L = 2, \ldots, 12$ to confirm that the number of independent $X$-stabilizers is $L^3 - 4$, and to determine that the number of independent $Z$-stabilizers is $5L^3 - 8$. This implies that the ground state degeneracy is $\log_2(GSD) = 6L^3 - (L^3 - 4) - (5L^3 - 8) = 12$, which is equal to the degeneracy of four copies of the toric code on a 3-torus. To check that the degeneracy is topological in nature, using the same numerical method as above, we return to the space $S_Z$ of $Z$-stabilizers described by its generating set. For each type of charge $\tau_{z}^i$, we add to the generating set three large string operators that transport the charge around the three cycles of the 3-torus. These string operators are products of $Z_{\mu\mu}$, and we choose them to run along straight lines, with their transverse position arbitrary. Upon adding these operators to the generating set, we now find $5L^3 + 4$ independent $Z$-operators, including both $Z$-stabilizers and string logical operators. Adding the string operators thus fully resolves the space of degenerate ground states. Moreover, because the transverse position of the string operators is arbitrary, they can be chosen to avoid the position of any local operator of interest, and it follows that the ground states are locally indistinguishable, as expected for four copies of the $d = 3$ toric code.

4.9 $\mathbb{Z}_2$ scalar charge theory in two dimensions

In this section, we construct the rank-2 $\mathbb{Z}_2$ scalar charge theory on the square lattice in two dimensions. The $\mathbb{Z}_2$ variables are defined in a similar way as in $d = 3$: each site has two diagonal
components while the off-diagonal components are defined on the plaquettes. The Hamiltonian is

\[ H_{Z_2}^{2d} = -K \sum_{r, \mu, \nu} F_{\mu\nu}^{2d} - u \sum_{r} G_{r}^{2d} \]  

(4.44)

where \( F_{\mu\nu} \) is a product of 4 \( Z_{\mu\nu} \) operators around each link and \( G_{r} \) is a product of 8 \( X_{\mu\nu} \) operators around a site, as shown in Fig. 4.11.

![Diagram of plaquette terms and gauge invariant terms](image)

Figure 4.11: (a) Plaquette terms generated by the Gauss law. (b) Gauge invariant terms.

We now show that this model is equivalent to three copies of the \( d = 2 \) toric code. Numerically, using the method we reviewed in Appendix 4.8, we obtained the ground state degeneracy of this model on a \( L \times L \) 2-torus, with even \( L = 2, \ldots , 20 \), finding \( \log_2(\text{GSD}) = 6 \). This is the expected result for three copies of the \( d = 2 \) toric code.

<table>
<thead>
<tr>
<th>Electric charges</th>
<th>Magnetic fluxes with ( \theta = \pi ) statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_1 )</td>
<td>( m_1, m_3 )</td>
</tr>
<tr>
<td>( \tau_2 )</td>
<td>( m_1, m_2 )</td>
</tr>
<tr>
<td>( \tau_3 )</td>
<td>( m_3 )</td>
</tr>
</tbody>
</table>

Table 4.2: In this table, for each type of electric charges as shown in Fig. 4.12(a), we list the magnetic fluxes \( m_{1,2,3} \) that acquire a statistical phase \( \theta = \pi \) when braided around the electric charge. Otherwise, the statistical phase is \( \theta = 0 \).

Point-like charge excitations are created at the sites by acting with \( Z_{\mu\nu} \). Acting with \( Z_{\mu\nu}(r) \), we create two charge excitations at the two neighbors of \( r \) in the \( \pm \mu \) direction. Charges can thus hop two lattice spacings in the \( x \) and \( y \) directions (Fig. 4.12a). Naively, it might seem that the four charges at corners of a plaquette are distinct types of excitations. But since these four charges can be created together by acting with \( Z_{xy} \), there are actually three different types of charges.
Acting with $X_{\mu\nu}$ creates magnetic flux excitations living on the links, denoted as $m_{1,2,3}$ (Fig. 4.12(b)). They are also mobile excitations. For example, $m_1$ can be transported along the $y$ direction by acting with a string operator that is a product of $X_{xx}$ operators along the $y$ direction. The same excitation can be transported along the $x$ direction by applying a “thick” string operator, as shown in Fig. 4.12(b). $m_1$ can hop one lattice spacing in the $y$-direction, and two lattice spacings in the $x$-direction. The other flux excitations have similar mobility. Similar to the charge excitations, four flux excitations on the edges of a plaquette are created together by acting with $X_{xy}$, which means that there are three rather than four distinct types of fluxes.

Next, we can check the braiding between magnetic fluxes and electric charges by simply considering the commutation relation between the corresponding string operators at a crossing point. The result is listed in Table 4.2. It is straightforward to verify that the flux type is fully resolved by braiding with the three different electric charges.

All the analysis above, including the ground state degeneracy, logical operators and properties of the excitations, shows that the rank-2 $\mathbb{Z}_2$ scalar charge theory is three copies of $d = 2$ toric code, which is of course a conventional topological order without any fracton or sub-dimensional excitations. This is consistent with our prediction from Higgsing the conservation law in Sec. 4.3.2. In three dimensions, we can gap the diagonal components of the tensor fields to get a hollow tensor gauge theory describing fracton topological order. However, in two dimensions, the corresponding
hollow gauge theory gives a classical state, because it lacks a gauge-invariant flux term.
Chapter 5

Higher Rank Deconfined Quantum Criticality at the Lifshitz Transition
and the Exciton Bose Condensate

5.1 Introduction

Phase transitions beyond the Landau-Ginzburg paradigm have attracted much attention due to the exotic phenomena they exhibit at critical points. The most well-known example is the quantum critical point between a Néel ordered antiferromagnet and a valence bond solid (VBS) in two spatial dimensions. Since these phases break different symmetries, Landau-Ginzburg theory predicts a first order phase transition between them. In contrast, modern studies have demonstrated a mechanism which allows for a continuous transition between these phases. [96, 92, 94, 62, 48, 89, 58, 52, 3, 90, 34, 83, 41, 63, 5, 65, 112] This critical point is described by a non-compact CP$^1$ theory in terms of emergent fractionalized excitations coupled with a gauge field. The deconfined excitations carry non-trivial quantum numbers and their distinct behaviors give rise to different symmetry breaking phases. For example, condensing spinons (spin-1/2 quasiparticles) results in Néel order. On the other hand, VBS order arises when spinons are confined by the proliferation of monopoles, which carry quantum numbers of lattice symmetries. Right at the critical point, spinons are gapless and coupled to a conventional $U(1)$ gauge field.

A similar deconfined quantum critical point can also occur between two VBS phases breaking different lattice symmetries. In particular, a system of spin-1/2s on the bilayer honeycomb lattice has been shown to support a second order phase transition between two VBS phases, with an emergent deconfined gauge theory arising at the critical point.[111] In contrast to the Néel-VBS
transition, this critical point has dynamical exponent $z = 2$, featuring low energy modes with quadratic dispersion, $\omega \sim k^2$. This transition, called Lifshitz transition, can be described by a compact $(2 + 1)$-dimensional $U(1)$ gauge theory, with low-energy Hamiltonian given by\[111]:

$$H = \kappa E^i E_i + K (\epsilon^{ij} \partial_i E_j)^2 + \frac{1}{2} (\epsilon^{ij} \partial_i A_j)^2$$  \hspace{1cm} (5.1)$$

where $A^i$ is the emergent gauge field and $E^i$ is its corresponding electric field. The deconfined quantum critical point occurs at $\kappa = 0$. (More precisely, this is a fixed line parametrized by $K$, which we will regard as fixed.) Here and below, all indices refer to spatial coordinates, and we sum over all the repeated indices in every equation. The gauge field also couples to deconfined spinon degrees of freedom, which are gapped at the critical point, unlike the Néel-VBS transition. Using a standard particle-vortex duality mapping (reviewed in Appendix 5.2), this Hamiltonian can also be conveniently written in terms of a compact scalar field $\phi$ as:

$$H = \kappa (\partial_i \phi)^2 + K (\partial_i \partial_j \phi)^2 + \frac{1}{2} n^2 + \ldots$$  \hspace{1cm} (5.2)$$

where $n$ is the canonical conjugate to $\phi$. The ellipsis includes interaction terms of $\phi$, such as a $\gamma \cos \phi$ term dual to the instantons of the gauge field $A^i$. At the critical point, $\kappa = 0$, the instantons are irrelevant (for a certain range of $K$ of interest), and we obtain a deconfined gauge theory with a quadratic photon. Away from the critical point, instantons proliferate and gap the theory resulting in confined phases.\[111]\]

The existence of deconfined spinons at the VBS-VBS$'$ critical point has been well-established. In the present work, however, we demonstrate that this critical point also features an even more exotic class of fractionalized excitations which have gone unnoticed in previous literature. In Sec. 5.3, we will study in detail that this critical theory is related by a duality transformation to a compact rank-two tensor gauge theory, with Hamiltonian:

$$H_{\kappa=0} = KE^{ij} E_{ij} + \frac{1}{2} (\epsilon^{ik} \epsilon^{jl} \partial_i \partial_j A_{kl})^2$$  \hspace{1cm} (5.3)$$

for symmetric tensor gauge field $A_{ij}$ and its corresponding electric tensor $E_{ij}$. The first terms in Eq. (5.1) and Eq. (5.2) map onto terms of creation and annihilation of magnetic fluxes in the
tensor gauge theory. Such tensor gauge theories, as we introduced in Sec. 1.3.2, have been studied in the context of fracton phases, where it has been found that the gauge charges subject severe restrictions on their motion. Specifically, the tensor gauge theory in Eq. (5.3) studied in this chapter has vector-valued charges via a generalized Gauss’s law:

\[ \partial_i E^{ij} = \rho^j \]  

(5.4)

In addition to charge conservation, these vector particles obey an extra conservation law which forces them to move only in the direction of their charge vector, resulting in one-dimensional behavior. We will show that the critical Hamiltonian of the VBS-VBS′ transition features such one-dimensional charge excitations. We note that a similar relation with tensor gauge theory has previously been noticed in the context of multicritical Rokhsar-Kivelson (RK) points of certain quantum dimer models[70, 22, 121], though without noting the existence of subdimensional particles.

Interestingly, the Hamiltonian of Eq. (5.2) can also describe a completely different physical situation, if the \( \cos \phi \) term is suppressed by a global \( U(1) \) symmetry. In this free theory, tuning \( \kappa \) across zero realizes the transition between a conventional superfluid and a finite-momentum Bose condensate of bosons \( e^{i\phi} \). This transition will be studied in Sec. 5.4. In this case, the system is invariant under translations of \( \phi \), and the dual rank-1 gauge theory of Eq. (5.1) is non-compact. In this situation, the phases on the two sides of the transition are no longer gapped, but gapless with a linear mode dual to rank-1 \( U(1) \) gauge theory by standard boson-vortex duality. This represents an entirely new type of deconfined quantum criticality, in which a tensor gauge structure emerges at a critical point between two conventional gauge theories. We will show that this critical point has a natural physical interpretation as a condensate of excitons formed by boson-hole pairs, even though isolated bosons remain uncondensed as studied in Ref. [24]. The one-dimensional particles can then be understood as the vortices of the exciton condensate. We refer to such a system as an “exciton Bose condensate” (EBC), in analogy with the closely-related exciton Bose liquid (EBL) phase. [71, 88, 61, 125, 123, 105]
In Sec. 5.5, we show that at the critical point, the one-dimensional particles carry a logarithmic energy cost, much like conventional vortices in a superfluid, which suggests that the exciton condensate can survive at nonzero temperatures. Like the normal BKT transition \[6, 7, 45\], where vortices proliferate at a critical temperature and destroy the low-energy quasi-long-range order, we argue for the existence of another phase transition at which the one-dimensional vortices undergo BKT-like unbinding and destroy the exciton condensate at the critical temperature, resulting in a completely disordered phase. We present the proposed phase diagram in Fig. 5.4, with a generic parameter regime featuring a finite-temperature EBC phase, which shrinks to the quantum critical point at zero temperature. We also establish some of the basic properties of this new finite-temperature region.

In Sec. 5.6, we provide a concrete lattice boson model which exhibits the physics described above. Additionally, in Sec. 5.7, we show that small modifications of the critical theory result in the stable quantum EBL phase studied in Ref. \[71\], which is also known to exhibit subdimensional particle excitations.\[71, 61, 125\]

### 5.2 Conventional Boson-Vortex Duality

We here review the standard boson-vortex duality in \((2 + 1)\) dimensions, which relates a superfluid of neutral bosons to a non-compact \(U(1)\) gauge theory describing an insulator of charged particles. These descriptions provide useful complementary ways of understanding not only the superfluid phase, but also the transition to a Mott insulator. In the superfluid, the primary dynamical variable is the phase field \(\phi\) of the microscopic boson field, \(i.e. \langle b \rangle = b_0 e^{i\phi}\). This phase field represents the gapless Goldstone mode of the theory, while all other excitations are gapped. The low-energy Hamiltonian describing the dynamics of this field takes the schematic form:

\[ H = K (\partial_i \phi)^2 + \frac{1}{2} n^2 \]  

(5.5)

where \(n\) is the boson number canonical conjugate to the angle \(\phi\). The system also supports topological excitations where \(\phi\) winds by \(2\pi\) around a point, corresponding to vortices of the superfluid.
Such vortex excitations will interact with each other through a logarithmic potential.

In parallel, let us consider the properties of a non-compact U(1) gauge theory coupled to gapped charges, which mirror those the superfluid. This theory features a gapless mode (the photon), and gapped charges interacting through a logarithmic potential. The Hamiltonian describing the gapless gauge sector takes the standard form:

\[ H = KE^iE_i + \frac{1}{2}B^2 \]  

where \( E_i \) is the two-dimensional electric vector field and \( B = \epsilon^{ij}\partial_iA_j \) is the one-component magnetic flux through the system. This Hamiltonian gives the gapless gauge mode a linear dispersion, matching with the properties of the Goldstone mode of the superfluid. The gapped charges act as sources for the electric field through Gauss’s law:

\[ \partial_iE^i = \rho \]

In two dimensions, this equation tells us that a point charge has an electric field scaling as \( 1/r \), leading to a logarithmic interaction potential between charges.

The above discussion indicates that the two theories, the superfluid and the U(1) gauge theory, have the same excitation spectrum. We can also directly map the two theories onto each other and match all physical observables. To begin, focus on the low-energy sector, where there are no charges, so that the electric field obeys the source-free Gauss’s law, \( \partial_iE^i = 0 \). The general solution to this equation takes the form:

\[ E^i = \epsilon^{ij}\partial_j\phi \]

for scalar field \( \phi \). The fields \( E_i \) and \( A_i \) obey canonical commutation relations:

\[ [E_i(x),A_j(y)] = -i\hbar\delta_{ij}\delta(x-y) \]

It then follows that \( \phi \) is canonically conjugate to \( B = \epsilon^{ij}\partial_iA_j \), which we now relabel as \( n = B \). Plugging these expressions into the gauge theory Hamiltonian in Equation 5.6, we obtain precisely...
the superfluid Hamiltonian of Equation 5.5. We can also directly derive the correspondence between
gauge charges and superfluid vortices. Consider the total charge enclosed within some curve \( C \):

\[
Q = \int d^2 x \, \partial_i E^i = \oint_C n^i E_i = -\oint_C ds^i \partial_i \phi = -\Delta \phi \tag{5.10}
\]

where \( \Delta \phi \) is the change in \( \phi \) going around the curve \( C \). This indicates that a unit of gauge charge
is equivalent to a winding of \( \phi \), which is the definition of a vortex of the superfluid.

### 5.3 Dual Tensor Gauge Theory of the Critical Point

We begin by showing that the critical theory of the VBS-VBS\(^\prime \) transition is a tensor gauge
theory. In order to obtain the desired mapping, it is simplest to start on the gauge theory side
of the duality and map it onto the scalar field Hamiltonian of Eq. (5.2) at \( \kappa = 0 \). (It is also
possible to derive the duality in the opposite direction, starting with the critical boson theory and
mapping onto the tensor gauge theory, as we show in Appendix 5.9.) We first review the basic
properties of the appropriate tensor gauge theory, known as the “vector charge theory” in the
fracton literature.[81, 79] The theory is formulated in terms of a rank-2 symmetric tensor \( U(1) \)
gauge field \( A_{ij} \), along with its canonical conjugate variable, which we call the electric tensor \( E_{ij} \).
The theory is defined in terms of its Gauss’s law:

\[
\partial_i E^{ij} = \rho^j \tag{5.11}
\]

which is sourced by vector-valued charges \( \rho^j \), assumed to be gapped. Notably, these charges obey
two separate conservation laws:

\[
Q^i = \int d^2 x \, \rho^i = \text{const.} \quad \mathbb{L} = \int d^2 x \, (\epsilon^{ijx} x_i \rho_j) = \text{const.} \tag{5.12}
\]

representing conservation of charge, \( Q^i \), and also the angular charge moment, \( \mathbb{L} \). (Note that \( \mathbb{L} \) is
analogous to, but distinct from, kinetic angular momentum.) This extra conservation law forces
the fundamental charges to move only in the direction of their charge vector, giving rise to one-
dimensional behavior. The theory also admits stable bound states with \( Q^i = 0 \), but \( \mathbb{L} \neq 0 \). These
bound states, which we refer to as $L$-particles, are fully mobile and correspond to ordinary spinons in the description of rank-1 gauge theory in Eq. (5.1).

The Gauss’s law of the theory implies that the low-energy charge-neutral sector (obeying $\partial_i E^{ij} = 0$) is invariant under the gauge transformation:

$$A_{ij} \to A_{ij} + \partial_i \alpha_j + \partial_j \alpha_i$$

(5.13)

where $\alpha_i$ is an arbitrary function of spatial coordinates. Gauge invariance then dictates the form of the low-energy Hamiltonian at critical point:

$$H = KE_{ij} E^{ij} + \frac{1}{2} B^2$$

(5.14)

where the magnetic field is a scalar quantity given by $B = \epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j A_{k\ell}$. Since the magnetic field contains two derivatives, the equations of motion yield a gapless gauge mode with quadratic dispersion, $\omega \sim k^2$. In order to find the dual description of this gauge theory, we begin in the charge-neutral sector, in which the source-free Gauss’s law, $\partial_i E^{ij} = 0$, has the general solution:

$$E_{ij} = \epsilon^{ik} \epsilon^{j\ell} \partial_k \partial_\ell \phi$$

(5.15)

for scalar field $\phi$. Since $E_{ij}$ is canonically conjugate to $A_{ij}$, it follows that $\phi$ is conjugate to $B = \epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j A_{k\ell}$, which we now relabel as $n$, for reasons which will become clear in the next section. Making the appropriate replacements in Eq. (5.14), we obtain the dual Hamiltonian:

$$H = K(\partial_i \partial_j \phi)^2 + \frac{1}{2} n^2$$

(5.16)

which is precisely the critical point of the Hamiltonian in Eq. (5.2). The relevant $\kappa(\partial_i \phi)^2$ perturbation cannot be written as a simple local term in terms of the tensor fields, but rather corresponds to a non-local flux created by an instanton event of the gauge field, as we will see shortly.

First, however, we determine the correspondence between the charges of the tensor gauge theory and gapped topological defects of the critical scalar field theory. To do this, we consider the total charge enclosed in a planar region $P$ with boundary $C$:

$$Q^i = \oint_P d^2 x \, p^i = \oint_C d\eta_i E^{ij} = \oint_C ds^k \partial_k (\epsilon^{ji} \partial_i \phi) = \epsilon^{ji} \Delta(\partial_i \phi)$$

(5.17)
Figure 5.1: All excitations and operators in the effective theory of the VBS-VBS$'$ transition can be mapped directly onto those of a tensor gauge theory with one-dimensional vector charges.

<table>
<thead>
<tr>
<th>Vortex of $\partial_i \phi$</th>
<th>$Q^i = \epsilon^{ij} \Delta(\partial_i \phi)$</th>
<th>Vector Charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vortex of $\phi$</td>
<td>$L = -\Delta \phi$</td>
<td>$L$-Particle</td>
</tr>
<tr>
<td>$\phi$</td>
<td></td>
<td>Flux Insertion</td>
</tr>
<tr>
<td>$\partial_i \phi$</td>
<td></td>
<td>Flux Dipole Insertion</td>
</tr>
<tr>
<td>$\partial_i \partial_j \phi$</td>
<td>$E^{ij}$</td>
<td>Electric Field</td>
</tr>
<tr>
<td>$n$</td>
<td>$B$</td>
<td>Magnetic Field</td>
</tr>
</tbody>
</table>

where the line element $ds^k$ is related to the normal vector via $ds^k = \epsilon^{ik} dn_i$, and $\Delta(\partial_i \phi)$ represents the change in $\partial_i \phi$ upon going around the closed curve $C$. Accordingly, the fundamental charges of the gauge theory correspond to singular points around which $\partial_i \phi$ has nontrivial winding. This type of singularity is much less familiar than a conventional winding of a compact scalar field $\phi$. Nevertheless, on a lattice, compactness of $\phi$ automatically implies compactness of $\partial_i \phi$. If $\phi$ is an angular variable, defined modulo $2\pi$, then $\partial_i \phi$ is only defined modulo $2\pi/a$, where $a$ is the lattice spacing. Whether or not such defects can still be sensibly discussed in a true continuum is unclear. But for a system with an underlying lattice, such as the bilayer honeycomb system under consideration[111], the compact field $\phi$ naturally hosts this type of defect. We will explicitly construct a configuration of $\phi$ which exhibits such a singularity (see Eq. (5.40)). The mobility of those one-dimensional gauge charges pick particular directions on the lattice according to the lattice derivative $\partial_i$. At the critical point, these defects have a logarithmic energy cost, analogous to conventional vortices of a superfluid, as we will see later.

Of course, we expect that our system also contains the usual windings of $\phi$, vortices which should be normal mobile particles. These conventional vortices map onto the $L$-particles of the gauge theory, with $Q^i = 0$ but $L \neq 0$, which have no constraints on their motion. To see this,
consider the total angular charge moment contained in the region \( P \) bounded by curve \( C \):

\[
\mathcal{L} = \int_P d^2x (\epsilon^{ik} x_j \partial_k) = \int_P d^2x (\epsilon^{ik} x_j \partial^i E_{ik}) = \oint_C ds^m x^j \partial_m \partial_j \phi = \oint_C ds^i \partial_i ((x^j \partial_j \phi) - \phi) \quad (5.18)
\]

When there are no net one-dimensional particles in the region \( P \), such that \( \partial_i \phi \) is single-valued on the boundary, \( \mathcal{L} \) can be written in terms of the winding of \( \phi \) as:

\[
\mathcal{L} = - \oint_C ds^i \partial_i \phi = -\Delta \phi \quad (5.19)
\]

Therefore, the \( \mathcal{L} \)-particles of the gauge theory correspond to the ordinary vortices of \( \phi \), which have been studied in previous treatments of the VBS-VBS' transition. These particles exist as gapped deconfined excitations with a \( 1/r^2 \) interaction at the \( \kappa = 0 \) critical point, but become confined on either side of the transition.

Finally, we discuss the role of instantons, arising from the compactness of the gauge field. For a noncompact theory, the definition of the magnetic field, \( B = \epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j A_{k\ell} \), would lead to two independent conserved quantities:

\[
\Phi = \int d^2x B = \text{const.} \quad \Pi^i = \int d^2x (Bx^i) = \text{const.} \quad (5.20)
\]

corresponding to the conservation of flux, and also the “dipole moment” of flux. In other words, magnetic flux would behave like a fracton. Since \( B \) is the canonical conjugate to \( \phi \), these conservation laws would map onto the following symmetries in the dual language:

\[
\phi \rightarrow \phi + \alpha, \quad \partial_i \phi \rightarrow \partial_i \phi + \lambda_i \quad (5.21)
\]

for constants \( \alpha \) and \( \lambda_i \). For a compact theory, however, these conservation laws and their associated symmetries will be broken. The gauge field \( A_{ij} \) is only defined up to some compactification radius, which we take to be \( 2\pi \). The path integral will then allow sudden changes in \( \Phi \) by \( 2\pi \), just as in a normal compact gauge theory. In the dual language, flux insertion corresponds to a symmetry-breaking perturbation to the Hamiltonian:

\[
H_\Phi = \gamma \cos \phi, \quad \text{(insertion of flux)} \quad (5.22)
\]
This term is irrelevant at the critical point[111], but gaps the gauge field on either side of the transition. In addition to this conventional flux slip event, the path integral will also admit an additional type of instanton, in which $\Phi$ is left unchanged but $\Pi^i$ changes by $2\pi a$ in some lattice direction. In other words, a dipole moment of flux is added to the system. Such a dipolar flux insertion corresponds to the following perturbation in the dual language:

$$H_{\Pi} = \kappa (\partial_i \phi)^2,$$

(5.23)

which is the only relevant operator at the critical point. All other perturbations can be shown to either be irrelevant or ruled out by symmetries of the bilayer system, leading to a generic second order phase transition.[111] We therefore see that, in the tensor gauge theory language, the transition away from the critical point is driven by the proliferation of “dipolar” instantons. On the two sides of the transition, single fluxes are mobile due to the background of the dipolar flux, then the “monopolar” instantons proliferate and lead to conventional confined gapped phases.

This completes the duality mapping between the bosonic critical theory and a tensor gauge theory, the details of which are summarized in Fig. 5.1.

5.4 Deconfined Quantum Criticality between Condensates

The physics of the VBS-VBS′ transition is well-captured by a Hamiltonian in terms of a scalar field $\phi$, as:

$$H = \kappa (\partial_i \phi)^2 + K (\partial_i \partial_j \phi)^2 + \frac{1}{2} n^2$$

(5.24)

with the critical point at $\kappa = 0$. Importantly, the $\cos \phi$ perturbation is irrelevant at the critical point, such that the critical theory can be written entirely in terms of derivatives of $\phi$. This motivates us to use this Hamiltonian to describe a completely different physical situation, if the monopolar instantons are forbidden by symmetry $\phi \rightarrow \phi + \alpha$. Then we can interpret Eq. (5.24) as a system of bosons $b_i^\dagger = e^{i\phi}$ with total number conservation. The variable $n$ then corresponds to the boson number operator, $n = b_i^\dagger b$ and the boson current is proportional to $\partial_i \phi$. The $K(\partial_i \partial_j \phi)^2$ operator of the critical theory describes two-boson hopping process (to be discussed in more detail.
later in Fig. 5.3), while the relevant $\kappa(\partial_i \phi)^2$ perturbation corresponds to single-boson hopping operators.

Unlike in the VBS-VBS$'$ transition, the $\gamma \cos \phi$ term is not allowed with the underlying $U(1)$ symmetry. But the critical Hamiltonian still possesses a relevant perturbation by the $\kappa(\partial_i \phi)^2$ term. (We will verify later that this remains the only relevant perturbation for a system with an underlying honeycomb lattice.) Equivalently, the dual vector gauge fields in the formulation:

$$H = \kappa E^i E_i + K(\epsilon^{ij} \partial_i E_j)^2 + \frac{1}{2}(\epsilon^{ij} \partial_i A_j)^2$$

(5.25)

should be regarded as noncompact. The end result is that the two sides of the phase transition are no longer gapped phases. Rather, both sides are dual to noncompact vector gauge theories, while the critical point remains a deconfined tensor gauge theory. This provides an example of an entirely new type of deconfined quantum criticality, in which a critical tensor gauge theory separates two stable vector gauge theories. The differences between the two types of deconfined quantum critical points are sketched in Fig. 5.2.

![Figure 5.2](image)

Figure 5.2: In the VBS-VBS$'$ transition, the critical tensor gauge theory separates two gapped confined phases. In contrast, the superfluid to finite-momentum condensate transition features a critical tensor gauge theory separating two stable noncompact vector gauge theories.

The phases on the two sides of this transition correspond to different types of Bose condensates. When the coefficient $\kappa$ is positive, we obtain a conventional superfluid phase. On the other hand, when $\kappa < 0$, it becomes energetically favorable for $\partial_i \phi$ to pick up an expectation value, $\langle \partial_i \phi \rangle = \lambda_i$, such that the field $\phi$ becomes “tilted,” behaving as $\langle \phi \rangle = \vec{\lambda} \cdot \vec{x}$. In terms of the microscopic boson field $b$, we then have $\langle b \rangle = b_0 \exp(i\vec{\lambda} \cdot \vec{x})$, corresponding to a condensate of the bosons.
at finite momentum.

In order to verify that the Hamiltonian of Eq. (5.24) describes a direct second order phase transition, we must check that the $\kappa(\partial_i\phi)^2$ operator is the only relevant perturbation at the $\kappa = 0$ critical point. The argument proceeds largely along the same lines as the analysis of the VBS-VBS' transition, but with the added advantage of a global $U(1)$ symmetry. This symmetry rules out all terms which involve bare $\phi$ operator (i.e. without derivatives), which significantly decreases the number of terms we need to consider. First, we focus on the rotationally-invariant terms, which are insensitive to the underlying lattice of the system. One worrisome operator of this sort is a quartic term, $u(\partial_i\phi)^4$, which is marginal at the power-counting level. This term was analyzed in the context of the VBS-VBS' transition, where it was shown to be marginally irrelevant for $u > 0$.[111, 22] For a system starting with positive $u$, this perturbation will be unimportant at low energies. We can also rule out all terms having odd powers of $\phi$ by imposing $\phi \rightarrow -\phi$ symmetry on the system, corresponding to particle-hole symmetry of the underlying bosons, which naturally arises at half-integer filling factors.[71] All other rotationally invariant terms are irrelevant by power-counting.

However, we must also worry about non-rotationally-invariant terms arising from the underlying lattice of the theory. (Recall that we focus on lattice systems, instead of a true continuum, in order to sensibly discuss windings of $\partial_i\phi$.) On the square lattice, there are relevant anisotropy terms which trigger the proliferation of instantons at the critical point and drive the transition first order.[111] If we consider a honeycomb lattice, however, then to fourth order in derivatives, we only need to consider the rotationally invariant terms, which we have already discussed.[111, 22]\(^1\)

Putting it all together, we see that the critical point on a honeycomb lattice has only a single relevant direction, namely the $\kappa(\partial_i\phi)^2$ term. Thus, up to marginally irrelevant corrections, the Hamiltonian of Eq. (5.24) describes a direct second order quantum phase transition between a superfluid and a finite momentum condensate.

Similar to the VBS-VBS' case, this transition can also be understood in the language of

---

\(^1\) This logic fails for dimer models with the symmetry properties considered in References [22], but these issues do not arise in the present context.
tensor gauge theory. At the \( \kappa = 0 \) critical point, the system is described by an emergent tensor gauge structure, with Hamiltonian given by:

\[
H_{\kappa=0} = KE_{ij}E_{ij} + \frac{1}{2}(\epsilon^{ij}_k\partial_i\partial_j A_{kt})^2
\]  

(5.26)

in which the one-dimensional vector charges, defined by \( \partial_i E_{ij} = \rho^i \), correspond to exotic vortices around which \( \partial_i \phi \) has nontrivial winding. The conventional mobile vortices (windings of \( \phi \)) correspond to the \( \mathbb{L} \)-particles discussed earlier, with \( Q^i = 0 \) but \( \mathbb{L} \neq 0 \). At the critical point, both types of vortices exist as well-defined excitations of the system, with logarithmic interaction between the one-dimensional vortices. Away from \( \kappa = 0 \), the \( \kappa(\partial_i \phi)^2 \) term, corresponding to proliferation of the dipolar instantons, will result in a linearly confining potential between the one-dimensional particles, leaving the conventional vortices as gapped logarithimically interacting particles, as expected. (Note that the monopolar instantons correspond to a \( \gamma \cos \phi \) perturbation to the Hamiltonian, which is ruled out by the global \( U(1) \) symmetry of the boson system.)

To study the behavior of the conventional vortices across the transition, and to recover a more familiar formulation of the superfluid phase, it is useful to rewrite the tensor gauge Hamiltonian in terms of the effective gauge field seen by the \( \mathbb{L} \)-particles. Since these particles are bound states of the fundamental vector charges, their effective gauge field takes the form[79]:

\[
A_k = \epsilon^{ij}\partial_j A_{jk}
\]  

(5.27)

The corresponding effective electric field \( E_i \) seen by the \( \mathbb{L} \)-particles satisfies \( E_{ij} = \epsilon_{ik}\partial^k E_j \).[79] In terms of these variables, we can rewrite the low-energy Hamiltonian as:

\[
H = \kappa E^i E_i + K(\epsilon^{ij}_k\partial_i E_j)^2 + \frac{1}{2}(\epsilon^{ij}_k\partial_i A_j)^2
\]  

(5.28)

where \( \kappa E^i E_i \) represents the creation/annihilation of dipolar fluxes of the tensor gauge theory. This is precisely the vector gauge formulation of Eq. (5.25), in which the electric field is related to the boson field by \( E^i = \epsilon^{ij}\partial_j \phi \). When combined with \( E^{ij} = \epsilon^{ik}\partial_k E_j \), we recover the expected relationship between the electric tensor and the boson field, \( E^{ij} = \epsilon^{ik}\epsilon^{jt}\partial_k \partial_t \phi \). When \( \kappa \neq 0 \), the \( K \) term is unimportant, and we recover the conventional gauge dual of a superfluid at \( \kappa > 0 \), with
photons of the gauge theory mapping onto the Goldstone modes. In the superfluid phase, separation of vortices have a logarithmic energy cost, as expected. Right at the critical point, however, the $\kappa E^2$ term of the Hamiltonian vanishes, leading to a finite energy for conventional vortices at large distance. When $\kappa < 0$, the logarithmic energy cost is restored. On this side of the transition, it is favorable for the vector electric field to pick up an expectation value, $\langle E^i \rangle = \epsilon^{ij} \langle \partial_j \phi \rangle = \epsilon^{ij} \lambda_j$, corresponding to a finite-momentum condensate of the microscopic bosons. The behavior of vortices in the finite-momentum condensate is then equivalent to logarithmically interacting charges moving in a background electric field.

5.5 Properties of the Critical Point

In the previous sections, we identified a quantum critical point described by a tensor gauge theory featuring subdimensional particles. In this section, we characterize some of the properties of this critical point, including the consequences of the critical tensor gauge structure for the surrounding parameter space. Most notably, we find a finite-temperature phase of matter which shrinks to the quantum critical point at zero temperature. This phase is distinct from both the infinite-temperature disordered phase and the zero-temperature ordered phases. For concreteness, we will phrase our discussion for the superfluid transition (though similar logic carries over to the VBS-VBS' transition). In the bosonic system, the new finite-temperature phase represents an exciton Bose condensate (EBC), in which excitons have condensed while single bosons have not. We will see in a later section how a small modification of the quantum critical point can also give rise to the quantum “exciton Bose liquid” (EBL) phase studied in Ref. [71].

5.5.1 Zero-Temperature Properties

5.5.1.1 Exciton Condensation

We now focus on the critical Hamiltonian taking the form:

$$H = K (\partial_i \partial_j \phi)^2 + \frac{1}{2} n^2 \quad (5.29)$$
where $\phi$ is the phase of the microscopic bosons, $b \sim e^{i\phi}$, and $n$ is the boson number, $b^{\dagger}b$. This Hamiltonian looks very similar to that of the superfluid phase, except that the first term features only second derivatives. This leads to a quadratic dispersion of the gapless mode, $\omega \sim k^2$, as opposed to the linearly dispersing Goldstone mode of the superfluid. In order to gain an intuitive understanding of this critical point, it is instructive to consider the microscopic origin of the derivative operators. The first derivative, $\partial_i \phi$, arises from single-boson hopping processes, since $b^{\dagger}(x^i + \epsilon^i)b(x^i) \sim \exp(ie^i\partial_i \phi)$. At the critical point, such first derivative terms are absent from the Hamiltonian, indicating zero hopping matrix elements for single bosons. In this sense, the fundamental bosons behave like fractons at the critical point. When the hopping matrix elements are turned back on, the system flows away from the critical point into a Bose-condensed phase.

Despite the absence of single-particle hopping in the Hamiltonian, the bosons are not completely nondynamical at the critical point. The second derivative operator, $\partial_i \partial_j \phi$, corresponds to two-boson hopping processes. More specifically, second derivatives correspond to processes in which two bosons move in opposite directions by the same amount of distance, thereby conserving center of mass. Importantly, we can also regard such a process as the motion of a particle-hole pair (“exciton”), as depicted in Fig. 5.3. We can therefore understand the critical point as a system in which excitons are the fundamental mobile particles, while single bosons are locked in place. At zero temperature, we then expect that the excitons will form a condensate, while single bosons remain uncondensed, leading us to call the system an exciton Bose condensate.

We can verify these expectations explicitly by checking for off-diagonal long-range order in the correlation functions of both bosons and excitons. Because the time correlations of both bosons and excitons have the same power law behavior even at finite temperature[24], below we focus on the spatial correlations which can distinguish the exciton and boson condensate. For single bosons, the appropriate correlator takes the form:

$$\langle b^{\dagger}(x) b(0) \rangle = \langle e^{i(\phi(x) - \phi(0))} \rangle = e^{-\frac{1}{2}\langle(\phi(x) - \phi(0))^2\rangle} \sim e^{(\phi(x)\phi(0))}$$

(5.30)
Figure 5.3: The $\partial_i \phi$ operators correspond to single-boson hopping processes. Similarly, the $\partial_i \partial_j \phi$ operators correspond to two-boson hopping processes conserving center of mass, which can equivalently be regarded as exciton hopping processes.

The correlation function of the phase field is given by:

$$\langle \phi(x) \phi(0) \rangle \sim \int d^2 k d\omega \frac{e^{i k \cdot x}}{\omega^2 + K k^4} \sim \frac{1}{\sqrt{K}} \int d^2 k \frac{e^{i k \cdot x}}{k^2} \sim -\frac{1}{\sqrt{K}} \log r \quad (5.31)$$

where $r = |x|$ and $k = |k|$. Note that, here and below, inside logarithms, $r$ should be taken in units of the lattice spacing, $a$. We then obtain the boson correlator as:

$$\langle b^\dagger(x) b(0) \rangle \sim \frac{1}{r^\zeta} \quad (5.32)$$

where $\zeta \sim K^{-1/2}$. This indicates that, even at zero temperature, there is no true long-range order (rather only quasi-long-range order) of the fundamental bosons. To find an operator exhibiting long-range order, we must consider the corresponding correlator for excitons:

$$\langle e^{i \partial_i \phi(x)} e^{-i \partial_i \phi(0)} \rangle \sim e^{\langle \partial_i \phi(x) \partial^i \phi(0) \rangle} \sim e^{-\frac{1}{\sqrt{K} \cdot r^2}} \quad (5.33)$$

where we used

$$\langle \partial_i \phi(x) \partial^i \phi(0) \rangle = \int d^2 k d\omega k_i k^i \langle \phi(k) \phi(0) \rangle e^{ik \cdot x} \sim -\frac{1}{\sqrt{K} r^2}. \quad (5.34)$$

The above correlation function approaches a nonzero constant as $r \to \infty$, indicating that, unlike the fundamental bosons, the excitons form a condensate with true off-diagonal long-range order at zero temperature. As such, it is appropriate to regard the quantum critical point as an exciton condensate, separating two conventional Bose-condensed phases.
5.5.1.2 Vortex Solutions

Having established the properties of the condensate, we now investigate the properties of its vortices, which correspond to charges in the dual tensor gauge theory. Since all charged excitations remain gapped at the critical point, charges will typically have much slower velocity than the gapless gauge mode. As such, the dominant interactions between charges will be electrostatic in origin. We here focus on this electrostatic limit, leaving retardation effects to future study. To this end, we first introduce a potential formulation, analogous to $E_i = -\partial_i \varphi$ in conventional electromagnetism. Similar potential formulations for tensor gauge theories have been studied in three dimensions in Ref. [79]. We begin by noting the form of the Faraday’s equation for this tensor gauge theory:

$$\partial_t B + \epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j E_{k\ell} = 0$$ (5.35)

following from the role of $E_{ij}$ as a function of the conjugate momentum to $A_{ij}$. In the static limit ($\partial_t B = 0$), the electric tensor must obey $\epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j E_{k\ell} = 0$. The general solution to this constraint takes the form:

$$E_{ij} = -(\partial_i \xi_j + \partial_j \xi_i)$$ (5.36)

where $\xi^i$ is a potential function representing the potential energy per unit vector charge.[79] We now consider the potential arising from a point particle with vector charge $q^i$, which must satisfy the Gauss’s law:

$$\partial_i E^{ij} = -(\partial^2 \xi^j + \partial^i (\partial_j \xi^i)) = q^i \delta^{(2)}(r)$$ (5.37)

Note that $q^i$ has units of (length)$^{-1}$, so $E^{ij}$ has units of (length)$^{-2}$. It can readily be checked that the following potential provides the appropriate solution:

$$\xi^i = \frac{1}{8\pi} \left( 3(\log r) q^i - \frac{(q \cdot r) r^i}{r^2} \right)$$ (5.38)

leading to a logarithmic interaction energy between the one-dimensional particles. The corresponding electric tensor is given by:

$$E^{ij} = \frac{1}{4\pi} \left( (q \cdot r) \delta^{ij} - 2 \frac{(q \cdot r) r^i r^j}{r^4} - \frac{(q_i r_j + q_j r_i)}{r^2} \right)$$ (5.39)
Using this form, and the relation $E^{ij} = \epsilon^{ik} \epsilon^{j\ell} \partial_k \partial_\ell \phi$, we can then determine the configuration of the $\phi$ field as:

$$\phi = \frac{1}{4\pi} \left( -2(\epsilon^{ij} q_i r_j) \theta + (q \cdot r) \log r \right)$$

(5.40)

which provides an explicit example of a singularity with winding of $\partial_i \phi$ around $r = 0$. We can check that

$$4\pi \partial_i \phi = 2\epsilon_{ij} q_j \theta + q_i (1 + \log r) + \frac{\epsilon_{ij} r_j \left( \epsilon_{k\ell} q_k r_\ell \right)}{r^2}$$

(5.41)

satisfying that the winding of $\partial_i \phi$ is $\epsilon_{ij} q_j$, where $\theta = \text{ArcTan} \frac{r_y}{r_x}$ and $r^2 = r_x^2 + r_y^2$.

We can also consider a situation where two opposite vector charges $q_j$ are created with a displacement $d$ perpendicular to the direction of vectors, i.e. an $\mathbb{L}$-particle with $\mathbb{L} = \epsilon^{jk} q_j d_k$. The potential generated by this charge configuration satisfies

$$\partial_i E^{ij} = -(\partial^2 \xi^j_L + \partial^j (\partial_i \xi^i_L)) = q^j \delta^{(2)}(r) - q^j \delta^{(2)}(r + d\hat{k}) = \epsilon^{jk} \mathbb{L} \partial_k \delta^{(2)}(r)$$

(5.42)

leading to the solution:

$$\xi^i_L = \frac{\mathbb{L}}{4\pi} \frac{\epsilon^{ik} r_k}{r^2}$$

(5.43)

This is the potential at distance $r$ away from the source $\mathbb{L}$-particle whose scale is much smaller than $r$. Notice this is also a vector potential acting on single vector charge. The potential between two $\mathbb{L}$-particles is given by $\partial_i \xi^i_L$. Then, the corresponding electric tensor takes the form:

$$E^{ij}_L = \frac{\mathbb{L}}{2\pi} \left( \frac{\epsilon^{ik} r_k r^j}{r^4} + \frac{\epsilon^{ik} r_k r^i}{r^4} \right)$$

(5.44)

which scales as $1/r^2$ leading to a finite energy cost for creating an isolated $\mathbb{L}$-particle, unlike the logarithmic energy cost for the one-dimensional particles. The configuration of the phase field $\phi$ for an $\mathbb{L}$-particle as source charge is given by:

$$\phi_L = -\frac{\mathbb{L}}{2\pi} \theta$$

(5.45)

which is the expected winding of $\phi$ for a normal vortex of a superfluid.
5.5.2 Finite-Temperature Behavior

5.5.2.1 Phase Diagram

We just found that the one-dimensional particles of the critical tensor gauge theory have a logarithmic interaction energy. By the usual logic of the BKT transition[6, 7, 45], we therefore expect a finite-temperature phase transition at which these particles proliferate. A similar argument applied for a single particle can be established. An isolated one-dimensional particle with fundamental charge \( q^i \) has an energy of order \( Kq^2 \log \ell \), where \( \ell \) is the system size. Similarly, the entropy per particles behaves as \( T \log \ell \) (working in units such that \( k_B = 1 \)). The resulting free energy per particle takes the schematic form:

\[
F = \mathcal{E} - TS \sim (Kq^2 - T) \log \ell
\]

(5.46)

At low temperatures, the energy term dominates and the free energy per particle is positive, indicating that it is unfavorable to form isolated one-dimensional particles, which serve as vortices of the exciton condensate. As such, the exciton condensate remains intact in this low-temperature regime. On the other hand, above a certain critical temperature:

\[
T_{c2} \sim Kq^2
\]

(5.47)

the free energy per particle becomes negative, signaling the creation of a particle is energetically favored. Addition to this single particle argument, a detailed study can be done for the many-body Hamiltonian:

\[
H_{vor} = \xi^i q_i = \frac{K}{8\pi} \sum_{r, r'} \left( 3q_r \cdot q_{r'} \log |r - r'| - \frac{[q_r \cdot (r - r')] [q_{r'} \cdot (r - r')]}{|r - r'|^2} \right) + y \sum_r |q_r|^2
\]

(5.48)

where \( y \) is the fugacity of the vortex of the exciton condensate. This Hamiltonian takes a similar form as that of the normal vortex in the superfluid, but with additional vector structure, also named vector Coulomb gas. A similar finite-temperature phase transition as the dislocation mediated melting transition, corresponding to an unbinding transition of the one-dimensional particles, is
indicated[13]. A group of similar scaling equations can be obtained on the hexagonal lattice[129, 69]:

\[
\frac{dK_R^{-1}}{d\ell} = \frac{9}{2} \pi y^2 \left[ I_0 \left( \frac{K_R}{8\pi} \right) - \frac{1}{2} I_1 \left( \frac{K_R}{8\pi} \right) \right] \\
\frac{dy}{d\ell} = (2 - \frac{3K_R}{8\pi})y + 2\pi y^2 I_0 \left( \frac{K_R}{8\pi} \right)
\]  

(5.49)

where \( I_0 \) and \( I_1 \) are modified Bessel functions, \( K_R = K/T \) and \( T \) is the temperature. We can obtain a fixed point where \( K_R(T_c) = \frac{16\pi}{3} \). According to the exciton correlation function at finite temperature \( \sim r^{-\eta} \) (see Eq. 5.54), we can get, at \( T_{c2} \), \( \eta = \frac{1}{4\pi K_R} = 0.004749 \) which is smaller than the exponent for the KT transition \( \eta_{KT} = 0.25 \), noticing in the conventional KT transition, the \( \eta_{KT} \) is defined for boson correlation function. We can also evaluate the other critical exponent \( \nu \) satisfying \( \xi \sim e^{[T-T_{c2}]^{-\nu}} \) where \( \xi \) is the correlation length. Using the method in Ref. [129], we can get \( \nu = 0.418099 \), which is between the value for the KT transition \( \nu_{KT} = 0.5 \) and the value for the dislocation mediated melting where \( \tilde{\nu} = 0.36963477 \).[129, 69, 13]

Above the critical temperature, one-dimensional vortices proliferate, the exciton condensate is destroyed, and the system enters the completely disordered normal phase. \textit{A priori}, this argument could be affected by \( L \)-particles, which cost finite energy and proliferate at any non-zero temperature. In Appendix 5.10, we verify that the \( L \)-particles do not significantly affect the transition properties of the one-dimensional particles. We note that the one-dimensional particles will acquire some limited mobility in their transverse direction at finite temperature, due to absorption of thermally excited \( L \)-particles. Nevertheless, the one-dimensional particles still have strongly anisotropic motion since they can only freely move along one direction. Motion along the other direction is a statistical process analogous to a random walk, occurring only upon the absorption of \( L \)-particles. This justifies our continued use of the term “one-dimensional particle” at finite temperature.

At the \( \kappa = 0 \) critical point, we have found that the system undergoes a finite-temperature phase transition corresponding to unbinding of one-dimensional vortices. Away from \( \kappa = 0 \), however, we also expect a BKT unbinding transition of the conventional superfluid vortices at some other critical temperature \( T_{c1} \), and the interplay of these two transitions is not immediately obvi-
Figure 5.4: The EBC quantum critical point between two conventional Bose condensates gives rise to a finite temperature EBC phase. For small nonzero $|\kappa|$, the EBC exists as an intermediate phase between the superfluid and disordered phases.

In order to build the picture of the overall phase diagram, together with $T_{c2}$, we also estimate the unbinding temperature $T_{c1}$ for conventional vortices. The same argument gives the free energy per vortex as $F \sim (|\kappa|L^2 - T) \log \ell$, where the energy cost for single vortex is $|\kappa|L^2 \log \ell$, where $L$ is fundamental charge of an $L$-particle. And the critical temperature behaves as:

$$T_{c1} \sim |\kappa|L^2$$

leading to a sharp suppression of $T_{c1}$ in the vicinity of the $\kappa = 0$ critical point. In contrast, the finite-temperature EBC phase has $T_{c2}$ almost independent of $\kappa$, remaining $Kq^2$. In Appendix 5.10.2, we also show that as long as the $L$-particles proliferate, the confined one-dimensional particles become logarithmically interacting. Therefore, their proliferation at $T_{c2}$ is unaffected by the non-zero $T_{c1}$. This leads to the phase diagram depicted in Fig. 5.4. For large $|\kappa|$ we recover the expected direct transition between the Bose condensate and the normal phase. In the vicinity of the critical point, however, (specifically for $|\kappa| < K(q/L)^2$) the system will undergo two phase transitions as the temperature is raised from zero, passing through a new intermediate finite-temperature phase. At $T_{c1}$, the conventional vortices proliferate, and the condensate of bosons is destroyed. However, even in the absence of condensation of the fundamental bosons, the excitons can remain condensed, leaving the system in a finite-temperature EBC phase. It is only at the higher temperature $T_{c2}$ that the one-dimensional vortices proliferate and the exciton condensate is destroyed, giving way
properties of the true disordered phase.

5.5.2.2 Properties of the Exciton Bose Condensate

Having established the existence of a new finite-temperature phase of bosons, we now describe some of its properties. This phase is characterized by unproliferated one-dimensional vortices, indicating that exciton condensation is still present at finite temperature. To see this explicitly, we repeat our calculation of correlation functions at finite temperature, where thermal fluctuations dominate quantum effects. As such, we calculate correlation functions based on the classical free energy:

\[ F = \beta \int d^2x \frac{K}{\beta} \left( \partial_i \partial_j \phi \right)^2 \]  

(5.51)

The phase correlator is then given by:

\[ \langle \phi(x) \phi(0) \rangle_\beta \sim \int d^2k \frac{e^{ik \cdot x}}{\beta K k^4} \sim -\frac{T}{K} r^2 \log r \]  

(5.52)

and the boson correlation function is:

\[ \langle e^{i\phi(x)} e^{-i\phi(0)} \rangle_\beta \sim e^{-\frac{T}{K} r^2 \log r} \]  

(5.53)

which decays exponentially, indicating the destruction of the boson condensate, as expected. In contrast, the corresponding exciton correlation function behaves as:

\[ \langle e^{i\partial^i \phi(x)} e^{-i\partial^i \phi(0)} \rangle_\beta \sim e^{-\frac{T}{\pi} \log r} \sim \frac{1}{r^\eta} \]  

(5.54)

where \( \eta = T/(4\pi K) \). We see that, at any finite temperature, the exciton condensate still exhibits quasi-long-range order contributed by the non-singular part of the field \( \phi \). By including the effect of vortices, this power-law correlation only persists until \( T_{c2} \), at which point the one-dimensional vortices unbind and the condensate will be completely destroyed, resulting in exponential decay of all correlation functions.

In addition to correlations functions, we can also characterize the finite-temperature EBC phase by an unusual thermodynamic property. The low-temperature thermodynamics will be dominated by the quadratically dispersing gapless mode. Generically, the specific heat contribution
from a gapless mode scales as $C \sim T^{d/z}$, where $z$ is the dynamical critical exponent, $\omega \sim k^z$, and $d$ is the spatial dimension. In the present case, $d = 2$ and $z = 2$, allowing us to conclude:

$$C \sim T$$ (5.55)

in the EBC phase. Such a $T$-linear specific heat is more commonly associated with a Fermi (or Bose) surface, and provides a clear distinction from conventional superfluid phases, where $C \sim T^2$.

Finally, we note that the exciton condensate should not lead to dissipationless transport of any nontrivial quantum numbers besides energy. Motion of an exciton corresponds to motion of a particle-hole pair, which does not carry typical quantum numbers of the fundamental bosons, such as charge. Nevertheless, a particle-hole pair does carry energy, so we expect that the exciton condensate will lead to dissipationless heat transport in the system, as proposed in the context of electronic exciton condensates.[19]

5.6 Lattice Model

In the previous sections, we have always assumed that the critical theory arises from an underlying lattice system, in order to have a well-defined vortex of $\partial_i \phi$. In this section, we show how to put the critical theory and its dual tensor gauge theory on the honeycomb lattice, which can host a continuous phase transition.

![Terms in the critical Hamiltonian of boson $e^{i\phi}$ on the honeycomb lattice. \( \hat{\phi} = \frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y} \) and \( \hat{\phi} = -\frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y} \).

Figure 5.5: Terms in the critical Hamiltonian of boson $e^{i\phi}$ on the honeycomb lattice. \( \hat{\phi} = \frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y} \) and \( \hat{\phi} = -\frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y} \).

On the honeycomb lattice, the bosons $e^{i\phi}$ live on the sites. The boson current lives on the
links along three directions: $\hat{+} = \frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y}$, $\hat{-} = -\frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y}$ and $\hat{x}$. There are six distinct terms in the critical Hamiltonian listed in Fig. 5.5.

![Diagram showing the boson field $\phi$ lives on the sites of the honeycomb lattice. The three diagonal components of the tensor field $E_{ij}$ live on the sites of the triangular lattice (center of hexagons of the honeycomb lattice), while the off-diagonal components live on the links of the triangular lattice (links of the honeycomb lattice).](image)

Figure 5.6: The boson field $\phi$ lives on the sites of the honeycomb lattice. The three diagonal components of the tensor field $E_{ij}$ live on the sites of the triangular lattice (center of hexagons of the honeycomb lattice), while the off-diagonal components live on the links of the triangular lattice (links of the honeycomb lattice).

Based on this bosonic model, we can define the gauge variables on the dual triangular lattice. Notice that $\epsilon^{ij} \partial_k \phi(r)$ is the rank-1 dual current perpendicular to the boson current along $k$ direction, which lives on the dual links. Then $E_{ij} = \epsilon^{il} \partial_l \epsilon^{jk} \partial_k \phi(r)$ is the difference of this rank-1 dual current along $l$ direction and thus it is defined on the site of the dual lattice if $i = j$ or at the center of the rhombus made up from two triangular plaquettes. For example, $E_{xx} = \partial_y \epsilon_{xj} \partial_j \phi = \Delta_y J_y^{\text{dual}}$ is defined as the difference of two dual currents living on the successive $y$-links and it lives at the site of the triangular lattice. Meanwhile, $E_{+x} = E_{x+} = \frac{1}{2} (\Delta_+ J_+^{\text{dual}} + \Delta_y J_y^{\text{dual}})$ where the first term is the $y$-directed dual current difference along the direction perpendicular to $+$ (denoted by $\hat{+}$) and the second terms is the $y$-directed difference of current along the link perpendicular to $+$. Therefore, on the dual triangular lattice, there are three diagonal components of the tensor $E_{ii}$ living on the sites while three off-diagonal components living on the three types of links as shown in Fig. (5.6).

The Gauss’s law $\partial_i E_{ij} = \rho_j$ in the vector charge theory now corresponds to rhombus terms
as shown in Fig. (5.8). Each rhombus term is a summation of six variables around a direct lattice link where a vector charge lives, involving four off-diagonal variables and two diagonal variables whose repeating subscript is the same as the direction of the link. The operation $E_{ij} \to E_{ij} + 1$ for off-diagonal components creates four vector charges at once. The same operation for diagonal components creates two vector charges. These charge configurations are listed in Fig. (5.7).

Based on the charge pattern created by adding one to a single $E_{ij}$, we can immediately write down the gauge transformation for its conjugate $A_{ij}$. Accordingly, we can write down the gauge invariant $B = \epsilon_{ij} \epsilon_{kl} \partial_i \partial_k A_{jl}$ which involves 21 variables within the orange hexagon as shown in Fig. (5.9).

5.7 Exciton Bose Liquid

Throughout this work, we have discussed the rank-two tensor gauge theory of Eq. 5.3, and its dual scalar formulation in Eq. 5.2, as a quantum critical point, either between two different VBS phases or between a superfluid and a finite-momentum condensate. However, since there is only a single relevant direction at the critical point, it seems plausible that some small modification of the theory could eliminate the instability, resulting in a stable two-dimensional quantum phase of matter described by a tensor gauge theory. In this section, we will describe a mechanism which
can promote the critical tensor gauge theory to a stable quantum phase protected by a subsystem symmetry. Below, by stable phase, we mean the phase is stable under perturbations preserving the subsystem symmetry. (This phase was originally proposed to be stable against symmetry breaking perturbations as well[71], but this claim remains controversial.)

The isotropic critical theory can become a stable phase on the square lattice through a slight modification introducing anisotropy. Accounting for square lattice anisotropy, our previously encountered critical Hamiltonian can be written in the form:

$$H = K(\eta(\partial_x^2 \phi)^2 + \eta(\partial_y^2 \phi)^2 + 2(\partial_x \partial_y \phi)^2) + \frac{1}{2} n^2$$

(5.56)

Previously, the boson model at $\eta = 0$ on the square lattice was studied in the context of “exciton Bose liquid” (EBL) phases.[71] The simplest EBL phase is obtained with Hamiltonian:

$$H_{EBL} = K(\partial_x \partial_y \phi)^2 + \frac{1}{2} n^2$$

(5.57)

This theory is similar to the isotropic critical theory, in that the gapless gauge mode has a quadratic dispersion. Notably, however, this theory has two lines ($k_x = 0$ and $k_y = 0$) along which the dispersion vanishes exactly, i.e. a “Bose surface.” Note that the Hamiltonian is invariant under the transformation:

$$\phi \rightarrow \phi + f(x) + g(y)$$

(5.58)

where $f(x)$ and $g(y)$ are functions of only a single coordinate. This symmetry on $\phi$ implies the
Figure 5.9: The gauge invariant is a summation of twelve $A_{ij}$ variables including six off-diagonal components and six diagonal components. The minus sign in front of the variable indicates its sign in the summation for the gauge invariant.

The following conservation law on the conjugate variable $n$:

$$\int dx\ n(x, y) = \text{constant} \quad \int dy\ n(x, y) = \text{constant} \quad (5.59)$$

representing the conservation of boson number on each row and column of the lattice. Previous studies on this model have shown that, unlike the isotropic theory, due to this subsystem symmetry[128], single-derivative perturbations to the Hamiltonian are irrelevant, along with all other perturbations, within a certain parameter regime.[71] As such, the EBL describes a stable phase of matter, not a critical point, as long as the subsystem symmetry is preserved.

Just like the EBC quantum critical point, we can also capture the stable EBL phase with a “tensor” gauge dual, which is a simple repackaging of the previously studied self-duality transformation of this model.[71] We obtained the EBL Hamiltonian by dropping diagonal derivatives from the isotropic theory. Similarly, we can obtain an appropriate gauge dual for the EBL by dropping diagonal elements of the tensor gauge field from the isotropic theory. The resulting tensor will only have a single component, the off-diagonal element $A_{xy}$, with its conjugate $E_{xy}$, the resulting Hamiltonian takes the form:

$$H = KE_{xy}^2 + \frac{1}{2} (\partial_y \partial_x A_{xy})^2 \quad (5.60)$$
This theory is invariant under the pseudo-gauge transformation:

\[ A_{xy} \rightarrow A_{xy} + f'(x) + g'(y) \]  

(5.61)

where \( f'(x) \) and \( g'(y) \) are functions of a single coordinate, as before. Note that this is not strictly a true gauge transformation, since the gauge parameter cannot be varied independently at all points in space. Correspondingly, the “Gauss’s law” of the theory no longer has a local expression. Instead, we only have the integral equations:

\[ \int dx \, E_{xy} = \text{constant} \quad \int dy \, E_{xy} = \text{constant} \]  

(5.62)

over each row and column of the lattice, closely mirroring Eq. (5.59). A generic \( E_{xy} \) configuration obeying these conditions takes the form:

\[ E_{xy} = \partial_x \partial_y \phi \]  

(5.63)

where \( \phi \) is conjugate to \( n \equiv \partial_x \partial_y A_{xy} \). Making these replacements in the tensor gauge theory of Eq. 5.60, we obtain precisely the EBL Hamiltonian of Eq. 5.57. Note that this duality transformation simply exchanges the two terms of the Hamiltonian, swapping the scalar field \( \phi \) for a “pseudoscalar” \( E_{xy} \). We can then regard the EBL phase as being effectively self-dual.

5.8 Conclusions

In this work, we have initiated the study of quantum critical points described by tensor gauge theories featuring subdimensional particles. We first showed that a previously studied quantum critical point between two valence bond solids maps exactly onto such a tensor gauge structure. We further demonstrated that a deconfined tensor gauge theory can exist at a critical point between two conventional gauge theories, representing an entirely new type of deconfined quantum criticality. Such a critical theory naturally describes the transition between a superfluid and a finite-momentum condensate. Furthermore, this critical point gives rise to a new finite-temperature phase of bosons, corresponding to an exciton Bose condensate. Our work opens a new door in the field of deconfined
quantum criticality, allowing for future study of exotic quantum critical points featuring deconfined tensor gauge theories.
5.9 Duality in Reverse

In the main text, we showed how to map from the rank-two tensor gauge theory onto the critical theory of the VBS-VBS' transition. For completeness, we here show how to obtain the duality in the opposite direction, starting from the critical theory in terms of the \( \phi \) variable. For this purpose, it will be most convenient to work in the Lagrangian formalism. The action for the critical theory takes the form:

\[
S = \int d^2x dt \frac{1}{2} \left( (\partial_t \phi)^2 - K (\partial_i \partial_j \phi)^2 \right)
\]  

(Note that \( K \) here differs by a factor of 2 from the definition in the main text, chosen for convenience.) We now decompose the field \( \phi \) into its smooth and singular pieces as \( \phi = \tilde{\phi} + \phi^{(s)} \), where \( \tilde{\phi} \) is a smooth single-valued function, and \( \phi^{(s)} \) is the singular contribution from vortices. For a system of normal superfluid vortices, the singular piece obeys:

\[
\epsilon^{ij} \partial_i \partial_j \phi^{(s)} = \rho
\]

with vortex density \( \rho \). For a system featuring the unconventional one-dimensional vortices discussed in the text (see Eq. 5.40), \( \phi \) will obey a modified source equation:

\[
\epsilon^{ij} \epsilon_{\ell k} \partial_i \partial_k \partial_j \phi^{(s)} = \rho^j
\]

where \( \rho^j \) is the vector charge density of the one-dimensional vortices. When these one-dimensional vortices are confined to bound states, such that only conventional vortices are present in the system,
this source equation will reduce to Eq. 5.65. We now introduce two Hubbard-Stratonovich fields, a scalar $B$ and a symmetric tensor $\chi_{ij}$, in terms of which we write the action as:

$$S = \int d^2 x dt \left( \frac{1}{2K} \chi^{ij} \chi_{ij} - \frac{1}{2} B^2 + \chi^{ij} \partial_i \partial_j \phi - B \partial_t \phi \right)$$

(5.67)

The action is now linear in the smooth function $\tilde{\phi}$, which can be integrated out, yielding the constraint:

$$\partial_t B + \partial_i \partial_j \chi^{ij} = 0$$

(5.68)

It is now useful to introduce the rotated field $E^{ij} = \epsilon^{ik} \epsilon^{j\ell} \chi_{k\ell}$, in terms of which the constraint:

$$\partial_t B + \epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j E_{k\ell} = 0$$

(5.69)

takes the form of the generalized Faraday’s equation of the two-dimensional vector charge theory.[79]

The general solution to this equation can be written in terms of two potential functions, a symmetric tensor $A_{ij}$ and a vector $\xi_i$:

$$B = \epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j A_{k\ell}$$

(5.70)

which is invariant under transformation $A_{k\ell} \rightarrow A_{k\ell} + \partial_k \lambda_\ell + \partial_\ell \lambda_k$ and

$$E^{ij} = -\partial_t A^{ij} - (\partial^i \xi^j + \partial^j \xi^i)$$

(5.71)

We can then write the action in the form:

$$S = \int d^2 x dt \left( \frac{1}{2K} E^{ij} E_{ij} - \frac{1}{2} B^2 - \rho^i \xi_i - J^{ij} A_{ij} \right)$$

(5.72)

where $J^{ij} = \epsilon^{ik} \epsilon^{j\ell} (\partial_i \partial_j \partial_k - \partial_k \partial_i \partial_j) \phi$ is a tensor current of the one-dimensional vortices. The action is now in precisely the form of the Lagrangian formulation of the two-dimensional vector charge theory[81, 79, 80], with the one-dimensional vortices playing the role of the vector charges. This action leads to one gapless gauge mode with quadratic dispersion, $\omega \sim k^2$, as expected from our original model. This completes the derivation of the duality between the critical theory of the VBS-VBS' transition and the two-dimensional vector charge tensor gauge theory.
5.10 Finite-Temperature Screening

In the main text, we established the electrostatic properties of isolated particles at the critical point. In particular, we found a logarithmic interaction potential between the one-dimensional particles. This hints that the system should undergo a finite-temperature phase transition at which the one-dimensional particles unbind. Unlike a system of normal logarithmically interacting particles, however, the tensor gauge theory also exhibits nontrivial bound states, namely the $L$-particles. These bound states have only a finite energy cost and therefore proliferate at arbitrarily low temperatures. Previous studies of three-dimensional fracton models\cite{78} have indicated that screening by a thermal bath of nontrivial bound states can often significantly modify the interactions between fundamental particles. We must therefore check carefully whether or not the logarithmic energy cost survives screening by the thermal bath of $L$-particles. The calculation will proceed in Appendix 5.10.1 as a straightforward extension of the screening analysis of Ref. [78] to the critical two-dimensional tensor gauge theory.

When $\kappa \neq 0$ in Eq. 5.24, the one dimensional particles are confined as we studied in Sec. 5.3. Meanwhile the two dimensional particles have logarithmical interaction. This case corresponds to the condensates at two sides of the critical point. Additionally, at finite temperature, previous study\cite{24} shows that the irrelevant perturbations, together with the temperature, would also generate correction to relevant terms, i.e. making the effective $\kappa$ finite although we tune it to zero to reach the critical point. In these cases, the one-dimensional particles are subject to interaction proportional to $r^2$ where $r$ is the separation between two of them. We are able to show in Appendix 5.10.2 that this strong confinement would be reduced to logarithmical interaction as long as the two dimensional particles condensate at any finite temperature.

5.10.1 Logarithmically interacting one dimensional particle

We showed earlier that the bare potential of an isolated vector charge $q^i$ takes the form:

$$\xi^i_{bare} = \frac{1}{8\pi} \left( 3(\log r)q^i - \frac{(q \cdot r)r^i}{r^2} \right)$$

(5.73)
In the presence of a screening cloud of L particles, however, the total potential surrounding a single vector charge will be modified to:

$$\xi^i[r] = \xi_{\text{bare}}^i[r] + \int d^2r' n_L(\xi^i[r'], T) \xi_i^i [r - r']$$

(5.74)

where $n_L(\xi^i, T)$ is the local density of L-particles at temperature $T$ and potential $\xi^i$, to be determined self-consistently, and $\xi_i^i(L)$ is the potential of an L-particle, from Eq. 5.43. We now assume that there is some finite thermal background density $n_0$ of the L-particles. These particles see an effective potential given by $L(\epsilon^{ij} \partial_i \xi_j)$. Therefore, in the presence of a potential $\xi^i$, the density will shift to:

$$n_L = n_0 e^{-\beta L(\epsilon^{ij} \partial_i \xi_j)} \approx n_0 (1 - \beta L(\epsilon^{ij} \partial_i \xi_j))$$

(5.75)

where $\beta = 1/T$, and we have assumed that the potential $\xi^i$ is small. (This assumption breaks down very close to the point charge, but will capture the correct long-distance physics.) Using this form of the density, we obtain:

$$\xi^i[r] = \xi_{\text{bare}}^i[r] - \frac{L^2}{4\pi} n_0 \beta \int d^2r' (\epsilon^{ij} \partial_i \xi_j[r']) \xi_i^i [r - r']$$

(5.76)

where $\xi_i^i[R] = \frac{\epsilon_{ik} R_k}{R^2}$. Taking a Fourier transform and solving for $\xi^i$, we obtain:

$$\xi^i = \left( \delta^{ij} - \frac{L^2 n_0 \beta}{8\pi^2 + L^4 n_0 \beta} \frac{\epsilon^{jk} k \epsilon_{ik} k_k}{k^2} \right) \xi_{\text{bare}}^i$$

(5.77)

In the limit of strong screening, $n_0 \to \infty$, the transverse component of the potential becomes completely projected out, leaving us with:

$$\xi^i = \left( \frac{k_i k_j}{k^2} \right) \xi_{\text{bare}}^i$$

(5.78)

With

$$\xi_{\text{bare}}^i = \frac{q_i}{k^2} - \frac{(k \cdot q) k_i}{2k^4},$$

(5.79)

we get $\xi^i = \frac{(k \cdot q) k^i}{2k^4}$. In real space, the potential corresponds to:

$$\xi^i = \frac{1}{8\pi} \left( (\log r) q^i + \frac{(q \cdot r) r^i}{r^2} \right)$$

(5.80)
From this equation, we can see the survival of the logarithmic behavior of the bare potential. We conclude that, even after accounting for screening by thermal L-particles, the one-dimensional particles still have a logarithmic interaction potential, leaving the finite-temperature unbinding transition intact. Furthermore, the composite object made up of the one-dimensional particle plus its screening cloud is still carrying a nonzero vector charge, indicating that the screened particle remains one-dimensional.

5.10.2 Confined interacting one dimensional particle

When the one-dimensional particles are confined. They are subjected to an interaction $\varepsilon_i$ proportional to $r^2$. (Note that the interaction energy is no longer equivalent to $\xi_i$ when $\kappa \neq 0$.) In the momentum space, it is proportional to $\varepsilon_{\text{bare}}^i \sim \kappa q_i/k^4$. Via a similar analysis to Eq. 5.76 with the L-particles now logarithmically interacting, i.e. $\partial_i \varepsilon_L^i \sim \log r$, it is straightforward to get the screened interaction of $d=1$ particles to be:

$$\varepsilon^j \sim k^2 \varepsilon_{\text{bare}}^i \sim \kappa q_i/k^4 \sim \kappa q_i \log(r/a) \quad (5.81)$$

From this calculation, we find that due to the proliferation of two-dimensional particles, the one-dimensional particles are no longer strongly confined and instead they interact logarithmically. Therefore, they can proliferate at finite temperature using Kosterlitz-Thouless criterion. When the proliferation temperature of one-dimensional particles is larger than that of the two-dimensional particles, which is true for small $\kappa$, the system will host an exciton Bose condensate phase.
Chapter 6

Summary and discussion

To summarize, in this thesis, we studied the mechanisms for the fracton phases, including fracton topological order and gapless U(1) fracton phase. We also discussed about their relation with ordinary topological order. Particularly, the fracton topological order can be constructed by coupling topological ordered layers through particle-loop condensation. Moreover, the U(1) fracton phase can turn into copies of ordinary topological order by Higgsing. The Higgs mechanism changes the immobile fracton into mobile excitations. Then, partial confinement transition gives constraints on the excitations of the resulting copies of topological order leading to fracton and other sub-dimensional excitations. Finally, we illustrated the possibility of observing sub-dimensional excitation, particularly, one-dimensional excitation, in reality.

As mentioned, it is natural to ask about the physics of the fracton topological order in the type II model, such as Haah’s code. In fact, the nature of this fracton topological order is still unclear. Due to the complicated dependence on the system size of its ground state degeneracy, it is very hard to propose a coupled layer construction. Also, we have not found a higher rank symmetric tensor gauge theory with all immobile gapped excitations. On a hypercubic lattice, it seems that we can always find a bound state which is mobile in sub-dimensional space. Therefore, the type II model with fractal operator is still mysterious. Current approach of understanding fracton order is unlikely to decode those models, which urges us to invent new approaches or consider other ingredients, such as lattice symmetries, determining the nature of the order.

Another interesting perspective of studying fracton phase is to study its entanglement prop-
erty. As we illustrated before, the fracton topological order has a long range entangled ground state which has an exotic correction on the entanglement entropy depending on system size. If we study the entanglement property further, we can find even more exotic properties. The branching structure generated by entanglement renormalization is an example. Both type I and type II models have this structure [30, 101, 100]. It may have some indication about the universal property of the fracton orders. But so far, it is not clear what the universal property is.

Lack of field theory in the continuum limit, it is hard to find the universal behavior of fracton phases. The main problem comes from the difficulty of doing ordinary renormalization to the fracton model and the symmetric tensor gauge theories. If we exclude the charge sector of the tensor gauge theory, then the renormalization process is well defined and there is a renormalization group fixed point in the low energy limit described by the gapless photon mode. But the essence of these theory, i.e., the fracton charge excitations, cannot be included compatibly. With the progress so far [30, 101, 100], we need a better understanding for the new renormalization scheme on the fracton physics.
Bibliography


